Antipodes, Preantipodes and Frobenius Functors

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PAOLO SARACCO

Abstract. We prove that a quasi-bialgebra admits a preantipode if and only if the associated free quasi-Hopf bimodule functor is Frobenius, if and only if the related (opmonoidal) monad is a Hopf monad. The same results hold in particular for a bialgebra, tightening the connection between Hopf and Frobenius properties.

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Introduction

It has been known for a long time that Hopf algebras (with some additional finiteness condition) are strictly related with Frobenius algebras. In fact, Larson and Sweedler proved

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in [24] that any finite-dimensional Hopf algebra over a PID is automatically Frobenius and Pareigis extended this result in [30] by proving that a bialgebra B over a commutative ring $\Bbbk$ is a finitely generated and projective Hopf algebra with $\int B^* \cong \Bbbk$ if and only if it is Frobenius as an algebra with Frobenius homomorphism $\psi \in \int B^*$. Afterwards, great attention has been devoted to those bialgebras that are also Frobenius and whose Frobenius homomorphism is an integral (see e.g. [22, 29]) and to the interactions between Frobenius and Hopf algebra theory in general (see [6, 7, 21, 25]). In particular, there exist a number of results that extend Larson-Sweedler's and Pareigis' theorems to more general classes of Hopf-like structures ([19, 20, 23, 39]).

Following their increasing importance, many extensions of Hopf and Frobenius algebras have arisen. Let us mention (co)quasi-Hopf algebras, Hopf algebroids, Hopf monads, Frobenius monads, Frobenius monoids and Frobenius functors. At the same time new results appeared ([5, 14, 20]), showing that there is a deeper connection between the Hopf and the Frobenius properties that deserves to be uncovered. In [32] we realised that Frobenius functors may play an important role in this. In fact, we proved that a bialgebra B is a one-sided Hopf algebra (in the sense of [17]) with anti-(co)multiplicative one-sided antipode if and only if the free Hopf module functor $-\otimes B : \mathfrak{M} \to \mathfrak{M}_B^B$ (key ingredient of the renowned Structure Theorem of Hopf modules) is Frobenius. In the finitely generated and projective case, this allowed us to prove a categorical extension of Pareigis' theorem (see [32, Theorem 3.14]). In the present paper we continue our investigation in this direction by analysing another important adjoint triple strictly connected with bialgebras and their representations, namely the one associated with the free two-sided Hopf module functor $-\otimes B : {}_B\mathfrak{M} \to {}_B\mathfrak{M}_B^B$. The study of the Frobenius property for this latter functor has proved to be more significant than the previous one for two main reasons. The first one is that being Frobenius for $-\otimes B : {}_B\mathfrak{M} \to {}_B\mathfrak{M}_B^B$ has proven to be in fact equivalent to B being a Hopf algebra. Even more generally, the following is our first main result.

Theorem (Theorem 2.9)**.** *The following are equivalent for a quasi-bialgebra A.*

(1) A admits a preantipode;
(2) $-\otimes A : ({}_A\mathfrak{M}, \otimes, \Bbbk) \to \left({}_A\mathfrak{M}_A^A, \otimes_A, A\right)$ is a monoidal equivalence of categories;
(3) $-\otimes A : {}_A\mathfrak{M} \to {}_A\mathfrak{M}_A^A$ is Frobenius;
(4) $\sigma_M : {}_A\mathsf{Hom}_A^A(A\otimes A, M) \to \overline{M}, f \mapsto \overline{f(1\otimes 1)}$ is an isomorphism for every $M \in {}_A\mathfrak{M}_A^A$, where $\overline{M} \cong M \otimes_A {}_\varepsilon\Bbbk$.

The second one is that the monad $T \coloneqq \overline{(-)} \otimes B$ on ${}_B\mathfrak{M}_B^B$ induced by the adjunction $\overline{(-)} \dashv -\otimes B$, being the functor $-\otimes B : {}_B\mathfrak{M} \to {}_B\mathfrak{M}_B^B$ a strong monoidal functor between monoidal categories, turns out to be an opmonoidal monad in the sense of [27]. As such, it allows us to relate our approach by means of Frobenius functors with the theory of Hopf monads developed by Bruguières, Lack and Virelizier in [9, 10]. In concrete, the following is our second main result.

Theorem (Theorem 3.2)**.** *The following are equivalent for a quasi-bialgebra A.*

(a) A admits a preantipode;

(b) *the natural transformation* $\psi_{M,N} : \overline{M \otimes_A N} \to \overline{M} \otimes \overline{N}$, $\overline{m \otimes_A n} \mapsto \overline{m_0} \otimes \overline{m_1 n}$, *for* $M, N \in {}_A\mathfrak{M}_A^A$, *is a natural isomorphism;*

(c) *the component* $\psi_{A\widehat{\otimes}A,A\otimes A}$ *of* ψ, *where* $A \,\widehat{\otimes}\, A = A \otimes A$ *with a suitable quasi-Hopf bimodule structure, is invertible;*

(d) $\left(\overline{(-)}, - \otimes A\right)$ *is a lax-lax adjunction;*

(e) $\left(\overline{(-)}, - \otimes A\right)$ *is a Hopf adjunction;*

(f) $T = \overline{(-)} \otimes A$ *is a Hopf monad on* ${}_A\mathfrak{M}_A^A$.

Let us highlight that a consequence of the previous theorem is that $\overline{(-)} \otimes A$ is an example of an opmonoidal monad which is Hopf if and only if it is Frobenius (see Remark 3.4).

Even if we are mainly interested in the Hopf algebra case, there are valid motivations for us to work in the more general context of quasi-bialgebras and preantipodes, despite the slight additional effort. Quasi-bialgebras, and in particular quasi-Hopf algebras (i.e. quasi-bialgebras with a quasi-antipode), naturally arise from the study of quantum groups and hence they are of general interest for the scientific community as well. Preantipodes, in turn, are proving to be in many situations a much better behaved analogue of antipodes for (co)quasi-bialgebras than quasi-antipodes (see [3, 4, 31, 34]). The results of the present paper are an additional confirmation of this fact and hence, either in case their existence turns out to be equivalent to the existence of quasi-antipodes or in case they prove to be a more general notion, preantipodes also are expected to be of interest for the community and they deserve to be investigated further.

The paper is organized as follows. In Section 1 we recall some general facts about adjoint triples, Frobenius functors, monoidal categories and quasi-bialgebras that will be needed in the sequel. Section 2 is devoted to the study of when the free quasi-Hopf bimodule functor $- \otimes A : {}_A\mathfrak{M} \to {}_A\mathfrak{M}_A^A$ for a quasi-bialgebra A is Frobenius. The main results of this section are Theorem 2.9, characterizing quasi-bialgebras A with preantipode as those for which $- \otimes A$ is Frobenius, and its consequence, Theorem 2.12, rephrasing this fact for bialgebras. A detailed example, in a context where computations are easily handled, follows and then the section is closed by a collection of results connecting the theory developed herein with some of those in [32] (§2.2) and in [12, 13] (§2.3). Finally, in Section 3 we investigate the connection between the Frobenius property for $- \otimes A : {}_A\mathfrak{M} \to {}_A\mathfrak{M}_A^A$ and the fact of being Hopf for the induced monad $T = \overline{(-)} \otimes A$. The main results here are Theorem 3.2 and its consequence, Corollary 3.3.

Notations and conventions. Throughout the paper, $\Bbbk$ denotes a base commutative ring (from time to time a field) and A a quasi-bialgebra over $\Bbbk$ with unit $u : \Bbbk \to A$ (the unit element of A is denoted by 1_A or simply 1), multiplication $m : A \otimes A \to A$ (often denoted by simple juxtaposition), counit $\varepsilon : A \to \Bbbk$ and comultiplication $\Delta : A \to A \otimes A$. We write $A^+ := \mathsf{ker}(\varepsilon)$ for the augmentation ideal of A. The category of all (central) $\Bbbk$-modules is denoted by $\mathfrak{M}$ and by $\mathfrak{M}_A$ (resp. ${}_A\mathfrak{M}$) and ${}_A\mathfrak{M}_A$ we mean the categories of right (resp. left) modules and bimodules over A. The unadorned tensor product $\otimes$ is the tensor product over $\Bbbk$ and the unadorned Hom stands for the space of $\Bbbk$-linear maps. The coaction of a

comodule is usually denoted by δ and the action of a module by μ, $\cdot$ or simply juxtaposition. In order to handle comultiplications and coactions, we resort to the following variation of Sweedler's sigma notation:

$$\Delta(a) = a_1 \otimes a_2 \qquad \text{and} \qquad \delta(n) = n_0 \otimes n_1 \qquad \text{(summation understood)}$$

for all $a \in A$, $n \in N$ comodule. We often shorten iterated tensor products $A \otimes A \otimes \cdots \otimes A$ of n copies of A by $A^{\otimes n}$. When specializing to the coassociative framework, we use B to denote a bialgebra over $\Bbbk$.

1. Preliminaries

1.1. **Adjoint triples.** Let us recall quickly some facts about adjoint triples and Frobenius functors that we are going to use in the paper. For further details on these objects in connection with our setting, see for example [32, §1]. Given categories $\mathcal{C}$ and $\mathcal{D}$, we say that functors $\mathcal{L}, \mathcal{R} : \mathcal{C} \to \mathcal{D}$, $\mathcal{F} : \mathcal{D} \to \mathcal{C}$ form an *adjoint triple* if $\mathcal{L}$ is left adjoint to $\mathcal{F}$ which is left adjoint to $\mathcal{R}$, in symbols $\mathcal{L} \dashv \mathcal{F} \dashv \mathcal{R}$. They form an *ambidextrous adjunction* if there is a natural isomorphism $\mathcal{L} \cong \mathcal{R}$. As a matter of notation, we set $\eta : \mathsf{Id} \to \mathcal{F}\mathcal{L}, \epsilon : \mathcal{L}\mathcal{F} \to \mathsf{Id}$ for the unit and counit of the left-most adjunction and $\gamma : \mathsf{Id} \to \mathcal{R}\mathcal{F}, \theta : \mathcal{F}\mathcal{R} \to \mathsf{Id}$ for the right-most one. If in addition $\mathcal{F}$ is fully faithful, that is, if ϵ and γ are natural isomorphisms (see [26, Theorem IV.3.1]), then we have a distinguished natural transformation

$$\sigma := \left(\mathcal{R} \xrightarrow{(\epsilon\mathcal{R})^{-1}} \mathcal{L}\mathcal{F}\mathcal{R} \xrightarrow{\mathcal{L}\theta} \mathcal{L} \right).$$

A *Frobenius pair* for the categories $\mathcal{C}$ and $\mathcal{D}$ is a couple of functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{G}$ is left and right adjoint to $\mathcal{F}$. A functor $\mathcal{F}$ is said to be *Frobenius* if there exists a functor $\mathcal{G}$ which is at the same time left and right adjoint to $\mathcal{F}$. The subsequent lemma collects some rephrasing of the Frobenius property for future reference.

Lemma 1.1. *The following are equivalent for a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$*

1. *$\mathcal{F}$ is Frobenius;*
2. *there exists $\mathcal{R} : \mathcal{D} \to \mathcal{C}$ such that $(\mathcal{F}, \mathcal{R})$ is a Frobenius pair;*
3. *there exists $\mathcal{L} : \mathcal{D} \to \mathcal{C}$ such that $(\mathcal{L}, \mathcal{F})$ is a Frobenius pair;*
4. *there exist $\mathcal{L}, \mathcal{R} : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{L} \dashv \mathcal{F} \dashv \mathcal{R}$ is an ambidextrous adjunction.*

Moreover, if $\mathcal{F}$ is fully faithful, anyone of the above conditions is equivalent to

5. *there exist $\mathcal{L}, \mathcal{R} : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{L} \dashv \mathcal{F} \dashv \mathcal{R}$ is an adjoint triple and $\sigma : \mathcal{R} \to \mathcal{L}$ is a natural isomorphism.*

Since we are interested in adjoint triples whose middle functor is fully faithful, Lemma 1.1 allows us to study the Frobenius property by simply looking at the invertibility of the canonical map σ. Observe that

$$\sigma\mathcal{F} = \epsilon^{-1} \circ \gamma^{-1} \qquad \text{and} \qquad \mathcal{F}\sigma = \eta \circ \theta, \tag{1}$$

whence, in particular, $\sigma\mathcal{F}$ is always a natural isomorphism.

1.2. **Monoidal categories.** Recall that a *monoidal category* $(\mathcal{M}, \otimes, \mathbb{I}, \mathfrak{a}, \mathfrak{l}, \mathfrak{r})$ is a category $\mathcal{M}$ endowed with a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (the *tensor product*), with a distinguished object $\mathbb{I}$ (the *unit*) and with three natural isomorphisms

$$\mathfrak{a} : \otimes \circ (\otimes \times \mathsf{Id}_{\mathcal{M}}) \to \otimes \circ (\mathsf{Id}_{\mathcal{M}} \times \otimes) \quad (\textit{associativity constraint})$$
$$\mathfrak{l} : \otimes \circ (\mathbb{I} \times \mathsf{Id}_{\mathcal{M}}) \to \mathsf{Id}_{\mathcal{M}} \quad (\textit{left unit constraint})$$
$$\mathfrak{r} : \otimes \circ (\mathsf{Id}_{\mathcal{M}} \times \mathbb{I}) \to \mathsf{Id}_{\mathcal{M}} \quad (\textit{right unit constraint})$$

that satisfy the *Pentagon* and the *Triangle Axioms*, that is,

$$\mathfrak{a}_{X,Y,Z\otimes W} \circ \mathfrak{a}_{X\otimes Y,Z,W} = (X \otimes \mathfrak{a}_{Y,Z,W}) \circ \mathfrak{a}_{X,Y\otimes Z,W} \circ (\mathfrak{a}_{X,Y,Z} \otimes W),$$
$$(X \otimes \mathfrak{l}_Y) \circ \mathfrak{a}_{X,\mathbb{I},Y} = \mathfrak{r}_X \otimes Y,$$

for all X, Y, Z, W objects in $\mathcal{M}$.

If the endofunctor $X \otimes - : Y \mapsto X \otimes Y$ (resp. $- \otimes X : Y \mapsto Y \otimes X$) has a right adjoint for every X in $\mathcal{M}$, then $\mathcal{M}$ is called a *left-closed* (resp. *right-closed*) monoidal category.

Given two monoidal categories $(\mathcal{M}, \otimes, \mathbb{I}, \mathfrak{a}, \mathfrak{l}, \mathfrak{r})$ and $(\mathcal{M}', \otimes', \mathbb{I}', \mathfrak{a}', \mathfrak{l}', \mathfrak{r}')$, a *quasi-monoidal functor* $(\mathcal{F}, \varphi_0, \varphi)$ between $\mathcal{M}$ and $\mathcal{M}'$ is a functor $\mathcal{F} : \mathcal{M} \to \mathcal{M}'$ together with an isomorphism $\varphi_0 : \mathbb{I}' \to \mathcal{F}(\mathbb{I})$ and a family of isomorphisms $\varphi_{X,Y} : \mathcal{F}(X) \otimes' \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$ for X, Y objects in $\mathcal{M}$, which are natural in both entrances. A quasi-monoidal functor $\mathcal{F}$ is said to be *neutral* if

$$\begin{aligned} \mathcal{F}(\mathfrak{l}_X) \circ \varphi_{\mathbb{I},X} \circ (\varphi_0 \otimes' \mathcal{F}(X)) &= \mathfrak{l}'_{\mathcal{F}(X)}, \\ \mathcal{F}(\mathfrak{r}_X) \circ \varphi_{X,\mathbb{I}} \circ (\mathcal{F}(X) \otimes' \varphi_0) &= \mathfrak{r}'_{\mathcal{F}(X)}, \end{aligned} \tag{2}$$

and it is said to be *strong monoidal* if, in addition,

$$\varphi_{X,Y\otimes Z} \circ (\mathcal{F}(X) \otimes' \varphi_{Y,Z}) \circ \mathfrak{a}'_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z)} = \mathcal{F}(\mathfrak{a}_{X,Y,Z}) \circ \varphi_{X\otimes Y,Z} \circ (\varphi_{X,Y} \otimes' \mathcal{F}(Z)) \tag{3}$$

for all X, Y, Z in $\mathcal{M}$. Furthermore, it is said to be *strict* if φ_0 and φ are the identities. A strong monoidal functor $(\mathcal{F}, \varphi_0, \varphi)$ such that $\mathcal{F}$ is an equivalence of categories is called a *monoidal equivalence*.

