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Point and differential C^1 quasi-interpolation on three direction meshes

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Abstract

In this paper we construct and analyse C^1 cubic and quartic quasi-interpolating splines on type-1 triangulations approximating regularly distributed data, without using minimal determining sets and without defining the approximating splines as linear combinations of compactly supported bivariate spanning functions. In particular, the C^1 cubic splines are directly determined by setting their Bernstein-Bézier coefficients to appropriate combinations of the given data values without using prescribed derivatives at any point of the domain, in such a way that the C^1-smoothness conditions are satisfied and approximation order three is guaranteed, for smooth functions. We also propose some numerical tests that confirm the theoretical results. Then, from the above C^1 cubic splines we obtain C^1 quartic splines exact on $\mathbb{P}_3$, achieving approximation order four. The associated differential quasi-interpolation operator involves the values of the first partial derivatives in its definition.

Keywords: Spline approximation, Quasi-interpolation, Bernstein-Bézier form, Type-1 triangulation

1 Introduction

Many approaches on quasi-interpolation and interpolation by bivariate splines are based on the construction of local and stable minimal determining sets (see e.g. [12] and references therein) or on the use of locally supported spanning functions like box splines (see e.g. [5, 7, 12, 18] and references therein). Moreover, some recent literature concerns the approximation in spaces of smooth splines of low degree on triangulations [1, 3, 4, 8, 13, 14].

In this paper, we construct and analyse C^1 cubic and quartic quasi-interpolating splines on type-1 triangulations approximating regularly distributed data, without using minimal determining sets and without defining the approximating splines as linear combinations of compactly supported bivariate spanning functions.

In particular, the C^1 cubic splines are directly determined by setting their Bernstein-Bézier (BB-) coefficients to appropriate combinations of the given data values without using prescribed derivatives at any point of the domain, in such a way that the C^1-smoothness conditions are satisfied and approximation order three is guaranteed, for smooth functions. We construct and analyse two families of cubic splines, based on two different sets of evaluation points. We want to remark that, although the data needed for our schemes have to be regularly distributed, the methods here proposed can be included in a two-step approach, where in the first step a polynomial approximant is computed locally on each triangle and then the data values on each triangle can be sampled from the approximant, as in the paper [9]. Moreover, the quasi-interpolation scheme here proposed is applicable to a compact domain in the plane, by considering special rules near the boundary (see [15]) or by extending the triangulation.

Afterwards, following the general method proposed in [2], from the above C^1 cubic splines we obtain C^1 quartic splines exact on $\mathbb{P}_3$, achieving approximation order four. The associated differential quasi-interpolation operator involves the values of the first partial derivatives in its definition.

Here is an outline of the paper. In Section 2, we give some preliminaries on the BB-form of splines on type-1 triangulations and we introduce some useful notation used throughout the paper.

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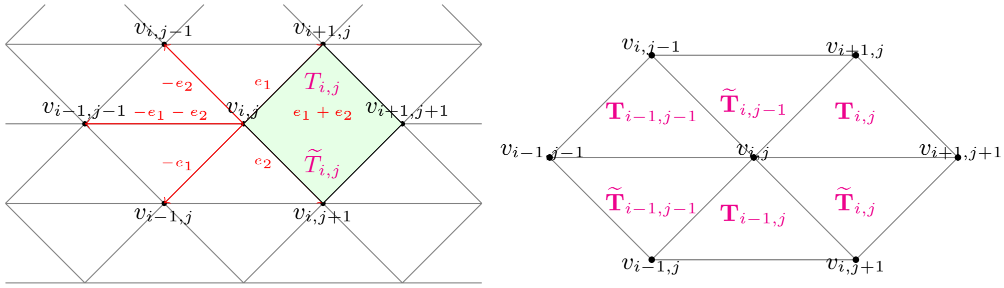


Figure 1: *The triangulation Δ (left) and the hexagon $H_{i,j}$ (right).*

In Section 3, we define families of C^1 cubic quasi-interpolating splines based on two different sets of points. We analyse the general schemes, in one case depending on two free parameters and we present some strategies in order to fix them. Moreover, we discuss the approximation properties of the corresponding operators and propose some numerical tests that confirm the theoretical results. Finally, in Section 4, from the C^1 cubic quasi-interpolant constructed in Section 3.1, we define a C^1 quartic differential quasi-interpolant exact on $\mathbb{P}_3$ and we describe it in the Bernstein basis.

2 Notations and preliminaries

Although the results presented in this paper are valid for any three-directional partition of the plane, following the notation of [16], given $h > 0$ and the two vectors $e_1 := (h, h)$ and $e_2 := (h, -h)$, the vertices $v_{i,j} := ie_1 + je_2$, $i, j \in \mathbb{Z}$, define the two-dimensional lattice $\mathcal{V} := \{v_{i,j} : i, j \in \mathbb{Z}\}$, that subdivides the plane into equal parallelograms $P_{i,j} := [v_{i,j}, v_{i,j+1}, v_{i+1,j+1}, v_{i+1,j}]$ (see Fig. 1(left)). Each parallelogram $P_{i,j}$ is split into two triangles

$$T_{i,j} := [v_{i,j}, v_{i+1,j+1}, v_{i+1,j}] \qquad \text{and} \qquad \widetilde{T}_{i,j} := [v_{i,j}, v_{i+1,j+1}, v_{i,,j+1}],$$

by drawing the diagonal $[v_{i,j}, v_{i+1,j+1}]$. Therefore, the triangulation Δ is defined in this way

$$\Delta := \bigcup_{i,j \in \mathbb{Z}} \left\{ T_{i,j}, \widetilde{T}_{i,j} \right\}.$$

The triangulation Δ can also be viewed as a collection of overlapping hexagons, as shown in Fig. 1(right), where $H_{i,j}$ is the hexagon centered at $v_{i,j}$.

We are interested in the construction of quasi-interpolating splines in the spaces

$$\mathcal{S}_\ell^1(\Delta) := \left\{ s \in C^1\left(\mathbb{R}^2\right) : s_{|T} \in \mathbb{P}_\ell, \text{ for all } T \in \Delta \right\},$$

with $\ell = 3, 4$. Here $\mathbb{P}_\ell := \text{span}\left\{ x_1^i x_2^j : 0 \le i + j \le \ell \right\}$ is the space of bivariate polynomials of total degree ℓ. Such splines will be defined by directly setting their BB-coefficients on the triangles of Δ (see e.g. [12]). Given a function $s \in \mathcal{S}_\ell^1(\Delta)$, its restriction to a triangle $T = [v_0, v_1, v_2] \in \Delta$ can be written as

$$s_{|T} = \sum_{i+j+k=\ell} c_{i,j,k}^T B_{i,j,k}^T,$$

where $B_{i,j,k}^T := \frac{\ell!}{i!j!k!} b_0^i b_1^j b_2^k$, $i, j, k \ge 0$, $i + j + k = \ell$, are the Bernstein polynomials of degree ℓ associated with T and (b_0, b_1, b_2) are the barycentric coordinates with respect to T, i.e. $x = b_0 v_0 + b_1 v_1 + b_2 v_2$, $b_0 + b_1 + b_2 = 1$ for $x := (x_1, x_2) \in T$. Notice that any reference to the triangle T has been omitted in the notation for the barycentric coordinates.

We associate the BB-coefficients $c_{i,j,k}^T$ of $s_{|T}$ relative to T with the domain points $\xi_{i,j,k}^\ell := (iv_0 + jv_1 + kv_2)/\ell$ in T. The union, without repetitions, of all domain points of each triangle in Δ gives rise to a set denoted by $\mathcal{D}_\ell$. For the construction of the quasi-interpolating splines, we also consider the subset $\mathcal{D}_2$ provided by the union, without repetitions, of the set of points $\xi_{i,j,k}^2 := (iv_0 + jv_1 + kv_2)/2$.

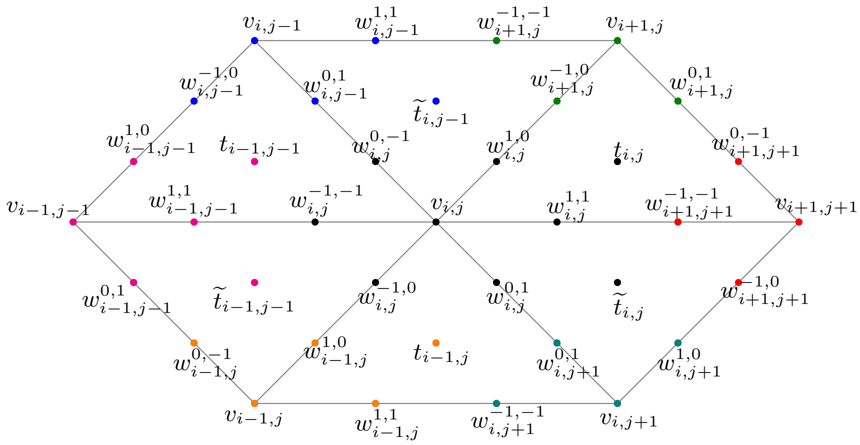


Figure 2: *The points of $\mathcal{D}_3$ relative to $H_{i,j}$.*

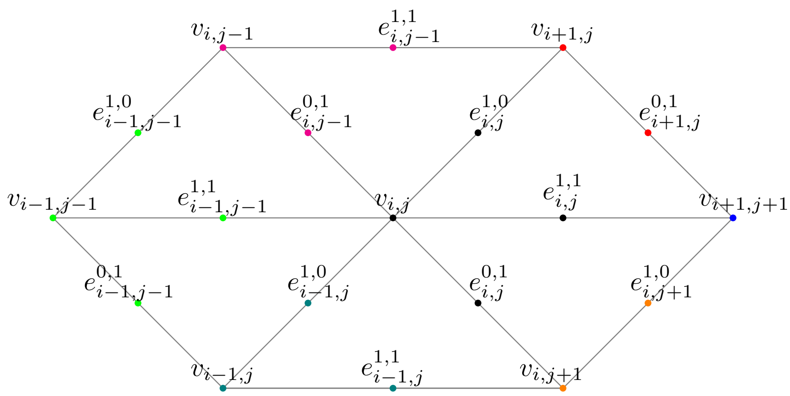


Figure 3: *The points of $\mathcal{D}_2$ relative to $H_{i,j}$.*

The proposed construction is based on appropriate partitions $\left\{\mathcal{D}^k_{i,j}, i,j \in \mathbb{Z}\right\}$ of $\mathcal{D}_k$, defined as follows: $k = 2,3$:

- $\mathcal{D}^3_{i,j} := \left\{v_{i,j}, t_{i,j}, \widetilde{t}_{i,j}\right\} \cup \left\{w^{k,m}_{i,j}, k,m \in \{-1,0,1\}, k+m \neq 0\right\}$, where

  – $t_{i,j}$ and $\widetilde{t}_{i,j}$ are the barycenters of $T_{i,j}$ and $\widetilde{T}_{i,j}$, respectively,

  – $w^{k,m}_{i,j} := \frac{1}{3}\left(2v_{i,j} + v_{i+k,j+m}\right)$,

- $\mathcal{D}^2_{i,j} := \left\{v_{i,j}, e^{1,0}_{i,j}, e^{0,1}_{i,j}, e^{1,1}_{i,j}\right\}$, with

$$e^{\ell,n}_{i,j} = \frac{1}{2}\left(e_{i,j} + e_{i+\ell,j+n}\right), \;\; \ell, h \in \{0,1\}, \;\; \ell + n \neq 0.$$

Therefore, $\mathcal{D}_k = \bigcup_{i,j} \mathcal{D}^k_{i,j}$, $k = 2,3$. Figs. 2 and 3 show the domain points in $\mathcal{D}_k$, $k = 3,2$, lying in the hexagon $H_{i,j}$, respectively.

