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A NOTE ON HERMITE–FEJÉR INTERPOLATION AT LAGUERRE ZEROS

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Abstract. In order to approximate functions defined on the real semiaxis, we introduce a new operator of Hermite–Fejér-type based on Laguerre zeros and prove its convergence in weighted uniform metric.

Keywords: Hermite–Fejér operator, weighted polynomial approximation, orthogonal polynomials, Laguerre zeros, real semiaxis.

MCS classification (2000): 41A05, 41A10.

1. Introduction and main results

The Lagrange or Hermite–Fejér interpolation based on the zeros of Laguerre polynomials has been considered in the literature by G. Szegő [11] and J. Szabados [10], who studied the uniform convergence of this interpolation process under proper hypotheses on the function (see also [6]).

Here we introduce a new operator of Hermite–Fejér-type, which is a slight modification of the one considered by the previous authors, and prove a uniform convergence theorem.

In the sequel $c, \mathcal{C}$ will stand for positive constants which can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ when $\mathcal{C}$ is independent of $a, b, \ldots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$. Finally, we will denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m. As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, will stand for the sets of all natural, integer, real numbers, while $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote the sets of positive integer and positive real numbers, respectively.

Let

$$w(x) = x^{\alpha} \mathrm{e}^{-x^{\beta}}, \quad \alpha > -1, \; \beta > 1/2, \qquad x > 0,$$

be a Laguerre-type weight and $\{p_m(w)\}_{m \in \mathbb{N}}$ the related sequence of orthonormal polynomials with positive leading coefficient. Let us denote by $x_k = x_{m,k}(w)$ the zeros of $p_m(w)$, located as follows [8]

$$\mathcal{C} \frac{a_m}{m^2} < x_1 < x_2 < \ldots < x_m < a_m \left(1 - \frac{\mathcal{C}}{m^{2/3}} \right), \tag{1.1}$$

where $a_m \sim m^{1/\beta}$ is the Mhaskar–Rakhmanov–Saff number related to $\sqrt{w}$ (see, e.g., [8]).

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Using an idea due to J. Szabados, we define the Hermite–Fejér polynomial based on these nodes and the extra point $x_{m+1} := a_m$ as follows

$$F_m(w,f,x) = \sum_{k=1}^{m+1} \ell_k^2(x) v_k(x) f(x_k)\,, \qquad x \geq 0$$

where f is a continuous function on $(0,\infty)$,

$$v_k(x) = 1 - 2\ell_k'(x_k)(x - x_k)\,,$$

$$\ell_k(x) = \frac{p_m(w,x)}{p_m'(w,x_k)(x-x_k)} \frac{a_m - x}{a_m - x_k}\,, \qquad k = 1,2,\ldots,m\,,$$

and

$$\ell_{m+1}(x) = \frac{p_m(w,x)}{p_m(w,a_m)}\,.$$

Let $\theta \in (0,1)$ be fixed, we define the index $j = j(m)$ as

$$x_j = \min_{1\leq k\leq m} \{x_k \ : \ x_k \geq \theta a_m\}$$

and denote by χ_j the characteristic function of the interval $[0, x_j]$. So, by using a procedure similar to that in [9] for Lagrange interpolation, we introduce the Hermite–Fejér-type operator $F_m^*(w)$ by

$$F_m^*(w,f,x) = F_m(w,\chi_j f, x) = \sum_{k=1}^{j} \ell_k^2(x) v_k(x) f(x_k)\,.$$

$F_m^*(w,f)$ is a polynomial of degree at most $2m+1$ and by definition we have

$$F_m^*(w,f,x_k) = \begin{cases} f(x_k)\,, & k = 1,2,\ldots,j\,; \\ 0\,, & k = j+1,\ldots,m+1\,. \end{cases}$$

