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Efficiency in Decentralized Markets with Aggregate Uncertainty*

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Abstract

We study efficiency in decentralized markets with common-value uncertainty and correlated asset values. There is an equal mass of buyers and sellers and payoffs from trade depend on an aggregate state, which only the sellers know. Buyers and sellers are randomly and anonymously matched in pairs over time, and buyers make the offers. We show that all equilibria become efficient as trading frictions vanish.

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1 Introduction

Market efficiency is a central concern in economics. In idealized markets, trade is centralized and information is perfect. In this case, the first welfare theorem shows that market outcomes are efficient. However, trade is often decentralized: rather than trade taking place at a single price that clears the market, in many markets buyers and sellers negotiate the terms of trade bilaterally. Moreover, information is typically asymmetric: rather than buyers and sellers being perfectly, and thus equally, informed, in many markets sellers often have better information about underlying features of their assets than buyers.

It is well-known that both decentralized trade and asymmetric information can, by themselves, hurt market efficiency. When trade is decentralized, it may take time for an agent to find a desirable trading partner. This delay in trade represents a loss of efficiency. On the other hand, when sellers are better informed about their assets than buyers, they can use their private information to extract rents from buyers, distorting the terms of trade. This distortion also leads to a loss of efficiency.

A question that has attracted substantial attention is how decentralization of trade and asymmetry of information interact to affect market efficiency. However, the literature on this topic has mainly focused on the case in which asset values are independent across *sellers*, i.e., the value of a seller's asset is independent of the value of any other seller's asset. While the assumption of independent asset values is reasonable in some markets, there are many relevant markets, both real and financial, in which asset values are correlated across sellers. Real-state markets and markets for asset-backed securities are prominent examples.[1]

In this paper, we study market efficiency in decentralized markets with common value uncertainty and correlated asset values. The environment we consider is as follows. There is an equal mass of buyers and sellers. Payoffs from trade depend on an aggregate state, which only the sellers know. Thus, asset values are perfectly correlated across sellers. The number of aggregate states is finite and gains from trade are nonnegative in all states. Time is discrete and in every period buyers

[1]For instance, in the case of mortgage-backed securities, the loans in the underlying pool of loans backing different securities could have been issued to borrowers with similar characteristics. In the housing market, sellers are typically better informed about neighborhood characteristics than buyers, which affects house values; see Kurlat and Stroebel (2015) for evidence on this.

and sellers in the market are randomly and anonymously matched in pairs with some probability. In a buyer-seller match, the buyer makes a take-it-or-leave-it offer to the seller, which the seller either accepts or rejects. If the seller accepts the offer, then trade takes place and both agents exit the market. Otherwise, the match is dissolved and both agents remain in the market.

Our main result is that, somewhat surprisingly, welfare in *any* equilibrium approaches welfare in the complete information case as trading frictions vanish, i.e., as the real time between two consecutive trading opportunities converges to zero. Thus, by itself, asymmetry of information between buyers and sellers is not enough to prevent market efficiency when asset values are perfectly correlated across sellers.

The rest of the paper is organized as follows. In the remainder of this section we discuss the related literature. In Section 2 we introduce our environment. In Section 3 we explicitly construct an equilibrium which achieves the first best in the limit as trading frictions vanish. In Section 4 we generalize the result of Section 3 and show that every equilibrium achieves the first best in the limit as trading frictions vanish. In Section 5 we discuss robustness and extensions of our efficiency result. In particular, we show that our efficiency result survives when we relax the assumption of perfect correlation among asset values by introducing private values. In Section 6 we conclude. The Appendix contains omitted details and proofs.

Related Literature The literature on market efficiency in decentralized markets with correlated asset values is scant. Similarly to us, Blouin and Serrano (2001) study this question in a market with aggregate uncertainty. There are important differences between our analysis and the analysis in Blouin and Serrano's paper, though. First, we focus on the case in which sellers know the aggregate state but buyers do not. Second, we allow for any finite number of aggregate states, instead of just two, and place no restrictions on payoffs from trade except that gains from trade are nonnegative in every state. Finally, and crucially, we depart from Blouin and Serrano (2001) in the bargaining protocol. They consider a stylized bargaining game which amounts to restricting the set of prices at which trade can take place and show that market outcomes remain inefficient even when trading frictions vanish. Our analysis thus shows that restricting prices at which trade can take place can have a critical impact on market efficiency; when buyers are not restricted in the offers they can make, market outcomes become efficient as trading frictions vanish.[2]

Golosov, Lorenzoni, and Tsyvinski (2014) study information aggregation in decentralized markets with aggregate uncertainty and divisible goods. They provide conditions under which in the long-run information is perfectly aggregated and trading outcomes become efficient.[3] In contrast, we show in an environment with indivisible goods that market efficiency is obtained as trading frictions vanish. However, as we discuss in Section 4, equilibria need not aggregate information perfectly in our setting.

Asriyan, Fuchs, and Green (2017a) study information spillovers in a dynamic market with imperfectly correlated asset values in which sellers are privately informed about the quality of their assets. They show that as long as asset values are sufficiently correlated, making transaction outcomes public, i.e., introducing transaction transparency, leads to multiple equilibria which are Pareto ranked. In our environment there are no information spillovers. The only way buyers can learn about the aggregate state is through their own experience in the market.

There are a number of papers that study decentralized trade with common-value uncertainty and independent asset values; see e.g., Blouin (2003), Camargo and Lester (2014), and Kim (2017). Our efficiency result contrasts strongly with inefficiency results in this literature. When asset values are independent, multiple types of assets co-exist in the market, allowing owners of lower-quality assets to extract informational rents from buyers. This is not possible in the presence of aggregate uncertainty. Our efficiency result also contrasts strongly with inefficiency results in the literature on bargaining with common-value uncertainty; see, e.g., Deneckere and Liang (2006) and Gerardi, Hörner, and Maestri (2014) for models of bargaining between two long-lived parties, and Hörner and Vieille (2009), Daley and Green (2012), Fuchs, Öry, and Skrzypacz (2016), and Kaya and Kim (2017) for models of bargaining between a long-lived seller and a sequence of short-lived buyers. Intuitively, unlike in a single-seller setting, an individual seller cannot affect aggregate behavior in

[2]One can show that even when there is no asymmetry of information between buyers and sellers, if buyers are restricted in the set of prices they can offer, then there exist equilibria in our environment which remain inefficient as market frictions disappear. Serrano and Yosha (1995) obtain the same result in a stationary version of our environment.

