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POLYNOMIAL APPROXIMATION WITH POLLACZECK–LAGUERRE WEIGHTS ON THE REAL SEMIAXIS. A SURVEY

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Abstract. The paper summarizes recent results on weighted polynomial approximation for functions defined on the real semiaxis, which can grow exponentially both at 0 and at $+\infty$: orthogonal polynomials, polynomial inequalities, function spaces with new moduli of smoothness, estimates for the best approximation, Gaussian rules and Lagrange interpolation with respect to the weight $w(x) = x^\gamma \mathrm{e}^{-x^{-\alpha}-x^\beta}$.

Keywords: orthogonal polynomials, weighted polynomial approximation, polynomial inequalities, Gaussian quadrature rules, Lagrange interpolation, Pollaczeck–Laguerre exponential weights.

MCS classification (2000): 41A05, 41A10, 41A17, 41A25, 65D05, 65D32.

1. Introduction

This paper is a short survey on weighted polynomial approximation of functions defined on the real semiaxis, which can grow exponentially both at 0 and at $+\infty$. As far as we know, this topic has received attention in the literature only recently (see [12, 13, 14, 15, 16]).

To this aim we consider weight functions of the form

$$w(x) = x^\gamma \mathrm{e}^{-x^{-\alpha}-x^\beta}, \qquad \alpha > 0,\ \beta > 1,\ \gamma \geq 0, \qquad x \in (0, +\infty). \tag{1.1}$$

Even if w can be seen as a combination of a Pollaczeck-type weight $\mathrm{e}^{-x^{-\alpha}}$ and a Laguerre-type weight $x^\gamma \mathrm{e}^{-x^\beta}$, one cannot investigate the problem reducing it to a combination of a Pollaczeck-type case (on $[0, 1]$, say) and a Laguerre-type case (on $[1, +\infty)$).

We are going to present the main results concerning orthogonal polynomials, polynomial inequalities, function spaces with new moduli of smoothness, estimates for the best polynomial approximation with respect to the weight w. We will also show due attention to Gaussian rules and Lagrange interpolation in weighted L^2 norm. The behaviour of the related Fourier sums and their discrete version, the Lagrange polynomials, in L^p norms remain an open problem.

In the sequel $c, \mathcal{C}$ will stand for positive constants which can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$, when $\mathcal{C}$ is independent of $a, b, \ldots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$. Finally, we will denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m. As

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usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, will stand for the sets of all natural, integer, real numbers, while $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote the sets of positive integer and positive real numbers, respectively.

2. Orthogonal polynomials

First of all we note that the weight w, defined by (1.1), can be reduced to a weight belonging to the class $\mathcal{F}(C^2+)$, introduced by Levin and Lubinsky in [7, pp. 7–8], by using a linear transformation. Let us recall the definition of this class for the reader's convenience.

Let $I = (c, d)$ be an interval, with $-\infty \le c < 0 < d \le +\infty$, and $\varrho : I \in \mathbb{R}$ be a weight function, with $\varrho = \mathrm{e}^{-Q}$, $Q : I \in [0, +\infty)$, satisfying the following properties:

(i) Q' is continuous in I and $Q(0) = 0$;
(ii) Q'' exists and is positive in $I \setminus \{0\}$;
(iii) $\lim_{x\to c^+} Q(x) = \lim_{x\to d^-} Q(x) = \infty$;
(iv) the function
$$T(x) = \frac{xQ'(x)}{Q(x)}\,, \qquad x \in I \setminus \{0\}\,,$$
is quasi-decreasing in $(c, 0)$ and quasi-increasing in $(0, d)$, with
$$T(x) \ge \Lambda > 1\,, \qquad x \in I \setminus \{0\}\,;$$
(v) there exist $C_1, C_2 > 0$ and a compact subinterval $J \subseteq I$, such that
$$\frac{Q''(x)}{|Q'(x)|} \le C_1 \frac{|Q'(x)|}{Q(x)}\,, \qquad \text{a.e.} \quad x \in I \setminus \{0\}\,,$$
and
$$\frac{Q''(x)}{|Q'(x)|} \ge C_2 \frac{|Q'(x)|}{Q(x)}\,, \qquad \text{a.e.} \quad x \in I \setminus J\,.$$

Then we say $\varrho \in \mathcal{F}(C^2+)$.

With the previous notation, we can state next lemma.

Lemma 1. *(see [16, pp. 817–818]). Letting w be the weight in (1.1), there exists a $\lambda > 0$ such that the weight $\widetilde{w}$ defined as*
$$\widetilde{w}(y) = \mathrm{e}^{-Q(y)}\,, \qquad y \in (-\lambda, +\infty)\,,$$
with
$$Q(y) = \frac{1}{(y+\lambda)^\alpha} + (y+\lambda)^\beta - \gamma \log(y+\lambda) - \lambda^{-\alpha} - \lambda^\beta + \gamma \log(\lambda)\,,$$
belongs to the class $\mathcal{F}(C^2+)$.

So, $w(y) = \mathcal{C}\widetilde{w}(y+\lambda)$, where λ is the unique positive zero of $q'(x) = -\alpha x^{-\alpha-1} + \beta x^{\beta-1} - \gamma x^{-1}$. Then we can deduce the properties of the orthogonal polynomials w.r.t. our weight w from the results obtained by Levin and Lubinsky, using the inverse transformation.

