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Published version:

DOI:10.1016/j.apnum.2019.06.009

POLYNOMIAL APPROXIMATION WITH POLLACZEK–TYPE WEIGHTS. A SURVEY

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Abstract. The paper deals with weighted polynomial approximation for functions defined on $(-1,1)$, which can grow exponentially both at -1 and at 1. We summarize recent results on function spaces with new moduli of smoothness, estimates for the best approximation, Lagrange interpolation, Fourier sums and Gaussian rules with respect to weights of the form $w(x) = (1-x^2)^\beta \mathrm{e}^{-(1-x^2)^{-\alpha}}$.

Keywords: weighted polynomial approximation; orthogonal polynomials; Lagrange interpolation at Pollaczek–type zeros; Fourier sums w.r.t. Pollaczek–type polynomials; Gaussian quadrature rules w.r.t. exponential weights; bounded intervals.

MCS classification (2000): 41A05, 41A10, 41A17, 41A25, 42C10, 42A20, 65D05, 65D32.

1. Introduction

There is an extensive literature concerning the trigonometric approximation of periodic functions. These results have been extended to the algebraic approximation on $(-1,1)$, w.r.t. "doubling" weights (for instance, Jacobi or generalized Jacobi weights). These processes are useful in the approximation of locally continuous functions, having algebraic singularities at the endpoints ± 1 and at some inner points. Nevertheless, these processes are not suitable in order to approximate functions having exponential growth close to ± 1.

In this paper we are going to propose a further extension, i.e., we will consider the algebraic approximation of functions defined on $(-1,1)$, which can grow exponentially both at -1 and at 1. To this aim we consider weight functions of the form

$$w(x) = (1-x^2)^\beta \mathrm{e}^{-(1-x^2)^{-\alpha}}, \qquad \alpha > 0,\ \beta \geq 0, \qquad x \in (-1,1). \tag{1.1}$$

We are going to present the main results concerning function spaces with new moduli of smoothness, estimates for the best polynomial approximation with respect to weights of the form (1.1), Lagrange interpolation, Fourier sums and Gaussian rules.

For reader's convenience and for a simpler comparison with the new results, we recall some basic facts on the trigonometric approximation of 2π-periodic functions. Let us denote by

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In memory of Prof. Sebastiano Seatzu.

The first author was partially supported by University of Basilicata (local funds). The second aurthor was partially supported by University of Basilicata (local funds) and by INdAM–GNCS..

$C^0 = L^\infty$ the space of all continuous 2π-periodic functions, with the norm

$$\|f\|_{C^0} = \|f\|_\infty = \sup_{x\in[0,2\pi]} |f(x)|\,,$$

and by L^p, $1 \le p < \infty$, the space of integrable functions, with

$$\|f\|_p^p = \int_0^{2\pi} |f(x)|^p \,\mathrm{d}x\,.$$

For smoother functions, we consider the Sobolev spaces

$$W_r^p = \left\{ f \in L^p \ : \ f^{(r-1)} \in AC \text{ and } \left\|f^{(r)}\right\|_p < \infty \right\}\,,$$

with $r \ge 1$ and

$$\|f\|_{W_r^p} = \|f\|_p + \left\|f^{(r)}\right\|_p\,.$$

For any $f \in L^p$, $1 \le p \le \infty$, the $K-$functional

$$K(f, t^r)_p = \inf_{g\in W_r^p} \left\{ \|f-g\|_p + t^r \left\|g^{(r)}\right\|_p \right\}$$

and the $r-$th modulus of smoothness

$$\omega^r(f,t)_p = \sup_{0<h\le t} \|\Delta_h^r f\|_p$$

$$\Delta_h f = f(x+h) - f(x), \quad \Delta_h^r = \Delta_h \Delta_h^{r-1}$$

fulfill

$$\lim_{t\to 0} \omega(f,t)_p = 0 \tag{1.2}$$

and

$$\omega^r(f,t)_p \sim K(f,t^r)_p\,, \tag{1.3}$$

where the constants in "$\sim$" are independent of f and t.

Further subspaces of L^p, $1 \le p \le \infty$, are the Zygmund spaces

$$Z_s^p = \left\{ f \in L^p \ : \ \sup_{t>0} \frac{\omega^r(f,t)_p}{t^s} < \infty \right\}, \quad r > s \in \mathbb{R}^+\,,$$

with

$$\|f\|_{Z_s^p} = \|f\|_p + \sup_{t>0} \frac{\omega^r(f,t)_p}{t^s}\,.$$

The error of best approximation of $f \in L^p$, $1 \le p \le \infty$,

$$E_m^*(f)_p = \inf_{T_m} \|f - T_m\|_p$$

by trigonometric polynomials of the form

$$T_m(x) = \frac{a_0}{2} + \sum_{k=1}^m \left[a_k \cos(kx) + b_k \sin(kx)\right],$$

can be estimated in terms of the $r-$th modulus of smoothness, i.e. the Jackson inequality

$$E_m^*(f)_p \le \mathcal{C}\omega^r\left(f, \frac{1}{m}\right)_p \tag{1.4}$$

holds with $\mathcal{C}$ independent of m and f. Moreover, applying the Bernstein inequality

$$\|T_m'\|_p \le m\|T_m\|_p \qquad \forall T_m\,,$$

one can deduce the Stechkin inequality

$$\omega^r\left(f, \frac{1}{m}\right)_p \le \frac{\mathcal{C}}{m^r}\sum_{k=1}^{m}(1+k)^{r-1}E_k^*(f)_p, \quad m > r\,, \tag{1.5}$$

where $\mathcal{C}$ is independent of m and f. It follows that the order of convergence of the best approximation characterizes the above defined function spaces, since

$$\lim_m E_m^*(f)_p = 0 \quad \Leftrightarrow \quad f \in L^p$$

and

$$\sup_{k\ge1} k^s E_k^*(f)_p < \infty \quad \Leftrightarrow \quad f \in Z_s^p\,.$$

Now, let us recall some approximation processes. For $f \in L^1$, the Fourier sums are given by

$$\begin{aligned} S_m(f,x) &= \frac{a_0}{2} + \sum_{k=1}^{m}\left[a_k\cos(kx) + b_k\sin(kx)\right] \\ &= \int_0^{2\pi} K_m(x-t)f(t)\,\mathrm{d}t\,, \end{aligned}$$

where K_m is the Darboux kernel and

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(t)\,\mathrm{d}t, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(t)\,\mathrm{d}t\,.$$

Denoting by

$$\|S_m\|_p := \|S_m\|_{L^p\to L^p}$$

the associated operator norm, it is well known that

$$\|S_m\|_1 = \|S_m\|_\infty \sim \log m \tag{1.6}$$

and

$$\|S_mf\|_p \le \mathcal{C}\|f\|_p\,, \quad 1 < p < \infty\,, \tag{1.7}$$

where $\mathcal{C}$ and the constants in "$\sim$" are independent of f and m.