[3]The seminal reference in the literature on information aggregation in decentralized markets is Wolinsky (1990). Serrano and Yosha (1993) shows that Wolinsky's negative result depends on the assumption of two-sided incomplete information. Blouin and Serrano (2001) extends the analysis in Wolinsky (1990) to non-stationary environments. Other recent papers in the literature on information aggregation in decentralized markets are Lauermann and Wolinsky (2016) and Asriyan, Fuchs, and Green (2017b).

a (large) market setting.

2 Environment

Time is discrete and indexed by $t \in \{0, 1, \ldots\}$. There is an equal mass of buyers and sellers with common discount factor $\delta \in (0, 1)$; trading frictions vanish as δ converges to one.[4] Each seller can produce one unit of an indivisible good and each buyer wants to consume one unit of the good. The set of (aggregate) states is $\mathcal{K} = \{1, \ldots, K\}$ and the probability that the state is $k \in \mathcal{K}$ is $\pi_k > 0$. Sellers know the state, but buyers do not. Agents have quasi-linear preferences. The value to a buyer from consuming the good in state k is v_k, while the cost to a seller of producing the good in the same state is $c_k \geq 0$. We assume nonnegative gains from trade in every state.

Assumption 1. $v_k - c_k \geq 0$ *for all* $k \in \mathcal{K}$.

Assumption 1 is fairly weak. In particular, single-crossing preferences, i.e., $v_k - c_k$ strictly increasing in k, is not necessary for our results. Moreover, as we show in Section 4, Assumption 1 cannot be relaxed. It implies that the first-best welfare is

$$W^* = \sum_{k=1}^{K} \pi_k (v_k - c_k).$$

Trade takes place as follows. In every period $t \geq 0$, a buyer in the market is randomly and anonymously matched to a seller with probability $\lambda \in (0, 1)$ and vice-versa. In each buyer-seller pair, the buyer makes a take-it-or-leave-it offer $p \in \mathbb{R}_+$ to the seller. If the seller accepts the offer, then trade occurs and both agents exit the market. Otherwise, the match is dissolved and both agents remain in the market. The assumption that $\lambda < 1$ ensures that there is a positive mass of agents in the market in every period.[5]

We now define strategies and equilibria. Let $\mathcal{H}_t$, with typical element h^t, be the set of private histories for an agent in the market in period t.[6] A behavior strategy for a buyer is a sequence

[4]More formally, one can think that agents discount the future at a common rate $\rho > 0$ and $\delta = e^{-\rho\Delta}$, where Δ is the time interval between two consecutive periods. Trading frictions vanish as Δ converges to zero.

[5]In Section 5 we discuss how to extend our analysis to the case in which $\lambda = 1$.

[6]A private history for a buyer in the market in period t is a sequence $h^t = (\tilde{p}_1, \ldots, \tilde{p}_{t-1})$, where $\tilde{p}_s \in \mathbb{R}_+$ is the buyer's (rejected) price offer in period s if he was matched in this period and $\tilde{p}_s = \emptyset$ if the buyer was not matched in period s. Private histories for sellers are defined similarly.

$\sigma^B = \{\sigma_t^B\}$, where $\sigma_t^B : \mathcal{H}_t \to \Delta(\mathbb{R}_+)$ and $\sigma_t^B(h^t)$ is the (random) price offer that the buyer makes in period t if he is matched to a seller when his private history is h^t. A behavior strategy for a seller is a sequence $\sigma^S = \{\sigma_t^S\}$, where $\sigma_t^S : \mathcal{H}_t \times \mathcal{K} \times \mathbb{R}_+ \to [0,1]$ and $\sigma_t^S(h^t, k, p)$ is the probability that the seller accepts an offer of p in period t when the state is k and his private history is h^t. A belief system for a buyer is a sequence $\mu = \{\mu_t\}$, where $\mu_t : \mathcal{H}_t \to \Delta(\mathcal{K})$ and $\mu_t(h^t)$ is the buyer's (posterior) belief about the state in period t when his private history is h^t. We let Σ and θ denote, respectively, a strategy profile and a profile of belief systems. We consider pairs (Σ, θ) which constitute a perfect Bayesian equilibrium (PBE).

We conclude by showing how to compute welfare for any strategy profile. An outcome for a seller is a triple $(k, \mathbf{T}, \mathbf{p})$, where $k \in \mathcal{K}$ is the aggregate state, $\mathbf{T} \in \mathbb{Z}_+ \cup \{\infty\}$ is the time at which the seller trades, and $\mathbf{p} \in P$ is the price at which the seller trades; $\mathbf{T} = \infty$ corresponds to the event in which the seller does not trade.[7] Together with the probability distribution over the set of aggregate states, a strategy profile Σ uniquely determines a probability distribution ξ over the set of outcomes for each seller. Let $\mathbb{E}_\xi$ denote the expectation with respect to ξ. Welfare under the strategy profile Σ is

$$W(\Sigma) = \sum_{k=1}^{K} \pi_k \mathbb{E}_\xi \left[\delta^{\mathbf{T}} | k\right] (v_k - c_k);$$

the term $\mathbb{E}_\xi \left[\delta^{\mathbf{T}} | k\right]$ is the discounted probability of trade in aggregate state k. Clearly, $W(\Sigma)$ is bounded above by W^*.

3 A Limit-Efficient Equilibrium

In our environment the first best is incentive feasible.[8] The question of interest is whether the outcome of decentralized trade always approaches the first best as trading frictions vanish. In this section we show that if discounting is sufficiently small, then there exists a PBE whose welfare approaches the first-best welfare as δ converges to one. For ease of exposition, we consider the case of two aggregate states and discuss how to extend our equilibrium construction to the case of

[7] When $\mathbf{T} = \infty$, the seller's transaction price is undetermined. We adopt the convention that $\mathbf{p} = 0$ in this case.

[8] Indeed, the direct mechanism in which trade occurs at price $p(\bar{k}) = \frac{1}{2}(v_{\bar{k}} + c_{\bar{k}})$ if, and only if, the average reported state $\bar{k}$ belongs to $\mathcal{K}$ is incentive compatible and individually rational. Truth-telling is immediate since no seller is pivotal when there are three or more of them in the population.

more than two aggregate states at the end of the section.

Assume, without any loss, that $c_1 < c_2$ and refer to a seller when the aggregate state is k as a type-k seller.[9] Consider the following symmetric assessment (Σ, θ): (*i*) a type-k seller accepts an offer of p if, and only if, $p \geq c_k$; (*ii*) a buyer offers $p = c_1$ the first time he is matched to a seller and offers $p = c_2$ afterwards; and (*iii*) a buyer in the market assigns probability π_1 to $k = 1$ if either he has not made any offer or the highest offer he has made is less than c_1, otherwise he assigns probability 0 to $k = 1$. Under Σ, all buyers trade after at most two offers, and so welfare approaches the first-best welfare as trading frictions vanish.

