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### This is the author's manuscript

*Original Citation:*

*Availability:*

This version is available <http://hdl.handle.net/2318/1734182> since 2020-03-23T15:02:34Z

*Published version:*

DOI:10.1007/s10468-019-09931-2

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(Article begins on next page)

# COQUASI-BIALGEBRAS WITH PREANTIPODE AND RIGID MONOIDAL CATEGORIES

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ABSTRACT. By a theorem of Majid, every monoidal category with a neutral quasi-monoidal functor to finitely generated and projective  $\mathbb{k}$ -modules gives rise to a coquasi-bialgebra. We prove that if the category is also rigid, then the associated coquasi-bialgebra admits a preantipode, providing in this way an analogue for coquasi-bialgebras of Ulbrich's reconstruction theorem for Hopf algebras. When  $\mathbb{k}$  is field, this allows us to characterize coquasi-Hopf algebras as well in terms of rigidity of finite-dimensional corepresentations.

## INTRODUCTION

A well-known result in the theory of Hopf algebras states that one can reconstruct, in a suitable way, a Hopf algebra from its category of finite-dimensional corepresentations. In details, if  $\mathcal{C}$  is a  $\mathbb{k}$ -linear, abelian, rigid symmetric monoidal category which is essentially small, and if  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  is a  $\mathbb{k}$ -linear, exact, faithful, monoidal functor, then there exists a commutative Hopf algebra  $H$ , unique up to isomorphism, such that  $\omega$  factorizes through an equivalence of categories  $\omega^H : \mathcal{C} \rightarrow \mathfrak{M}_f^H$  followed by the forgetful functor; in fact  $H$  represents the functor

$$R \rightarrow \text{Aut}^\otimes(\omega \otimes R)$$

which associates any commutative  $\mathbb{k}$ -algebra  $R$  with the group of monoidal natural automorphisms of  $\omega \otimes R : \mathcal{C} \rightarrow \text{Mod}_R$  sending  $X$  to  $\omega(X) \otimes R$ , see [R], [DM] and [JS]. In particular, if  $\mathcal{C}$  is already the category of finite-dimensional right comodules over a commutative Hopf algebra  $A$ , then one can show that  $A \cong H$  as Hopf algebras. In [U], Ulbrich showed that even in case the symmetry condition is dropped, it is still possible to construct an associated Hopf algebra  $H$ .

In [M2], Majid extended this result to coquasi-bialgebras (or dual quasi-bialgebras), proving that if  $\mathcal{C}$  is an essentially small monoidal category endowed with a functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  that respects the tensor product in a suitable way (but that is not necessarily monoidal), then there is a coquasi-bialgebra  $H$  such that  $\omega$  factorizes through a monoidal functor  $\omega^H : \mathcal{C} \rightarrow {}^H\mathfrak{M}_f$  followed by the forgetful functor.

In [AP1], Ardizzoni and Pavarin introduced preantipodes to characterize those coquasi-bialgebras over a field for which a (suitable) structure theorem for coquasi-Hopf bicomodules holds and in [Sc4, Theorem 2.6] Schauenburg proved (in a

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2010 *Mathematics Subject Classification.* 18D10 (16T15).

*Key words and phrases.* Coquasi-bialgebra, preantipode, coquasi-Hopf algebra, rigid monoidal category, tensor functor, reconstruction.

This paper was written while the author was member of the "National Group for Algebraic and Geometric Structures and their Applications" (GNSAGA-INdAM). The author is sincerely grateful to Alessandro Ardizzoni and Claudia Menini for their contribution and to the referee for her/his useful suggestions.

non-constructive way) that preantipodes characterize also those coquasi-bialgebras whose category of finite-dimensional comodules is rigid.

Inspired by these results, we are going to show in Section 2 that if  $\mathcal{C}$  is an essentially small right rigid monoidal category together with a quasi-monoidal functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  to the category of finitely generated and projective  $\mathbb{k}$ -modules, then there exists a preantipode for the coendomorphism coquasi-bialgebra  $H$  of  $\omega$  (Proposition 2.13). In particular, this will allow us to reconstruct a coquasi-bialgebra with preantipode from its category of finite-dimensional left comodules, in the spirit of the classical Tannaka-Krein duality. Our approach presents three remarkable advantages. First of all, nowhere we will assume to have an isomorphism between the “underlying”  $\mathbb{k}$ -module  $\omega(X^*)$  of a dual object and the dual  $\mathbb{k}$ -module  $\omega(X)^*$  of the “underlying” object (as it is done for example in [M1, §9.4.1] or [H, §3]). In fact, we will see that natural isomorphisms  $\omega(X^*) \cong \omega(X)^*$  are in bijection with coquasi-Hopf algebra structures on  $H$ . Secondly, we will develop our main construction working over a generic commutative ring  $\mathbb{k}$  instead of over a field. Thirdly, we will not only show that a preantipode exists, but we will show how to construct it explicitly.