While for the Lagrange interpolating polynomial

$$L_m^*(f,x) = \frac{2}{2m+1}\sum_{k=0}^{2m} K_m(x-t_k)f(t_k)\,, \qquad t_k = \frac{2\pi k}{2m+1}\,,$$

denoting by $\|L_m^*\|_\infty$ the norm of the associated operator $L_m : C^0 \to C^0$, one has

$$\|S_m\|_\infty \le \|L_m^*\|_\infty \le (1+\pi)\|S_m\|_\infty$$

and so the Lagrange intepolation converges with the order of the best approximation times $\log m$, as well as the Fourier sums. Moreover, from the Marcinkiewicz equivalence

$$\|L_m^*(f)\|_p \sim \left(\frac{2\pi}{2m+1}\sum_{k=0}^{2m}|f(t_k)|^p\right)^{\frac{1}{p}}, \quad 1<p<\infty\,, \tag{1.8}$$

we can deduce that the Lagrange interpolation converges with the order of the best approximation for functions belonging to W_r^p or Z_s^p, with $r \ge 1$, $s > 1/p$ and $1 < p < \infty$.

The results briefly exposed above can be found in every approximation theory book, among others we recall [32, 36]. Our aim is to extend these results to the case of weighted polynomial approximation of functions defined on $(-1,1)$, growing exponentially at -1 and 1.

In the sequel $c, \mathcal{C}$ will stand for positive constants which can assume different values in each formula and we shall write $\mathcal{C} \ne \mathcal{C}(a, b, \ldots)$ when $\mathcal{C}$ is independent of $a, b, \ldots$ or $\mathcal{C}_a$ when $\mathcal{C}$ depends on a. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \le \mathcal{C}$. Finally, we will denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m. As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, will stand for the sets of all natural, integer, real numbers, while $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote the sets of positive integer and positive real numbers, respectively.

2. Weighted function spaces and best approximation

In the last two decades E. Levin and D.S. Lubinsky have extensively studied orthogonal polynomials related to exponential weights. Among other topics, they have considered weight functions of the form $\mathrm{e}^{-Q(x)}$, $|x| \le 1$, where Q is an even function which satisfies suitable assumptions. The reader can find their numerous results in the monograph [9] (see also [8]).

In [10, 11] D.S. Lubinsky considered the polynomial approximation in $[-1,1]$ with this class of weights. He proved the Jackson inequality using the modulus of smoothness $\omega_{\Phi_t}^r(f,t)_{u,p}$, whose definition involves the finite difference $\Delta_{\Phi_t}^r$, where the step function Φ_t depends not only on x but also on t. Namely

$$\Phi_t(x) = \sqrt{\left|1 - \frac{|x|}{a_{1/t}}\right|} + \frac{1}{\sqrt{T(a_{1/t})}}$$

where $a_{1/t}$ is the Mhaskar–Rahmanov–Saff number related to u, $T(x) = Q'(x)/Q(x)$ and the second summand tends to 0 as $t \to 0$. However, since

$$\omega_{\Phi_t}^r(f,t)_{u,p} \le \mathcal{C}t^r\|f^{(r)}\Phi_t^r u\|_p, \qquad \mathcal{C} \ne \mathcal{C}(t,f),$$

in order to prove the equivalence between this modulus of smoothness and some $K-$functional, one is bound to define Sobolev spaces with seminorms containing a parameter t extraneus w.r.t. the class of functions. Moreover, a Bernstein inequality of the form

$$\|P_m'\Phi_{1/m}u\|_p \le \mathcal{C}m\|P_m u\|_p \qquad \forall P_m \in \mathbb{P}_m, \qquad \mathcal{C} \ne \mathcal{C}(m, P_m),$$

is also needed. But the function Φ_t creates some further difficulties in iterating this last inequality and in proving the Stechkin inequality.

Subsequently, in [21], extending the ideas of Z. Ditzian–V. Totik and B. Della Vecchia–G. Mastroianni–J. Szabados in [7, 6], we introduced different moduli of smoothness involving the step function $\varphi(x) = \sqrt{1-x^2}$, equivalent to suitable K-functionals, and proved the Jackson theorem, also in its weaker form.

2.1. **Function spaces and moduli of smoothness.** Let us consider the weight function

$$u(x) = (1-x^2)^{\gamma} \mathrm{e}^{-\frac{1}{2}(1-x^2)^{-\alpha}}\,, \tag{2.1}$$

where $\alpha > 0$, $\gamma \ge 0$, $x \in (-1, 1)$.

We point out that the weight u does not satisfy the doubling condition and, for $\alpha \ge 1/2$, does not belong to the Szegő class (see [30] and [34]). Nevertheless, the weight u belongs to a wide class of exponential weights defined by Levin and Lubinsky in [8] and [9], as it was checked in [21]. In particular, setting $Q(x) = -\log u(x)$, we can define the Mhaskar–Rakhmanov–Saff number $\bar{a}_\tau = \bar{a}_\tau(u)$, $1 \le \tau \in \mathbb{R}$, as the positive root of

$$\tau = \frac{2}{\pi}\int_0^1 \bar{a}_\tau t Q'(\bar{a}_\tau t)\frac{\mathrm{d}t}{\sqrt{1-t^2}}\,.$$

The number $\bar{a}_\tau$ is an increasing function of τ, with $\lim_{\tau\to+\infty} \bar{a}_\tau = 1$ and

$$C_1\tau^{-\frac{1}{\alpha+1/2}} \le 1 - \bar{a}_\tau \le C_2\tau^{-\frac{1}{\alpha+1/2}},$$

where C_1 and C_2 are positive constants independent of τ and α is fixed (see [9, pp. 13,31]).

We can associate to the weight u the following function spaces. For $1 \le p < \infty$, by L^p_u we denote the set of all measurable functions f such that

$$\|f\|_{L^p_u} := \|fu\|_p = \left(\int_{-1}^1 |fu|^p(x)\,\mathrm{d}x\right)^{1/p} < \infty\,.$$

For $p = \infty$, by a slight abuse of notation, we set

$$L^\infty_u := C_u = \left\{ f \in C^0(-1,1) : \lim_{x\to\pm1} f(x)u(x) = 0 \right\},$$

and we equip this space with the norm

$$\|f\|_{L^\infty_u} := \|fu\|_\infty = \sup_{x\in(-1,1)} |f(x)u(x)|\,.$$

Note that the limit conditions in the definition of C_u are necessary and sufficient for the validity of the Weierstrass theorem in C_u.

We emphasize that the functions belonging to the spaces L^p_u can grow exponentially at the endpoints ± 1. For instance, the function $f(x) = \frac{e^{\frac{1}{(1-x)^{1/2}}}}{\sqrt{1+x}} \sin(x)$ belongs to the spaces C_u, with parameters $\alpha, \gamma > \frac{1}{2}$.