We claim that there exists $\delta^* \in (0, 1)$ such that (Σ, θ) is a PBE if $\delta \geq \delta^*$. First notice that buyer beliefs are consistent with Bayes' rule on the path of play. Moreover, sellers' behavior is sequentially rational. Clearly, a type-2 seller has no incentive to deviate. A type-1 seller also has no incentive to deviate by rejecting an offer $p \geq c_1$. Indeed, a type-1 seller who rejects an offer knows that in the future he is matched with probability 1 to a buyer who has not had the chance to make an offer (and so will offer $p = c_1$). Finally, notice that the only possibly profitable deviation for a buyer is to offer a price $p \geq c_2$ and trade immediately if the highest price he has offered so far is smaller than c_1. The expected payoff from this deviation is bounded above by $\overline{u} = \pi_1 v_1 + \pi_2 v_2 - c_2$. On the other hand, if the highest price offer a buyer has made so for is smaller than c_1, then the buyer's expected payoff from following the equilibrium strategy is

$$\pi_1(v_1 - c_1) + \pi_2 \delta \sum_{s=0}^{\infty} \delta^s (1 - \lambda)^s \lambda (v_2 - c_2) = \pi_1(v_1 - c_1) + \pi_2 \frac{\delta \lambda}{1 - \delta(1 - \lambda)} (v_2 - c_2),$$

which is greater than $\overline{u}$ as long as δ is sufficiently close to one. This establishes the desired result.

We conclude this section by discussing how to extend the above equilibrium construction to the case in which there are more than two aggregate states. We provide a sketch of the argument in what follows; the details are in the Appendix. Let $\mathcal{C}$ be the set of possible production costs for sellers. Then $\mathcal{C} = \{\overline{c}_1, \ldots, \overline{c}_L\}$, with $L \leq K$ and $\overline{c}_1 < \cdots < \overline{c}_L$. Now consider the symmetric strategy profile in which buyers and sellers behave as follows: (*i*) a type-k seller accepts an offer of p if, and only if $p \geq c_k$; and (*ii*) a buyer offers $\overline{c}_1$ the first time he is matched to a seller and offers

[9] By re-ordering the states if necessary, we have that $c_1 \leq c_2$. When $c_1 = c_2$, it is immediate to see that there exists a symmetric PBE where buyers offer $p = c_1$ when matched to a seller and sellers accept and offer of p if, and only if, $p \geq c_1$. Welfare in this equilibrium approaches the first-best welfare as trading frictions vanish.

the smallest element in $\mathcal{C}$ that is greater than the greatest offer he has made so far any other time he is matched to a seller. Notice that on the path of play a buyer offers $\bar{c}_k$, with $k \in \{1, \ldots, L\}$, the kth time he is matched to a seller and trades after making at most L offers. So, welfare approaches the first-best welfare as trading frictions vanish. This strategy profile reduces to the strategy profile of the PBE we constructed above when there are two aggregate states.

As in the case with two aggregate states, the sellers' behavior is sequentially rational in the strategy profile under consideration, and for the same reason: with probability one, a type-k seller who rejects an offer of $p \geq c_k$ receives an offer of at most c_k in any subsequent period. Likewise, the only profitable deviation for a buyer is to offer a price that is higher than the price he is supposed to offer in order to increase the probability of trade. In the Appendix we show that there exists a (symmetric) belief system consistent with Bayes' rule on the path of play such that the buyers' behavior is also sequentially rational if they are patient enough. The intuition is straightforward: in order for a buyer to increase his probability of trade in any period, he needs to increase the price he offers by a discontinuous amount, which does not pay when trading frictions are small enough, as the buyer trades with probability one after making at most L offers.

4 Market Efficiency

In Section 3 we constructed a PBE that approaches the first best as trading frictions vanish. This, of course, does not rule out the possibility that there are PBE whose welfare is bounded away from W^* no matter how small trading frictions are. In this section we show that this is not the case.

Theorem 1. *Let $\{\delta_n\}$ be a sequence of discount factors such that $\lim_n \delta_n = 1$. For any sequence $\{(\Sigma_n, \theta_n)\}$ such that (Σ_n, θ_n) is a PBE when $\delta = \delta_n$, the sequence $\{W(\Sigma_n)\}$ converges to W^*.*

In what follows we present a sketch of the proof of Theorem 1. For simplicity, we consider the limiting case in which λ is close enough to one that the probability that a buyer who stays in the market is matched to a seller is essentially equal to one.[10] The proof of the general case—when

[10]Formally, we show that if $\{\delta_n\}$ and $\{\lambda_n\}$ are, respectively, a sequence of discount factors and a sequence of matching probabilities such that $\lim_n \delta_n = \lim_n \lambda_n = 1$, then $\lim_n W(\Sigma_n) = W^*$ for any sequence $\{(\Sigma_n, \theta_n)\}$ such that (Σ_n, θ_n) is a PBE when $\delta = \delta_n$ and $\lambda = \lambda_n$.

λ assumes any value in $(0, 1)$—is in the Appendix; we discuss how to extend the argument that follows to the general case at the end of the section.

Let (Σ, θ) be a PBE when the discount factor is $\delta \in (0, 1)$. Notice that even though (Σ, θ) need not be symmetric, all agents on a given side of the market obtain the same payoff; this property of equilibria holds regardless of the value of λ. In fact, since there is a continuum of buyers and matching is random and anonymous, a buyer can obtain the same payoff as any other buyer by mimicking the other buyer's behavior. The same applies for sellers. Let V^B be the buyers' ex-ante equilibrium payoff and V^k be the type-k seller's equilibrium payoff. Moreover, let V_t^k be the type-k sellers' payoff in period t; by construction, $V_0^t = V^k$. Since sellers know the aggregate state, V_t^k does not depend on the private history of a seller, only on the period t.

Now observe that for every $k \in \mathcal{K}$ and $s > t \geq 0$, we have that $V_t^k \geq \delta^{s-t} V_s^k$. This follows immediately from the fact that a possibility for a seller in period t is to reject all offers between periods t and $s - 1$, and then follow the equilibrium strategy from period s on. Moreover, for every $k \in \mathcal{K}$ and $t \geq 0$, we have that $V_t^k \leq z = \max_k v_k$. This follows from the fact buyers do not find it optimal to offer $p > z$ on the path of play if such an offer is accepted with positive probability. Both these facts about the payoffs V_t^k hold regardless of λ.