Then, we will apply this result to recover uniqueness of preantipodes and the fact that coquasi-bialgebra morphisms automatically preserves them, from their categorical counterparts. We will also recover the characterization of coquasi-bialgebras with preantipode as those coquasi-bialgebras whose category of finite-dimensional corepresentations is rigid and that of coquasi-Hopf algebras as those for which in addition  $\omega(-^*)$  and  $\omega(-)^*$  are isomorphic (Theorem 2.21). In conclusion, we will see in §3 how we can endow the finite dual coalgebra of a quasi-bialgebra with preantipode with a structure of coquasi-bialgebra with preantipode.

## 1. COQUASI-BIALGEBRAS AND PREANTIPODES

We extend here the notion of preantipode as it has been introduced in [AP1] to the case of coquasi-bialgebras over a commutative ring. We prove that when it exists, it has to be unique and that coquasi-bialgebra morphisms have to preserve it *under the additional assumption that the coquasi-bialgebras are  $\mathbb{k}$ -flat*.

**1.1. Monoidal categories and coquasi-bialgebras.** A *monoidal category* is a category  $\mathcal{C}$  endowed with a functor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , called the *tensor product*, with a distinguished object  $\mathbb{I}$ , called the *unit*, and with three natural isomorphisms

$$\alpha : \otimes (\otimes \times \text{Id}_{\mathcal{C}}) \rightarrow \otimes (\text{Id}_{\mathcal{C}} \times \otimes) \quad (\text{associativity constraint})$$

$$\iota : \otimes (\mathbb{I} \times \text{Id}_{\mathcal{C}}) \rightarrow \text{Id}_{\mathcal{C}}, \quad \tau : \otimes (\text{Id}_{\mathcal{C}} \times \mathbb{I}) \rightarrow \text{Id}_{\mathcal{C}} \quad (\text{left and right unit constraints})$$

that satisfy the *Pentagon* and the *Triangle Axioms*, that is, for all  $X, Y, Z, W$  in  $\mathcal{C}$

$$\begin{aligned} (X \otimes \alpha_{Y,Z,W}) \circ \alpha_{X,Y \otimes Z,W} \circ (\alpha_{X,Y,Z} \otimes W) &= \alpha_{X,Y,Z \otimes W} \circ \alpha_{X \otimes Y,Z,W}, \\ (X \otimes \iota_Y) \circ \alpha_{X,\mathbb{I},Y} &= \tau_X \otimes Y. \end{aligned}$$

A *quasi-monoidal functor* between  $(\mathcal{C}, \otimes, \mathbb{I}, \alpha, \iota, \tau)$  and  $(\mathcal{C}', \otimes', \mathbb{I}', \alpha', \iota', \tau')$  is a functor  $\omega : \mathcal{C} \rightarrow \mathcal{C}'$  together with an isomorphism  $\varphi_0 : \mathbb{I}' \rightarrow \omega(\mathbb{I})$  and a natural isomorphism  $\varphi = (\varphi_{X,Y} : \omega(X) \otimes' \omega(Y) \rightarrow \omega(X \otimes Y))_{X,Y \in \mathcal{C}}$  in  $\mathcal{C}'$ . Omitting the composition symbols, a quasi-monoidal functor  $\omega$  is said to be *neutral* if

$$(1) \quad \omega(\iota_X) \varphi_{\mathbb{I},X} (\varphi_0 \otimes' \omega(X)) = \iota'_{\omega(X)}, \quad \omega(\tau_X) \varphi_{X,\mathbb{I}} (\omega(X) \otimes' \varphi_0) = \tau'_{\omega(X)}$$

for all  $X$  in  $\mathcal{C}$ . Furthermore,  $\omega$  is said to be *monoidal*<sup>(1)</sup> if

$$(2) \quad \omega(\mathbf{a}_{X,Y,Z})\varphi_{X \otimes Y, Z}(\varphi_{X,Y} \otimes' \omega(Z)) = \varphi_{X,Y \otimes Z}(\omega(X) \otimes' \varphi_{Y,Z})\mathbf{a}'_{\omega(X), \omega(Y), \omega(Z)}$$

for all  $X, Y, Z$  in  $\mathcal{C}$ . It is said to be *strict* if  $\varphi_0$  and  $\varphi$  are the identities.

The notions of (co)algebra and (co)module over a (co)algebra can be introduced in the general setting of monoidal categories (see e.g. [AM, §1.2], where (co)algebras are called (co)monoids). Given an algebra  $A$  in  $\mathcal{C}$ , one can define the categories  ${}_A\mathcal{C}$ ,  $\mathcal{C}_A$  and  ${}_A\mathcal{C}_A$  of left, right and two-sided modules over  $A$ , respectively. Similarly, given a coalgebra  $C$  in  $\mathcal{C}$ , one can define the categories of  $C$ -comodules  ${}^C\mathcal{C}$ ,  $\mathcal{C}^C$ ,  ${}^C\mathcal{C}^C$ .

