A trivariate near-best blending quadratic quasi-interpolant

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the construction of such operators appears in [4], where univariate quasi-interpolants are based on a point or derivative linear functional (see [5] for the nonuniform case). The bivariate case was considered in [1, 3, 6] by using C^2-quartic B-splines on the four-direction mesh, H-spline and a Ω-spline, respectively (the case of quadratic box spline appears in [17] and the use of cubic multi-box spline is considered in [19]). The extension to the three-dimensional case is done in [8, 13, 18, 20].

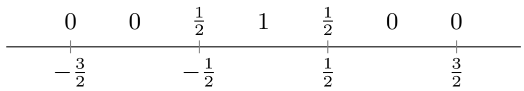
In order to obtain a better upper bound to be minimized it is possible to bound the Lebesgue function associated with Q from the Bernstein-Bézier coefficients of φ. This approach has been considered in [2, 9].

In this paper, we deal with the construction of a new near-best trivariate spline quasi-interpolation operator by blending 1D and 2D C^1 quadratic spline quasi-interpolants and minimizing an objective function constructed from the Bernstein-Bézier coefficients of the Lebesgue function of the resulting operator. In particular, in Section 2, we introduce the univariate and bivariate spline spaces, quasi-interpolation operators in such spaces and we define the blending trivariate operator. In Section 3, we study the problem of the construction of near-best quasi-interpolants, by defining the objective function characterizing the minimization problem and providing the explicit solution. Finally, in Section 4, some conclusions are presented.

2 Spline spaces and quasi-interpolation operators

Let B be the quadratic B-spline supported on the interval $\left[-\frac{3}{2}, \frac{3}{2}\right]$. It is a C^1 quadratic B-spline on the real line having knots at the half-integers. Its Bernstein-Bézier (BB-) coefficients in every sub-interval of its support appear in Figure 1 (see e.g. [12]).

Let M be the quadratic box spline on the four-directional triangulation τ of the plane generated by the directions $d_1 := (1,0)$, $d_2 := (0,1)$, $d_3 := d_1 + d_2$, $d_4 := d_2 - d_1$. It is a $C^1(\mathbb{R}^2)$ function whose restriction to every triangle in τ is a quadratic polynomial (see e.g. [11, 12]). Figure 1 shows the support of the box spline and provides the BB-coefficients of $8M$ in the triangles of τ included in the polygon with vertices $(0,0)$, $(1,-1)$, $\left(\frac{3}{2}, -\frac{1}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}\right)$, and $(1,1)$. The BB-coefficients relative to the other triangles in the support of M are determined by the symmetries of the octagon.



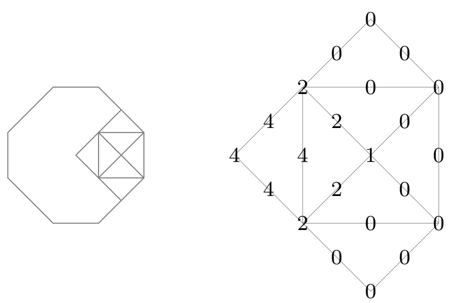


Figure 1: (Left) BB-coefficients of the quadratic B-spline B. (Right) BB-coefficients of the box spline $8M$.

From the B-spline B and the box spline M, we consider the spaces $\mathcal{B}_1 := \operatorname{span}\{B(\cdot - k) : k \in \mathbb{Z}\}$ and $\mathcal{B}_2 := \operatorname{span}\{M(\cdot - i_1, \cdot - i_2) : (i_1, i_2) \in \mathbb{Z}^2\}$. They contain the spaces of univariate and bivariate quadratic polynomials, respectively (cf. [11, p. 53], [12, p. 19]).

The main goal is to construct a near-best trivariate quasi-interpolation operator (QIO) from some univariate and bivariate QIOs properly chosen. Firstly, we consider the univariate Schoenberg QIO $\overline{S}$ defined by

$$\overline{S}f(z) := \sum_{k \in \mathbb{Z}} f(k)\, B(z-k), \tag{2.1}$$

and the operator $\overline{Q}$ defined by

$$\overline{Q}f(z) := \sum_{k \in \mathbb{Z}} \left(\sum_{\ell=-2}^{2} a_\ell f(k-\ell) \right) B(z-k), \tag{2.2}$$

where the coefficients a_ℓ are chosen to produce an operator exact on $P_2(\mathbb{R})$, the space of univariate quadratic polynomials. The operator $\overline{S}$ given in (2.1) is exact only on $P_1(\mathbb{R})$, the space of univariate linear polynomials. Regarding $\overline{Q}$, it is well-known that it is possible to define a QIO exact on $P_2(\mathbb{R})$ using only three coefficients instead of five like in (2.2) (see e.g. [21]). Explicitly, $a_{-2} = a_2 = 0$, $a_{-1} = a_1 = -\frac{1}{8}$, and $a_0 = \frac{5}{2}$. However, some oversampling is allowed in order to be able to reduce the infinity norm of the operator. We have introduced the minimum number of freedom degrees and suppose that $a_{-2} = a_2$ and $a_{-1} = a_1$ to produce an even fundamental function. For the sequence of coefficients we will write $\mathbf{a} := (a_0; a_1; a_2)$.