We proceed in three steps. First, we construct prices $\hat{p}_1$ to $\hat{p}_K$ such that for all $k \in \mathcal{K}$ a type-k seller accepts an offer of $\hat{p}_k$ in the first K periods. Then, we use these "reservation" prices to construct a lower bound to V^B. Finally, we use the lower bound to V^B to construct a lower bound to $W(\Sigma)$ that is independent of Σ and show that the lower bound converges to the first-best welfare as δ converges to one.

Step 1. Reservation Prices for Sellers

Fix $\varepsilon > 0$ and let $T \geq K$. For the argument that follows it suffices to set $T = K$. Letting T be arbitrary helps us when we discuss the extension of our argument to the case in which λ can assume any value in $(0, 1)$. Now, for each $k \in \mathcal{K}$, let

$$\hat{p}_k = c_k + \frac{V_1^k}{\delta^{T-2}} + \frac{\varepsilon}{4}.$$

We claim that a type-k seller accepts an offer of $\hat{p}_k$ in the first $T - 1$ periods. Indeed, it is strictly optimal for a type-k seller to accept an offer of p in period t if $p - c_k > \delta V_{t+1}^k$. Now observe that if $t \in \{0, \ldots, T-1\}$, then $V_1^k \geq \delta^t V_{t+1}^k \geq \delta^{T-1} V_{t+1}^k$. Hence, $t \in \{0, \ldots, T-1\}$ implies that $\hat{p}_k - c_k \geq \delta V_{t+1}^k + \varepsilon/4 > \delta V_{t+1}^k$, and so a type-$k$ seller accepts an offer of $\hat{p}_k$ in period t.

Step 2. Lower Bound to Buyers' Payoff

Re-label the aggregate states so that $\hat{p}_k$ is (weakly) increasing in k. Consider now the following alternative strategy $\hat{\sigma}_B$ for a buyer: offer $\hat{p}_{t+1}$ if matched to a seller in period $t \in \{0, \ldots, K-1\}$ and offer $\hat{p}_K$ if matched to a seller in period $t \geq K$. Denote by $u(\hat{\sigma}^B; (\Sigma, \theta))$ the payoff to a buyer who follows $\hat{\sigma}^B$ when all other agents behave according to Σ and the belief system is θ. Since $\lambda \approx 1$, the buyer transacts with probability essentially equal to one in at most $K \leq T$ periods and pays at most $\hat{p}_k$ for the good when the aggregate state is k. Thus, a lower bound to the buyer's payoff in aggregate state k when he follows $\hat{\sigma}^B$ is $\delta^{T-1} v_k - \hat{p}_k - \varepsilon/4$, and so

$$
\begin{aligned}
u(\hat{\sigma}^B; (\Sigma, \theta)) &= \sum_{k=1}^{K} \pi_k \big(\delta^{T-1} v_k - \hat{p}_k\big) - \frac{\varepsilon}{4} \\
&= \sum_{k=1}^{K} \pi_k \big(\delta^{T-1} v_k - c_k\big) - \frac{1}{\delta^{T-2}} \sum_{k=1}^{K} \pi_k V_1^k - \frac{\varepsilon}{2}.
\end{aligned}
$$

Given that (Σ, θ) is an equilibrium, $V^B \geq u(\hat{\sigma}^B; (\Sigma, \theta))$, otherwise buyers would have a profitable deviation. Consequently,

$$
\begin{aligned}
V^B &\geq \sum_{k=1}^{K} \pi_k (v_k - c_k) - (1 - \delta^{T-1}) \sum_{k=1}^{K} \pi_k v_k - \frac{1}{\delta^{T-2}} \sum_{k=1}^{K} \pi_k V_1^k - \frac{\varepsilon}{2} \\
&\geq W^* - (1 - \delta^{T-1}) z - \frac{1}{\delta^{T-1}} \sum_{k=1}^{K} \pi_k V^k - \frac{\varepsilon}{2}.
\end{aligned}
$$

where the second inequality follows from the fact that $V^k \leq z = \max_k v_k$ and $V^k = V_0^k \geq \delta V_1^k$ for every aggregate state k.

Step 3. Lower Bound to Welfare

Since preferences are quasi-linear,

$$
W(\Sigma) = V^B + \sum_{k=1}^{K} \pi_k V^k.
$$

Thus, from Step 2 and using again the fact that $V^k \leq z$ for all $k \in \mathcal{K}$, we have that

$$
W(\Sigma) \geq W^* - (1 - \delta^{T-1}) \Big(z + \frac{z}{\delta^{T-1}}\Big) - \frac{\varepsilon}{2}.
$$

Consequently, there exists $\overline{\delta} \in (0,1)$ such that $W(\Sigma) \geq W^* - \varepsilon$ if $\delta \geq \overline{\delta}$. The desired result follows from the fact that ε is arbitrary.

Extending the above argument to the case in which λ is bounded away from one requires changing the strategy $\hat{\sigma}^B$ used to compute the lower bound to the buyers' equilibrium payoff to account for the fact that a buyer might not be matched to a seller in every period. Loosing speaking, when λ is bounded away from one, the strategy $\hat{\sigma}^B$ must be such that a buyer first attempts to trade with the type of seller with the lowest reservation price for sufficiently many periods, then attempts to trade with the type of seller with the second lowest reservation price for sufficiently many periods, and so on. This requires taking T in the above definition of reservation prices to be large enough. In the limit as δ converges to one, the delay in trading implied by the modified strategy $\hat{\sigma}^B$ converges to zero and one still obtains a lower bound to the equilibrium welfare that is independent of the equilibrium under play and converges to the first-best welfare. The details are in the Appendix. We conclude this section with a couple of remarks about our results.

Role of Aggregate Uncertainty and Random Matching. Our efficiency result contrasts strongly with inefficiency results in dynamic decentralized markets with common-value uncertainty but uncorrelated asset values. In such environments, multiple types of seller co-exist in the market. As is well-known, the incentive that sellers with low valuation have to mimic the behavior of sellers with high valuation ensures that equilibria remain inefficient even as trading frictions vanish—a reduction in delay costs reduces the cost for the former type of seller to imitate the latter type of seller. In our environment, a single type of seller is present in the market at any point in time. As such, an option for a buyer is to offer the reservation prices of the different types of sellers in ascending order, thus extracting the residual surplus from sellers in a finite number of periods regardless of the aggregate state. In the limit as δ converges to one, the inefficiency resulting from this strategy converges to zero. Since the buyer's payoff in any equilibrium is bounded below by the payoff they obtain using the strategy described above, all equilibria become efficient as trading frictions vanish.

