**Fourier integral operators algebra and fundamental solutions to hyperbolic systems with polynomially bounded coefficients on R^n**

# FOURIER INTEGRAL OPERATORS ALGEBRA AND FUNDAMENTAL SOLUTIONS TO HYPERBOLIC SYSTEMS WITH POLYNOMIALLY BOUNDED COEFFICIENTS ON $\mathbb{R}^n$

ALESSIA ASCANELLI AND SANDRO CORIASCO

Abstract. We study the composition of an arbitrary number of Fourier integral operators $A_j$, $j = 1, \dots, M$, $M \geqslant 2$, defined through symbols belonging to the so-called SG classes. We give conditions ensuring that the composition $A_1 \circ \cdots \circ A_M$ of such operators still belongs to the same class. Through this, we are then able to show well-posedness in weighted Sobolev spaces for first order hyperbolic systems of partial differential equations in SG classes, by constructing the associated fundamental solutions. These results expand the existing theory for the study of the properties "at infinity" of the solutions to hyperbolic Cauchy problems on $\mathbb{R}^n$ with polynomially bounded coefficients.

Contents



## 1. Introduction

We deal with a class of Fourier integral operators globally defined on $\mathbb{R}^n$, namely, the SG Fourier integral operators (SG FIOs, for short, in the sequel), that is, the class of FIOs defined through symbols belonging to the so-called SG classes.

The class $S^{m,\mu}(\mathbb{R}^{2n})$ of SG symbols of order $(m, \mu) \in \mathbb{R}^2$ is given by all the functions $a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with the property that, for any multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$, there exist constants $C_{\alpha\beta} > 0$ such that the conditions

$$|D_\xi^\alpha D_x^\beta a(x, \xi)| \leqslant C_{\alpha\beta} \langle x \rangle^{m-|\beta|} \langle \xi \rangle^{\mu-|\alpha|}, \qquad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{1.1}$$

hold. Here $\langle x \rangle = (1 + |x|^2)^{1/2}$ when $x \in \mathbb{R}^n$, and $\mathbb{Z}_+$ is the set of non-negative integers. These classes, together with corresponding classes of pseudo-differential operators $\mathrm{Op}(S^{m,\mu})$, were first introduced in the '70s by H.O. Cordes [10] and C. Parenti [27], see also R. Melrose [26]. They form a graded algebra with respect to composition, i.e.,

$$\mathrm{Op}(S^{m_1,\mu_1}) \circ \mathrm{Op}(S^{m_2,\mu_2}) \subseteq \mathrm{Op}(S^{m_1+m_2,\mu_1+\mu_2}),$$

2010 *Mathematics Subject Classification.* Primary: 58J40; Secondary: 35S05, 35S30, 47G30, 58J45.

*Key words and phrases.* Fourier integral operator, multi-product, phase function, hyperbolic first order systems.

whose residual elements are operators with symbols in

$$S^{-\infty,-\infty}(\mathbb{R}^{2n}) = \bigcap_{(m,\mu)\in\mathbb{R}^2} S^{m,\mu}(\mathbb{R}^{2n}) = \mathcal{S}(\mathbb{R}^{2n}),$$

that is, those having kernel in $\mathcal{S}(\mathbb{R}^{2n})$, continuously mapping $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Operators in $\mathrm{Op}(S^{m,\mu})$ are continuous on $\mathcal{S}(\mathbb{R}^n)$, and extend uniquely to continuous operators on $\mathcal{S}'(\mathbb{R}^n)$ and from $H^{s,\sigma}(\mathbb{R}^n)$ to $H^{s-m,\sigma-\mu}(\mathbb{R}^n)$, where $H^{r,\varrho}(\mathbb{R}^n)$, $r,\varrho\in\mathbb{R}$, denotes the weighted Sobolev (or Sobolev-Kato) space

$$H^{r,\varrho}(\mathbb{R}^n) = \{u\in\mathcal{S}'(\mathbb{R}^n)\colon \|u\|_{r,\varrho} = \|\langle .\rangle^r\langle D\rangle^\varrho u\|_{L^2} < \infty\}.$$

An operator $A = \mathrm{Op}(a)$, is called *elliptic* (or $S^{m,\mu}$*-elliptic*) if $a\in S^{m,\mu}(\mathbb{R}^{2n})$ and there exists $R\geqslant 0$ such that

$$C\langle x\rangle^m\langle\xi\rangle^\mu \leqslant |a(x,\xi)|, \qquad |x|+|\xi|\geqslant R,$$

for some constant $C>0$. An elliptic SG operator $A\in\mathrm{Op}(S^{m,\mu})$ admits a parametrix $P\in\mathrm{Op}(S^{-m,-\mu})$ such that

$$PA = I+K_1, \quad AP = I+K_2,$$

for suitable $K_1,K_2\in\mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^{2n}))$, where $I$ denotes the identity operator. In such a case, $A$ turns out to be a Fredholm operator on the scale of functional spaces $H^{r,\varrho}(\mathbb{R}^n)$, $r,\varrho\in\mathbb{R}$.

In 1987, E. Schrohe [29] introduced a class of non-compact manifolds, the so-called SG manifolds, on which a version of SG calculus can be defined. Such manifolds admit a finite atlas, whose changes of coordinates behave like symbols of order $(0,1)$ (see [29] for details and additional technical hypotheses). A relevant example of SG manifolds are the manifolds with cylindrical ends, where also the concept of classical SG operator makes sense, see, e. g. [7, 15, 20, 23, 25, 26]. With $\widehat{u}$ denoting the Fourier transform of $u\in\mathcal{S}(\mathbb{R}^n)$, given by

$$\widehat{u}(\xi) = \int e^{-ix\cdot\xi}u(x)\,dx, \tag{1.2}$$

for any $a\in S^{m,\mu}(\mathbb{R}^{2n})$, $\varphi\in\mathcal{P}$ – the set of SG phase functions, see Section 2 below –, the SG FIOs are defined, for $u\in\mathcal{S}(\mathbb{R}^n)$, as

$$u\mapsto (\mathrm{Op}_\varphi(a)u)(x) = (2\pi)^{-n}\int e^{i\varphi(x,\xi)}a(x,\xi)\widehat{u}(\xi)\,d\xi, \tag{1.3}$$

and

$$u\mapsto (\mathrm{Op}^*_\varphi(a)u)(x) = (2\pi)^{-n}\iint e^{i(x\cdot\xi-\varphi(y,\xi))}\overline{a(y,\xi)}u(y)\,dyd\xi. \tag{1.4}$$

Here the operators $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(a)$ are sometimes called SG FIOs of type I and type II, respectively, with symbol $a$ and SG phase function $\varphi$. Note that a type II operator satisfies $\mathrm{Op}^*_\varphi(a) = \mathrm{Op}_\varphi(a)^*$, that is, it is the formal $L^2$-adjoint of the type I operator $\mathrm{Op}_\varphi(a)$.

The analysis of SG FIOs started in [11], where composition results with the corresponding classes of pseudodifferential operators, and of SG FIOs of type I and type II with regular phase functions, have been proved, as well as the basic continuity properties in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ of operators in the class. A version of the Asada-Fujiwara $L^2(\mathbb{R}^n)$-continuity theorem was also proved there, for operators $\mathrm{Op}_\varphi(a)$ with symbol $a\in S^{0,0}(\mathbb{R}^{2n})$ and regular SG phase function $\varphi\in\mathcal{P}_r$, see Definition 2.4. Applications to SG hyperbolic Cauchy problems were initially given in [12, 17].

Many authors have, since then, expanded the SG FIOs theory in various directions. To mention a few, see, e.g., G.D. Andrews [1], M. Ruzhansky, M. Sugimoto

[28], E. Cordero, F. Nicola, L Rodino [9], and the recent works by S. Coriasco and M. Ruzhansky [18], S. Coriasco and R. Schulz [19, 20]. Concerning applications to SG hyperbolic problems and propagation of singularities, see, e.g., A. Ascanelli and M. Cappiello [2, 3, 4], M. Cappiello [8], S. Coriasco, K. Johansson, J, Toft [13], S. Coriasco, L. Maniccia [14]. Concerning applications to anisotropic evolution equations of Schrödinger type see, e.g., A. Ascanelli, M. Cappiello [5].

Here our aim is to expand the results in [11, 12], through the study of the composition of $M \geqslant 2$ SG FIOs $A_j := \mathrm{Op}_{\varphi_j}(a_j)$ with *regular* SG phase functions $\varphi_j \in \mathcal{P}_r(\tau_j)$ – see Definition 2.4 below – and symbols $a_j \in S^{m_j,\mu_j}(\mathbb{R}^{2n})$, $j = 1, \dots, M$. To our best knowledge, the composition of SG FIOs with different phase functions of the type that we consider in this paper has not been studied by other authors.

First, we shall prove, under suitable assumptions, the existence of a SG phase function $\phi \in \mathcal{P}_r(\tau)$, called the *multi-product* of the SG phase functions $\varphi_1, \dots, \varphi_M$, and of a symbol $a \in S^{m,\mu}(\mathbb{R}^{2n})$, with $m := m_1 + \cdots + m_M$, $\mu := \mu_1 + \cdots + \mu_M$, such that

$$A = \mathrm{Op}_{\phi}(a) := A_1 \circ \cdots \circ A_M, \tag{1.5}$$

see Theorem 4.3 below for the precise statement.

Subsequently, we apply such result to study a class of hyperbolic Cauchy problems. We focus on first order systems of partial differential equations of hyperbolic type with $(t,x)-$depending coefficients in SG classes. By means of Theorem 4.3, we construct the fundamental solution $\{E(t,s)\}_{0\leqslant s\leqslant t\leqslant T}$ to the system. The existence of the fundamental solution provides, via Duhamel's formula, existence and uniqueness of the solution to the system, for any given Cauchy data in the weighted Sobolev spaces $H^{r,\varrho}(\mathbb{R}^n)$. A remarkable feature, typical for these classes of hyperbolic problems, is the *well-posedness with loss/gain of decay at infinity*, observed for the first time in [2], see also Section 5 below. We need these results in the study of certain stochastic equations, which will be treated in the forthcoming paper [6].

This paper is organized as follows. Section 2 is devoted to fixing notation and recalling some basic definitions and known results on SG symbols and Fourier integral operators, which will be used throughout the paper. In Section 3 we perform the first step of the proof of our main result, Theorem 4.3, defining and studying the multi-product of $M \geqslant 2$ regular SG phase functions. In Section 4 we prove Theorem 4.3, showing the existence, under suitable hypotheses, of $\phi \in \mathcal{P}_r$ and $a \in S^{m,\mu}$ such that (1.5) holds. Finally, in Section 5 we obtain the fundamental solution to SG hyperbolic first order systems.

## Acknowledgements

The authors were supported by the INdAM-GNAMPA grant "Equazioni Differenziali a Derivate Parziali di Evoluzione e Stocastiche" (Coordinator: S. Coriasco, Dep. of Mathematics "G. Peano", University of Turin).

## 2. SG symbols and Fourier integral operators

In this section we fix some notation and recall some of the results proved in [11], which will be used below. SG pseudodifferential operators $a(x,D) = \mathrm{Op}(a)$ can be introduced by means of the usual left-quantization

$$(\mathrm{Op}(a)u)(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

with $\widehat{u}$ the Fourier transform of $u$ defined in (1.2), starting from symbols $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying (1.1). Symbols of this type belong to the class denoted by $S^{m,\mu}(\mathbb{R}^{2n})$, and the corresponding operators constitute the class $\mathrm{Op}(S^{m,\mu}(\mathbb{R}^{2n}))$. In

the sequel we will often simply write $S^{m,\mu}$, fixing the dimension of the base space to $n$. For $m, \mu \in \mathbb{R}$, $l \in \mathbb{Z}_+$, $a \in S^{m,\mu}$, the quantities

$$|||a|||_l^{m,\mu} = \max_{|\alpha+\beta|\leqslant l} \sup_{x,\xi\in\mathbb{R}^n} \langle x\rangle^{-m+|\alpha|}\langle\xi\rangle^{-\mu+|\beta|}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|$$

are a family of seminorms, defining the Fréchet topology of $S^{m,\mu}$. The continuity properties of the elements of $\mathrm{Op}(S^{m,\mu})$ on the scale of spaces $H^{r,\rho}$, $m, \mu, r, \rho \in \mathbb{R}$, is expressed more precisely in the next Theorem 2.1 (see [10] and the references quoted therein for the result on more general classes of SG type symbols).

**Theorem 2.1.** *Let $a \in S^{m,\mu}(\mathbb{R}^n)$, $m, \mu \in \mathbb{R}$. Then, for any $r, \rho \in \mathbb{R}$, $\mathrm{Op}(a) \in \mathcal{L}(H^{r,\rho}(\mathbb{R}^n), H^{r-m,\rho-\mu}(\mathbb{R}^n))$, and there exist a constant $C > 0$ and an integer $\ell$, depending only on $n, m, \mu, r, \rho$, such that*

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(H^{r,\rho}(\mathbb{R}^n),H^{r-m,\rho-\mu}(\mathbb{R}^n))} \leqslant C|||a|||_\ell^{m,\mu}. \tag{2.1}$$

We now introduce the class of SG phase functions. Here and in what follows, $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$, where $A \lesssim B$ means that $A \leqslant c \cdot B$, for a suitable constant $c > 0$.

**Definition 2.2** (*SG* phase function)**.** *A real valued function $\varphi \in C^\infty(\mathbb{R}^{2n})$ belongs to the class $\mathcal{P}$ of SG phase functions if it satisfies the following conditions:*

(1) $\varphi \in S^{1,1}(\mathbb{R}^{2n})$;
(2) $\langle\varphi'_x(x,\xi)\rangle \asymp \langle\xi\rangle$ *as* $|(x,\xi)| \to \infty$;
(3) $\langle\varphi'_\xi(x,\xi)\rangle \asymp \langle x\rangle$ *as* $|(x,\xi)| \to \infty$.

Functions of class $\mathcal{P}$ are those used in the construction of the SG FIOs calculus. The SG FIOs of type I and type II, $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(b)$, are defined as in (1.3) and (1.4), respectively, with $\varphi \in \mathcal{P}$ and $a, b \in S^{m,\mu}$. The next Theorem 2.3 about composition between SG pseudodifferential operators and SG FIOs was originally proved in [11], see also [13, 16, 22].

**Theorem 2.3.** *Let $\varphi \in \mathcal{P}$ and assume $p \in S^{t,\tau}(\mathbb{R}^{2n})$, $a, b \in S^{m,\mu}(\mathbb{R}^{2n})$. Then,*

$$\mathrm{Op}(p)\circ\mathrm{Op}_\varphi(a) = \mathrm{Op}_\varphi(c_1 + r_1) = \mathrm{Op}_\varphi(c_1) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^{2d})),$$

$$\mathrm{Op}(p)\circ\mathrm{Op}^*_\varphi(b) = \mathrm{Op}^*_\varphi(c_2 + r_2) = \mathrm{Op}^*_\varphi(c_2) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^{2d})),$$

$$\mathrm{Op}_\varphi(a)\circ\mathrm{Op}(p) = \mathrm{Op}_\varphi(c_3 + r_3) = \mathrm{Op}_\varphi(c_3) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^{2d})),$$

$$\mathrm{Op}^*_\varphi(b)\circ\mathrm{Op}(p) = \mathrm{Op}^*_\varphi(c_4 + r_4) = \mathrm{Op}^*_\varphi(c_4) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^{2d})),$$

*for some $c_j \in S^{m+t,\mu+\tau}(\mathbb{R}^{2n})$, $r_j \in S^{-\infty,-\infty}(\mathbb{R}^{2d})$, $j = 1, \dots, 4$.*

To obtain the composition of SG FIOs of type I and type II, some more hypotheses are needed, leading to the definition of the classes $\mathcal{P}_r$ and $\mathcal{P}_r(\tau)$ of regular SG phase functions.

**Definition 2.4** (Regular *SG* phase function)**.** *Let $\tau \in [0, 1)$ and $r > 0$. A function $\varphi \in \mathcal{P}$ belongs to the class $\mathcal{P}_r(\tau)$ if it satisfies the following conditions:*

(1) $|\det(\varphi''_{x\xi})(x,\xi)| \geqslant r$, $\forall(x,\xi)$;
(2) *the function $J(x,\xi) := \varphi(x,\xi) - x\cdot\xi$ is such that*

$$\sup_{\substack{x,\xi\in\mathbb{R}^n\\ |\alpha+\beta|\leqslant 2}} \frac{|D_\xi^\alpha D_x^\beta J(x,\xi)|}{\langle x\rangle^{1-|\beta|}\langle\xi\rangle^{1-|\alpha|}} \leqslant \tau. \tag{2.2}$$

*If only condition (1) holds, we write $\varphi \in \mathcal{P}_r$.*

**Remark 2.5.** *Notice that condition (2.2) means that $J(x,\xi)/\tau$ is bounded with constant 1 in $S^{1,1}$. Notice also that condition (1) in Definition 2.4 is authomatically fulfilled when condition (2) holds true for a sufficiently small $\tau \in [0,1)$.*

For $\ell \in \mathbb{N}$, we also introduce the seminorms

$$\|J\|_{2,\ell} := \sum_{2\leqslant|\alpha+\beta|\leqslant 2+\ell} \sup_{(x,\xi)\in\mathbb{R}^{2n}} \frac{|D_\xi^\alpha D_x^\beta J(x,\xi)|}{\langle x\rangle^{1-|\beta|}\langle\xi\rangle^{1-|\alpha|}},$$

and

$$\|J\|_{\ell} := \sup_{\substack{x,\xi\in\mathbb{R}^n\\ |\alpha+\beta|\leqslant 1}} \frac{|D_\xi^\alpha D_x^\beta J(x,\xi)|}{\langle x\rangle^{1-|\beta|}\langle\xi\rangle^{1-|\alpha|}} + \|J\|_{2,\ell}.$$

We notice that $\varphi \in \mathcal{P}_r(\tau)$ means that (1) of Definition 2.4 and $\|J\|_0 \leqslant \tau$ hold, and then we define the following subclass of the class of regular SG phase functions:

**Definition 2.6.** *Let $\tau \in [0,1)$, $r > 0$, $\ell \geqslant 0$. A function $\varphi$ belongs to the class $\mathcal{P}_r(\tau,\ell)$ if $\varphi \in \mathcal{P}_r(\tau)$ and $\|J\|_\ell \leqslant \tau$ for the corresponding $J$.*

Theorem 2.7 below shows that the composition of SG FIOs of type I and type II with the same regular SG phase functions is a SG pseudodifferential operator, see [11] for a detailed proof.

**Theorem 2.7.** *Let $\varphi \in \mathcal{P}_r$ and assume $a \in S^{m,\mu}(\mathbb{R}^{2n})$, $b \in S^{t,\tau}(\mathbb{R}^{2n})$. Then,*

$$\mathrm{Op}_\varphi(a)\circ\mathrm{Op}_\varphi^*(b) = \mathrm{Op}(c_5 + r_5) = \mathrm{Op}(c_5) \mod \mathrm{Op}(S^{-\infty,-\infty}),$$

$$\mathrm{Op}_\varphi^*(b)\circ\mathrm{Op}_\varphi(a) = \mathrm{Op}(c_6 + r_6) = \mathrm{Op}(c_6) \mod \mathrm{Op}(S^{-\infty,-\infty}),$$

*for some $c_j \in S^{m+t,\mu+\tau}(\mathbb{R}^{2n})$, $r_j \in S^{-\infty,-\infty}(\mathbb{R}^{2d})$, $j = 5,6$.*

Furthermore, asymptotic formulae can be given for $c_j$, $j = 1,\dots,6$, in terms of $\varphi$, $p$, $a$ and $b$, see [11]. A generalization of Theorems 2.3 and 2.7 to operators defined by means of broader, generalized SG classes was proved in [13, 22], together with similar asymptotic expansions, studied by means of the criteria obtained in [21].

**Remark 2.8.** *In particular, in Section 5 we will make use of the following (first order) expansion of the symbol of $c_1$, coming from [11]:*

$$c_1(x,\xi) = p(x,\varphi_x'(x,\xi))a(x,\xi) + s(x,\xi), \quad s \in S^{m+t-1,\mu+\tau-1}(\mathbb{R}^{2n}).$$

Finally, when $a \in S^{m,\mu}$ is elliptic and $\varphi \in \mathcal{P}_r$, the corresponding SG FIOs admit a parametrix, that is, there exist $b_1, b_2 \in S^{-m,-\mu}$ such that

$$\mathrm{Op}_\varphi(a)\circ\mathrm{Op}_\varphi^*(b_1) = \mathrm{Op}_\varphi^*(b_1)\circ\mathrm{Op}_\varphi(a) = I \mod \mathrm{Op}(S^{-\infty,-\infty}), \tag{2.3}$$

$$\mathrm{Op}_\varphi^*(a)\circ\mathrm{Op}_\varphi(b_2) = \mathrm{Op}_\varphi(b_2)\circ\mathrm{Op}_\varphi^*(a) = I \mod \mathrm{Op}(S^{-\infty,-\infty}), \tag{2.4}$$

where $I$ is the identity operator, see again [11, 13, 22].