3 C^1-cubic quasi-interpolation

In this section, we construct and analyse quasi-interpolating splines $Q_{3,\sigma} f \in \mathcal{S}^1_3(\Delta)$, $\sigma = 2,3$, to a given function $f \in C\left(\mathbb{R}^2\right)$, by assuming to know the values $f(v)$, $v \in \mathcal{D}_\sigma$.

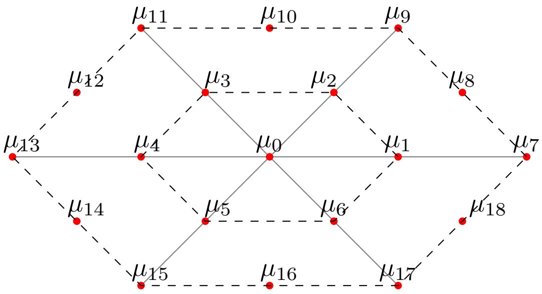


Figure 4: Ordering a general sequence μ representing $f_{i,j}(\mathcal{D}_2)$ or any of masks α, β, γ or ω_k, $k = 0, \ldots, 5$.

3.1 C^1-cubic quasi-interpolation based on $\mathcal{D}_2$

Given the values $f(v)$, $v \in \mathcal{D}_2$, we construct the spline $Q_{3,2}f \in \mathcal{S}_3^1(\Delta)$ by setting its BB-coefficients on each triangle $T \in \Delta$, taking into account that Δ is a uniform triangulation. For example, we write the restrictions of $Q_{3,2}f$ to the triangle $T_{i,j}$ as

$$\begin{aligned} Q_{3,2}f_{|T_{i,j}} &= c_3(v_{i,j}) B^{T_{i,j}}_{3,0,0} + c_3\left(w^{1,1}_{i,j}\right) B^{T_{i,j}}_{2,1,0} + c_3\left(w^{1,0}_{i,j}\right) B^{T_{i,j}}_{2,0,1} + c_3\left(w^{-1,-1}_{i+1,j+1}\right) B^{T_{i,j}}_{1,2,0} \\ &+ c_3(t_{i,j}) B^{T_{i,j}}_{1,1,1} + c_3\left(w^{-1,0}_{i+1,j}\right) B^{T_{i,j}}_{1,0,2} + c_3(v_{i+1,j+1}) B^{T_{i,j}}_{0,3,0} + c_3\left(w^{0,-1}_{i+1,j+1}\right) B^{T_{i,j}}_{0,2,1} \\ &+ c_3\left(w^{0,1}_{i+1,j}\right) B^{T_{i,j}}_{0,1,2} + c_3(v_{i+1,j}) B^{T_{i,j}}_{0,0,3}, \end{aligned} \tag{3.1}$$

with $c_3(p)$ denoting the BB-coefficient associated with the domain point $p \in D^3_{i,j}$ of the cubic polynomial $Q_{3,2}f_{|T_{i,j}}$. Notice that the three vertices defining $T_{i,j}$ are counter-clockwise ordered starting from $v_{i,j}$.

The BB-coefficients corresponding to the domain points are expressed as linear combinations of the values of f at the 19 domain points of $\mathcal{D}_2$ lying in $H_{i,j}$ (see Fig. 3). For example, the BB-coefficient associated with the domain point $v_{i,j}$ has the following form:

$$\begin{aligned} c_3(v_{i.j}) &= \alpha_0 f(v_{i,j}) + \alpha_1 f\left(e^{1,1}_{i,j}\right) + \alpha_2 f\left(e^{1,0}_{i,j}\right) + \alpha_3 f\left(e^{0,1}_{i,j-1}\right) + \alpha_4 f\left(e^{1,1}_{i-1,j-1}\right) \\ &+ \alpha_5 f\left(e^{1,0}_{i-1,j}\right) + \alpha_6 f\left(e^{0,1}_{i,j}\right) + \alpha_7 f(v_{i+1,j+1}) + \alpha_8 f\left(e^{0,1}_{i+1,j}\right) + \alpha_9 f(v_{i+1,j}) \\ &+ \alpha_{10} f\left(e^{1,1}_{i,j-1}\right) + \alpha_{11} f(v_{i,j-1}) + \alpha_{12} f\left(e^{1,0}_{i-1,j-1}\right) + \alpha_{13} f(v_{i-1,j-1}) + \alpha_{14} f\left(e^{0,1}_{i-1,j-1}\right) \\ &+ \alpha_{15} f(v_{i-1,j}) + \alpha_{16} f\left(e^{1,1}_{i-1,j}\right) + \alpha_{17} f(v_{i,j+1}) + \alpha_{18} f\left(e^{1,0}_{i,j+1}\right). \end{aligned} \tag{3.2}$$

In order to simplify the notations, let $f_{i,j}(\mathcal{D}_2) \in \mathbb{R}^{19}$ be the vector of the values of f at the 19 domain points of $\mathcal{D}_2$ lying in $H_{i,j}$, enumerated as in Fig. 4, and let $\alpha \in \mathbb{R}^{19}$ be the vector whose elements are enumerated in the same way. We call α a *mask*. Therefore, we write

$$c_3(v_{i,j}) = f_{i,j}(\mathcal{D}_2) \cdot \alpha,$$

where $A \cdot B := \sum_{k=1}^n A_k B_k$, with n the cardinality of A and B. The BB-coefficients associated with the $w-$points $w^{1,1}_{i,j}$, $w^{1,0}_{i,j}$, $w^{0,-1}_{i,j}$, $w^{-1,-1}_{i,j}$, $w^{-1,0}_{i,j}$, and $w^{0,1}_{i,j}$ are defined in a similar way by using masks ω_k, $0 \leq k \leq 5$, respectively.

Analogously, the BB-coefficients $c_3(t_{i,j})$ and $c_3\left(\widetilde{t}_{i,j}\right)$ are defined by considering masks β and γ, respectively:

$$c_3(t_{i,j}) = f_{i,j}(\mathcal{D}_2) \cdot \beta \qquad \text{and} \qquad c_3\left(\widetilde{t}_{i,j}\right) = f_{i,j}(\mathcal{D}_2) \cdot \gamma.$$

The main goal in this section is to define masks α, β, γ and ω_k, $0 \leq k \leq 5$, such that

$$Q_{3,2}f \in C^1\left(\mathbb{R}^2\right) \quad \text{and} \quad Q_{3,2}f = f \text{ for all } f \in \mathbb{P}_2. \tag{3.3}$$

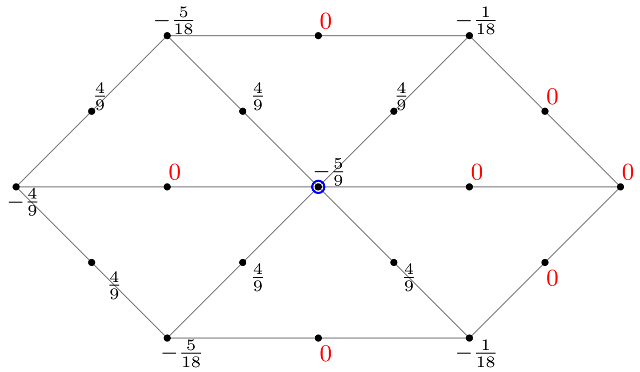


Figure 5: *Mask α for the evaluation of the BB-coefficient associated with the point $v_{i,j}$.*

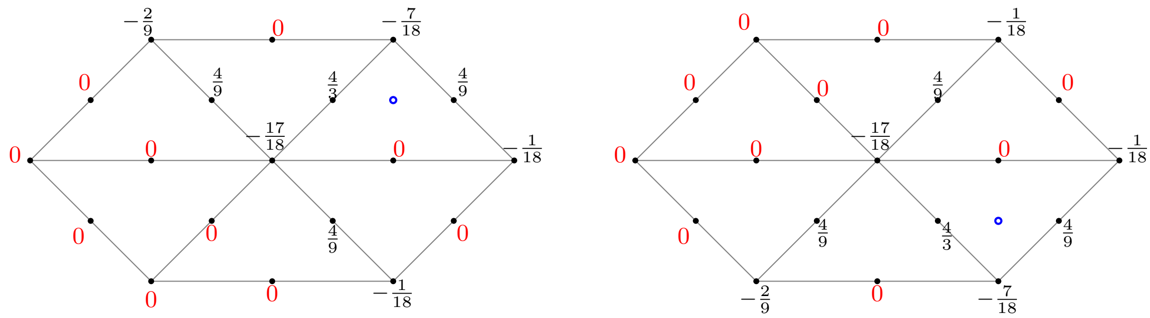


Figure 6: *Masks β (left) and γ (right) for the evaluation of the BB-coefficients associated with the points $t_{i,j}$ and $\widetilde{t}_{i,j}$, respectively.*

Proposition 1 *The problem (3.3) has a unique solution. The associated masks appear in Figs. 5, 6 and 7. The quasi-interpolation operator $\mathcal{Q}_3 : C(\mathbb{R}^2) \longrightarrow S_3^1(\Delta)$ defined by $\mathcal{Q}_{3,2}(f) := Q_{3,2}f$ is bounded in the uniform norm and*

$$\|\mathcal{Q}_{3,2}\|_\infty \leq \max\{\|\alpha\|_1, \|\beta\|_1, \|\gamma\|_1, \|\omega_k\|_1, 0 \leq k \leq 5\} = \frac{13}{3}. \tag{3.4}$$

Proof. Thanks to the symmetry properties of the partition, we can fix a general vertex $v_{i,j}$. Firstly we impose the C^1-smoothness conditions on every segment emanating from $v_{i,j}$. Then, we impose the reproduction of $\mathbb{P}_2$ on every triangle having $v_{i,j}$ as vertex, by requiring $Q_{3,2}m_{\gamma_1,\gamma_2}(x,y) = m_{\gamma_1,\gamma_2}(x,y)$, $\gamma_1 + \gamma_2 \leq 2$, $\gamma_1, \gamma_2 \geq 0$, with $m_{\gamma_1,\gamma_2}(x,y) = x^{\gamma_1}y^{\gamma_2}$.