The Sobolev-type subspaces of L^p_u are given by

$$W^p_r(u) = \left\{ f \in L^p_u : \ f^{(r-1)} \in AC(-1,1), \ \|f^{(r)}\varphi^r u\|_p < \infty \right\}, \quad 1 \le r \in \mathbb{N},$$

where $1 \le p \le \infty$, $\varphi(x) := \sqrt{1-x^2}$ and $AC(-1,1)$ denotes the set of all functions which are absolutely continuous on every closed subinterval of $(-1,1)$. We equip these spaces with the norm

$$\|f\|_{W^p_r(u)} = \|fu\|_p + \|f^{(r)}\varphi^r u\|_p .$$

The $K-$ functional, connecting L^p_u and $W^p_r(u)$, is defined as

$$K(f,t^r)_{u,p} = \inf_{g \in W^p_r} \left\{ \|(f-g)u\|_p + t^r \left\| g^{(r)}\varphi^r u \right\|_p \right\} .$$

In order to introduce some further subspaces of L^p_u, for $1 \le p \le \infty$, $r \ge 1$ and for all sufficiently small $t > 0$ (say $t < t_0$), we define the main part of the $r-$th modulus of smoothness as

$$\Omega^r_\varphi(f,t)_{u,p} = \sup_{0<h\le t} \left\| \Delta^r_{h\varphi}(f)\, u \right\|_{L^p(\mathcal{I}_h)} ,$$

where $\mathcal{I}_h = [-h^*, h^*]$, $h^* = 1 - \mathrm{A}\, h^{1/(\alpha+1/2)}$, $\mathrm{A} > 0$ is a fixed constant, and

$$\Delta^r_{h\varphi} f(x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i f\left(x + (r-2i)\frac{h\varphi(x)}{2} \right) .$$

Then the complete $r-$th modulus of smoothness is given by

$$\omega^r_\varphi(f,t)_{u,p} = \Omega^r_\varphi(f,t)_{u,p} + \inf_{P \in \mathbb{P}_{r-1}} \|(f-P)\,u\|_{L^p[-1,-t^*]} + \inf_{P \in \mathbb{P}_{r-1}} \|(f-P)\,u\|_{L^p[t^*,1]}$$

with $t^* = 1 - \mathrm{A}\, t^{1/(\alpha+1/2)}$ and $\mathrm{A} > 0$. We emphasize that the behavior of $\omega^r_\varphi(f,t)_{u,p}$ is independent of the constant A.

We also remark that

$$\lim_{t \to 0} \omega_\varphi(f,t)_{u,p} = 0 \quad \Leftrightarrow \quad f \in L^p_u$$

and

$$\omega^r_\varphi(f,t)_{u,p} \sim K(f,t^r)_{u,p} , \tag{2.2}$$

by analogy with (1.2) and (1.3). Moreover, for any $f \in W^p_r(u)$, with $r \ge 1$ and $1 \le p \le \infty$, we have

$$\begin{aligned} \Omega^r_\varphi(f,t)_{u,p} &\sim \sup_{0<h\le t} \inf_{g \in W^p_r} \left\{ \|(f-g)u\|_{L^p(\mathcal{I}_h)} + h^r \left\| g^{(r)}\varphi^r u \right\|_{L^p(\mathcal{I}_h)} \right\} \\ &\le \mathcal{C} \sup_{0<h\le t} h^r \left\| f^{(r)}\varphi^r u \right\|_{L^p(\mathcal{I}_h)} \end{aligned}$$

where $\mathcal{C} \neq \mathcal{C}(f,t)$.

By means of the $r-$th modulus of smoothness, for $1 \le p \le \infty$, we can define the Zygmund spaces

$$Z_s^p(u) := Z_{s,r}^p(u) = \left\{ f \in L_u^p : \sup_{t>0} \frac{\omega_\varphi^r(f,t)_{u,p}}{t^s} < \infty,\ r > s \right\}, \quad 0 < s \in \mathbb{R},$$

equipped with the norm

$$\|f\|_{Z_{s,r}^p(u)} = \|f\|_{L_u^p} + \sup_{t>0} \frac{\omega_\varphi^r(f,t)_{u,p}}{t^s}\,.$$

In the sequel we will denote these subspaces briefly by $Z_s^p(u)$, without the second index r and with the assumption $r > s$. We note that $\Omega_\varphi^r(f,t)_{u,p} \sim \omega_\varphi^r(f,t)_{u,p}$ for any $f \in Z_s^p(u)$, $r > s$.

It is useful to observe that the spaces L_u^p, $W_r^p(u)$ and $Z_s^p(u)$ are analogous to L^p, W_r^p and Z_s^p used in the trigonometric approximation of periodic functions, whereas the moduli of smoothness have a different nature, since the step of the finite difference $\Delta_{h\varphi}^r$ is variable.

2.2. **Error of best weighted approximation.** Let us denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m and by

$$E_m(f)_{u,p} = \inf_{P \in \mathbb{P}_m} \|(f-P)\,u\|_p$$

the error of best polynomial approximation in L_u^p, $1 \le p \le \infty$. A polynomial realizing the infimum in the previous definition is called polynomial of best approximation for $f \in L_u^p$.

The next theorem collects the Jackson and Stechkin type inequalities and it can be deduced from the results proved in [21] for the weight $\sigma(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, taking into account that the weight u has a similar behaviour (see also [22, Proposition 2.3, p. 627]).

Theorem 2.1. (cfr. [21, Theorems 3.4, 3.5 and 3.6, p. 175] and [29, Theorems 4.1 and 4.2, p. 297]) *Let $u(x) = (1-x^2)^\gamma \mathrm{e}^{-\frac{1}{2}(1-x^2)^{-\alpha}}$, with $\alpha > 0$ and $\gamma \ge 0$. For any $f \in L_u^p$, $1 \le p \le \infty$, the inequalities*

$$E_m(f)_{u,p} \le \mathcal{C}\, \omega_\varphi^r \left(f, \frac{1}{m} \right)_{u,p}, \tag{2.3}$$

$$E_m(f)_{u,p} \le \mathcal{C} \int_0^{\frac{1}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t} dt \tag{2.4}$$

and

$$\omega_\varphi^r \left(f, \frac{1}{m} \right)_{u,p} \le \frac{\mathcal{C}}{m^r} \sum_{i=0}^{m} (1+i)^{r-1} E_i(f)_{u,p}\,, \tag{2.5}$$

hold with $\mathcal{C}$ independent of m and f.

Note that (2.3) and (2.5) are analogues to (1.4) and (1.5), respectively, while (2.4) is a weak form of the Jackson inequality. By analogy with the trigonometric case, the proof of Stechkin-type inequality (2.5) is based on the Bernstein inequality (see [29])

$$\|P_m'\varphi u\|_p \le \mathcal{C}m\|P_m u\|_p\,.$$

With the help of Theorem 2.1, we can characterize the weighted function spaces L^p_u, namely by

$$\lim_m E_m(f)_{u,p} = 0 \quad \Leftrightarrow \quad f \in L^p_u\,.$$

Moreover, from (2.3) and (2.2) we deduce the following estimates for the error of best approximation

$$E_m(f)_{u,p} \le \frac{\mathcal{C}}{m^r}\|f\|_{W^p_r(u)}\,, \qquad \forall f \in W^p_r(u)\,,\ r \ge 1\,, \tag{2.6}$$

and

$$E_m(f)_{u,p} \le \frac{\mathcal{C}}{m^s}\|f\|_{Z^p_s(u)}\,, \qquad \forall f \in Z^p_s(u)\,,\ s > 0\,, \tag{2.7}$$

where $\mathcal{C} \ne \mathcal{C}(m,f)$ and $1 \le p \le \infty$.