In this paper we extend the existing theory of SG FIOs, dealing with the composition of SG FIOs of type I with different phase functions. We then apply it to compute the fundamental solution to SG hyperbolic systems with coefficients of polynomial growth.

The following result is going to be used in Sections 3 and 5. Given a symbol $a \in C([0,T];S^{\epsilon,1})$ with $\epsilon \in [0,1]$, let us consider the eikonal equation

$$\begin{cases} \partial_t\varphi(t,s,x,\xi) = a(t,x,\varphi_x'(t,s,x,\xi)), & t \in [0,T_0]\\ \varphi(s,s,x,\xi) = x\cdot\xi, & s \in [0,T_0], \end{cases} \tag{2.5}$$

with $0 < T_0 \leqslant T$. By an extension of the theory developed in [12], it is possible to prove that the following Proposition 2.9 holds true.

**Proposition 2.9.** *For any small enough $T_0 \in [0,T]$, equation (2.5) admits a unique solution $\varphi \in C^1([0,T_0]^2_{t,s}, S^{1,1}(\mathbb{R}^n_{x,\xi}))$, satisfying $J \in C^1([0,T_0]^2_{t,s}, S^{\epsilon,1}(\mathbb{R}^n_{x,\xi}))$ and*

$$\partial_s \varphi(t,s,x,\xi) = -a(s,\varphi'_\xi(t,s,x,\xi),\xi), \tag{2.6}$$

*for any $t,s \in [0,T_0]$. Moreover, for every $h \geqslant 0$ there exist $c_h \geqslant 1$ and $T_h \in [0,T_0]$ such that $\varphi(t,s,x,\xi) \in \mathcal{P}_r(c_h|t-s|)$, with $\|J\|_{2,h} \leqslant c_h|t-s|$ for all $0 \leqslant s \leqslant t \leqslant T_h$.*

In the sequel we will sometimes write $\varphi_{ts}(x,\xi) := \varphi(t,s,x,\xi)$, for a solution $\varphi$ of (2.5).

## 3. Multiproducts of SG phase functions

The first step in our construction is to define the multi-product of regular SG phase functions and to analyze its properties, which we perform in the present section, following mainly [24].

Let us consider a sequence $\{\varphi_j\}_{j\geqslant 1}$ of regular SG phase functions $\varphi_j(x,\xi) \in \mathcal{P}_r(\tau_j)$ with

$$\sum_{j=1}^{\infty} \tau_j =: \tau_0 < 1/4. \tag{3.1}$$

By Definition 2.4 and assumption (3.1) we have that the sequence $\{J_k(x,\xi)/\tau_k\}_{k\geqslant 1}$ is bounded in $S^{1,1}$ and for every $\ell \in \mathbb{N}$ that there exists a constant $c_\ell > 0$ such that

$$\|J_k\|_{2,\ell} \leqslant c_\ell \tau_k \quad \text{and} \quad \sum_{k=1}^{\infty} \|J_k\|_{2,\ell} \leqslant c_\ell \tau_0. \tag{3.2}$$

Notice that from (2.2) we have $c_0 = 1$. This will be useful in the proof of Theorem 3.10 at the end of the present section.

**Example 3.1.** *A simple realization of a sequence $\{\varphi_j\}_{j\geqslant 1}$ satisfying (3.1) and (2.2) can be obtained using the phase function $\varphi(t,s,x,\xi)$ solving the eikonal equation (2.5). Indeed, it is sufficient to take a partition*

$$s = t_{\ell+1} \leqslant t_\ell \leqslant \cdots \leqslant t_1 \leqslant t_0 = t,$$

*of the interval $[s,t]$ and define*

$$\varphi_j(x,\xi) = \begin{cases} \varphi(t_{j-1},t_j,x,\xi) & 1 \leqslant j \leqslant \ell+1 \\ x\cdot\xi & j \geqslant \ell+2. \end{cases}$$

*In fact, from Proposition 2.9 we know that $\varphi_j \in \mathcal{P}_r(\tau_j)$ with $\tau_j = c_0(t_{j-1}-t_j)$ for $1 \leqslant j \leqslant \ell+1$ and with $\tau_j = 0$ for $j \geqslant \ell+2$. Condition (3.1) is fulfilled if we choose $T_0$ small enough, since*

$$\sum_{j=1}^{\infty} \tau_j = \sum_{j=1}^{\ell+1} c_0(t_{j-1}-t_j) = c_0(t-s) \leqslant c_0 T_0 < \frac{1}{4}$$

*if $T_0 < (4c_0)^{-1}$. Moreover, again from Proposition 2.9, we know that $\|J_j\|_{2,0} \leqslant c_0|t_j - t_{j-1}| = \tau_j$ for all $1 \leqslant j \leqslant \ell+1$ and $J_j = 0$ for $j \geqslant \ell+2$, so each one of the $J_j$ satisfies (2.2).*

With a fixed integer $M \geqslant 1$, we denote

$$\begin{aligned}(X,\Xi) &= (x_0,x_1,\dots,x_M,\xi_1,\dots,\xi_M,\xi_{M+1}) := (x,T,\Theta,\xi),\\ (T,\Theta) &= (x_1,\dots,x_M,\xi_1,\dots,\xi_M),\end{aligned}$$

and define the function of $2(M+1)n$ real variables

$$\psi(X,\Xi) := \sum_{j=1}^{M} \big(\varphi_j(x_{j-1},\xi_j) - x_j\cdot\xi_j\big) + \varphi_{M+1}(x_M,\xi_{M+1}). \tag{3.3}$$

For every fixed $(x,\xi)\in\mathbb{R}^{2n}$, the critical points $(Y,N)=(Y,N)(x,\xi)$ of the function of $2Mn$ variables $\widetilde{\psi}(T,\Theta)=\psi(x,T,\Theta,\xi)$ are the solutions to the system

$$\begin{cases}\psi'_{\xi_j}(X,\Xi)=\varphi'_{j,\xi}(x_{j-1},\xi_j)-x_j=0 & j=1,\dots,M,\\ \psi'_{x_j}(X,\Xi)=\varphi'_{j+1,x}(x_j,\xi_{j+1})-\xi_j=0 & j=1,\dots,M,\end{cases}$$

in the unknowns $(T,\Theta)$. That is $(Y,N)=(Y_1,\dots,Y_M,N_1,\dots,N_M)(x,\xi)$ satisfies, if $M=1$,

$$\begin{cases}Y_1=\varphi'_{1,\xi}(x,N_1)\\ N_1=\varphi'_{2,x}(Y_1,\xi),\end{cases}\tag{3.4}$$

or, if $M\geqslant 2$,

$$\begin{cases}Y_1=\varphi'_{1,\xi}(x,N_1)\\ Y_j=\varphi'_{j,\xi}(Y_{j-1},N_j), & j=2,\dots,M\\ N_j=\varphi'_{j+1,x}(Y_j,N_{j+1}), & j=1,\dots,M-1\\ N_M=\varphi'_{M+1,x}(Y_M,\xi).\end{cases}\tag{3.5}$$

In the sequel we will only refer to the system (3.5), tacitly meaning (3.4) when $M=1$. Definition 3.2 below of the multi product of SG phase functions is analogous to the one given in [24] for (local) symbols of Hörmander type.

**Definition 3.2** (Multi-product of SG phase functions)**.** *If, for every fixed $(x,\xi)\in\mathbb{R}^{2n}$, the system (3.5) admits a unique solution $(Y,N)=(Y,N)(x,\xi)$, we define*

$$\phi(x,\xi)=(\varphi_1\sharp\cdots\sharp\varphi_{M+1})(x,\xi):=\psi(x,Y(x,\xi),N(x,\xi),\xi).\tag{3.6}$$

*The function $\phi$ is called multi-product of the SG phase functions $\varphi_1,\dots,\varphi_{M+1}$.*

**Example 3.3.** *The simplest case of a well-defined multi-product of SG phase functions is given by the sharp product $\varphi\sharp\varphi_0$, where $\varphi\in\mathcal{P}_r$ and $\varphi_0(x,\xi)=x\cdot\xi$. Indeed, the critical points $(Y,N)$ of the function*

$$\widetilde{\psi}(x_1,\xi_1)=\psi(x,x_1,\xi_1,\xi)=\varphi(x,\xi_1)-x_1\cdot\xi_1+x_1\cdot\xi$$

*are given by $(Y,N)(x,\xi)=(\varphi'_\xi(x,\xi),\xi)$. The multi-product $\varphi\sharp\varphi_0$ is so defined by*

$$\phi(x,\xi)=\psi(x,\varphi'_\xi(x,\xi),\xi,\xi)=\varphi(x,\xi)-\varphi'_\xi(x,\xi)(\xi-\xi)=\varphi(x,\xi).$$

*Similarly, the multi-product $\varphi_0\sharp\varphi$ is well defined. Indeed, the function*

$$\widetilde{\psi}(x_1,\xi_1)=\psi(x,x_1,\xi_1,\xi)=x\cdot\xi_1-x_1\cdot\xi_1+\varphi(x_1,\xi)$$

*has critical points $(Y,N)(x,\xi)=(x,\varphi'_x(x,\xi))$, and*

$$\phi(x,\xi)=\psi(x,x,\varphi'_x(x,\xi),\xi)=(x-x)\cdot\varphi'_x(x,\xi)+\varphi(x,\xi)=\varphi(x,\xi).$$

*Notice that we have proved here above that for every $\varphi\in\mathcal{P}_r$ the identity*

$$\varphi\sharp\varphi_0=\varphi_0\sharp\varphi=\varphi$$

*holds true. That is, the multi-product of SG phase functions defined in (3.6) admits the trivial phase function $\varphi_0(x,\xi)=x\cdot\xi$ as identity element.*

**Example 3.4.** *A situation where (3.6) is well defined, which is interesting for applications, see Section 5, is given by the multi-product of solutions to the eikonal equation (2.5) on different, neighboring time intervals. Indeed, the critical points $(Y,N)(x,\xi)$ of the function*

$$\widetilde{\psi}_{tsr}(x_1,\xi_1):=\psi_{tsr}(x,x_1,\xi_1,\xi)=\varphi(t,s,x,\xi_1)-x_1\cdot\xi_1+\varphi(s,r,x_1,\xi)$$

*are given by*

$$\begin{cases}\psi'_{rst,x_1}(x,x_1,\xi_1,\xi)=-\xi_1+\varphi'_x(s,r,x_1,\xi)=0\\ \psi'_{rst,\xi_1}(x,x_1,\xi_1,\xi)=\varphi'_\xi(t,s,x,\xi_1)-x_1=0.\end{cases}\tag{3.7}$$

*The Jacobian matrix with respect to $(x_1, \xi_1)$ of the system (3.7) is*

$$J(t,s,r,x,x_1,\xi_1,\xi) = \begin{pmatrix} \varphi''_{xx}(s,r,x_1,\xi) & -I \\ -I & \varphi''_{\xi\xi}(t,s,x,\xi_1) \end{pmatrix},$$

*where $I$ is the $(n\times n)$-dimensional unit matrix. By (2.5), $\det J(t,r,r,x,x_1,\xi_1,\xi) = 1$. Thus, taking a small interval $[0,T_0]$ such that $\det J(t,s,r,x,x_1,\xi_1,\xi) > 0$ for all $r,s,t$ such that $0 \leqslant r \leqslant s \leqslant t \leqslant T_0$ and all $(X,\Xi) \in \mathbb{R}^{4n}$, by the implicit function theorem it follows that the system (3.7) admits a unique solution $(Y,N)_{tsr} = (Y_{tsr}, N_{tsr})(x,\xi) = (Y(t,s,r,x,\xi), N(t,s,r,x,\xi))$. The multi-product*

$$\begin{aligned}\phi_{tsr}(x,\xi) &= \phi(t,s,r,x,\xi) = (\varphi_{ts} \sharp \varphi_{sr})(x,\xi) = \psi_{tsr}(x, Y_{tsr}(x,\xi), N_{tsr}(x,\xi),\xi)\\ &= \varphi(t,s,x,N_{tsr}(x,\xi)) - Y_{tsr}(x,\xi)\cdot N_{tsr}(x,\xi) + \varphi(s,r,Y_{tsr}(x,\xi),\xi)\end{aligned}$$

*is then well defined. Moreover, it is quite simple to show, in view of to Proposition 2.9, that the multi-product $\varphi_{ts} \sharp \varphi_{sr}$ satisfies the associative law*

$$\varphi_{ts} \sharp \varphi_{sr} = \varphi_{tr}, \quad 0 \leqslant r \leqslant s \leqslant t \leqslant T_0. \tag{3.8}$$

*Indeed, $\phi(t,s,r,x,\xi)$ does not depend on $s$:*

$$\begin{aligned}\frac{d}{ds}[\phi(t,s,r,x,\xi)] &= (\partial_s\varphi)(t,s,x,N_{tsr}(x,\xi)) + \varphi'_\xi(t,s,x,N_{tsr}(x,\xi))\cdot(\partial_s N)(t,s,r,x,\xi)\\ &\quad - (\partial_s Y)(t,s,r,x,\xi)\cdot N(t,s,r,x,\xi) - Y(t,s,r,x,\xi)\cdot(\partial_s N)(t,s,r,x,\xi)\\ &\quad + (\partial_t\varphi)(s,r,Y_{tsr}(x,\xi),\xi) + \varphi'_x(s,r,Y_{tsr}(x,\xi),\xi)\cdot(\partial_s Y)(t,s,r,x,\xi) = 0,\end{aligned}$$

*since, by (2.5), (2.6) and the definition (3.7) of the critical point $(Y,N)_{tsr}$, we have*

$$\begin{aligned}\varphi'_x(s,r,Y_{tsr}(x,\xi),\xi) &= N(t,s,r,x,\xi),\\ \varphi'_\xi(t,s,x,N_{tsr}(x,\xi)) &= Y(t,s,r,x,\xi),\\ (\partial_t\varphi)(s,r,Y_{tsr}(x,\xi),\xi) &= a(s,Y_{tsr}(x,\xi),\varphi'_x(s,r,Y_{tsr}(x,\xi),\xi))\\ &= a(s,Y_{tsr}(x,\xi),N_{tsr}(x,\xi)),\\ (\partial_s\varphi)(t,s,x,N_{tsr}(x,\xi)) &= -a(s,\varphi'_\xi(t,s,x,N_{tsr}(x,\xi)),N_{tsr}(x,\xi))\\ &= -a(s,Y_{tsr}(x,\xi),N_{tsr}(x,xi)).\end{aligned}$$

*This gives, with $\varphi_0(x,\xi) = x\cdot\xi$,*

$$\begin{aligned}(\varphi_{ts}\sharp\varphi_{sr})(x,\xi) &= \phi(t,s,r,x,\xi) = \phi(t,r,r,x,\xi) = (\varphi_{tr}\sharp\varphi_{rr})(x,\xi) = (\varphi_{tr}\sharp\varphi_0)(x,\xi)\\ &= \varphi_{tr}(x,\xi),\end{aligned}$$

*by Example 3.3, as claimed.*

Now we want to show that under assumption (3.1) the multi-product $\phi(x,\xi)$ of Definition 3.2 is well defined on $\mathbb{R}^{2n}$, and it is a regular SG phase function itself. To this aim, we switch from the system (3.5) in the unknown $(Y,N)$ to the equivalent system (3.10) in the unknown $(\tilde{Y},\tilde{N}) = (y_1,\dots,y_M,\eta_1,\dots,\eta_M) \in \mathbb{R}^{2Mn}$ as follows. Define

$$\begin{cases} z^0 := 0\\ z^j := \sum_{k=1}^{j} y_k\,, & j = 1,\dots,M\\ \zeta^j := \sum_{k=j}^{M}\eta_k\,, & j = 1,\dots,M\\ \zeta^{M+1} := 0,\end{cases} \tag{3.9}$$

and then consider the system

$$\begin{cases} y_k = J'_{k,\xi}(x+z^{k-1},\xi+\zeta^k), & k = 1,\dots,M\\ \eta_k = J'_{k+1,x}(x+z^k,\xi+\zeta^{j+1}), & k = 1,\dots,M.\end{cases} \tag{3.10}$$

We have that:

**Lemma 3.5.** *For every fixed $(x,\xi)\in\mathbb{R}^{2n}$, $(Y,N)(x,\xi)$ is a solution of (3.5) if and only if $(\bar Y,\bar N)(x,\xi)=(y_1,\dots,y_M,\eta_1,\dots,\eta_m)(x,\xi)$, defined by*

$$\begin{cases} y_1 = Y_1 - x \\ y_j = Y_j - Y_{j-1} \quad j=2,\dots,M \\ \eta_j = N_j - N_{j+1} \quad j=1,\dots,M-1 \\ \eta_M = N_M - \xi, \end{cases} \tag{3.11}$$

*is a solution of (3.10).*

*Proof.* Substituting (3.11) in (3.9), we immediately get the relation

$$\begin{cases} Y_j = x + z^j \\ N_j = \xi + \zeta^j. \end{cases} \tag{3.12}$$

By this, it follows that $(Y,N)$ is a solution of (3.5) if and only if

$$\begin{cases} x + z^j = \varphi'_{j,\xi}(x+z^{j-1},\xi+\zeta^j) & j=1,\dots,M \\ \xi + \zeta^j = \varphi'_{j+1,x}(x+z^j,\xi+\zeta^{j+1}) & j=1,\dots,M; \end{cases}$$

by substituting $\varphi_j(x,\xi)=J_j(x,\xi)+x\cdot\xi$ we obtain

$$\begin{cases} z^j - z^{j-1} = J'_{j,\xi}(x+z^{j-1},\xi+\zeta^j) & j=1,\dots,M \\ \zeta^j - \zeta^{j+1} = J'_{j+1,x}(x+z^j,\xi+\zeta^{j+1}) & j=1,\dots,M, \end{cases}$$

which is exactly (3.10), in view of (3.9). □

We are then reduced to prove the following Theorem 3.6.

**Theorem 3.6.** *Under the assumption (3.1), for every fixed $(x,\xi)\in\mathbb{R}^{2n}$ there exists a unique solution $(\bar Y,\bar N)(x,\xi)$ of (3.10). Moreover, the solution $(\bar Y,\bar N)$ satisfies*

$$|y_k| \leqslant \frac{4}{3}\tau_k\langle x\rangle, \qquad |\eta_k| \leqslant \frac{4}{3}\tau_{k+1}\langle\xi\rangle, \quad k=1,\dots,M, \tag{3.13}$$

*and the functions $z_j$ and $\zeta_j$ in (3.9) satisfy*

$$|z^j| \leqslant \frac{1}{3}\langle x\rangle, \qquad |\zeta^j| \leqslant \frac{1}{3}\langle\xi\rangle, \quad j=1,\dots,M. \tag{3.14}$$

**Remark 3.7.** *We aim at obtaining a solution $(Y,N)$ such that $\phi=\psi(.,Y,N,.)\in\mathcal{P}_r(\tau)$. By Definition 3.2, recalling that $\psi$ a smooth function, it is enough to show that $(Y,N)$ is of class $C^\infty(\mathbb{R}^{2n})$, that $Y_j\in S^{1,0}$, $N_j\in S^{0,1}$, and that $\langle Y_j(x,\xi)\rangle\asymp\langle x\rangle$ as $|x|\to\infty$, $\langle N_j(x,\xi)\rangle\asymp\langle\xi\rangle$ as $|\xi|\to\infty$. To get these last equivalences, it is sufficient to prove the existence of a constant $k\in(0,1)$ such that $|Y_j(x,\xi)-x|\leqslant k\langle x\rangle$ and $|N_j(x,\xi)-\xi|\leqslant k\langle\xi\rangle$. Indeed, the following implication holds:*

$$|b|\leqslant k\langle a\rangle,\ k\in(0,1),\ a,b\in\mathbb{R}^n \Longrightarrow (1-k)\langle a\rangle\leqslant\langle a+b\rangle\leqslant(1+k)\langle a\rangle. \tag{3.15}$$

*Formula (3.14) gives precisely the desired estimates, with $k=1/3$, owing to (3.12). Theorem 3.6 then ensures that the multi-product is well-defined. We show that $(Y,N)\in C^\infty(\mathbb{R}^{2n})$ in the subsequent Theorem 3.8.*

*Proof of Theorem 3.6.* We divide the proof into two steps. In step one we suppose the existence of a solution $(\bar Y,\bar N)$ of (3.10) and prove that such solution satisfies (3.13) and that (3.14) holds. In step two we show, by a fixed point argument, the existence and uniqueness of the solution $(\bar Y,\bar N)$.
*Step 1.* If $(\bar Y,\bar N)$ is a solution of (3.10), then by (3.10) and (2.2) we get, for any $(x,\xi)\in\mathbb{R}^{2n}$,

$$\begin{cases} |y_k| \leqslant \tau_k\langle x+z^{k-1}\rangle \\ |\eta_k| \leqslant \tau_{k+1}\langle\xi+\zeta^{k+1}\rangle \end{cases}$$

for $k = 1, \dots, M$. Now, using the inequality

$$\langle x + y \rangle \leqslant \langle x \rangle + |y| \quad \forall x, y \in \mathbb{R}^n \tag{3.16}$$

and definition (3.9), we get, for $k = 1, \dots, M$ and any $(x, \xi) \in \mathbb{R}^{2n}$,

$$\begin{cases} |y_k| \leqslant \tau_k \left( \langle x \rangle + |z^{k-1}| \right) \leqslant \tau_k \left( \langle x \rangle + \sum_{j=1}^{M} |y_j| \right), \\ |\eta_k| \leqslant \tau_{k+1} \left( \langle \xi \rangle + |\zeta^{k+1}| \right) \leqslant \tau_{k+1} \left( \langle \xi \rangle + \sum_{j=1}^{M} |\eta_j| \right), \end{cases} \tag{3.17}$$

so that

$$\begin{cases} \sum_{k=1}^{M} |y_k| \leqslant \sum_{k=1}^{M} \tau_k \left( \langle x \rangle + \sum_{k=1}^{M} |y_k| \right) =: \bar{\tau}_M \left( \langle x \rangle + \sum_{k=1}^{M} |y_k| \right), \\ \sum_{k=1}^{M} |\eta_k| \leqslant \sum_{k=1}^{M} \tau_{k+1} \left( \langle \xi \rangle + \sum_{k=1}^{M} |\eta_k| \right) =: \bar{\tau}_{M+1} \left( \langle \xi \rangle + \sum_{k=1}^{M} |\eta_k| \right). \end{cases} \tag{3.18}$$

The two inequalities here above are of the form $\alpha \leqslant \tau(\langle x \rangle + \alpha)$ with $\tau < \tau_0 < 1/4$ by assumption (3.1), so they give

$$\alpha \leqslant \frac{\tau}{1 - \tau} \langle x \rangle < \frac{1}{3} \langle x \rangle,$$

and, coming back to (3.18), we have, for any $(x, \xi) \in \mathbb{R}^{2n}$,

$$|z^k| \leqslant \sum_{j=1}^{k} |y_j| < \frac{1}{3} \langle x \rangle, \qquad |\zeta^k| \leqslant \sum_{j=k}^{M} |\eta_j| < \frac{1}{3} \langle \xi \rangle,$$

that is (3.14). Substituting in (3.17) we obtain

$$|y_k| \leqslant \tau_k \left( \langle x \rangle + \frac{1}{3} \langle x \rangle \right) = \frac{4}{3} \tau_k \langle x \rangle, \qquad |\eta_k| \leqslant \tau_{k+1} \left( \langle \xi \rangle + \frac{1}{3} \langle \xi \rangle \right) = \frac{4}{3} \tau_{k+1} \langle \xi \rangle,$$

that is (3.14).