Some of the driving forces present in our model with a continuum of agents are also not present in bargaining models in which a single seller dynamically meets a with a sequence of short-run buyers or the same buyer. The main difference between our model and the aforementioned ones is that in our environment a single trader cannot influence the aggregate dynamics of the economy. On the other hand, when there is a single seller in the market, his behavior can affect the future behavior of buyers. In this case, the only way to provide incentives for a low-valuation seller to trade at a low price is to have delay in trade with a high-valuation seller.

Information Aggregation. A question that has attracted a lot of attention is whether markets fully aggregate the information dispersed among agents. While the equilibrium of Section 3 aggregates information perfectly, it is easy to construct examples of PBE which become efficient in the limit as trading frictions vanish but fail to aggregate information perfectly. For instance, in the case of three aggregate states, if $\pi_1 v_1 + \pi_2 v_2 + \pi_3 v_3 < c_3$ but

$$\tilde{v} = \frac{\pi_2}{\pi_2 + \pi_3} v_2 + \frac{\pi_3}{\pi_2 + \pi_3} v_3 > c_3,$$

then as long δ is close to one there exists a PBE such that: (*i*) buyers offer c_1 the first time they are matched to a seller, and type-1 sellers accept this offer; and (*ii*) buyers make the pooling offer $\tilde{v}$ any other time they are matched to a seller, and type-2 and type-3 sellers accept this offer.

5 Robustness and Extensions

In this section we discuss robustness and extensions of our efficiency result.

Gains From Trade. Theorem 1 shows that the assumption of nonnegative gains from trade in every state is sufficient for welfare in all PBE to approach the first-best welfare as trading frictions disappear. The example below shows that this assumption is also necessary for limit efficiency.

Suppose that $K = 2$ and $v_1 < c_1 < c_2 < v_2$, so that gains from trade are negative when $k = 1$. In this case, $W^* = \pi_2(v_2 - c_2)$. Take a sequence $\{\delta_n\}$ of discount factors that converges to one and, for each $n \in \mathbb{N}$, let (Σ_n, θ_n) be a PBE when $\delta = \delta_n$. Assume towards a contradiction that $W(\Sigma_n)$ converges to W^*. Then $\lim_n \mathbb{E}_{\xi_n}\left[\delta_n^{\mathbf{T}}|k=1\right] = 0$ and $\lim_n \mathbb{E}_{\xi_n}\left[\delta_n^{\mathbf{T}}|k=2\right] = 1$, where ξ_n is the probability distribution over the set of outcomes induced by Σ_n. Now let $\mathbf{Q}$ be the first (random) period in which a buyer makes an offer $p \geq c_2$. Then $\lim_n \mathbb{E}_{\xi_n}\left[\delta_n^{\mathbf{Q}}|k=2\right] = 1$. It is possible to show that $\lim_n \mathbb{E}_{\xi_n}\left[\delta_n^{\mathbf{T}}|k=1\right] = 0$ and $\lim \mathbb{E}_{\xi_n}\left[\delta_n^{\mathbf{Q}}|k=2\right] = 1$ together imply that $\lim_n \mathbb{E}_{\xi_n}\left[\delta_n^{\mathbf{Q}}|k=1\right] = 1$. So, a seller in state 1 can secure a limit payoff of at least $c_2 - c_1 > 0$ by following the strategy in which he accepts an offer p if, and only if, $p \geq c_2$, a contradiction.

Bargaining Protocol. It is possible to extend Theorem 1 to the case in which in every buyer-seller pair the buyer makes a take-it-or-leave-it offer to the seller with positive probability. A sketch of the proof—which is similar to the proof of Theorem 1—is as follows. We again consider the limiting case in which λ is close enough to one that the probability that a buyer in the market is matched to a seller is essentially equal to one; the extension to the case in which λ is bounded away from one is the same as in the previous section. A lower bound to a buyer's payoff in any equilibrium is obtained when the buyer: (*i*) rejects any offer that he receives from a seller; and (*ii*) offers the type-k seller's reservation price in the kth period in which the buyer gets to make an offer. This strategy ensures that the buyer trades after making at most K offers to sellers. As trading frictions vanish, this lower bound on the buyers's equilibrium payoff converges to the first-best welfare net of the sellers' ex-ante equilibrium payoff, which establishes the desired result.

Theorem 1 is not true when sellers have all the bargaining power, though. Signalling opens up the possibility of equilibria which remain inefficient even as trading frictions vanish.

Uninformed Sellers. We assume that all sellers are informed about the aggregate state. As it turns out, this assumption cannot be relaxed. When some sellers are uninformed about the aggregate state, it is possible to construct equilibria in which signalling sustains inefficient outcomes even as trading frictions vanish: a buyer who deviates by making an offer to attract an uninformed seller changes the uninformed seller's belief in a way that precludes trade.

Matching Probabilities We can extend Theorem 1 to the case in which $\lambda = 1$, and so in every period all buyers and sellers in the market are matched in pairs with probability one. In this case, however, we need to consider the stronger concept of sequential equilibrium. This is the natural equilibrium concept to consider when $\lambda = 1$, as in any sequential equilibrium the payoffs to buyers and sellers are well-defined even if there is a zero mass of agents in the market. In particular, the payoff to a buyer or a seller is well-defined if aggregate behavior is such that the market clears but the agent behaves in a such a way that he does not trade.[11]

Since the existence of a sequential equilibrium cannot be guaranteed when action sets are infinite, we make the additional assumption that buyers are restricted to make offers in a finite grid $P = \{p_0, p_1, \ldots, p_M\}$ of prices, where p_i is strictly increasing in i, $p_0 < \min_k c_k$, and $p_M > \max_k c_k$.[12] The assumption that $p_0 < \min_k c_k$ is natural, as it implies that buyers can make offers that are always rejected. The assumption that buyers can make offers that are greater than the highest cost of production is also natural; otherwise, it is trivial to generate inefficient equilibria. For any price grid P, let $\mathcal{C}(P) = \max_{0 \in \{1,\ldots,M-1\}} |p_{i+1} - p_i|$ be the coarseness of P. Our efficiency result is obtained in the limit as $\mathcal{C}(P)$ converges to zero, and so the grid of possible price offers becomes arbitrarily fine. This limiting case approaches the case in which buyers are not restricted in the offers they can make to sellers.