Henceforth and unless stated otherwise, we will fix a base commutative ring  $\mathbb{k}$  and we will assume to work in the monoidal category  $\mathfrak{M}$  of  $\mathbb{k}$ -modules: all (co)algebras will be  $\mathbb{k}$ -(co)algebras, the unadorned tensor product  $\otimes$  will denote the tensor product over  $\mathbb{k}$  and  $\text{Hom}(V, W)$  will be the set of  $\mathbb{k}$ -linear morphisms from  $V$  to  $W$ . We will often omit the composition symbols between maps as we did above. In order to deal with comultiplications and coactions, we will use the following variation of *Sweedler's Sigma Notation* (cf. [Sw, §1.2])

$$\Delta(x) := \sum x_1 \otimes x_2, \quad \rho_V^r(v) := \sum v_0 \otimes v_1, \quad \rho_W^l(w) := \sum w_{-1} \otimes w_0$$

for every coalgebra  $C$ , right  $C$ -comodule  $V$ , left  $C$ -comodule  $W$  and for all  $x \in C$ ,  $v \in V$  and  $w \in W$ . Recall that

- if  $W$  is a left  $C$ -comodule finitely generated and projective over  $\mathbb{k}$ , then its linear dual  $W^* := \text{Hom}(W, \mathbb{k})$  is naturally a right  $C$ -comodule with  $\sum f_0 \otimes f_1$  uniquely determined by  $\sum f_0(w)f_1 = \sum w_{-1}f(w_0)$  for all  $w \in W$ ;
- if  $A$  is an algebra, then  $\text{Hom}(C, A)$  is an algebra with composition law defined by  $(f * g)(x) = \sum f(x_1)g(x_2)$  for all  $f, g \in \text{Hom}(C, A)$  and  $x \in C$  (the *convolution product*) and if  $M$  is an  $A$ -bimodule then we may as well consider  $(f * \phi * g)(x) = \sum f(x_1) \cdot \phi(x_2) \cdot g(x_3)$  for all  $f, g \in \text{Hom}(C, A)$ ,  $\phi \in \text{Hom}(C, M)$  and  $x \in C$ .

The following result is formally dual to [ABM, Theorem 1] and has already been mentioned in [Sc2, §2.3]. Since the proof is quite long, technical and not of particular interest, it is omitted.

**Proposition 1.1.** *For a coalgebra  $(C, \Delta, \varepsilon)$  there is a bijective correspondence between*

- *monoidal structures on  ${}^C\mathfrak{M}$  such that the underlying functor  $\mathcal{U} : {}^C\mathfrak{M} \rightarrow \mathfrak{M}$  is quasi-monoidal;*
- *sets of morphisms  $\{m, u, \omega, l, r\}$  such that  $\omega : C \otimes C \otimes C \rightarrow \mathbb{k}$ ,  $l, r : C \rightarrow \mathbb{k}$  are convolution invertible linear maps,  $m : C \otimes C \rightarrow C$ ,  $u : \mathbb{k} \rightarrow C$  are coalgebra morphisms and*

$$(3) \quad \omega(C \otimes C \otimes m) * \omega(m \otimes C \otimes C) = (\varepsilon \otimes \omega) * \omega(C \otimes m \otimes C) * (\omega \otimes \varepsilon),$$

$$(4) \quad \omega(C \otimes u \otimes C) = r^{-1} \otimes l, \quad m(u \otimes C) * l = l * C, \quad m(C \otimes u) * r = r * C, \\ m(C \otimes m) * \omega = \omega * m(m \otimes C).$$

A *coquasi-bialgebra* (or *dual quasi-bialgebra*) is a coassociative and counital coalgebra  $(H, \Delta, \varepsilon)$  endowed with a multiplication  $m : H \otimes H \rightarrow H$ , a unit  $u : \mathbb{k} \rightarrow H$  and three linear maps  $\omega : H \otimes H \otimes H \rightarrow \mathbb{k}$ ,  $l, r : H \rightarrow \mathbb{k}$  such that the conditions

<sup>(1)</sup>In [AM, Definition 3.5], these are called *strong* monoidal functors.

of Proposition 1.1 are satisfied. We set  $1_H := u(1_{\mathbb{k}})$  and we refer to  $\omega$  as the *reassociator* of the coquasi-bialgebra. A *morphism of coquasi-bialgebras*

$$f : (H, m, u, \Delta, \varepsilon, \omega, l, r) \rightarrow (H', m', u', \Delta', \varepsilon', \omega', l', r')$$

is a coalgebra homomorphism  $f : (H, \Delta, \varepsilon) \rightarrow (H', \Delta', \varepsilon')$  such that

$$m'(f \otimes f) = f m, \quad f u = u', \quad \omega'(f \otimes f \otimes f) = \omega, \quad l' f = l, \quad r' f = r.$$