The Mhaskar–Rakhmanov–Saff (MRS) numbers related to $w(x) = \mathrm{e}^{-q(x)}$, with $q(x) = x^{-\alpha} + x^{\beta} - \gamma \log(x)$, are $\varepsilon_\tau = \varepsilon_\tau(w)$ and $a_\tau = a_\tau(w)$, are defined by

$$\tau = \frac{1}{\pi} \int_{\varepsilon_\tau}^{a_\tau} \frac{x q'(x)}{\sqrt{(a_\tau - x)(x - \varepsilon_\tau)}} \, \mathrm{d}x$$

and

$$0 = \frac{1}{\pi} \int_{\varepsilon_\tau}^{a_\tau} \frac{q'(x)}{\sqrt{(a_\tau - x)(x - \varepsilon_\tau)}} \, \mathrm{d}x \, .$$

Proposition 2. *(see* [16, pp. 820] *and* [7, p. 13]*).For $\tau > 0$, ε_τ is a decreasing function and a_τ is an increasing function of τ, and*

$$\lim_{\tau \to +\infty} \varepsilon_\tau = 0 \, , \qquad \lim_{\tau \to +\infty} a_\tau = +\infty \, ,$$

with

$$\varepsilon_\tau \sim \left(\frac{\sqrt{a_\tau}}{\tau} \right)^{\frac{1}{\alpha + 1/2}} \tag{2.1}$$

and

$$a_\tau \sim \tau^{1/\beta} \, . \tag{2.2}$$

Let us denote by $\{p_m(w)\}_{m \in \mathbb{N}}$ the sequence of the orthonormal polynomials defined by

$$p_m(w, x) = \gamma_m x^m + \text{ lower degree terms} \, , \qquad \gamma_m = \gamma_m(w) > 0 \, ,$$

and

$$\int_0^{+\infty} p_m(w, x) p_n(w, x) w(x) \, \mathrm{d}x = \delta_{m,n} \, .$$

The zeros of $p_m(w)$ lie in the MRS interval associated with $\sqrt{w}$. So, here and in the rest of the paper, we use the notation $\varepsilon_\tau = \varepsilon_\tau(\sqrt{w})$ and $a_\tau = a_\tau(\sqrt{w})$, taking into account that, by definition, $\varepsilon_\tau(\sqrt{w}) = \varepsilon_{2\tau}(w)$ and $a_\tau(\sqrt{w}) = a_{2\tau}(w)$. The next proposition provides further information concerning the distribution of these zeros.

Proposition 3. *(see* [14, pp. 1656–1657] *and* [7, pp. 312–324]*). The zeros of $p_m(w)$ are located as*

$$\varepsilon_m < x_1 < x_2 < \cdots < x_m < a_m \, ,$$

with

$$x_1 - \varepsilon_m \sim \delta_m \, , \qquad \delta_m \sim \left(\frac{\sqrt{a_m}}{m} \right)^{\frac{2}{3}\left(\frac{2\alpha+3}{2\alpha+1}\right)} \sim m^{-\frac{2}{3}\left(\frac{2\alpha+3}{2\alpha+1}\right)\left(1 - \frac{1}{2\beta}\right)} \, ,$$

and

$$a_m - x_m \sim a_m m^{-2/3} \sim m^{\frac{1}{\beta} - \frac{2}{3}} \, ,$$

where the constants in "$\sim$" are independent of m.

The distance between two consecutive zeros $\Delta x_k = x_{k+1} - x_k$ can be estimated by

$$\Delta x_k \sim \Psi_m(x_k) \, , \qquad k = 1, \ldots, m-1,$$

where

$$\Psi_m(x_k) = \frac{a_m\, x_k}{m\sqrt{(x_k - \varepsilon_m)(a_m - x_k)}}$$

and the constants in "$\sim$" are independent of k and m.

Now, letting $\theta \in (0,1)$ be fixed, we define two indexes $j_1 = j_1(m)$ and $j_2 = j_2(m)$ as

$$x_{j_1} = \max_{1\le k\le m} \{x_k \;:\; x_k \le \varepsilon_{\theta m}\} \quad \text{and} \quad x_{j_2} = \min_{1\le k\le m} \{x_k \;:\; x_k \ge a_{\theta m}\}\,. \tag{2.3}$$

For the sake of completeness, if $\{x_k \;:\; x_k \le \varepsilon_{\theta m}\}$ or $\{x_k \;:\; x_k \ge a_{\theta m}\}$ are empty, we set $x_{j_1} = x_1$ or $x_{j_2} = x_m$, respectively.

From Proposition 3, it follows that

$$\Delta x_k \sim \frac{\sqrt{a_m}}{m}\sqrt{x_k}\,, \qquad k = j_1, \ldots, j_2\,.$$

Let

$$\lambda_m(w,x) = \left(\sum_{k=0}^{m-1} p_m^2(w,x)\right)^{-1}$$

be the mth Christoffel function and

$$\lambda_k(w) = \lambda_m(w, x_k)\,, \qquad k = 1, \ldots, m\,,$$

be the Christoffel numbers related to w.

Proposition 4. *(see [7, p. 257]). We have*

$$\lambda_m(w,x) \sim \Psi_m(x) w(x)\,, \qquad x \in [\varepsilon_m, a_m]\,,$$

where Ψ_m is given by

$$\Psi_m(x) = \frac{a_m\, x}{m\sqrt{(x - \varepsilon_m + \delta_m)(a_m - x + a_m m^{-2/3})}}$$

and the constants in "$\sim$" are independent of m.

In particular, for $\theta \in (0,1)$, we get

$$\lambda_m(w,x) \sim \frac{\sqrt{a_m}}{m}\sqrt{x}\, w(x)\,, \qquad x \in [\varepsilon_{\theta m}, a_{\theta m}]\,,$$

From the numerical point of view, in order to compute the zeros of $p_m(w)$ and the Christoffel numbers, we use a procedure given in [14] and the MATHEMATICA package `OrthogonalPolynomials` (cf. [3] and [18]), which is freely downloadable from the Web Site: `http://www.mi.sanu.ac.rs/~gvm/`.

For the sake of brevity we omit the description of the numerical procedures for the computation of the zeros of $p_m(w)$, the Christoffel numbers and the Mhaskar–Rahmanov-Saff numbers ε_m and a_m. The interested reader can find all the details about these procedures in [14, pp. 1676–1680] (cf. [15]).

The following estimates are crucial tools in order to study the convergence of several approximation processes.

Proposition 5. *(see [7, pp. 325 and 360]). We have*

$$\sup_{x\in(0,+\infty)} |p_m(w,x)| \sqrt{w(x)} \sqrt[4]{|(a_m-x)(x-\varepsilon_m)|} \sim 1 \,,$$

$$\sup_{x\in(0,+\infty)} |p_m(w,x)| \sqrt{w(x)} \sim m^{\frac{1}{6}\left(1-\frac{1}{2\beta}\right)\left(\frac{2\alpha+3}{2\alpha+1}\right)} \,,$$

and

$$\frac{1}{|p_m'(w,x_k)| \sqrt{w(x_k)}} \sim \Delta x_k \sqrt[4]{(a_m-x)(x-\varepsilon_m)}$$

where the constants in "$\sim$" are independent of m.