If $\mathcal{F}$ comes together with a morphism $\varphi_0 : \mathbb{I}' \to \mathcal{F}(\mathbb{I})$ and a natural transformation $\varphi_{X,Y} : \mathcal{F}(X) \otimes' \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$ that are not necessarily invertible but that satisfy (2) and (3) then it is called a *lax monoidal functor* in [1, Definition 3.1] (also termed *monoidal functor* in [9, 10]). If instead $\mathcal{F}$ comes together with a morphism $\psi_0 : \mathcal{F}(\mathbb{I}) \to \mathbb{I}'$ and a natural transformation $\psi_{X,Y} : \mathcal{F}(X \otimes Y) \to \mathcal{F}(X) \otimes' \mathcal{F}(Y)$ (not necessarily invertible) satisfying the analogues of (2) and (3) then it is called a *colax monoidal functor* in [1, Definition 3.2] (also termed *opmonoidal functor* in [27] and *comonoidal functor* in [9, 10]).

In [1, Definition 3.8], a natural transformation γ between monoidal functors $(\mathcal{F}, \varphi_0, \varphi)$ and $(\mathcal{G}, \phi_0, \phi)$ from a monoidal category $(\mathcal{M}, \otimes, \mathbb{I}, \mathfrak{a}, \mathfrak{l}, \mathfrak{r})$ to $(\mathcal{M}', \otimes', \mathbb{I}', \mathfrak{a}', \mathfrak{l}', \mathfrak{r}')$ is said to be a *morphism of lax monoidal functors* (also called *monoidal natural transformation* in [9, 10]) if

$$(\gamma_X \otimes' \gamma_Y) \circ \varphi_{X,Y} = \phi_{X,Y} \circ \gamma_{X\otimes Y} \quad \text{and} \quad \gamma_{\mathbb{I}} \circ \varphi_0 = \phi_0. \tag{4}$$

Similarly, one defines *morphisms of colax monoidal functors* (also called *transformations of opmonoidal functors* in [27] and *comonoidal natural transformations* in [9, 10]). An adjoint

pair of monoidal functors is called a *lax-lax adjunction* in [1, Definition 3.87] (also termed a *monoidal adjunction* in [10]) if the unit and the counit are morphisms of lax monoidal functors. Analogously, see [1, Definition 3.88], one defines *colax-colax adjunctions* (also termed *comonoidal adjunctions* in [9]) as those for which the unit and the counit are morphisms of colax monoidal functors.

By adhering to the suggestions of the referee and because results from [1] are widely used, we will adopt the terminology of [1] all over the paper. The unique exception will be the use of the term *opmonoidal monad* in §3, because the latter is, as far as the author knows, the most widely used in the study of Hopf monads and related constructions (see for example [8], in particular [8, Chapter 3], and the references therein).

Henceforth, we will often omit the constraints when referring to a monoidal category.

1.3. **Quasi-bialgebras and quasi-Hopf bimodules.** Let $\Bbbk$ be a commutative ring. Recall from [15, §1, Definition] that a *quasi-bialgebra* over $\Bbbk$ is an algebra A endowed with two algebra maps $\Delta : A \to A \otimes A$, $\varepsilon : A \to \Bbbk$ and a distinguished invertible element $\Phi \in A \otimes A \otimes A$ such that

$$(A \otimes A \otimes \Delta)(\Phi)(\Delta \otimes A \otimes A)(\Phi) = (1 \otimes \Phi)(A \otimes \Delta \otimes A)(\Phi)(\Phi \otimes 1), \tag{5}$$

$$(A \otimes \varepsilon \otimes A)(\Phi) = 1 \otimes 1, \tag{6}$$

Δ is counital with counit ε and it is coassociative up to conjugation by Φ, that is,

$$\Phi(\Delta \otimes A)(\Delta(a)) = (A \otimes \Delta)(\Delta(a))\Phi. \tag{7}$$

As a matter of notation, we will write $\Phi = \Phi^1 \otimes \Phi^2 \otimes \Phi^3 = \Psi^1 \otimes \Psi^2 \otimes \Psi^3 = \cdots$ and $\Phi^{-1} = \varphi^1 \otimes \varphi^2 \otimes \varphi^3 = \phi^1 \otimes \phi^2 \otimes \phi^3 = \cdots$ (summation understood). A *preantipode* (see [34, Definition 1]) for a quasi-bialgebra is a $\Bbbk$-linear endomorphism $S : A \to A$ such that

$$S(a_1 b)a_2 = \varepsilon(a)S(b) = a_1 S(ba_2) \quad \text{and} \quad \Phi^1 S(\Phi^2)\Phi^3 = 1 \tag{8}$$

for all $a, b \in A$. An *antipode* (from time to time also called *quasi-antipode*, to distinguish it from the ordinary antipode of Hopf algebras) is a triple (s, α, β), where $s : A \to A$ is an algebra anti-homomorphism and $\alpha, \beta \in A$ are elements, such that for all $a \in A$ we have

$$\begin{aligned} s(a_1)\alpha a_2 &= \varepsilon(a)\alpha, & a_1 \beta s(a_2) &= \varepsilon(a)\beta, \\ \Phi^1 \beta s(\Phi^2)\alpha \Phi^3 &= 1, & s(\varphi^1)\alpha\varphi^2\beta s(\varphi^3) &= 1. \end{aligned}$$

A quasi-bialgebra admitting an antipode is called a *quasi-Hopf algebra* (see [15, Definition, page 1424]). By comparing [34, Theorem 4] and [37, Theorem 3.1], we have that the following holds.

Proposition 1.2. *Over a field $\Bbbk$, a finite-dimensional quasi-bialgebra A admits a preantipode if and only if it is a quasi-Hopf algebra.*

The subsequent lemma gives an equivalent characterization of quasi-bialgebras in terms of their categories of modules (see [2, Theorem 1]).

Lemma 1.3. *A $\Bbbk$-algebra A is a quasi-bialgebra if and only if its category of left (equivalently, right) modules is a monoidal category with neutral quasi-monoidal underlying functor to $\Bbbk$-modules. The associativity constraint is given by $\mathfrak{a}_{M,N,P}(m\otimes n\otimes p)=\Phi\cdot(m\otimes n\otimes p)$ for every $M,N,P\in {}_A\mathfrak{M}$ and for all $m\in M, n\in N, p\in P$.*

As a matter of notation, if the context requires to stress explicitly the (co)module structures on a particular $\Bbbk$-module V, we will adopt the following conventions. With a full bullet, such as $V_\bullet$ or $V^\bullet$, we will denote a given right action or coaction respectively (analogously for the left ones). For example, a left comodule V over a bialgebra B in the category of B-bimodules will be also denoted by ${}^\bullet_\bullet V_\bullet$. With $V^u := V\otimes\Bbbk^u$ and $V_\varepsilon := V\otimes\Bbbk_\varepsilon$ we will denote the trivial right comodule and right module structures on V, respectively (analogously for the left ones).

Remark 1.4. Also the category ${}_A\mathfrak{M}_A$ of A-bimodules over a quasi-bialgebra A is monoidal with neutral quasi-monoidal underlying functor to $\Bbbk$-modules. In particular, the tensor product of two A-bimodules M, N is (up to isomorphism) their tensor product over $\Bbbk$ with bimodule structure given by the *diagonal actions*

$$a\cdot(m\otimes n)\cdot b = a_1\cdot m\cdot b_1\otimes a_2\cdot n\cdot b_2 \tag{9}$$

for all $a,b\in A$, $m\in M$, $n\in N$. The unit is $\Bbbk$ with two-sided action given by restriction of scalars along ε. The associativity constraint is given by conjugation by Φ: for every $M,N,P\in {}_A\mathfrak{M}_A$ and for all $m\in M, n\in N, p\in P$,

$$\mathfrak{a}_{M,N,P}(m\otimes n\otimes p)=\Phi\cdot(m\otimes n\otimes p)\cdot\Phi^{-1}. \tag{10}$$

One may check that $(\varepsilon\otimes A\otimes A)(\Phi)=1\otimes 1=(A\otimes A\otimes\varepsilon)(\Phi)$ and the same for Φ^{-1}:

$$(\varepsilon\otimes A\otimes A)\left(\Phi^{-1}\right)=1\otimes 1=(A\otimes\varepsilon\otimes A)\left(\Phi^{-1}\right)=1\otimes 1=(A\otimes A\otimes\varepsilon)\left(\Phi^{-1}\right). \tag{11}$$

As a consequence, if for example ${}_\bullet M$ is a left A-module and ${}_\bullet N_\bullet, {}_\bullet P_\bullet$ are A-bimodules, then we may look at ${}_\bullet M_\varepsilon\in {}_A\mathfrak{M}_A$ and $\mathfrak{a}_{M,N,P}(m\otimes n\otimes p)=\Phi\cdot(m\otimes n\otimes p)$ for all $m\in M, n\in N, p\in P$. Therefore, we will use the notation $\mathfrak{a}$ for the associativity constraint in the category of left, right and A-bimodules indifferently. In the same way, the tensor product of a left A-module ${}_\bullet M$ and an A-bimodule ${}_\bullet N_\bullet$ is a bimodule with two-sided action given by (9), i.e.

$$a\cdot(m\otimes n)\cdot b = a_1\cdot m\varepsilon(b_1)\otimes a_2\cdot n\cdot b_2 = a_1\cdot m\otimes a_2\cdot n\cdot b$$

for all $a,b\in A$, $m\in M$, $n\in N$. We will denote $M\otimes N$ with the latter structures by ${}_\bullet M\otimes {}_\bullet N_\bullet$. Furthermore, it can be checked that A, as a bimodule over itself and together with Δ and ε, is a comonoid in ${}_A\mathfrak{M}_A$, so that we may consider the category ${}_A\mathfrak{M}_A^A=({}_A\mathfrak{M}_A)^A$ of the so-called *quasi-Hopf bimodules*. It is important to highlight that:

(a) The coassociativity of the coaction $\delta: M\to M\otimes A$ of a quasi-Hopf bimodule $M\in {}_A\mathfrak{M}_A^A$ is expressed by $\mathfrak{a}_{M,A,A}\circ(\delta\otimes A)\circ\delta=(M\otimes\Delta)\circ\delta$, i.e., for all $m\in M$,

$$\Phi^1\cdot m_{0_0}\cdot\varphi^1\otimes\Phi^2 m_{0_1}\varphi^2\otimes\Phi^3 m_1\varphi^3 = m_0\otimes m_{1_1}\otimes m_{1_2}. \tag{12}$$

(b) If $N \in {}_A\mathfrak{M}_A$ then ${}_\bullet N_\bullet \otimes {}_\bullet A^\bullet_\bullet \in {}_A\mathfrak{M}^A_A$ with diagonal actions and $\delta := \mathfrak{a}^{-1}_{N,A,A} \circ (N \otimes \Delta)$, i.e., for all $n \in N$, $a, b, c \in A$,

$$\begin{aligned} a \cdot (n \otimes b) \cdot c &= a_1 \cdot n \cdot c_1 \otimes a_2 b c_2 \qquad \text{and} \\ \delta(n \otimes a) &= \varphi^1 \cdot n \cdot \Phi^1 \otimes \varphi^2 a_1 \Phi^2 \otimes \varphi^3 a_2 \Phi^3. \end{aligned} \tag{13}$$

It is straightforward to check that the category of left modules over a quasi-bialgebra A is not only monoidal, but in fact a right (and left) closed monoidal category with internal hom-functor ${}_A\mathsf{Hom}\,(A \otimes N, -)$ for all $N \in {}_A\mathfrak{M}$ (for a proof, see [33, Lemma 2.1.2]).

Lemma 1.5. *Let A be a quasi-bialgebra. Then the category ${}_A\mathfrak{M}$ of left A-modules is left and right-closed. Namely, we have bijections*

$$\begin{aligned} {}_A\mathsf{Hom}\,(M \otimes N, P) &\underset{\varkappa}{\overset{\varpi}{\rightleftarrows}} {}_A\mathsf{Hom}\,(M, {}_A\mathsf{Hom}\,(A \otimes N, P))\,, \\ {}_A\mathsf{Hom}\,(N \otimes M, P) &\underset{\varkappa'}{\overset{\varpi'}{\rightleftarrows}} {}_A\mathsf{Hom}\,(M, {}_A\mathsf{Hom}\,(N \otimes A, P))\,, \end{aligned} \tag{14}$$

natural in M and P, given explicitly by

$$\begin{aligned} \varpi(f)(m) &: a \otimes n \mapsto f(a \cdot m \otimes n), \qquad & \varkappa(g) &: m \otimes n \mapsto g(m)(1 \otimes n), \\ \varpi'(f)(m) &: n \otimes a \mapsto f(n \otimes a \cdot m), \qquad & \varkappa'(g) &: n \otimes m \mapsto g(m)(n \otimes 1), \end{aligned}$$

where the left A-module structures on ${}_A\mathsf{Hom}\,(N \otimes A, P)$ and ${}_A\mathsf{Hom}\,(A \otimes N, P)$ are induced by the right A-module structure on A itself:

$$(b \cdot f)(n \otimes a) = f(n \otimes ab) \qquad \text{and} \qquad (b \cdot g)(a \otimes n) = g(ab \otimes n) \tag{15}$$

for all $a, b \in A$, $n \in N$, $f \in {}_A\mathsf{Hom}\,(N \otimes A, P)$ and $g \in {}_A\mathsf{Hom}\,(A \otimes N, P)$.

Finally, let us recall that the category ${}_A\mathfrak{M}^A_A$ is a monoidal category in such a way that the forgetful functor ${}_A\mathfrak{M}^A_A \to {}_A\mathfrak{M}_A$ is strong monoidal, that is to say, the tensor product is $\otimes_A$ and the unit object A itself. Given $M, N \in {}_A\mathfrak{M}^A_A$, the quasi-Hopf bimodule structure on $M \otimes_A N$ is the following: for every $a, b \in A$, $m \in M$ and $n \in N$

$$a \cdot (m \otimes_A n) \cdot b = (a \cdot m) \otimes_A (n \cdot b) \qquad \text{and} \qquad \delta(m \otimes_A n) = m_0 \otimes_A n_0 \otimes m_1 n_1.$$

Moreover, in light of [36, Proposition 3.6] the functor $- \otimes A$ is a strong monoidal functor from $({}_A\mathfrak{M}, \otimes, \Bbbk)$ to $\left({}_A\mathfrak{M}^A_A, \otimes_A, A\right)$. In a nutshell, the argument revolves around the fact that

$$\begin{aligned} (V \otimes A) \otimes_A (W \otimes A) &\xrightarrow{\qquad\qquad} (V \otimes W) \otimes A \\ (v \otimes a) \otimes_A (w \otimes b) &\longmapsto \varphi^1 \cdot v \otimes \varphi^2 a_1 \cdot w \otimes \varphi^3 a_2 b \\ (\Phi^1 \cdot v \otimes 1) \otimes_A (\Phi^2 \cdot w \otimes \Phi^3 a) &\longmapsfrom v \otimes w \otimes a \end{aligned} \tag{16}$$

is an isomorphism of quasi-Hopf bimodules, natural in V and W objects of ${}_A\mathfrak{M}$.