In particular, the category  ${}^H\mathfrak{M}$  of left comodules over a coquasi-bialgebra  $H$  comes endowed with a monoidal structure such that the underlying functor  $\mathcal{U} : {}^H\mathfrak{M} \rightarrow \mathfrak{M}$  is a strict quasi-monoidal functor. Explicitly, given two left  $H$ -comodules  $V$  and  $W$ , their tensor product  $V \otimes W$  is an  $H$ -comodule via the diagonal coaction  $\rho_{V \otimes W}(v \otimes w) = \sum v_{-1} w_{-1} \otimes v_0 \otimes w_0$ . The unit is  $\mathbb{k}$ , regarded as a left  $H$ -comodule via the trivial coaction  $\rho_{\mathbb{k}}(k) = 1_H \otimes k$ . The constraints are given by

$$\begin{aligned} \mathbf{a}_{U,V,W}(u \otimes v \otimes w) &:= \sum \omega^{-1}(u_{-1} \otimes v_{-1} \otimes w_{-1}) u_0 \otimes v_0 \otimes w_0, \\ \mathbf{l}_V(1_{\mathbb{k}} \otimes v) &= \sum l(v_{-1}) v_0 \quad \mathbf{r}_V(v \otimes 1_{\mathbb{k}}) = \sum r(v_{-1}) v_0 \end{aligned}$$

for every  $U, V, W \in {}^H\mathfrak{M}$  and all  $u \in U, v \in V, w \in W$ . Moreover, every morphism of coquasi-bialgebras  $f : H \rightarrow H'$  induces a strict monoidal functor  ${}^f\mathfrak{M} : {}^H\mathfrak{M} \rightarrow {}^{H'}\mathfrak{M}$ , which is given by the assignments

$${}^f\mathfrak{M}(X, \rho_X : X \rightarrow H \otimes X) = (X, (f \otimes X) \rho_X), \quad {}^f\mathfrak{M}(\gamma : X \rightarrow Y) = \gamma.$$

**Remark 1.2** (Compare with [D]). Let  $H$  be a coquasi-bialgebra and consider a convolution invertible element  $F : H \otimes H \rightarrow \mathbb{k}$ . Define

$$\begin{aligned} \omega_F &:= (\varepsilon \otimes F) * F(H \otimes m) * \omega * F^{-1}(m \otimes H) * (F^{-1} \otimes \varepsilon) \\ m_F &:= F * m * F^{-1}, \quad l_F := F(u \otimes H) * l, \quad r_F := F(H \otimes u) * r. \end{aligned}$$

Then  $H_F := (H, \Delta, \varepsilon, m_F, u, \omega_F, l_F, r_F)$  is still a coquasi-bialgebra and we say that it has been obtained from  $H$  by *twisting via the element  $F$* . Notice that, since the coalgebra structure has not been touched,  ${}^H\mathfrak{M} = {}^{H_F}\mathfrak{M}$ . What is changing is the monoidal structure on  ${}^H\mathfrak{M}$ . In fact, it can be checked that a coquasi-bialgebra  $H'$  has been obtained from another coquasi-bialgebra  $H$  by twisting via an element  $F$  (i.e.  $H' \cong H_F$  as coquasi-bialgebras) if and only if there exists a monoidal equivalence  $\mathcal{G} : {}^H\mathfrak{M} \rightarrow {}^{H'}\mathfrak{M}$  (i.e. a monoidal functor which is also an equivalence of categories) such that  $\mathcal{U}' \circ \mathcal{G} = \mathcal{U}$ , where  $\mathcal{U}, \mathcal{U}'$  are the obvious forgetful functors. In particular, there is a bijective correspondence between convolution invertible elements  $F \in (H \otimes H)^*$  and monoidally isomorphic monoidal structures on  ${}^H\mathfrak{M}$ .

If  $H'$  has been obtained from a coquasi-bialgebra  $H$  by twisting via an element  $F$ , then we say that  $H'$  is *(twist) equivalent* to  $H$ . The interested reader may verify that being (twist) equivalent for coquasi-bialgebras is an equivalence relation: (i) every coquasi-bialgebra is equivalent to itself via  $F = \varepsilon \otimes \varepsilon$ ; (ii) if  $g : H_F \rightarrow H'$  is an isomorphism of coquasi-bialgebras and  $F' := F^{-1}(g^{-1} \otimes g^{-1})$ , then  $g^{-1} : H'_{F'} \rightarrow H$  is an isomorphism of coquasi-bialgebras; (iii) if  $g : H_F \rightarrow H'$  and  $g' : H'_{F'} \rightarrow H''$  are isomorphisms of coquasi-bialgebras and  $F'' := F'(g \otimes g) * F$ , then  $g'g : H_{F''} \rightarrow H''$  is an isomorphism of coquasi-bialgebras. In particular, a coquasi-bialgebra  $H$  is equivalent to  $H_F$  for every convolution invertible element  $F \in (H \otimes H)^*$ .