*Step 2.* Since we have shown that every solution $(\bar{Y}, \bar{N})$ of (3.10) satisfies (3.14) for any $(x, \xi) \in \mathbb{R}^{2n}$, to show existence and uniqueness of a solution to (3.10) in $\mathbb{R}^{2Mn}$ it is sufficient to show existence and uniqueness of $(\bar{Y}, \bar{N})$ in the space

$$\Sigma = \Sigma_{x,\xi} := \left\{ (y_1, \dots, y_M, \eta_1, \dots, \eta_M) \in \mathbb{R}^{2Mn} : \sum_{k=1}^{M} |y_k| \leqslant \frac{1}{3} \langle x \rangle, \ \sum_{k=1}^{M} |\eta_k| \leqslant \frac{1}{3} \langle \xi \rangle \right\},$$

$(x, \xi) \in \mathbb{R}^{2n}$, which is a metric space with norm

$$\|(y_1, \dots, y_M, \eta_1, \dots, \eta_M)\|_\Sigma := \sum_{k=1}^{M} \left( \langle x \rangle^{-1} |y_k| + \langle \xi \rangle^{-1} |\eta_k| \right).$$

We define the map

$$T = T_{x,\xi} : \Sigma \longrightarrow \Sigma$$

by $T(y_1, \dots, y_M, \eta_1, \dots, \eta_M) := (w_1, \dots, w_M, \omega_1, \dots, \omega_M)$, where, for $k = 1, \dots, M$, $(x, \xi) \in \mathbb{R}^{2n}$,

$$\begin{cases} w_k = J'_{k,\xi}(x + z^{k-1}, \xi + \zeta^k) \\ \omega_k = J'_{k+1,x}(x + z^k, \xi + \zeta^{k+1}). \end{cases}$$

The map $T$ is well defined. Indeed, by (2.2), (3.16) and (3.14) we have, for any $(x,\xi)\in\mathbb{R}^{2n}$,

$$\begin{cases} |w_k| \leqslant \tau_k\langle x+z^{k-1}\rangle \leqslant \tau_k(\langle x\rangle+\frac{1}{3}\langle x\rangle) = \frac{4}{3}\tau_k\langle x\rangle \\ |\omega_k| \leqslant \tau_{k+1}\langle \xi+\zeta^{k+1}\rangle \leqslant \tau_{k+1}(\langle \xi\rangle+\frac{1}{3}\langle \xi\rangle) = \frac{4}{3}\tau_{k+1}\langle \xi\rangle, \end{cases} \tag{3.19}$$

so that

$$\sum_{k=1}^{M}|w_k| \leqslant \frac{4}{3}\langle x\rangle\cdot\sum_{k=1}^{M}\tau_k < \frac{1}{3}\langle x\rangle, \quad \text{and} \quad \sum_{k=1}^{M}|\omega_k| \leqslant \frac{4}{3}\langle \xi\rangle\cdot\sum_{k=1}^{M}\tau_{k+1} < \frac{1}{3}\langle \xi\rangle.$$

By (3.10), to show existence and uniqueness of $(\bar{Y},\bar{N}) = (\bar{Y},\bar{N})(x,\xi)$ is equivalent to show existence and uniqueness of a fixed point $(\bar{Y},\bar{N})$ of the map $T$. We show here below that, under assumption (3.1), $T$ is a contraction on $\Sigma$, so it admits a unique fixed point $(\bar{Y},\bar{N})$.

Let us consider two arbitrary points

$$(Y,N) = (y_1,\dots,y_M,\eta_1,\dots,\eta_M),\ (\widetilde{Y},\widetilde{N}) = (\widetilde{y}_1,\dots,\widetilde{y}_M,\widetilde{\eta}_1,\dots,\widetilde{\eta}_M)\in\Sigma,$$

and let

$$T(Y,N) = (w_1,\dots,w_M,\omega_1,\dots,\omega_M),\ T(\widetilde{Y},\widetilde{N}) = (\widetilde{w}_1,\dots,\widetilde{w}_M,\widetilde{\omega}_1,\dots,\widetilde{\omega}_M).$$

For every fixed $k=1,\dots,M$, $(x,\xi)\in\mathbb{R}^{2n}$, we have

$$\begin{aligned} \widetilde{w}_k - w_k &= J'_{k,\xi}(x+\widetilde{z}^{k-1},\xi+\widetilde{\zeta}^k) - J'_{k,\xi}(x+z^{k-1},\xi+\zeta^k) \\ &= (\widetilde{z}^{k-1}-z^{k-1})\int_0^1 J''_{k,\xi x}(x+z^{k-1}+\theta(\widetilde{z}^{k-1}-z^{k-1}),\xi+\zeta^k)d\theta \\ &+ (\widetilde{\zeta}^k-\zeta^k)\int_0^1 J''_{k,\xi\xi}(x+z^{k-1},\xi+\zeta^k+\theta(\widetilde{\zeta}^k-\zeta^k))d\theta \end{aligned}$$

and from (2.2) we get

$$|\widetilde{w}_k - w_k| \leqslant \tau_k\left(|\widetilde{z}^{k-1}-z^{k-1}| + |\widetilde{\zeta}^k-\zeta^k|\langle x+z^{k-1}\rangle\int_0^1\langle \xi+\zeta^k+\theta(\widetilde{\zeta}^k-\zeta^k)\rangle^{-1}d\theta\right).$$

By inequality (3.15) with $b=z^k$ and $k=1/3$ we get $\frac{2}{3}\langle x\rangle \leqslant \langle x+z^k\rangle \leqslant \frac{4}{3}\langle x\rangle$; the same inequality with $b=\zeta^k+\theta(\widetilde{\zeta}^k-\zeta^k)$ and $k=1/3$ gives $\frac{2}{3}\langle \xi\rangle \leqslant \langle \xi+\zeta^k+\theta(\widetilde{\zeta}^k-\zeta^k)\rangle \leqslant \frac{4}{3}\langle \xi\rangle$; substituting these inequalities into the estimate of $|\widetilde{w}_k-w_k|$ we come to

$$\begin{aligned} |\widetilde{w}_k - w_k| &\leqslant \tau_k\left(|\widetilde{z}^{k-1}-z^{k-1}| + |\widetilde{\zeta}^k-\zeta^k|2\langle x\rangle\int_0^1\langle \xi\rangle^{-1}d\theta\right) \\ &\leqslant \tau_k\sum_{j=1}^{M}\left(|\widetilde{y}_j-y_j| + |\widetilde{\eta}_j-\eta_j|2\langle x\rangle\langle \xi\rangle^{-1}\right). \end{aligned}$$

Similarly:

$$\begin{aligned}
|\widetilde{\omega}_k - \omega_k| &\leqslant |\widetilde{z}^k - z^k| \left| \int_0^1 J''_{k+1,xx}(x + z^k + \theta(\widetilde{z}^k - z^k), \xi + \zeta^{k+1}) d\theta \right| \\
&\quad + |\widetilde{\zeta}^{k+1} - \zeta^{k+1}| \left| \int_0^1 J''_{k+1,x,\xi}(x + z^k, \xi + \zeta^{k+1} + \theta(\widetilde{\zeta}^{k+1} - \zeta^{k+1})) d\theta \right| \\
&\leqslant \tau_{k+1} \left( |\widetilde{z}^k - z^k| 2\langle x \rangle^{-1} \langle \xi \rangle + |\widetilde{\zeta}^{k+1} - \zeta^{k+1}| \right) \\
&\leqslant \tau_{k+1} \sum_{j=1}^{M} \left( |\widetilde{y}_j - y_j| 2\langle x \rangle^{-1} \langle \xi \rangle + |\widetilde{\eta}_{k+1} - \eta_{k+1}| \right).
\end{aligned}$$

Thus

$$\begin{aligned}
\|T(Y,N) - T(\widetilde{Y},\widetilde{N})\|_\Sigma &= \sum_{k=1}^{M} \left( \langle x \rangle^{-1} |\widetilde{w}_k - w_k| + \langle \xi \rangle^{-1} |\widetilde{\omega}_k - \omega_k| \right) \\
&\leqslant \sum_{k=1}^{M} \left( \tau_k \sum_{j=1}^{M} \left( \langle x \rangle^{-1} |\widetilde{y}_j - y_j| + 2\langle \xi \rangle^{-1} |\widetilde{\eta}_j - \eta_j| \right) \right. \\
&\qquad \left. + \tau_{k+1} \sum_{j=1}^{M} \left( |\widetilde{y}_j - y_j| 2\langle x \rangle^{-1} + |\widetilde{\eta}_j - \eta_j| \langle \xi \rangle^{-1} \right) \right) \\
&\leqslant \sum_{k=1}^{M} \max\{\tau_k, \tau_{k+1}\} 3 \sum_{j=1}^{M} \left( |\widetilde{y}_j - y_j| \langle x \rangle^{-1} + |\widetilde{\eta}_j - \eta_j| \langle \xi \rangle^{-1} \right) \\
&\leqslant 3\tau_0 \|(Y,N) - (\widetilde{Y},\widetilde{N})\|_\Sigma.
\end{aligned}$$

This shows that the map $T$ is Lipschitz continuous, with Lispchitz constant $3\tau_0 < 1$. It follows that $T$ is a strict contraction on $\Sigma$, which then admits a unique fixed point $(\bar{Y},\bar{N}) \in \Sigma$, for any $(x,\xi) \in \mathbb{R}^{2n}$. Such fixed point obviously gives the unique solution of (3.10). The proof is complete. □

**Theorem 3.8.** *The unique solution* $(\bar{Y},\bar{N}) = (\bar{Y},\bar{N})(x,\xi)$ *of* (3.10) *is of class* $C^\infty(\mathbb{R}^{2n})$.

*Proof.* For $(Y,N) \in \mathbb{R}^{2Mn}$ and $(x,\xi) \in \mathbb{R}^{2n}$, we define the function

$$F(Y,N;x,\xi) := (F_1,\dots,F_M)(Y,N;x,\xi),$$

with values in $\mathbb{R}^{2M}$, where for all $k = 1,\dots,M$,

$$F_k(Y,N;x,\xi) := \left( y_k - J'_{k,\xi}(x + z^{k-1}, \xi + \zeta^k),\ \eta_k - J'_{k+1,x}(x + z^k, \xi + \zeta^{k+1}) \right).$$

We apply the implicit function Theorem to the function $F$, which is clearly of class $C^\infty$ with respect to all variables, being $J_k$ a $C^\infty$ function for all $k = 1,\dots,M$. For every fixed $(x,\xi)$ we have that

$$F((\bar{Y},\bar{N})(x,\xi);x,\xi) = 0,$$

since $(\bar{Y},\bar{N})$ is the solution of (3.10). Moreover, we are going to prove here below that

$$\det\left( \frac{\partial F}{\partial(Y,N)}((\bar{Y},\bar{N})(x,\xi);x,\xi) \right) \neq 0. \tag{3.20}$$

This means that the implicitly defined function $(\bar{Y},\bar{N})(x,\xi)$ has the same regularity as $F$, so it is of class $C^\infty(\mathbb{R}^{2n})$. To complete the proof, it remains only to show that (3.20) holds true.

Let us compute the entries of the $2M \times 2M$ matrix $\frac{\partial F}{\partial(Y,N)}(Y,N;x,\xi)$. For every fixed $k = 1,\dots,M$, $(x,\xi) \in \mathbb{R}^{2n}$, we have

$$F'_{k,y_j}(Y,N;x,\xi)=\begin{cases}\Big(-J''_{k,\xi x}(x+z^{k-1},\xi+\zeta^k),\ -J''_{k+1,xx}(x+z^k,\xi+\zeta^{k+1})\Big), & 1\leqslant j\leqslant k-1\\ \Big(1,\ -J''_{k+1,xx}(x+z^k,\xi+\zeta^{k+1})\Big), & j=k\\ (0,\ 0), & k+1\leqslant j\leqslant M,\end{cases}$$

and

$$F'_{k,\eta_j}(Y,N;x,\xi)=\begin{cases}(0,\ 0), & 1\leqslant j\leqslant k-1\\ \Big(-J''_{k,\xi\xi}(x+z^{k-1},\xi+\zeta^k),\ 1\Big), & j=k\\ \Big(-J''_{k,\xi\xi}(x+z^{k-1},\xi+\zeta^k),\ -J''_{k+1,x\xi}(x+z^k,\xi+\zeta^{k+1})\Big), & k+1\leqslant j\leqslant M,\end{cases}$$

so we can write

$$\frac{\partial F}{\partial(Y,N)}(Y,N;x,\xi) = \begin{pmatrix} I-H_{11}(Y,N;x,\xi) & -H_{12}(Y,N;x,\xi)\\ -H_{21}(Y,N;x,\xi) & I-H_{22}(Y,N;x,\xi)\end{pmatrix},$$

where $I$ stands for the identity $M\times M$ matrix, and

$$H_{1,1} = \begin{pmatrix} 0 & 0 & \cdots & 0\\ J''_{2,\xi x} & 0 & \ddots & 0\\ \cdots & \cdots & \ddots & \vdots\\ J''_{M,\xi x} & \cdots & J''_{M,\xi x} & 0\end{pmatrix},\quad H_{1,2} = \begin{pmatrix} J''_{1,\xi\xi} & \cdots & \cdots & J''_{1,\xi\xi}\\ 0 & J''_{2,\xi\xi} & \cdots & J''_{2,\xi\xi}\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & J''_{M,\xi\xi}\end{pmatrix}$$

$$H_{2,1} = \begin{pmatrix} J''_{2,xx} & 0 & \cdots & 0\\ J''_{3,xx} & J''_{3,xx} & \cdots & 0\\ \vdots & \cdots & \ddots & \vdots\\ J''_{M+1,xx} & \cdots & \cdots & J''_{M+1,xx}\end{pmatrix},\quad H_{2,2} = \begin{pmatrix} 0 & J''_{2,x\xi} & \cdots & J''_{2,x\xi}\\ 0 & 0 & \cdots & \vdots\\ \vdots & \vdots & \ddots & J''_{M,x\xi}\\ 0 & \cdots & \cdots & 0\end{pmatrix}.$$

Let us estimate the matrix norm of each one of the $H_{ij}$:

$$\|H_{11}(Y,N;x,\xi)\| = \max_{j=1,\dots,M}\sum_{i=1}^{M}|(h_{11})_{ij}| \leqslant \max_{j=1,\dots,M}\sum_{i=j+1}^{M}\tau_i \leqslant \sum_{j=1}^{M}\tau_j$$

$$\|H_{12}(Y,N;x,\xi)\| = \max_{j=1,\dots,M}\sum_{i=1}^{M}|(h_{12})_{ij}| \leqslant \max_{j=1,\dots,M}\sum_{i=1}^{j}\tau_i\langle x+z^{i-1}\rangle\langle\xi+\zeta^i\rangle^{-1}$$

$$\|H_{21}(Y,N;x,\xi)\| = \max_{j=1,\dots,M}\sum_{i=1}^{M}|(h_{21})_{ij}| \leqslant \max_{j=1,\dots,M}\sum_{i=j}^{M}\tau_{i+1}\langle x+z^{i}\rangle^{-1}\langle\xi+\zeta^{i+1}\rangle$$

$$\|H_{22}(Y,N;x,\xi)\| = \max_{j=1,\dots,M}\sum_{i=1}^{M}|(h_{22})_{ij}| \leqslant \max_{j=1,\dots,M}\sum_{i=1}^{j-1}\tau_{i+1} \leqslant \sum_{j=1}^{M}\tau_j.$$

With the choice $(Y,N) = (\bar{Y},\bar{N})(x,\xi)$ these estimates become, via formula (3.12) and Remark 3.7,

$$\|H_{11}((\bar{Y},\bar{N})(x,\xi);x,\xi)\| \leqslant \sum_{j=1}^{M}\tau_j,\qquad \|H_{12}((\bar{Y},\bar{N})(x,\xi);x,\xi)\| \leqslant 2\langle x\rangle\langle\xi\rangle^{-1}\sum_{i=1}^{M}\tau_i,$$

$$\|H_{21}((\bar{Y},\bar{N})(x,\xi);x,\xi)\| \leqslant 2\langle x\rangle^{-1}\langle\xi\rangle\sum_{i=1}^{M}\tau_i,\qquad \|H_{22}((\bar{Y},\bar{N})(x,\xi);x,\xi)\| \leqslant \sum_{j=1}^{M}\tau_j.$$

Now, since $\det(I - H_{11}) = 1$, being $H_{11}$ triangular with null diagonal, we have

$$\begin{aligned}
&\det \frac{\partial F}{\partial(Y,N)}((\bar{Y},\bar{N})(x,\xi);x,\xi)\\
&= \det\begin{pmatrix} I - H_{11}((\bar{Y},\bar{N})(x,\xi);x,\xi) & -\langle\xi\rangle\langle x\rangle^{-1}H_{12}((\bar{Y},\bar{N})(x,\xi);x,\xi)\\ -\langle x\rangle\langle\xi\rangle^{-1}H_{21}((\bar{Y},\bar{N})(x,\xi);x,\xi) & I - H_{22}((\bar{Y},\bar{N})(x,\xi);x,\xi)\end{pmatrix}\\
&= \det\left(I - \begin{pmatrix} H_{11}((\bar{Y},\bar{N})(x,\xi);x,\xi) & \langle\xi\rangle\langle x\rangle^{-1}H_{12}((\bar{Y},\bar{N})(x,\xi);x,\xi)\\ \langle x\rangle\langle\xi\rangle^{-1}H_{21}((\bar{Y},\bar{N})(x,\xi);x,\xi) & H_{22}((\bar{Y},\bar{N})(x,\xi);x,\xi)\end{pmatrix}\right)\\
&= \det(I - A(x,\xi)),
\end{aligned}$$

with

$$\begin{aligned}
\|A(x,\xi)\| &= \max\{\|H_{11}((\bar{Y},\bar{N})(x,\xi);x,\xi)\| + \|\langle x\rangle\langle\xi\rangle^{-1}H_{21}((\bar{Y},\bar{N})(x,\xi);x,\xi)\|,\\
&\qquad \|H_{22}((\bar{Y},\bar{N})(x,\xi);x,\xi)\| + \|\langle\xi\rangle\langle x\rangle^{-1}H_{12}((\bar{Y},\bar{N})(x,\xi);x,\xi)\|\}\\
&\leqslant 3\sum_{j=1}^{M}\tau_j \leqslant 3\tau_0 < \frac{3}{4},
\end{aligned}$$

and applying Proposition 3.9 below, cfr. [24], we get $\det(I - A(x,\xi)) \geqslant 4^{-2M} > 0$. That is, (3.20) holds true, and the proof is complete. □

**Proposition 3.9** (Proposition 5.3, page 336 in [24]). *Let $A = (a_{ij})_{1\leqslant i,j\leqslant \ell}$ be a real matrix and suppose that there exists a constant $c_0 \in [0,1)$ such that*

$$\|A\| := \max_{j=1,\dots,\ell}\sum_{i=1}^{\ell}|a_{ij}| \leqslant c_0.$$

*Then,*

$$(1 - c_0)^{\ell} \leqslant \det(I - A) \leqslant (1 + c_0)^{\ell}.$$

The following theorem gives crucial estimates of the unique $C^\infty$ solution $(Y,N)$ of (3.5).