Remark 1.6. For the sake of the interested reader, there is a categorical reason behind the monoidality of ${}_A\mathfrak{M}_A^A$. For a quasi-bialgebra A, the category ${}_A\mathfrak{M}_A$ is a *duoidal* (or *2-monoidal*, in the terminology of [1, Definition 6.1]) category with monoidal structures $(\otimes_A, A)$ and $(\otimes, \Bbbk)$. The structure morphisms connecting the two are

$$\varepsilon : A \to \Bbbk, \qquad \Delta : A \to A \otimes A, \qquad \Bbbk \otimes_A \Bbbk \to \Bbbk : h \otimes_A k \mapsto hk,$$
$$\zeta : (M \otimes P) \otimes_A (N \otimes Q) \to (M \otimes_A N) \otimes (P \otimes_A Q)$$
$$(m \otimes p) \otimes_A (n \otimes q) \mapsto (m \otimes_A n) \otimes (p \otimes_A q).$$

The quintuple $(A, \mu : A \otimes_A A \cong A, \mathsf{Id}_A, \Delta, \varepsilon)$ is a *bimonoid* in $({}_A\mathfrak{M}_A, \otimes_A, A, \otimes, \Bbbk)$, that is to say, (A, μ, Id_A) is a monoid in $({}_A\mathfrak{M}_A, \otimes_A, A)$ and (A, Δ, ε) is a comonoid in $({}_A\mathfrak{M}_A, \otimes, \Bbbk)$, plus certain compatibility conditions between the two structures. By [1, Proposition 6.41], the category ${}_A\mathfrak{M}_A^A$ of right comodules over the bimonoid A in ${}_A\mathfrak{M}_A$ is a monoidal category with tensor product, unit object and constraints induced from $({}_A\mathfrak{M}_A, \otimes_A, A)$.

2. Preantipodes and Frobenius functors

This section is devoted to the study of a distinguished adjoint triple that naturally arises when dealing with the so-called Structure Theorem for quasi-Hopf bimodules over a quasi-bialgebra A (see [18, §3], [33, §2.2.1], [34, §2.1]). We will see that being Frobenius for the functors involved is equivalent to being equivalences and hence to the existence of a preantipode for A. As a by-product, we will find a new equivalent condition for a bialgebra to admit an antipode.

2.1. **The main result.** For every quasi-Hopf bimodule M, the quotient $\overline{M} = M/MA^+$ is a left A-module with $a \cdot \overline{m} \coloneqq \overline{a \cdot m}$ for all $a \in A, m \in M$. On the other hand, for every left A-module N the tensor product $N \otimes A$ is a quasi-Hopf bimodule with

$$a \cdot (n \otimes b) \cdot c = a_1 \cdot n \otimes a_2 bc \quad \text{and} \quad \delta(n \otimes b) = \varphi^1 \cdot n \otimes \varphi^2 b_1 \otimes \varphi^3 b_2 \tag{17}$$

for all $m \in M, n \in N$ and $a, b, c \in A$ (see Remark 1.4 and [33, §2.2.1], [34, §2.1] for additional details). It is known that these constructions induce an adjunction

$$\overline{(-)} \left(\begin{array}{c} {}_A\mathfrak{M}_A^A \\ \downarrow \uparrow \\ {}_A\mathfrak{M} \end{array} \right) -\otimes A \ .$$

Moreover, the bijection ${}_A\mathsf{Hom}\left({}_\bullet M \otimes {}_\bullet N, {}_\bullet P\right) \cong {}_A\mathsf{Hom}\left({}_\bullet M, {}_A\mathsf{Hom}\left({}_\bullet A \otimes {}_\bullet N, {}_\bullet P\right)\right)$ from (14) induces a natural bijection

$${}_A\mathsf{Hom}_A^A\left({}_\bullet M \otimes {}_\bullet N^\bullet_\bullet, {}_\bullet P^\bullet_\bullet\right) \cong {}_A\mathsf{Hom}\left({}_\bullet M, {}_A\mathsf{Hom}_A^A\left({}_\bullet A \otimes {}_\bullet N^\bullet_\bullet, {}_\bullet P^\bullet_\bullet\right)\right)$$

that makes of ${}_A\mathsf{Hom}_A^A(A \otimes A, -)$ the right adjoint of the free quasi-Hopf bimodule functor $- \otimes A$. Therefore we have an adjoint triple

$$\overline{(-)} \ \dashv \ - \otimes A \ \dashv \ {}_A\mathsf{Hom}_A^A(A \otimes A, -) \tag{18}$$

between ${}_A\mathfrak{M}$ and ${}_A\mathfrak{M}_A^A$, with units and counits given by

$$\begin{aligned} &\eta_M : M \to \overline{M} \otimes A, \quad m \mapsto \overline{m_0} \otimes m_1, \qquad \epsilon_N : \overline{(N \otimes A)} \xrightarrow{\cong} N, \quad \overline{n \otimes a} \mapsto n\varepsilon(a), \\ &\gamma_N : N \xrightarrow{\cong} {}_A\mathsf{Hom}_A^A\left(A \otimes A, N \otimes A\right), \quad n \mapsto \left[a \otimes b \mapsto a \cdot n \otimes b\right], \\ &\theta_M : {}_A\mathsf{Hom}_A^A\left(A \otimes A, M\right) \otimes A \to M, \quad f \otimes a \mapsto f(1 \otimes 1) \cdot a. \end{aligned} \tag{19}$$

Remark 2.1. Observe that, either because ϵ is a natural isomorphism or because γ is so, the functor $- \otimes A : {}_A\mathfrak{M} \to {}_A\mathfrak{M}_A^A$ is fully faithful (see [26, Theorem IV.3.1]). The interested reader may also refer to [36, Proposition 3.6].

Since we are in the situation of §1.1, we may consider the natural transformation σ whose component at $M \in {}_A\mathfrak{M}_A^A$ is the A-linear map

$$\sigma_M : {}_A\mathsf{Hom}_A^A\left(A \otimes A, M\right) \to \overline{M}; \quad f \mapsto \overline{f\left(1 \otimes 1\right)}. \tag{20}$$

Remark 2.2. Three things deserve to be observed before proceeding.

(a) A admits a preantipode S if and only if either the left-most or the right-most adjunction in (18) is an equivalence (whence both are). See [33, Theorem 2.2.7] and [34, Theorem 4] for further details. In particular, an inverse for σ_M in this case is given by

$$\sigma_M^{-1} : \overline{M} \to {}_A\mathsf{Hom}_A^A\left(A \otimes A, M\right), \ \overline{m} \mapsto \left[(a \otimes b) \mapsto \Phi^1 a_1 \cdot m_0 \cdot S\left(\Phi^2 a_2 m_1\right) \Phi^3 b\right].$$

(b) In light of equation (1) with $\mathcal{F} = - \otimes A$ and since $A \cong \Bbbk \otimes A$ in ${}_A\mathfrak{M}_A^A$, the component $\sigma_A : {}_A\mathsf{Hom}_A^A\left(A \otimes A, A\right) \to \overline{A}$ is always an isomorphism with inverse given by $\Bbbk \to {}_A\mathsf{Hom}_A^A\left(A \otimes A, A\right), 1_\Bbbk \mapsto [x \otimes y \mapsto \varepsilon(x)y]$.

(c) For every $M \in {}_A\mathfrak{M}_A^A$, the relation $\overline{m \cdot a} = \overline{m}\,\varepsilon(a)$ holds in $\overline{M}$ for all $a \in A$, $m \in M$. We will make often use of it in what follows.

By Lemma 1.1, the functor $- \otimes A$ is Frobenius if and only if σ of (20) is a natural isomorphism. Thus, let us start by having a closer look at ${}_A\mathsf{Hom}_A^A\left(A \otimes A, M\right)$.

Remark 2.3. Let $M \in {}_A\mathfrak{M}_A^A$ and consider $f \in {}_A\mathsf{Hom}_A^A\left(A \otimes A, M\right)$. Due to right A-linearity, f is uniquely determined by the elements $f(a \otimes 1)$ for $a \in A$. Consider the assignment $T_f : A \to M$, $a \mapsto f(a \otimes 1)$, so that $f(a \otimes b) = T_f(a) \cdot b$ for all $a, b \in A$. From A-colinearity of f it follows that

$$\delta_M(T_f(a)) \overset{(17)}{=} f\left(\varphi^1 a \otimes \varphi^2\right) \otimes \varphi^3 = T_f(\varphi^1 a) \cdot \varphi^2 \otimes \varphi^3 \tag{21}$$

and from left A-linearity it follows that

$$T_f(a_1 b) \cdot a_2 = f\left(a_1 b \otimes a_2\right) \overset{(17)}{=} a \cdot T_f(b) \tag{22}$$

for all $a, b \in A$. Denote by ${}^\dagger\mathsf{Hom}\left(A, M\right)$ the $\Bbbk$-submodule of $\mathsf{Hom}\left(A, M\right)$ of those $\Bbbk$-linear maps satisfying (21) and (22). It is an A-submodule with respect to the action $(a \triangleright g)(b) \coloneqq g(ba)$ for $a, b \in A$, $g \in {}^\dagger\mathsf{Hom}\left(A, M\right)$. The assignment

$${}_A\mathsf{Hom}_A^A\left(A \otimes A, M\right) \to {}^\dagger\mathsf{Hom}\left(A, M\right), \quad f \mapsto T_f$$

is an isomorphism of left A-modules. Let now N be any right A-module and let $N \widehat{\otimes} A$ be the quasi-Hopf bimodule $N_\bullet \otimes {}_\bullet A^\bullet_\bullet$. In light of Remark 1.4, the coaction is given by the composition $\mathfrak{a}^{-1}_{N,A,A} \circ (N \otimes \Delta)$, which means that

$$\delta_{N \widehat{\otimes} A}(n \otimes a) = n \cdot \Phi^1 \otimes a_1 \Phi^2 \otimes a_2 \Phi^3.$$

In light of (21), for every $f \in {}_A\mathsf{Hom}^A_A\left(A \otimes A, N \widehat{\otimes} A\right)$ we have

$$(N \otimes \Delta)\left(T_f(a)\right) \cdot \Phi = \delta_{N \widehat{\otimes} A}\left(T_f(a)\right) = T_f(\varphi^1 a) \cdot \varphi^2 \otimes \varphi^3 \tag{23}$$

and, by applying $N \otimes \varepsilon \otimes A$ to both sides of (23),

$$T_f(a) \overset{(6)}{=} (N \otimes \varepsilon \otimes A)\left((N \otimes \Delta)\left(T_f(a)\right) \cdot \Phi\right) = (N \otimes \varepsilon \otimes A)\left(T_f(\varphi^1 a) \cdot \varphi^2 \otimes \varphi^3\right)$$
$$\overset{(*)}{=} (N \otimes \varepsilon)\left(T_f(\varphi^1 a)\right) \cdot \varphi^2 \otimes \varphi^3$$

where $(*)$ follows from the fact that the right A-action on $N \widehat{\otimes} A$ is given by $(n \otimes a) \cdot b = n \cdot b_1 \otimes a b_2$ for all $a, b \in A$, $n \in N$. If we define $\tau_f : A \to N$ by $\tau_f(a) \coloneqq (N \otimes \varepsilon) T_f(a)$ for all $a \in A$, then it follows that

$$f(a \otimes b) = T_f(a) \cdot b = \tau_f(\varphi^1 a) \cdot \varphi^2 b_1 \otimes \varphi^3 b_2 \tag{24}$$

for all $a, b \in A$, $f \in {}_A\mathsf{Hom}^A_A\left(A \otimes A, N \widehat{\otimes} A\right)$. Moreover, in view of (11) and since the left A-action on $N \widehat{\otimes} A$ is given by $a \cdot (n \otimes b) = n \otimes ab$, applying $N \otimes \varepsilon$ to both sides of (22) gives

$$\tau_f(a_1 b) \cdot a_2 = \varepsilon(a) \tau_f(b) \tag{25}$$

for all $a, b \in A$. Denote by ${}^\star\mathsf{Hom}\,(A, N)$ the family of $\Bbbk$-linear morphisms $g : A \to N$ that satisfy (25). Then we have an isomorphism of left A-modules

$$\begin{aligned} \tau : {}_A\mathsf{Hom}^A_A\left(A \otimes A, N \widehat{\otimes} A\right) &\longleftrightarrow {}^\star\mathsf{Hom}\,(A, N) \\ f &\longmapsto \tau_f \\ \left[a \otimes b \mapsto g(\varphi^1 a) \cdot \varphi^2 b_1 \otimes \varphi^3 b_2\right] &\longmapsfrom g \end{aligned} \tag{26}$$

where the A-module structure on ${}^\star\mathsf{Hom}\,(A, N)$ is the one induced by $\mathsf{Hom}\,(A, N)$, that is, $(a \triangleright g)(b) = g(ba)$ for all $a, b \in A$ and $g \in \mathsf{Hom}\,(A, N)$.

Our first aim is to show that if σ_M is invertible for every $M \in {}_A\mathfrak{M}^A_A$, then A admits a preantipode. Let us keep the notation introduced in Remark 2.3 and consider the distinguished quasi-Hopf bimodule $\widehat{A} = A \widehat{\otimes} A \coloneqq A_\bullet \otimes {}_\bullet A^\bullet_\bullet$ and the component

$$\sigma_{\widehat{A}} : {}_A\mathsf{Hom}^A_A\left(A \otimes A, A \widehat{\otimes} A\right) \to \overline{A \widehat{\otimes} A}, \qquad f \mapsto \overline{\tau_f(\varphi^1)\varphi^2 \otimes \varphi^3}. \tag{27}$$

Observe that, in light of the structures on $A \otimes A$ and $A \widehat{\otimes} A$, bilinearity and colinearity of f can be expressed explicitly by

$$\begin{gathered} f\left(a_1 x \otimes a_2 y b\right) = (1 \otimes a) f\left(x \otimes y\right) \Delta(b) \quad \text{and} \\ (A \otimes \Delta)\left(f\left(a \otimes b\right)\right) \cdot \Phi = f\left(\varphi^1 a \otimes \varphi^2 b_1\right) \otimes \varphi^3 b_2 \end{gathered} \tag{28}$$

for every $f \in {}_A\mathsf{Hom}_A^A\left(A\otimes A, A\,\widehat{\otimes}\, A\right)$ and $a,b,x,y\in A$.