Let us now introduce a couple of function-spaces associated to the weights

$$u(x) = x^\gamma \mathrm{e}^{-x^\beta}\,, \quad \beta > 1/2\,,\ \gamma \geq 0\,, \quad x > 0$$

and

$$\bar{u}(x) = \log(2+x) u(x)\,.$$

With $C^0(0,\infty)$ the set of all continuous functions on $(0,\infty)$, we consider the spaces

$$C_u = \left\{ f \in C^0(0,\infty) : \lim_{x\to 0} f(x)u(x) = \lim_{x\to\infty} f(x)u(x) = 0 \right\}$$

with norm

$$\|f\|_{C_u} = \sup_{x\in(0,\infty)} |f(x)u(x)| =: \|fu\|$$

and

$$C_{\bar{u}} = \left\{ f \in C^0(0,\infty) : \lim_{x\to 0} f(x)\bar{u}(x) = \lim_{x\to\infty} f(x)\bar{u}(x) = 0 \right\}$$

with norm

$$\|f\|_{C_{\bar{u}}} = \sup_{x\in(0,\infty)} |f(x)\bar{u}(x)| =: \|f\bar{u}\| .$$

Obviously $C_{\bar{u}} \subset C_u$.

In order to introduce the $r-$th modulus of smoothness in $C_{\bar{u}}$, proceeding as in [7], we define

$$\Omega_\varphi^r(f,t)_{\bar{u}} = \sup_{0<h\le t} \left\|\Delta_{h\varphi}^r (f)\, \bar{u}\right\|_{\mathcal{I}_h} ,$$

where $\mathcal{I}_h = \left[Ah^2, Ah^*\right]$, $A > 1$ is a fixed constant, $h^* = h^{-\frac{1}{\beta-1/2}}$

$$\Delta_{h\varphi}^r(f,x) = \sum_{k=0}^{r}(-1)^k \binom{r}{k} f\left(x + (r-k)h\varphi(x)\right)$$

and $\varphi(x) = \sqrt{x}$. Then we set

$$\omega_\varphi^r(f,t)_{\bar{u}} = \Omega_\varphi^r(f,t)_{\bar{u}} + \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)\,\bar{u}\|_{[0,At^2]}$$

$$+ \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)\,\bar{u}\|_{[At^*,\infty)}$$

Proceeding as in [7] we can easily prove that

$$E_m(f)_{\bar{u}} = \inf_{P_m\in\mathbb{P}_m} \|(f-P_m)\,\bar{u}\| \le \mathcal{C}\omega_\varphi^r\left(f, \frac{\sqrt{a_m}}{m}\right)_{\bar{u}} .$$

Considering $F_m^*(w)$ as a map from $C_{\bar{u}}$ into C_u, we can prove the following theorems.

Theorem 1. *If the parameters of the weights w and u satisfy*

$$0 \le \gamma - \alpha - \frac{1}{2} \le 1$$

then, for any function $f \in C_{\bar{u}}$, we have

$$\|F_m^*(w,f)\,u\| \le \mathcal{C}\|f\bar{u}\|_{[x_1,x_j]} ,$$

where $\mathcal{C} \ne \mathcal{C}(m,f)$ depends only on the parameters α, γ and θ.

Theorem 2. *Under the assumptions of Theorem 1, we get*

$$\|[f - F_m^*(w,f)\,u]\| \le \mathcal{C}\omega_\varphi\left(f, \frac{\sqrt{a_m}\log m}{m}\right)_{\bar{u}} + \mathcal{C}\mathrm{e}^{-cm}\|f\bar{u}\|$$

with $\mathcal{C} \ne \mathcal{C}(m,f)$ and $c \ne c(m,f)$ depending only on the parameters α, γ and θ.

For the sake of simplicity, we have considered the orthonormal system related to the weight $w(x) = x^\alpha \mathrm{e}^{-x^\beta}$. We can obtain similar results replacing w with a weight of the form $x^\alpha\mathrm{e}^{-Q(x)}$, where $\mathrm{e}^{-Q(x)}$ belongs to the class $\mathcal{F}(C^2+)$ introduced by Levin and Lubinsky (see [2, p.109] or [3, p.109]).