Now, notice that  $m(1_H \otimes 1_H) = l(1_H) 1_H l^{-1}(1_H) = 1_H$  and hence, by resorting to (3),  $\omega(1_H \otimes 1_H \otimes 1_H)^3 = \omega(1_H \otimes 1_H \otimes 1_H)^2$ . Since  $\omega$  is convolution invertible,  $\omega(1_H \otimes 1_H \otimes 1_H)$  is invertible in  $\mathbb{k}$  and so we conclude that  $\omega(1_H \otimes 1_H \otimes 1_H) = 1_{\mathbb{k}}$ .

As a consequence, from (4) we deduce that  $r^{-1}(1_H)l(1_H) = 1_{\mathbb{k}}$  and therefore  $p := r(1_H) = l(1_H)$  is a well-defined invertible element in  $\mathbb{k}$ . If we consider  $F := pr^{-1} \otimes l^{-1}$ , this is a convolution invertible element such that  $l_F = \varepsilon = r_F$ . Thus, any coquasi-bialgebra is (twist) equivalent to one in which  $l = \varepsilon = r$ .

In light of Remark 1.2, we will focus only on the latter case and from now on all coquasi-bialgebras will satisfy

$$(4') \quad \omega(C \otimes u \otimes C) = \varepsilon \otimes \varepsilon, \quad m(u \otimes C) = C = m(C \otimes u)$$

instead of relations (4). Moreover, all quasi-monoidal functors will be neutral and hence we will omit to specify it.

Dually to coquasi-bialgebras we have *quasi-bialgebras*, that is to say, ordinary algebras  $A$  with a counital comultiplication which is coassociative up to conjugation by a suitable invertible element  $\Phi \in A \otimes A \otimes A$ .

**1.2. Preantipodes for coquasi-bialgebras.** The following definition traces word by word [AP1, Definition 3.6].

**Definition 1.3.** A *preantipode* for a coquasi-bialgebra  $H$  is a  $\mathbb{k}$ -linear endomorphism  $S : H \rightarrow H$  such that, for all  $h \in H$ ,

$$(5) \quad \begin{aligned} \sum S(h_1)_1 h_2 \otimes S(h_1)_2 &= 1_H \otimes S(h), \\ \sum S(h_2)_1 \otimes h_1 S(h_2)_2 &= S(h) \otimes 1_H, \\ \sum \omega(h_1 \otimes S(h_2) \otimes h_3) &= \varepsilon(h). \end{aligned}$$

**Remark 1.4.** Let  $H$  be a coquasi-bialgebra with a preantipode  $S$ . Then

$$\sum h_1 S(h_2) = \varepsilon S(h) 1_H = \sum S(h_1) h_2$$

for all  $h \in H$ . In particular, if  $\omega = \varepsilon \otimes \varepsilon \otimes \varepsilon$  (i.e. if  $H$  is an ordinary bialgebra) then  $\varepsilon S(h) = \varepsilon(h)$  and  $S$  is an ordinary antipode.

A coquasi-bialgebra  $H$  turns out to be an algebra in the monoidal category  ${}^H\mathfrak{M}^H$  [Sc4, §2]. Thus we may consider the so-called category of *right coquasi-Hopf  $H$ -bicomodules*  ${}^H\mathfrak{M}_H^H := ({}^H\mathfrak{M}^H)_H$ .

Assume that  $H$  is flat over  $\mathbb{k}$ . Then the functor  $F : {}^H\mathfrak{M} \rightarrow {}^H\mathfrak{M}_H^H$  given by  $F(V) := V \otimes H$  admits a right adjoint  $G : {}^H\mathfrak{M}_H^H \rightarrow {}^H\mathfrak{M}$ ,  $G(M) := M^{\text{co}H}$ , where  $M^{\text{co}H} := \{m \in M \mid m_0 \otimes m_1 = m \otimes 1_H\}$  is the space of right  $H$ -coinvariant elements in  $M$ . The counit  $\epsilon : FG \rightarrow \text{id}$  and the unit  $\eta : \text{id} \rightarrow GF$  of the adjunction are given respectively by  $\epsilon_M(x \otimes h) := xh$  and  $\eta_N(n) := n \otimes 1_H$  for every  $M \in {}^H\mathfrak{M}_H^H$ ,  $N \in {}^H\mathfrak{M}$  and for all  $m \in M$ ,  $n \in N$ ,  $h \in H$  (we refer to [AP1] for details). Then one can mimic step by step the proof of [AP1, Theorem 3.9] to prove the following.