Remark 2.4. For all $a,b,x,y\in A$, $f\in {}_A\mathsf{Hom}_A^A\left(A\otimes A, A\,\widehat{\otimes}\, A\right)$, the rules

$$(a\otimes b)\triangleright \overline{x\otimes y} \coloneqq \overline{ax\otimes by} \quad \text{and} \quad ((a\otimes b)\triangleright f)(x\otimes y)\coloneqq (a\otimes 1)\, f\,(xb\otimes y) \tag{29}$$

provide actions of $A\otimes A$ on $\overline{A\,\widehat{\otimes}\, A}$ and ${}_A\mathsf{Hom}_A^A\left(A\otimes A, A\,\widehat{\otimes}\, A\right)$, respectively, and the ordinary left A-action on $\overline{A\,\widehat{\otimes}\, A}$ satisfies $a\cdot(\overline{x\otimes y}) = (1\otimes a)\triangleright \overline{x\otimes y}$. Since

$$\sigma_{\widehat{A}}((a\otimes b)\triangleright f) \overset{(20)}{=} \overline{((a\otimes b)\triangleright f)\,(1\otimes 1)} \overset{(29)}{=} \overline{(a\otimes 1)\, f\,(b\otimes 1)} \overset{(15)}{=} (a\otimes 1)\triangleright\left(\overline{(b\cdot f)\,(1\otimes 1)}\right)$$
$$\overset{(20)}{=} (a\otimes 1)\triangleright\sigma_{\widehat{A}}(b\cdot f) \overset{(*)}{=} (a\otimes 1)\triangleright\left(b\cdot\sigma_{\widehat{A}}(f)\right) = (a\otimes 1)\triangleright\left((1\otimes b)\triangleright\sigma_{\widehat{A}}(f)\right) = (a\otimes b)\triangleright\sigma_{\widehat{A}}(f)$$

for all $a,b\in A$ (where $(*)$ follows by A-linearity of $\sigma_{\widehat{A}}$), we have that $\sigma_{\widehat{A}}$ is $A\otimes A$-linear with respect to these action. If $\sigma_{\widehat{A}}$ is invertible, then $\sigma_{\widehat{A}}^{-1}$ is $A\otimes A$-linear as well and hence

$$\sigma_{\widehat{A}}^{-1}(\overline{a\otimes b}) = (a\otimes b)\triangleright\sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1}). \tag{30}$$

As a consequence, and by right A-linearity of $\sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})$,

$$\begin{aligned}\sigma_{\widehat{A}}^{-1}\left(\overline{a\otimes b}\right)(x\otimes y) &\overset{(30)}{=} \left((a\otimes b)\triangleright\sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})\right)(x\otimes y)\\ &\overset{(29)}{=} (a\otimes 1)\sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})\,(xb\otimes y) \overset{(28)}{=} (a\otimes 1)\sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})\,(xb\otimes 1)\,\Delta(y)\end{aligned} \tag{31}$$

for all $a,b,x,y\in A$. In particular, we have

$$\sigma_{\widehat{A}}^{-1}(\overline{1\otimes a})\,(1\otimes 1) = \sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})\,(a\otimes 1)\,. \tag{32}$$

Proposition 2.5. *If $\sigma_{\widehat{A}}$ is invertible, then $S\coloneqq\tau\left(\sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})\right)$, given by*

$$S(a) = (A\otimes\varepsilon)\left(\sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})\,(a\otimes 1)\right) \tag{33}$$

for every $a\in A$, satisfies $S(a_1b)a_2 = \varepsilon(a)S(b) = a_1S(ba_2)$ for all $a,b\in A$.

Proof. Since $\sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})$ belongs to ${}_A\mathsf{Hom}_A^A\left(A\otimes A, A\,\widehat{\otimes}\, A\right)$, it follows from relation (24) that $\sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})\,(a\otimes 1) = S(\varphi^1a)\varphi^2\otimes\varphi^3$ and hence

$$\sigma_{\widehat{A}}^{-1}\left(\overline{a\otimes b}\right)(x\otimes y) \overset{(31)}{=} (a\otimes 1)\sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})\,(xb\otimes 1)\Delta(y) = aS(\varphi^1xb)\varphi^2y_1\otimes\varphi^3y_2 \tag{34}$$

for all $a,b,x,y\in A$. Now, $S(a_1b)\,a_2 = \varepsilon(a)S(b)$ is relation (25) for $f = \sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})$. Moreover, since $\overline{a_1\otimes a_2} = \overline{(1\otimes 1)\cdot a} = (\overline{1\otimes 1})\,\varepsilon(a)$ by definition of $\overline{A\,\widehat{\otimes}\, A}$, we have

$$\begin{aligned}a_1S(ba_2) &\overset{(11)}{=} (A\otimes\varepsilon)\Big(a_1S(\varphi^1ba_2)\varphi^2\otimes\varphi^3\Big) \overset{(34)}{=} (A\otimes\varepsilon)\Big(\sigma_{\widehat{A}}^{-1}(\overline{a_1\otimes a_2})\,(b\otimes 1)\Big)\\ &= (A\otimes\varepsilon)\Big(\sigma_{\widehat{A}}^{-1}(\overline{1\otimes 1})\,(b\otimes 1)\Big)\varepsilon(a) = \varepsilon(a)S(b)\end{aligned}$$

for all $a,b\in A$ and the proof is complete. □

It view of Proposition 2.5, the endomorphism $S = \tau\left(\sigma_{\widehat{A}}^{-1}\left(\overline{1\otimes 1}\right)\right)$ is a preantipode if and only if $\Phi^1 S(\Phi^2)\Phi^3 = 1$ (see (8)). The forthcoming lemmata are intermediate steps toward the proof of this latter identity.

Lemma 2.6. *For $M \in {}_A\mathfrak{M}$ and $N \in {}_A\mathfrak{M}_A$ we have $\overline{M\otimes N} \cong M\otimes\overline{N}$ in ${}_A\mathfrak{M}$ via the assignment $\overline{m\otimes n}\mapsto m\otimes\overline{n}$.*

Proof. Since $\overline{N} = N/NA^+$ and $A/A^+ \cong \Bbbk$, the thesis follows from the isomorphisms

$$\overline{M\otimes N} \cong (M\otimes N)\otimes_A \Bbbk \cong M\otimes(N\otimes_A \Bbbk) \cong M\otimes\overline{N}. \qquad \square$$

Lemma 2.7. *If σ is a natural isomorphism, then for any $M \in {}_A\mathfrak{M}_A^A$, $m\in M$ and $x,y\in A$,*

$$\sigma_M^{-1}(\overline{m})(x\otimes y) = \Phi^1 x_1\cdot m_0\cdot S(\Phi^2 x_2 m_1)\Phi^3 y. \tag{35}$$

Proof. Set $A_2 \coloneqq {}_\bullet A\otimes A_\bullet\otimes {}_\bullet A^\bullet \in {}_A\mathfrak{M}_A^A$ with explicit structures

$$\begin{aligned} a\cdot(u\otimes v\otimes w)\cdot b &= a_1 u\otimes v b_1\otimes a_2 w b_2 \qquad \text{and}\\ \delta(u\otimes v\otimes w) &= \varphi^1 u\otimes v\Phi^1\otimes\varphi^2 w_1\Phi^2\otimes\varphi^3 w_2\Phi^3 \end{aligned} \tag{36}$$

for all $a,b,u,v,w\in A$. Denote by $\iota : A\otimes\overline{A\mathbin{\widehat{\otimes}}A}\to\overline{A_2}$ the isomorphism of Lemma 2.6. Consider also the left A-linear morphism

$$\begin{aligned} \tilde{\iota} : A\otimes {}_A\mathsf{Hom}_A^A\left(A\otimes A, A\mathbin{\widehat{\otimes}}A\right) &\to {}_A\mathsf{Hom}_A^A\left(A\otimes A, A_2\right)\\ a\otimes f &\longmapsto \left[x\otimes y\mapsto \Phi^1 x_1 a\otimes f(\Phi^2 x_2\otimes\Phi^3 y)\right]. \end{aligned} \tag{37}$$

It is well-defined because the following direct computation

$$\begin{aligned} \tilde{\iota}(a\otimes f)(b_1 x\otimes b_2 y c) &= \Phi^1 b_{1_1} x_1 a\otimes f(\Phi^2 b_{1_2} x_2\otimes\Phi^3 b_2 y c) \overset{(7)}{=} b_1\Phi^1 x_1 a\otimes f(b_{2_1}\Phi^2 x_2\otimes b_{2_2}\Phi^3 y c)\\ &\overset{(28)}{=} b_1\Phi^1 x_1 a\otimes(1\otimes b_2) f(\Phi^2 x_2\otimes\Phi^3 y)\Delta(c) \overset{(36)}{=} b\cdot(\tilde{\iota}(a\otimes f)(x\otimes y))\cdot c \end{aligned}$$

entails that $\tilde{\iota}(a\otimes f)$ is A-bilinear and

$$\begin{aligned} \delta(\tilde{\iota}(a\otimes f)(x\otimes y)) &\overset{(37)}{=} \delta\left(\Phi^1 x_1 a\otimes f(\Phi^2 x_2\otimes\Phi^3 y)\right)\\ &\overset{(36)}{=} \varphi^1\Phi^1 x_1 a\otimes(1\otimes\varphi^2\otimes\varphi^3)(A\otimes\Delta) f(\Phi^2 x_2\otimes\Phi^3 y)\cdot\Phi\\ &\overset{(28)}{=} \varphi^1\Phi^1 x_1 a\otimes(1\otimes\varphi^2\otimes\varphi^3)\left(f(\phi^1\Phi^2 x_2\otimes\phi^2\Phi_1^3 y_1)\otimes\phi^3\Phi_2^3 y_2\right)\\ &\overset{(28)}{=} \varphi^1\Phi^1 x_1 a\otimes f(\varphi_1^2\phi^1\Phi^2 x_2\otimes\varphi_2^2\phi^2\Phi_1^3 y_1)\otimes\varphi^3\phi^3\Phi_2^3 y_2\\ &\overset{(5)}{=} \Phi^1\varphi_1^1 x_1 a\otimes f(\Phi^2\varphi_2^1 x_2\otimes\Phi^3\varphi^2 y_1)\otimes\varphi^3 y_2\\ &\overset{(37)}{=} \tilde{\iota}(a\otimes f)(\varphi^1 x\otimes\varphi^2 y_1)\otimes\varphi^3 y_2 \end{aligned}$$

implies that it is colinear for all $a\in A$, $f\in {}_A\mathsf{Hom}_A^A\left(A\otimes A, A\mathbin{\widehat{\otimes}}A\right)$. The A-linearity of $\tilde{\iota}$ follows from

$$\tilde{\iota}(b_1 a\otimes b_2\cdot f)(x\otimes y) \overset{(37)}{=} \Phi^1 x_1 b_1 a\otimes(b_2\cdot f)(\Phi^2 x_2\otimes\Phi^3 y)$$

$$\overset{(15)}{=} \Phi^1 x_1 b_1 a \otimes f(\Phi^2 x_2 b_2 \otimes \Phi^3 y) \overset{(37)}{=} \tilde{\iota}(a \otimes f)(xb \otimes y) \overset{(15)}{=} (b \cdot \tilde{\iota}(a \otimes f))(x \otimes y)$$

for all $a, b, x, y \in A$ and $f \in {}_A\mathsf{Hom}_A^A\left(A \otimes A, A \widehat{\otimes} A\right)$.

Let us show that the diagram

$$\begin{array}{ccc} A \otimes {}_A\mathsf{Hom}_A^A\left(A \otimes A, A \widehat{\otimes} A\right) & \xrightarrow{A \otimes \sigma_{\widehat{A}}} & A \otimes \overline{A \widehat{\otimes} A} \\ \downarrow{\scriptstyle \tilde{\iota}} & & \downarrow{\scriptstyle \iota} \\ {}_A\mathsf{Hom}_A^A\left(A \otimes A, A_2\right) & \xrightarrow[\sigma_{A_2}]{} & \overline{A_2} \end{array} \tag{38}$$

is commutative. For every $a \in A$ and $f \in {}_A\mathsf{Hom}_A^A\left(A \otimes A, A \widehat{\otimes} A\right)$ compute

$$\begin{aligned} \sigma_{A_2}\left(\tilde{\iota}(a \otimes f)\right) &\overset{(20)}{=} \overline{\tilde{\iota}(a \otimes f)(1 \otimes 1)} \overset{(37)}{=} \overline{\Phi^1 a \otimes f(\Phi^2 \otimes \Phi^3)} \\ &\overset{(28)}{=} \overline{\Phi^1 a \otimes f(\Phi^2 \otimes 1)\Delta(\Phi^3)} \overset{(36)}{=} \overline{(\Phi^1 a \otimes f(\Phi^2 \otimes 1)) \cdot \Phi^3} \\ &\overset{(*)}{=} \overline{a \otimes f(1 \otimes 1)} = \iota((A \otimes \sigma_{\widehat{A}})(a \otimes f)), \end{aligned}$$

where $(*)$ follows from the fact that $(A \otimes A \otimes \varepsilon)(\Phi) = 1 \otimes 1$ and the definition of $\overline{A_2}$. Therefore, in light of (34), for all $a, b, c, x, y \in A$ we have

$$\begin{aligned} \sigma_{A_2}^{-1}\left(\overline{a \otimes b \otimes c}\right)(x \otimes y) &\overset{(38)}{=} \left(\tilde{\iota} \circ \left(A \otimes \sigma_{\widehat{A}}^{-1}\right) \circ \iota^{-1}\right)\left(\overline{a \otimes b \otimes c}\right)(x \otimes y) \\ &= \tilde{\iota}\left(a \otimes \sigma_{\widehat{A}}^{-1}\left(\overline{b \otimes c}\right)\right)(x \otimes y) \overset{(37)}{=} \Phi^1 x_1 a \otimes \sigma_{\widehat{A}}^{-1}\left(\overline{b \otimes c}\right)(\Phi^2 x_2 \otimes \Phi^3 y) \\ &\overset{(34)}{=} \Phi^1 x_1 a \otimes bS(\varphi^1 \Phi^2 x_2 c)\varphi^2 \Phi_1^3 y_1 \otimes \varphi^3 \Phi_2^3 y_2. \end{aligned} \tag{39}$$

Now, for any $N \in {}_A\mathfrak{M}_A$ consider $N \otimes A = {}_\bullet N_\bullet \otimes {}_\bullet A_\bullet^\bullet \in {}_A\mathfrak{M}_A^A$. For every $n \in N$, the assignment $f_n : {}_\bullet A \otimes A_\bullet \to N, a \otimes b \mapsto a \cdot n \cdot b$, is a well-defined A-bilinear morphism. Naturality of σ^{-1} implies then that

$$\begin{aligned} \sigma_{N \otimes A}^{-1}(\overline{n \otimes b})(x \otimes y) &\overset{(\mathrm{nat})}{=} (f_n \otimes A)\left(\sigma_{A_2}^{-1}\left(\overline{1 \otimes 1 \otimes b}\right)(x \otimes y)\right) \\ &\overset{(39)}{=} (f_n \otimes A)\left(\Phi^1 x_1 \otimes S(\varphi^1 \Phi^2 x_2 b)\varphi^2 \Phi_1^3 y_1 \otimes \varphi^3 \Phi_2^3 y_2\right) \\ &= \Phi^1 x_1 \cdot n \cdot S(\varphi^1 \Phi^2 x_2 b)\varphi^2 \Phi_1^3 y_1 \otimes \varphi^3 \Phi_2^3 y_2 \end{aligned} \tag{40}$$

for all $n \in N$, $b, x, y \in A$. Finally, the coaction $\delta_M : {}_\bullet M_\bullet^\bullet \to {}_\bullet M_\bullet \otimes {}_\bullet A_\bullet^\bullet$ is a well-defined morphism in ${}_A\mathfrak{M}_A^A$ and hence we may resort again to naturality of σ^{-1} to get that

$$\begin{aligned} \delta_M\left(\sigma_M^{-1}(\overline{m})(x \otimes y)\right) &\overset{(\mathrm{nat})}{=} \sigma_{M \otimes A}^{-1}(\overline{m_0 \otimes m_1})(x \otimes y) \\ &\overset{(40)}{=} \Phi^1 x_1 \cdot m_0 \cdot S(\varphi^1 \Phi^2 x_2 m_1)\varphi^2 \Phi_1^3 y_1 \otimes \varphi^3 \Phi_2^3 y_2 \end{aligned}$$

for all $m \in M$, $x, y \in A$. Applying $M \otimes \varepsilon$ to both sides and recalling (11) give the result. □

Proposition 2.8. *If σ is a natural isomorphism, then $\Phi^1 S(\Phi^2)\Phi^3 = 1$.*

Proof. By Lemma 2.7, for every $M \in {}_A\mathfrak{M}_A^A$ and for all $m \in M$, $x, y \in A$ we have $\sigma_M^{-1}(\overline{m})(x \otimes y) = \Phi^1 x_1 \cdot m_0 \cdot S(\Phi^2 x_2 m_1)\Phi^3 y$. For $M = A$ and $m = x = y = 1$ this implies

$$1 \overset{(*)}{=} \sigma_A^{-1}(\overline{1})(1 \otimes 1) = \Phi^1 S(\Phi^2)\Phi^3$$

where $(*)$ follows by (b) of Remark 2.2. □

Summing up, we have the following central result.