In [29], using the Nikolskii inequalities

$$\|P_m u\|_q \le \mathcal{C}\left(\frac{m}{\sqrt{1-a_m}}\right)^{\frac{1}{p}-\frac{1}{q}}\|P_m u\|_p\,, \qquad \mathcal{C} \ne \mathcal{C}(m,P_m),$$

and

$$\|P_m\varphi^{\frac{1}{p}-\frac{1}{q}}u\|_q \le \mathcal{C}m^{\frac{1}{p}-\frac{1}{q}}\|P_m u\|_p\,, \qquad \mathcal{C} \ne \mathcal{C}(m,P_m),$$

for any $P_m \in \mathbb{P}_m$ and for $1 \le p < q \le \infty$, some embedding theorems among the function spaces related to u have been proved, extending the results proved by P.L. Ul'yanov in [37] for the periodic case.

Theorem 2.2. (see [29, Theorem 4.3 and Corollary 4.5, p. 298–299]) *If $f \in L^p_u$, $1 \le p < \infty$, is such that*

$$\int_0^1 \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t < \infty\,, \qquad r \ge 1\,,$$

then f is continuous on $(-1,1)$, while if

$$\int_0^1 \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+\nu/p}}\,\mathrm{d}t < \infty\,, \qquad r \ge 1\,,$$

where $\nu = (2\alpha+2)/(2\alpha+1)$ then $f \in C_u$.

Finally, the following equivalence holds

$$\omega^r_\varphi\left(f,\frac{1}{m}\right)_{u,p} \sim \inf_{P_m}\left\{\|(f-P_m)u\|_p + \frac{1}{m^r}\left\|P_m^{(r)}\varphi^r u\right\|_p\right\}\,.$$

3. Approximation operators

For $\alpha = 0$ then u and w are Jacobi weights and the results related to Fourier sums and Lagrange interpolation are known since over twenty years and can be found for instance in [15, 4]. These approximation processes are useful in the (weighted) polynomial approximation of locally continuous functions, having algebraic singularities at the endpoints ± 1 and at some inner points. Nevertheless, these processes are not suitable in order to approximate functions having exponential growth close to ± 1 . This last topic has received few attention and, as far as we know, we recall [2, 12, 13, 31, 35].

Let us denote by $S_m(w,f)$ the $m-$th Fourier sum of $f \in L^1_w$ in the orthonormal system $p_m(w)_m$ w.r.t. the weight w in (1.1), and by $L_m(w,f)$ the Lagrange interpolation polynomial of $f \in C^0(-1,1)$ based on the zeros of $p_m(w)$. Unfortunately, as in the case of exponential weights on unbounded intervals (see, e.g., [19, 20, 15, 26, 27]), the sequence $S_m(w,f)_m$ converges to f in L^p_u for a restricted class of functions (see [22, 23]). Therefore, we cannot expect good approximation properties for the polynomial $L_m(w,f)$, which is the discrete version of $S_m(w,f)$. In fact, the associated Lebesgue constants in L^p_u are "big" (see [2, 12]). On the other hand, bounded projectors, or projectors having the minimal order $\log m$, are required in several contexts. So, in this Section we are going to consider some modified Lagrange-type and Fourier-type operators having optimal convergence order.

3.1. **Lagrange interpolation.** Let us consider w defined as in (1.1), i.e.,

$$w(x) = (1-x^2)^\beta \mathrm{e}^{-(1-x^2)^{-\alpha}}\,, \qquad \beta \ge 0\,, \alpha > 0\,,$$

and the related sequence of orthonormal polynomials $\{p_m(w)\}_m$ with positive leading coefficient. The zeros of $p_m(w)$ are ordered as

$$-a_m < x_1 < \ldots < x_m < a_m$$

where $a_m = a_m(\sqrt{w})$ is the Mhaskar–Rahmanov–Saff number and $1 - a_m \sim \left(\frac{1}{m}\right)^{\frac{1}{\alpha+1/2}}$.

We want to study the Lagrange interpolation based on the zeros of $p_m(w)$ in the spaces related to the weight u defined in (2.1), i.e.,

$$u(x) = (1-x^2)^\gamma \mathrm{e}^{-\frac{1}{2}(1-x^2)^{-\alpha}}\,, \qquad \gamma \ge 0\,, \alpha > 0\,.$$

First of all we recall that, denoting by $L_m(w,f)$ the Lagrange polynomial interpolating $f \in C_u$ at zeros of $p_m(w)$, the related Lebesgue constant is not optimal. For instance, if w and u are replaced by $\sigma(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$ and $\sqrt{\sigma}$,

$$\|L_m(\sigma)\|_\infty = \|L_m(\sigma)\|_{C_{\sqrt{\sigma}} \to C_{\sqrt{\sigma}}} \sim m^{\frac{1}{6}\left(\frac{2\alpha+3}{2\alpha+1}\right)}\,,$$

as can be deduced from a result of S.B. Damelin in [2]. On the other hand, using an idea of J. Szabados in [33], if we consider $\widetilde{L}_{m+2}(w,f)$, interpolating f in the additional nodes $\pm a_m$, we obtain (see [24])

$$\|\widetilde{L}_{m+2}(w)\|_\infty = \|\widetilde{L}_{m+2}(w)\|_{C_u \to C_u} \sim \log m\,.$$

Now, with a procedure used for different exponential weights (see, e.g., [17, 19, 25, 26, 27]), we are going to introduce a "truncated" interpolation process. Namely, we define the Lagrange–type polynomial

$$L^*_{m+2}(w,f,x) = \sum_{|x_k|\le a_{\theta m}} \ell_k(x) f(x_k)\,,$$

with

$$\ell_k(x) = \frac{(a_m^2-x^2)p_m(w,x)}{(a_m^2-x_k^2)p'_m(w,x_k)(x-x_k)} \qquad |x_k|\le a_{\theta m}\,.$$

The polynomial $L^*_{m+2}(w,f)$ does not interpolate the function f at all the nodes, but

$$L^*_{m+2}(w,f,x_k) = \begin{cases} f(x_k) & |x_k|\le a_{\theta m} \\ 0 & |x_k| > a_{\theta m} \end{cases}\,.$$

So the operator $L^*_{m+2}(w)$ does not preserves all polynomials of $\mathbb{P}_{m+1}$. Indeed,

$$L^*_{m+2}(w,\mathbf{1},x) \neq 1\,.$$

Nevertheless, if we introduce the polynomial space

$$\mathcal{P}^*_{m+1} = \{Q\in\mathbb{P}_{m+1}:\ 0 = Q(\pm a_m) = Q(x_k),\ |x_k|>a_{\theta m}\}$$

we have

$$\mathcal{P}^*_{m+1} = L^*_{m+2}(w)\,(\mathbb{P}_{m+1})$$

and, moreover,

$$E^*_m(f)_{u,p} = \inf_{Q\in\mathcal{P}^*_{m+1}} \|(f-Q)\,u\|_p \le \mathcal{C}\left[E_M(f)_{u,p} + \mathrm{e}^{-cM^\eta}\|fu\|_p\right],$$

where $M=\left\lfloor\left(\frac{\theta}{\theta+1}\right)\frac{m}{s}\right\rfloor \sim m$, $s>1$ fixed, and $\eta = 2\alpha/(2\alpha{+}1)$, as shown in [24, formula (2.17), p. 71].