Theorem 2. *Let $\{\delta_n\}$ be a sequence of discount factors such that $\lim_n \delta_n = 1$ and $\{P_n\}$ be a sequence of finite price grids such that $\min P_n < \min_k c_k$ and $\max P_n > \max_k v_k$ for all $n \in \mathbb{N}$ and $\lim_n \mathcal{C}(P_n) = 0$. For any sequence $\{(\Sigma_n, \theta_n)\}$ such that (Σ_n, θ_n) is a sequential equilibrium when $\delta = \delta_n$ and $P = P_n$, the sequence $\{W(\Sigma_n)\}$ converges to W^*.*

The proof of the above theorem is very similar to the proof of Theorem 1 and thus is omitted. Loosely speaking, when buyers are restricted to make offers in a finite set P, the reservation prices used to derive a lower bound to the buyers' equilibrium payoff need to be adjusted so that they are elements of P. As $\mathcal{C}(P)$ converges to zero, this adjustment becomes arbitrarily small and one obtains a lower bound to the buyers' equilibrium payoff that approaches the lower bound we obtain when buyers are not restricted in the offers they can make.

Private Values We assume that asset values are perfectly correlated across buyers and sellers. It turns out that this assumption is not necessary for our results. As we now discuss, an efficiency

[11]Indeed, first notice that if the pair (Σ, θ) is such that Σ has full support, then there is a positive mass of agents in the market in every period, in which case payoffs are well-defined after any history. Now observe that payoffs in a sequential equilibrium are the limits of payoffs when the pair (Σ, θ) is such that Σ has full support.

[12]A straightforward fixed-point argument shows that a sequential equilibrium exists when buyers are restricted to make offers in a finite price grid.

result is possible when asset values have a common- and a private-value component.

The environment is the same as before, except that now buyers and sellers have idiosyncratic tastes. The set of possible buyer types is $\mathcal{D} = \{1, \ldots, D\}$, with $D \geq 1$, while the set of possible seller types is $\mathcal{L} = \{1, \ldots, L\}$, with $L \geq 1$. We denote a typical element of $\mathcal{D}$ by d and a typical element of $\mathcal{L}$ by ℓ. An agent's type is his private information and is independent of any other agent's type. The probability that a buyer is of type d is $\varphi_d \in [0, 1]$, while the probability that a seller is of type ℓ when the aggregate state is k is $\gamma_\ell^k \in [0, 1]$. The payoff to a type-d buyer from consuming the good in state k is $v_{k,d}$, while the cost to a type-ℓ seller of producing the good in state k is $c_{k,\ell}$. As in the setting of Theorem 1, we assume nonnegative gains from trade in every state, in which case the first-best welfare is given by

$$W^* = \sum_{k=1}^{K} \sum_{\ell=1}^{L} \sum_{d=1}^{D} \pi_k \gamma_\ell^k \varphi_d (v_{k,d} - c_{k,\ell}).$$

Assumption 2. $v_{k,d} - c_{k,\ell} \geq 0$ *for all* $(k, d, \ell) \in \mathcal{K} \times \mathcal{D} \times \mathcal{L}$.

It is possible to extend Theorem 1 to the case described here.[13] While the idea behind this more general efficiency result is similar to the idea behind Theorem 1, the proof needs to be adapted to take into account the fact that when there is more than one type of seller in each aggregate state, these different types of seller can exit the market at different rates in equilibrium.

6 Concluding Remarks

Most of the literature that studies the impact of asymmetric information on market efficiency in decentralized markets with common-value uncertainty focuses on the case of independent asset values. In this paper, we show that allowing correlated asset values—a more realistic assumption in many relevant markets—can lead to starkly different results. While our assumption of perfectly correlated asset values is strong, it constitutes a useful first step in relaxing the assumption of independent asset values. The question of how asymmetric information affects market efficiency in decentralized markets with common-value uncertainty and *imperfectly* correlated asset values remains an important open question.

[13] The proof is available from the authors upon request.

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7 Appendix

In this Appendix we first extend the equilibrium construction of Section 3 to the case of any finite number of aggregate states. We then provide the proof of Theorem 1 in the general case in which λ can assume any value in $(0,1)$.

Equilibrium of Section 3

Let $\mathcal{C} = \{\bar{c}_1, \dots, \bar{c}_L\}$, with $\bar{c}_1 < \cdots < \bar{c}_L$ and $L \leq K$, be the set of possible production costs for sellers and define $f : \mathcal{C} \rightrightarrows \mathcal{K}$ to be the correspondence such that $f(\bar{c}_\ell) = \{k \in \mathcal{K} : c_k = \bar{c}_\ell\}$. Now consider the symmetric assessment (Σ, θ) in which the sellers' strategy, the buyers' strategy, and the buyers' belief system are as follows:

Sellers' Strategy σ^S: A type-k seller in a match accepts an offer p if, and only if, $p \geq c_k$.

Buyers' Strategy σ^B: For any history h^t for a buyer, let: (*i*) $m(h^t) \in \mathbb{N}_+$ be the number of times the buyer was matched in the market so far; (*ii*) $\bar{p}(h^t)$ be the largest offer the buyer has made so far, where $\bar{p}(h^t) = -\infty$ if $m(h^t) = 0$; and (*iii*) $\mathcal{C}(\bar{p}(h^t)) = \{c \in \mathcal{C} : c > \bar{p}(h^t)\}$. Notice that $\mathcal{C}(\bar{p}(h^t)) = \mathcal{C}$ if $m(h^t) = 0$. A buyer's behavior after a history h^t is as follows. If $\mathcal{C}(\bar{p}(h^t)) \neq \emptyset$, then the buyer offers the smallest element of $\mathcal{C}(\bar{p}(h^t))$ if he is matched in the market. If, instead, $\mathcal{C}(\bar{p}(h^t)) = \emptyset$, then the buyer offers $\bar{c}_L$ if he is matched in the market.

Buyers' Belief System μ^B: If $m(h^t) = 0$, then $\mu(h^t)$ assigns probability π_k to $k \in \mathcal{K}$. Now suppose that $m(h^t) > 0$. If $\mathcal{C}(\bar{p}(h^t)) \neq \emptyset$, then $\mathcal{C}(\bar{p}(h^t)) = \{\bar{c}_j, \dots, \bar{c}_L\}$ for some $j \in \{1, \dots, L\}$. In this case, $\mu(h^t)$ assigns probability 0 to every $k \in \bigcup_{s=1}^{j-1} f(\bar{c}_s)$ and assigns probability

$$\frac{\pi_k}{\sum_{\ell \in \bigcup_{s=j}^{L} f(\bar{c}_s)} \pi_\ell}$$

to every other aggregate state. On the other hand, if $\mathcal{C}(\bar{p}(h^t)) = \emptyset$, then $\mu(h^t)$ assigns probability 0 to every $k \in \bigcup_{s=1}^{L-1} f(\bar{c}_s)$ and assigns probability

$$\frac{\pi_k}{\sum_{\ell \in f(\bar{c}_L)} \pi_\ell}$$

to the remaining aggregate states.