Theorem 2.9. *The following are equivalent for a quasi-bialgebra A:*

(1) A admits a preantipode;
(2) $- \otimes A : ({}_A\mathfrak{M}, \otimes, \Bbbk) \to \left({}_A\mathfrak{M}_A^A, \otimes_A, A\right)$ is a monoidal equivalence of categories;
(3) $- \otimes A : {}_A\mathfrak{M} \to {}_A\mathfrak{M}_A^A$ is Frobenius;
(4) $\sigma_M : {}_A\mathsf{Hom}_A^A(A \otimes A, M) \to \overline{M}, f \mapsto \overline{f(1 \otimes 1)}$ is an isomorphism for every $M \in {}_A\mathfrak{M}_A^A$.

Proof. The proof of the equivalence between *(1)* and *(2)* is contained in [34, Theorem 3 and subsequent discussion], but without explicit mention to the monoidality of the functor $- \otimes A : ({}_A\mathfrak{M}, \otimes, \Bbbk) \to \left({}_A\mathfrak{M}_A^A, \otimes_A, A\right)$. A more exhaustive proof can be found in [33, Theorem 2.2.7]. The implication from *(2)* to *(3)* is clear and the equivalence between *(3)* and *(4)* follows from Lemma 1.1. Finally, the implication *(4)* $\Rightarrow$ *(1)* follows from Proposition 2.5 and Proposition 2.8. □

The subsequent corollary improves considerably [31, Proposition A.3].

Corollary 2.10. *Let A be a quasi-bialgebra with preantipode S. For all $a, b \in A$ we have $S(ab) = S(\varphi^1 b)\varphi^2 S(a\varphi^3)$.*

Proof. For every $f \in {}_A\mathsf{Hom}_A^A\left(A \otimes A, A \widehat{\otimes} A\right)$ and $a, b, c \in A$ we have

$$\begin{aligned}
\tau_f(\varphi^1 ab)\varphi^2 c_1 \otimes \varphi^3 c_2 &\overset{(24)}{=} f(ab \otimes c) = \sigma_{\widehat{A}}^{-1}\left(\sigma_{\widehat{A}}(f)\right)(ab \otimes c) = \sigma_{\widehat{A}}^{-1}\left(\overline{f(1 \otimes 1)}\right)(ab \otimes c) \\
&\overset{(29)}{=} \left((1 \otimes b) \triangleright \sigma_{\widehat{A}}^{-1}\left(\overline{f(1 \otimes 1)}\right)\right)(a \otimes c) \overset{(30)}{=} \sigma_{\widehat{A}}^{-1}\left((1 \otimes b) \triangleright \overline{f(1 \otimes 1)}\right)(a \otimes c) \\
&\overset{(29)}{=} \sigma_{\widehat{A}}^{-1}\left(\overline{(1 \otimes b) f(1 \otimes 1)}\right)(a \otimes c) \overset{(28)}{=} \sigma_{\widehat{A}}^{-1}\left(\overline{f(b_1 \otimes b_2)}\right)(a \otimes c) \\
&\overset{(28)}{=} \sigma_{\widehat{A}}^{-1}\left(\overline{f(b_1 \otimes 1) \cdot b_2}\right)(a \otimes c) = \sigma_{\widehat{A}}^{-1}\left(\overline{f(b \otimes 1)}\right)(a \otimes c) \\
&\overset{(24)}{=} \sigma_{\widehat{A}}^{-1}\left(\overline{\tau_f(\psi^1 b)\psi^2 \otimes \psi^3}\right)(a \otimes c) \overset{(34)}{=} \tau_f(\psi^1 b)\psi^2 S(\varphi^1 a \psi^3)\varphi^2 c_1 \otimes \varphi^3 c_2,
\end{aligned}$$

so that, by applying $A \otimes \varepsilon$ to both sides and taking $c = 1$, $\tau_f(ab) = \tau_f(\varphi^1 b)\varphi^2 S(a\varphi^3)$. Since τ is bijective and $S \in {}^\star\mathsf{Hom}(A, A)$, there exists $f \in {}_A\mathsf{Hom}_A^A\left(A \otimes A, A \widehat{\otimes} A\right)$ such that $\tau_f = S$ and so $S(ab) = S(\varphi^1 b)\varphi^2 S(a\varphi^3)$ for all $a, b \in A$. □

Remark 2.11. At the present moment it is not clear to us if there exists a quasi-Hopf bimodule M such that σ is a natural isomorphism if and only if σ_M is an isomorphism.

Recall that a bialgebra B is in particular a quasi-bialgebra with $\Phi = 1 \otimes 1 \otimes 1$. Moreover, B is a Hopf algebra if and only if, as a quasi-bialgebra, it admits a preantipode. Therefore, from Theorem 2.9 descends the following result.

Theorem 2.12. *The following are equivalent for a bialgebra B:*

(1) B is a Hopf algebra;
(2) $-\otimes B : ({}_B\mathfrak{M}, \otimes, \Bbbk) \to \left({}_B\mathfrak{M}_B^B, \otimes_B, B\right)$ is a monoidal equivalence of categories;
(3) $-\otimes B : {}_B\mathfrak{M} \to {}_B\mathfrak{M}_B^B$ is Frobenius;
(4) $\sigma_M : {}_B\mathsf{Hom}_B^B(B\otimes B, M) \to \overline{M}, f \mapsto \overline{f(1\otimes 1)}$ is an isomorphism for all $M \in {}_B\mathfrak{M}_B^B$.

Example 2.13. This "toy example" is intended to show, in an easy-handled context, some of the facts and the computations presented so far. We point out that it already appeared in this setting in [34, Example 1] and previously in [16, Preliminaries 2.3]. Let $G := \langle g\rangle$ be the cyclic group of order 2 with generator g and let $\Bbbk$ be a field of characteristic different from 2. Consider the group algebra $A := \Bbbk G$, which is a commutative algebra of dimension 2. An A-bimodule is a $\Bbbk$-vector space V endowed with two distinguished commuting automorphisms α, β such that $\alpha^2 = \mathsf{Id}_V = \beta^2$ (which are left and right action by g respectively). Consider the distinguished elements $t := \frac{1}{2}(1+g)$ (total integral in A) and $p = \frac{1}{2}(1-g)$. They form a pair of pairwise orthogonal idempotents and $A \cong \Bbbk t \oplus \Bbbk p$ as $\Bbbk$-algebras. Moreover, with respect to this new basis, $g = t - p$ and $1 = t + p$.

Now, endow A with the group-like comultiplication $\Delta(g) = g\otimes g$ and counit $\varepsilon(g) = 1$ and consider the element

$$\Phi := 1\otimes 1\otimes 1 - 2p\otimes p\otimes p.$$

This is invertible (with inverse itself) and it satisfies the conditions (5), (6) and (7). These make of A a genuine quasi-bialgebra (with $A^+ = \Bbbk p$), so that the category of A-bimodules is now a monoidal category. Observe that

$$\Delta(t) = \frac{1}{2}1\otimes 1 + \frac{1}{2}g\otimes g = t\otimes g + 1\otimes p = t\otimes t + p\otimes p,$$
$$\Delta(p) = \frac{1}{2}1\otimes 1 - \frac{1}{2}g\otimes g = p\otimes g + 1\otimes p = p\otimes t + t\otimes p.$$

A bimodule M is a quasi-Hopf bimodule if it comes endowed with an A-bilinear coassociative and counital A-coaction in ${}_A\mathfrak{M}_A$. For every $m \in M$, write $\delta(m) := m_1\otimes t + m_2\otimes p$. The counitality condition already implies that $m_1 = m$, so that we may write $\delta(m) = m\otimes t + m'\otimes p$ and $\delta(m') = m'\otimes t + m''\otimes p$. Concerning the coassociativity condition, compute

$$\begin{aligned}\Phi\cdot(\delta\otimes A)(\delta(m)) &= \Phi\cdot(m\otimes t\otimes t + m'\otimes p\otimes t + m'\otimes t\otimes p + m''\otimes p\otimes p)\\ &= m\otimes t\otimes t + m'\otimes p\otimes t + m'\otimes t\otimes p + m''\otimes p\otimes p - 2pm''\otimes p\otimes p,\end{aligned}$$

$$\begin{aligned}(M\otimes\Delta)(\delta(m))\cdot\Phi &= (m\otimes t\otimes t + m\otimes p\otimes p + m'\otimes p\otimes t + m'\otimes t\otimes p)\cdot\Phi\\ &= m\otimes t\otimes t + m\otimes p\otimes p + m'\otimes p\otimes t + m'\otimes t\otimes p - 2mp\otimes p\otimes p.\end{aligned}$$

By equating the right-most terms we find

$$(1-2p)m''\otimes p\otimes p = m''\otimes p\otimes p - 2pm''\otimes p\otimes p = m\otimes p\otimes p - 2mp\otimes p\otimes p = m(1-2p)\otimes p\otimes p$$

so that $gm'' = (1-2p)m'' = m(1-2p) = mg$ and hence $m'' = gmg$. This allows us to define a $\Bbbk$-linear automorphism $\nu : M \to M, m \mapsto m'$, which satisfies $\nu^2(m) = gmg$ and,

since

$$\delta(gm) = (g \otimes g)\delta(m) = (g \otimes g)(m \otimes t + m' \otimes p) = gm \otimes t - gm' \otimes p,$$
$$\delta(mg) = (m \otimes t + m' \otimes p)(g \otimes g) = mg \otimes t - m'g \otimes p,$$

it satisfies $\nu(gm) = -g\nu(m)$ and $\nu(mg) = -\nu(m)g$ as well. In particular, $\nu(mt) = \nu(m)p$ and $\nu(mp) = \nu(m)t$ for all $m \in M$. Thus, a quasi-Hopf bimodule over A is essentially a vector space with three distinguished automorphisms α, β, ν such that $\alpha^2 = \mathsf{Id}_V = \beta^2$, $\alpha \circ \beta = \beta \circ \alpha = \nu^2$, $\nu \circ \alpha = -\alpha \circ \nu$ and $\nu \circ \beta = -\beta \circ \nu$.

Let $M \in {}_A\mathfrak{M}_A^A$ and pick $f \in {}_A\mathsf{Hom}_A^A(A \otimes A, M)$. As we have seen, such an f is uniquely determined by $T_f : A \to M$ satisfying (21) and (22). In particular, since from (22) it follows that $T_f(g) = T_f(g)g^2 = gT_f(1)g$, f is uniquely determined by an element $\omega := T_f(1)$ satisfying (21). If we compute first $2T_f(p) = T_f(1 - g) = \omega - g\omega g$, then (21) becomes

$$\omega \otimes t + \nu(\omega) \otimes p = \delta(\omega) \overset{(21)}{=} T_f(1) \otimes 1 - 2T_f(p)p \otimes p = \omega \otimes 1 - \omega p \otimes p + g\omega gp \otimes p$$
$$= \omega \otimes t + \omega \otimes p - \omega p \otimes p - g\omega p \otimes p = \omega \otimes t + (\omega - \omega p - g\omega p) \otimes p.$$

Thus, $\nu(\omega) = \omega - \omega p - g\omega p = \omega - 2t\omega p$. As a consequence, observe that $\nu(\omega)p = \omega p - 2t\omega p = -g\omega p$ and so $\omega p = -g\nu(\omega)p$. Therefore, $\omega = \omega t + \omega p = \omega t - g\nu(\omega)p = \omega t + \nu(g\omega)p = \omega t + \nu(g\omega t)$ and ω is uniquely determined by ωt. The converse is true as well: if $\omega \in M$ satisfies $\omega = \omega t + \nu(g\omega)p$ then the morphism $f_\omega : A \to M$ given by $f_\omega(a) = f_\omega(a_e 1 + a_g g) := a_e\omega + a_g g\omega g$ satisfies (21) and (22). This means that $T_f(1)$ is uniquely determined by its image via the projection $M \to Mt, m \mapsto mt$, which in turn induces an isomorphism of left A-modules $\overline{M} \cong Mt$. Summing up, the existence of the bijective correspondence $T : {}_A\mathsf{Hom}_A^A(A \otimes A, M) \to Mt, f \mapsto T_f(1)t$, shows that the canonical map $\sigma_M : {}_A\mathsf{Hom}_A^A(A \otimes A, M) \to \overline{M}$ is an isomorphism for every $M \in {}_A\mathfrak{M}_A^A$. Note that we explicitly have

$$T^{-1}(mt)(1 \otimes 1) = mt + \nu(gmt) = mt - g\nu(m)p,$$
$$T^{-1}(mt)(g \otimes 1) = g(mt + \nu(gmt))g = gmt + \nu(mt). \tag{41}$$

To see how the preantipode looks like, first we compute ν for $A \widehat{\otimes} A$. Since the coaction on the elements of the basis behave as follows:

$$1 \otimes 1 \mapsto 1 \otimes 1 \otimes 1 - 2p \otimes p \otimes p = (1 \otimes 1) \otimes t + (1 \otimes 1 - 2p \otimes p) \otimes p,$$
$$g \otimes 1 \mapsto g \otimes 1 \otimes 1 - 2gp \otimes p \otimes p = (g \otimes 1) \otimes t + (g \otimes 1 + 2p \otimes p) \otimes p,$$
$$1 \otimes g \mapsto 1 \otimes g \otimes g - 2p \otimes gp \otimes gp = (1 \otimes g) \otimes t - (1 \otimes g + 2p \otimes p) \otimes p,$$
$$g \otimes g \mapsto g \otimes g \otimes g - 2gp \otimes gp \otimes gp = (g \otimes g) \otimes t - (g \otimes g - 2p \otimes p) \otimes p,$$

we see that, by definition of $\nu : M \to M, m \mapsto m'$,

$$\nu(1 \otimes 1) = 1 \otimes 1 - 2p \otimes p, \qquad \nu(g \otimes 1) = g \otimes 1 + 2p \otimes p,$$
$$\nu(1 \otimes g) = -1 \otimes g - 2p \otimes p, \qquad \nu(g \otimes g) = -g \otimes g + 2p \otimes p.$$

Thus, by resorting to (41) and by writing $a = a_e 1 + a_g g$ for all $a \in A$, we find out that

$$S(a) \overset{(33)}{=} (A \otimes \varepsilon)\left(\sigma_{\widehat{A}}^{-1}\left(\overline{1 \otimes 1}\right)(a \otimes 1)\right) = (A \otimes \varepsilon)\left(T^{-1}\left((1 \otimes 1) \cdot t\right)(a \otimes 1)\right)$$

$$= a_e(A\otimes\varepsilon)\left(T^{-1}\left((1\otimes 1)\cdot t\right)(1\otimes 1)\right) + a_g(A\otimes\varepsilon)\left(T^{-1}\left((1\otimes 1)\cdot t\right)(g\otimes 1)\right)$$
$$\overset{(41)}{=} a_e(A\otimes\varepsilon)\left(t_1\otimes t_2 - g(1\otimes 1 - 2p\otimes p)p\right) + a_g(A\otimes\varepsilon)\left(g(t_1\otimes t_2) + (1\otimes 1 - 2p\otimes p)p\right)$$
$$= a_e(A\otimes\varepsilon)\left(t_1\otimes t_2 - p_1\otimes gp_2 - 2pp_1\otimes gpp_2\right) + a_g(A\otimes\varepsilon)\left(t_1\otimes gt_2 + p_1\otimes p_2 - 2pp_1\otimes pp_2\right)$$
$$= a_e(t-p) + a_g(t+p) = a_e g + a_g = ag,$$

for every $a\in A$, which coincides with the one constructed in [34, Example 1] as expected.