Now, let us investigate the convergence of the sequence of operators $\{L^*_{m+2}(w)\}_m$ in the weighted spaces associated with u.

Theorem 3.1. (see [24, Theorem 3.7, p. 75]) *We have*

$$\left\|L^*_{m+2}(w,f)u\right\|_\infty \le \mathcal{C}(\log m)\|\chi f u\|_\infty \qquad \forall f\in C_u\,,$$

where $\mathcal{C}\neq\mathcal{C}(m,f)$ and χ is the characteristic function of $[-a_{\theta m},a_{\theta m}]$, if and only if

$$0\le \gamma - \frac{\beta}{2} + \frac{3}{4} \le 1\,. \tag{3.1}$$

Moreover, under the assumptions (3.1), we have

$$\left\|\left[f-L^*_{m+2}(w,f)\right]u\right\|_\infty \le \mathcal{C}\left[(\log m)E_M(f)_{u,\infty} + \mathrm{e}^{-cM^\eta}\|fu\|_\infty\right]$$

for any $f\in C_u$, where $M=\left\lfloor\left(\frac{\theta}{\theta+1}\right)\frac{m}{s}\right\rfloor\sim m$, $s>1$ fixed, $\eta=2\alpha/(2\alpha+1)$, $\mathcal{C}\neq\mathcal{C}(m,f)$ and $c\neq c(m,f)$ in both cases.

To study the operator $L^*_{m+2}(w)$ in some subspaces of L^p_u, with $p<\infty$, we set

$$v(x)=1-x^2\,.$$

So we can state the following

Theorem 3.2. (see [24, Theorem 3.8, p. 75]) *Let $1\le p<\infty$. We have*

$$\left\|L^*_{m+2}(w,f)u\right\|_p \le \mathcal{C}\|\chi f u\|_\infty \qquad \forall f\in C_u\,,$$

with $\mathcal{C}\ne\mathcal{C}(m,f)$ if and only if

$$\frac{v^{\gamma+1}}{\sqrt{v^\beta\varphi}}\in L^p \quad \textit{and} \quad \frac{\sqrt{v^\beta\varphi}}{v^{\gamma+1}}\in L^1\,.$$

In order to show a more complete result, we consider the Dini-type spaces

$$D^p_u=\left\{f\in L^p_u:\ \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\in L^1(0,1)\right\}\quad 1<p<\infty\,,$$

with the norm

$$\|f\|_{D^p_u}=\|fu\|_p+\int_0^1\frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\,.$$

We remark that, by Theorem 2.2 if $f\in D^p_u$ then f is continuous on $(-1,1)$. Moreover, the space D^p_u is a wide subspace of L^p_u, since it includes the Sobolev spaces $W^p_r(u)$ and the Zygmund spaces $Z^p_s(u)$ for $s>1/p$.

Theorem 3.3. (see [24, Theorem 3.8, p. 75]) *Let $1<p<\infty$. Then, with $\Delta x_k=x_{k+1}-x_k$, we have*

$$\left\|L^*_{m+2}(w,f)u\right\|_p\sim\left(\sum_{|x_k|\le a_{\theta m}}\Delta x_k|fu|^p(x_k)\right)^{1/p}\qquad\forall f\in D^p_u\,, \tag{3.2}$$

with the constants in "$\sim$" independent of f and m if and only if

$$\frac{v^{\gamma+1}}{\sqrt{v^\beta\varphi}}\in L^p \quad \textit{and} \quad \frac{\sqrt{v^\beta\varphi}}{v^{\gamma+1}}\in L^q\,,\quad \frac{1}{p}+\frac{1}{q}=1\,. \tag{3.3}$$

Moreover, under the assumptions (3.3), we have

$$\left\|\left[f-L^*_{m+2}(w,f)\right]u\right\|_p\le\frac{\mathcal{C}}{m^{1/p}}\int_0^{1/m}\frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t+\mathcal{C}\mathrm{e}^{-cM^\eta}\|fu\|_\infty$$

for any $f\in D^p_u$, where $M=\left\lfloor\left(\frac{\theta}{\theta+1}\right)\frac{m}{s}\right\rfloor\sim m$, $s>1$ fixed, $\eta=2\alpha/(2\alpha+1)$, $\mathcal{C}\ne\mathcal{C}(m,f)$ and $c\ne c(m,f)$.

We note that inequality (3.2), if $f = Q \in \mathcal{P}^*_{m+1}$ is the analogue to the Marcinkiewicz equivalence (1.8), holding for trigonometric polynomials.

For example, if $f \in W^p_r(u)$, by Theorem 3.3, we have

$$\left\| \left[f - L^*_{m+2}(w,f) \right] u \right\|_p \leq \frac{\mathcal{C}}{m^r} \|f\|_{W^p_r(u)} ,$$

for m sufficiently large, as in the periodic case.

Moreover, the operators $L^*_{m+2}(w)$ are uniformly bounded in Sobolev and Zygmund spaces, under the assumptions (3.3). More in general, from the results in [24] we can deduce

Corollary 3.4. *Under the assumptions* (3.3)*, we have*

$$\sup_m \|L^*_{m+2}(w,f)\|_{D^p_u} \leq \mathcal{C} \|f\|_{D^p_u}$$

for any $f \in D^p_u$, $1 < p < \infty$, *where* $\mathcal{C} \neq \mathcal{C}(f)$.

To conclude this section, we want to emphasize that Theorem 3.3 is false if we replace $L^*_{m+2}(w)$ by $\widetilde{L}_{m+2}(w)$, since the constant $\mathcal{C}$ depends on the parameter θ, and in particular on $\log^{-1}(1/\theta)$. So, the "truncation" is crucial in our results, since parameter θ cannot assume the value 1 (for further details see [24]).

3.2. **Fourier sums.** By analogy with the Lagrange interpolation, the behaviour of the classical Fourier sums in the orthonormal system associated with the weight w

$$S_m(w,f,x) = \sum_{k=0}^{m-1} c_k p_k(w,x) , \qquad c_k = \int_{-1}^{1} f(t) p_k(w,t) w(t) \, \mathrm{d}t ,$$

is not optimal. For instance, with $\sigma(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, we have (see [22, Proposition 3.1, p. 628])

$$\left\| S_m(\sigma,f)\sqrt{\sigma} \right\|_p \leq \mathcal{C} \|f\sqrt{\sigma}\|_p \quad \forall f \in L^p_{\sqrt{\sigma}} \quad \forall m \in \mathbb{N} \qquad \Rightarrow \qquad \frac{4}{3} < p < 4 ,$$

and so the Fourier sums converge for a restricted class of functions.