2. Proofs

of Theorem 1. Taking into account that (see, e.g., [8])

$$\|F_m^*(w,f)\,u\| = \|F_m^*(w,f)\,u\|_{I_m}\ ,$$

where $I_m = \left[\dfrac{a_m}{m^2}, a_m - c\dfrac{a_m}{m^{2/3}}\right]$, $c>0$, and for $x \in I_m$, letting x_d be a zero closest to x, we have

$$\ell_k^2(x)\frac{|v_k(x)|}{\log(2+x_k)}\frac{u(x)}{u(x_k)} \le \mathcal{C} \qquad k = d-1, d, d+1$$

whence

$$|F_m^*(w,f,x)|\,u(x) \le \mathcal{C}\,\|f\bar{u}\|_{[x_1,x_j]}\left\{1 + \sum_{\substack{1\,\le\, k\,\le\, j\\ k\,\ne\, d,\, d\,\pm\, 1}} \ell_k^2(x)\frac{|v_k(x)|}{\log(2+x_k)}\frac{u(x)}{u(x_k)}\right\}. \tag{2.1}$$

Using (see [8, p. 126])

$$|p_m(w,x)| \le \frac{\mathcal{C}}{\sqrt{w(x)\sqrt{x(a_m-x)}}}\,, \qquad x \in I_m\,, \tag{2.2}$$

$$\frac{1}{|p_m'(w,x_k)|} \sim \Delta x_k\sqrt{w(x_k)\sqrt{x_k(a_m-x_k)}}\,, \qquad x_1 \le x_k \le x_j\,, \tag{2.3}$$

where

$$\Delta x_k = x_{k+1} - x_k \sim \frac{\sqrt{a_m}}{m}\sqrt{x_k}\,, \qquad k = 1,2,\ldots,j\,, \tag{2.4}$$

for $k \ne d, d\pm 1$, by (1), we obtain

$$\begin{aligned}
\ell_k^2(x)\frac{u(x)}{u(x_k)} &= \left|\frac{p_m(w,x)}{p_m'(w,x_k)(x-x_k)}\frac{a_m-x}{a_m-x_k}\right|^2 \frac{w(x)}{w(x_k)}\left(\frac{x}{x_k}\right)^{\gamma-\alpha}\\
&\le \mathcal{C}\left(\frac{\Delta x_k}{x-x_k}\right)^2\left(\frac{x}{x_k}\right)^{\gamma-\alpha-1/2}\left(\frac{a_m-x}{a_m-x_k}\right)^{3/2}\\
&\le \mathcal{C}\left(\frac{\Delta x_k}{x-x_k}\right)^2\left(\frac{x}{x_k}\right)^{\gamma-\alpha-1/2}
\end{aligned}$$

and (2.1) becomes

$$\begin{aligned}
|F_m^*(w,f,x)|\,u(x) &\le \mathcal{C}\,\|f\bar{u}\|_{[x_1,x_j]}\left\{1 + \sum_{\substack{1\,\le\, k\,\le\, j\\ k\,\ne\, d,\, d\,\pm\, 1}} \left(\frac{\Delta x_k}{x-x_k}\right)^2\left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac{1}{2}}\frac{|v_k(x)|}{\log(2+x_k)}\right\}\\
&=: \mathcal{C}\,\|f\bar{u}\|_{[x_1,x_j]}\left\{1+\sigma(x)\right\}.
\end{aligned}$$

Let us now estimate the term

$$v_k(x) = 1 - 2\ell'_k(w, x_k)(x - x_k)\,.$$

We can write

$$\ell'_k(x) = \left(\frac{a_m - x}{a_m - x_k}\right)' \tilde{\ell}_k(x) + \left(\frac{a_m - x}{a_m - x_k}\right) \tilde{\ell}'_k(x)\,,$$

where $\tilde{\ell}_k$ are the fundamental Lagrange polynomials based on the nodes $x_1, x_2, \ldots, x_m$. Since

$$\tilde{\ell}'_k(x_k) = \frac{p''_m(w, x_k)}{p'_m(w, x_k)}$$

we have

$$v_k(x) = 1 + 2\left[\frac{1}{a_m - x_k} - \frac{p''_m(w, x_k)}{p'_m(w, x_k)}\right](x - x_k)\,.$$