We claim that there exists $\delta^* \in (0,1)$ such that (Σ, θ) is a PBE if $\delta > \delta^*$. It is easy to see that μ is consistent with Bayes' rule on the path of play. The same argument as in the case of two aggregate states shows that σ^S is sequentially rational regardless of δ: with probability one, a type-k seller who rejects an offer in a given period receives no offer greater than c_k in the future.

Consider now a buyer with history h^t. From above, there exists $j \in \{1, \ldots, L\}$ such that $\mu(h^t)$ assigns probability

$$\widetilde{\pi}_k = \frac{\pi_k}{\sum_{\ell \in \bigcup_{s=j}^{L} f(\overline{c}_s)} \pi_\ell}$$

to every aggregate state $k \in \bigcup_{s=j}^{L} f(\overline{c}_s)$ and assigns probability 0 to every other aggregate state. If $j = L$, then the buyer does not have a profitable deviation as he believes that $k \in f(\overline{c}_L)$ and expects that any seller accepts an offer of $\overline{c}_L$. Suppose then that $j < L$ and let $\tau \geq L - 1$ be the random time at which the buyer is matched for the Lth time in the market. If the buyer follows the equilibrium strategy, then his expected payoff is bounded below by

$$u_j = \sum_{\ell \in \bigcup_{k=j}^{L} f(\overline{c}_k)} \mathbb{E}[\delta^{\tau-1}] \widetilde{\pi}_\ell (v_\ell - c_\ell).$$

Clearly, the most profitable deviation for the buyer consists in offering a price $\overline{c}_k \in \{\overline{c}_{j+1}, \ldots, \overline{c}_L\}$ to trade earlier. The payoff from this deviation is bounded above by

$$u'_j = \underbrace{\sum_{\ell \in \bigcup_{k=j}^{L} f(\overline{c}_k)} \widetilde{\pi}_\ell (v_\ell - c_\ell)}_{\overline{u}} - \underbrace{\mathbb{E}[\delta^{\tau-1}](\overline{c}_{j+1} - \overline{c}_j) \sum_{\ell \in f(\overline{c}_j)} \widetilde{\pi}_\ell}_{\varepsilon}.$$

The payoff $\overline{u}$ is the agent's expected payoff from trading immediately at the correct price, while ε is a lower bound to the agent's expected loss from trading at a higher price; with probability at least $\sum_{\ell \in f(\overline{c}_j)} \widetilde{\pi}_\ell$ there is an aggregate state in which the buyer purchases the good at a price at least $(\overline{c}_{j+1} - \overline{c}_j) > 0$ greater than $\overline{c}_j$. Since $\lim_{\delta \to 1} \mathbb{E}[\delta^{\tau-1}] = 1$ by the dominated convergence theorem, there exists $\delta_j \in (0, 1)$ such that $\delta > \delta_j$ implies that $u_j > u'_j$.

Letting $\delta^* = \max\{\delta_1, \ldots, \delta_{L-1}\}$, we can then conclude that σ^B is sequentially rational whenever $\delta > \delta^*$, and so (Σ, θ) is a PBE if $\delta > \delta^*$. Given that under Σ the buyers trade after at most L offers, it follows that $\lim_{\delta \to 1} W(\Sigma) = W^*$.

Proof of Theorem 1

We show that for all $\varepsilon \in (0, z)$, there exists $\overline{\delta} \in (0, 1)$ such that if $\delta > \overline{\delta}$, then $W(\Sigma) > W^* - \varepsilon$ for every assessment (Σ, θ) which constitutes a PBE when the agents' discount factor is δ. Recall that $z = \max_k v_k$.

Fix $\varepsilon \in (0, z)$ and let (Σ, θ) be a PBE for some discount factor δ. We know from the main text that buyers obtain the same payoff V^B and each type k of seller obtains the same payoff V^k. Moreover, if we let V_t^k denote a type-k seller's payoff in period t, then this payoff does not depend on a seller's private history. Finally, the payoffs V_t^k are such that: (*i*) $V_t^k \geq \delta^{s-t} V_s^k$ for every $k \in \mathcal{K}$ and $s > t \geq 0$; and (*ii*) $V_t^k \leq z$ for every $k \in \mathcal{K}$ and $t \geq 0$.

Step 1. Reservation Prices for Sellers

We proceed as in the main text and first identify for each $k \in \mathcal{K}$ a set of offers that, in equilibrium, a type-k seller accepts with probability one. Let $\kappa = \varepsilon/16z > 0$ and define $T(\kappa)$ as the smallest positive integer such that $(1-\lambda)^{T(\kappa)} < \kappa$.

Lemma 1. *Consider the equilibrium* (Σ, θ)*. For every* $k \in \mathcal{K}$ *and* $t \geq 0$*, let*

$$\underline{p}_{k,t} = c_k + \frac{V_1^k}{\delta^{t-1}}.$$

If a type-k seller receives an offer $p > \underline{p}_{k,t}$ in period t, then he accepts it with probability one.

Proof. Consider a type-k seller in period $t \geq 0$. He accepts an offer of p if $p - c_k > \delta V_{t+1}^k$. The desired result follows from the fact that $\underline{p}_{k,t} - c_k = V_1^k/\delta^{t-1} \geq \delta V_{t+1}^k$, where the inequality follows from the fact that $V_t^t \geq \delta^t V_{t+1}^k$ for all $k \in \mathcal{K}$ and $t \geq 0$. □

Now, for each $k \in \mathcal{K}$, define $\widehat{p}_k$ to be such that

$$\hat{p}_k = c_k + \frac{V_1^k}{\delta^{T(\kappa)K-2}} + \frac{\varepsilon}{4}.$$

Since, by construction, $\hat{p}_k > \underline{p}_{k,t}$ for all $t \in \{0, \ldots, T(\kappa)K - 1\}$, Lemma 1 implies that a type-k seller accepts an offer of $\hat{p}_k$ in any period $t \in \{0, \ldots, T(\kappa)K - 1\}$.