**Theorem 1.5.** *Under the standing assumption that  $H$  is  $\mathbb{k}$ -flat, the adjunction  $(F, G)$  is an equivalence of categories if and only if  $H$  admits a preantipode.*

We might have given now a direct proof of the fact that coquasi-bialgebra morphisms preserve preantipodes, but we opted for a less direct approach relying on the subsequent proposition suggested by Alessandro Ardizzoni. The effort is the same and we think that the general result in Proposition 1.6 deserves to be highlighted, as it may find applications in other contexts.

**Proposition 1.6.** *Let  $(C, \Delta_C, \varepsilon_C)$  be a coalgebra and let  $(H, \Delta_H, \varepsilon_H, m, u, \omega, S)$  be a  $\mathbb{k}$ -flat coquasi-bialgebra with preantipode  $S$ . Assume that  $g, h : C \rightarrow H$  are  $\mathbb{k}$ -linear maps such that  $g$  is a coalgebra morphism and  $g$  and  $h$  satisfy:*

$$(6) \quad \sum h(z_2)_1 \otimes g(z_1)h(z_2)_2 = h(z) \otimes 1_H,$$

$$(7) \quad \sum h(z_1)_1 g(z_2) \otimes h(z_1)_2 = 1_H \otimes h(z),$$

$$(8) \quad \sum \omega(g(z_1) \otimes h(z_2) \otimes g(z_3)) = \varepsilon(z),$$

for all  $z \in C$ . Then  $h = Sg$ .

*Proof.* As in [AP1, §3.5], consider the coquasi-Hopf bicomodule  $H \hat{\otimes} H := H \otimes H$  with explicit structures given by

$$\begin{aligned} \rho^r(x \otimes y) &= \sum x_1 \otimes y_1 \otimes x_2 y_2, & \rho^l(x \otimes y) &= \sum y_1 \otimes x \otimes y_2, \\ (x \otimes y)h &= \sum x_1 \otimes y_1 h_1 \omega(x_2 \otimes y_2 \otimes h_2), \end{aligned}$$

for all  $x, y, h \in H$ . Consider also the distinguished component  $\hat{\varepsilon} : (H \hat{\otimes} H)^{\text{co}H} \otimes H \rightarrow H \hat{\otimes} H$  of the counit of the adjunction  $(F, G)$ , which is given explicitly by  $\hat{\varepsilon}(x \otimes y \otimes h) = \sum x_1 \otimes y_1 h_1 \omega(x_2 \otimes y_2 \otimes h_2)$ . Since  $H$  admits a preantipode, it is invertible with inverse  $\hat{\varepsilon}^{-1}(x \otimes y) = \sum ((x_1 \otimes S(x_2)) \otimes x_3) y$  for all  $x, y, h \in H$ . Finally, consider the assignment  $\beta : C \rightarrow H \otimes H \otimes H$  given by

$$\beta(z) = \sum g(z_1) \otimes h(z_2) \otimes g(z_3)$$

for all  $z \in C$ . Observe that

$$\begin{aligned} \rho^r\left(\sum g(z_1) \otimes h(z_2)\right) &= \sum g(z_1)_1 \otimes h(z_2)_1 \otimes g(z_1)_2 h(z_2)_2 \\ &\stackrel{(*)}{=} \sum g(z_1) \otimes h(z_3)_1 \otimes g(z_2) h(z_3)_2 \stackrel{(6)}{=} \sum g(z_1) \otimes h(z_2) \otimes 1_H, \end{aligned}$$

where in  $(*)$  we used the hypothesis that  $g$  is comultiplicative, whence  $\sum g(z_1) \otimes h(z_2) \in (H \hat{\otimes} H)^{\text{co}H}$  for all  $z \in C$ . Therefore for all  $z \in C$  we can compute

$$\begin{aligned} \hat{\varepsilon}\beta(z) &= \hat{\varepsilon}\left(\sum g(z_1) \otimes h(z_2) \otimes g(z_3)\right) \\ &= \sum g(z_1)_1 \otimes h(z_2)_1 g(z_3)_1 \omega(g(z_1)_2 \otimes h(z_2)_2 \otimes g(z_3)_2) \\ &= \sum g(z_1) \otimes h(z_3)_1 g(z_4) \omega(g(z_2) \otimes h(z_3)_2 \otimes g(z_5)) \\ &\stackrel{(7)}{=} \sum g(z_1) \otimes 1_H \omega(g(z_2) \otimes h(z_3) \otimes g(z_4)) \stackrel{(8)}{=} g(z) \otimes 1_H \end{aligned}$$

so that

$$\beta(z) = \hat{\varepsilon}^{-1}(g(z) \otimes 1_H) = \sum g(z)_1 \otimes S(g(z)_2) \otimes g(z)_3 = \sum g(z_1) \otimes S(g(z_2)) \otimes g(z_3)$$

and, by applying  $\varepsilon \otimes H \otimes \varepsilon$  to both sides,

$$\begin{aligned} h(z) &= \sum \varepsilon(g(z_1)) h(z_2) \varepsilon(g(z_3)) = (\varepsilon \otimes H \otimes \varepsilon) \left( \sum g(z_1) \otimes h(z_2) \otimes g(z_3) \right) \\ &= (\varepsilon \otimes H \otimes \varepsilon) (\beta(z)) = (\varepsilon \otimes H \otimes \varepsilon) \left( \sum g(z_1) \otimes S(g(z_2)) \otimes g(z_3) \right) = S(g(z)). \quad \square \end{aligned}$$