In order to estimate $\frac{p''_m(w,x_k)}{p'_m(w,x_k)}$, we consider the generalized Freud weight $\bar{w}(x) = |x|^{2\alpha+1}\mathrm{e}^{-|x|^{2\beta}}$ and the associated orthonormal system $\{q_m(\bar{w})\}_m$. We denote by $\bar{x}_k = x_{m,k}(\bar{w})$ the zeros of $q_m(\bar{w})$ and by $\bar{a}_m = a_m(\sqrt{\bar{w}})$ the Mhaskar–Rahmanov–Saff number related to $\sqrt{\bar{w}}$. Since $q_{2m}(\bar{w}, x) = p_m(w, x^2)$ and $a^2_m(\sqrt{\bar{w}}) \sim a_m(\sqrt{w})$ (see [8]),

$$\frac{q''_{2m}(\bar{w}, x)}{q'_{2m}(\bar{w}, x)} = \frac{1}{x} + 2x\frac{p''_m(w, x^2)}{p'_m(w, x^2)}\,,$$

and so

$$\frac{p''_m(w, x^2)}{p'_m(w, x^2)} = \frac{q''_{2m}(\bar{w}, x)}{2xq'_{2m}(\bar{w}, x)} - \frac{1}{2x^2}\,,$$

from the inequality (see [1, Theorem 3.6 at p. 42])

$$\left|\frac{q''_{2m}(\bar{w}, x)}{q'_{2m}(\bar{w}, x)}\right| \le \mathcal{C}\left[\frac{|\bar{x}_k|}{a^2_m(\sqrt{\bar{w}})} + |x_k|^{2\beta-1} + \frac{1}{|\bar{x}_k|}\right]$$

we deduce

$$\left|\frac{p''_m(w, x_k)}{p'_m(w, x_k)}\right| \le \mathcal{C}\left[1 + x_k^{\beta-1} + \frac{1}{x_k}\right].$$

So we get

$$|v_k(x)| \le \mathcal{C}\left[1 + (1 + x_k)^{\beta-1}|x - x_k| + \frac{|x - x_k|}{x_k}\right]. \tag{2.5}$$

Let us estimate $\sigma(x)$, considering first the case $x > 2$. Setting

$$A_k(x) = \left(\frac{\Delta x_k}{x - x_k}\right)^2 \left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac{1}{2}} \frac{|v_k(x)|}{\log(2 + x_k)}$$

we can write

$$\begin{aligned}\sigma(x) &= \left\{\sum_{x_1\le x_k\le 1} + \sum_{1<x_k\le x_{d-2}} + \sum_{x_{d+2}\le x_k\le x_j}\right\} A_k(x)\\ &=: \sigma_1(x)+\sigma_2(x)+\sigma_3(x)\,.\end{aligned}$$

For $x_1 \le x_k \le 1$ from (2.5) we get $|v_k(x)| \le \mathcal{C}\frac{x}{x_k}$. Since $x > 2$, whence $x - x_k > \frac{x}{2}$, we have

$$A_k(x) \le \mathcal{C}\frac{x^{\gamma-\alpha+\frac12}}{x^2}\,\frac{\Delta x_k}{x_k}\,x_k^{-\gamma+\alpha+\frac12}\Delta x_k \le \mathcal{C}x_k^{-\gamma+\alpha+\frac12}\Delta x_k$$

using (1.1) and $1 \le \gamma-\alpha+\frac12 \le 2$. It follows that

$$\sigma_1(x) \le \mathcal{C}\int_0^1 t^{-\gamma+\alpha+\frac12}\,\mathrm{d}t\ = \mathcal{C}\,.$$

For $1 < x_k \le x_{d-2}$ from (2.5) we obtain

$$|v_k(x)| \le \mathcal{C}\left[1 + x_k^{\beta-1}(x-x_k)\right]$$

and then

$$\begin{aligned}A_k(x) &\le \mathcal{C}\left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac12}\frac{1+x_k^{\beta-1}(x-x_k)}{\log(2+x_k)}\left(\frac{\Delta x_k}{x-x_k}\right)^2\\ &\le \mathcal{C}\left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac12}\left(\frac{\Delta x_k}{x-x_k}\right)^2 + \mathcal{C}\left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac12}\frac{\Delta x_k\, x_k^{\beta-1}}{\log(2+x_k)}\frac{\Delta x_k}{x-x_k}\\ &\le \mathcal{C}\Delta x_d\left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac12}\frac{\Delta x_k}{(x-x_k)^2} + \frac{\mathcal{C}}{\log m}\left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac12}\frac{\Delta x_k}{x-x_k}\\ &=: A_k^*(x) + A_k^{**}(x)\end{aligned}$$

since, for $\beta > 1/2$, by (2.4)

$$\frac{\Delta x_k\, x_k^{\beta-1}}{\log(2+x_k)} \sim \frac{x_k^{\beta-1/2}}{\log(2+x_k)}\frac{\sqrt{a_m}}{m} \le \frac{\mathcal{C}}{\log m}\frac{a_m^\beta}{m} \sim \frac{1}{\log m}\,.$$