Proposition 6. *(cfr. [7, p. 25]). For the leading coefficient of $p_m(w)$ we have*

$$\gamma_m = \frac{1}{\sqrt{2\pi}} \left(\frac{4}{a_m+\varepsilon_m}\right)^{m+\frac{1}{2}} \exp\left(\frac{1}{\pi}\int_{\varepsilon_m}^{a_m} \frac{q(x)}{\sqrt{(a_m-x)(x-\varepsilon_m)}}\,\mathrm{d}x\right)(1+o(1)) \,,$$

where $q(x) = \frac{1}{2}\left(x^{-\alpha} + x^{\beta} - \gamma \log x\right)$.

3. Polynomial inequalities

Letting w be given by (1.1), $x \in \mathbb{R}_+$, we introduce the weight function

$$u(x) = x^{\delta}\sqrt{w(x)}\,, \quad \delta \in \mathbb{R}\,. \tag{3.1}$$

In the sequel, by a slight abuse of notation, we denote by $\|\cdot\|_p$ the quasinorm of the L^p-spaces for $0 < p < 1$, defined in the usual way.

Lemma 7. *(see [16, p. 809]). Let $\delta \in \mathbb{R}$ and $n = m + \lceil|\delta|\rceil$. For any $P_m \in \mathbb{P}_m$, with $0 < p \le \infty$, we have*

$$\|P_m u\|_p \le \mathcal{C}\, \|P_m u\|_{L^p[\varepsilon_n, a_n]} \,,$$

where $\mathcal{C} \ne \mathcal{C}(m, P_m)$, and ε_n, a_n are defined by (2.1) and (2.2).

On the other hand, for any $s > 1$, we have

$$\|P_m u\|_{L^p(\mathbb{R}_+\setminus[\varepsilon_{sm}, a_{sm}])} \le \mathcal{C}\mathrm{e}^{-cm^{\nu}} \|P_m u\|_p \,,$$

where

$$\nu = \left(1 - \frac{1}{2\beta}\right)\frac{2\alpha}{2\alpha+1}\,, \tag{3.2}$$

and $\mathcal{C}$ and c are independent of m and P_m.

For the rest of the paper, let

$$\varphi(x) = \sqrt{x}\,.$$

Next lemma has interest in itself and gives rise to a useful procedure for proving polynomial inequalities.

Lemma 8. *(see [16, p. 809]). For a sufficiently large m (say $m \geq m_0$), there exists a polynomial $R_{\ell m} \in \mathbb{P}_{\ell m}$, with ℓ a fixed integer, such that*

$$R_{\ell m}(x) \sim w(x)$$

and

$$|R'_{\ell m}(x)|\, \varphi(x) \leq \mathcal{C} \frac{m}{\sqrt{a_m}}\, w(x)$$

for $x \in [\varepsilon_m, a_m]$, where $\varepsilon_m = \varepsilon_m(w)$ and $a_m = a_m(w)$ are defined by (2.1) and (2.2). The constants in "$\sim$" and $\mathcal{C}$ are independent of m.

By Lemmas 7 and 8 we reduce the problem of the polynomial inequalities related to the weight u on $(0, +\infty)$, to analogous inequalities on bounded intervals with Jacobi weights. In fact, we get:

Theorem 9. *(see [16, p. 810]). Let $0 < p \leq \infty$. Then, for any $P_m \in \mathbb{P}_m$, we have*

$$\|P'_m\, \varphi\, u\|_p \leq \mathcal{C} \frac{m}{\sqrt{a_m}}\, \|P_m\, u\|_p \tag{3.3}$$

and

$$\|P'_m\, u\|_p \leq \mathcal{C} \frac{m}{\sqrt{\varepsilon_m a_m}}\, \|P_m\, u\|_p\ , \tag{3.4}$$

where $\mathcal{C} \neq \mathcal{C}(m, P_m)$.

We want to emphasize that the presence of the algebraic factor x^δ in the definition of u allows us to iterate the Bernstein inequality (3.3) as follows

$$\left\|P_m^{(r)} \varphi^r u\right\|_p \leq \mathcal{C} \left(\frac{m}{\sqrt{a_m}}\right)^r \|P_m\, u\|_p\ ,$$

for $1 \leq r \in \mathbb{Z}$.

Also, the factor

$$\frac{m}{\sqrt{\varepsilon_m a_m}} \sim \left(\frac{m}{\sqrt{a_m}}\right)^{\frac{2\alpha+2}{2\alpha+1}} = \left(\frac{m}{\sqrt{a_m}}\right)^{1+\frac{1}{2\alpha+1}}\ .$$

in the Markoff inequality (3.4) is smaller than the one appearing in the analogous inequality (see [17])

$$\|P'_m\, w_\beta\|_p \leq \mathcal{C} \left(\frac{m}{\sqrt{a_m}}\right)^2 \|P_m\, w_\beta\|_p$$

with the generalized Laguerre weight $w_\beta(x) = \mathrm{e}^{-x^\beta}$ on $(0, +\infty)$. Whereas, the factors of the Bernstein inequalities for the weights u and w_β are the same.

Using standard arguments, the Markoff inequality (3.4) can be deduced from the Bernstein inequality (3.3) and the Schur inequality stated in the following theorem.

Theorem 10. *(see [16, p. 810]). Let $0 < p \le \infty$. Then, for any $P_m \in \mathbb{P}_m$, we have*

$$\|P_m\, u\|_p \le \mathcal{C} \left(\frac{m}{\sqrt{a_m}} \right)^{\frac{\delta}{\alpha+1/2}} \|P_m\, v_\delta\, u\|_p \ ,$$

where $v_\delta(x) = x^\delta$ and $\mathcal{C} \ne \mathcal{C}(m, P_m)$.

In analogy with the Bernstein and Markoff inequalities, we give two versions of the Nikolskii inequality.