**Theorem 3.10.** *Under the assumptions (3.1) and (2.2), the unique $C^\infty$ solution $(Y,N)(x,\xi)$ of (3.5) satisfies:*

$$|\partial_\xi^\alpha\partial_x^\beta(Y_j - Y_{j-1})(x,\xi)| \leqslant c_{\alpha,\beta}\tau_j\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{1-|\beta|}, \tag{3.21}$$

$$|\partial_\xi^\alpha\partial_x^\beta(N_j - N_{j+1})(x,\xi)| \leqslant c_{\alpha,\beta}\tau_{j+1}\langle\xi\rangle^{1-|\alpha|}\langle x\rangle^{-|\beta|}, \tag{3.22}$$

*for all $\alpha,\beta \in \mathbb{Z}_+^n$, $j = 1,\dots,M$, $x,\xi \in \mathbb{R}^n$, with constants $c_{\alpha,\beta}$ not depending on $j$ and $M$. Moreover,*

$$\{(Y_j - Y_{j-1})(x,\xi)/\tau_j\}_{j\geqslant 1} \text{ is bounded in } S^{0,1}, \tag{3.23}$$

$$\{(N_j - N_{j+1})(x,\xi)/\tau_{j+1}\}_{j\geqslant 1} \text{ is bounded in } S^{1,0}. \tag{3.24}$$

*Proof.* Estimates (3.21), (3.22) in the case $\alpha = \beta = 0$ have already been proved, see (3.13) and (3.11). To prove the same estimates for $|\alpha + \beta| \geqslant 1$, it is sufficient, by (3.13), (3.11) and (3.2), to show that the solution $(\bar{Y},\bar{N})(x,\xi)$ of (3.10) is such that

$$|\partial_\xi^\alpha\partial_x^\beta y_k(x,\xi)| \leqslant c_{\alpha,\beta}\|J_k\|_{2,|\alpha+\beta|-1}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{1-|\beta|}, \tag{3.25}$$

$$|\partial_\xi^\alpha\partial_x^\beta\eta_k(x,\xi)| \leqslant c_{\alpha,\beta}\|J_{k+1}\|_{2,|\alpha+\beta|-1}\langle\xi\rangle^{1-|\alpha|}\langle x\rangle^{-|\beta|}, \tag{3.26}$$

for $|\alpha + \beta| \geqslant 1$, $k = 1,\dots,M$, $x,\xi \in \mathbb{R}^n$. Estimates (3.25), (3.26) are going to be proved by induction on $N = |\alpha + \beta|$.
*Step $N = 1$.* We need to check (3.25), (3.26) for the first order derivatives. Let us start

with the derivatives with respect to $x$. By definition (3.10) of $y_k, \eta_k$, $k = 1, \dots, M$, $x, \xi \in \mathbb{R}^n$, we have

$$(3.27)\quad \begin{cases} y'_{k,x} = J''_{k,\xi x}(. + z^{k-1}, .. + \zeta^k)(1 + (z^{k-1})'_x) + J''_{k,\xi\xi}(. + z^{k-1}, .. + \zeta^k)(\zeta^k)'_x \\ \eta'_{k,x} = J''_{k+1,xx}(. + z^k, .. + \zeta^{k+1})(1 + (z^k)'_x) + J''_{k+1,x\xi}(. + z^k, .. + \zeta^{k+1})(\zeta^{k+1})'_x. \end{cases}$$

By (2.2), setting $h(x,\xi) = \langle x\rangle\langle\xi\rangle^{-1}$, we obtain

$$\begin{aligned} \|y'_{k,x}\| + h\cdot\|\eta'_{k,x}\| &\leqslant \tau_k\left\{1 + \|(z^{k-1})'_x\| + \langle . + z^{k-1}\rangle\langle .. + \zeta^k\rangle^{-1}\|(\zeta^k)'_x\|\right\} \\ &\quad + \tau_{k+1}\cdot h\cdot\left\{\langle . + z^k\rangle^{-1}\langle .. + \zeta^{k+1}\rangle(1 + \|(z^k)'_x\|) + \|(\zeta^{k+1})'_x\|\right\}; \end{aligned}$$

from (3.14) we have $\frac{2}{3}\langle x\rangle \leqslant \langle x + z^{k-1}\rangle \leqslant \frac{4}{3}\langle x\rangle$ and $\frac{2}{3}\langle\xi\rangle \leqslant \langle\xi + \zeta^k\rangle \leqslant \frac{4}{3}\langle\xi\rangle$, so we come to

$$\begin{aligned} \|y'_{k,x}\| + h\cdot\|\eta'_{k,x}\| &\leqslant \tau_k\left\{1 + \|(z^{k-1})'_x\| + 2\cdot h\cdot\|(\zeta^k)'_x\|\right\} \\ &\quad + \tau_{k+1}\left\{2 + 2\|(z^k)'_x\| + h\cdot\|(\zeta^{k+1})'_x\|\right\} \\ &\leqslant \tau_k\left\{1 + \sum_{k=1}^M\|y'_{k,x}\| + 2\cdot h\cdot\sum_{k=1}^M\|\eta'_{k,x}\|\right\} \\ &\quad + \tau_{k+1}\left\{2 + 2\sum_{k=1}^M\|y'_{k,x}\| + h\cdot\sum_{k=1}^M\|\eta'_{k,x}\|\right\}, \end{aligned}$$

where we have used also definition (3.9). Summing for $k = 1, \dots, M$, we get, for any $x, \xi \in \mathbb{R}^n$,

$$\begin{aligned} \sum_{k=1}^M\left(\|y'_{k,x}\| + h\cdot\|\eta'_{k,x}\|\right) &\leqslant \bar\tau_M\left\{1 + \sum_{k=1}^M\|y'_{k,x}\| + 2\cdot h\cdot\sum_{k=1}^M\|\eta'_{k,x}\|\right\} \\ &\quad + \bar\tau_{M+1}\left\{2 + 2\sum_{k=1}^M\|y'_{k,x}\| + h\cdot\sum_{k=1}^M\|\eta'_{k,x}\|\right\} \\ &\leqslant 3\bar\tau_{M+1}\left\{1 + \sum_{k=1}^M\left(\|y'_{k,x}\| + h\cdot\|\eta'_{k,x}\|\right)\right\}. \end{aligned}$$

This last inequality immediately gives

$$(3.28)\qquad \sum_{k=1}^M\left(\|y'_{k,x}\| + h\cdot\|\eta'_{k,x}\|\right) \leqslant \frac{3\tau_{M+1}}{1 - 3\tau_{M+1}} \leqslant \frac{3\tau_0}{1 - 3\tau_0}$$

with $1 - 3\tau_0 > 1 - 3/4 = 1/4 > 0$, so that the amount (3.28) is finite (bounded by 3). Coming back to (3.27) and substituting there the estimate here above we get

$$\begin{aligned} \|y'_{k,x}\| &\leqslant \|J_k\|_{2,0}\left\{1 + \|(z^{k-1})'_x\| + 2\cdot h\cdot\|\eta'_{k,x}\|\right\} \\ &\leqslant 2\|J_k\|_{2,0}\left\{1 + \sum_{k=1}^M\left(\|y'_{k,x}\| + h\cdot\|\eta'_{k,x}\|\right)\right\} \\ &\leqslant 2\|J_k\|_{2,0}\left(1 + \frac{3\tau_0}{1 - 3\tau_0}\right) =: c_{0,1}\|J_k\|_{2,0}, \end{aligned}$$

that is (3.25) with $\alpha = 0$ and $|\beta| = 1$. With similar computations we obtain

$$\begin{aligned}
\|\eta'_{k,x}(x,\xi)\| &\leqslant \|J_{k+1}\|_{2,0}\left(\langle .+z^k\rangle^{-1}\langle .+\zeta^{k+1}\rangle(1+\|(z^k)'_x\|)+\|(\zeta^{k+1})'_x\|\right)(x,\xi)\\
&\leqslant \|J_{k+1}\|_{2,0}\left(2\cdot h^{-1}(1+\|(z^k)'_x\|)+\|(\zeta^{k+1})'_x\|\right)(x,\xi)\\
&\leqslant 2\|J_{k+1}\|_{2,0}\left[h^{-1}\left(1+\|(z^k)'_x\|+h\cdot\|(\zeta^{k+1})'_x\|\right)\right](x,\xi)\\
&\leqslant 2\|J_{k+1}\|_{2,0}\left[h^{-1}\left(1+\sum_{k=1}^{M}\left(\|y'_{k,x}\|+h\cdot\|\eta'_{k,x}\|\right)\right)\right](x,\xi)\\
&\leqslant 2\langle x\rangle^{-1}\langle\xi\rangle\|J_{k+1}\|_{2,0}\left(1+\frac{3\tau_0}{1-3\tau_0}\right)\\
&= C_{0,1}\|J_{k+1}\|_{2,0}\langle x\rangle^{-1}\langle\xi\rangle,\quad x,\xi\in\mathbb{R}^n,
\end{aligned}$$

and also

$$\|y'_{k,\xi}(x,\xi)\|\leqslant C_{1,0}\|J_k\|_{2,0}\langle x\rangle\langle\xi\rangle^{-1},\qquad \|\eta'_{k,\xi}(x,\xi)\|\leqslant C_{1,0}\|J_{k+1}\|_{2,0},\quad x,\xi\in\mathbb{R}^n.$$

The step $N=1$ is complete.
*Step $N\rightsquigarrow N+1$.* Let us now suppose that (3.25), (3.26) hold for $1\leqslant|\alpha+\beta|\leqslant N$, $N\geqslant 1$, $x,\xi\in\mathbb{R}^n$, and prove the same estimates for $|\alpha+\beta|=N+1$. If we substitute (3.2) into (3.25), (3.26) we immediately get

$$|\partial_\xi^\alpha\partial_x^\beta y_k(x,\xi)|\leqslant c'_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{1-|\beta|},\tag{3.29}$$

$$|\partial_\xi^\alpha\partial_x^\beta \eta_k(x,\xi)|\leqslant c'_{\alpha,\beta}\langle\xi\rangle^{1-|\alpha|}\langle x\rangle^{-|\beta|},\tag{3.30}$$

for $1\leqslant|\alpha+\beta|\leqslant N$ and $k=1,\dots,M$. These estimates are going to be used to bound the derivatives $\partial_x^\beta\partial_\xi^\alpha$ with $|\alpha+\beta|=N$ of the functions $y'_{k,x}$, $y'_{k,\xi}$, $\eta'_{k,x}$, $\eta'_{k,\xi}$ (i.e. the derivatives $\partial_x^\beta\partial_\xi^\alpha$ with $|\alpha+\beta|=N+1$ of the functions $y_k,\eta_k$). Let us start by computing, from (3.27), the derivative

$$\begin{aligned}
\partial_x^\beta\partial_\xi^\alpha y'_{k,x} &= \partial_x^\beta\partial_\xi^\alpha\left[J''_{k,\xi x}(.+z^{k-1},..+\zeta^k)\cdot\left(1+(z^{k-1})'_x\right)\right]\\
&\quad+\partial_x^\beta\partial_\xi^\alpha\left[J''_{k,\xi\xi}(.+z^{k-1},..+\zeta^k)\cdot(\zeta^k)'_x\right].
\end{aligned}\tag{3.31}$$

To obtain an estimate of (3.31), we use Faá di Bruno formula, write the derivatives of $z^k$ and $\zeta^k$ as derivatives with respect to $y_k$ and $\eta_k$ by (3.9), and finally we apply (3.29), (3.30), obtaining

$$\begin{aligned}
&|\partial_x^\beta\partial_\xi^\alpha\left(J''_{k,\xi x}(x+z^{k-1}(x,\xi),\xi+\zeta^k(x,\xi))\right)|\\
&\leqslant \sum_{\substack{\beta_1+\cdots+\beta_r=\beta\\ \beta_i\neq 0}}\ \sum_{\substack{\alpha_1+\cdots+\alpha_q=\alpha\\ \alpha_i\neq 0}} C_{q,r,\alpha,\beta}\|J_k\|_{2,q+r}\langle\xi\rangle^{-q}\langle x\rangle^{-r}\cdot\\
&\qquad\cdot\langle\xi\rangle^{(1-|\alpha_1|)+\cdots+(1-|\alpha_q|)}\langle x\rangle^{(1-|\beta_1|)+\cdots+(1-|\beta_r|)}\\
&\leqslant C_{\alpha,\beta}\|J_k\|_{2,|\alpha+\beta|}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}
\end{aligned}$$

and

$$|\partial_x^\beta\partial_\xi^\alpha\left(J''_{k,\xi\xi}(x+z^{k-1}(x,\xi),\xi+\zeta^k(x,\xi))\right)|\leqslant C_{\alpha,\beta}\|J_k\|_{2,|\alpha+\beta|}\langle\xi\rangle^{-1-|\alpha|}\langle x\rangle^{1-|\beta|}.$$

Thus, coming back to (3.31), substituting these last two estimates and using (3.9) we come to

$$\begin{aligned}|\partial_x^\beta\partial_\xi^\alpha y'_{k,x}(x,\xi)| &\leqslant \|J_k\|_{2,0}\sum_{j=1}^M\Big(|\partial_x^\beta\partial_\xi^\alpha y'_{j,x}(x,\xi)| + 2\langle x\rangle\langle\xi\rangle^{-1}|\partial_x^\beta\partial_\xi^\alpha \eta'_{j,x}(x,\xi)|\Big)\\ &\quad + C'_{\alpha,\beta}\|J_k\|_{2,|\alpha+\beta|}\Big(\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|} + C''_{\alpha,\beta}\langle\xi\rangle^{-1-|\alpha|}\langle x\rangle^{1-|\beta|}\langle\xi\rangle\langle x\rangle^{-1}\Big)\\ &\leqslant \|J_k\|_{2,0}\sum_{j=1}^M\Big(|\partial_x^\beta\partial_\xi^\alpha y'_{j,x}(x,\xi)| + 2\langle x\rangle\langle\xi\rangle^{-1}|\partial_x^\beta\partial_\xi^\alpha \eta'_{j,x}(x,\xi)|\Big)\\ &\quad + \widetilde{C}_{\alpha,\beta}\|J_k\|_{2,|\alpha+\beta|}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}. \qquad (3.32)\end{aligned}$$

Working similarly on the terms $\partial_x^\beta\partial_\xi^\alpha\eta'_{k,x}$ coming from the derivatives in (3.27), we get the corresponding estimate:

$$\begin{aligned}|\partial_x^\beta\partial_\xi^\alpha \eta'_{k,x}(x,\xi)| &\leqslant \|J_{k+1}\|_{2,0}\langle x\rangle^{-1}\langle\xi\rangle\sum_{j=1}^M\Big(|\partial_x^\beta\partial_\xi^\alpha y'_{j,x}(x,\xi)| + 2\langle x\rangle\langle\xi\rangle^{-1}|\partial_x^\beta\partial_\xi^\alpha \eta'_{j,x}(x,\xi)|\Big)\\ &\quad + \widetilde{C}'_{\alpha,\beta}\|J_{k+1}\|_{2,|\alpha+\beta|}\langle\xi\rangle^{1-|\alpha|}\langle x\rangle^{-1-|\beta|}. \qquad (3.33)\end{aligned}$$

Now summing up for $k = 1,\dots,M$ inequalities (3.32) and (3.33) we have

$$\begin{aligned}&\sum_{k=1}^M\Big(|\partial_x^\beta\partial_\xi^\alpha y'_{k,x}(x,\xi)| + 2\langle x\rangle\langle\xi\rangle^{-1}|\partial_x^\beta\partial_\xi^\alpha \eta'_{k,x}(x,\xi)|\Big) \leqslant\\ &\leqslant\left(\sum_{k=1}^M\|J_k\|_{2,0} + 2\sum_{k=1}^M\|J_{k+1}\|_{2,0}\right)\cdot\sum_{k=1}^M\Big(|\partial_x^\beta\partial_\xi^\alpha y'_{k,x}(x,\xi)| + 2\langle x\rangle\langle\xi\rangle^{-1}|\partial_x^\beta\partial_\xi^\alpha \eta'_{k,x}(x,\xi)|\Big)\\ &\quad+\bar{C}_{\alpha,\beta}\left(\sum_{k=1}^M\|J_k\|_{2,|\alpha+\beta|} + 2\sum_{k=1}^M\|J_{k+1}\|_{2,|\alpha+\beta|}\right)\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}\\ &\leqslant 3c_0\tau_0\sum_{k=1}^M\Big(|\partial_x^\beta\partial_\xi^\alpha y'_{k,x}(x,\xi)| + 2\langle x\rangle\langle\xi\rangle^{-1}|\partial_x^\beta\partial_\xi^\alpha \eta'_{k,x}(x,\xi)|\Big) + 3c_{|\alpha+\beta|}\tau_0\bar{C}_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|},\end{aligned}$$

where $c_0, c_{|\alpha+\beta|}$ are the constants defined in (3.2). In particular, notice that, by (2.2), we have $c_0 = 1$. From this, we finally obtain

$$\begin{aligned}\sum_{k=1}^M\Big(|\partial_x^\beta\partial_\xi^\alpha y'_{k,x}(x,\xi)| + 2\langle x\rangle\langle\xi\rangle^{-1}|\partial_x^\beta\partial_\xi^\alpha \eta'_{k,x}(x,\xi)|\Big) &\leqslant \bar{C}'_{\alpha,\beta}\frac{\tau_0}{1-3\tau_0}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}\\ &< \bar{C}'_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|} \qquad (3.34)\end{aligned}$$

by the choice of $\tau_0$ in (3.1). Substituting (3.34) in (3.32) and (3.33) we get

$$|\partial_x^\beta\partial_\xi^\alpha y'_{k,x}(x,\xi)| \leqslant C_{\alpha,\beta}\|J_k\|_{2,|\alpha+\beta|}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|} \qquad (3.35)$$

$$|\partial_x^\beta\partial_\xi^\alpha \eta'_{k,x}(x,\xi)| \leqslant C_{\alpha,\beta}\|J_{k+1}\|_{2,|\alpha+\beta|}\langle\xi\rangle^{1-|\alpha|}\langle x\rangle^{-1-|\beta|}. \qquad (3.36)$$

All the computations from (3.31) to (3.36) on the functions $y'_{k,x}$ and $\eta'_{k,x}$ can be repeated on the functions $y'_{k,\xi}$ and $\eta'_{k,\xi}$ with minor changes. In this way we finally obtain the estimates corresponding to (3.35) and (3.36), namely

$$|\partial_x^\beta\partial_\xi^\alpha y'_{k,\xi}(x,\xi)| \leqslant C_{\alpha,\beta}\|J_k\|_{2,|\alpha+\beta|}\langle\xi\rangle^{-1-|\alpha|}\langle x\rangle^{1-|\beta|} \qquad (3.37)$$

$$|\partial_x^\beta\partial_\xi^\alpha \eta'_{k,\xi}(x,\xi)| \leqslant C_{\alpha,\beta}\|J_{k+1}\|_{2,|\alpha+\beta|}\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}. \qquad (3.38)$$

The proof is complete, since (3.35)-(3.38) are the desired estimates (3.25) and (3.26) for all the derivatives of order $N+1$ of the functions $y_k$ and $\eta_k$. □

We conclude with a Theorem that summarizes what we have proved throughout the present section, and gives the main properties of the multi-products of regular SG phase functions.