**Proposition 1.7.** *If  $(H, S_H), (L, S_L)$  are  $\mathbb{k}$ -flat coquasi-bialgebras with preantipode and  $f : H \rightarrow L$  is a morphism of coquasi-bialgebras, then  $fS_H = S_L f$ . In particular, the preantipode for a  $\mathbb{k}$ -flat coquasi-bialgebra  $H$  is unique.*

*Proof.* Since  $f$  is a morphism of coquasi-bialgebras, it is in particular a coalgebra morphism and  $fS_H = h$  satisfies (6), (7) and (8) of Proposition 1.6, whence  $fS_H = S_L f$ . Now, assume that  $S$  and  $T$  are two preantipodes for  $H$ . The first claim applied to the case  $(H, S), (H, T)$  and  $f = \text{Id}_H$  entails that  $S = T$ .  $\square$

**Remark 1.8.** Notice that the functor  $(-)^{\text{co}H} : {}^H\mathfrak{M}_H^H \rightarrow {}^H\mathfrak{M}$  needs not to be well-defined if the functor  $H \otimes -$  does not preserve, at least, coreflexive equalizers (i.e. equalizers of parallel arrows admitting a common retraction).

## 2. COQUASI-BIALGEBRAS WITH PREANTIPODE AND RIGID MONOIDAL CATEGORIES

It is well-known that every rigid monoidal category together with a monoidal functor to the category of finitely generated and projective  $\mathbb{k}$ -modules gives rise to a Hopf algebra. Via a variant of the same Tannaka-Kreĭn reconstruction process, it has been shown by Majid in [M2] that every monoidal category  $\mathcal{C}$  together with a quasi-monoidal functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  gives rise to a coquasi-bialgebra  $H$  instead. A very natural question then is what happens if the category  $\mathcal{C}$  is also rigid.

Our aim in this section is to show how this rigidity is related with the existence of a preantipode on  $H$ . We will do this without any additional assumption on  $\mathcal{C}$ . In particular, we will implicitly allow  $\omega(X^*)$  and  $\omega(X)^*$  to be non-isomorphic objects, as it happens for example in [Sc1, Example 4.5.1].

**Remark 2.1.** The following observation about a previous version of the present paper has been brought to our attention and deserves to be highlighted. Assume that  $\mathbb{k}$  is a field and consider a category  $\mathcal{C}$  together with a functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$ . Let  $C$  be the reconstructed coalgebra and denote by  $\omega^C : \mathcal{C} \rightarrow {}^C\mathfrak{M}_f$  the induced functor. Then every finite-dimensional  $C$ -comodule can be recovered from comodules of the form  $\omega^C(X)$  by taking finite direct sums, kernels and cokernels (see [Sc3, Corollary 2.2.9]). Since in an abelian monoidal category with exact tensor product (e.g. the category of comodules over a  $\mathbb{k}$ -coquasi-bialgebra), the family of rigid objects is closed under finite biproducts, kernels and cokernels, if  $\mathcal{C}$  is rigid then  ${}^C\mathfrak{M}_f$  has to be rigid as well and hence  $C$  admits a preantipode in light of [Sc4, Theorem 2.6].

This purely categorical argument shows that, at least when  $\mathbb{k}$  is a field, the validity of our reconstruction theorem should not be surprising. However, the main focus in the present paper is not only on proving the existence of a preantipode for the reconstructed coquasi-bialgebra (even when  $\mathbb{k}$  is just a commutative ring), but also to provide an explicit construction of it (thing that, up to our knowledge, cannot be obtained from the foregoing approach).

**2.1. The classical reconstruction.** The results in this subsection are well-known. Nevertheless, we retrieve the main steps of the classical reconstruction process for the sake of the unaccustomed reader. We refer to [M2] and [Sc3] for further details.