Theorem 11. *(see [16, p. 810]). Let $0 < p < q \le \infty$. Then, for any $P_m \in \mathbb{P}_m$, we get*

$$\left\| P_m\, \varphi^{\frac{1}{p}-\frac{1}{q}}\, u \right\|_q \le \mathcal{C} \left(\frac{m}{\sqrt{a_m}} \right)^{\frac{1}{p}-\frac{1}{q}} \|P_m\, u\|_p \ , \tag{3.5}$$

and

$$\|P_m\, u\|_q \le \mathcal{C} \left(\frac{m}{\sqrt{\varepsilon_m a_m}} \right)^{\frac{1}{p}-\frac{1}{q}} \|P_m\, u\|_p \tag{3.6}$$

where $\mathcal{C} \ne \mathcal{C}(m, P_m)$.

In analogy with different weighted polynomial inequalities, the factor $m/\sqrt{\varepsilon_m a_m}$ in the second Nikolskii inequality is the same as the one appearing in the Markoff inequality.

4. Function spaces, K-functionals and moduli of smoothness

Let us now define some function spaces related to the weight u (see [13, pp. 168–172]). By L^p_u, $1 \le p < \infty$, we denote the set of all measurable functions f such that

$$\|f\|_{L^p_u} := \|fu\|_p = \left(\int_0^{+\infty} |fu|^p\,(x)\, \mathrm{d}x \right)^{1/p} < \infty \,,$$

while, for $p = \infty$, by a slight abuse of notation, we set

$$L^\infty_u = C_u = \left\{ f \in C^0(0, +\infty) : \lim_{x \to 0^+} f(x)u(x) = 0 = \lim_{x \to +\infty} f(x)u(x) \right\}$$

with the norm

$$\|f\|_{L^\infty_u} := \|fu\|_\infty = \sup_{x \in (0,+\infty)} |f(x)u(x)| \ .$$

For smoother functions we introduce the Sobolev-type spaces

$$W^p_r(u) = \left\{ f \in L^p_u : \ f^{(r-1)} \in AC(0, +\infty),\ \|f^{(r)}\varphi^r u\|_p < \infty \right\},$$

where $1 \le p \le \infty$, $1 \le r \in \mathbb{Z}^+$, $\varphi(x) := \sqrt{x}$ and $AC(0, +\infty)$ denotes the set of all absolutely continuous functions on $(0, +\infty)$. We equip these spaces with the norm

$$\|f\|_{W^p_r(u)} = \|fu\|_p + \|f^{(r)}\varphi^r u\|_p \,.$$

To characterize some subspaces of L^p_u, we introduce the following moduli of smoothness. Let us consider the intervals

$$\mathcal{I}_h(c) = \left[h^{1/(\alpha+1/2)}, \frac{c}{h^{1/(\beta-1/2)}}\right],$$

with α and β in (3.1), $h > 0$ sufficiently small, and $c > 1$ an arbitrary but fixed constant. For any $f \in L^p_u$, $1 \le p \le \infty$, $r \ge 1$ and $t > 0$ sufficiently small (say $t < t_0$), we set

$$\Omega^r_\varphi(f,t)_{u,p} = \sup_{0<h\le t} \left\| \Delta^r_{h\varphi}(f)\, u \right\|_{L^p(\mathcal{I}_h(c))},$$

where

$$\Delta^r_{h\varphi} f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f\left(x + (r-i)h\varphi(x)\right), \quad \varphi(x) = \sqrt{x}\,.$$

Moreover, we introduce the following K-functional

$$K(f,t^r)_{u,p} = \inf_{g\in W^p_r(u)} \left\{ \|(f-g)\,u\|_p + t^r \|g^{(r)}\varphi^r u\|_p \right\}$$

and its main part

$$\widetilde{K}(f,t^r)_{u,p} = \sup_{0<h\le t} \inf_{g\in W^p_r(u)} \left\{ \|(f-g)\,u\|_{L^p(\mathcal{I}_h(c))} + h^r \|g^{(r)}\varphi^r u\|_{L^p(\mathcal{I}_h(c))} \right\}.$$

The main part of the $K-$functional is equivalent to the main part of the previous modulus of smoothness, as the following lemma shows.

Lemma 12. *(see [13, p. 171]). Let $r \ge 1$ and $0 < t < t_0$ for some $t_0 < 1$. Then, for any $f \in L^p_u$, $1 \le p \le \infty$, we have*

$$\Omega^r_\varphi(f,t)_{u,p} \sim \widetilde{K}(f,t^r)_{u,p}$$

where the constants in "$\sim$" are independent of f and t.

Then we define the complete rth modulus of smoothness by

$$\begin{aligned} \omega^r_\varphi(f,t)_{u,p} &= \Omega^r_\varphi(f,t)_{u,p} + \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)\,u\|_{L^p(0,t^{1/(\alpha+1/2)}]} \\ &\quad + \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)\,u\|_{L^p[c\,t^{-1/(\beta-1/2)},+\infty)} \end{aligned}$$

with $c > 1$ a fixed constant. We emphasize that the behaviour of $\omega^r_\varphi(f,t)_{u,p}$ is independent of the constant c. Moreover, the following lemma shows that this modulus of smoothness is equivalent to the K-functional.

Lemma 13. *(see [13, p. 172]). Let $r \ge 1$ and $0 < t < t_0$ for some $t_0 < 1$. Then, for any $f \in L^p_u$, $1 \le p \le \infty$, we have*

$$\omega^r_\varphi(f,t)_{u,p} \sim K(f,t^r)_{u,p}\,,$$

where the constants in "$\sim$" are independent of f and t.

By means of the main part of the modulus of smoothness, for $1 \le p \le \infty$, we can define the Zygmund-type spaces

$$Z_s^p(u) = \left\{ f \in L_u^p : \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s} < \infty, \ r > s \right\},$$

$s \in \mathbb{R}^+$, with the norm

$$\|f\|_{Z_s^p(u)} = \|f\|_{L_u^p} + \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s}.$$

We remark that, in the definition of $Z_s^p(u)$, the main part of the rth modulus of smoothness $\Omega_\varphi^r(f,t)_{u,p}$ can be replaced by the complete modulus $\omega_\varphi^r(f,t)_{u,p}$, as can be deduced from Theorem 5.1 in next section.