2.2. **Connections with one-sided Hopf modules and Hopf algebras.** Given a bialgebra B, one can consider its category of (right) Hopf modules $\mathfrak{M}_B^B$ and we always have an adjoint triple $\overline{(-)}^B \dashv -\otimes B \dashv (-)^{\mathrm{co}B}$ between $\mathfrak{M}$ and $\mathfrak{M}_B^B$, where $\overline{M}^B = M/MB^+$ and $M^{\mathrm{co}B} = \{m\in M \mid \delta(m) = m\otimes 1\}$. In [32, Theorem 2.7] we proved that the functor $-\otimes B : \mathfrak{M}\to\mathfrak{M}_B^B$ is Frobenius if and only if the canonical map $\varsigma_M : M^{\mathrm{co}B}\to\overline{M}^B, m\mapsto\overline{m}$, is an isomorphism for every $M\in\mathfrak{M}_B^B$, if and only if B is a right Hopf algebra (i.e. it admits a right convolution inverse of the identity).

By working with left Hopf modules instead, recall that the counit of the adjunction $B\otimes(-) : \mathfrak{M}\rightleftarrows {}^B_B\mathfrak{M} : {}^{coB}(-)$ on the Hopf module $\check{B} := {}^\bullet_\bullet B\otimes {}_\bullet B$ induces

$$\mathfrak{can} : \left(\begin{array}{ccccc} {}^\bullet_\bullet B\otimes B & \longrightarrow & {}^\bullet_\bullet B\otimes {}^{coB}({}^\bullet_\bullet B\otimes {}_\bullet B) & \xrightarrow{\epsilon_{\check{B}}} & {}^\bullet_\bullet B\otimes {}_\bullet B \\ a\otimes b & \longmapsto & a\otimes(1\otimes b) & \longmapsto & a_1\otimes a_2 b \end{array}\right)$$

Lemma 2.14. *Let B be a bialgebra. The canonical morphism*

$$\mathfrak{can} : B\otimes B\to B\otimes B : \quad a\otimes b\mapsto a_1\otimes a_2 b$$

can be considered as a morphism $\mathfrak{can} : {}_\bullet B^\bullet\otimes B^\bullet_\bullet\to {}_\bullet B\otimes {}_\bullet B^\bullet_\bullet$ in ${}_B\mathfrak{M}_B^B$.

Proof. The B-bilinearity is clear. For colinearity, we compute

$$(\mathfrak{can}\otimes B)\,\delta(a\otimes b) = \mathfrak{can}(a_1\otimes b_1)\otimes a_2 b_2 = a_{1_1}\otimes a_{1_2}b_1\otimes a_2 b_2 = a_1\otimes a_{2_1}b_1\otimes a_{2_2}b_2$$
$$= (a_1\otimes a_2 b)_0\otimes(a_1\otimes a_2 b)_1 = \delta\left(\mathfrak{can}(a\otimes b)\right). \qquad \square$$

Lemma 2.15. *The assignments $M^\bullet_\bullet\mapsto B^\bullet\otimes M^\bullet_\bullet$ and $f\mapsto B^\bullet\otimes f$ provide a monad $B^\bullet\otimes -$ on $\mathfrak{M}_B^B$. Its Eilenberg-Moore category of algebras is ${}_B\mathfrak{M}_B^B$. In particular, the functor $B\otimes - : \mathfrak{M}_B^B\to {}_B\mathfrak{M}_B^B, M^\bullet_\bullet\mapsto {}_\bullet B^\bullet\otimes M^\bullet_\bullet$ is left adjoint to the forgetful functor ${}_BU : {}_B\mathfrak{M}_B^B\to\mathfrak{M}_B^B$:*

$$ {}_B\mathsf{Hom}_B^B\left({}_\bullet B^\bullet\otimes M^\bullet_\bullet, {}_\bullet N^\bullet_\bullet\right)\cong \mathsf{Hom}_B^B\left(M^\bullet_\bullet, N^\bullet_\bullet\right) \tag{42}$$

Proof. In a nutshell, the comodule $B^\bullet$ is an algebra in the monoidal category $\left(\mathfrak{M}^B,\otimes,\Bbbk\right)$. Given any monoidal category $\mathcal{C}$ and A, A' two algebras, the endofunctors $A\otimes -$ and $-\otimes A'$ provide monads on $\mathcal{C}_{A'}$ and ${}_A\mathcal{C}$ respectively. In particular, $B^\bullet\otimes -$ does on $\mathfrak{M}_B^B$. A direct check is also possible: recall that the B-comodule structure on the tensor product of two comodules is given by the diagonal coaction, i.e. $\delta(n\otimes p) = n_0\otimes p_0\otimes n_1 p_1$ for all $n\in N^\bullet$, $p\in P^\bullet$. Consider the assignments

$$\rho : B^\bullet\otimes(B^\bullet\otimes M^\bullet_\bullet)\to B^\bullet\otimes M^\bullet_\bullet : a\otimes b\otimes m\mapsto ab\otimes m$$

$$M^\bullet_\bullet \to B^\bullet \otimes M^\bullet_\bullet : m \mapsto 1 \otimes m$$

for every M in $\mathfrak{M}^B_B$. They are morphism of right Hopf modules since

$$(\rho \otimes B)\,\delta\,(a \otimes (b \otimes m)) = a_1 b_1 \otimes m_0 \otimes a_2\,(b_2 m_1) = (ab)_1 \otimes m_0 \otimes (ab)_2\, m_1 = \delta(ab \otimes m)$$

(the other three compatibilities are trivial). Therefore $B^\bullet \otimes -$ is indeed a monad on $\mathfrak{M}^B_B$. An algebra (M, μ) for this monad is an object M in $\mathfrak{M}^B_B$, whose underlying vector space admits a left B-module structure $b \triangleright m \coloneqq \mu\,(b \otimes m)$ which is B-linear and B-colinear:

$$b \triangleright (ma) = (b \triangleright m)\,a, \qquad b_1 \triangleright m_0 \otimes b_2 m_1 = (b \triangleright m)_0 \otimes (b \triangleright m)_1\,,$$

i.e. it is an object in ${}_B\mathfrak{M}^B_B$, and viceversa. □

Remark 2.16. The fact that $B^\bullet$ is an algebra in the monoidal category $\left(\mathfrak{M}^B, \otimes, \Bbbk\right)$, mentioned in the proof of Lemma 2.15, implies also that the functor $- \otimes B : \mathfrak{M}^B \to \mathfrak{M}^B_B$ is left adjoint to the corresponding forgetful functor $\mathfrak{M}^B_B \to \mathfrak{M}^B$, forgetting the module structure (see, for example, [26, §VII.4]).

As a consequence of Lemma 2.15 and Remark 2.16, for all M in ${}_B\mathfrak{M}^B_B$ we have a $\Bbbk$-linear map $\Lambda_M : {}_B\mathsf{Hom}^B_B\,(B \otimes B, M) \to M^{\mathrm{co}B}$, natural in M, given by the composition of the chain of isomorphisms

$${}_B\mathsf{Hom}^B_B\,({}_\bullet B^\bullet \otimes B^\bullet_\bullet, {}_\bullet M^\bullet_\bullet) \overset{(42)}{\cong} \mathsf{Hom}^B_B\,(B^\bullet_\bullet, M^\bullet_\bullet) \cong \mathsf{Hom}^B\,(\Bbbk^\bullet, M^\bullet) \cong M^{\mathrm{co}B}$$

with the morphism ${}_B\mathsf{Hom}^B_B\,({}_\bullet B \otimes {}_\bullet B^\bullet_\bullet, {}_\bullet M^\bullet_\bullet) \to {}_B\mathsf{Hom}^B_B\,({}_\bullet B^\bullet \otimes B^\bullet_\bullet, {}_\bullet M^\bullet_\bullet)$ induced by $\mathfrak{can}$. It is given by the assignment $f \mapsto f\,(1 \otimes 1)$, whence the following diagram in $\mathfrak{M}$ commutes

$$\begin{array}{ccc} {}_B\mathsf{Hom}^B_B\,(B \otimes B, M) & & \\ {\scriptstyle \Lambda_M}\swarrow & \searrow{\scriptstyle \sigma_M} & \\ M^{\mathrm{co}B} & \xrightarrow[\varsigma_M]{} & \overline{M}^B \end{array}\,.$$

Recall that a bialgebra B is a Hopf algebra if and only if $\mathfrak{can}$ is invertible (in light, for example, of a left-handed version of [35, Example 2.1.2]).

Proposition 2.17. *The following are equivalent for a bialgebra B:*

(1) B is a Hopf algebra;
(2) σ is a natural isomorphism;
(3) Λ is a natural isomorphism.

If any of the foregoing conditions holds, then ς is a natural isomorphism.

Proof. The equivalence between *(1)* and *(2)* comes from Theorem 2.12. Concerning the equivalence between *(1)* and *(3)*, Λ is a natural isomorphism if and only if ${}_B\mathsf{Hom}^B_B\,(\mathfrak{can}, -)$ is a natural isomorphism, if and only if $\mathfrak{can}$ is an isomorphism. □

Remark 2.18. Note however that being ς_M an isomorphism for every $M \in {}_B\mathfrak{M}^B_B$ is not enough to have that B is a Hopf algebra. In fact, denote by ${}_BU : {}_B\mathfrak{M}^B_B \to \mathfrak{M}^B_B$ and by $U : {}_B\mathfrak{M} \to \mathfrak{M}$ the forgetful functors. If B is a right Hopf algebra, then ς_N an isomorphism

for every $N \in \mathfrak{M}_B^B$ and hence, in particular, $\varsigma_M : ({}_BU(M))^{\mathrm{co}B} \to U\left(\overline{M}\right)$ is an isomorphism for every $M \in {}_B\mathfrak{M}_B^B$. Since there exist right Hopf algebras that are not Hopf, the latter cannot imply that B is Hopf.

2.3. **Frobenius functors and unimodularity.** It would be interesting, in light of the similarity between Theorem 2.9 and [32, Theorem 2.7], to look for an analogue of [32, Theorem 3.12]. Let us report briefly on some partial achievements in this direction.

For a quasi-bialgebra A one can consider its space of *right integrals* $\int_r A$, which is the the $\Bbbk$-module $\{t \in A \mid ta = t\varepsilon(a) \text{ for all } a \in A\}$, and its space of *left integrals* $\int_l A$, which is the the $\Bbbk$-module $\{s \in A \mid as = \varepsilon(a)s \text{ for all } a \in A\}$. As in [18, page 14], we say that A is *unimodular* if $\int_l A = \int_r A$. The following fact will be used in the forthcoming results.

Lemma 2.19. *If $\Bbbk$ is a field and A is a finite-dimensional quasi-bialgebra with preantipode over $\Bbbk$, then* $\dim_{\Bbbk}\left(\int_l A\right) = 1 = \dim_{\Bbbk}\left(\int_r A\right)$.

Proof. In view of Proposition 1.2, A is a finite-dimensional quasi-Hopf algebra. Thus the result follows from [11, Theorem 2.2]. □

Consider the adjunctions

$$\begin{array}{ccc} {}_A\mathfrak{M}_A^A & {}_A\mathfrak{M}_A^A & {}_A\mathfrak{M}_A \\ \overline{(-)}\downarrow\uparrow -\otimes A & U\downarrow\uparrow -\widetilde{\otimes}A & -\otimes_A\Bbbk\downarrow\uparrow -\otimes\Bbbk \\ {}_A\mathfrak{M} & {}_A\mathfrak{M}_A & {}_A\mathfrak{M} \end{array}$$

where for every A-bimodule N, $N\widetilde{\otimes}A$ denotes the quasi-Hopf bimodule ${}^{\bullet}N_{\bullet} \otimes {}^{\bullet}A^{\bullet}_{\bullet}$, U is the functor forgetting the coaction and $\Bbbk$ is considered as a left or right A-module via ε.

For $V \in {}_A\mathfrak{M}$, recall that we set $V_\varepsilon := V \otimes \Bbbk \in {}_A\mathfrak{M}_A$. An easy observation allows us to conclude that $\overline{M} = U(M) \otimes_A \Bbbk$ and that $V \otimes A = V_\varepsilon \widetilde{\otimes} A$ for all $M \in {}_A\mathfrak{M}_A^A, V \in {}_A\mathfrak{M}$. Therefore, similarly to what was proven in [32, Proposition 3.3], if $\left(U, -\widetilde{\otimes}A\right)$ is Frobenius and if ${}_A\mathsf{Hom}_A\left(V_\varepsilon, U(M)\right) \cong {}_A\mathsf{Hom}\left(V, \overline{M}\right)$ naturally in $V \in {}_A\mathfrak{M}$ and $M \in {}_A\mathfrak{M}_A^A$, then

$${}_A\mathsf{Hom}_A^A\left(V\otimes A, M\right) = {}_A\mathsf{Hom}_A^A\left(V_\varepsilon\widetilde{\otimes}A, M\right) \cong {}_A\mathsf{Hom}_A\left(V_\varepsilon, U(M)\right) \cong {}_A\mathsf{Hom}\left(V, \overline{M}\right)$$

and so $\left(\overline{(-)}, -\otimes A\right)$ is Frobenius, which in turn implies that A admits a preantipode.

Lemma 2.20. *Any bijection ${}_A\mathsf{Hom}_A\left(V_\varepsilon, U(M)\right) \cong {}_A\mathsf{Hom}\left(V, \overline{M}\right)$ natural in $V \in {}_A\mathfrak{M}$ and $M \in {}_A\mathfrak{M}_A^A$ is a $\Bbbk$-linear natural isomorphism*

$$\Theta_{V,M} : {}_A\mathsf{Hom}\left(V, \overline{M}\right) \to {}_A\mathsf{Hom}_A\left(V_\varepsilon, U(M)\right), \qquad f \mapsto \tau_M \circ f_\varepsilon$$

where $\tau_M := \Theta_{\overline{M},M}\left(\mathsf{Id}_{\overline{M}}\right) : \overline{M} \to M$. Moreover, when a $\Bbbk$-linear natural isomorphism $\Theta_{V,M}$ exists, then A is unimodular and $\int_l A = \int_r A \cong \Bbbk$.