**Theorem 3.11.** *Under assumptions (3.1) and (2.2), the multi-product $\phi(x,\xi)$ of Definition 3.2 is well defined for every $M \geqslant 1$ and has the following properties.*

(1) *There exists $k \geqslant 1$ such that $\phi(x,\xi) = (\varphi_1 \sharp \cdots \sharp \varphi_{M+1})(x,\xi) \in \mathcal{P}_r(k\bar{\tau}_{M+1})$ and, setting*
$$J_{M+1}(x,\xi) := (\varphi_1 \sharp \cdots \sharp \varphi_{M+1})(x,\xi) - x\cdot\xi,$$
*the sequence $\{J_{M+1}/\bar{\tau}_{M+1}\}_{M\geqslant 1}$ is bounded in $S^{1,1}(\mathbb{R}^{2n})$.*

(2) *The following relations hold:*
$$\begin{cases} \phi'_x(x,\xi) = \varphi'_{1,x}(x, N_1(x,\xi)) \\ \phi'_\xi(x,\xi) = \varphi'_{M+1,\xi}(Y_M(x,\xi),\xi), \end{cases}$$
*where $(Y,N)$ is the critical point (3.5).*

(3) *The associative law holds: $\varphi_1 \sharp (\varphi_2 \sharp \cdots \sharp \varphi_{M+1}) = (\varphi_1 \sharp \cdots \sharp \varphi_M) \sharp \varphi_{M+1}$.*

(4) *For any $\ell \geqslant 0$ there exist $0 < \tau^* < 1/4$ and $c^* \geqslant 1$ such that, if $\varphi_j \in \mathcal{P}_r(\tau_j,\ell)$ for all $j$ and $\tau_0 \leqslant \tau^*$, then $\phi \in \mathcal{P}_r(c^*\bar{\tau}_{M+1},\ell)$.*

*Proof.* By theorems 3.6 and 3.8 we know that, for any $M \geqslant 1$, $\phi$ is a well-defined smooth function on $\mathbb{R}^{2n}$. We start by showing (1). We write, with $Y_0(x,\xi) = Y_{M+1}(x,\xi) := x$, $N_{M+1}(x,\xi) := \xi$,
$$\begin{aligned} J_{M+1}(x,\xi) &= \sum_{j=1}^{M} \big(\varphi_j(Y_{j-1}(x,\xi), N_j(x,\xi)) - Y_j(x,\xi)\cdot N_j(x,\xi)\big) \\ &\qquad + \varphi_{M+1}(Y_M(x,\xi),\xi) - x\cdot\xi \\ &= \sum_{j=1}^{M+1} \big(\varphi_j(Y_{j-1}, N_j) - Y_j\cdot N_j\big)(x,\xi) \\ &= \sum_{j=1}^{M+1} \big(J_j(Y_{j-1},N_j) - (Y_j - Y_{j-1})\cdot N_j\big)(x,\xi). \end{aligned}$$
This gives that
$$\frac{J_{M+1}}{\bar{\tau}_{M+1}} = \sum_{j=1}^{M+1} \frac{\tau_j}{\bar{\tau}_{M+1}} \left( \frac{J_j(Y_{j-1},N_j)}{\tau_j} - \frac{Y_j - Y_{j-1}}{\tau_j}\cdot N_j \right) \text{ is bounded in } S^{1,1}$$
since $\{J_j/\tau_j\}_{j\geqslant 1}$ is bounded in $S^{1,1}$, (3.23) holds, and $\langle N_j(x,\xi)\rangle \asymp \langle\xi\rangle$. Now, the boundedness proved here above implies the existence of a positive constant $k$ such that
$$\|J_{M+1}\|_2 \leqslant k\bar{\tau}_{M+1} < k\tau_0, \tag{3.39}$$
and taking $\tau_0$ small enough, so that $k\tau_0 < 1$, we obtain that $\phi \in \mathcal{P}_r(k\bar{\tau}_{M+1})$. Statement (1) is proved. Statement (4) immediately follows. Indeed, if $\varphi_j \in \mathcal{P}_r(\tau_j,\ell)$, then we have $\|J_{M+1}\|_\ell \leqslant (k+1)\bar{\tau}_{M+1}$, with $k$ coming from (3.39), and we obtain $\|J_{M+1}\|_\ell \leqslant c^*\bar{\tau}_{M+1}$ and $\phi \in \mathcal{P}_r(c^*\bar{\tau}_{M+1},\ell)$ if we choose $c^*$ such that $c^*\tau_0 < 1$. Let us now come to (2), which is quite simple. Indeed, from (3.2) and (3.5), we have
$$\phi(x,\xi) := \sum_{j=1}^{M} \big(\varphi_j(Y_{j-1}(x,\xi), N_j(x,\xi)) - Y_j(x,\xi)\cdot N_j(x,\xi)\big) + \varphi_{M+1}(Y_M(x,\xi),\xi).$$

A derivation of the expression above with respect to $x$ and the use of (3.5) give

$$\begin{aligned}\phi'_x(x,\xi) &= \sum_{j=1}^{M}\Big(\varphi'_{j,x}(Y_{j-1}(x,\xi),N_j(x,\xi))\cdot Y'_{j-1,x}(x,\xi)\\ &\qquad +\varphi'_{j,\xi}(Y_{j-1}(x,\xi),N_j(x,\xi))\cdot N'_{j,x}(x,\xi)\\ &\qquad -Y'_{j,x}(x,\xi)\cdot N_j(x,\xi)-Y_j(x,\xi)\cdot N'_{j,x}(x,\xi)\Big)\\ &+\varphi'_{M+1,x}(Y_M(x,\xi),\xi)\cdot Y'_{M,x}(x,\xi)\\ &=\varphi'_{1,x}(x,N_1(x,\xi))-Y'_{1,x}(x,\xi)\cdot N_1(x,\xi)\\ &+\sum_{j=2}^{M}\Big((Y'_{j-1,x}\cdot N_{j-1}-Y'_{j,x}\cdot N_j)+N_M\cdot Y'_{M,x}\Big)(x,\xi)\\ &=\varphi'_{1,x}(x,N_1(x,\xi)),\end{aligned}$$

which is exactly the first equality in (2). The second equality can be obtained similarly, by derivation with respect to $\xi$ of $\phi(x,\xi)$.
Finally, we deal with (3). We want to show that

$$(\varphi_1\sharp\varphi_2\sharp\cdots\sharp\varphi_M)\sharp\varphi_{M+1}=\varphi_1\sharp\cdots\sharp\varphi_{M+1}. \tag{3.40}$$

To this aim, let us denote

$$\widetilde{\phi}:=\varphi_1\sharp\cdots\sharp\varphi_M,$$

and compute by (3.3), with $M=1$, the product

$$(\widetilde{\phi}\sharp\varphi_{M+1})(x,\xi)=\widetilde{\phi}(x,\widetilde{N}(x,\xi))-\widetilde{Y}(x,\xi)\cdot\widetilde{N}(x,\xi)+\varphi_{M+1}(\widetilde{Y}(x,\xi),\xi), \tag{3.41}$$

where $(\widetilde{Y},\widetilde{N})=(\widetilde{Y},\widetilde{N})(x,\xi)$ is the $2n-$dimensional critical point given by

$$\begin{cases}\widetilde{Y}=\widetilde{\phi}'_\xi(x,\widetilde{N}),\\ \widetilde{N}=\varphi'_{M+1,x}(\widetilde{Y},\xi).\end{cases} \tag{3.42}$$

Notice that $\widetilde{\phi}\sharp\varphi_{M+1}$ is well-defined by (1) (eventually, with a smaller $\tau_0$). Now, we compute the value of $\widetilde{\phi}(x,\widetilde{N}(x,\xi))=(\varphi_1\sharp\cdots\sharp\varphi_M)(x,\widetilde{N}(x,\xi))$ in (3.41), using (3.3) with $M-1$ in place of $M$ and $\widetilde{N}$ in place of $\xi$, obtaining

$$\begin{aligned}&\widetilde{\phi}(x,\widetilde{N}(x,\xi))=\\ &=\sum_{j=1}^{M-1}\Big(\varphi_j(\bar{Y}_{j-1}(x,\widetilde{N}(x,\xi)),\bar{N}_j(x,\widetilde{N}(x,\xi)))-\bar{Y}_j(x,\widetilde{N}(x,\xi))\cdot\bar{N}_j(x,\widetilde{N}(x,\xi))\Big)\\ &+\varphi_M(\bar{Y}_{M-1}(x,\widetilde{N}(x,\xi)),\widetilde{N}(x,\xi)),\end{aligned} \tag{3.43}$$

with the $2(M-1)n-$dimensional critical point $(\bar{Y},\bar{N})$ given by

$$\begin{cases}\bar{Y}_0=x\\ \bar{Y}_j=\varphi'_{j,\xi}(\bar{Y}_{j-1},\bar{N}_j) & j=1,\dots,M-1\\ \bar{N}_j=\varphi'_{j+1,x}(\bar{Y}_j,\bar{N}_{j+1}) & j=1,\dots,M-1\\ \bar{N}_M=N,\end{cases} \tag{3.44}$$

obtained from (3.5), with $M-1$ in place of $M$ and $\widetilde{N}$ in place of $\xi$. Moreover, we have from (3.42) and (2), with $M-1$ in place of $M$, that

$$\begin{aligned}\widetilde{Y}(x,\xi)&=\widetilde{\phi}'_\xi(x,\widetilde{N}(x,\xi))=(\varphi_1\sharp\cdots\sharp\varphi_M)'_\xi(x,\widetilde{N}(x,\xi))\\ &=\varphi'_{M,\xi}(\bar{Y}_{M-1}(x,\widetilde{N}(x,\xi)),\widetilde{N}(x,\xi)).\end{aligned} \tag{3.45}$$

Summing up, from (3.45), the second equation in (3.42), and (3.44), we have that $(\bar{Y}_1, \cdots, \bar{Y}_{M-1}, \widetilde{Y}, \bar{N}_1, \cdots, \bar{N}_{M-1}, \widetilde{N})$ solves system (3.5), and thus it is the $2Mn-$dimensional critical point needed to define the multi-product $\varphi_1 \sharp \cdots \sharp \varphi_{M+1}$, which turns out to be given, in view of (3.3), by

$$
\begin{aligned}
&(\varphi_1 \sharp \cdots \sharp \varphi_{M+1})(x,\xi) = \\
&\quad = \sum_{j=1}^{M-1} \Big( \varphi_j(\bar{Y}_{j-1}(x,\widetilde{N}(x,\xi)), \bar{N}_j(x,\widetilde{N}(x,\xi))) - \bar{Y}_j(x,\widetilde{N}(x,\xi)) \cdot \bar{N}_j(x,\widetilde{N}(x,\xi)) \Big) \\
&\quad + \varphi_M(\bar{Y}_{M-1}(x,\widetilde{N}(x,\xi)), \widetilde{N}(x,\xi)) - \bar{Y}(x,\xi) \cdot \bar{N}(x,\xi) + \varphi_{M+1}(\widetilde{Y}(x,\xi),\xi).
\end{aligned}
$$

We observe that this last expression coincides with (3.41) after substituting (3.43) in it. This gives that $\varphi_1 \sharp \cdots \sharp \varphi_{M+1} = \widetilde{\phi} \sharp \varphi_{M+1}$, that is (3.40). Similarly, we can prove the corresponding law $\varphi_1 \sharp (\varphi_2 \sharp \cdots \sharp \varphi_{M+1}) = \varphi_1 \sharp \cdots \sharp \varphi_{M+1}$, completing the proof of (3). □

## 4. Composition of SG Fourier integral operators

We can now prove our main theorem on compositions of regular SG FIOs. We start with an invertibility result for $I_\varphi = \mathrm{Op}_\varphi(1)$ and $I_\varphi^* = \mathrm{Op}_\varphi^*(1)$ when $\varphi$ is a regular phase function. Theorem 4.1 below gives more precise versions of (2.3), (2.4), with a slight additional restriction on $\varphi$, for FIOs with constant, nonvanishing symbol.

**Theorem 4.1.** *Assume that $\varphi \in \mathcal{P}_r(\tau)$ with $0 < \tau < \frac{1}{4}$ sufficiently small. Then, there exists $q \in S^{0,0}(\mathbb{R}^{2n})$ such that*

$$
I_\varphi \circ \mathrm{Op}_\varphi^*(q) = \mathrm{Op}_\varphi^*(q) \circ I_\varphi = I, \tag{4.1}
$$

$$
I_\varphi^* \circ \mathrm{Op}_\varphi(q) = \mathrm{Op}_\varphi(q) \circ I_\varphi^* = I. \tag{4.2}
$$

*Moreover, if the family of SG phase functions $\{\varphi_s(x,\xi)\}$ is such that the family $\{J_s(x,\xi)\} = \{\varphi_s(x,\xi) - x\cdot\xi\}$ is bounded in $S^{1,1}$, then the corresponding family $\{q_s\}$ is also bounded in $S^{0,0}$.*

*Proof.* For $u \in \mathcal{S}(\mathbb{R}^n)$ we have, by definition of type I and type II SG FIOs,

$$
((I_\varphi \circ I_\varphi^*)u)(x) = (2\pi)^{-n} \iint e^{i(\varphi(x,\xi)-\varphi(y,\xi))}\, u(y)\, dy d\xi. \tag{4.3}
$$

The map

$$
\Xi_{x,y} \colon \xi \mapsto \Xi_{x,y}(\xi) = \Xi(x,y,\xi) = \int_0^1 \varphi_x'(x + t(y-x),\xi)\, dt
$$

is globally invertible on $\mathbb{R}^n$. In fact, its Jacobian is given by the matrix

$$
\int_0^1 \varphi_{x\xi}''(x + t(y-x),\xi)\, dt = I + \int_0^1 J_{x\xi}''(x + t(y-x),\xi)\, dt
$$

which has nonvanishing determinant, in view of the hypothesis $\varphi \in \mathcal{P}_r(\tau)$, $0 \leqslant \tau < \frac{1}{4}$. Moreover, condition (2) in Definition 2.2 implies that $\Xi$ is coercive, and these two properties give its global invertibility on $\mathbb{R}^n$, see [11, Theorems 11 and 12] and the references quoted therein. Finally, $\Xi_{x,y}$ is also a *SG diffeomorphism with 0-order parameter-dependence*, that is both $\Xi(x,y,\xi)$ and $\Xi^{-1}(x,y,\eta)$ belong to $S^{0,0,1}(\mathbb{R}^{3n})$, the space of *SG amplitudes* of order $(0,0,1)$, see [10, 11], and satisfy $\langle \Xi(x,y,\xi)\rangle \asymp \langle \xi\rangle$, $\langle \Xi^{-1}(x,y,\eta)\rangle \asymp \langle \eta\rangle$, uniformly with respect to $x, y \in \mathbb{R}^n$. In (4.3) we can then change variable, setting

$$
\eta = \Xi(x,y,\xi) \Leftrightarrow \xi = \Xi^{-1}(x,y,\eta),
$$

and obtain

$$((I_\varphi \circ I_\varphi^*)u)(x) = u(x) + (2\pi)^{-n} \iint e^{i(x-y)\cdot\eta} a_0(x,y,\eta)u(y)\,dyd\eta = ((I+A_0)u)(x),$$

with

$$a_0(x,y,\eta) = \det(I + J''_{x\xi}(x,y,\xi))^{-1}|_{\xi=\Xi^{-1}(x,y,\eta)} - 1.$$

By the results on composition of *SG functions* in [11, 22], we find that $a_0 \in S^{0,0,0}(\mathbb{R}^{3d})$, the space of SG-amplitudes of order $(0,0,0)$. Since the seminorms of $a_0$ can be controlled by means of the parameter $\tau$, and the map associating $a_0$ with the symbol $a \in S^{0,0}$ such that $A_0 = \mathrm{Op}(a)$ is continuous, the same holds for the seminorms of $a$. By general arguments, see [10, 24, 30, 31], it turns out that $(I + \mathrm{Op}(a))^{-1}$ exists in $\mathrm{Op}(S^{0,0})$. Then, setting $Q_\varphi^* = I_\varphi^* \circ (I + \mathrm{Op}(a))^{-1}$, using Theorem 2.3 we find $Q_\varphi^* = \mathrm{Op}_\varphi^*(q)$ for some $q \in S^{0,0}$ and $I_\varphi \circ \mathrm{Op}_\varphi^*(q) = I$, which is the first part of (4.1). The remaining statements follow by arguments analogous to those used in the proof of [24, Theorem 6.1]. □

The next Theorem 4.2 is one of our main results.

**Theorem 4.2.** *Let $\varphi_j \in \mathcal{P}_r(\tau_j)$, $j = 1,2$, be such that $0 \leqslant \tau_1 + \tau_2 \leqslant \tau \leqslant \frac{1}{4}$ for some sufficiently small $\tau > 0$. Then, there exists $p \in S^{0,0}(\mathbb{R}^{2n})$ such that*

$$I_{\varphi_1} \circ I_{\varphi_2} = \mathrm{Op}_{\varphi_1\sharp\varphi_2}(p), \tag{4.4}$$

$$I^*_{\varphi_2} \circ I^*_{\varphi_1} = \mathrm{Op}^*_{\varphi_1\sharp\varphi_2}(p). \tag{4.5}$$

*Moreover, if the families of SG phase functions $\{\varphi_{js}(x,\xi)\}$, $j = 1,2$, are such that the families $\{J_{js}(x,\xi)\} = \{\varphi_{js}(x,\xi) - x\cdot\xi\}$ are bounded in $S^{1,1}$, $j = 1,2$, then the corresponding family $\{p_s(x,\xi)\}$ is also bounded in $S^{0,0}$.*

We will achieve the proof of Theorem 4.2 through various intermediate results, adapting the analogous scheme in [24]. Before getting to that, let us first state and prove our main Theorem 4.3, which is obtained as a consequence of Theorems 4.1 and 4.2.

**Theorem 4.3.** *Let $\varphi_j \in \mathcal{P}_r(\tau_j)$, $j = 1,2,\dots,M$, $M \geqslant 2$, be such that $\tau_1 + \cdots + \tau_M \leqslant \tau \leqslant \frac{1}{4}$ for some sufficiently small $\tau > 0$, and set*

$$\begin{aligned}
\Phi_0(x,\xi) &= x\cdot\xi,\\
\Phi_1 &= \varphi_1,\\
\Phi_j &= \varphi_1\sharp\cdots\sharp\varphi_j,\ j = 2,\dots,M\\
\Phi_{M,j} &= \varphi_j\sharp\varphi_{j+1}\sharp\cdots\sharp\varphi_M, j = 1,\dots,M-1,\\
\Phi_{M,M} &= \varphi_M,\\
\Phi_{M,M+1}(x,\xi) &= x\cdot\xi.
\end{aligned}$$

*Assume also $a_j \in S^{m_j,\mu_j}(\mathbb{R}^{2n})$, and set $A_j = \mathrm{Op}_{\varphi_j}(a_j)$, $j = 1,\dots,M$. Then, the following holds true.*

(1) *Given $q_j, q_{M,j} \in S^{0,0}(\mathbb{R}^{2n})$, $j = 1,\dots,M$, such that*

$$\mathrm{Op}^*_{\Phi_j}(q_j) \circ I_{\Phi_j} = I, \quad I^*_{\Phi_{M,j}} \circ \mathrm{Op}_{\Phi_{M,j}}(q_{M,j}) = I,$$

*set $Q_j^* = \mathrm{Op}^*_{\Phi_j}(q_j)$, $Q_{M,j} = \mathrm{Op}_{\Phi_{M,j}}(q_{M,j})$, and*

$$R_j = I_{\Phi_{j-1}} \circ A_j \circ Q_j^*, \quad R_{M,j} = Q_{M,j} \circ A_j \circ I^*_{\Phi_{M,j+1}}, \quad j = 1,\dots,M.$$

*Then, $R_j, R_{M,j} \in \mathrm{Op}(S^{0,0}(\mathbb{R}^{2n}))$, $j = 1,\dots,M$, and*

$$A = A_1 \circ \cdots \circ A_M = R_1 \circ \cdots \circ R_M \circ I_{\Phi_M} = I^*_{\Phi_{M,1}} \circ R_{M,1} \circ \cdots \circ R_{M,M}. \tag{4.6}$$

(2) *There exists* $a \in S^{m,\mu}(\mathbb{R}^{2n})$, $m = m_1 + \cdots + m_M$, $\mu = \mu_1 + \cdots + \mu_M$ *such that, setting* $\phi = \varphi_1 \sharp \cdots \sharp \varphi_M$,

$$A = A_1 \circ \cdots \circ A_M = \mathrm{Op}_\phi(a).$$

(3) *For any* $l \in \mathbb{Z}_+$ *there exist* $l' \in \mathbb{Z}_+$, $C_l > 0$ *such that*

$$|||a|||_l^{m,\mu} \leqslant C_l \prod_{j=1}^{M} |||a_j|||_{l'}^{m_j,\mu_j}. \tag{4.7}$$

*Proof.* The existence of $q_j, q_{M,j} \in S^{0,0}$, $j = 1, \ldots, M$, with the desired properties follows from Theorem 4.1. We also notice that, trivially, $I_{\Phi_0} = I_{\Phi_{M,M+1}} = I$, so that, inserting either $I = Q_1^* \circ I_{\Phi_1} = \cdots = Q_M^* \circ I_{\Phi_M}$ or $I = I_{\Phi_{M,1}}^* \circ Q_{M,1} = \cdots = I_{\Phi_{M,M}}^* \circ Q_{M,M}$, we indeed find

$$\begin{aligned} A_1 \circ \cdots \circ A_M &= I_{\Phi_0} \circ A_1 \circ Q_1^* \circ I_{\Phi_1} \circ A_2 \circ \cdots \circ I_{\Phi_M} \circ A_M \circ Q_M^* \circ I_{\Phi_M} \\ &= R_1 \circ \cdots \circ R_M \circ I_{\Phi_M} \\ &= I_{\Phi_{M,1}}^* \circ Q_{M,1} \circ A_1 \circ I_{\Phi_{M,2}}^* \circ Q_{M,2} \circ A_2 \circ \cdots \circ I_{\Phi_{M,M}}^* \circ Q_{M,M} \circ A_M \circ I_{\Phi_{M,M+1}} \\ &= I_{\Phi_{M,1}}^* \circ R_{M,1} \circ \cdots \circ R_{M,M}, \end{aligned}$$

as claimed. Now, we observe that, again in view of Theorem 4.1, there exists $p_j \in S^{0,0}$ such that $I_{\varphi_j} \circ \mathrm{Op}_{\varphi_j}^*(p_j) = I$, $j = 1, \ldots, M$. Setting $P_j^* = \mathrm{Op}_{\varphi_j}^*(p_j)$, and inserting it into the definition of $R_j$, by Theorem 4.2 we then find, for $j = 1, \ldots, M$,

$$R_j = (I_{\Phi_{j-1}} \circ I_{\varphi_j}) \circ (P_j^* \circ A_j) \circ Q_j^* = I_{\Phi_{j-1} \sharp \varphi_j} \circ (P_j^* \circ A_j) \circ Q_j^* = I_{\Phi_j} \circ (P_j^* \circ A_j) \circ Q_j^*.$$