Here, we want to discuss whether bounds of the form

$$\|S_m(w,f)u\|_p \leq \mathcal{C} \|fu\|_p$$

hold with

$$w(x) = v^\beta(x) \mathrm{e}^{-(1-x^2)^{-\alpha}} , \qquad u(x) = v^\gamma(x) \mathrm{e}^{-\frac{(1-x^2)^{-\alpha}}{2}}$$

and $v(x) = 1 - x^2$.

We point out that, since w does not belong to the Szegő class, it would seem that the Pollard decomposition could not hold. Neverthless, in [22, Proposition 2.2, p. 627, and formula (27) p. 632] we proved that this decomposition holds true. Namely, the Christoffel–Darboux kernel

$$K_m(w,x,t) := \sum_{k=0}^{m-1} p_k(w,x) p_k(w,t)$$

can be written as

$$K_m(w,x,t) = -\alpha_m p_m(w,x)p_m(w,t) \\ + \beta_m \frac{p_m(w,x)p_{m-1}(w\varphi^2,t)\varphi^2(t) - p_{m-1}(w\varphi^2,x)\varphi^2(x)p_m(w,t)}{x-t},$$

where $\varphi^2(t) = 1-t^2$ and $\alpha_m \sim 1 \sim \beta_m$.

Denoting by χ the characteristic function of $[-a_{\theta m}, a_{\theta m}]$ we obtain the following

Theorem 3.5. (see [22, Theorem 3.2, pp. 628-629]) *Let $1<p<\infty$. Then*

$$\|\chi S_m(w,\chi f)u\|_p \le \mathcal{C}_\theta \|\chi f u\|_p \qquad \forall f \in L^p_u,$$

with $\mathcal{C}_\theta = \mathcal{O}(\frac{1}{\theta}, \frac{1}{1-\theta})$, if and only if

$$\frac{v^\gamma}{\sqrt{v^\beta \varphi}} \in L^p, \quad \frac{1}{v^\gamma}\sqrt{\frac{v^\beta}{\varphi}} \in L^q, \quad \frac{1}{p}+\frac{1}{q}=1. \tag{3.4}$$

Moreover, under the assumptions (3.4), we have

$$\|[f-\chi S_m(w,\chi f)]\,u\|_p \le \mathcal{C}_\theta \left\{E_M(f)_{u,p} + \mathrm{e}^{-cM^\eta}\|fu\|_p\right\}$$

for any $f \in L^p_u$, where $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right)\frac{m}{s}\right\rfloor \sim m$, $s>1$ fixed, $\eta = 2\alpha/(2\alpha+1)$, C_θ and c are independent of f and m.

If we truncate only the function f, we obtain

Theorem 3.6. (see [22, Theorem 3.3, p. 629]) *For $1<p<4$, the conditions*

$$\frac{v^\gamma}{\sqrt{v^\beta \varphi}} \in L^p, \quad \frac{1}{v^\gamma}\sqrt{\frac{v^\beta}{\varphi}} \in L^q, \quad \frac{1}{p}+\frac{1}{q}=1, \tag{3.5}$$

are equivalent to

$$\|S_m(w,\chi f)u\|_p \le \mathcal{C}_\theta \|\chi f u\|_p \qquad \forall f \in L^p_u.$$

Moreover, under the assumptions (3.5), we have

$$\|[f-S_m(w,\chi f)]\,u\|_p \le \mathcal{C}_\theta \left\{E_M(f)_{u,p} + \mathrm{e}^{-cM^\eta}\|fu\|_p\right\},$$

for any $f \in L^p_u$, where $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right)\frac{m}{s}\right\rfloor \sim m$, $s>1$ fixed, $\eta = 2\alpha/(2\alpha+1)$, $\mathcal{C} \ne \mathcal{C}(m,f)$ and $c \ne c(m,f)$.

While, for $p=\infty$ and $p=1$, we get

Theorem 3.7. (see [23, Theorem 1.1, p. 1677]) *The inequality*

$$\|\chi S_m(w,\chi f)u\|_\infty \le \mathcal{C}_\theta (\log m)\|\chi f u\|_\infty$$

holds for any $f \in C_u$ if and only if

$$\frac{1}{4} \le \gamma - \frac{\beta}{2} \le \frac{3}{4}. \tag{3.6}$$

Moreover, under the assumptions (3.6), *we have*

$$\|[f-\chi S_m(w,\chi f)]\,u\|_\infty \le \mathcal{C}_\theta\left\{(\log m)E_M(f)_{u,\infty}+\mathrm{e}^{-cM^\eta}\|fu\|_\infty\right\}$$

for any $f\in C_u$, *where* $M=\left\lfloor\left(\frac{\theta}{\theta+1}\right)\frac{m}{s}\right\rfloor\sim m$, $s>1$ *fixed*, $\eta=2\alpha/(2\alpha+1)$, $\mathcal{C}\ne\mathcal{C}(m,f)$ *and* $c\ne c(m,f)$ *in both cases.*

Theorem 3.8. (see [23, Theorem 1.2, pp. 1677–1678]) *The inequality*

$$\|\chi S_m(w,\chi f)u\|_1\le \mathcal{C}_\theta(\log m)\|\chi f u\|_1$$

holds for any $f\in L^1_u$ *if and only if*

$$\frac{v^\gamma}{\sqrt{v^\beta\varphi}}\in L^1,\quad \frac{1}{v^\gamma}\sqrt{\frac{v^\beta}{\varphi}}\in L^\infty\,. \tag{3.7}$$

Moreover, under the assumptions (3.7), *we have*

$$\|[f-\chi S_m(w,\chi f)]\,u\|_1 \le \mathcal{C}_\theta\left\{(\log m)E_M(f)_{u,1}+\mathrm{e}^{-cM^\eta}\|fu\|_1\right\}$$

for any $f\in L^1_u$, *where* $M=\left\lfloor\left(\frac{\theta}{\theta+1}\right)\frac{m}{s}\right\rfloor\sim m$, $s>1$ *fixed*, $\eta=2\alpha/(2\alpha+1)$, $\mathcal{C}\ne\mathcal{C}(m,f)$ *and* $c\ne c(m,f)$ *in both cases.*

We note that, Theorems 3.5, 3.7 and 3.8 are only partially analogous to the inequalities (1.7) and (1.6), since the elements of the sequence $\{\chi S_m(w,f)\}_m$ are not polynomials on the whole interval $(-1,1)$, but are "truncated" polynomials, in contrast to $\{L^*_{m+2}(w,f)\}_m$ which is a sequence of polynomials on $(-1,1)$.

Nevertheless, as in the trigonometric case, under proper assumptions on the weights w and u, the sequence $\{\chi S_m(w,f)\}_m$ converges essentially with the order of the best approximation in L^p_u, $1<p<\infty$, and with the order of the best approximation times an extra factor $\log m$ for $p=1,\infty$.

4. Gaussian quadrature rules

In this Section we are going to consider the Gaussian quadrature rule related to the weight w in (1.1), in order to approximate integrals on $(-1,1)$ containing functions dacaying exponentially at the endpoints. This topic has received attention in the literature only recently, although being of interest in several contexts (see [3, 18]).