Proof. Assume firstly that a natural bijection $\Theta_{V,M} : {}_A\mathsf{Hom}\left(V, \overline{M}\right) \cong {}_A\mathsf{Hom}_A\left(V_\varepsilon, U(M)\right)$ exists. Since $\overline{M} \in {}_A\mathfrak{M}$, for every $f \in {}_A\mathsf{Hom}\left(V, \overline{M}\right)$ we may compute

$$\Theta_{V,M}(f) = \Theta_{V,M}(\mathsf{Id}_{\overline{M}} \circ f) = \Theta_{V,M}\left({}_A\mathsf{Hom}\left(f, \overline{M}\right)(\mathsf{Id}_{\overline{M}})\right)$$
$$= {}_A\mathsf{Hom}_A\left(f_\varepsilon, U(M)\right)\left(\Theta_{\overline{M},M}(\mathsf{Id}_{\overline{M}})\right) = \Theta_{\overline{M},M}(\mathsf{Id}_{\overline{M}}) \circ f_\varepsilon.$$

By setting $\tau_M \coloneqq \Theta_{\overline{M},M}(\mathsf{Id}_{\overline{M}}) : \overline{M}_\varepsilon \to U(M)$, we have that $\Theta_{V,M}(f) = \tau_M \circ f_\varepsilon$ for all $f \in {}_A\mathsf{Hom}\left(V, \overline{M}\right)$, which is $\Bbbk$-linear. Now, assume that a natural isomorphism $\Theta_{V,M}$ exists and consider the particular case $V = A \in {}_A\mathfrak{M}$ and $M = A \in {}_A\mathfrak{M}_A^A$. On the one hand, the assignment ${}_A\mathsf{Hom}_A\left(A_\varepsilon, U(A)\right) \to \int_r A : f \mapsto f(1)$ is invertible with explicit inverse $\int_r A \to {}_A\mathsf{Hom}_A\left(A_\varepsilon, U(A)\right) : t \mapsto [a \mapsto at]$. On the other hand,

$${}_A\mathsf{Hom}\left(A, \overline{A}\right) \cong {}_A\mathsf{Hom}\left(A, \Bbbk\right) \cong \mathsf{Hom}\left(\Bbbk, \Bbbk\right) = \Bbbk,$$

so that any element $f \in {}_A\mathsf{Hom}\left(A, \overline{A}\right)$ is of the form $f_k : a \mapsto \varepsilon(a)k\,\overline{1_A}$ for some $k \in \Bbbk$. Therefore, since $\Theta_{A,A}$ is an isomorphism, for every $t \in \int_r A$ there exists a (unique) $k \in \Bbbk$ such that

$$at = \Theta_{A,A}(f_k)(a) = \tau_A\left(f_k(a)\right) = \varepsilon(a)k\tau_A(1_A)$$

for every $a \in A$. In particular, for $a = 1_A$, $t = k\tau_A(1_A)$ and so it is a left integral as well, showing that A is unimodular. Moreover, we have the $\Bbbk$-linear isomorphism

$$\int_r A \cong {}_A\mathsf{Hom}_A\left(A_\varepsilon, U(A)\right) \cong {}_A\mathsf{Hom}\left(A, \overline{A}\right) \cong \Bbbk$$

and hence $\int_r A$ is free of rank 1 over $\Bbbk$. □

Remark 2.21. The interested reader may check that there is a bijection between natural transformations $\Theta_{V,M} : {}_A\mathsf{Hom}\left(V, \overline{M}\right) \to {}_A\mathsf{Hom}_A\left(V_\varepsilon, U(M)\right)$ and $\Bbbk$-linear morphisms $\partial : A \to A \otimes A, a \mapsto \partial^{(1)}(a) \otimes \partial^{(2)}(a)$ satisfying, for all $a, b \in A$,

$$a\partial^{(1)}(b) \otimes \partial^{(2)}(b) = \partial^{(1)}(a_2 b)a_1 \otimes \partial^{(2)}(a_2 b),$$
$$\partial^{(1)}(a) \otimes \partial^{(2)}(a)\varepsilon(b) = \partial^{(1)}(a) \otimes \partial^{(2)}(a)b = \partial^{(1)}(ab_2) \otimes b_1\partial^{(2)}(ab_2).$$

This is given by $\Theta \mapsto \left[a \mapsto ((A \otimes A) \otimes \varepsilon)\left(\tau_{(A\otimes A)\otimes A}\left(\overline{(1 \otimes 1) \otimes a}\right)\right)\right]$ and $\partial \mapsto \Theta^{(\partial)}$, where

$$\Theta^{(\partial)}_{V,M}(f) : V_\varepsilon \to U(M), \quad v \mapsto \partial^{(1)}(f(v)_1) \cdot f(v)_0 \cdot \partial^{(2)}(f(v)_1).$$

Proposition 2.22. *Assume that $\Bbbk$ is a field. Then the following assertions are equivalent for a quasi-bialgebra A over $\Bbbk$:*

(1) $\left(U, - \widetilde{\otimes} A\right)$ *is Frobenius and* ${}_A\mathsf{Hom}_A\left(V_\varepsilon, U(M)\right) \cong {}_A\mathsf{Hom}\left(V, \overline{M}\right)$ *naturally in* $V \in {}_A\mathfrak{M}$ *and* $M \in {}_A\mathfrak{M}_A^A$*;*
(2) $- \otimes A$ *is Frobenius,* A *is finite-dimensional and unimodular, and* $\int_l A = \int_r A \cong \Bbbk$*;*
(3) A *is a finite-dimensional unimodular quasi-bialgebra with preantipode;*
(4) A *is a finite-dimensional unimodular quasi-Hopf algebra.*

Moreover, any one of the above implies

(5) A is a unimodular Frobenius $\Bbbk$-algebra whose Frobenius homomorphism ψ is a left cointegral in the sense of [18, Definition 4.2].

Proof. We know that $-\otimes A$ is Frobenius if and only if A admits a preantipode by Theorem 2.9 and that the spaces of integrals over a finite-dimensional quasi-bialgebra with preantipode are always 1-dimensional (see Lemma 2.19), whence *(2)* $\Leftrightarrow$ *(3)*. The equivalence *(3)* $\Leftrightarrow$ *(4)* follows from the fact that, in the finite-dimensional case, quasi-Hopf algebras and quasi-bialgebras with preantipode are equivalent notions (see Proposition 1.2). It follows from Lemma 2.20 and our observations preceding it that if *(1)* holds then $-\otimes A$ is Frobenius, A is unimodular and $\int_l A = \int_r A \cong \Bbbk$. In addition, in view of [12, Theorem 5.8] and [13, Proposition 1.3], if $\left(U, -\widetilde{\otimes} A\right)$ is Frobenius then A is finite-dimensional. Therefore *(1)* $\Rightarrow$ *(2)*. Let us conclude by showing that *(4)* implies *(1)*. (**a**) In light of [13, Theorem 3.4(iv)], since A is a finite-dimensional unimodular quasi-Hopf algebra, the pair $\left(U, -\widetilde{\otimes} A\right)$ is Frobenius. (**b**) Since A is a quasi-Hopf algebra, in particular it is a quasi-bialgebra with preantipode (see [34, Theorem 6]) and hence $-\otimes A$ is an equivalence of categories. Therefore

$$ {}_A\mathsf{Hom}_A\left(V_\varepsilon, U(M)\right) \overset{\mathbf{(a)}}{\cong} {}_A\mathsf{Hom}_A^A\left(V_\varepsilon \widetilde{\otimes} A, M\right) = {}_A\mathsf{Hom}_A^A\left(V\otimes A, M\right) \overset{\mathbf{(b)}}{\cong} {}_A\mathsf{Hom}\left(V, \overline{M}\right). $$

Finally, *(4)* $\Rightarrow$ *(5)* follows from [18, Theorem 4.3] (together with [11, Theorem 2.2]. See also [20, Lemma 3.2]). $\square$

Remark 2.23. It is still an open question if *(5)* implies any of the other assertions or which additional conditions on A in *(5)* would allow us to prove that.

In this direction, and for the sake of future investigations on the subject, let us provide the explicit details of an equivalent description of when the pair $\left(U, -\widetilde{\otimes} A\right)$ is Frobenius.

Theorem 2.24 ([12, Theorem 5.8])**.** *For a quasi-bialgebra A, the pair $(U, -\widetilde{\otimes} A)$ is Frobenius if and only if there exists $z := z^{(1)}\otimes z^{(2)}\otimes z^{(3)} \in A\otimes A\otimes A$ and $\omega : A\otimes A \to A\otimes A, a\otimes b \mapsto \omega^{(1)}(a\otimes b)\otimes\omega^{(2)}(a\otimes b)$ such that for all $a, b \in A$*

$$ a_1 z^{(1)}\otimes z^{(2)} b_1\otimes a_2 z^{(3)} b_2 = z^{(1)} a\otimes b z^{(2)}\otimes z^{(3)}, $$

$$ \omega^{(1)}(x_{1_2} a y_{1_2}\otimes x_2 b y_2) x_{1_1}\otimes y_{1_1}\omega^{(2)}(x_{1_2} a y_{1_2}\otimes x_2 b y_2) = x\omega^{(1)}(a\otimes b)\otimes\omega^{(2)}(a\otimes b) y, $$

$$ \begin{aligned} \omega^{(1)}(\varphi^3 a_2\Phi^3\otimes b)_1\varphi^1\otimes\Phi^1\omega^{(2)}(\varphi^3 a_2\Phi^3\otimes b)_1\otimes\omega^{(1)}(\varphi^3 a_2\Phi^3\otimes b)_2\varphi^2 a_1\Phi^2\omega^{(2)}(\varphi^3 a_2\Phi^3\otimes b)_2 \\ = \omega^{(1)}(\varphi_2^1 a\Phi_2^1\otimes\varphi^2 b_1\Phi^2)\varphi_1^1\otimes\Phi_1^1\omega^{(2)}(\varphi_2^1 a\Phi_2^1\otimes\varphi^2 b_1\Phi^2)\otimes\varphi^3 b_2\Phi^3, \end{aligned} $$

$$ \omega^{(1)}(z_2^{(1)} a z_2^{(2)}\otimes z^{(3)}) z_1^{(1)}\otimes z_1^{(2)}\omega^{(2)}(z_2^{(1)} a z_2^{(2)}\otimes z^{(3)}) = \varepsilon(a) 1\otimes 1, $$

$$ \omega^{(1)}(\varphi_2^1 z^{(3)}\Phi_2^1\otimes\varphi^2 a_1\Phi^2)\varphi_1^1 z^{(1)}\otimes z^{(2)}\Phi_1^1\omega^{(2)}(\varphi_2^1 z^{(3)}\Phi_2^1\otimes\varphi^2 a_1\Phi^2)\otimes\varphi^3 a_2\Phi^3 = 1\otimes 1\otimes a. $$

Proof. We refer to [13] for the notations. In view of [13, Proposition 3.2],

$$ \left(A, \psi : A\otimes(A^{\mathrm{op}}\otimes A)\to(A^{\mathrm{op}}\otimes A)\otimes A,\ x\otimes(a\otimes b)\mapsto(a_1\otimes b_1)\otimes a_2 x b_2\right) $$

is a coalgebra in the category $\mathcal{T}^{\#}_{A^{\mathrm{op}}\otimes A}$ and the associated category of Doi-Hopf modules is exactly $\mathcal{M}(A^{\mathrm{op}}\otimes A)^A_{A^{\mathrm{op}}\otimes A} \cong {}_A\mathfrak{M}_A^A$. According to [12, Theorem 5.8], the forgetful functor U

is Frobenius if and only if (A,ψ) is a Frobenius coalgebra in $\mathcal{T}^{\#}_{A^{\mathrm{op}}\otimes A}$. By writing explicitly the conditions reported in [13, §1.2], one finds exactly the ones in the statement, with $z^{(1)}\otimes z^{(2)}\otimes z^{(3)}$ playing the role fo the Frobenius element and ω the role of the Casimir morphism. □

3. Preantipodes and Hopf monads

We conclude this paper with one last condition equivalent to the existence of a preantipode for a quasi-bialgebra. It showed up while addressing the question in §2, but it is independent from the results therein and hence we dedicate to it this small section.

Recall from [9, §2.7] that a *Hopf monad* on a monoidal category $(\mathcal{M},\otimes,\mathbb{I})$ is a monad (T,μ,ν) on $\mathcal{M}$ such that the functor T is a colax monoidal functor with $\phi_0 : T(\mathbb{I})\to\mathbb{I}$, $\phi_{X,Y} : T(X\otimes Y)\to T(X)\otimes T(Y)$, the natural transformations $\mu : T^2\to T, \nu : T\to \mathsf{Id}_{\mathcal{M}}$ are morphisms of colax monoidal functors and the *fusion operators*

$$H^l_{X,Y} \coloneqq \left(T(X\otimes T(Y)) \xrightarrow{\phi_{X,T(Y)}} T(X)\otimes T^2(Y) \xrightarrow{T(X)\otimes\mu_Y} T(X)\otimes T(Y) \right),$$
$$H^r_{X,Y} \coloneqq \left(T(T(X)\otimes Y) \xrightarrow{\phi_{T(X),Y}} T^2(X)\otimes T(Y) \xrightarrow{\mu_X\otimes T(Y)} T(X)\otimes T(Y) \right)$$

are natural isomorphisms in $X,Y\in\mathcal{M}$.

Similarly, consider a colax-colax adjunction $\mathcal{L}:\mathcal{M}\rightleftharpoons\mathcal{M}':\mathcal{R}$ between monoidal categories $(\mathcal{M},\otimes,\mathbb{I}),(\mathcal{M}',\otimes',\mathbb{I}')$, with colax monoidal structures $(\mathcal{L},\psi_0,\psi)$ and $(\mathcal{R},\varphi_0,\varphi)$. In [9, §2.8], the pair $(\mathcal{L},\mathcal{R})$ is called a *Hopf adjunction* if the *Hopf operators*

$$\begin{aligned}
\mathbb{H}^l_{X,X'} &\coloneqq \left(\mathcal{L}(X\otimes\mathcal{R}(X')) \xrightarrow{\psi_{X,\mathcal{R}(X')}} \mathcal{L}(X)\otimes'\mathcal{L}\mathcal{R}(X') \xrightarrow{\mathcal{L}(X)\otimes'\epsilon_{X'}} \mathcal{L}(X)\otimes' X' \right),\\
\mathbb{H}^r_{X',X} &\coloneqq \left(\mathcal{L}(\mathcal{R}(X')\otimes X) \xrightarrow{\psi_{\mathcal{R}(X'),X}} \mathcal{L}\mathcal{R}(X')\otimes'\mathcal{L}(X) \xrightarrow{\epsilon_{X'}\otimes'\mathcal{L}(X)} X'\otimes'\mathcal{L}(X) \right),
\end{aligned} \tag{43}$$

are natural isomorphisms in $X\in\mathcal{M}, X'\in\mathcal{M}'$.

Let A be a quasi-bialgebra over a commutative ring $\Bbbk$. In view of [1, Proposition 3.84] and the fact that $-\otimes A : {}_A\mathfrak{M}\to{}_A\mathfrak{M}^A_A$ is strong monoidal with $\xi_{V,W} : (V\otimes A)\otimes_A(W\otimes A)\to(V\otimes W)\otimes A$ as in (16), the functor $\overline{(-)}$ enjoys a colax monoidal structure (unique such that the adjunction $\left(\overline{(-)},-\otimes A\right)$ is colax-lax) where $\epsilon_{\Bbbk}$ provides the (iso)morphism $\overline{A}\cong\Bbbk$ connecting the unit objects and

$$\psi_{M,N} : \overline{M\otimes_A N}\to\overline{M}\otimes\overline{N},\quad \overline{m\otimes_A n}\mapsto\overline{m_0}\otimes\overline{m_1 n} \tag{44}$$

provides the natural transformation connecting the tensor products, where $M,N\in{}_A\mathfrak{M}^A_A$. This, in particular, makes of $\left(\overline{(-)},-\otimes A\right)$ a colax-colax adjunction (in light of [1, Proposition 3.93], for example).