Theorem 2.7 implies that $P_j^* \circ A_j \in \mathrm{Op}(S^{m_j,\mu_j})$, and Theorem 2.3 then implies that $(P_j^* \circ A_j) \circ Q_j^* = \mathrm{Op}_{\Phi_j}^*(d_j)$, for some $d_j \in S^{m_j,\mu_j}$, $j = 1, \ldots, M$. Another application of Theorem 2.3 gives that

$$R_j = I_{\Phi_j} \circ \mathrm{Op}_{\Phi_j}^*(d_j) \in \mathrm{Op}(S^{m_j,\mu_j}),\ j = 1, \ldots, M,$$

so that the standard composition rules for SG pseudodifferential operators and a further application of Theorem 2.3 imply, for $\phi = \varphi_1 \sharp \cdots \sharp \varphi_M$ and a suitable $a \in S^{m,\mu}$,

$$A = A_1 \circ \cdots \circ A_M = \mathrm{Op}_\phi(a),$$

as claimed. Similar considerations hold for $R_{M,j}$, $j = 1, \ldots, M$ and the representation formula

$$A_1 \circ \cdots \circ A_M = I_{\Phi_{M,1}}^* \circ R_{M,1} \circ \cdots \circ R_{M,M}.$$

The estimate (4.7) follows from the composition results in [11], applied repeatedly to (4.6), observing that the amplitudes of the resulting operators depend continuously on those of the involved factors. The proof is complete. □

To start proving Theorem 4.2, with two SG phase functions $\varphi_1, \varphi_2$ as in the corresponding hypotheses and $u \in \mathcal{S}(\mathbb{R}^n)$, let us write, as it is possible,

$$[(I_{\varphi_1} \circ I_{\varphi_2})u](x) = \iiint e^{i(\varphi_1(x,\xi') - x' \cdot \xi' + \varphi_2(x',\xi))}\, \widehat{u}(\xi)\, \bar{d}\xi' dx' \bar{d}\xi.$$

Now, with $\phi = \varphi_1 \sharp \varphi_2$, set

$$\varphi_0(x, x', \xi', \xi) = \varphi_1(x, \xi') - x' \cdot \xi' + \varphi_2(x', \xi) - \phi(x, \xi), \tag{4.8}$$

and consider, in the sense of oscillatory integrals,

$$p(x, \xi) = \iint e^{i\varphi_0(x,x',\xi',\xi)}\, \bar{d}\xi' dx'. \tag{4.9}$$

Then, we can write

$$[(I_{\varphi_1}\circ I_{\varphi_2})u](x)=\int e^{i\phi(x,\xi)}\,p(x,\xi)\,\widehat{u}(\xi)\,đ\xi,\quad u\in\mathcal{S}(\mathbb{R}^n),$$

which gives the desired claim, if we show that (4.9) indeed defines a symbol $p\in S^{0,0}(\mathbb{R}^{2n})$. Let us now define the adapted cut-off functions which will be needed for the proof of this fact.

**Definition 4.4.** *We set*

$$\chi(x,x',\xi',\xi)=\chi_a(x,x')\cdot\chi_a(\xi,\xi'),$$

*where, with $a>0$ to be fixed later and $w,w'\in\mathbb{R}^n$, we assume*

$$\chi_a(w,w')=\psi(a(w-w')\langle w\rangle^{-1}),$$

*for a fixed cut-off function $\psi\in C_0^\infty(\mathbb{R}^n)$. In particular, we also assume that, for all $w\in\mathbb{R}^n$, $0\leqslant\psi(w)\leqslant 1$, $\operatorname{supp}\psi=B_{\frac{2}{3}}(0)$, $\psi|_{B_{\frac{1}{2}}(0)}\equiv 1$, $w\notin B_{\frac{1}{2}}(0)\Rightarrow 0\leqslant\psi(w)<1$, where $B_r(w_0)$ is the closed ball in $\mathbb{R}^n$ centered at $w_0$ with radius $r>0$.*

For the proof of the next lemma see, e.g., [11].

**Lemma 4.5.** *i) For any multi-indices $\gamma_1,\gamma_2\in\mathbb{Z}_+^n$, the function $\chi_a(w,w')$ introduced in Definition 4.4 satisfies, for all $w,w'\in\mathbb{R}^n$,*

$$|\partial_{w'}^{\gamma_1+\gamma_2}\chi_a(w,w')|\lesssim\langle w\rangle^{-|\gamma_1|}\langle w'\rangle^{-|\gamma_2|}. \tag{4.10}$$

*ii) For any multi-indices $\alpha_1\alpha_2,\beta_1,\beta_2\in\mathbb{Z}_+^n$, the function $\chi(x,x',\xi',\xi)$ introduced in Definition 4.4 satisfies, for all $x,x',\xi,\xi'$, the estimates*

$$|\partial_{x'}^{\alpha_1+\alpha_2}\partial_{\xi'}^{\beta_1+\beta_2}\chi(x,x',\xi',\xi)|\lesssim\langle x\rangle^{-|\alpha_1|}\langle x'\rangle^{-|\alpha_2|}\langle\xi\rangle^{-|\beta_1|}\langle\xi'\rangle^{-|\beta_2|}. \tag{4.11}$$

**Remark 4.6.** *In view of Definition 4.4,*

$$\begin{aligned}1-\chi(x,x',\xi',\xi)&=1-\chi_a(x,x')+\chi_a(x,x')-\chi_a(x,x')\cdot\chi_a(\xi,\xi')\\&=1-\chi_a(x,x')+\chi_a(x,x')\cdot(1-\chi_a(\xi,\xi')),\end{aligned}$$

*which implies that on* $\operatorname{supp}(1-\chi(x,x',\xi',\xi))$ *either* $|x-x'|\geqslant\dfrac{1}{2a}\langle x\rangle$ *or* $|\xi-\xi'|\geqslant\dfrac{1}{2a}\langle\xi\rangle$.

Now write $p$ in (4.9) as $p=p_0+p_\infty$ with

$$p_0(x,\xi)=\iint e^{i\varphi_0(x,x',\xi',\xi)}\,\chi(x,x',\xi',\xi)\,đ\xi'dx', \tag{4.12}$$

$$p_\infty(x,\xi)=\iint e^{i\varphi_0(x,x',\xi',\xi)}\,(1-\chi(x,x',\xi',\xi))\,đ\xi'dx'. \tag{4.13}$$

We analyze separately $p_0$ and $p_\infty$.

**Proposition 4.7.** *Under the hypotheses of Theorem 4.2, for $p_\infty$ defined in (4.13) we have $p_\infty\in S^{-\infty,-\infty}(\mathbb{R}^{2n})$.*

*Proof.* Define

$$\varphi_\infty(x,x',\xi',\xi)=\varphi_1(x,\xi')-x'\cdot\xi'+\varphi_2(x',\xi)-x\cdot\xi,$$

so we have from (4.8)

$$\varphi_0(x,x',\xi',\xi)=\varphi_\infty(x,x',\xi',\xi)+x\cdot\xi-\phi(x,\xi)$$

and

$$p_\infty(x,\xi)=e^{-iJ(x,\xi)}p'_\infty(x,\xi),$$

where we have set $J(x,\xi)=\phi(x,\xi)-x\cdot\xi$ and

$$\widetilde{p}_\infty(x,\xi)=\iint e^{i\varphi_\infty(x,x',\xi',\xi)}(1-\chi(x,x',\xi',\xi))\,đ\xi'dx'.$$

It is straightforward, since $J \in S^{1,1}$ for small $\tau > 0$, that it is enough to prove that $\widetilde{p}_\infty \in S^{-\infty,-\infty}$ to achieve the desired result. Also, in view of the definition of $\varphi_\infty$,

$$\begin{aligned}
\varphi'_{\infty,x}(x,x',\xi',\xi) &= \xi' - \xi + J'_{1,x}(x,\xi'),\\
\varphi'_{\infty,\xi'}(x,x',\xi',\xi) &= x - x' + J'_{1,\xi}(x,\xi'),\\
\varphi'_{\infty,x'}(x,x',\xi',\xi) &= \xi - \xi' + J'_{2,x}(x',\xi),\\
\varphi'_{\infty,\xi}(x,x',\xi',\xi) &= x' - x + J'_{2,\xi}(x',\xi).
\end{aligned}$$

Then, on $\operatorname{supp}(1-\chi(x,x',\xi',\xi))$, for a known $c > 0$ and a sufficiently small $\tau > 0$, depending on $\varphi_1$, $\varphi_2$, and $\chi$, there exist suitable $k_1, k_2 > 0$, such that either

$$\begin{aligned}
|\varphi'_{\infty,x'}(x,\xi',x',\xi)| &\geqslant |\xi - \xi'| - c\tau\langle\xi\rangle \geqslant |\xi - \xi'| - c\tau|\xi - \xi'| = (1-c\tau)|\xi - \xi'|\\
&\geqslant k_1(\langle\xi\rangle + \langle\xi'\rangle) > 0,
\end{aligned}$$

or

$$\begin{aligned}
|\varphi'_{\infty,\xi'}(x,\xi',x',\xi)| &\geqslant |x - x'| - c\tau\langle x\rangle \geqslant |x - x'| - c\tau|x - x'| = (1-c\tau)|x - x'|\\
&\geqslant k_2(\langle x\rangle + \langle x'\rangle) > 0.
\end{aligned}$$

Let us set, for $b > 2a > 0$,

(4.14)
$$\widetilde{p}_{1\infty}(x,\xi) = \iint e^{i\varphi_\infty(x,\xi',x',\xi)}\,(1-\chi(x,x',\xi',\xi))\cdot\chi_b(x,x')\,\bar{d}\xi' dx',$$

(4.15)
$$\widetilde{p}_{2\infty}(x,\xi) = \iint e^{i\varphi_\infty(x,\xi',x',\xi)}\,(1-\chi(x,x',\xi',\xi))\cdot(1-\chi_b(x,x'))\cdot\chi_b(\xi,\xi')\,\bar{d}\xi' dx',$$

(4.16)
$$\widetilde{p}_{3\infty}(x,\xi) = \iint e^{i\varphi_\infty(x,\xi',x',\xi)}\,(1-\chi(x,x',\xi',\xi))\cdot(1-\chi_b(x,x'))\cdot(1-\chi_b(\xi,\xi'))\,\bar{d}\xi' dx',$$

so that

$$\widetilde{p}_\infty(x,\xi) = \widetilde{p}_{1\infty}(x,\xi) + \widetilde{p}_{2\infty}(x,\xi) + \widetilde{p}_{3\infty}(x,\xi).$$

Then, the operator

$$T_V = -i|\varphi'_{\infty,x'}(x,x',\xi',\xi)|^{-2}\,\varphi'_{\infty,x'}(x,x',\xi',\xi)\cdot\nabla_{x'} = V(x,x',\xi',\xi)\cdot\nabla_{x'}$$

such that

$$T_V e^{i\varphi_\infty(x,\xi',x',\xi)} = e^{i\varphi_\infty(x,\xi',x',\xi)}$$

is well defined on the support if the integrand of (4.14), and, respectively, the operator

$$T_C = -i|\varphi'_{\infty,\xi'}(x,x',\xi',\xi)|^{-2}\,\varphi'_{\infty,\xi'}(x,x',\xi',\xi)\cdot\nabla_{\xi'} = C(x,x',\xi',\xi)\cdot\nabla_{\xi'}$$

such that

$$T_C e^{i\varphi_\infty(x,\xi',x',\xi)} = e^{i\varphi_\infty(x,\xi',x',\xi)}$$

is well defined on the support of the integrand of (4.15). Both $T_V$ and $T_C$ are well defined on the support of the integrand of (4.16). Notice also that the coefficients of $T_V$ satisfy, on the support of the integrand of (4.14), estimates of the type

$$|\partial^\alpha_{x'}\partial^\beta_{\xi'}V(x,x',\xi',\xi)| \lesssim \langle x'\rangle^{-|\alpha|}\langle\xi'\rangle^{-|\beta|}(\langle\xi\rangle + \langle\xi'\rangle)^{-1}. \tag{4.17}$$

Since there $\langle x\rangle \asymp \langle x'\rangle$, the same holds with $x$ in place of $x'$. Similarly, the coefficients of $T_C$ satisfy, on the support of the integrand of (4.15), estimates of the type

$$|\partial^\alpha_{x'}\partial^\beta_{\xi'}C(x,x',\xi',\xi)| \lesssim \langle x'\rangle^{-|\alpha|}\langle\xi'\rangle^{-|\beta|}(\langle x\rangle + \langle x'\rangle)^{-1}, \tag{4.18}$$

as well as the analogous ones with $\xi$ in place of $\xi'$, since $\langle\xi\rangle \asymp \langle\xi'\rangle$ there. Moreover, both (4.17) and (4.18) hold on the support of the integrand in (4.16). The claim then

follows by repeated integration by parts, using $T_C$ and/or $T_V$ in the expressions of $p_{3\infty}$, $p_{2\infty}$, and $p_{1\infty}$, and recalling Lemma 4.5. □

**Proposition 4.8.** *Under the hypotheses of Theorem 4.2, for $p_0$ defined in (4.12) we have $p_0 \in S^{0,0}(\mathbb{R}^{2n})$.*

To prove Proposition 4.8, we will use the change of variables

$$\begin{cases} x' = Y(x,\xi) + y\cdot\omega(x,\xi)^{-1} \\ \xi' = N(x,\xi) + \eta\cdot\omega(x,\xi), \end{cases} \tag{4.19}$$

where $\omega(x,\xi) = \langle x\rangle^{-\frac{1}{2}}\langle\xi\rangle^{\frac{1}{2}} \in S^{-\frac{1}{2},\frac{1}{2}}$ and $(Y,N) = (Y(x,\xi),N(x,\xi))$ is the unique solution of

$$\begin{cases} Y(x,\xi) = \varphi'_{1\xi}(x,N(x,\xi)) \\ N(x,\xi) = \varphi'_{2x}(Y(x,\xi),\xi), \end{cases}$$

see (3.4) of Section 3 above. With $\chi$ as in Definition 4.4, let

$$\rho(y,\eta;x,\xi) = \chi(x,Y(x,\xi)+y\cdot\omega(x,\xi)^{-1},N(x,\xi)+\eta\cdot\omega(x,\xi),\xi),$$
$$\varphi(y,\eta;x,\xi) = \varphi_0(x,Y(x,\xi)+y\cdot\omega(x,\xi)^{-1},N(x,\xi)+\eta\cdot\omega(x,\xi),\xi),$$

so that

$$p_0(x,\xi) = \iint e^{\varphi(y,\eta;x,\xi)}\rho(y,\eta;x,\xi)\,dy\text{đ}\eta.$$

By construction, on supp $\rho$,

$$|Y(x,\xi)+y\cdot\omega(x,\xi)^{-1}-x| \leqslant \frac{2}{3a}\langle x\rangle, \quad |N(x,\xi)+\eta\cdot\omega(x,\xi)-\xi| \leqslant \frac{2}{3a}\langle\xi\rangle,$$

which implies that, for a sufficiently large $a>0$ and a suitable $\tilde{k}\in(0,1)$, on supp $\rho$ we also have by (3.12) and (3.14)

$$|y|\cdot\omega(x,\xi)^{-1} \leqslant \tilde{k}\langle x\rangle \text{ and } |\eta|\cdot\omega(x,\xi)\leqslant\tilde{k}\langle\xi\rangle \Rightarrow |y|,|\eta| \leqslant \tilde{k}\,(\langle x\rangle\langle\xi\rangle)^{\frac{1}{2}}.$$

Furthermore, recalling that $\langle\varphi'_{1\xi}(x,\xi)\rangle \asymp \langle x\rangle$ and $\langle\varphi'_{2x}(x,\xi)\rangle\asymp\langle\xi\rangle$, we find that, on supp $\rho$, for any $\theta\in[0,1]$,

$$\langle Y(x,\xi)+\theta\cdot y\cdot\omega(x,\xi)^{-1}\rangle \asymp \langle x\rangle, \quad \langle N(x,\xi)+\theta\cdot\eta\cdot\omega(x,\xi)\rangle\asymp\langle\xi\rangle. \tag{4.20}$$

The next Lemma 4.9 can be proved analysing the Taylor expansions of $\varphi(y,\eta;x,\xi)$.

**Lemma 4.9.** *Let*

$$A_1(\eta;x,\xi) = \omega(x,\xi)^2\int_0^1(1-\theta)J''_{1\xi\xi}(x,N(x,\xi)+\theta\cdot\eta\cdot\omega(x,\xi))\,d\theta,$$
$$A_2(y;x,\xi) = \omega(x,\xi)^{-2}\int_0^1(1-\theta)J''_{2xx}(Y(x,\xi)+\theta\cdot y\cdot\omega(x,\xi)^{-1},\xi)\,d\theta,$$
$$B_1(\eta;x,\xi) = \omega(x,\xi)^2\int_0^1 J''_{1\xi\xi}(x,N(x,\xi)+\theta\cdot\eta\cdot\omega(x,\xi))\,d\theta,$$
$$B_2(y;x,\xi) = \omega(x,\xi)^{-2}\int_0^1 J''_{1xx}(Y(x,\xi)+\theta\cdot y\cdot\omega(x,\xi)^{-1},\xi)\,d\theta.$$

*Then*

$$\begin{aligned}\varphi(y,\eta;x,\xi) =& -y\cdot\eta+(\varphi_1(x,N(x,\xi)+\eta\cdot\omega(x,\xi))-\varphi_1(x,N(x,\xi)))\\ &-\varphi'_{1\xi}(x,N(x,\xi))\cdot\eta\cdot\omega(x,\xi)\\ &+(\varphi_2(Y(x,\xi)+y\cdot\omega(x,\xi)^{-1},\xi)-\varphi_2(Y(x,\xi),\xi))\\ &-\varphi'_{2x}(Y(x,\xi),\xi))\cdot y\cdot\omega(x,\xi)^{-1}\\ =& -y\cdot\eta+[A_1(\eta;x,\xi)\eta]\cdot\eta+[A_2(y;x,\xi)y]\cdot y,\end{aligned}\tag{4.21}$$

$$\begin{aligned}\varphi'_y(y,\eta;x,\xi) =& -\eta+[\varphi'_{2x}(Y(x,\xi)+y\cdot\omega(x,\xi)^{-1},\xi)-\varphi'_{2x}(Y(x,\xi),\xi)]\cdot\omega(x,\xi)^{-1}\\ =& -\eta+B_2(y;x,\xi)y,\end{aligned}\tag{4.22}$$

$$\begin{aligned}\varphi'_\eta(y,\eta;x,\xi) =& -y+[\varphi'_{1\xi}(x,N(x,\xi)+\eta\cdot\omega(x,\xi))-\varphi'_{1\xi}(x,N(x,\xi))]\cdot\omega(x,\xi)\\ =& -y+B_1(\eta;x,\xi)\eta.\end{aligned}\tag{4.23}$$

*Proof.* By the definition (4.8) of $\varphi_0$ and of the multi-product of phase functions (3.3) and (3.6), recalling (4.19), we can write

$$\begin{aligned}\varphi_0(x,x',\xi',\xi) =& \varphi_1(x,\xi')-x'\cdot\xi'+\varphi_2(x',\xi)\\ &-\varphi_1(x,N(x,\xi))+Y(x,\xi)\cdot N(x,\xi)-\varphi_2(Y(x,\xi),\xi),\end{aligned}$$

which implies

$$\begin{aligned}&\varphi_0(x,Y(x,\xi)+y\cdot\omega(x,\xi)^{-1},N(x,\xi)+\eta\cdot\omega(x,\xi),\xi)\\ &=\varphi_1(x,N(x,\xi)+\eta\cdot\omega(x,\xi))-(Y(x,\xi)+y\cdot\omega(x,\xi)^{-1})\cdot(N(x,\xi)+\eta\cdot\omega(x,\xi))\\ &+\varphi_2(Y(x,\xi)+y\cdot\omega(x,\xi)^{-1},\xi)-\varphi_1(x,N(x,\xi))-\varphi_2(Y(x,\xi),\xi)+Y(x,\xi)\cdot N(x,\xi)\\ &=-y\cdot\eta+(\varphi_1(x,N(x,\xi)+\eta\cdot\omega(x,\xi))-\varphi_1(x,N(x,\xi)))-Y(x,\xi)\cdot\eta\cdot\omega(x,\xi)\\ &\quad+(\varphi_2(Y(x,\xi)+y\cdot\omega(x,\xi)^{-1},\xi)-\varphi_2(Y(x,\xi),\xi))-y\cdot N(x,\xi)\cdot\omega(x,\xi)^{-1}.\end{aligned}$$