By using a symbolic computation software, the results stated in Proposition 1 are established.

Moreover, for any $f \in C(\mathbb{R}^2)$, all the BB-coefficients of $Q_{3,2}f$ on a triangle T are determined by using values of f at the points located in the hexagons containing T only. Therefore, since the Bernstein polynomials form a partition of unity, according to Figs. 5, 6 and 7, (3.4) follows.

Thanks to standard results in approximation theory (see e.g. [5, 12]), we can state the following result.

Theorem 2 *Let T be an arbitrary triangle in Δ, and let Ω_T be the union of the triangles in Δ having a non-empty intersection with T. Then, there exist constants $\bar{K}_{|\gamma|}$, independent on h, such that for every $f \in C^{m+1}(\mathbb{R}^2)$, $0 \leq m \leq 2$,*

$$\|D^\gamma(f - Q_{3,2}f)\|_{\infty,T} \leq \bar{K}_{|\gamma|}h^{m+1-|\gamma|}\left\|D^{m+1}f\right\|_{\infty,\Omega_T},$$

for all $0 \leq |\gamma| \leq m$, $\gamma = (\gamma_1, \gamma_2)$.

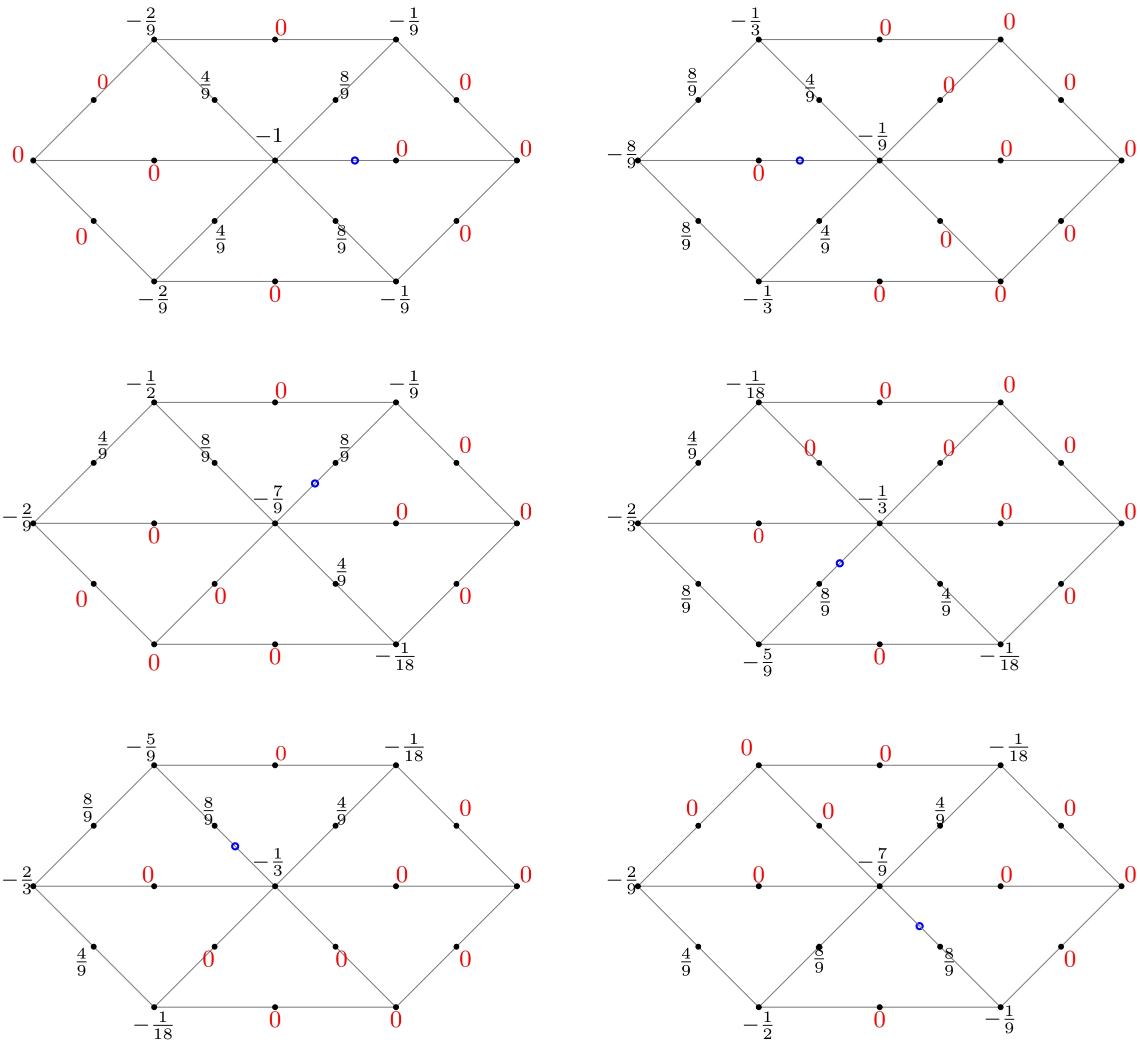


Figure 7: *Masks* ω_k, $k = 0, \dots, 5$, *for the evaluation of the BB-coefficients associated with the* $w-$*points.*

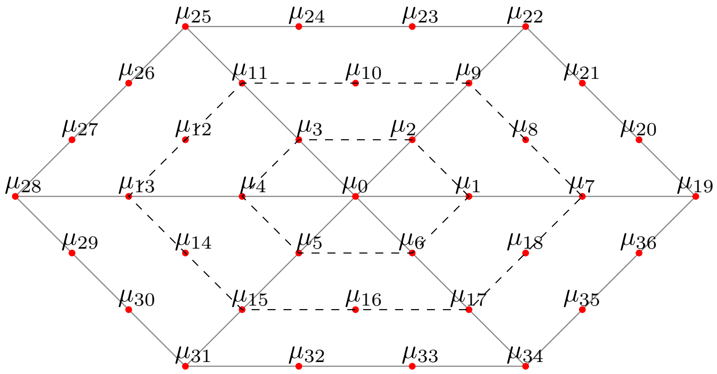


Figure 8: *Ordering a general sequence μ representing $f_{i,j}(\mathcal{D}_3)$, or any of the masks $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\omega}_k$, $0 \le k \le 5$.*

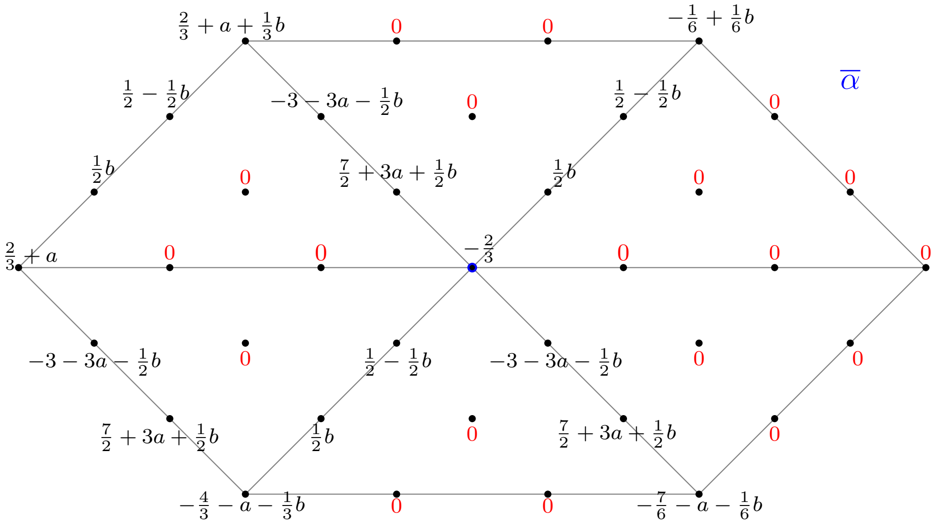


Figure 9: *Mask $\overline{\alpha}$ for the construction based on $\mathcal{D}_3$.*

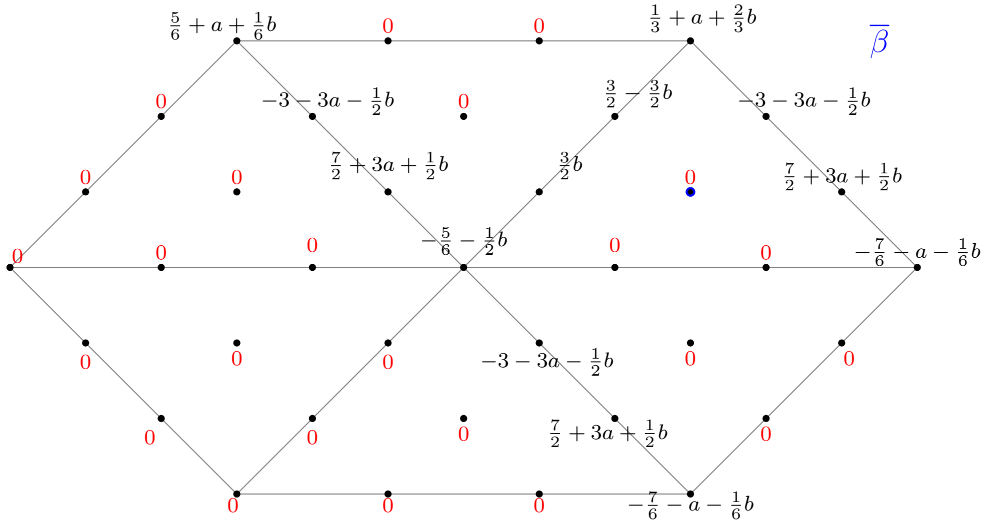
3.2 C^1-cubic quasi-interpolation based on $\mathcal{D}_3$

The construction above can be carried out in a similar way if the BB-coefficient of every point in $T_{i,j}$ and $\widetilde{T}_{i,j}$ is defined as a linear combination of the values of f at the points in $D^3_{i,j}$. Their coefficients come from masks $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ and $\overline{\omega}_k$, $0 \le k \le 5$, in $\mathbb{R}^{37}$. In Fig. 8 it is shown how to order a general mask.

After imposing the C^1 class and the reproduction of quadratic polynomials a general solution depending on the free parameters $a := \overline{\omega}_{0,0}$ and $b := \overline{\omega}_{2,0}$ is obtained by using the software Mathematica. The masks associated with $\overline{\alpha}$ and $\overline{\beta}$ appear in Fig. 9 and 10.