First of all, we are going to show that the Gaussian rule has not an optimal behavior in order to approximate integrals of the form

$$\int_{-1}^{1} f(x)w(x)\,\mathrm{d}x\,,$$

where $f\in W^1_1(w)$. This phenomenon occurs also in the case of exponential weights on unbounded intervals and in this regard the reader can consult, for instance, [5, 17, 19, 16] and the references therein. On the other hand, this fact contrasts with what happens on bounded intervals for Jacobi weights: in such a case, the error of the Gaussian rule converges to zero with the same order of the best approximation in weighted L^1-Sobolev spaces (see

[15, p. 338]). Therefore, also following an idea in [17], we are going to propose a quadrature rule that is as simple as the Gaussian rule but requires a lower computational cost and converges with the order of the best polynomial approximation if $f \in W_1^1(w)$. We point out that the results in this section can be deduced from those proved in [3] for the weight $\sigma(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, taking into account that the weight w has a similar behaviour (see also [22, Proposition 2.3, p. 627]).

Let us consider w defined as in (1.1), i.e.

$$w(x) = (1-x^2)^{\beta}\mathrm{e}^{-(1-x^2)^{-\alpha}} \qquad \beta \geq 0\,, \alpha > 0\,,$$

and the related sequence of orthonormal polynomials $\{p_m(w)\}_m$ with positive leading coefficient.

The Gaussian rule related to w is defined as

$$\int_{-1}^{1} f(x)w(x)\,\mathrm{d}x = \sum_{k=1}^{m} \lambda_k(w) f(x_k) + e_m(f) =: G_m(f) + e_m(f)\,,$$

where x_k are the zeros of $p_m(w)$ and $\lambda_k(w)$ are the Christoffel numbers.

We note that the considered weights w are nonclassical and the coefficients of the three-term recurrence relation of the corresponding orthonormal polynomials are unknown. Therefore, we computed the moments

$$\mu_k = \int_{-1}^{1} x^k w(x)\mathrm{d}x, \quad k = 0, 1, \ldots,$$

in variable-precision arithmetic to approximate these coefficients and then we have calculated the zeros of $p_m(w)$ and the Christoffel numbers, using the functions "aChebyshevAlgorithm" and "aGaussianNodesWeights" of the MATHEMATICA package "`OrthogonalPolynomials`" (see [1, 28]).

It is useful to remark that applying a Gauss–Legendre formula to approximate integrals of the form

$$\int_{-1}^{1} f(x)w(x)\,\mathrm{d}x$$

is a bad idea, since the distance between -1 and the first Legendre zeros is $\sim m^{-2}$, while for the Pollaczek-type zeros we have $\sim m^{-\frac{1}{\alpha+1/2}}$. For example, let us consider (see [3, Example 1])

$$\int_{-1}^{1} f(x)w(x)\,\mathrm{d}x = \int_{-1}^{1} \cos(\pi x)\,\mathrm{e}^{-(1-x^2)^{-50}}\,\mathrm{d}x\,.$$

In Table 1, we compare the Gauss–Legendre formula applied to fw and the Gauss–Pollaczek-type rule applied to f with the weight w. We write only the correct digits and the symbol "–" means that the required precision has been achieved. In Figure 1 we show the graph of

m	Legendre rule	Pollaczek-type rule
4	0.0	0.07236909
8	0.0	0.072369091024665
16	0.07	–
32	0.07	–
64	0.07236	–
128	0.07236909	–
256	0.07236909102466	–
512	0.072369091024665	–

TABLE 1. Comparison with the Gauss–Legendre rule

the integrand function fw.

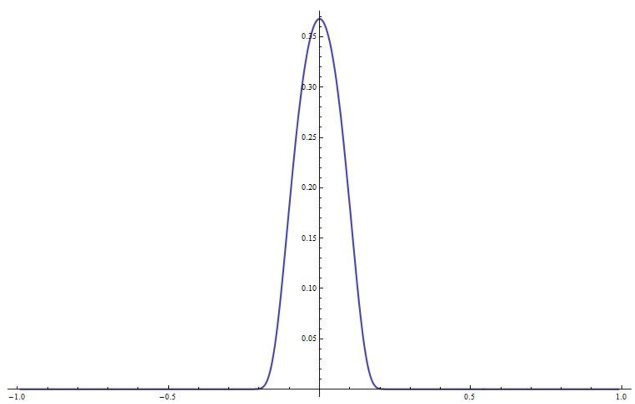

FIGURE 1. Graph of fw

Another comparison between the Gauss–Legendre and the Gauss–Pollaczek-type rule has been given by M. Masjed-Jamei and G.V. Milovanović in [14, Example 4.1, p. 187] for the integral

$$\int_{-1}^{1} f(x)w(x)\,\mathrm{d}x = \int_{-1}^{1} \frac{3\mathrm{e}^{-\frac{1}{\sqrt{1-x^2}}} - 2\sin(3x) - x^2}{(1-x^2)^2}\,\mathrm{e}^{-(1-x^2)^{-\alpha}}\mathrm{d}x\,,$$

with $\alpha = 1/2$ and $\alpha = 10$. In Tables 2 and 3, we rewrite the relative errors they obtained in the Gauss–Legendre formula applied to fw and the Gauss–Pollaczek-type rule applied to f with the weight w In Figure 2 we show the graphs of the integrand functions fw. We note that in this case f contains $\mathrm{e}^{-\frac{1}{\sqrt{1-x^2}}}$ and for $\alpha = 1/2$ the two Gaussian rules behave similarly, while for $\alpha = 10$ the Gauss–Pollaczek-type rule converges much faster.

m	Legendre rule	Pollaczek-type rule
10	1.01	1.66
20	$1.43 \cdot 10^{-1}$	$2.38 \cdot 10^{-1}$
30	$1.12 \cdot 10^{-2}$	$4.54 \cdot 10^{-2}$
40	$4.87 \cdot 10^{-3}$	$1.04 \cdot 10^{-2}$
50	$7.09 \cdot 10^{-4}$	$2.71 \cdot 10^{-3}$

TABLE 2. Relative errors for $\alpha = 1/2$

m	Legendre rule	Pollaczek-type rule
10	$3.52 \cdot 10^{-2}$	$4.32 \cdot 10^{-13}$
20	$1.21 \cdot 10^{-3}$	$2.94 \cdot 10^{-24}$
30	$1.57 \cdot 10^{-5}$	$5.27 \cdot 10^{-35}$
40	$2.93 \cdot 10^{-6}$	$1.86 \cdot 10^{-45}$
50	$1.82 \cdot 10^{-7}$	$1.09 \cdot 10^{-55}$