Then, recalling that $Y(x,\xi)=\varphi'_{1\xi}(x,N(x,\xi))$ and $N(x,\xi)=\varphi'_{2x}(Y(x,\xi),\xi)$, we get

$$\begin{aligned}\varphi(y,\eta;x,\xi) =& -y\cdot\eta+(\varphi_1(x,N(x,\xi)+\eta\cdot\omega(x,\xi))-\varphi_1(x,N(x,\xi)))\\ &-Y(x,\xi)\cdot\eta\cdot\omega(x,\xi)+(\varphi_2(Y(x,\xi)+y\cdot\omega(x,\xi)^{-1},\xi)-\varphi_2(Y(x,\xi),\xi))\\ &-y\cdot N(x,\xi)\cdot\omega(x,\xi)^{-1}\\ =& -y\cdot\eta+(\varphi_1(x,N(x,\xi)+\eta\cdot\omega(x,\xi))-\varphi_1(x,N(x,\xi)))\\ &-\varphi'_{1\xi}(x,N(x,\xi))\cdot\eta\cdot\omega(x,\xi)\\ &+(\varphi_2(Y(x,\xi)+y\cdot\omega(x,\xi)^{-1},\xi)-\varphi_2(Y(x,\xi),\xi))\\ &-\varphi'_{2x}(Y(x,\xi),\xi))\cdot y\cdot\omega(x,\xi)^{-1}\\ =& -y\cdot\eta+[A_1(\eta;x,\xi)\eta]\cdot\eta+[A_2(y;x,\xi)y]\cdot y,\end{aligned}$$

that is (4.21) and its subsequent expression in terms of $A_1, A_2$. Then (4.22) and (4.23) immediately follow taking derivatives with respect to $y,\eta$ in (4.21), and then looking at the definitions of $B_1$, $B_2$. □

**Lemma 4.10.** *For $A_1, A_2, B_1, B_2$ defined in Lemma 4.9 we have, for all $x,y,\xi,\eta\in\mathbb{R}^n$ in* supp $\rho$,

$$\begin{aligned}\|\partial_x^\beta\partial_\xi^\alpha\partial_\eta^{\alpha'}(A_1,B_1)(\eta;x,\xi)\| &\lesssim \tau\langle\xi\rangle^{-|\alpha|-\frac{|\alpha'|}{2}}\langle x\rangle^{-|\beta|-\frac{|\alpha'|}{2}}\langle y,\eta\rangle^{|\alpha+\beta|},\\ \|\partial_x^\beta\partial_\xi^\alpha\partial_y^{\beta'}(A_2,B_2)(y;x,\xi)\| &\lesssim \tau\langle\xi\rangle^{-|\alpha|-\frac{|\beta'|}{2}}\langle x\rangle^{-|\beta|-\frac{|\beta'|}{2}}\langle y,\eta\rangle^{|\alpha+\beta|},\end{aligned}$$

*where $\langle y,\eta\rangle := \sqrt{1+|y|^2+|\eta|^2}$, $y,\eta\in\mathbb{R}^n$.*

*Proof.* The result follows from the Faá di Bruno formula for the derivatives of the composed functions, the properties of $X \in S^{1,0}$, $N \in S^{0,1}$ stated above, the fact that, on $\operatorname{supp}\rho$, (4.20) holds for any $\theta \in [0,1]$, as well as

$$Y(x,\xi) + \theta \cdot y \cdot \omega(x,\xi)^{-1} \in S^{1,0} \cdot \langle y,\eta\rangle, \quad N(x,\xi) + \theta \cdot \eta \cdot \omega(x,\xi) \in S^{0,1} \cdot \langle y,\eta\rangle,$$

recalling that the seminorms of $J_1$ and $J_2$ involving their derivatives up to order 2 are proportional to $\tau \in (0,1)$.

The proof works by induction on the order of the derivatives. Let us give an idea of the step $|\alpha+\beta+\alpha'| = 1$. Let $e_j$ be the multi-index such that $|e_j| = 1$, with all the components equal to 0 but the $j$-th. Then, for instance, on $\operatorname{supp}\rho$,

$$\begin{aligned}\partial_x^{e_j} B_1(\eta;x,\xi) &= (\partial_x^{e_j}\omega^2)\int_0^1 J''_{1\xi\xi}(\dots)\,d\theta + \omega^2\int_0^1 J'''_{1x\xi\xi}(\dots)\,d\theta\\ &\quad + \omega^2\int_0^1 J'''_{1\xi\xi\xi}(\dots)\,d\theta\cdot\partial_x^{e_j}(N(x,\xi)+\theta\cdot\eta\cdot\omega(x,\xi))\\ &\quad \in S^{-1,0} + S^{-1,0}\cdot\langle y,\eta\rangle \subset S^{-1,0}\cdot\langle y,\eta\rangle,\end{aligned}$$

since $\omega^2 \in S^{-1,1}$, $\int_0^1 J''_{1\xi\xi}(\dots)\,d\theta \in S^{1,-1}$, $\int_0^1 J'''_{1x\xi\xi}(\dots)\,d\theta \in S^{0,-1}$, $\int_0^1 J'''_{1\xi\xi\xi}(\dots)\,d\theta \in S^{1,-2}$, and $N(x,\xi)+\theta\cdot\eta\cdot\omega(x,\xi) \in S^{0,1}|\eta|$. Similarly,

$$\begin{aligned}\partial_\xi^{e_j} B_1(\eta;x,\xi) &= (\partial_\xi^{e_j}\omega^2)\int_0^1 J''_{1\xi\xi}(\dots)\,d\theta\\ &\quad + \omega^2\int_0^1 J'''_{1\xi\xi\xi}(\dots)\,d\theta\cdot\partial_\xi^{e_j}(N(x,\xi)+\theta\cdot\eta\cdot\omega(x,\xi))\\ &\quad \in S^{0,-1} + S^{0,-1}\cdot\langle y,\eta\rangle \subset S^{0,-1}\cdot\langle y,\eta\rangle,\\ \partial_\eta^{e_j} B_1(\eta;x,\xi) &= \omega^2\int_0^1 J'''_{1\xi\xi\xi}(\dots)\,d\theta\cdot(\theta\cdot\omega(x,\xi)) \in S^{-1/2,-1/2}.\end{aligned}$$

The estimates for general multi-indices follow by induction. □

**Lemma 4.11.** *On* $\operatorname{supp}\rho$,

$$|\varphi'_y(y,\eta;x,\xi)| + |\varphi'_\eta(y,\eta;x,\xi)| \asymp |y| + |\eta|.$$

*Proof.* From Lemmas 4.9 and 4.10, on $\operatorname{supp}\rho$, for $\tau \in (0,1)$,

$$\begin{aligned}\|B_1(\eta;x,\xi)\| \lesssim \tau \Rightarrow \|B_1(\eta;x,\xi)\eta\| \lesssim \tau|\eta|,\\ \|B_2(y;x,\xi)\| \lesssim \tau \Rightarrow \|B_2(y;x,\xi)y\| \lesssim \tau|y|,\end{aligned}$$

which imply

$$\begin{aligned}|\varphi'_y(y,\eta;x,\xi)| \lesssim |\eta| + \tau|y|, \quad |\varphi'_y(y,\eta;x,\xi)| \gtrsim |\eta| - \tau|y|,\\ |\varphi'_\eta(y,\eta;x,\xi)| \lesssim |y| + \tau|\eta|, \quad |\varphi'_\eta(y,\eta;x,\xi)| \gtrsim |y| - \tau|\eta|.\end{aligned}$$

These give

$$\begin{aligned}|\varphi'_y(y,\eta;x,\xi)| + |\varphi'_\eta(y,\eta;x,\xi)| \lesssim (1+\tau)(|y|+|\eta|),\\ |\varphi'_y(y,\eta;x,\xi)| + |\varphi'_\eta(y,\eta;x,\xi)| \gtrsim (1-\tau)(|y|+|\eta|),\end{aligned}$$

as claimed. □

**Lemma 4.12.** *On* supp $\rho$*, for any multi-indices* $\alpha, \beta, \alpha', \beta'$*, and all* $x, y, \xi, \eta$*,*

$$|\partial_x^\beta \partial_y^{\beta'} \partial_\xi^\alpha \partial_\eta^{\alpha'} \varphi'_y(y,\eta;x,\xi)| \lesssim \begin{cases} 0 & \text{if } |\alpha'| \geqslant 2, \\ 1 & \text{if } |\alpha'| = 1, \\ \tau\langle x\rangle^{-|\beta|-\frac{|\beta'|}{2}}\langle\xi\rangle^{-|\alpha|-\frac{|\beta'|}{2}}\langle y,\eta\rangle^{1+|\alpha+\beta|} & \text{if } |\alpha'| = 0, \\ & \quad |\alpha+\beta+\beta'| > 0; \end{cases}$$

$$|\partial_x^\beta \partial_y^{\beta'} \partial_\xi^\alpha \partial_\eta^{\alpha'} \varphi'_\eta(y,\eta;x,\xi)| \lesssim \begin{cases} 0 & \text{if } |\beta'| \geqslant 2, \\ 1 & \text{if } |\beta'| = 1, \\ \tau\langle x\rangle^{-|\beta|-\frac{|\alpha'|}{2}}\langle\xi\rangle^{-|\alpha|-\frac{|\alpha'|}{2}}\langle y,\eta\rangle^{1+|\alpha+\beta|} & \text{if } |\beta'| = 0, \\ & \quad |\alpha+\alpha'+\beta| > 0. \end{cases}$$

*Proof.* The results follow from Lemma 4.10 and the estimates (4.20). □

**Lemma 4.13.** *On* supp $\rho$*, for any multi-indices* $\alpha, \beta, \alpha', \beta'$*, and all* $x, y, \xi, \eta$*,*

$$|\partial_x^\beta \partial_y^{\beta'} \partial_\xi^\alpha \partial_\eta^{\alpha'} \varphi'_x(y,\eta;x,\xi)| \lesssim \begin{cases} \tau\langle\xi\rangle^{-|\alpha|-\frac{|\beta'|}{2}}\langle x\rangle^{-1-|\beta|-\frac{|\beta'|}{2}}\langle y,\eta\rangle^{3+|\alpha+\beta|} & \text{if } |\beta'| > 0, \\ \tau\langle\xi\rangle^{-|\alpha|-\frac{|\alpha'|}{2}}\langle x\rangle^{-1-|\beta|-\frac{|\alpha'|}{2}}\langle y,\eta\rangle^{3+|\alpha+\beta|} & \text{if } |\alpha'| > 0, \\ \tau\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-1-|\beta|}\langle y,\eta\rangle^{3+|\alpha+\beta|} & \text{if } \alpha' = \beta' = 0; \end{cases}$$

$$|\partial_x^\beta \partial_y^{\beta'} \partial_\xi^\alpha \partial_\eta^{\alpha'} \varphi'_\xi(y,\eta;x,\xi)| \lesssim \begin{cases} \tau\langle\xi\rangle^{-1-|\alpha|-\frac{|\beta'|}{2}}\langle x\rangle^{-|\beta|-\frac{|\beta'|}{2}}\langle y,\eta\rangle^{3+|\alpha+\beta|} & \text{if } |\beta'| > 0, \\ \tau\langle\xi\rangle^{-1-|\alpha|-\frac{|\alpha'|}{2}}\langle x\rangle^{-|\beta|-\frac{|\alpha'|}{2}}\langle y,\eta\rangle^{3+|\alpha+\beta|} & \text{if } |\alpha'| > 0, \\ \tau\langle\xi\rangle^{-1-|\alpha|}\langle x\rangle^{-|\beta|}\langle y,\eta\rangle^{3+|\alpha+\beta|} & \text{if } \alpha' = \beta' = 0. \end{cases}$$

*Proof.* The results follow from Lemma 4.10, observing that

$$\varphi'_x(y,\eta;x,\xi) = d_x[(A_1(\eta;x,\xi)\eta)\cdot\eta] + d_x[(A_2(y;x,\xi)y)\cdot y],$$
$$\varphi'_\xi(y,\eta;x,\xi) = d_\xi[(A_1(\eta;x,\xi)\eta)\cdot\eta] + d_\xi[(A_2(y;x,\xi)y)\cdot y].$$

□

**Lemma 4.14.** *For any multi-indices* $\alpha, \beta, \alpha', \beta'$*, and all* $x, y, \xi, \eta$*,*

$$|\partial_\xi^\alpha \partial_\eta^{\alpha'} \partial_x^\beta \partial_y^{\beta'} \rho(y,\eta;x,\xi)| \lesssim \langle\xi\rangle^{-|\alpha|-\frac{|\alpha'|}{2}}\langle x\rangle^{-|\beta|-\frac{|\beta'|}{2}}.$$

*Proof.* Immediate, by the definition of $\rho$, the hypotheses on $\psi$, the properties $Y(x,\xi) \in S^{1,0}$, $N(x,\xi) \in S^{0,1}$, and the estimates (4.20). □

**Lemma 4.15.** *Let*

$$\Gamma = \Gamma(y,\eta;x,\xi) = 1 + |\varphi'_y(y,\eta;x,\xi)|^2 + |\varphi'_\eta(y,\eta;x,\xi)|^2.$$

*Then, on* supp $\rho$*, for any multi-indices* $\alpha, \beta, \alpha', \beta'$*, and all* $x, y, \xi, \eta$*,*

$$\left|\partial_\xi^\alpha \partial_\eta^{\alpha'} \partial_x^\beta \partial_y^{\beta'} \left(\frac{1}{\Gamma(y,\eta;x,\xi)}\right)\right| \lesssim \tau\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}\langle y,\eta\rangle^{-2+|\alpha+\beta|}.$$

*Proof.* Immediate, by Lemmas 4.9, 4.10, 4.11, and 4.12. □

The next Lemma 4.16 is a straightforward consequence of Lemma 4.15 and the definition of transpose operator.

**Lemma 4.16.** *Let us define the operator*

$$M = \frac{1}{\Gamma}(1 - i\varphi'_y(y,\eta;x,\xi)\cdot\nabla_y - i\varphi'_\eta(y,\eta;x,\xi)\cdot\nabla_\eta)$$

*such that* $Me^{i\varphi(y,\eta;x,\xi)} = e^{i\varphi(y,\eta;x,\xi)}$*. Then,*

$${}^tM = M_0 + M_1\cdot\nabla_y + M_2\cdot\nabla_\eta,$$

*where, on* supp $\rho$, *for any multi-indices* $\alpha, \beta, \alpha', \beta'$, *and all* $x, y, \xi, \eta$,

$$\|\partial_\xi^\alpha \partial_\eta^{\alpha'} \partial_x^\beta \partial_y^{\beta'} [(M_0, M_1, M_2)(y, \eta; x, \xi)]\| \lesssim \langle \xi \rangle^{-|\alpha|} \langle x \rangle^{-|\beta|} \langle y, \eta \rangle^{-1+|\alpha+\beta|}.$$

*Proof of Proposition 4.8.* Using the operator $M$ defined in Lemma 4.16, we have, for arbitrary $k \in \mathbb{Z}_+$,

$$p_0(x, \xi) = \iint e^{i\varphi(y,\eta;x,\xi)} ((^tM)^k \rho)(y, \eta; x, \xi)\, dy\!\!\bar{}\, d\eta.$$

Notice that, from the analysis above, for any $k \in \mathbb{Z}_+$, any multi-indices $\alpha', \beta'$, and all $x, y, \xi, \eta$,

$$|(\partial_\xi^{\alpha'} \partial_x^{\beta'} ((^tM)^k \rho))(y, \eta; x, \xi)| \lesssim \langle x \rangle^{-|\beta'|} \langle \xi \rangle^{-|\alpha'|} \langle y, \eta \rangle^{-k+|\alpha'+\beta'|}.$$

Then, for any fixed $\alpha, \beta \in \mathbb{Z}_+^n$, and arbitrary $k \in \mathbb{Z}_+$, we find

$$\partial_\xi^\alpha \partial_x^\beta p_0(x, \xi) =$$
$$= \sum_{\alpha_1+\alpha_2=\alpha} \sum_{\beta_1+\beta_2=\beta} \binom{\alpha}{\alpha_1} \binom{\beta}{\beta_1} \iint \left( \partial_\xi^{\alpha_1} \partial_x^{\beta_1} e^{i\varphi(y,\eta;x,\xi)} \right) \cdot (\partial_\xi^{\alpha_2} \partial_x^{\beta_2} ((^tM)^k \rho)(y, \eta; x, \xi))\, dy\!\!\bar{}\, d\eta.$$

Choosing $k$ such that $-k + 6|\alpha + \beta| \leqslant -(2n + 1)$, from the results in Lemmas 4.13, 4.14, and 4.16 above, we get

$$|\partial_x^\alpha \partial_\xi^\beta p_0(x, \xi)| \lesssim \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|} \iint \langle y, \eta \rangle^{-(2n+1)}\, dy d\eta \lesssim \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|},$$

as claimed. □

**Remark 4.17.** *Let us notice that we have proved here above that the seminorms of* $p_0$ *are controlled by those of* $\varphi_1$ *and* $\varphi_2$. *This implies that, if* $J_1$ *and* $J_2$ *are bounded in* $S^{1,1}$, *so is* $p_0$ *in* $S^{0,0}$. *The boundedness conditions of Theorem 4.2 are so fulfilled, and the proof of Theorem 4.2 is complete.*

## 5. Fundamental solution to hyperbolic systems in SG classes

In the present section we apply the results of Sections 3 and 4 to construct the fundamental solution $E(t, s)$ to the Cauchy problem for a first order system of partial differential equations of hyperbolic type, with coefficients in SG classes and roots of (possibly) variable multiplicity. A standard argument, which we omit here, gives then the solution, via $E(t, s)$ and Duhamel's formula, see Theorem 5.1 below. We follow the approach in [24, Section 10.7].

Let us consider the Cauchy problem

$$\begin{cases} LW(t, x) = F(t, x) & (t, x) \in (0, T] \times \mathbb{R}^n, \\ W(0, x) = W_0(x) & x \in \mathbb{R}^n, \end{cases} \tag{5.1}$$

where

$$L(t, x, D_t, D_x) = D_t + \Lambda(t, x, D_x) + R(t, x, D_x), \tag{5.2}$$

$\Lambda$ is an $m \times m$ diagonal operator matrix whose entries $\lambda_j(t, x, D_x)$, $j = 1, \dots, m$, are pseudo-differential operators with symbols $\lambda_j(t, x, \xi) \in C([0, T]; S^{\epsilon,1})$, $\epsilon \in [0, 1]$, and $R$ is an $m \times m$-operator matrix with elements in $C([0, T], S^{\epsilon-1,0})$. The case $\epsilon = 0$ corresponds to symbols uniformly bounded in the space variable, while the case $\epsilon = 1$ is the standard situation of SG symbols with equal order components.

Assume also that the system (5.2) is of hyperbolic type, that is, $\lambda_j(t, x, \xi) \in \mathbb{R}$, $j = 1, \dots, m$. Notice that, differently from [12, 17], here we do not impose any "separation condition at infinity" on the $\lambda_j$, $j = 1, \dots, m$. Indeed, the results

presented below apply both to the constant as well as the variable multiplicities cases.

For $0 < T_0 \leqslant T$, we define $\Delta_{T_0} := \{(t,s)|\ 0 \leqslant s \leqslant t \leqslant T_0\}$. The fundamental solution of (5.1) is a family $\{E(t,s)|(t,s) \in \Delta_{T_0}\}$ of SG FIOs, satisfying

$$\begin{cases} LE(t,s) = 0 & (t,s) \in \Delta_{T_0}, \\ E(s,s) = I & s \in [0,T_0]. \end{cases} \tag{5.3}$$

In this section we aim to show that, if $T_0$ is small enough, it is possible to construct the family $\{E(t,s)\}$ satisfying (5.3).