5. Weighted approximation and embedding theorems

5.1. **Estimates for the best weighted approximation.** Let us denote by

$$E_m(f)_{u,p} = \inf_{P \in \mathbb{P}_m} \|(f-P)\,u\|_p$$

the error of best polynomial approximation of a function $f \in L_u^p$, $1 \le p \le \infty$. The following Jackson, weak Jackson and Stechkin inequalities hold true.

Theorem 14. *(see [13, p. 173]). For any $f \in L_u^p$, $1 \le p \le \infty$, and $m > r \ge 1$, we have*

$$E_m(f)_{u,p} \le \mathcal{C}\,\omega_\varphi^r\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p}, \tag{5.1}$$

and, assuming $\Omega_\varphi^r(f,t)_{u,p}\,t^{-1} \in L^1[0,1]$,

$$E_m(f)_{u,p} \le \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t}\,\mathrm{d}t, \qquad r < m.$$

Finally for any $f \in L_u^p$, $1 \le p \le \infty$, we get

$$\omega_\varphi^r\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p} \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \sum_{i=0}^m \left(\frac{i}{\sqrt{a_i}}\right)^r \frac{E_i(f)_{u,p}}{i}. \tag{5.2}$$

In any case $\mathcal{C}$ is independent of m and f.

In particular, for any $f \in W_r^p(u)$, $1 \le p \le +\infty$, we obtain

$$E_m(f)_{u,p} \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \left\|f^{(r)}\varphi^r u\right\|_p, \qquad \mathcal{C} \ne \mathcal{C}(m,f). \tag{5.3}$$

Whereas, for any $f \in Z_s^p(u)$, $1 \le p \le +\infty$, we get

$$E_m(f)_{u,p} \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s}, \quad r > s, \quad \mathcal{C} \ne \mathcal{C}(m,f). \tag{5.4}$$

From (5.1), (5.2) and (5.4), we deduce the following equivalences

$$\lim_m \omega_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p} = 0 \qquad \Leftrightarrow \qquad \lim_m E_m(f)_{u,p} = 0$$

and

$$\|fu\|_p + \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s} \sim \|fu\|_p + \sup_{m\ge 1}\left(\frac{m}{\sqrt{a_m}}\right)^s E_m(f)_{u,p}$$

for $1 \le p \le \infty$ and $r > s$.

5.2. **Embedding theorems.** Now, using Theorem 14, the dyadic decomposition, the Nikolskii inequalities (3.5) and (3.6), we can prove some embedding theorems, connecting different subspaces of L_u^p.

Theorem 15. *(see* [12, p. 159]*). For any $f \in L_u^p$, $1 \le p < \infty$, such that*

$$\int_0^1 \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\eta/p}}\,\mathrm{d}t < \infty\,,$$

where $\eta = (2\alpha+2)/(2\alpha+1)$, we have

$$E_m(f)_{u,\infty} \le \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\eta/p}}\,\mathrm{d}t\,,$$

$$\Omega_\varphi^r\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,\infty} \le \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\eta/p}}\,\mathrm{d}t$$

and

$$\|fu\|_\infty \le \mathcal{C}\left\{\|fu\|_p + \int_0^1 \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\eta/p}}\,\mathrm{d}t\right\},$$

where $\mathcal{C}$ depends only on r.

In the following theorem, we replace η/p by $1/p$.

Theorem 16. *(see* [12, pp. 159–160]*). For any $f \in L_u^p$, $1 \le p < \infty$, such that*

$$\int_0^1 \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t < \infty\,,$$

we have

$$E_m(f)_{\varphi^{1/p}u,\infty} \le \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\,,$$

$$\Omega_\varphi^r\left(f, \frac{\sqrt{a_m}}{m}\right)_{\varphi^{1/p}u,\infty} \le \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t$$

and

$$\|f\varphi^{1/p}u\|_\infty \le \mathcal{C}\left\{\|fu\|_p + \int_0^1 \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\right\},$$

where $\mathcal{C}$ depends only on r.

From Theorem 16 we can easily deduce the following corollary, useful in several contexts.

Corollary 17. *(see [12, p. 160]). If $f \in L^p_u$, $1 \le p < \infty$, is such that*

$$\int_0^1 \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t < \infty\,,$$

then f is continuous on $(0,+\infty)$.

6. Quadrature rules and Lagrange interpolation

Here we are going to show a slight extension of the results proved [14] for $\gamma = 0$.

6.1. **Gaussian formulas.** The Gaussian rule related to the weight $w(x) = x^\gamma \mathrm{e}^{-x^{-\alpha}-x^\beta}$ can be defined by the equality

$$\int_0^{+\infty} P_{2m-1}(x)w(x)\,\mathrm{d}x = \sum_{k=1}^m \lambda_k(w)P_{2m-1}(x_k) \tag{6.1}$$

where x_k are the zeros of $p_m(w)$, $\lambda_k(w)$ are the Christoffel numbers, which holds for any polynomial $P_{2m-1} \in \mathbb{P}_{2m-1}$.

Thus the error of the Gaussian rule for any continuous function f is given by

$$e_m(f) = \int_0^{+\infty} f(x)w(x)\,\mathrm{d}x - \sum_{k=1}^m \lambda_k(w)f(x_k)\,.$$

Let us consider the weight

$$\sigma(x) = (1+x)^\delta w^a(x)\,, \quad \delta \ge 0\,,\ 0 < a \le 1\,. \tag{6.2}$$

Naturally, taking also into account Lemma 7, the results of Sections 3 and 4 hold with u replaced by σ.

If we assume $f \in C_\sigma$, then we can write

$$\left|\sum_{k=1}^m \lambda_k(w)f(x_k)\right| \le \|f\sigma\|_\infty \sum_{k=1}^m \frac{\lambda_k(w)}{\sigma(x_k)} \le \mathcal{C}\,\|f\sigma\|_\infty \int_0^{+\infty} \frac{w(x)}{\sigma(x)}\,\mathrm{d}x$$

and the next proposition easily follows.