Lemma 1 *The quasi-interpolant (QI) $\overline{Q}f(z)$ given by (2.2) can be written as*

$$\overline{Q}f(z) = \sum_{k \in \mathbb{Z}} f(k)\, L_{\mathbf{a}}(z-k),$$

where the fundamental function $L_{\mathbf{a}}$ is the linear combination of integer translates of the B-spline B given by the expression

$$L_{\mathbf{a}}(z) := \sum_{\ell=-2}^{2} a_\ell B(z-\ell). \tag{2.3}$$

Moreover, the operator $\overline{Q}$ is exact on $P_2(\mathbb{R})$ if and only if

$$a_0 + 2a_1 + 2a_2 = 1 \quad \textit{and} \quad a_1 + 4a_2 = -\frac{1}{8}.$$

Proof. The first claim is derived easily. Regarding the exactness, we will use a general result in [7, p. 274] that implies in the quadratic case the exactness on $P_2(\mathbb{R})$ of the differential operator D_3 given by

$$D_3 f := \sum_{k \in \mathbb{Z}} \left(f(k) - \frac{1}{8} f''(k) \right) B(\cdot - z).$$

It is obvious to prove that $\overline{Q}$ reproduces the monomial $m_0(z) := 1$ if and only if $a_0 + 2a_1 + 2a_2 = 1$. This constraint on the coefficients defining $\overline{Q}$ also yields that $\overline{Q}$ reproduces the monomial $m_1(z) := z$. As far as monomial $m_2(z) := z^2$ is concerned, $\overline{Q}$ provides equality

$$\overline{Q}m_2(z) = \sum_{k \in \mathbb{Z}} k^2 B(z-k) + 2(a_1 + 4a_2).$$

Since

$$D_3 m_2(z) = \sum_{k \in \mathbb{Z}} k^2 B(z-k) - \frac{1}{4},$$

then $\overline{Q}$ will reproduce m_2 if and only if $a_1 + 4a_2 = -\frac{1}{8}$, and the proof is complete.

Now, let S be the Schoenberg QIO associated with the box spline M (see e.g. [22]), which is defined by

$$Sf(x,y) := \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2)\, M(x-i_1, y-i_2)\,. \tag{2.4}$$

It is exact on the space of bilinear polynomials. Once again, in order to obtain a trivariate operator with small infinity norm, we consider the QIO Q defined by

$$Qf(x,y) := \sum_{(i_1,i_2)\in\mathbb{Z}^2} \left(\sum_{(j_1,j_2)\in J} c_{j_1,j_2} f(i_1-j_1, i_2-j_2) \right) M(x-i_1, y-i_2)\,, \tag{2.5}$$

where

$$J := \{(0,0),(1,0),(0,1),(-1,0),(0,-1),(2,0),(0,2),(-2,0),(0,-2),(1,1),(-1,1),(-1,-1),(1,-1)\},$$

and $c := \{(c_{j_1,j_2}),\ (j_1,j_2)\in J\}$ is a lozenge sequence [2] such that Q is exact on $P_2(\mathbb{R}^2)$, the space of bivariate polynomials of total degree two, i.e.

$$c_{0,1} = c_{-1,0} = c_{0,-1} = c_{1,0},\ c_{0,2} = c_{-2,0} = c_{0,-2} = c_{2,0},\ c_{-1,1} = c_{-1,-1} = c_{1,-1} = c_{1,1}.$$

We will write $c = (c_{0,0}; c_{1,0}; c_{2,0}, c_{1,1})$.

Lemma 2 *The spline Qf in (2.5) can be written as*

$$Qf(x,y) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2)\, L_c(x-i_1, y-i_2)\,,$$

where the fundamental function L_c is expressed as the linear combination of the integer translates of M given by

$$L_c(x,y) := \sum_{(j_1,j_2)\in J} c_{j_1,j_2} M(x-j_1, y-j_2)\,. \tag{2.6}$$

Moreover, Q is exact on $P_2(\mathbb{R}^2)$ if and only if

$$c_{0,0} + 4c_{1,0} + 4c_{2,0} + 4c_{1,1} = 1 \quad \textit{and} \quad c_{1,0} + 4c_{2,0} + 2c_{1,1} = \frac{1}{8}.$$

Proof. As in the proof of Lemma 1, it is straightforward to prove the first claim. With respect to the exactness of Q, we will use a method similar to that described in Lemma 1. Now the starting point is the differential operator D exact on $P_2(\mathbb{R}^2)$ given by

$$Df := \sum_{i\in\mathbb{Z}^2} \left(f(i) - \frac{1}{8}\Delta f(i) \right) M(\cdot - i)\,,$$

where Δ stands for the Laplacian of f. It is obtained from a general method described in [14]. As in the univariate case, the operator Q reproduces the monomial $m_{0,0}(x,y) := 1$ if and only if $c_{0,0} + 4c_{1,0} + 4c_{2,0} + 4c_{1,1} = 1$, and under this constraint the monomials $m_{1,0}(x,y) := x$ and $m_{0,1}(x,y) := y$ are automatically reproduced since for α equal to $(1,0)$ or $(0,1)$ it holds

$$Qm_\alpha(x,y) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} m_\alpha(i_1,i_2)\, M(x-i_1, y-i_2) = Dm_\alpha(x,y) = m_\alpha(x,y)\,.$$