Consider the monad $T = \overline{(-)} \otimes A$ on ${}_A\mathfrak{M}_A^A$ associated to the adjunction $\Big(\overline{(-)}, - \otimes A\Big)$. The natural transformations μ and ν are provided by

$$\mu_M : \overline{\overline{M} \otimes A} \otimes A \xrightarrow{\cong} \overline{M} \otimes A; \quad \overline{\overline{m} \otimes a} \otimes b \mapsto \overline{m} \otimes \varepsilon(a)b \tag{45}$$
$$\text{and} \qquad \nu_M : M \to \overline{M} \otimes A; \quad m \mapsto \overline{m_0} \otimes m_1,$$

where μ is invertible because the counit ϵ from (19) is so. It is an *opmonoidal monad* by [9, §2.5] with

$$\phi_{M,N} \coloneqq \left(\begin{array}{l} \overline{M \otimes_A N} \otimes A \xrightarrow{\psi_{M,N}\otimes A} \overline{M} \otimes \overline{N} \otimes A \xrightarrow{\xi^{-1}_{\overline{M},\overline{N}}} \Big(\overline{M} \otimes A\Big) \otimes_A \Big(\overline{N} \otimes A\Big) \\ \overline{m \otimes_A n} \otimes a \longmapsto \Big(\overline{\Phi^1 \cdot m_0} \otimes 1\Big) \otimes_A \Big(\overline{\Phi^2 m_1 \cdot n} \otimes \Phi^3 a\Big) \end{array} \right)$$

$$\text{and} \qquad \phi_0 \coloneqq \left(\begin{array}{l} \overline{A} \otimes A \xrightarrow{\epsilon_{\Bbbk}\otimes A} \Bbbk \otimes A \xrightarrow{\cong} A \\ \overline{a} \otimes b \longmapsto \varepsilon(a)b \end{array} \right). \tag{46}$$

Remark 3.1. Opmonoidal monads are monads and colax monoidal functors at the same time such that the multiplication and unit of the monad are morphisms of colax monoidal functors. They have been called *Hopf monads* in [28, Definition 1.1] and *bimonads* in [9, §2.5], [10, §2.3], but we decided to adhere to the terminology introduced by [27, page 472] because it is nowadays the most widely used in the subject (see, for example, [8, Chapter 3]). In particular, a Hopf monad here is an opmonoidal monad whose fusion operators are natural isomorphisms.

The following is the main result of the present section.

Theorem 3.2. *For a quasi-bialgebra A the following are equivalent*

(a) A admits a preantipode;
(b) the natural transformation Ψ of equation (44) is a natural isomorphism;
(c) the component $\Psi_{A\widehat{\otimes}A,A\otimes A}$ of Ψ is invertible;
(d) $\Big(\overline{(-)}, - \otimes A\Big)$ is a lax-lax adjunction;
(e) $\Big(\overline{(-)}, - \otimes A\Big)$ is a Hopf adjunction;
(f) $T = \overline{(-)} \otimes A$ is a Hopf monad on ${}_A\mathfrak{M}_A^A$.

The proof of Theorem 3.2 is postponed to §3.1. We decided to split it in some smaller intermediate results for the sake of clearness.

Corollary 3.3. *For a bialgebra B the following are equivalent*

(a) B is a Hopf algebra;
(b) the natural transformation Ψ of equation (44) is an isomorphism;
(c) the component $\Psi_{B\widehat{\otimes}B,B\otimes B}$ of Ψ is invertible;
(d) $\Big(\overline{(-)}, - \otimes B\Big)$ is a lax-lax adjunction;
(e) $\Big(\overline{(-)}, - \otimes B\Big)$ is a Hopf adjunction;

(f) $T = \overline{(-)} \otimes B$ *is a Hopf monad on* ${}_B\mathfrak{M}_B^B$.

Remark 3.4. Let A be a quasi-bialgebra. Observe that we implicitly proved the following noteworthy fact: the monad $T = \overline{(-)} \otimes A$ is a Frobenius monad if and only if it is a Hopf monad, if and only if it is naturally isomorphic to the identity functor. In fact, on the one hand T is a Frobenius monad if and only if $- \otimes A$ is a Frobenius functor (since we know from Remark 2.1 that $- \otimes A$ is fully faithful, it is monadic. Therefore the claim can be easily deduced from [38, Theorem 1.6]. For the details, see [32, Proposition 1.5]). On the other hand, by Theorem 2.9 the latter is equivalent to the existence of a preantipode for A and this, in turn, is equivalent to T being Hopf by Theorem 3.2.

3.1. **Proof of Theorem 3.2.** In this subsection, we will often make use of the following isomorphism of left A-modules

$$\chi_{M,N} \coloneqq \left(\overline{M \otimes_A N} \cong (M \otimes_A N) \otimes_A \Bbbk \cong M \otimes_A (N \otimes_A \Bbbk) \cong M \otimes_A \overline{N}\right), \tag{47}$$

natural in $M, N \in {}_A\mathfrak{M}_A^A$, which closely resembles the one we used to prove Lemma 2.6.

Lemma 3.5. *Let* $\mathcal{L} : \mathcal{M} \rightleftarrows \mathcal{M}' : \mathcal{R}$ *be a colax-colax adjunction between monoidal categories* $(\mathcal{M}, \otimes, \mathbb{I}), (\mathcal{M}', \otimes', \mathbb{I}')$*, with colax monoidal structures* $(\mathcal{L}, \psi_0, \psi)$ *and* $(\mathcal{R}, \varphi_0, \varphi)$*. Then* ψ_0 *and* ψ *are natural isomorphisms if and only if* $(\mathcal{L}, \mathcal{R})$ *is a lax-lax adjunction.*

Proof. The proof is already contained in [1, Propositions 3.93 and 3.96]. Let us sketch it anyway, for the sake of the reader. Since $\mathcal{R}$ is right adjoint to an colax monoidal functor, in light of [1, Proposition 3.84] it naturally inherits a unique lax monoidal structure such that the pair $(\mathcal{L}, \mathcal{R})$ is a colax-lax adjunction. Moreover, by the (dual of the) proof of [1, Proposition 3.96], this unique lax monoidal structure is provided by the inverses of φ_0 and φ, thus making of $\mathcal{R}$ a strong monoidal functor. Now, if ψ_0 and ψ are natural isomorphisms, then $\mathcal{L}$ is a strong monoidal functor. By the direct implication of [1, Proposition 3.93 (1)], $(\mathcal{L}, \mathcal{R})$ is a lax-lax adjunction. Conversely, assume that $(\mathcal{L}, \mathcal{R})$ is a lax-lax adjunction where the lax monoidal structure on $\mathcal{L}$ is denoted by $(\mathcal{L}, \gamma_0, \gamma)$. As left adjoint of a lax monoidal functor, $\mathcal{L}$ inherits a unique colax monoidal structure such that $(\mathcal{L}, \mathcal{R})$ is a colax-lax adjunction, by [1, Proposition 3.84] again, and this has to be provided by the inverses of γ_0 and γ. However, $\mathcal{L}$ already has a colax monoidal structure such that $(\mathcal{L}, \mathcal{R})$ is a colax-lax adjunction: $(\mathcal{L}, \psi_0, \psi)$. Therefore, $\gamma_0^{-1} = \psi_0$ and $\gamma^{-1} = \psi$. □

Since in the context of Theorem 3.2 we have that $\phi_0 = \epsilon_{\Bbbk}$ is always invertible, the equivalence between *(b)* and *(d)* follows from Lemma 3.5: $\left(\overline{(-)}, - \otimes A\right)$ is a lax-lax adjunction if and only if $\phi = \xi^{-1} \circ \psi$ is a natural isomorphism, if and only if ψ is.

Proposition 3.6. *The following assertions are equivalent for a quasi-bialgebra* A*.*

(1) The natural transformation ψ *of equation* (44) *is a natural isomorphism;*
(2) $\left(\overline{(-)}, - \otimes A\right)$ *is a Hopf adjunction between* ${}_A\mathfrak{M}_A^A$ *and* ${}_A\mathfrak{M}$*;*
(3) $T = \overline{(-)} \otimes A$ *is a Hopf monad on* ${}_A\mathfrak{M}_A^A$*.*

Proof. Since the functor $-\otimes A$ is fully faithful (see Remark 2.1), the counit ϵ of the adjunction $\left(\overline{(-)}, -\otimes A\right)$ is a natural isomorphism. Thus the implication from *(1)* to *(2)* follows by looking at the explicit form (43) of the Hopf operators: if ϵ and ψ are natural isomorphisms, then $\mathbb{H}^l$ and $\mathbb{H}^r$ are natural isomorphisms as well. The implication from *(2)* to *(3)* is [9, Proposition 2.14]. Finally, let us show that *(3)* implies *(1)*. By using the explicit form (46) of ϕ, the left fusion operator can be rewritten as

$$H^l_{M,N} = \left(\left(\overline{M}\otimes A\right)\otimes_A \mu_N\right)\circ \phi_{M,\overline{N}\otimes A} \overset{(46)}{=} \left(\left(\overline{M}\otimes A\right)\otimes_A \mu_N\right)\circ \xi^{-1}_{\overline{M},\overline{\overline{N}\otimes A}}\circ\left(\psi_{M,\overline{N}\otimes A}\otimes A\right),$$

whence if $H^l_{M,N}$ is invertible then $\psi_{M,\overline{N}\otimes A}\otimes A$ is invertible as well (because both ξ of (16) and μ of (45) are). Now, consider the following facts: for every M, N quasi-Hopf A-bimodules,

(i) we have that $\psi_{M,\overline{N}\otimes A}\circ\epsilon_{M\otimes_A(\overline{N}\otimes A)} = \epsilon_{\overline{M}\otimes\overline{\overline{N}\otimes A}}\circ\overline{\psi_{M,\overline{N}\otimes A}\otimes A}$ by naturality of ϵ, so that if $\psi_{M,\overline{N}\otimes A}\otimes A$ is an isomorphism then $\psi_{M,\overline{N}\otimes A}$ is an isomorphism (because ϵ is always an isomorphism);

(ii) in view of the triangular identity $\epsilon_{\overline{N}}\circ\overline{\eta_N} = \mathsf{Id}_{\overline{N}}$ for the adjunction $\overline{(-)}\dashv -\otimes A$, $\overline{\eta_N}$ is an isomorphism with inverse $\epsilon_{\overline{N}}$;

(iii) we have that $\chi_{M,\overline{N}\otimes A}\circ\overline{M\otimes_A\eta_N} = (M\otimes_A\overline{\eta_N})\circ\chi_{M,N}$ by naturality of χ of (47), so that $\overline{M\otimes_A\eta_N}$ is an isomorphism by (ii) and

(iv) $\left(\overline{M}\otimes\overline{\eta_N}\right)\circ\psi_{M,N} = \psi_{M,\overline{N}\otimes A}\circ\overline{M\otimes_A\eta_N}$ by naturality of ψ. Thus, by (ii) and (iii), if $\psi_{M,\overline{N}\otimes A}$ is an isomorphism then $\psi_{M,N}$ is an isomorphism.

Therefore, by (i) and (iv), if $\psi_{M,\overline{N}\otimes A}\otimes A$ is invertible for all $M, N\in {}_A\mathfrak{M}^A_A$ then $\psi_{M,N}$ is invertible as well, concluding the proof. □

As a consequence of Proposition 3.6, we have that *(b)* ⇔ *(e)* ⇔ *(f)* in Theorem 3.2.

Proposition 3.7. *The natural transformation* ψ *of equation* (44) *is a natural isomorphism if and only if the unit* η *of the adjunction* $\left(\overline{(-)}, -\otimes A\right)$ *is a natural isomorphism. Moreover, the component* $\psi_{A\widehat{\otimes}A,A\otimes A}$ *is invertible if and only if the component* $\eta_{A\widehat{\otimes}A}$ *is.*

Proof. Denote by $\kappa_{V,W}$ the obvious isomorphism $(V\otimes A)\otimes_A W\cong V\otimes W$, which is natural in $V, W\in {}_A\mathfrak{M}$. One can check by a direct computation that

$$\psi_{M,N} = \kappa_{\overline{M},N}\circ\left(\eta_M\otimes_A\overline{N}\right)\circ\chi_{M,N}, \tag{48}$$

so that ψ is a natural isomorphism if η is. Conversely, take $N = {}_\bullet A\otimes {}_\bullet A^\bullet_\bullet$. For every $\overline{m\otimes_A(a\otimes b)}\in\overline{M\otimes_A(A\otimes A)}$, compute

$$\begin{aligned}
&(\eta_M\circ(M\otimes_A\epsilon_A)\circ\chi_{M,A\otimes A})\left(\overline{m\otimes_A(a\otimes b)}\right) \overset{(47)}{=} (\eta_M\circ(M\otimes_A\epsilon_A))\left(m\otimes_A\overline{(a\otimes b)}\right)\\
&\overset{(19)}{=} \eta_M(m\cdot a\varepsilon(b)) \overset{(19)}{=} \overline{m_0\cdot a_1}\otimes m_1a_2\varepsilon(b) = \overline{m_0}\otimes m_1a\varepsilon(b)\\
&\overset{(19)}{=} \left(\overline{M}\otimes\epsilon_A\right)\left(\overline{m_0}\otimes\overline{m_{1_1}a\otimes m_{1_2}b}\right) \overset{(44)}{=} \left(\left(\overline{M}\otimes\epsilon_A\right)\circ\psi_{M,A\otimes A}\right)\left(\overline{m\otimes_A(a\otimes b)}\right).
\end{aligned}$$

Therefore $\eta_M \circ (M \otimes_A \epsilon_A) \circ \chi_{M,A\otimes A} = \left(\overline{M} \otimes \epsilon_A\right) \circ \psi_{M,A\otimes A}$ and hence η is a natural isomorphism if ψ is. In particular, for $M = A \,\widehat{\otimes}\, A = A_\bullet \otimes {}_\bullet A^\bullet_\bullet$, $\eta_{A\widehat{\otimes}A}$ is invertible if and only if $\psi_{A\widehat{\otimes}A,A\otimes A}$ is. □

In light of [34, Theorem 4], A admits a preantipode if and only if η is a natural isomorphism (because the counit ϵ is always a natural isomorphism), if and only if the distinguished component $\eta_{A\widehat{\otimes}A}$ is invertible. Therefore, if follows from Proposition 3.7 that *(a)* $\Leftrightarrow$ *(b)* $\Leftrightarrow$ *(c)* in Theorem 3.2 and this concludes its proof.

Remark 3.8. Concerning the implication from *(a)* to *(b)*, it follows from [34, Equations (16) and (28)] that if A admits a preantipode S, then $\eta_M^{-1}(\overline{m} \otimes a) = \Phi^1 m_0 S(\Phi^2 m_1)\Phi^3 a$ for all $m \in M, a \in A$. In this case, an explicit inverse for $\psi_{M,N}$ is given by

$$\overline{M} \otimes \overline{N} \to \overline{M \otimes_A N}: \qquad \overline{m} \otimes \overline{n} \mapsto \overline{\Phi^1 m_0 S(\Phi^2 m_1)\Phi^3 \otimes_A n}.$$

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