For the non-zero values of $\overline{\omega}_0$ the following expressions hold:

$$
\begin{aligned}
\overline{\omega}_{3,0} &= \tfrac{7}{2}+3a+\tfrac{1}{2}b, & \overline{\omega}_{5,0} &= \tfrac{1}{2}-\tfrac{1}{2}a, & \overline{\omega}_{6,0} &= -6-6a-b, & \overline{\omega}_{9,0} &= 1-a,\\
\overline{\omega}_{11,0} &= -3-3a-\tfrac{1}{2}b, & \overline{\omega}_{15,0} &= \tfrac{1}{2}b, & \overline{\omega}_{17,0} &= 7+6a+b, & \overline{\omega}_{22,0} &= -\tfrac{1}{3}+\tfrac{1}{3}b,\\
\overline{\omega}_{25,0} &= \tfrac{5}{6}+a+\tfrac{1}{6}b, & \overline{\omega}_{31,0} &= -\tfrac{1}{6}-\tfrac{1}{6}b, & \overline{\omega}_{34,0} &= -\tfrac{7}{3}-2a-\tfrac{1}{3}b.
\end{aligned}
$$



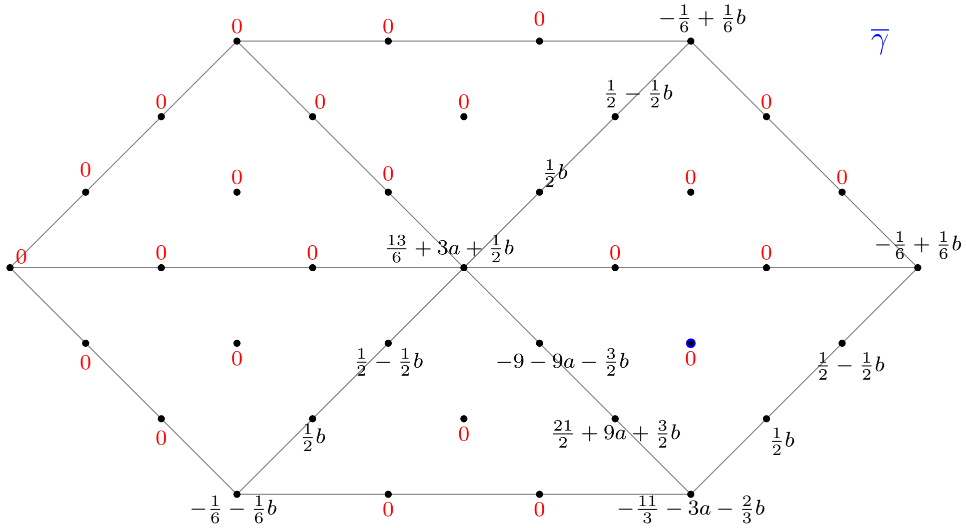


Figure 10: *Masks* $\overline{\beta}$ *and* $\overline{\gamma}$ *for the construction based on* $\mathcal{D}_3$.

The following results are obtained for the masks ω_k, $1 \leq k \leq 5$:

$$
\begin{array}{llllllll}
\overline{\omega}_{0,1} & = -\frac{11}{6} - a - \frac{1}{2}b, & \overline{\omega}_{3,1} & = 7 + 6a + b, & \overline{\omega}_{6,1} & = -3 - 3a - \frac{1}{2}b, & \overline{\omega}_{9,1} & = 1 - b, \\
\overline{\omega}_{11,1} & = -6 - 6a - b, & \overline{\omega}_{17,1} & = \frac{7}{2} + 3a + \frac{1}{2}b, & \overline{\omega}_{22,1} & = -\frac{1}{3} + \frac{1}{3}b, & \overline{\omega}_{25,1} & = \frac{3}{2} + 2a + \frac{1}{2}b, \\
\overline{\omega}_{26,1} & = \frac{1}{2} - \frac{1}{2}b, & \overline{\omega}_{27,1} & = \frac{1}{2}b, & \overline{\omega}_{28,1} & = -\frac{1}{6} - \frac{1}{6}b, & \overline{\omega}_{34,1} & = -\frac{7}{6} - a - \frac{1}{6}b, \\
\overline{\omega}_{0,2} & = -\frac{5}{2} - 2a - \frac{1}{2}b, & \overline{\omega}_{2,2} & = \frac{1}{2}b, & \overline{\omega}_{3,2} & = 7 + 6a + b, & \overline{\omega}_{9,2} & = \frac{1}{2} - \frac{1}{2}b, \\
\overline{\omega}_{11,2} & = -6 - 6a - b, & \overline{\omega}_{22,2} & = -\frac{1}{6} + \frac{1}{6}b, & \overline{\omega}_{25,2} & = \frac{4}{3} + 2a + \frac{2}{3}b, & & \\
\overline{\omega}_{26,2} & = 1 - b, & \overline{\omega}_{27,2} & = b, & \overline{\omega}_{28,2} & = a - \frac{1}{6}b, & & \\
\overline{\omega}_{29,2} & = -3 - 3a - \frac{1}{2}b, & \overline{\omega}_{30,2} & = \frac{7}{2} + 3a + \frac{1}{2}b, & \overline{\omega}_{31,2} & = -\frac{7}{6} - a - \frac{1}{6}b, & & \\
\overline{\omega}_{0,3} & = -\frac{4}{3} - a, & \overline{\omega}_{3,3} & = \frac{7}{2} + 3a + \frac{1}{2}b, & \overline{\omega}_{5,3} & = \frac{1}{2} - \frac{1}{2}b, & \overline{\omega}_{11,3} & = -3 - 3a - \frac{1}{2}b, \\
\overline{\omega}_{15,3} & = \frac{1}{2}b, & \overline{\omega}_{25,3} & = \frac{1}{2} + a + \frac{1}{2}b, & \overline{\omega}_{26,3} & = 1 - b, & \overline{\omega}_{27,3} & = b, \\
\overline{\omega}_{28,3} & = \frac{4}{3} + 2a, & \overline{\omega}_{29,3} & = -6 - 6a - b, & \overline{\omega}_{30,3} & = 7 + 6a + b, & \overline{\omega}_{31,3} & = -\frac{5}{2} - 2a - \frac{1}{2}b, \\
\overline{\omega}_{0,4} & = \frac{1}{2} + a + \frac{1}{2}b, & \overline{\omega}_{5,4} & = 1 - b, & \overline{\omega}_{5,6} & = -3 - 3a - \frac{1}{2}b, & \overline{\omega}_{15,4} & = b, \\
\overline{\omega}_{17,4} & = \frac{7}{2} + 3a + \frac{1}{2}b, & \overline{\omega}_{25,4} & = -\frac{1}{6} + \frac{1}{6}b, & \overline{\omega}_{26,4} & = \frac{1}{2} - \frac{1}{2}b, & & \\
\overline{\omega}_{27,4} & = \frac{1}{2}b, & \overline{\omega}_{28,4} & = \frac{3}{2} + 2a + \frac{1}{6}b, & \overline{\omega}_{29,4} & = -6 - 6a - b, & & \\
\overline{\omega}_{30,4} & = 7 + 6a + b, & \overline{\omega}_{31,4} & = -\frac{8}{3} - 2a - \frac{2}{3}b, & \overline{\omega}_{34,4} & = -\frac{7}{6} - a - \frac{1}{6}b, & & \\
\overline{\omega}_{0,5} & = \frac{7}{6} + 2a + \frac{1}{2}b, & \overline{\omega}_{2,5} & = \frac{1}{2}b, & \overline{\omega}_{5,5} & = 1 - b, & \overline{\omega}_{6,5} & = -6 - 6a - b, \\
\overline{\omega}_{9,5} & = \frac{1}{2} - \frac{1}{2}b, & \overline{\omega}_{15,5} & = b, & \overline{\omega}_{17,5} & = 7 + 6a + b, & & \\
\overline{\omega}_{22,5} & = -\frac{1}{6} + \frac{1}{2}b, & \overline{\omega}_{28,5} & = \frac{5}{6} + a + \frac{1}{6}b, & \overline{\omega}_{29,5} & = -3 - 3a - \frac{1}{2}b, & & \\
\overline{\omega}_{30,5} & = \frac{7}{2} + 3a + \frac{1}{2}b, & \overline{\omega}_{31,5} & = -\frac{3}{2} - a - \frac{1}{2}b, & \overline{\omega}_{34,5} & = -\frac{7}{3} - 2a - \frac{1}{3}b. & &
\end{array}
$$

The infinity norm of the operator $\mathcal{Q}_{3,3}$ defined by the masks above is bounded by

$$
\max\left\{\|\overline{\alpha}\|_1, \|\overline{\beta}\|_1, \|\overline{\gamma}\|_1, \|\overline{\omega}_k\|_1, 0 \leq k \leq 5\right\}.
$$

This is a function depending on the variables a and b, so it is natural to determine the values of such parameters providing the minimum value of that function. It can be proved that its minimum value, that it is equal to 5, is reached at all points lying in the triangle with vertices $(-1, 0)$, $\left(-\frac{7}{6}, 0\right)$ and $\left(-\frac{7}{6}, 1\right)$.

We can notice that the values $(a, b) = (-1, 0)$ provide symmetrical masks $\overline{\beta}$ and $\overline{\gamma}$ and therefore we consider such a choice in the numerical tests proposed in Section 3.3.

Also for $Q_{3,3}f$ the approximation error Theorem 2 holds.

3.3 Numerical Results

Now, we show the results of some numerical tests, developed in the Matlab environment, on Franke's function

$$
\begin{aligned}
f_1(z_1, z_2) = {} & 0.75 \exp\left(-\frac{(9z_1-2)^2}{4} - \frac{(9z_2-2)^2}{4}\right) + 0.75 \exp\left(-\frac{(9z_1+1)^2}{49} - \frac{9z_2+1}{10}\right) \\
& + 0.5 \exp\left(-\frac{(9z_1-7)^2}{4} - \frac{(9z_2-3)^2}{4}\right) - 0.2 \exp\left(-(9z_1-4)^2 - (9z_2-7)^2\right),
\end{aligned}
$$

and the highly oscillating test function

$$
f_2(z_1, z_2) = 0.1\left(1 + \cos\left(12\pi \cos\left(\pi\sqrt{z_1^2 + z_2^2}\right)\right)\right),
$$

both defined on the unit square $[0, 1]^2$.