It follows that

$$\begin{aligned}\sigma_2(x) &\le \mathcal{C}\sum_{1<x_k\le x_{d-2}}\left(A_k^*(x)+A_k^{**}(x)\right)\\ &\le \mathcal{C}\Delta x_d\int_1^{x-\Delta x_d}\left(\frac{x}{t}\right)^{\gamma-\alpha-\frac12}\frac{\mathrm{d}t}{(x-t)^2} + \frac{\mathcal{C}}{\log m}\int_1^{x-\Delta x_d}\left(\frac{x}{t}\right)^{\gamma-\alpha-\frac12}\frac{\mathrm{d}t}{x-t}\,.\end{aligned}$$

The first integral, with $t = xy$, is equal to

$$\begin{aligned} \frac{\Delta x_d}{x}\int_{1/x}^{1-\Delta x_d/x} y^{-\gamma+\alpha+\frac12}\frac{\mathrm{d}y}{(1-y)^2} &\le \mathcal{C}\frac{\Delta x_d}{x}\left[\int_0^{1/2} y^{-\gamma+\alpha+\frac12}\,\mathrm{d}y + \int_{1/2}^{1-\Delta x_d/x}\frac{\mathrm{d}y}{(1-y)^2}\right] \\ &\le \mathcal{C}\frac{\Delta x_d}{x}\left[1+\frac{x}{\Delta x_d}\right] \le \mathcal{C}\,. \end{aligned}$$

Using the same substitution, the second integral is dominated by

$$\frac{\mathcal{C}}{\log m}\int_{1/x}^{1-\Delta x_d/x} y^{-\gamma+\alpha+\frac12}\frac{\mathrm{d}y}{1-y} \le \frac{\mathcal{C}}{\log m}\left[\int_0^{1/2} y^{-\gamma+\alpha+\frac12}\,\mathrm{d}y + \int_{1/2}^{1-\Delta x_d/x}\frac{\mathrm{d}y}{1-y}\right] \le \mathcal{C}\,.$$

Finally, if $x_{d+2} \le x_k \le x_j$, from (2.5) we get again $|v_k(x)| \le \mathcal{C}\left[1 + x_k^{\beta-1}(x-x_k)\right]$. Hence

$$A_k(x) \le \mathcal{C}\left(\frac{\Delta x_k}{x-x_k}\right)^2 + \mathcal{C}\frac{x_k^{\beta-1}\Delta x_k}{\log(2+x_k)}\frac{\Delta x_k}{x_k-x}$$

and

$$\sigma_3(x) \le \mathcal{C}\sum_{x_{d+2}\le x_k\le x_j}\left(\frac{\Delta x_k}{x-x_k}\right)^2 + \frac{\mathcal{C}}{\log m}\sum_{x_{d+2}\le x_k\le x_j}\frac{\Delta x_k}{x_k-x} \le \mathcal{C}\,.$$

Then, for $2 < x \le \mathcal{C}a_m$, we have

$$|F_m^*(w,f,x)|\,u(x) \le \mathcal{C}\,\|f\bar{u}\|_{[x_1,x_j]}\,.$$

Let us now consider the case $\frac{a_m}{m^2} \le x \le 2$. We can write

$$\begin{aligned} \sigma(x) &= \left\{\sum_{x_1\le x_k\le\frac{x}{2}} + \sum_{\frac{x}{2}<x_k\le x_{d-2}} + \sum_{x_{d+2}\le x_k\le 2x} + \sum_{2x<x_k\le x_j}\right\}A_k(x) \\ &=: \sigma_1(x)+\sigma_2(x)+\sigma_3(x)+\sigma_4(x)\,. \end{aligned}$$