Proposition 18. *(cfr. [14, p. 1660]). If $w/\sigma \in L^1$, then, for any $f \in C_\sigma$, we have*

$$|e_m(f)| \le \mathcal{C}E_{2m-1}(f)_{\sigma,\infty}\,, \tag{6.3}$$

where $\mathcal{C} \ne \mathcal{C}(m,f)$.

This proposition generalizes a result due to Uspensky [19], who first proved the convergence of Gaussian rules on unbounded intervals related to Laguerre and Hermite weights (see also [9, pp. 341–345] and [11]).

Notice that the assumption $w/\sigma \in L^1$ in Proposition 18 is fulfilled if $a = 1$ and $\delta > 1$, or if $a < 1$ and δ is arbitrary. The error estimate (6.3) implies the convergence of the Gaussian rule for any $f \in C_\sigma$. For smoother function, for instance $f \in W_r^\infty(\sigma)$, by (6.3) and (5.3), we obtain

$$|e_m(f)| \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \left\|f^{(r)}\varphi^r\sigma\right\|_\infty,$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$ and $a_m \sim m^{1/\beta}$.

Thus, a natural question is to establish the degree of convergence of $e_m(f)$ if the function f is infinitely differentiable, i.e., $f \in C^\infty(\mathbb{R}_+)$. We recall that [1, 2] proved estimates of $e_m(f)$ related to Hermite or Freud weights for analytic functions in some domains of the complex plane containing the quadrature nodes. For precise estimates, considering the same class of functions and different weights we refer to [8]. Here we consider the case of infinitely differentiable functions on $\mathbb{R}_+$, with the condition that $(f^{(m)}\sigma)(x)$ is uniformly bounded w.r.t. m and x. We note that the derivatives of the function can increase exponentially for $x \to 0$ and $x \to +\infty$.

Theorem 19. *(cfr. [14, p. 1660]). Let σ be the weight in (6.2) with $0 < a < 1$ and δ arbitrary. For any infinitely differentiable function f, if $K(f) := \sup_m \left\|f^{(m)}\sigma\right\|_\infty < +\infty$, we have*

$$|e_m(f)| \le \mathcal{C}K(f)\,\Gamma_m, \qquad \lim_m \sqrt[2m]{\Gamma_m} = 0.$$

In order to study the behaviour of the Gaussian rule in Sobolev spaces $W_r^1(w)$, it is natural to investigate whether estimates of the form

$$|e_m(f)| \le \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi w\|_1, \quad \mathcal{C} \neq \mathcal{C}(m,f), \quad f \in W_1^1(w), \tag{6.4}$$

hold true.

We recall that, as shown in the previous Section, for the error of best approximation we have

$$E_m(f)_{w,1} \le \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi w\|_1, \quad \mathcal{C} \neq \mathcal{C}(m,f), \quad f \in W_1^1(w).$$

On the other hand, inequality (6.4) holds, mutatis mutandis, for Gaussian rules on bounded intervals related to Jacobi weights. But, as for many exponential weights (see, e.g., [4, 5, 10, 11]), inequality (6.4) is false in the sense of the following theorem.

Theorem 20. *(cfr. [14, p. 1661]). Let $w(x) = x^\gamma \mathrm{e}^{-x^{-\alpha}-x^\beta}$, $\alpha > 0$, $\beta > 1$ and $\gamma \ge 0$. Then, for any $f \in W_1^1(w)$, we have*

$$|e_m(f)| \le \mathcal{C}m^{1/3}\frac{\sqrt{a_m}}{m}\|f'\varphi w\|_1,$$

where $\mathcal{C}$ is independent of m and f. Moreover, for a sufficiently large m (say $m \ge m_0$), there exists a function f_m, with $0 < \|f_m'\varphi w\|_1 < +\infty$, and a constant $\mathcal{C} \neq \mathcal{C}(m, f_m)$ such that

$$|e_m(f_m)| \ge \mathcal{C}m^{1/3}\frac{\sqrt{a_m}}{m}\|f_m'\varphi w\|_1.$$

Nevertheless, estimates of the form (6.4) are required in different contexts. So, in order to obtain this kind of error estimates, using also an idea from [10], we are going to modify the Gaussian rule.

With $\theta \in (0,1)$ fixed, we define two indexes $j_1 = j_1(m)$ and $j_2 = j_2(m)$ as in (2.3). Then, for a sufficiently large N, let $\mathbb{P}_N^*$ denote the following subset of all polynomials of degree at most N

$$\mathbb{P}_N^* = \{P \in \mathbb{P}_N : \ P(x_i) = 0\,, \ x_i < x_{j_1} \ \mathit{or} \ x_i > x_{j_2}\}\,.$$

Naturally, $p_m(w) \in \mathbb{P}_N^*$, for $N \geq m$ and $\theta \in (0,1)$ arbitrary.

Now, in analogy with (6.1), we define the new Gaussian rule, by means of the equality

$$\int_0^{+\infty} Q_{2m-1}(x)w(x)\,\mathrm{d}x = \sum_{k=1}^{m} \lambda_k(w) Q_{2m-1}(x_k) = \sum_{k=j_1}^{j_2} \lambda_k(w) Q_{2m-1}(x_k)\,,$$

which holds for every $Q_{2m-1} \in \mathbb{P}_{2m-1}^*$.

Then, for any continuous function f, the "truncated" Gaussian rule is defined as

$$\int_0^{+\infty} f(x)w(x)\,\mathrm{d}x = \sum_{k=j_1}^{j_2} \lambda_k(w) f(x_k) + e_m^*(f)\;, \tag{6.5}$$

whose error $e_m^*(f)$ is the difference between the integral and the quadrature sum.

Compared to the Gaussian rule (6.1), in the formula (6.5) the terms of the quadrature sum corresponding to the zeros which are "close" to the MRS numbers are dropped. From the numerical point of view, this fact has two consequences. First, it avoids overflow phenomena (taking into account that, in general, the function f is exponentially increasing at the endpoints of $\mathbb{R}_+$). Moreover, it produces a computational saving, which is evident in the numerical treatment of linear functional equations (see [15]).