For a step length h, the maximal error (ME) for a given function f and a quasi-interpolation operator Q is estimated as the value ME_h given by maximum of the quasi-interpolation error $|f - Qf|$ on a finite subset $G = \{(g_{1,i}, g_{2,j}) : (i, j) \in J\}$ of points lying in the unit square, and the root mean square error (RMSE) as

$$
RMSE_h := \sqrt{\frac{\sum_{(i,j)\in J}\left(f(g_{1,i}, g_{2,j}) - Qf(g_{1,i}, g_{2,j})\right)^2}{\operatorname{card} J}},
$$

	Test function f_1			Test function f_2		
h	ME_h	NCO	$RMSE_h$	ME_h	NCO	$RMSE_h$
1/4	4.35×10^{-1}	–	1.11×10^{-1}			
1/8	7.46×10^{-2}	2.54	2.41×10^{-2}			
1/16	2.31×10^{-2}	1.69	4.39×10^{-3}			
1/32	3.55×10^{-3}	2.70	6.31×10^{-4}			
1/64	4.46×10^{-4}	2.99	8.23×10^{-5}	8.70×10^{-2}	–	1.56×10^{-2}
1/128	5.70×10^{-5}	2.97	1.04×10^{-5}	1.42×10^{-2}	2.62	2.56×10^{-3}
1/256	7.14×10^{-5}	3.00	1.31×10^{-6}	1.89×10^{-3}	2.90	3.45×10^{-4}
1/512	8.92×10^{-7}	3.00	1.64×10^{-7}	2.43×10^{-4}	2.96	4.40×10^{-5}

Table 1: *Numerical results for functions f_1 and f_2 using the operator $\mathcal{Q}_{3,2}$.*

	Test function f_1			Test function f_2		
h	ME_h	NCO	$RMSE_h$	ME_h	NCO	$RMSE_h$
1/4	4.02×10^{-1}	–	1.07×10^{-1}			
1/8	8.51×10^{-2}	2.24	2.27×10^{-2}			
1/16	2.14×10^{-2}	2.0	3.82×10^{-3}			
1/32	3.02×10^{-3}	2.83	5.32×10^{-4}			
1/64	3.63×10^{-4}	3.05	6.86×10^{-5}	8.45×10^{-2}	–	1.43×10^{-2}
1/128	4.66×10^{-5}	2.96	8.66×10^{-6}	1.31×10^{-2}	2.69	2.18×10^{-3}
1/256	5.77×10^{-6}	3.02	1.09×10^{-6}	1.72×10^{-3}	2.92	2.87×10^{-4}
1/512	7.19×10^{-7}	3.00	1.36×10^{-7}	2.21×10^{-4}	2.96	3.64×10^{-5}

Table 2: *Numerical results for functions f_1 and f_2 using the operator $\mathcal{Q}_{3,3}$.*

with card J standing for the cardinality of J.

In order to evaluate these values we have sampled the splines on 300 points in each triangle of Δ, for every considered value of h. The evaluation of the quasi-interpolating splines is carried out by the de Casteljau's algorithm [12, p. 25].

The numerical convergence orders are computed by the formula

$$NCO := \log_2 \frac{ME_h}{ME_{h/2}}.$$

We have omitted any reference to f and Q in denoting these quantities.

For the more difficult function f_2, we started the computations by using a larger set of data points, i.e. by considering an initial value of h smaller than the one used for the test function f_1.

Here we consider $Q = \mathcal{Q}_{3,2}$, $\mathcal{Q}_{3,3}$ and we report the results in Table 1 and 2, respectively. We can notice that the obtained results confirm the theoretical value for the convergence order.

4 Defining C^1-quartic differential quasi-interpolating splines

The main goal in this section is to define from the C^1-cubic quasi-interpolant $Q_{3,2}f$ provided by the operator $\mathcal{Q}_{3,2}$, which is exact on $\mathbb{P}_2$, a C^1-quartic differential quasi-interpolant $H_4 f$ in such a way that the associated operator $\mathcal{H}_4$ is exact on $\mathbb{P}_3$, and to describe it in the Bernstein basis.

To do that, we recall the results in [6] (see also [17, 10, 11]). Given a compact convex domain Ω of $\mathbb{R}^n$ with a non-empty interior, let $\mathcal{L}$ be a linear operator defined on the subspace $C^k(\Omega)$ of $C(\Omega) := C^0(\Omega)$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, endowed with the norm given by the expression

$$\|L\| := \sup \left\{ \sup_{x \in \Omega} |\mathcal{L}[f](x)| : f \in C^k(\Omega), \max_{0 \le j \le k} \left\| D^j f \right\|_\Omega = 1 \right\}.$$

As in [11], here $C^k(\Omega)$ is the subspace of all functions that are k times continuously differentiable in the following sense: for each $x \in \Omega$ and any $y \in \mathbb{R}^n$ such that $x + y \in \Omega$, the directional derivatives

$$D_y^j f(x) := \frac{d^j}{dt^j} f(x + ty)_{|t=0}, \ 0 \leq j \leq k,$$

exist and depend continuously on x. Moreover,

$$\left\| D^j f \right\|_{\Omega} := \sup_{x \in \Omega} \sup \left\{ \left| D_y^j f \right| : y \in \mathbb{R}^n, \|y\| = 1 \right\}.$$

If $\mathcal{L}$ is exact on $\mathbb{P}_m$, then for $r \in \mathbb{N}$ the operator $\mathcal{H}_{m,r}$ defined by

$$\mathcal{H}_{m,r}[f](x) := \mathcal{L}\left[\sum_{j=0}^{r} \frac{a_{m,r,j}}{j!} D_{x-\cdot}^j f \right](x)$$

for $f \in C^{k+r}(\Omega)$ with

$$a_{m,r,j} := \frac{(m + r - j)!}{(m + r)!\,(r - j)!}$$

is exact on $\mathbb{P}_{m+r}$. The conditions above allow to derive an expression for the error $f - \mathcal{H}_{m,r}[f]$.

Since $\mathcal{Q}_{3,2}$ is exact on $\mathbb{P}_2$, this construction can be applied to the operator $\mathcal{Q}_{3,2}$ with $r = 1$ and $m = 2$ to define the action $H_4 f$ of $\mathcal{H}_4$ on a given function f:

$$H_4 f(x) := \mathcal{Q}_{3,2}\left[f + \frac{1}{3} D_{x-\cdot} f \right](x). \tag{4.1}$$

The operator $\mathcal{H}_4$ is exact on $\mathbb{P}_3$. $H_4 f$ is a differential quasi-interpolant to f involving the values of the first order partial derivatives of f at the points in a subset of $\mathbb{R}^2$ that will be described later.

To obtain the explicit expression of $H_4 f$ on every triangle in Δ, we will consider the generic triangles $T_{i,j}$ and $\widetilde{T}_{i,j}$ defining the parallelogram $P_{i,j}$. The domain points involved in the expression of $Q_{3,2} f$ on $T_{i,j}$ (see Fig. 2) are ordered as $v_{i,j}$, $w_{i,j}^{1,1}$, $w_{i,j}^{1,0}$, $w_{i+1,j+1}^{-1,-1}$, $t_{i,j}$, $w_{i+1,j}^{-1,0}$, $v_{i+1,j+1}$, $w_{i+1,j+1}^{0,-1}$, $w_{i+1,j}^{0,1}$, and $v_{i+1,j}$. With this order in mind, let

$$\mathcal{T}_{i,j}^3 := \left\{ v_{i,j}, w_{i,j}^{1,1}, w_{i,j}^{1,0}, w_{i+1,j+1}^{-1,-1}, t_{i,j}, w_{i+1,j}^{-1,0}, v_{i+1,j+1}, w_{i+1,j+1}^{0,-1}, w_{i+1,j}^{0,1}, v_{i+1,j} \right\}.$$

Analogously, let

$$\widetilde{\mathcal{T}}_{i,j}^3 := \left\{ v_{i,j}, w_{i,j}^{1,1}, w_{i,j}^{0,1}, w_{i+1,j+1}^{-1,-1}, \widetilde{t}_{i,j}, w_{i,j+1}, v_{i+1,j+1}, w_{i+1,j+1}^{-1,0}, w_{i,j+1}^{1,0}, v_{i,j+1} \right\}.$$

For every domain point p in $\mathcal{T}_{i,j}^3$ or $\widetilde{\mathcal{T}}_{i,j}^3$, let $P := (P_0, P_1, P_2)$ stand for the index (i, j, k) associated with p (see Fig. 11)

Then, by (3.1) for all $x \in T_{i,j}$ it holds

$$Q_{3,2} f(x) = \sum_{p \in \mathcal{T}_{i,j}^3} c_3(p) B_P,$$

where $B_P = \frac{3}{P_0!P_1!P_2!} b_0^{P_0} b_1^{P_1} b_2^{P_2}$ is the Bernstein polynomial associated with p an $b_s := b_s(x)$, $0 \leq s \leq 2$, are the barycentric coordinates of x with respect to $T_{i,j}$. We have omitted any reference to $T_{i,j}$ in the notation for the Bernstein polynomials and the barycentric coordinates and also the dependence of B_P of the variable x because it follows from the relationship between x and b_s. Notice that (b_0, b_1, b_2) is given by the solution of the system of equations

$$x = b_0 v_{i,j} + b_1 v_{i+1,j+1} + b_2 v_{i+1,j}, \quad b_0 + b_1 + b_2 = 1,$$

i.e.

$$x_1 = h\left((i + j)\, b_0 + (i + j + 2)\, b_1 + (i + j + 1)\, b_2 \right),$$

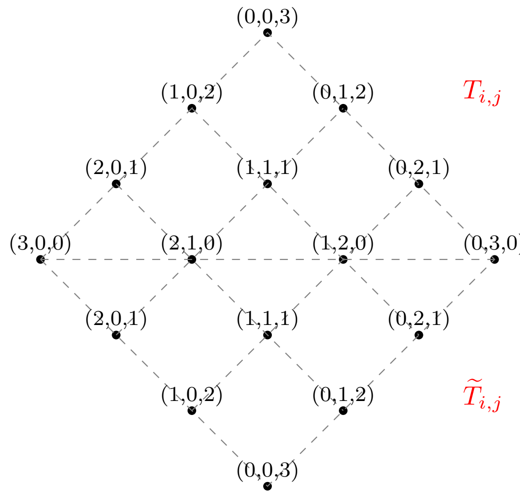


Figure 11: *Indices P for the domain points in $T_{i,j}$ and $\widetilde{T}_{i,j}$.*

$$x_2 = h\left((i-j)\,b_0 + (i-j)\,b_1 + (i-j+1)\,b_2\right),$$

and

$$b_0 = i+1-\frac{x_1+x_2}{2h},\ b_1 = -j+\frac{x_1-x_2}{2h},\ b_2 = -i+j+\frac{x_2}{h}.$$

Similarly, for all $x \in \widetilde{T}_{i,j}$ it holds

$$Q_{3,2}f\,(x) = \sum_{p\in\widetilde{\mathcal{T}}^3_{i,j}} c_3\,(p)\,B_P,$$

with the barycentric coordinates given by

$$x = b_0 v_{i,j} + b_1 v_{i+1,j+1} + b_2 v_{i,j+1}, \quad b_0+b_1+b_2 = 1,$$

i.e.

$$x_1 = h\left((i+j)\,b_0 + (i+j+2)\,b_1 + (i+j+1)\,b_2\right)$$
$$x_2 = h\left((i-j)\,b_0 + (i-j)\,b_1 + (i-j-1)\,b_2\right),$$

and

$$b_0 = j+1-\frac{x_1-x_2}{2h},\ b_1 = -i+\frac{x_1+x_2}{2h},\ b_2 = i-j-\frac{x_2}{h}.$$