Moreover, after some calculations, for $m_{2,0}(x,y) := x^2$ it follows that

$$Qm_{2,0}(x,y) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} \left(i_1^2 + 2\left(c_{1,0} + 2c_{1,1} + 4c_{2,0}\right) \right) M(x-i_1, y-i_2)\,.$$

Therefore, it is equal to

$$x^2 = Dm_{2,0}(x,y) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} \left(i_1^2 - \frac{1}{4}\right) M(x-i_1, y-i_2)$$

if and only if $c_{1,0} + 2c_{1,1} + 4c_{2,0} = -\frac{1}{8}$.

Once again, both equalities imply that also the monomials $m_{0,2}(x,y) := y^2$ and $m_{1,1}(x,y) := xy$ are automatically reproduced, and the proof is complete.

Now, we define trivariate extensions of the operators above.

Definition 3 *Once defined in (2.1) and (2.4) the univariate and bivariate Schoenberg operators and the QIOs* $\overline{Q}$ *and* Q *in (2.4) and (2.5), we consider the trivariate extensions of these operators. They are given by*

$$\overline{S}f(x,y,z) = \sum_{k\in\mathbb{Z}} f(x,y,k)\, B(z-k), \tag{2.7}$$

$$\overline{Q}f(x,y,z) = \sum_{k\in\mathbb{Z}} \left(\sum_{\ell=-2}^{2} a_\ell f(x,y,k-\ell)\right) B(z-k) = \sum_{k\in\mathbb{Z}} f(x,y,k)\, L_{\overline{Q}}(z-k), \tag{2.8}$$

$$Sf(x,y,z) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2,z)\, M(x-i_1, y-i_2), \tag{2.9}$$

$$Qf(x,y,z) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} \left(\sum_{(j_1,j_2)\in\Omega} c_{j_1,j_2} f(i_1-j_1, i_2-j_2, z)\right) M(x-i_1, y-i_2) \tag{2.10}$$

$$= \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2,z)\, L_Q(x-i_1, y-i_2).$$

We are now in position to define the type of operator we are interested in (see [9, 15, 20]).

Definition 4 *From the operators given by (2.7), (2.8), (2.9) and (2.10), the trivariate blending operator* R *is defined as*

$$R := S\overline{Q} + Q\overline{S} - S\overline{S}. \tag{2.11}$$

The operator R is a linear map into the tensor product spline space spanned by the trivariate piecewise polynomial functions $M(\cdot - i_1, \cdot - i_2)\, B(\cdot - k)$, $(i_1, i_2, k) \in \mathbb{Z}^3$. The QI Rf provided by the operator in (2.11) can be expressed from the fundamental functions relative to the operators $\overline{Q}$ and Q given in (2.3) and (2.6).

Lemma 5 *It holds*

$$Rf(x,y,z) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} \sum_{k\in\mathbb{Z}} f(i_1,i_2,k)\, L(x-i_1, y-i_2, z-k),$$

where

$$L(x,y,z) := M(x,y)\, L_{\overline{Q}}(z) + L_Q(x,y)\, B(z) - M(x,y)\, B(z) \tag{2.12}$$

$$= L_Q(x,y)\, B(z) + M(x,y)\, (L_{\overline{Q}}(z) - B(z)). \tag{2.13}$$

Moreover, R *reproduces the monomials* 1, x, y, z, x^2, y^2, z^2, xy, xz, yz, x^2z, xz^2, y^2z, yz^2, xyz, *and* xyz^2.

Proof. The proof of the first statement is trivial. For the second one, see [20, Theorem 4].

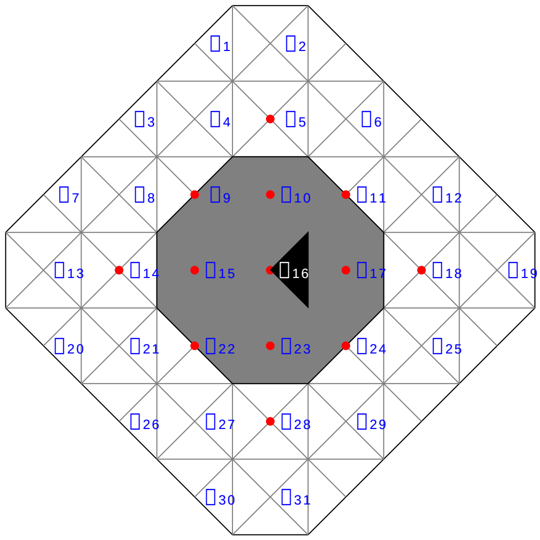


Figure 2: The supports Ω and ω of L_γ and M, respectively, the T-like triangles T_i, $1 \le i \le 31$, and the points associated with the lozenge sequence c.