For $x_1 \le x_k \le \frac{x}{2}$, since $v_k(x) \le \mathcal{C}\frac{x}{x_k}$ and

$$\frac{|v_k(x)|}{\log(2+x_k)}\cdot\frac{\Delta x_k}{x-x_k} \le \mathcal{C}\,,$$

we get

$$\begin{aligned} \sigma_1(x) &\le \mathcal{C}\sum_{x_1\le x_k\le\frac{x}{2}}\left(\frac{x}{x_k}\right)^{\gamma-\alpha-\frac12}\frac{\Delta x_k}{x-x_k} \\ &\sim \int_0^{x/2}\left(\frac{x}{t}\right)^{\gamma-\alpha-\frac12}\frac{\mathrm{d}t}{x-t} \\ &= \int_0^{1/2} y^{-\gamma+\alpha+\frac12}\frac{\mathrm{d}y}{1-y} = \mathcal{C}\,. \end{aligned}$$

For $\frac{x}{2} < x_k \le x_{d-2}$ we have $|v_k(x)| \le \mathcal{C}$ and $x \sim x_k$, whence

$$\sigma_2(x) \le \mathcal{C} \sum_{\frac{x}{2} < x_k \le x_{d-2}} \left(\frac{\Delta x_k}{x - x_k} \right)^2 = \mathcal{C}\,.$$

For $x_{d+2} \le x_k \le 2x$ we get $|v_k(x)| \le \mathcal{C}$ and, proceeding as for $\sigma_2(x)$, we get

$$\sigma_3(x) \le \mathcal{C}\,.$$

For $2x \le x_k \le x_j$, since $|v_k(x)| \le \mathcal{C}(x_k - x)x_k^{\beta-1}$ and $x_k - x \ge \frac{x_k}{2}$, it follows that

$$A_k(x) \le \mathcal{C} \frac{x_k^{\beta-1} \Delta x_k}{\log(2 + x_k)} \frac{\Delta x_k}{x_k} \le \frac{\mathcal{C}}{\log m} \frac{\Delta x_k}{x_k}$$

and then

$$\sigma_4(x) \le \mathcal{C}$$

which completes the proof. $\square$

In order to prove Theorem 2 we need some preliminary results. We recall that if g is a continuous function having a continuous derivative, the Hermite polynomial based on the nodes $x_1, x_2, \ldots, x_{m+1}$ can be written as

$$\begin{aligned} H_{2m}(w, g, x) &= F_m(w, g, x) + \sum_{k=1}^{m} \ell_k^2(x)\,(x - x_k)\,g'(x_k) \\ &=: F_m(w, g, x) + G_m(w, g, x)\,. \end{aligned}$$

Setting $G_m^*(w, g) = G_m(w, \chi_j g)$, we can prove the following lemma.

Lemma 3. *If the parameters of the weights w and u satisfy*

$$0 \le \gamma - \alpha - \frac{1}{2} \le 1$$

then, for any function $g \in C_u$ such that $\|g'\varphi u\| < \infty$, we have

$$\|G_m^*\,(w, g)\,u\| \le \mathcal{C} \frac{\sqrt{a_m}}{m} (\log m) \|g'\varphi u\|_{[0, x_j]}\,,$$

where $\mathcal{C} \ne \mathcal{C}(m, f)$.

Proof. Using (2.2), (2.3) and (2.4) we easily get

$$\begin{aligned} |G_m^*\,(w, g, x)|\,u(x) &\le \mathcal{C} \frac{\sqrt{a_m}}{m} \|g'\varphi u\| \left\{ 1 + \sum_{\substack{1 \le k \le j \\ k \ne d,\, d \pm 1}} \left(\frac{x}{x_k} \right)^{\gamma - \alpha - 1/2} \frac{\Delta x_k}{|x - x_k|} \right\} \\ &\sim \frac{\sqrt{a_m}}{m} (\log m) \|g'\varphi u\| \end{aligned}$$

for $x \in I_m = \left[\frac{a_m}{m^2}, a_m - c \frac{a_m}{m^{2/3}} \right]$. $\square$

We now set $N = \left\lfloor \frac{M}{\log M} \right\rfloor$, where $M = \left\lfloor \frac{\theta m}{1+\theta} \right\rfloor$, $0 < \theta < 1$, $N < M$, and prove the following lemma.