Every BB-coefficient in the restrictions of $Q_{3,2}f$ to $T_{i,j}$ and $\widetilde{T}_{i,j}$ is a linear combination of the values of f at the 19 domain points in $\mathcal{D}^2_{i,j}$ (see Fig. 2). To simplify the notation, let us write

$$\mathcal{D}^2_{i,j} = \{q_m, 0 \le m \le 18\}.$$

The coefficients of every linear combination are the values given by the corresponding mask (α for a vertex, ω for a $w-$point, and β for $t_{i,j}$). Let $\mu\,(p) := (\mu_m\,(p))_{0\le m\le 18}$ stand for the mask associated with p. Then,

$$c_3\,(p) = \sum_{m=0}^{18} \mu_m\,(p)\,f\,(q_m).$$

For a C^1-function f, the restriction of H_4f to $P_{i,j}$ becomes

$$H_4f\,(x) = \begin{cases} \displaystyle\sum_{p\in\mathcal{T}^3_{i,j}} \left(\sum_{m=0}^{18}\mu_m\,(p)\left(f\,(q_m)+\tfrac{1}{3}\nabla f\,(q_m)\cdot(x-q_m)\right)\right) B_P, & x\in T_{i,j},\\ \displaystyle\sum_{p\in\widetilde{\mathcal{T}}^3_{i,j}} \left(\sum_{m=0}^{18}\mu_m\,(p)\left(f\,(q_m)+\tfrac{1}{3}\nabla f\,(q_m)\cdot(x-q_m)\right)\right) B_P, & x\in \widetilde{T}_{i,j},\end{cases}$$

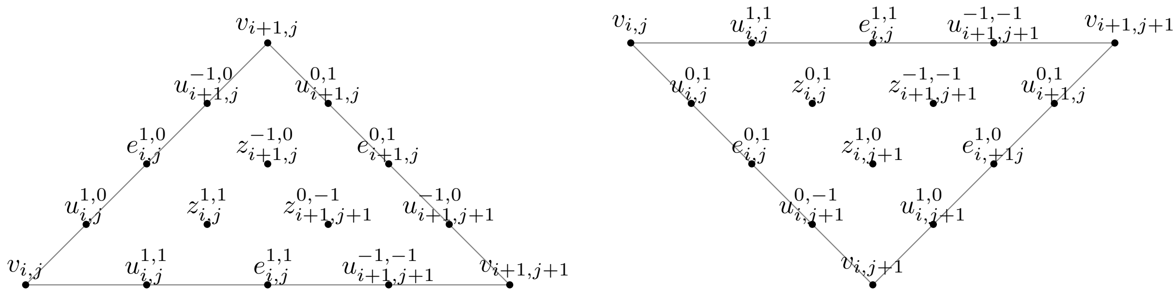


Figure 12: The domain points in $T_{i,j}$ and $\widetilde{T}_{i,j}$.

with ∇ denoting the gradient operator.

Notice that the computation of the BB-coefficients of $H_4 f$ on $T_{i,j}$ can be done symbolically taking into account that

$$H_4 f(x) = \sum_{p\in\mathcal{T}_{i,j}^3} \left(\sum_{m=0}^{18} \mu_m(p) \left(f(q_m)(b_0+b_1+b_2) + \frac{1}{3}\nabla f(q_m)\cdot(x - q_m(b_0+b_1+b_2)) \right) \right) B_P,$$

and using a symbolic computation software to expand the expression above to determine the coefficient of each one of the 15 Bernstein polynomials relative to $T_{i,j}$.

The BB-coefficients of $H_4 f$ depend on the values of f and its partial derivatives $f^{(1,0)}$ and $f^{(0,1)}$ of the first order at the points in $D\left(T_{i,j}^3\right) := D_{i,j}^3 \cup D_{i+1,j+1}^3 \cup D_{i+1,j}^3$ (see Fig. 13). The notation $f_{i,j}(\mathcal{D}_2)$ in Section 3 is extended to be applied to define $f_{i,j}^{(1,0)}(\mathcal{D}_2)$ and $f_{i,j}^{(0,1)}(\mathcal{D}_2)$.

The BB-coefficient of the quartic polynomial $H_4 f_{|T_{i,j}}$ relative to the vertex $v_{i,j}$ (see Fig. 12) is given by the expression

$$c_4(v_{i,j}) := f_{i,j}(\mathcal{D}_2)\cdot\alpha + \frac{h}{6} f_{i,j}^{(1,0)}(\mathcal{D}_2)\cdot\alpha*\Xi_1 + \frac{h}{6} f_{i,j}^{(0,1)}(\mathcal{D}_2)\cdot\alpha*\Xi_2, \tag{4.2}$$

where $\alpha * \Xi$ denotes pointwise product and

$$\Xi_1 := (0,-2,-1,1,2,1,-1,-4,-3,-2,0,2,3,4,3,2,0,-2,-3),$$
$$\Xi_2 := (0,0,-1,-1,0,1,1,0,-1,-2,-2,-2,-1,0,1,2,2,2,1).$$

Also these masks are ordered as indicated in Fig. 4.

With respect to the point $u_{i,j}^{1,1}$ and $u_{i,j}^{1,0}$ in Fig. 12, it holds

$$\begin{aligned} c_4\left(u_{i,j}^{1,1}\right) &:= \frac{1}{4} f_{i,j}(\mathcal{D}_2)\cdot(\alpha+3\omega_0) + \frac{h}{24} f_{i,j}^{(1,0)}(\mathcal{D}_2)\cdot(\alpha*\Xi_3+3\omega_0*\Xi_1) \\ &\quad + \frac{h}{24} f_{i,j}^{(0,1)}(\mathcal{D}_2)\cdot(\alpha+3\omega_0)*\Xi_2, \end{aligned} \tag{4.3}$$

$$\begin{aligned} c_4\left(u_{i,j}^{1,0}\right) &:= \frac{1}{4} f_{i,j}(\mathcal{D}_2)\cdot(\alpha+3\omega_1) + \frac{h}{24} f_{i,j}^{(1,0)}(\mathcal{D}_2)\cdot(\alpha*\Xi_4+3\omega_0*\Xi_1) \\ &\quad + \frac{h}{24} f_{i,j}^{(0,1)}(\mathcal{D}_2)\cdot(\alpha*\Xi_5+3\omega_1*\Xi_2), \end{aligned} \tag{4.4}$$

with

$$\Xi_3 := (4,2,3,5,6,5,3,0,1,2,4,6,7,8,7,6,4,2,1),$$
$$\Xi_4 := (2,0,1,3,4,3,1,2,-1,0,2,4,5,6,5,4.2,0,-1).$$

The masks $\Xi_1,\ldots,\Xi_4$ are represented in Fig. 14.

For the BB-coefficients associated with the remaining points in Fig. 12, we get the following expressions:

$$c_4\left(e_{i,j}^{1,1}\right) := \frac{1}{2}\left(f_{i,j}(\mathcal{D}_2)\cdot\omega_0 + f_{i+1,j+1}(\mathcal{D}_2)\cdot\omega_3\right) \tag{4.5}$$

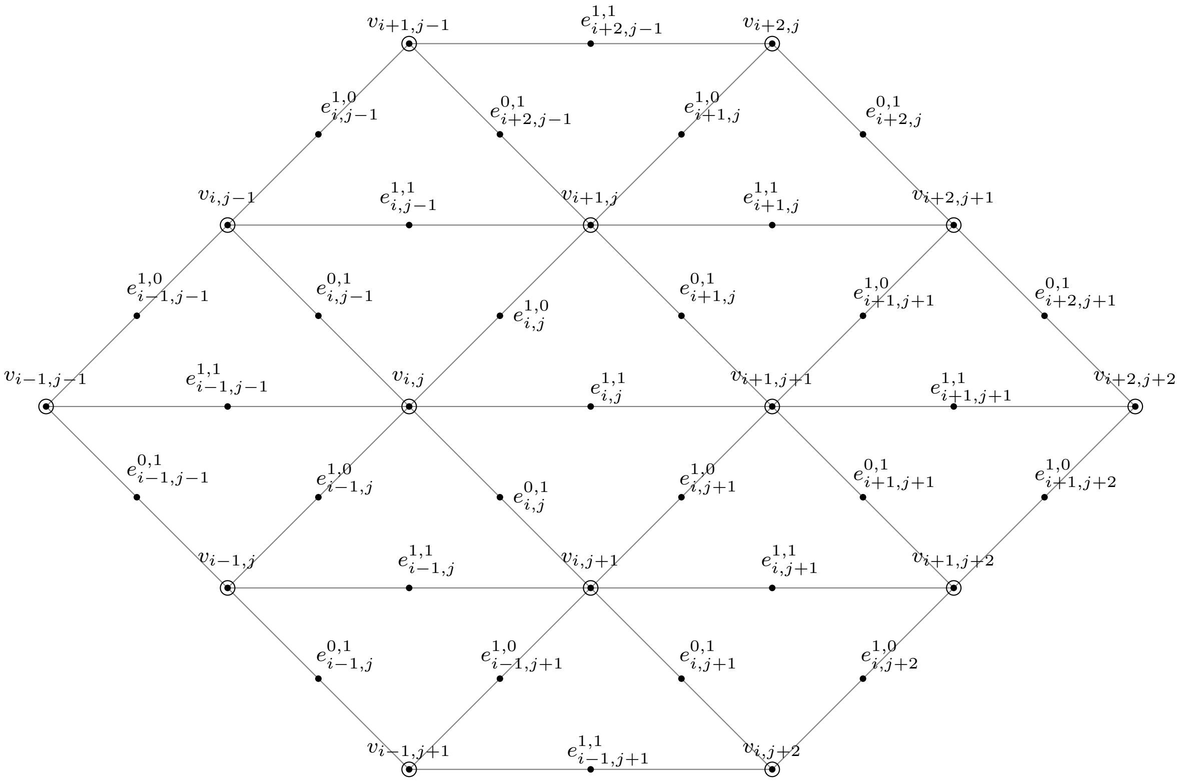


Figure 13: The subset $D\left(T^3_{i,j}\right)$.