3 Near-best trivariate quadratic quasi-interpolation

The infinity norm of the QIO R is provided by the maximum of the associated Lebesgue function

$$\Lambda(x,y,z) := \sum_{(i_1,i_2)\in\mathbb{Z}^2} \sum_{k\in\mathbb{Z}} |L(x-i_1, y-i_2, z-k)|. \tag{3.1}$$

Since we are dealing with a uniform partition of the three dimensional space, Λ is a 1-periodic function, so that to determine its maximum value it is sufficient to consider its restriction to the cube $\left[-\frac{1}{2},\frac{1}{2}\right]^3$. Moreover, due to the symmetries of B and M and those of the coefficients in (2.2) and (2.5), the maximum is attained in a subset of the prism $P := T \times I$ with triangular horizontal sections, where T is the triangle defined in Section 2 and I is the interval $\left[-\frac{1}{2},\frac{1}{2}\right]$ (see [2, Lemma 3]).

Considering the complex structure of Λ, which also depends on a and c, it is very difficult to determine its maximum in P and the points at which it is reached. Therefore, we look for a good upper bound of $\|R\|_\infty$ by examining carefully the contribution of every term $L(x-i_1, y-i_2, z-k)$ to $\Lambda(x,y,z)$, $(x,y,z) \in P$.

According to (2.13), the fundamental function L in (2.12) is decomposed into two terms. The first one, $L_\gamma(x,y)B(z)$, is supported on $\Omega \times \left[-\frac{3}{2},\frac{3}{2}\right]$, Ω being the octagon with vertices

$$\left(\frac{7}{2},\frac{1}{2}\right), \left(\frac{1}{2},\frac{7}{2}\right), \left(-\frac{1}{2},\frac{7}{2}\right), \left(-\frac{7}{2},\frac{1}{2}\right), \left(-\frac{7}{2},-\frac{1}{2}\right), \left(-\frac{1}{2},-\frac{7}{2}\right), \left(\frac{1}{2},-\frac{7}{2}\right) \text{ and } \left(\frac{7}{2},-\frac{1}{2}\right).$$

However, the second term, $M(x,y)(L_\gamma(z) - B(z))$, is supported on $\omega \times \left[-\frac{7}{2},\frac{7}{2}\right]$, where ω is the octagon included in Ω with vertices

$$\left(\frac{3}{2},\frac{1}{2}\right), \left(\frac{1}{2},\frac{3}{2}\right), \left(-\frac{1}{2},\frac{3}{2}\right), \left(-\frac{3}{2},\frac{1}{2}\right), \left(-\frac{3}{2},-\frac{1}{2}\right), \left(-\frac{1}{2},-\frac{3}{2}\right), \left(\frac{1}{2},-\frac{3}{2}\right) \text{ and } \left(\frac{3}{2},-\frac{1}{2}\right).$$

Both octagons are represented in Figure 2, where also the points associated with the lozenge sequence c and the T-like triangles involved in (3.1) are shown.

To facilitate the calculation of the upper bound to the Lebesgue function, we will consider the polynomials defining the B-spline B on $\left[-\frac{7}{2},\frac{7}{2}\right]$ and the box spline M on Ω instead of $\left[-\frac{3}{2},\frac{3}{2}\right]$ and ω, respectively.

It is straightforward to prove the following result (see Figure 1).

Proposition 6 *The restrictions* b_ℓ *of the B-spline* B *to the sub-intervals* $\left[k - \frac{9}{2}, k - \frac{7}{2}\right]$, $1 \le k \le 7$, *are given by* $b_1(z) = b_2(z) = 0$, $b_3(z) = \frac{1}{8}(3+2z)^2$, $b_4(z) = \frac{3}{4} - z^2$, $b_5(z) = \frac{1}{8}(3-2z)^2$, $b_6(z) = b_7(z) = 0$. *Moreover, those of* $L_\ell(z) - B(z)$ *are*

$$Q_1(z) = a_{-2}b_1(z+2) - b_1(z), \quad Q_2(z) = a_{-1}b_1(z+1) + a_{-2}b_2(z+2) - b_2(z),$$
$$Q_3(z) = a_0b_1(z) + a_{-1}b_2(z+1) + a_{-2}b_3(z+2) - b_3(z),$$
$$Q_4(z) = a_1b_1(z-1) + a_0b_2(z) + a_{-1}b_3(z+1) - b_4(z),$$
$$Q_5(z) = a_2b_1(z-2) + a_1b_2(z-1) + a_0b_3(z) - b_5(z),$$
$$Q_6(z) = a_2b_2(z-2) + a_1b_3(z-1) - b_6(z), \quad Q_7(z) = a_2b_3(z-2) - b_7(z).$$

A similar result is easily stated regarding the restrictions of M and L_ℓ to the T-like triangles.