Lemma 4. *For any polynomial $P_N \in \mathbb{P}_N$ we have*

$$\|H_{2m}\left(w, (1-\chi_j)P_N\right)u\| \le \mathcal{C}\mathrm{e}^{-cm}\|P_N u\|\,,$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$ and $c \neq c(m,f)$.

Proof. Taking into account that

$$H_{2m}\left(w, (1-\chi_j)P_N\right) = F_m\left(w, (1-\chi_j)P_N\right) + G_m\left(w, (1-\chi_j)P_N\right)\,,$$

we are going to prove only that

$$\|F_m\left(w, (1-\chi_j)P_N\right)u\| \le \mathcal{C}\mathrm{e}^{-cm}\|P_N u\|\,,$$

since the other term can be handled in a similar way and implies only a Bernstein inequality.

We have

$$\begin{aligned} |F_m\left(w, (1-\chi_j)P_N, x\right)|\,u(x) &\le \mathcal{C}\|P_N u\|_{[x_j,\infty)} \sum_{k=j+1}^{m+1} \ell_k^2(x)\frac{v(x_k)}{u(x_k)}u(x) \\ &\le \mathcal{C}m^\tau \|P_N u\|_{[\theta a_m,\infty)} \\ &\le \mathcal{C}m^\tau \mathrm{e}^{-cm}\|P_N u\| \le \mathcal{C}\mathrm{e}^{-cm}\|P_N u\| \end{aligned}$$

for some $\tau > 0$, having used (see, [3] or [8])

$$\|P_m u\|_{[sa_m,\infty)} \le \mathcal{C}\mathrm{e}^{-cm}\|P_m u\|\,, \qquad s > 1\,.$$

□

We are now able to prove Theorem 2.

of Theorem 2. For any polynomial $P_N \in \mathbb{P}_N$, where $N = \left\lfloor \frac{M}{\log M} \right\rfloor$, $M = \left\lfloor \frac{\theta m}{1+\theta} \right\rfloor$, $0 < \theta < 1$, we can write

$$\begin{aligned} f - F_m^*(w,f) &= f - P_N - F_m^*(w,f) + H_{2m}(w,P_N) \\ &= f - P_N - F_m^*(w, f-P_N) + G_m^*(w,P_N) + H_{2m}(w,(1-\chi_j)P_N)\,. \end{aligned}$$

Hence, using Theorem 1, we get

$$\|[f - F_m^*(w,f)]u\| \le \mathcal{C}\|(f-P_N)\bar{u}\|_{[x_1,x_j]} + \|G_m^*(w,P_N)u\| + \|H_{2m}(w,(1-\chi_j)P_N)u\|$$

whence, by Lemma 3 and Lemma 4, we obtain

$$\|[f - F_m^*(w,f)]u\| \le \mathcal{C}\left[\|(f-P_N)\bar{u}\|_{[x_1,x_j]} + \frac{1}{N}\|P_N'\varphi\bar{u}\|_{[x_1,x_j]} + \mathrm{e}^{-cm}\|P_N\bar{u}\|\right],$$

since $u \le \bar{u}$ and $\frac{\sqrt{a_m}}{m}(\log m) \le \frac{1}{N}$.

Taking the infimum on $P_N \in \mathbb{P}_N$ we have (see, [4, Theorem 3.5] for a similar argument)

$$\begin{aligned}\inf_{P_N \in \mathbb{P}_N} \left\{ (f - P_N)\bar{u}\|_{[x_1,x_j]} + \mathcal{C}\frac{\sqrt{a_N}}{N}\|P_N'\varphi\bar{u}\|_{[x_1,x_j]} \right\} \ &\sim \ \omega_\varphi\left(f, \frac{\sqrt{a_N}}{\sqrt{N}}\right)_{\bar{u}} \\ &\sim \ \omega_\varphi\left(f, \frac{\sqrt{a_m}}{m}(\log m)\right)_{\bar{u}}\end{aligned}$$

and $\|P_N\bar{u}\| \le 2\|f\bar{u}\|$, which completes the proof. □

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