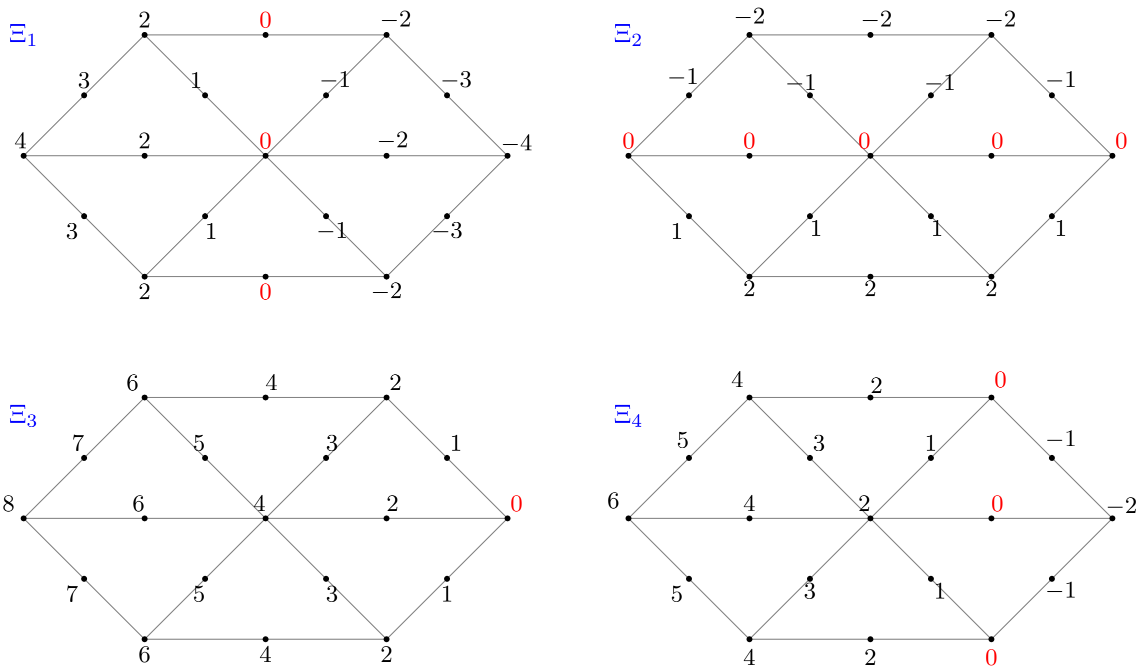


Figure 14: The masks $\Xi_1, \ldots, \Xi_4$.

$$
\begin{aligned}
& + \frac{h}{12}\left(f_{i,j}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\omega_0 * \Xi_3 + f_{i+1,j+1}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\omega_3 * \Xi_6\right)\\
& + \frac{h}{12}\left(f_{i,j}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\omega_0 * \Xi_2 + f_{i+1,j+1}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\omega_3 * \Xi_2\right),
\end{aligned}
$$

$$
\begin{aligned}
c_4\left(z_{i,j}^{1,1}\right) := {} & \frac{1}{4} f_{i,j}\left(\mathcal{D}_2\right)\cdot\left(2\beta+\omega_0+\omega_1\right)\\
& + \frac{h}{24} f_{i,j}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\left(2\beta * \Xi_1 + \omega_0 * \Xi_4 + \omega_1 * \Xi_3\right)\\
& + \frac{h}{24} f_{i,j}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\left(2\beta * \Xi_2 + \omega_0 * \Xi_5 + \omega_1 * \Xi_2\right),
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
c_4\left(e_{i,j}^{1,0}\right) := {} & \frac{1}{2}\left(f_{i,j}\left(\mathcal{D}_2\right)\cdot\omega_1 + f_{i+1,j}\left(\mathcal{D}_2\right)\cdot\omega_4\right)\\
& + \frac{h}{12}\left(f_{i,j}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\omega_1 * \Xi_4 + f_{i+1,j}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\omega_4 * \Xi_7\right)\\
& + \frac{h}{12}\left(f_{i,j}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\omega_1 * \Xi_5 + f_{i+1,j}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\omega_4 * \Xi_8\right),
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
c_4\left(u_{i+1,j+1}^{-1,-1}\right) := {} & \frac{1}{4} f_{i+1,j+1}\left(\mathcal{D}_2\right)\cdot\left(\alpha+3\omega_3\right) + \frac{h}{24} f_{i+1,j+1}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\left(\alpha * \Xi_6 + 3\omega_3 * \Xi_1\right)\\
& + \frac{h}{24} f_{i+1,j+1}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\left(\alpha * \Xi_2 + 3\omega_3 * \Xi_2\right),
\end{aligned} \tag{4.8}
$$

$$
\begin{aligned}
c_4\left(z_{i+1,j+1}^{0,-1}\right) := {} & \frac{1}{4}\left(f_{i,j}\left(\mathcal{D}_2\right)\cdot 2\beta + f_{i+1,j+1}\left(\mathcal{D}_2\right)\cdot\left(\omega_2+\omega_3\right)\right)\\
& + \frac{h}{24}\left(f_{i,j}^{(1,0)}\left(\mathcal{D}_2\right)\cdot 2\beta * \Xi_3 + f_{i+1,j+1}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\left(\omega_2 * \Xi_6 + \omega_3 * \Xi_7\right)\right)\\
& + \frac{h}{24}\left(f_{i,j}^{(0,1)}\left(\mathcal{D}_2\right)\cdot 2\beta * \Xi_2 + f_{i+1,j+1}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\left(\omega_2 * \Xi_2 + \omega_3 * \Xi_5\right)\right),
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
c_4\left(z_{i+1,j}^{-1,0}\right) := {} & \frac{1}{4}\left(f_{i,j}\left(\mathcal{D}_2\right)\cdot 2\beta + f_{i+1,j}\left(\mathcal{D}_2\right)\cdot\left(\omega_4+\omega_5\right)\right)\\
& + \frac{h}{24}\left(f_{i,j}^{(1,0)}\left(\mathcal{D}_2\right)\cdot 2\beta * \Xi_4 + f_{i+1,j}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\left(\omega_4 * \Xi_4 + \omega_5 * \Xi_7\right)\right)\\
& + \frac{h}{24}\left(f_{i,j}^{(0,1)}\left(\mathcal{D}_2\right)\cdot 2\beta * \Xi_5 + f_{i+1,j}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\left(\omega_4+\omega_5\right) * \Xi_8\right),
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
c_4\left(u_{i+1,j}^{-1,0}\right) := {} & \frac{1}{4} f_{i+1,j}\left(\mathcal{D}_2\right)\cdot\left(\alpha+3\omega_4\right) + \frac{h}{24} f_{i+1,j}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\left(\alpha * \Xi_6 + 3\omega_3 * \Xi_1\right)\\
& + \frac{h}{24} f_{i+1,j}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\left(\alpha * \Xi_8 + 3\omega_4 * \Xi_2\right),
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
c_4\left(u_{i+1,j+1}^{0,-1}\right) := {} & \frac{1}{4} f_{i+1,j+1}\left(\mathcal{D}_2\right)\cdot\left(\alpha+3\omega_2\right) + \frac{h}{24} f_{i+1,j+1}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\left(\alpha * \Xi_7 + 3\omega_2 * \Xi_1\right)\\
& + \frac{h}{24} f_{i+1,j+1}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\left(\alpha * \Xi_5 + 3\omega_2 * \Xi_2\right),\\
c_4\left(e_{i+1,j}^{0,1}\right) := {} & \frac{1}{2}\left(f_{i+1,j+1}\left(\mathcal{D}_2\right)\cdot\omega_2 + f_{i+1,j}\left(\mathcal{D}_2\right)\cdot\omega_5\right)\\
& + \frac{h}{24}\left(f_{i+1,j+1}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\omega_2 * \Xi_7 + f_{i+1,j}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\omega_5 * \Xi_4\right)\\
& + \frac{h}{24}\left(f_{i+1,j+1}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\omega_2 * \Xi_5 + f_{i+1,j}^{(0,1)}\left(\mathcal{D}_2\right)\cdot\omega_5 * \Xi_8\right),\\
c_4\left(u_{i+1,j}^{0,1}\right) := {} & \frac{1}{4} f_{i+1,j}\left(\mathcal{D}_2\right)\cdot\left(\alpha+3\omega_5\right) + \frac{h}{24} f_{i+1,j}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\left(\alpha * \Xi_4 + 3\omega_5 * \Xi_1\right)\\
& + \frac{h}{24} f_{i+1,j}^{(1,0)}\left(\mathcal{D}_2\right)\cdot\left(\alpha * \Xi_8 + 3\omega_5 * \Xi_2\right).
\end{aligned}
$$

The values of the BB-coefficients for $v_{i+1,j+1}$ and $v_{i+1,j}$ follow from the previous results with the appropriate changes of indices.

With respect to the domain points in $\widetilde{T}_{ij}$ shown in Fig. 12, the BB-coefficients associated with $v_{i,j}$, $u_{i,j}^{1,1}$, $e_{i,j}^{1,1}$, $u_{i+1,j+1}^{-1,-1}$ and $v_{i+1,j+1}$ are given by their counterparts in the triangle $T_{i,j}$. The BB-coefficient $c_4\left(v_{i,j+1}\right)$ is provided by equality (4.2) with the appropriate index change.

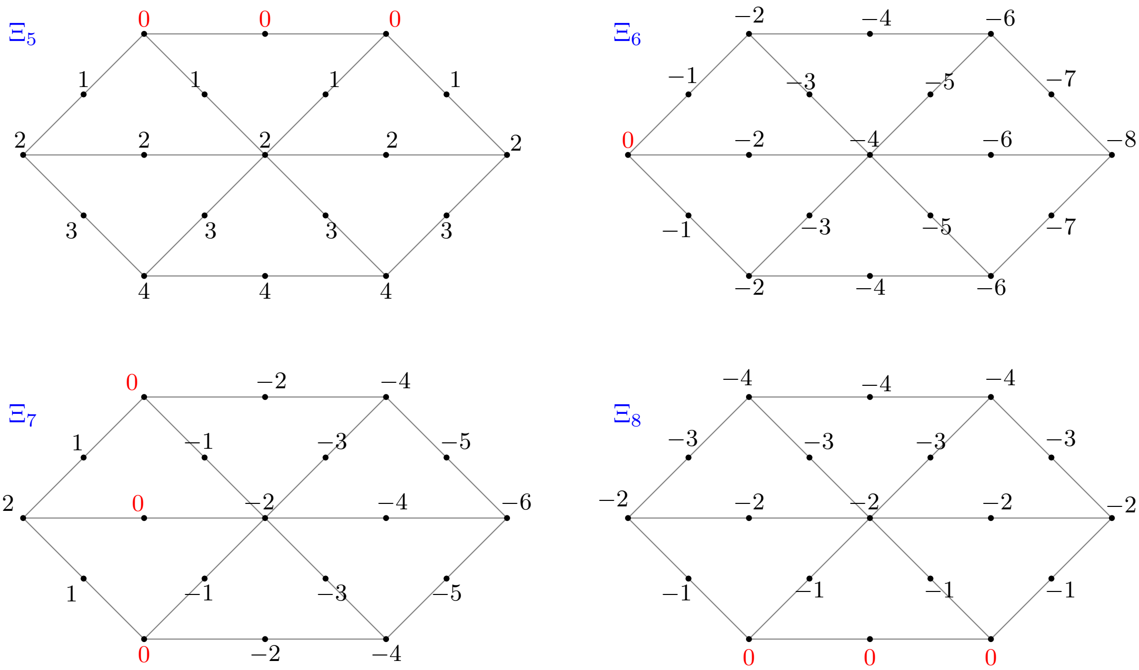


Figure 15: The masks $\Xi_5, \ldots, \Xi_8$.