Proposition 7 *The nonzero restrictions* p_ℓ *of* M *to the triangles* T_ℓ, $i \in \{9, 10, 15, 16, 17, 22, 23\}$, *are given by the following polynomials:*

$$p_9(x,y) = \frac{1}{4}\left(4 + 4x + x^2 - 4y - 2xy + y^2\right), \quad p_{10}(x,y) = \frac{1}{8}\left(7 - 4x - 2x^2 - 8y + 4xy + 2y^2\right),$$
$$p_{15}(x,y) = \frac{1}{8}\left(5 + 4x - 4y^2\right), p_{16}(x,y) = \frac{1}{2}\left(1 - x^2 - y^2\right), p_{17}(x,y) = \frac{1}{8}\left(9 - 12x + 4x^2\right),$$
$$p_{22}(x,y) = \frac{1}{4}\left(4 + 4x + x^2 + 4y + 2xy + y^2\right), p_{23}(x,y) = \frac{1}{8}\left(7 - 4x - 2x^2 + 8y - 4xy + 2y^2\right).$$

If q_ℓ *stands for the restriction of* L_ℓ *to* T_ℓ, *then it holds*

$$q_1(x,y) = c_{0,2}p_1(x,y-2), \quad q_2(x,y) = c_{0,2}p_2(x,y-2), \quad q_3(x,y) = c_{-1,1}p_1(x+1,y-1),$$
$$q_4(x,y) = c_{0,1}p_1(x,y-1) + c_{-1,1}p_2(x+1,y-1) + c_{0,2}p_3(x,y-2),$$
$$q_5(x,y) = c_{1,1}p_1(x-1,y-1) + c_{0,1}p_2(x,y-1) + c_{0,2}p_4(x,y-2),$$
$$q_6(x,y) = c_{1,1}p_2(x-1,y-1) + c_{0,2}p_5(x,y-2), \quad q_7(x,y) = c_{-2,0}p_1(x+2,y),$$
$$q_8(x,y) = c_{-1,0}p_1(x+1,y) + c_{-2,0}p_2(x+2,y) + c_{-1,1}p_3(x+1,y-1),$$
$$q_9(x,y) = c_{0,0}p_1(x,y) + c_{-1,0}p_2(x+1,y) + c_{0,1}p_3(x,y-1) + c_{-1,1}p_4(x+1,y-1) + c_{0,2}p_6(x,y-2),$$
$$q_{10}(x,y) = c_{1,0}p_1(x-1,y) + c_{0,0}p_2(x,y) + c_{1,1}p_3(x-1,y-1) + c_{0,1}p_4(x,y-1) + c_{-1,1}p_5(x+1,y-1) + c_{0,2}p_7(x,y-2),$$
$$q_{11}(x,y) = c_{2,0}p_1(x-2,y) + c_{1,0}p_2(x-1,y) + c_{1,1}p_4(x-1,y-1) + c_{0,1}p_5(x,y-1),$$
$$q_{12}(x,y) = c_{2,0}p_2(x-2,y) + c_{1,1}p_5(x-1,y-1), \quad q_{13}(x,y) = c_{-2,0}p_3(x+2,y),$$
$$q_{14}(x,y) = c_{-1,-1}p_1(x+1,y+1) + c_{-1,0}p_3(x+1,y) + c_{-2,0}p_4(x+2,y) + c_{-1,1}p_6(x+1,y-1),$$
$$q_{15}(x,y) = c_{0,-1}p_1(x,y+1) + c_{-1,-1}p_2(x+1,y+1) + c_{0,0}p_3(x,y) + c_{-1,0}p_4(x+1,y) + c_{-2,0}p_5(x+2,y) + c_{0,1}p_6(x,y-1) + c_{-1,1}p_7(x+1,y-1),$$
$$q_{16}(x,y) = c_{1,-1}p_1(x-1,y+1) + c_{0,-1}p_2(x,y+1) + c_{1,0}p_3(x-1,y) + c_{0,0}p_4(x,y) + c_{-1,0}p_5(x+1,y) + c_{1,1}p_6(x-1,y-1) + c_{0,1}p_7(x,y-1)$$
$$q_{17}(x,y) = c_{1,-1}p_2(x-1,y+1) + c_{2,0}p_3(x-2,y) + c_{1,0}p_4(x-1,y) + c_{0,0}p_5(x,y) + c_{1,1}p_7(x-1,y-1),$$
$$q_{18}(x,y) = c_{2,0}p_4(x-2,y) + c_{1,0}p_5(x-1,y), \quad q_{19}(x,y) = c_{2,0}p_5(x-2,y), \quad q_{20}(x,y) = c_{-2,0}p_6(x+2,y),$$
$$q_{21}(x,y) = c_{-1,-1}p_3(x+1,y+1) + c_{-1,0}p_6(x+1,y) + c_{-2,0}p_7(x+2,y),$$
$$q_{22}(x,y) = c_{0,-2}p_1(x,y+2) + c_{0,-1}p_3(x,y+1) + c_{-1,-1}p_4(x+1,y+1) + c_{0,0}p_6(x,y) + c_{-1,0}p_7(x+1,y),$$
$$q_{23}(x,y) = c_{0,-2}p_2(x,y+2) + c_{1,-1}p_3(x-1,y+1) + c_{0,-1}p_4(x,y+1) + c_{-1,-1}p_5(x+1,y+1) + c_{1,0}p_6(x-1,y) + c_{0,0}p_7(x,y),$$
$$q_{24}(x,y) = c_{1,-1}p_4(x-1,y+1) + c_{0,-1}p_5(x,y+1) + c_{2,0}p_6(x-2,y) + c_{1,0}p_7(x-1,y),$$
$$q_{25}(x,y) = c_{1,-1}p_5(x-1,y+1) + c_{2,0}p_7(x-2,y), \quad q_{26}(x,y) = c_{-1,-1}p_6(x+1,y+1),$$
$$q_{27}(x,y) = c_{0,-2}p_3(x,y+2) + c_{0,-1}p_6(x,y+1) + c_{-1,-1}p_7(x+1,y+1),$$
$$q_{28}(x,y) = c_{0,-2}p_4(x,y+2) + c_{1,-1}p_6(x-1,y+1) + c_{0,-1}p_7(x,y+1),$$