We are now going to study the behaviour $e_m^*(f)$ in C_σ and $W_r^1(w)$. We will see that the errors $e_m(f)$ and $e_m^*(f)$ have essentially the same behaviour in C_σ, but not in $W_r^1(w)$, since $e_m^*(f)$ satisfies (6.4), while $e_m(f)$ does not.

The behaviour of $e_m^*(f)$ in C_σ is given by the following proposition.

Proposition 21. *(cfr. [14, p. 1662]). Assume $w/\sigma \in L^1$. Then, for any $f \in C_\sigma$, we get*

$$|e_m^*(f)| \leq \mathcal{C}\left\{E_M(f)_{\sigma,\infty} + \mathrm{e}^{-cm^\nu}\|f\sigma\|_\infty\right\}, \tag{6.6}$$

where $M = \left\lfloor\left(\frac{\theta}{\theta+1}\right)m\right\rfloor$, $\theta \in (0,1)$, ν is given by (3.2), $\mathcal{C} \neq \mathcal{C}(m,f)$ and $c \neq c(m,f)$.

In particular, if $f \in W_r^\infty(\sigma)$, inequality (6.6) becomes

$$|e_m^*(f)| \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W_r^\infty(\sigma)}\,.$$

For smoother functions, the analogue of Theorem 19 is given by the following statement.

Theorem 22. *(cfr. [14, p. 1662]). If the weight σ and the function f satisfy the assumption of Theorem 19, then, for any $0 < \mu < \alpha(1 - 1/(2\beta))/(\alpha + 1/2)$ fixed, we get*

$$|e_m^*(f)| \leq \mathcal{C}\left[\|f\sigma\|_\infty + K(f)\right]\bar{\Gamma}_m,$$

where $\lim_m \bar{\Gamma}_m^{1/m^\mu} = 0$ and $\mathcal{C} \neq \mathcal{C}(m, f)$.

For functions $f \in W_1^1(w)$ or $f \in Z_s^1(w)$, $1 < s \in \mathbb{R}_+$, the following theorem states the required estimates.

Theorem 23. *(cfr. [14, pp. 1662–1663]). For any $f \in W_1^1(w)$, we have*

$$|e_m^*(f)| \leq \mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi w\|_1 + \mathcal{C}\mathrm{e}^{-cm^\nu}\|fw\|_1. \tag{6.7}$$

Moreover, for any $f \in Z_s^1(w)$, with $s > 1$, we get

$$|e_m^*(f)| \leq \mathcal{C}\frac{\sqrt{a_m}}{m}\int_0^{\sqrt{a_m}/m}\frac{\Omega_\varphi^r(f,t)_{w,1}}{t^2}\mathrm{d}t + \mathcal{C}\mathrm{e}^{-cm^\nu}\|fw\|_1, \tag{6.8}$$

where $r > s > 1$. In both cases $\mathcal{C}$ and c do not depend on m and f, and ν is given by (3.2).

In conclusion, inequality (6.7) is the required estimate and, by (6.8), it can be generalized as

$$|e_m^*(f)| \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s \|f\|_{Z_s^1(w)}, \quad \mathcal{C} \neq \mathcal{C}(m, f),$$

for $f \in Z_s^1(w)$, $s > 1$. In particular, if s is an integer number, recalling (6.7), the Zygmund norm can be replaced by the Sobolev norm.

Finally, we emphasize that the previous estimate cannot be improved, since, in these function spaces, $e_m^*(f)$ converges to 0 with the order of the best polynomial approximation.

6.2. **Lagrange interpolation in $L^2_{\sqrt{w}}$.** Here we want to apply the results in Section 6.1 to estimate the error of the Lagrange interpolation process based on the zeros of $p_m(w)$. If $f \in C^0(\mathbb{R}_+)$, then the Lagrange polynomial interpolating f at the zeros of $p_m(w)$ is defined by

$$L_m(w, f, x) = \sum_{k=1}^m l_k(w, x) f(x_k), \quad l_k(w, x) = \frac{p_m(w, x)}{p_m'(w, x_k)(x - x_k)},$$

and we are going to study the error $\|[f - L_m(w, f)]\sqrt{w}\|_2$ for different function classes.

Since

$$\left\|L_m(w, f)\sqrt{w}\right\|_2^2 = \sum_{k=1}^m \frac{\lambda_k(w)}{w(x_k)}(f\sqrt{w})^2(x_k) \tag{6.9}$$

and we are dealing with an unbounded interval, we cannot expect an analogue of the theorem by Erdős and Turán [6]. On the other hand, if $f \in C_{\tilde{u}}$, with $\tilde{u}(x) = (1+x)^\delta\sqrt{w(x)})$, $\delta > 1/2$, it is easily seen that

$$\left\|[f - L_m(w, f)]\sqrt{w}\right\|_2 \leq \mathcal{C}E_{m-1}(f)_{\tilde{u},\infty}, \quad \mathcal{C} \neq \mathcal{C}(m, f).$$

Nevertheless, as for the Gaussian formula, if $f \in W_1^2(\sqrt{w})$, then $L_m(w,f)$ has not an optimal behaviour, i.e., an estimate of the form

$$\left\|[f - L_m(w,f)]\sqrt{w}\right\|_2 \le \mathcal{C}\frac{\sqrt{a_m}}{m}\left\|f'\varphi\sqrt{w}\right\|_2, \qquad \mathcal{C} \neq \mathcal{C}(m,f),$$

does not hold. In order to overcome this gap, for any $f \in C^0(\mathbb{R}_+)$ we introduce the following "truncated" Lagrange polynomial

$$L_m^*(w,f,x) = \sum_{k=j_1}^{j_2} l_k(w,x) f(x_k),$$

where j_1, j_2 are given by (2.3).

Naturally, in general $L_m^*(w,P) \neq P$ for arbitrary polynomials $P \in \mathbb{P}_{m-1}$ (for example, $L_m(w,\mathbf{1}) \neq \mathbf{1}$). But $L_m^*(w,Q) = Q$ for any $Q \in \mathbb{P}_{m-1}^*$ and $L_m^*(w,f) \in \mathbb{P}_{m-1}^*$ for any $f \in C^0(\mathbb{R}_+)$. So, the operator $L_m^*(w)$ is a projector from $C^0(\mathbb{R}_+)$ into $\mathbb{P}_{m-1}^*$.