As a consequence of (5.3), it is quite easy to get the following:

**Theorem 5.1.** *For every $F \in C([0,T];H^{r,\varrho}(\mathbb{R}^n))$ and $G \in H^{r,\varrho}(\mathbb{R}^n)$, the solution $W(t,x)$ of the Cauchy problem (5.1) exists uniquely, it belongs to the class $C([0,T_0],H^{r-(\epsilon-1),\varrho}(\mathbb{R}^n))$, and it is given by*

$$W(t) = E(t,0)G + i\int_0^t E(t,s)F(s)ds, \quad t \in [0,T_0].$$

**Remark 5.2.** *Theorem 5.1 gives well-posedness of the Cauchy problem (5.1) in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$; moreover it gives "well posedness with loss/gain of decay" (depending on the sign of r) of (5.1) in weighted Sobolev spaces $H^{r,\varrho}(\mathbb{R}^n)$. This phenomenon is quite common in the theory of hyperbolic partial differential equations with SG type coefficients, see [2, 4, 5]. We remark that in the symmetric case $\epsilon = 1$ the Cauchy problem (5.1) turns out to be well-posed also in $H^{r,\varrho}(\mathbb{R}^n)$.*

To begin, consider SG phase functions $\varphi_j = \varphi_j(t,s,x,\xi)$, $1 \leqslant j \leqslant m$, defined on $\Delta_{T_0} \times \mathbb{R}^{2n}$, and define the operator matrix

$$I_\varphi(t,s) = \begin{pmatrix} I_{\varphi_1}(t,s) & & 0 \\ & \ddots & \\ 0 & & I_{\varphi_m}(t,s) \end{pmatrix},$$

where $I_{\varphi_j} := Op_{\varphi_j}(1)$, $1 \leqslant j \leqslant m$. From Theorem 2.3 (see Remark 2.8) we see that

$$\begin{aligned} D_t I_{\varphi_j} + \lambda_j(t,x,D_x)I_{\varphi_j} &= \int e^{i\varphi_j(t,s,x,\xi)} \frac{\partial \varphi_j}{\partial t}(t,s,x,\xi)\text{đ}\xi \\ &+ \int e^{i\varphi_j(t,s,x,\xi)} \lambda_j(t,x,\varphi'_{j,x}(t,s,x,\xi))\text{đ}\xi \\ &+ \int e^{i\varphi_j(t,s,x,\xi)} b_{0,j}(t,s,x,\xi)\text{đ}\xi, \end{aligned}$$

where $b_{0,j}(t,s) \in S^{\epsilon-1,0} \subseteq S^{0,0}$. The first two integrals in the right-hand side of the equation here above cancel if we choose $\varphi_j$, $j = 1,\dots,m$, to be the solution to the eikonal equation (2.5) associated with the symbol $a = \lambda_j$, $j = 1,\dots,m$. By Proposition 2.9, this is possible, provided that $T_0$ is small enough. Writing $B_{0,j} := Op_{\varphi_j}(b_{0,j})$, we define the family $\{W_1(t,s);(t,s) \in \Delta_{T_0}\}$ of SG FIOs by

$$W_1(t,s,x,D_x) := -i\left(\begin{pmatrix} B_{0,1}(t,s,x,D_x) & & 0 \\ & \ddots & \\ 0 & & B_{0,m}(t,s,x,D_x) \end{pmatrix} + R(t,x,D_x)\right) I_\varphi(t,s,x,D_x),$$

and we denote by $w_1(t,s,x,\xi)$ the symbol of $W_1(t,s,x,\xi)$. Notice that

$$L(t,x,D_x)I_\varphi(t,s,x,D_x) = iW_1(t,s,x,D_x), \tag{5.4}$$

that is, $iW_1$ is the residual of system (5.1) for $I_\varphi$. We define then by induction the sequence of $m \times m$-matrices of SG FIOs, denoted by $\{W_\nu(t,s); (t,s) \in \Delta_{T_0}\}_{\nu\in\mathbb{N}}$, as

$$W_{\nu+1}(t,s,x,D_x) = \int_s^t W_1(t,\theta,x,D_x)W_\nu(\theta,s,x,D_x)d\theta, \tag{5.5}$$

and we denote by $w_{\nu+1}(t,s,x,\xi)$ the symbol of $W_{\nu+1}(t,s,x,D_x)$. We are now going to prove that the operator norms of $W_\nu$, seen as operators from the Sobolev space $H^{r,\varrho}$ into $H^{r-(\nu-1)(\epsilon-1),\varrho}$ for any fixed $(r,\varrho) \in \mathbb{R}^2$ can be estimated from above by

$$\|W_\nu(t,s)\|_{\mathcal{L}(H^{r,\varrho},H^{r-(\nu-1)(\epsilon-1),\varrho})} \leqslant \frac{C_{r,\varrho}^{\nu-1}|t-s|^{\nu-1}}{(\nu-1)!} \leqslant \frac{C_{r,\varrho}^{\nu-1}T_0^{\nu-1}}{(\nu-1)!}, \tag{5.6}$$

for all $(t,s) \in \Delta_{T_0}$ and $\nu \in \mathbb{N}$, where $C_{r,\varrho}$ is a constant which only depends on $r,\varrho$.

To deal with the operator norms in (5.6), we need to explicitly write the matrices $W_\nu$. An induction in (5.5) easily shows that

$$W_\nu(t,s) = \int_s^t \int_s^{\theta_1} \dots \int_s^{\theta_{\nu-2}} W_1(t,\theta_1)\dots W_1(\theta_{\nu-2},\theta_{\nu-1})d\theta_{\nu-1}\dots d\theta_1. \tag{5.7}$$

The integrand is a product of $\nu-1$ $m\times m$-matrices of SG FIOs, therefore it is an operator matrix whose entries consist of $m^{\nu-2}$ summands of compositions of $\nu-1$ SG FIOs. Denoting by $Q_1 \circ \dots \circ Q_{\nu-1}$ one of these compositions, where each of the $Q_j$ is one of the $m^2$ entries of the $m\times m$-matrix of SG FIOs $W_1$, we have from Example 3.3 and (2) of Theorem 4.3 that $Q_1 \circ \dots \circ Q_{\nu-1}$ is again a SG FIO with symbol $q_{1,\dots,\nu-1} \in S^{(\nu-1)(\epsilon-1),0} \subseteq S^{0,0}$. Moreover, from (3) of Theorem 4.3, for all $\ell \in \mathbb{N}$ there exists $C_\ell > 0$ and $\ell' \in \mathbb{N}_0$ such that

$$\begin{aligned}|||q_{1,\dots,\nu-1}(t,\theta_1,\dots,\theta_{\nu-1})|||_\ell^{(\nu-1)(\epsilon-1),0} \\ \leqslant C_\ell^{\nu-2}|||q_1(t,\theta_1)|||_{\ell'}^{\epsilon-1,0}\dots|||q_{\nu-1}(\theta_{\nu-2},\theta_{\nu-1})|||_{\ell'}^{\epsilon-1,0},\end{aligned}$$

where for $j=1,\dots,\nu-1$, $q_j(t,s)$ denotes the symbol of the SG FIO $Q_j(t,s)$, $(t,s) \in \Delta_{T_0}$. Now we set

$$\bar\sigma := \sup_{j=1,\dots,\nu-1}\sup_{(t,s)\in\Delta_{T_0}} |||q_j(t,s)|||_{\ell'}^{\epsilon-1,0} < \infty,$$

so that

$$|||q_{1,\dots,\nu-1}(t,\theta_1,\dots,\theta_{\nu-1})|||_\ell^{(\nu-1)(\epsilon-1),0} \leqslant C_\ell^{\nu-2}\bar\sigma^{\nu-1}.$$

The continuity of the SG FIOs $Q_1\circ\dots\circ Q_{\nu-1}(t,\theta_1,\cdots,\theta_{\nu-1}) : H^{r,\varrho} \longrightarrow H^{r-(n-1)(\epsilon-1),\varrho}$ (see Theorem 2.1) and the previous inequality give that for every $r,\varrho$ there exist constants $C_{r,\varrho} > 0$ (depending only on the indeces of the Sobolev space) and $\ell_{r,\varrho} \in \mathbb{N}_0$ such that for all $u \in H^{r,\varrho}$

$$\begin{aligned}\|Q_1(t,\theta_1)\circ\dots\circ Q_{\nu-1}(\theta_{\nu-2}\theta_{\nu-1})u\|_{r-(n-1)(\epsilon-1),\varrho} \\ \leqslant C_{r,\varrho}|||q_{1,\dots,\nu-1}(t,\theta_1,\dots,\theta_{\nu-1})|||_{\ell_{r,\varrho}}^{(\nu-1)(\epsilon-1),0}\|u\|_{r,\varrho} \\ \leqslant C_{r,\varrho}C_{\ell_{r,\varrho}}^{\nu-2}\bar\sigma^{\nu-1}\|u\|_{r,\varrho}.\end{aligned} \tag{5.8}$$

Therefore, in the operator matrix $W_1(t,\theta_1)\dots W_1(\theta_{\nu-2},\theta_{\nu-1})$, the operator norm of each entry can be bounded from above by $m^{\nu-2}C_{r,\varrho}C_{\ell_{r,\varrho}}^{\nu-2}\bar\sigma^{\nu-1}$. Now by (5.7) and

(5.8) we deduce that

$$
\begin{aligned}
&\|W_\nu(t,s)\|_{\mathcal{L}(H^{r,\varrho},H^{r-(\nu-1)(\epsilon-1),\varrho})}\\
&\quad\leqslant \int_s^t\int_s^{\theta_1}\dots\int_s^{\theta_{\nu-2}}\|W_1(t,\theta_1)\dots W_1(\theta_{\nu-2},\theta_{\nu-1})\|_{\mathcal{L}(H^{r,\varrho},H^{r-(\nu-1)(\epsilon-1),\varrho})}d\theta_{\nu-1}\dots d\theta_1\\
&\quad\leqslant m^{\nu-2}C_{r,\varrho}C_{\ell_{r,\varrho}}^{\nu-2}\bar{\sigma}^{\nu-1}\int_s^t\int_s^{\theta_1}\dots\int_s^{\theta_{\nu-2}}d\theta_{\nu-1}\dots d\theta_1\\
&\quad\leqslant \frac{m^{\nu-2}C_{r,\varrho}C_{\ell_{r,\varrho}}^{\nu-2}\bar{\sigma}^{\nu-1}|t-s|^{\nu-1}}{(\nu-1)!}=\frac{\tilde{C}_{r,\varrho}^{\nu-1}|t-s|^{\nu-1}}{(\nu-1)!}
\end{aligned}
\tag{5.9}
$$

for a new constant $\tilde{C}_{r,\varrho}$ depending only on $r,\varrho$, which yields the claim (5.6).

Now, using the estimate (5.6), we can show that the sequence of SG FIOs, defined for all $(t,s)\in\Delta_{T_0}$ and all $N\in\mathbb{N}$ by

$$
E_N(t,s)=I_\varphi(t,s)+\int_s^t I_\varphi(t,\theta)\sum_{\nu=1}^{N}W_\nu(\theta,s)d\theta,
\tag{5.10}
$$

is a well-defined SG FIO in $\mathcal{L}(H^{r,\varrho},H^{r-\epsilon+1,\varrho})$ for every $r,\varrho$, and converges, as $N\to\infty$, to the well-defined SG FIO, belonging to $\mathcal{L}(H^{r,\varrho},H^{r-\epsilon+1,\varrho})$, given by

$$
E(t,s)=I_\varphi(t,s)+\int_s^t I_\varphi(t,\theta)\sum_{\nu=1}^{\infty}W_\nu(\theta,s)d\theta.
\tag{5.11}
$$

$E(t,s)$ in (5.11) is the fundamental solution to the system (5.1) in the sense that it satisfies (5.3). Indeed, at symbols level, with the notations $E_N=Op_\varphi(e_N)$, $E=Op_\varphi(e)$ and $W_1\circ\dots\circ W_1=Op_\varphi(\sigma_{\nu-1})$, for every $l\in\mathbb{N}$ and $|\alpha+\beta|\leqslant\ell$, we have

$$
\begin{aligned}
&|\partial_\xi^\alpha\partial_x^\beta e_N(t,s,x,\xi)|\\
&\leqslant\int_s^t\sum_{\nu=1}^{N}|\partial_\xi^\alpha\partial_x^\beta w_\nu(\theta,s,x,\xi)|d\theta\\
&\leqslant\sum_{\nu=1}^{N}\int_s^t\int_s^\theta\int_s^{\theta_1}\dots\int_s^{\theta_{\nu-2}}\big|\partial_\xi^\alpha\partial_x^\beta\sigma_{\nu-1}(t,\theta_1,\dots,\theta_{\nu-1},x,\xi)\big|d\theta_{n-1}\dots d\theta_1d\theta\\
&\leqslant\sum_{\nu=1}^{N}\int_s^t\dots\int_s^{\theta_{\nu-2}}|||\sigma_{\nu-1}(t,\theta_1,\dots,\theta_{\nu-1})|||_\ell^{(\nu-1)(\epsilon-1),0}\langle x\rangle^{(\nu-1)(\epsilon-1)-|\beta|}\langle\xi\rangle^{-|\alpha|}d\theta_{\nu-1}\dots d\theta\\
&\leqslant\langle x\rangle^{\epsilon-1-|\beta|}\langle\xi\rangle^{-|\alpha|}\sum_{\nu=1}^{N}\frac{m^{\nu-2}C_\ell^{\nu-2}\bar{\sigma}^{\nu-1}|t-s|^{\nu-1}}{(\nu-1)!},
\end{aligned}
$$

so

$$
|||e_N(t,s)|||_\ell^{\epsilon-1,0}\leqslant\sum_{\nu=0}^{N-1}\frac{(C'_\ell|t-s|)^\nu}{\nu!},
$$

for a new constant $C'_\ell>0$. Then, for $N\to\infty$ we get

$$
|||e(t,s)|||_\ell^{\epsilon-1,0}\leqslant\exp(C'_\ell(t-s))<\infty.
$$

Thus, the SG FIO (5.11) has a well-defined symbol. On the other hand, at operator's level, by definitions (5.10) and (5.2) we have

$$
LE_N=LI_\phi-i\sum_{\nu=1}^{N}W_\nu(t,s)+\int_s^t LI_\phi(t,\theta)\sum_{\nu=1}^{N}W_\nu(\theta,s)d\theta.
\tag{5.12}
$$

An induction shows that

$$\sum_{\nu=1}^{N} W_\nu(t,s) = -i(LI_\phi)(t,s) - i\int_s^t (LI_\phi)(t,\theta)\sum_{\nu=1}^{N-1} W_\nu(\theta,s)d\theta. \tag{5.13}$$

Indeed, for $N=2$ we have by (5.4) and (5.5)

$$W_1(t,s) + W_2(t,s) = -i(LI_\phi)(t,s) - i\int_s^t (LI_\phi)(t,\theta)W_1(\theta,s)d\theta;$$

the induction step $N \mapsto N+1$ works as follows:

$$\begin{aligned}\sum_{\nu=1}^{N+1} W_\nu(t,s) &= W_{N+1}(t,s) + \sum_{\nu=1}^{N} W_\nu(t,s)\\ &= -i\int_s^t (LI_\phi)(t,\theta)W_N(\theta,s)d\theta - i(LI_\phi)(t,s) - i\int_s^t (LI_\phi)(t,\theta)\sum_{\nu=1}^{N-1} W_\nu(\theta,s)d\theta\\ &= -i(LI_\phi)(t,s) - i\int_s^t (LI_\phi)(t,\theta)\sum_{\nu=1}^{N} W_\nu(\theta,s)d\theta.\end{aligned}$$

Substituting (5.13) into (5.12) we get

$$(LE_N)(t,s) = \int_s^t (LI_\phi)(t,\theta)W_N(\theta,s)d\theta.$$

Now, for $N\to\infty$, $\|W_N(t,s)\|_{\mathcal{L}(H^{r,\varrho},H^{r-(N-1)(\epsilon-1),\varrho})} \to 0$ because of (5.9); thus $LE_N \to LE = 0$. Moreover, it is easy to verify that $E(s,s) = I$. So, (5.3) is fulfilled, and we have constructed the fundamental solution to $L$. As it concerns the dependence of the fundamental solution on the parameters $(t,s)$, we finally notice that the SG FIO-valued map $(t,s)\mapsto E(t,s)$ belongs to $C(\Delta_{T_0})$, since $E$ is obtained by continuous operations of operators which are continuous in $t,s$, see (5.11).

## References


[1] G.D. Andrews, SG Fourier integral operators with closure under composition, Dissertation. Imperial College, London, 2009.

[2] Ascanelli, A., Cappiello, M., Log-Lipschitz regularity for *SG* hyperbolic systems. J. Differential Equations, 230 (2006), 556–578.

[3] Ascanelli, A., Cappiello, M., Hölder continuity in time for *SG* hyperbolic systems. J. Differential Equations, 244 (2008), 2091–2121.

[4] Ascanelli, A., Cappiello, M., The Cauchy problem for finitely degenerate hyperbolic equations with polynomial coefficients. Osaka J.Math., 47, n.2 (2010), 423–438.

[5] Ascanelli, A., Cappiello, M., Weighted energy estimates for p-evolution equations in SG classes. Journal of Evolution Equations, (2015), DOI: 10.1007/s00028-015-0274-6

[6] Ascanelli, A., Coriasco, S., Süß, A., Solution theory of hyperbolic stochastic partial differential equations with polynomially bounded coefficients (2015), in preparation.

[7] Battisti, U., Coriasco, S., Wodzicki residue for operators on manifolds with cylindrical ends. Ann. Global Anal. Geom., Vol. 40, No. 2 (2011), 223–249.

[8] Cappiello, M., Fourier integral operators of infinite order and applications to *SG*-hyperbolic equations. Tsukuba J. Math., 28 (2004), 311–361.

[9] Cordero, E., Nicola, F., Rodino, L., On the global boundedness of Fourier integral operators. Ann. Global Anal. Geom., 38 (2010), 373–398.

[10] H. O. Cordes *The technique of pseudodifferential operators.* Cambridge Univ. Press, 1995.

[11] S. Coriasco. Fourier integral operators in *SG* classes I. Composition theorems and action on *SG* Sobolev spaces. *Rend. Sem. Mat. Univ. Pol. Torino*, 57, 4:249–302, 1999.

[12] S. Coriasco. Fourier integral operators in *SG* classes II. Application to *SG* hyperbolic Cauchy problems. *Ann. Univ. Ferrara*, 47:81–122, 1998.

[13] Coriasco, S., Johansson, K., Toft, J., Calculus, continuity and global wave-front properties for Fourier integral operators on $\mathbb{R}^d$. Preprint arXiv:1307.6249 (2014).

[14] Coriasco, S., Maniccia, L., Wave front set at infinity and hyperbolic linear operators with multiple characteristics. Ann. Global Anal. Geom., 24 (2003), 375–400.

[15] Coriasco, S., Maniccia, L., On the spectral asymptotics of operators on manifolds with ends. Abstr. Appl. Anal., Vol. 2013, Article ID 909782 (2013).
[16] Coriasco, S., Panarese, P., Fourier integral operators defined by classical symbols with exit behaviour. Math. Nachr., 242 (2002), 61-78.
[17] S. Coriasco and L. Rodino. Cauchy problem for *SG*-hyperbolic equations with constant multiplicities. *Ric. di Matematica*, 48, (Suppl.):25–43, 1999.
[18] Coriasco, S., Ruzhansky, M., Global $L^p$-continuity of Fourier integral operators. Trans. Amer. Math. Soc., Vol. 366, No. 5 (2014), 2575-2596.
[19] Coriasco, S., Schulz, R., Global wave front set of tempered oscillatory integrals with inhomogeneous phase functions. J. Fourier Anal. Appl., Vol. 19, No. 5 (2013), 1093–1121.
[20] Coriasco, S., Schulz, R., **SG**-Lagrangian submanifolds and their parametrization. Preprint arXiv:1406.1888 (2014).
[21] Coriasco, S., Toft, J., Asymptotic expansions for Hörmander symbol classes in the calculus of pseudo-differential operators. J. Pseudo-Differ. Oper. Appl., Vol. 5, No. 1 (2014), 27–41.
[22] Coriasco, S., Toft, J., Calculus for Fourier integral operators in generalized **SG** classes. Preprint, arXiv:1412.8050 (2014).
[23] Y. V. Egorov and B.-W. Schulze. *Pseudo-differential operators, singularities, applications*, volume 93 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997.
[24] H. Kumano-go. *Pseudo-differential operators*. MIT Press, 1981.
[25] L. Maniccia and P. Panarese. Eigenvalue asymptotics for a class of md-elliptic $\psi$do's on manifolds with cylindrical exits. *Ann. Mat. Pura Appl. (4)*, 181, 3:283–308, 2002.
[26] R. Melrose. *Geometric scattering theory.* Stanford Lectures. Cambridge University Press, Cambridge, 1995.
[27] C. Parenti. Operatori pseudodifferenziali in $\mathbb{R}^n$ e applicazioni. *Ann. Mat. Pura Appl.*, 93:359–389, 1972.
[28] Ruzhansky, M., Sugimoto, M., Global $L^2$ boundedness theorems for a class of Fourier integral operators. Comm. Partial Differential Equations, 31 (2006), 547–569.
[29] Schrohe, E., Spaces of weighted symbols and weighted Sobolev spaces on manifolds In: Proceedings, Oberwolfach. (H.O. Cordes, B. Gramsch, H. Widom, eds.), pp. 360-377. Springer LMN, Vol. 1256. New York, 1986.
[30] B.-W. Schulze. *Boundary value problems and singular pseudo-differential operators*. Pure and Applied Mathematics (New York). John Wiley & Sons Ltd., Chichester, 1998.
[31] M. A. Shubin. *Pseudodifferential operators and spectral theory*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1987.

Dipartimento di Matematica, Università degli Studi di Ferrara, via Machiavelli 30, 44121 Ferrara, Italy
*Email address*: alessia.ascanelli@unife.it

Dipartimento di Matematica "G. Peano", Università degli Studi di Torino, via Carlo Alberto 10. 10123 Torino, Italy
*Email address*: sandro.coriasco@unito.it