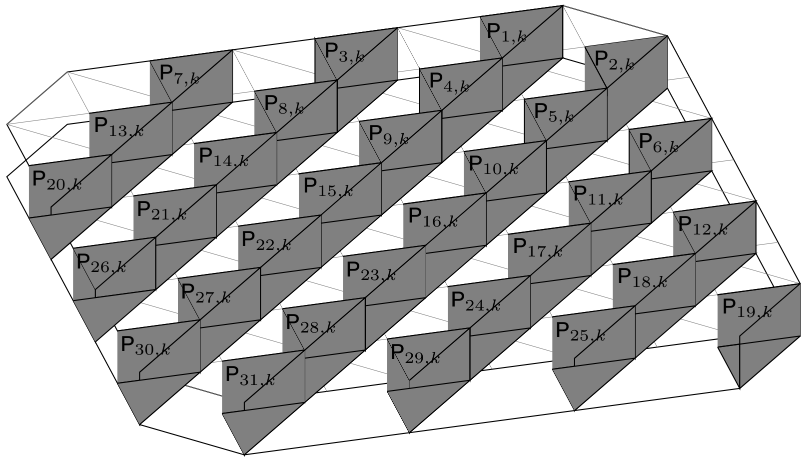


Figure 3: Prisms $P_{j,k}$ of the k-th level of the set $\Omega\left(\frac{7}{2}, \frac{7}{2}\right)$.

$q_{29}(x,y) = c_{0,-2}\, p_5(x, y+2) + c_{1,-1}\, p_7(x-1, y+1)$, $\quad q_{30}(x,y) = c_{0,-2}\, p_6(x, y+2)$,
$q_{31}(x,y) = c_{0,-2}\, p_7(x, y+2)$.

Once determined the polynomial structure of B, $L_h B$, M and $L_h M$, it only remains to restrict the fundamental function of the operator R to every P-like prism $P_{j,k}$, $1 \le j \le 31$, $1 \le k \le 7$ (see Figure 3) and to translate the resulting functions as indicated below to produce the terms whose supports contain the prism P. Concerning the translation in the z-direction, we go through the set $\Omega\left(\frac{7}{2}, \frac{7}{2}\right)$ from interval $\left[-\frac{7}{2}, -\frac{5}{2}\right]$ to interval $\left[\frac{5}{2}, \frac{7}{2}\right]$. Regarding the translation in the x and y directions, the centers $y_i := (y_{i,1}, y_{i,2})$ form the subset $\Gamma := \{y_i, 1 \le i \le 31\}$ given by

$$\Gamma = \{(1,-3), (0,-3), (2,-2), (1,-2), (0,-2), (-1,-2), (3,-1), (2,-1), (1,-1), (0,-1), (-1,-1), (-2,-1), (3,0), (2,0), (1,0), (0,0), (-1,0), (-2,0), (-3,0), (3,1), (2,1), (1,1), (0,1), (-1,1)(-2,1), (2,2), (1,2), (0,2), (-1,2), (1,3), (0,3)\}.$$

For every $1 \le j \le 31$ and $1 \le k \le 7$, the trivariate function

$$S_{j,k}(x,y,z) := q_j(x - y_{j,1}, y - y_{j,2})\, b_k(z + k - 4) + p_j(x - y_{j,1}, y - y_{j,2})\, Q_k(z + k - 4)$$

is the restriction of $L(\cdot - y_{j,1}, \cdot - y_{j,2})$ to the prism P. Therefore, we get

$$\Lambda(x,y,z) \le \sum_{\substack{1 \le j \le 31 \\ 1 \le k \le 7}} |S_{j,k}(x,y,z)| =: U(x,y,z), \quad (x,y,z) \in P. \tag{3.2}$$

Now, we decompose the prism P into the following three tetrahedra. If the vertices of P are $V_{1,1} = \left(0, 0, \frac{1}{2}\right)$, $V_{1,2} = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$, $V_{1,3} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $V_{2,1} = \left(0, 0, -\frac{1}{2}\right)$, $V_{2,2} = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ and $V_{2,3} = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$, the tetrahedra are $T_1 := [V_{1,1}, V_{2,1}, V_{2,2}, V_{2,3}]$, $T_2 := [V_{1,1}, V_{1,2}, V_{2,2}, V_{2,3}]$ and $T_3 := [V_{2,3}, V_{1,1}, V_{1,2}, V_{1,3}]$ (see Figure 4).