Step 2. Lower Bound to Buyers' Payoff

As in the main text, we now use the prices $\hat{p}_1$ to $\hat{p}_K$ to derive a lower bound to the buyers' equilibrium payoff. Re-label the aggregate states so that $\hat{p}_k$ is (weakly) increasing in k. Consider the following alternative strategy $\hat{\sigma}^B$ for a buyer: offer $\hat{p}_k$ if matched to a seller in periods $t \in \{T(\kappa)(k-1), \ldots, T(\kappa)k - 1\}$ and offer $\hat{p}_K$ if matched to a seller in any period $t \geq T(\kappa)K$. Denote by $u\left(\hat{\sigma}^B; (\Sigma, \theta)\right)$ the payoff the buyer obtains when all other agents behave according to the strategy profile Σ and the belief system is θ. Notice that $V^B \geq u\left(\hat{\sigma}^B; (\Sigma, \theta)\right)$, otherwise the buyer would have a profitable deviation.

We obtain a lower bound for $u\left(\hat{\sigma}^B; (\Sigma, \theta)\right)$, and thus V^B, as follows. Suppose the aggregate state is k. There are two mutually exclusive and exhaustive events to consider: the buyer transacts in period $t < T(\kappa)(k-1)$ or the buyer is still in the market in period $T(\kappa)(k-1)$. In the first event, the buyer's expected payoff is bounded below by $\delta^{T(\kappa)K-1}v_k - \hat{p}_k$; this is because $\hat{p}_k$ is increasing in k and $v_k \geq 0$. Consider now the second event. Either the buyer is matched with a seller in some period $t \in \{T(\kappa)(k-1), \ldots, T(\kappa)k - 1\}$ and obtains a payoff of at least $\delta^{T(\kappa)K-1}v_k - \hat{p}_k$, or the buyer is not matched with a seller in any of these periods and obtains a payoff of a least $-\hat{p}_K$. Given that $(1-\lambda)^{T(\kappa)} < \kappa$, the probability the buyer does not meet a seller in some period $t \in \{T(\kappa)(k-1), \ldots, T(\kappa)k-1\}$ is at most κ. So, the buyer's expected payoff in the second event is bounded below by $(1-\kappa)\left(\delta^{T(\kappa)K-1}v_k - \widehat{p}_k\right) - \kappa\widehat{p}_K$. Given that the lower bound to the buyers' expected payoff is lower in the second event, we then have that

$$u\left(\hat{\sigma}^B; (\Sigma, \theta)\right) \geq (1-\kappa)\sum_{k=1}^{K}\pi_k\left(\delta^{T(\kappa)K-1}v_k - \hat{p}_k\right) - \kappa\hat{p}_K.$$

Now observe that $\max_k\left\{c_k, V_1^k\right\} \leq z$ and $\varepsilon \in (0, z)$ implies that

$$\kappa\hat{p}_K \leq \kappa\left(c_k + V_1^K + \varepsilon + \frac{\left(1-\delta^{T(\kappa)K-2}\right)V_1^K}{\delta^{T(\kappa)K-2}}\right) \leq 3\kappa z + z\left(\frac{1-\delta^{T(\kappa)K-2}}{\delta^{T(\kappa)K-2}}\right). \tag{1}$$

Moreover, we also have that

$$\begin{aligned}
(1-\kappa)\sum_{k=1}^{K}\pi_k\left(\delta^{T(\kappa)K-1}v_k - \hat{p}_k\right) &= (1-\kappa)\sum_{k=1}^{K}\pi_k\left(\delta^{T(\kappa)K-1}v_k - c_k - \frac{\delta V_1^k}{\delta^{T(\kappa)K-1}} - \frac{\varepsilon}{4}\right) \\
&= (1-\kappa)\left[\sum_{k=1}^{K}\pi_k(v_k - c_k) - \left(1-\delta^{T(\kappa)K-1}\right)\sum_{k=1}^{K}\pi_k v_k - \frac{\varepsilon}{4} - \sum_{k=1}^{K}\pi_k\frac{\delta V_1^k}{\delta^{T(\kappa)K-1}}\right] \\
&\geq \sum_{k=1}^{K}\pi_k(v_k - c_k) - \kappa\sum_{k=1}^{K}\pi_k v_k - \left(1-\delta^{T(\kappa)K-1}\right)\sum_{k=1}^{K}\pi_k v_k - \frac{\varepsilon}{4} - \sum_{k=1}^{K}\pi_k\frac{\delta V_1^k}{\delta^{T(\kappa)K-1}} \\
&\geq \sum_{k=1}^{K}\pi_k(v_k - c_k) - \kappa z - \left(1-\delta^{T(\kappa)K-1}\right)z - \frac{\varepsilon}{4} - z\left(\frac{1-\delta^{T(\kappa)K-1}}{\delta^{T(\kappa)K-1}}\right) - \sum_{k=1}^{K}\pi_k\delta V_1^k; \quad (2)
\end{aligned}$$

the first inequality follows from the fact that c_k, v_k, and V_1^k are non-negative for all $k \in \mathcal{K}$, while the second inequality follows from the fact that $\sum_{k=1}^{K}\pi_k v_k \leq z$ and $\sum_{k=1}^{K}\pi_k\delta V_1^k \leq z$.

Using inequalities (1) and (2) and the facts that $4\kappa z = \varepsilon/4$ and $(1-\delta^t)/\delta^t$ is increasing in t, we can then conclude that

$$V^B \geq W^* - \sum_{k=1}^{K} \pi_k \delta V_1^k - \left(1 - \delta^{T(\kappa)K-1}\right) z - \frac{\varepsilon}{2} - 2z \left(\frac{1-\delta^{T(\kappa)K-1}}{\delta^{T(\kappa)K-1}}\right).$$

Step 3. Lower Bound to Welfare

We conclude by using the above lower bound to V^B to obtain a lower bound to welfare. Since $W(\Sigma) = V^B + \sum_{k=1}^{K} \pi_k V^k$ and, for all $k \in \mathcal{K}$, $V^k = V_0^k \geq \delta V_1^k$, we have

$$W(\Sigma) \geq W^* - \left(1 - \delta^{T(\kappa)K-1}\right) z - \frac{\varepsilon}{2} - 2z \left(\frac{1-\delta^{T(\kappa)K-1}}{\delta^{T(\kappa)K-1}}\right).$$

Taking $\overline{\delta} \in (0,1)$ such that $\delta > \overline{\delta}$ implies that

$$\left(1 - \delta^{T(\kappa)K-1}\right) z + 2z \left(\frac{1-\delta^{T(\kappa)K-1}}{\delta^{T(\kappa)K-1}}\right) < \frac{\varepsilon}{2},$$

we can then conclude that $W(\sigma, \mu) > W^* - \varepsilon$ whenever $\delta > \overline{\delta}$. The desired result follows from the fact that ε was arbitrary.