TABLE 3. Relative errors for $\alpha = 10$

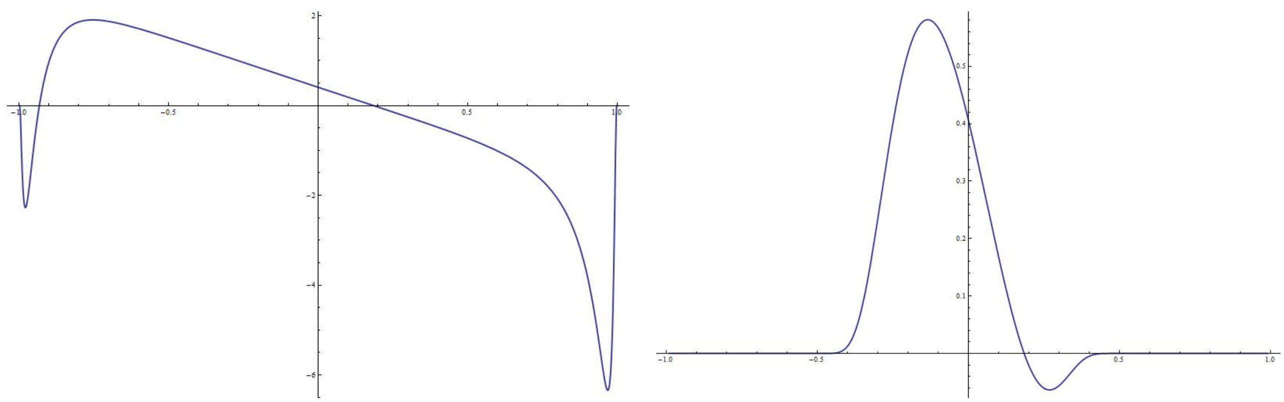

FIGURE 2. Graph of fw for $\alpha = 1/2$ (left) and $\alpha = 10$ (right)

Naturally, if $f \in C^0[-1,1]$ the error of the associated Gaussian rules satisfies the estimate

$$|e_m(f)| \le 2\|w\|_1 E_{2m-1}(f)_\infty ,$$

where $E_m(f)_\infty$ denotes the unweighted error of best polynomial approximation. Moreover, if $f \in C_w$, it is easily seen that (cfr. [3, p. 439])

$$|e_m(f)| \le \mathcal{C} E_{2m-1}(f)_{w,\infty} , \quad \mathcal{C} \neq \mathcal{C}(m, f) .$$

So, by (2.3), the error of the Gaussian rule converges to zero with the order of the best approximation in C_w. Moreover, using arguments analogous to those in [16], for infinitely differentiable functions we obtain

Theorem 4.1. (cfr. [16, Theorem 3.2, p. 1660]) *Let $u(x)=[w(x)]^\delta$, with $0<\delta<1$. For any infinitely differentiable function f such that*

$$K(f):=\sup_m \|f^{(m)}u\|_\infty<\infty\,,$$

we have

$$\lim_m \sqrt[m]{\frac{|e_m(f)|}{K(f)}}=0\,.$$

Nevertheless, if we want to estimate the error of this quadrature rule for functions $f\in W_1^1(w)$, we obtain only

$$|e_m(f)|\le \mathcal{C}\frac{m^{\frac{2\alpha}{2\alpha+3}}}{m}\|f'\varphi w\|_1\,,$$

and this estimate cannot be improved (see [3, Theorem 2, p. 440]). So the error of the rule does not converge with the order of best approximation for $f\in W_1^1(w)$ and this is in contrast with what happens for Jacobi weights (see [15, pp. 170, 338]). This phenomenon has led many authors to consider "truncated" Gaussian rules.

Now, let us fix $0<\theta<1$ and consider the interval

$$A_{\theta m}:=[-a_{\theta m},a_{\theta m}]\subset[-a_m,a_m]=:A_m\,.$$

So, introducing the "truncated" Gaussian rule

$$\int_{-1}^1 f(x)w(x)\,\mathrm{d}x=\sum_{|x_k|\le a_{\theta m}}\lambda_k(w)f(x_k)+e_m^*(f)\,.$$

This is the ordinary Gaussian formula, in which we drop the terms related to zeros closest to the endpoint of the interval of integration. This produces a reduction of the computational cost, in terms of evaluation of the integrand function, which became more evident in the numerical treatment of integral equations (see [3]).

With the "truncated" Gaussian rule we obtain the required error estimate for $f\in W_1^1(w)$, as shown in the next theorem.

Theorem 4.2. (cfr. [3, Theorem 3, p. 443]) *We have*

$$|e_m^*(f)|\le\frac{\mathcal{C}}{M}\|f'\varphi w\|_1+\mathcal{C}\mathrm{e}^{-cM^\eta}\|fw\|_1\qquad \forall f\in W_1^1(w)\,,$$

and

$$|e_m^*(f)|\le\frac{\mathcal{C}}{M}\int_0^{1/M}\frac{\omega_\varphi^r(f,t)_{w,1}}{t^2}\mathrm{d}t+\mathcal{C}\mathrm{e}^{-cM^\eta}\|fw\|_1\qquad \forall f\in Z_s^1(w)\,,\ r>s>1\,,$$

where in both cases $M=\left\lfloor\left(\frac{2\theta}{\theta+1}\right)m\right\rfloor$, $\eta=\frac{2\alpha}{2\alpha+1}$, $\mathcal{C},c$ do not depend on m and f.

In particular, from Theorem 4.2, for m sufficiently large, we deduce the estimates

$$|e_m^*(f)|\le\frac{\mathcal{C}}{m^r}\|f\|_{W_r^1(w)},\qquad \forall f\in W_r^1(w),\quad r\ge 1,\tag{4.1}$$

and

$$|e_m^*(f)| \le \frac{\mathcal{C}}{m^s}\|f\|_{Z_s^1(w)}, \qquad \forall f \in Z_s^1(w), \quad s>1 \ (s\in\mathbb{R}), \tag{4.2}$$

where $\mathcal{C}$ is independent of m and f in both cases. Therefore, in these function spaces, $e_m^*(f)$ converges to 0 with the order of the best polynomial approximation, taking into avccount by (2.6) and (2.7). As a consequence, inequalities (4.1) and (4.2), which are not true for the error of the ordinary Gaussian rule, cannot be improved from the order point of view.

Finally, we want to give the main idea that justifies the 'truncation". For any $P_m \in \mathbb{P}_m$ $1 \le p \le \infty$, the restricted range inequalities

$$\|P_m u\|_p \le \mathcal{C}\|P_m u\|_{L^p(A_m)}$$

and

$$\|P_m u\|_{L^p(A'_{sm})} \le \mathcal{C}\mathrm{e}^{-c\,m^\eta}\|P_m u\|_p, \quad A'_{sm}=[-1,1]\setminus[-a_{sm},a_{sm}],\ s>1,$$

hold with $c, \mathcal{C}$ independent of m and P_m, $\eta = 2\alpha/(2\alpha+1)$. From the second inequality, for any $f \in L_u^p$, we deduce

$$\|fu\|_p \le \mathcal{C}\|fu\|_{L^p(A_{\theta m})} + E_M(f)_{u,p}\,.$$

So, the main part of $\|fu\|_p$ is $\|fu\|_{L^p(A_{\theta m})} = \|\chi f u\|_p$. This suggests to apply the Gaussian rule only to χf, χ is the characteristic function of $A_{\theta m}$.

Acknowledgments

The authors wish to thank the referees for their remarks, which allowed to improve the previous version of this paper.

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