Let $T := [V_1, V_2, V_3, V_4]$ be one of the tetrahedra above, and let $\lambda := (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be the barycentric coordinates of a point (x,y,z) with respect to T, i.e. it holds

$$(x,y,z) = \sum_{i=1}^{4} \lambda_i V_i, \quad |\lambda| := \sum_{i=1}^{4} \lambda_i = 1.$$

Since every function $S_{j,k}(x,y,z)$ in the upper bound (3.2) is a trivariate quartic polynomial on T that can be represented in terms of the Bernstein polynomials

$$B^4_{\alpha,T}(\lambda) := \frac{4!}{\alpha!}\lambda^\alpha = \frac{4!}{\alpha_1!\alpha_2!\alpha_3!\alpha_4!}\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4},$$

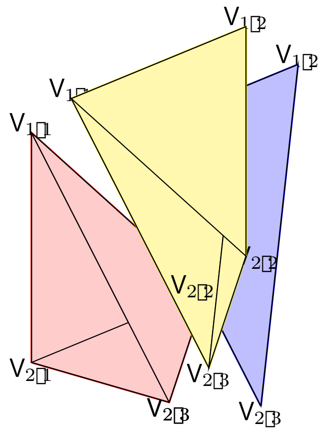


Figure 4: Decomposition of prism P into tetrahedra.

where the length $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ of the multi-index $\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (\mathbb{N} \cup \{0\})^4$ is equal to four. Therefore, there exist coefficients $g^{\mathbb{T}}_{\alpha} := g^{\mathbb{T}}_{\alpha}(a, c)$ such that

$$S(x, y, z) = \sum_{|\alpha|=4} g^{\mathbb{T}}_{\alpha} B^{4,\mathbb{T}}_{\alpha}(\lambda).$$

Thus, since $\left\{B^{4,\mathbb{T}}_{\alpha}\right\}_{|\alpha|=4}$ is a non-negative partition of unity, for all $(x, y, z) \in T$, it follows from (3.2) that

$$U(x, y, z) \leq \sum_{\substack{1 \leq i \leq 31 \\ 1 \leq j \leq 7}} \sum_{|\alpha|=4} \left|g^{\mathbb{T}}_{\alpha}\right| B^{4,\mathbb{T}}_{\alpha}(\lambda) = \sum_{|\alpha|=4} \left(\sum_{\substack{1 \leq i \leq 31 \\ 1 \leq j \leq 7}} \left|g^{\mathbb{T}}_{\alpha}\right| \right) B^{4,\mathbb{T}}_{\alpha}(\lambda) \leq \max_{|\alpha|=4} \sum_{\substack{1 \leq i \leq 31 \\ 1 \leq j \leq 7}} \left|g^{\mathbb{T}}_{\alpha}\right|.$$

When this construction is carried out for T_1, T_2 and T_3, the following result is obtained.

Proposition 8 *Under the conditions above, the function*

$$F(a, b) := \max_{1 \leq \ell \leq 3} \max_{|\alpha|=4} \sum_{\substack{1 \leq i \leq 31 \\ 1 \leq j \leq 7}} \left|g^{\mathbb{T}_\ell}_{\alpha}(a, c)\right| \tag{3.3}$$

is an upper bound to the infinity norm of the trivariate blending QIO R.

Hence, we state the following minimization problem in order to determine a near-best blending QIO.

Problem 9 *Minimize the objective function* $F(a, c)$ *given in (3.3) on*

$$A := \left\{ c \mid Q \textit{ is exact on } P_2\left(\mathbb{R}^2\right) \right\} \times \left\{ a \mid \overline{Q} \textit{ is exact on } P_2(\mathbb{R}) \right\}$$

Every coefficient g_{α} is a linear function of a and c, hence $|g_{\alpha}|$ is also a convex function. Thus, F is a convex function on A since it is the maximum of a set of convex functions, and the existence of a solution for Problem 9 is guaranteed (see e.g. [23]). If the minimum value of F is attained at a point $(c, a) \in A$, then corresponding operator R is said to be a near-best QIO.

The nonlinear minimization Problem 9 can be solved by converting it into a linear programming one with inequality and equality constraints.

An arduous work that uses the symbolic computation software *Mathematica* allows to prove the following result.

Proposition 10 *The minimum value of* $F(a,c)$ *on* A *is equal to* $\frac{532653}{393376} \approx 1.35406$*. It is attained uniquely at* $a = \left(\frac{12907}{12293}; \frac{823}{98344}; -\frac{3279}{98344}\right)$ *and* $c = \left(\frac{50881}{49172}; \frac{441}{12293}; -\frac{441}{12293}, -\frac{1709}{196688}\right)$.