For the BB-coefficients of the other domain points, we get the following expressions:

$$
\begin{aligned}
c_4\left(u_{i,j}^{0,1}\right) &= \frac{1}{4} f_{i,j}\left(\mathcal{D}_2\right) \cdot \left(\alpha + 3\omega_5\right) + \frac{h}{24} f_{i,j}^{(1,0)}\left(\mathcal{D}_2\right) \cdot \left(\alpha * \Xi_4 + 3\omega_5 * \Xi_1\right) \qquad (4.12)\\
&\quad + \frac{h}{24} f_{i,j}^{(0,1)}\left(\mathcal{D}_2\right) \cdot \left(\alpha * \Xi_8 + 3\omega_5 * \Xi_2\right),\\
c_4\left(z_{i,j}^{0,1}\right) &= \frac{1}{4} f_{i,j}\left(\mathcal{D}_2\right) \cdot \left(2\gamma + \omega_0 + \omega_5\right) \qquad (4.13)\\
&\quad + \frac{h}{24} f_{i,j}^{(1,0)}\left(\mathcal{D}_2\right) \cdot \left(2\gamma * \Xi_1 + \omega_0 * \Xi_4 + \omega_5 * \Xi_3\right)\\
&\quad + \frac{h}{24} f_{i,j}^{(0,1)}\left(\mathcal{D}_2\right) \cdot \left(2\gamma * \Xi_2 + \omega_0 * \Xi_8 + \omega_5 * \Xi_2\right),\\
c_4\left(e_{i,j}^{0,1}\right) &= \frac{1}{2} \left(f_{i,j}\left(\mathcal{D}_2\right) \cdot \omega_5 + f_{i,j+1}\left(\mathcal{D}_2\right) \cdot \omega_2\right) \qquad (4.14)\\
&\quad + \frac{h}{12} \left(f_{i,j}^{(1,0)}\left(\mathcal{D}_2\right) \cdot \omega_5 * \Xi_4 + f_{i,j+1}^{(1,0)}\left(\mathcal{D}_2\right) \cdot \omega_2 * \Xi_7\right)\\
&\quad + \frac{h}{12} \left(f_{i,j}^{(0,1)}\left(\mathcal{D}_2\right) \cdot \omega_5 * \Xi_8 + f_{i,j+1}^{(0,1)}\left(\mathcal{D}_2\right) \cdot \omega_2 * \Xi_5\right),\\
c_4\left(z_{i+1,j+1}^{-1,-1}\right) &= \frac{1}{4} \left(f_{i,j}\left(\mathcal{D}_2\right) \cdot 2\gamma + f_{i+1,j+1}\left(\mathcal{D}_2\right) \cdot \left(\omega_3 + \omega_4\right)\right) \qquad (4.15)\\
&\quad + \frac{h}{24} \left(f_{i,j}^{(1,0)}\left(\mathcal{D}_2\right) \cdot 2\gamma * \Xi_3 + f_{i+1,j+1}^{(1,0)}\left(\mathcal{D}_2\right) \cdot \left(\omega_3 * \Xi_7 + \omega_4 * \Xi_6\right)\right)\\
&\quad + \frac{h}{24} \left(f_{i,j}^{(0,1)}\left(\mathcal{D}_2\right) \cdot 2\gamma * \Xi_2 + f_{i+1,j+1}^{(0,1)}\left(\mathcal{D}_2\right) \cdot \left(\omega_3 * \Xi_8 + \omega_4 * \Xi_2\right)\right),\\
c_4\left(z_{i,j+1}^{1,0}\right) &= \frac{1}{4} \left(f_{i,j}\left(\mathcal{D}_2\right) \cdot 2\gamma + f_{i,j+1}\left(\mathcal{D}_2\right) \cdot \left(\omega_1 + \omega_2\right)\right) \qquad (4.16)\\
&\quad + \frac{h}{24} \left(f_{i,j}^{(1,0)}\left(\mathcal{D}_2\right) \cdot 2\gamma * \Xi_4 + f_{i,j+1}^{(1,0)}\left(\mathcal{D}_2\right) \cdot \left(\omega_1 * \Xi_7 + \omega_2 * \Xi_4\right)\right)\\
&\quad + \frac{h}{24} \left(f_{i,j}^{(0,1)}\left(\mathcal{D}_2\right) \cdot 2\gamma * \Xi_8 + f_{i,j+1}^{(0,1)}\left(\mathcal{D}_2\right) \cdot \left(\omega_1 + \omega_2\right) * \Xi_5\right),
\end{aligned}
$$

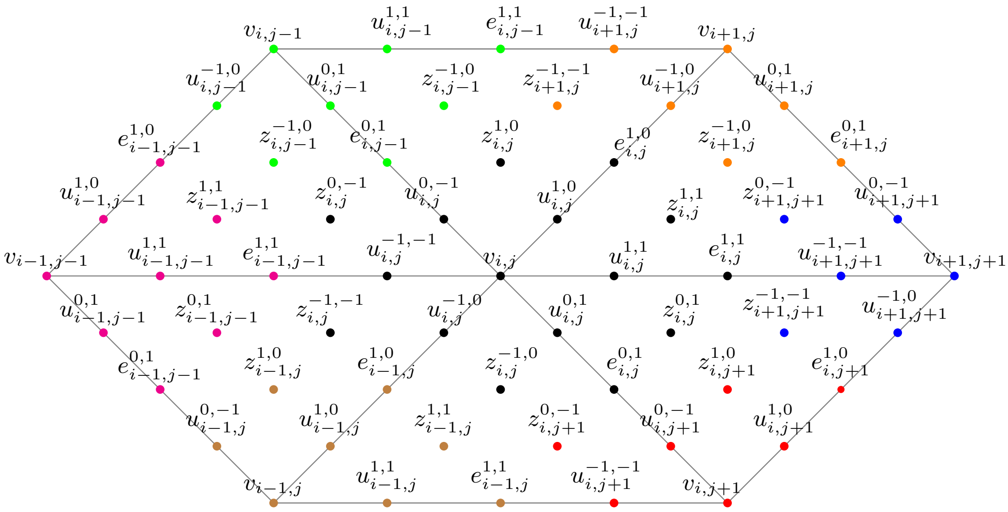


Figure 16: *The points of $\mathcal{D}_4$ relative to $H_{i,j}$.*

$$
\begin{aligned}
c_4\left(u^{0,-1}_{i,j+1}\right) &= \frac{1}{4} f_{i,j+1}\left(\mathcal{D}_2\right)\cdot\left(\alpha+3\omega_2\right)+\frac{h}{24} f^{(1,0)}_{i,j+1}\left(\mathcal{D}_2\right)\cdot\left(\alpha*\Xi_7+3\omega_2*\Xi_1\right) \\
&+\frac{h}{24} f^{(0,1)}_{i,j+1}\left(\mathcal{D}_2\right)\cdot\left(\alpha*\Xi_5+3\omega_2*\Xi_2\right), \\
c_4\left(u^{-1,0}_{i+1,j+1}\right) &= \frac{1}{4} f_{i+1,j+1}\left(\mathcal{D}_2\right)\cdot\left(\alpha+3\omega_4\right)+\frac{h}{24} f^{(1,0)}_{i+1,j+1}\left(\mathcal{D}_2\right)\cdot\left(\alpha*\Xi_7+3\omega_4*\Xi_1\right) \\
&+\frac{h}{24} f^{(0,1)}_{i+1,j+1}\left(\mathcal{D}_2\right)\cdot\left(\alpha*\Xi_8+3\omega_4*\Xi_2\right), \\
c_4\left(e^{1,0}_{i,j+1}\right) &= \frac{1}{2}\left(f_{i+1,j+1}\left(\mathcal{D}_2\right)\cdot\omega_4+f_{i,j+1}\left(\mathcal{D}_2\right)\cdot\omega_1\right) \\
&+\frac{h}{12}\left(f^{(1,0)}_{i+1,j+1}\left(\mathcal{D}_2\right)\cdot\omega_4*\Xi_7+f^{(1,0)}_{i,j+1}\left(\mathcal{D}_2\right)\cdot\omega_1*\Xi_4\right) \\
&+\frac{h}{12}\left(f^{(0,1)}_{i+1,j+1}\left(\mathcal{D}_2\right)\cdot\omega_4*\Xi_8+f^{(0,1)}_{i,j+1}\left(\mathcal{D}_2\right)\cdot\omega_1*\Xi_5\right), \\
c_4\left(u^{1,0}_{i,j+1}\right) &= \frac{1}{4} f_{i,j+1}\left(\mathcal{D}_2\right)\cdot\left(\alpha+3\omega_1\right)+\frac{h}{24} f^{(1,0)}_{i,j+1}\left(\mathcal{D}_2\right)\cdot\left(\alpha*\Xi_4+3\omega_1*\Xi_1\right) \\
&+\frac{h}{24} f^{(0,1)}_{i,j+1}\left(\mathcal{D}_2\right)\cdot\left(\alpha*\Xi_5+3\omega_1*\Xi_2\right).
\end{aligned}
\tag{4.17}
$$

The rules (4.2)-(4.13), along with the needed changes of indices, provide the values of BB-coefficients associated with the domain points in

$$
\mathcal{D}^4_{i,j} := \left\{v_{i,j}, u^{1,1}_{i,j}, u^{1,0}_{i,j}, u^{0,-1}_{i,j}, u^{-1,-1}_{i,j}, u^{-1,0}_{i,j}, u^{0,1}_{i,j}, e^{1,1}_{i,j}, z^{1,1}_{i,j}, e^{1,0}_{i,j}, z^{1,0}_{i,j}, z^{0,-1}_{i,j}, z^{-1,-1}_{i,j}, z^{-1,0}_{i,j}, e^{0,1}_{i,j}, z^{0,1}_{i,j}\right\}.
$$

The subset $\mathcal{D}^4_{i,j}$ is shown in Fig. 16. It has been defined in such a way that $\left\{\mathcal{D}^4_{i,j}, i,j\in\mathbb{Z}\right\}$ is a partition of $\mathcal{D}_4$, the union, without repetitions, of all domain points $\xi^4_{i,j,k} := \left(iv_0+jv_1+kv_2\right)/4$ of each triangle in Δ.

A similar construction can be applied to the cubic quasi-interpolant $\mathcal{Q}_{3,3}$.

5 Conclusions

We have analyzed the construction of C^1 cubic quasi-interpolants defined on a type-1 triangulation based on point values without imposing a structure based on the translation of one or more compactly supported functions. Instead, the quasi-interpolating splines are determined by setting their BB-coefficients to appropriate combinations of the given data only using values of the function to be approximated. The associated operator reproduces quadratic polynomials.

We have proved that the proposed problem has a general solution depending of three parameters, and determined quasi-interpolation operators with the minimal uniform norm. Moreover, from a general quasi-interpolation exact on quadratics and providing C^1 quadratic approximating splines we have defined a C^1 quartic differential quasi-interpolant by expressing its BB-coefficients from the masks of the point operator.

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