Moreover, considering the weight

$$\tilde{u}(x) = (1+x)^\delta\sqrt{w(x)}, \qquad \delta > 0, \tag{6.10}$$

we can show that every function $f \in L_{\tilde{u}}^p$ can be approximated by polynomials of $\mathbb{P}_m^*$. To this aim we define

$$\widetilde{E}_m(f)_{\tilde{u},p} = \inf_{P \in \mathbb{P}_m^*}\|(f-P)\tilde{u}\|_p, \quad 1 \le p \le +\infty.$$

Lemma 24. *(cfr. [14, p. 1664]). For any $f \in L_{\tilde{u}}^p$, where $\tilde{u}$ is given by (6.10) and $1 \le p \le +\infty$, we have*

$$\widetilde{E}_m(f)_{\tilde{u},p} \le \mathcal{C}\left\{E_M(f)_{\tilde{u},p} + \mathrm{e}^{-cm^\nu}\|f\tilde{u}\|_p\right\},$$

where $M = \left\lfloor\left(\frac{\theta}{\theta+1}\right)m\right\rfloor$, $\theta \in (0,1)$, ν is given by (3.2), $\mathcal{C} \neq \mathcal{C}(m,f)$ and $c \neq c(m,f)$.

As an immediate consequence of the previous lemma and equality (6.9), we get the following

Proposition 25. *(cfr. [14, p. 1664]). For any $f \in C_{\tilde{u}}$, with $\tilde{u}$ as (6.10), $\delta > 1/2$, we have*

$$\left\|[f - L_m^*(w,f)]\sqrt{w}\right\|_2 \le \mathcal{C}\left\{E_M(f)_{\tilde{u},\infty} + \mathrm{e}^{-cm^\nu}\|f\tilde{u}\|_\infty\right\},$$

where $M = \left\lfloor\left(\frac{\theta}{\theta+1}\right)m\right\rfloor$, $\theta \in (0,1)$, ν is given by (3.2), $\mathcal{C} \neq \mathcal{C}(m,f)$ and $c \neq c(m,f)$.

We are going to study the behaviour of the sequence $\{L_m^*(w)\}_m$ in the Sobolev spaces $W_r^2(\sqrt{w})$, which is interesting in different contexts.

To this regard, we observe that, since no results concerning the sequence of the Fourier sum $\{S_m(w)\}_m$ are known, we cannot deduce the behaviour of $\{L_m^*(w)\}_m$ from that of $\{S_m(w)\}_m$. Therefore, we need a different approach.

The following theorem describes the behaviour of the operator $L_m^*(w)$ in different function spaces.

Theorem 26. *(cfr. [14, p. 1664]). Assume $f \in L^2_{\sqrt{w}}$ and*

$$\int_0^1 \frac{\Omega_\varphi^r(f,t)_{\sqrt{w},2}}{t^{3/2}}\mathrm{d}t < +\infty\,, \qquad r \geq 1\,, \tag{6.11}$$

then we have

$$\left\|[f - L_m^*(w,f)]\sqrt{w}\right\|_2 \leq \mathcal{C}\left\{\left(\frac{\sqrt{a_m}}{m}\right)^{1/2}\int_0^{\sqrt{a_m}/m}\frac{\Omega_\varphi^r(f,t)_{\sqrt{w},2}}{t^{3/2}}\mathrm{d}t + \mathrm{e}^{-cm^\nu}\left\|f\sqrt{w}\right\|_2\right\}, \tag{6.12}$$

where ν is given by (3.2) and the constants $\mathcal{C}, c$ are independent of m and f.

Note that, by Corollary 17, the assumption (6.11) implies $f \in C^0(\mathbb{R}_+)$.

The error estimate (6.12) has interesting consequences.

Firstly, if

$$\sup_{t>0}\frac{\Omega_\varphi^r(f,t)_{\sqrt{w},2}}{t^s}\mathrm{d}t < +\infty\,, \qquad r > s > 1/2\,,$$

i.e., $f \in Z_s^2(\sqrt{w})$, then the order of convergence of the process is $\mathcal{O}\left(\left(\sqrt{a_m}/m\right)^s\right)$. While, if $f \in W_r^2(\sqrt{w})$, $r \geq 1$ is integer, we have

$$\left\|[f - L_m^*(w,f)]\sqrt{w}\right\|_2 \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r\|f\|_{W_r^2(\sqrt{w})}\,.$$

This means that the process converges with the error of the best approximation for the considered classes of functions.

Secondly, we are now able to show the uniform boundedness of the sequence $\{L_m^*(w)\}$ in the Sobolev spaces.

Theorem 27. *(cfr. [14, p. 1665]). With the previous notation, for any $f \in W_r^2(\sqrt{w})$, $r \geq 1$, we have*

$$\sup_m \|L_m^*(w,f)\|_{W_r^2(\sqrt{w})} \leq \mathcal{C}\,\|f\|_{W_r^2(\sqrt{w})}\,, \qquad \mathcal{C} \neq \mathcal{C}(f)\,.$$

Moreover, for any $f \in W_s^2(\sqrt{w})$, $s > r$, we have

$$\|f - L_m^*(w,f)\|_{W_r^2(\sqrt{w})} \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{s-r}\|f\|_{W_s^2(\sqrt{w})}\,, \qquad \mathcal{C} \neq \mathcal{C}(m,f)\,.$$

Remark 28. *In all the estimates for $e_m^*(f)$ and $(f - L_m^*(w,f))$, a constant $\mathcal{C} \neq \mathcal{C}(m,f)$ appears. We have not pointed out the dependence on the parameter $\theta \in (0,1)$, since θ is fixed. Nevertheless, it is useful to observe that $\mathcal{C} = \mathcal{C}(\theta) = \mathcal{O}\left((\theta/(1-\theta))^2\right)$. So, it is clear that the parameter θ cannot assume the value 0 or 1 and the "truncation" is necessary in this sense (see [14] for more details).*

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