Proof. By replacing the solutions $(a_0, a_1, a_2) = \left(\frac{1}{16}(17-24x), x, -\frac{1}{32}(1+8x)\right)$ and $(c_{0,0}, c_{1,0}, c_{2,0}, c_{1,1}) = \left(y, z, \frac{1}{16}(-5+4y+8y), \frac{1}{16}(9-8y-24z)\right)$ of the linear systems in Lemmas 1 and 2 into the objective function of Problem 9, we get an unconstrained minimization problem with objective function

$$\tilde{F}(x,y,z) = \max_{1 \le \ell \le 3} \max_{|j|=4} \sum_{\substack{1 \le i \le 31 \\ 1 \le k \le 7}} \left| G^{\mathbb{T}_\ell}_{i,j,k}(x,y,z) \right|,$$

where

$$G^{\mathbb{T}_\ell}_{i,j,k}(x,y,z) := g^{\mathbb{T}_\ell}_{i,j,k}\left(\frac{1}{16}(17-24x), x, -\frac{1}{32}(1+8x), y, z, \frac{1}{16}(-5+4y+8y), \frac{1}{16}(9-8y-24z)\right)$$

Function $\tilde{F}$ has been constructed from the BB-coefficients relative to the three tetrahedra in which the prism P has been decomposed of the integer translates of the fundamental function associated with the operator R. A large number of BB-coefficients are equal to zero because the B-spline B is nonzero only on the interval $\left[-\frac{3}{2}, \frac{3}{2}\right]$, and the box spline M is zero on $\Omega \setminus \omega$. It is also possible to eliminate the repetitions of coefficients that occur in the three tetrahedra. An explicit calculation shows that $\tilde{F}$ is defined from 34 expressions depending on a maximum of 121 terms, so

$$\tilde{F}(x,y,z) = \max_{1 \le p \le 34} \frac{1}{e_p} \sum_{n=1}^{121} \left| \tilde{d}_{p,n} \left(\tilde{\alpha}_{p,n} x + \tilde{\beta}_{p,n} y + \tilde{\gamma}_{p,n} z\right) \right|,$$

for integers $\tilde{\alpha}_{p,n}$, $\tilde{\beta}_{p,n}$, $\tilde{\gamma}_{p,n}$, and $e_p, \tilde{d}_{p,n} \in \mathbb{N}$. Therefore, the minimization of $\tilde{F}$ is equivalent to the following linear programming problem:

$$\begin{aligned}
&\text{Minimize } \mu \\
&\text{such that } \begin{cases}
\displaystyle\sum_{n=1}^{121} \tilde{d}_{p,n}(u_{p,n} + v_{p,n}) - e_p \mu \le 0, & 1 \le p \le 34, \\
\tilde{\alpha}_{p,n}(X_1 - X_2) + \tilde{\beta}_{p,n}(Y_1 - Y_2) + \tilde{\gamma}_{p,n}(Z_1 - Z_2) - u_{p,n} + v_{p,n} = 0, & 1 \le n \le 121, \\
u_{p,n}, v_{p,n}, X_1, X_2, Y_1, Y_2, Z_1, Z_2, \mu \ge 0.
\end{cases}
\end{aligned}$$

The solution of this problem is then determined with the symbolic calculation software, and the minimum value $\mu = \frac{532653}{393376}$ is reached at

$$X_1 = \frac{823}{98344}, \; X_2 = 0, \; Y_1 = \frac{50881}{49172}, \; Y_2 = 0, \; Z_1 = \frac{441}{12293} \quad \text{and} \quad Z_2 = 0,$$

i.e. $x = \frac{823}{98344}$, $y = \frac{50881}{49172}$ and $z = \frac{441}{12293}$. Analyzing the $\tilde{F}$ function in a neighbourhood of $\left(\frac{823}{98344}, \frac{50881}{49172}, \frac{441}{12293}\right)$ it is concluded that it is the unique point at which the minimum is attained.

If $Q_{a,c}$ denotes the operator given by the solution of Problem 9, then the evaluation of its Lebesgue function at the points resulting in dividing the subset $\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$ into $20 \times 20 \times 20$ equal parts provides the value 1.34899 as a lower bound to $\|Q_{a,c}\|_\infty$. This shows that the proposed construction has allowed to improve the result in [9], where the near-best blending QIO is obtained by minimizing an objective function also established from the BB-coefficients of the B-spline and the box spline, and has a uniform norm equal to $\frac{11}{8} \simeq 1.375$.

4 Conclusions

In this paper, we have constructed a new trivariate near-best spline quasi-interpolation operator by blending 1D and 2D C^1 quadratic spline quasi-interpolants and minimizing an upper bound of its infinity norm. It is derived from the Bernstein-Bézier coefficients of its Lebesgue function. In particular, the new operator has a smaller norm with respect to the blending quasi-interpolant obtained in [9]. Such a technique can also be generalized by considering different spline spaces.

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