Let  $(\mathcal{C}, \boxtimes, \mathbb{I}, \mathbf{a}, \mathbf{l}, \mathbf{r})$  be an essentially small monoidal category equipped with a quasi-monoidal functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  from  $\mathcal{C}$  into the category of finitely generated and projective  $\mathbb{k}$ -modules. This means that in  $\mathfrak{M}$  we have a family of isomorphisms  $\varphi_{X,Y} : \omega(X) \otimes \omega(Y) \rightarrow \omega(X \boxtimes Y)$ , which is natural in both components, and an isomorphism  $\varphi_0 : \mathbb{k} \rightarrow \omega(\mathbb{I})$  compatible with the left and right unit constraints as in (1). For every  $\mathbb{k}$ -module  $V$  and every  $n \geq 1$ , denote by  $\omega^n : \mathcal{C}^n \rightarrow \mathfrak{M}_f$  the functor mapping every  $n$ -uple of objects  $(X_1, \dots, X_n)$  in  $\mathcal{C}^n$  to the tensor product  $\omega(X_1) \otimes \dots \otimes \omega(X_n)$  in  $\mathfrak{M}$  and by  $\text{Nat}(\omega^n, V \otimes \omega^n)$  the set of natural transformations between  $\omega^n$  and the functor  $V \otimes \omega^n : \mathcal{C}^n \rightarrow \mathfrak{M}$ , sending  $(X_1, \dots, X_n)$



to  $V \otimes \omega^n(X_1, \dots, X_n)$ . It turns out the functor  $\text{Nat}(\omega^n, - \otimes \omega^n) : \mathfrak{M} \rightarrow \text{Set}$  is represented by the  $n$ -fold tensor product  $H_{\omega}^{\otimes n}$  of a suitable coquasi-bialgebra  $H_{\omega}$  via a natural isomorphism

$$(9) \quad \vartheta^n : \text{Hom}(H_{\omega}^{\otimes n}, -) \cong \text{Nat}(\omega^n, - \otimes \omega^n).$$

For all  $X_1, \dots, X_n$  in  $\mathcal{C}$ ,  $V$  in  $\mathfrak{M}$  and  $f \in \text{Hom}(H_{\omega}^{\otimes n}, V)$ , this is given explicitly by

$$\vartheta_V^n(f)_{X_1, \dots, X_n} = (f \otimes \omega^n(X_1, \dots, X_n)) \tau_n(\delta_{X_1} \otimes \dots \otimes \delta_{X_n})$$

where  $\delta := \vartheta_H(\text{id}_H) : \omega \rightarrow H \otimes \omega$ ,  $\tau_n := \tau_{\omega^{n-1}(X_1, \dots, X_{n-1}), H} \circ \dots \circ \tau_{\omega(X_1), H}$  and  $\tau_{V, W} : V \otimes W \rightarrow W \otimes V$  denotes the natural transformation acting as  $\tau_{V, W}(v \otimes w) = w \otimes v$  for every pair of objects  $V, W$  in  $\mathfrak{M}$ . Since  $\omega$  is fixed, we may write  $H$  instead of  $H_{\omega}$  and we refer to it as the *coendomorphism coquasi-bialgebra* of  $\omega$ . As a  $\mathbb{k}$ -module, it is defined to be the coend<sup>(2)</sup> of the functor  $\omega \otimes \omega^*$  from  $\mathcal{C} \times \mathcal{C}^{\text{op}}$  to  $\mathfrak{M}$ . The comultiplication  $\Delta$  and the counit  $\varepsilon$  are the unique linear maps such that  $\vartheta_{H \otimes H}(\Delta) = (H \otimes \delta) \delta$  and  $\vartheta_{\mathbb{k}}(\varepsilon) = \text{id}_{\omega}$ . The multiplication  $m : H \otimes H \rightarrow H$  is uniquely given by the relation  $(H \otimes \varphi_{X, Y}) \vartheta_H^2(m)_{X, Y} = \delta_{X \boxtimes Y} \varphi_{X, Y}$  while the reassociator  $\omega \in (H \otimes H \otimes H)^*$  satisfies

$$(10) \quad \varphi_{X \boxtimes Y, Z}(\varphi_{X, Y} \otimes \omega(Z)) \vartheta_{\mathbb{k}}^3(\omega)_{X, Y, Z} = \omega(\mathfrak{a}_{X, Y, Z}^{-1}) \varphi_{X, Y \boxtimes Z}(\omega(X) \otimes \varphi_{Y, Z})$$

for all  $X, Y, Z$  in  $\mathcal{C}$ . The unit is the unique morphism  $u : \mathbb{k} \rightarrow H$  such that

$$(11) \quad (H \otimes \varphi_0)(u \otimes \mathbb{k}) = \delta_{\mathbb{I}} \varphi_0.$$

Observe that every  $\omega(X)$  is an  $H$ -comodule via  $\rho_{\omega(X)} = \delta_X$  and that  $\vartheta_H^2(m)_{X, Y} = \rho_{\omega(X) \otimes \omega(Y)}$  and  $u \otimes \mathbb{k} = \rho_{\mathbb{k}}$  are exactly the coactions that makes of  ${}^H\mathfrak{M}$  a monoidal category. Summing up, we have the following central result.

**Theorem 2.2** ([M2, Theorem 2.2]). *Let  $(\mathcal{C}, \boxtimes, \mathbb{I}, \mathfrak{a}, \mathfrak{l}, \mathfrak{r})$  be an essentially small monoidal category and let  $(\omega, \varphi, \varphi_0) : \mathcal{C} \rightarrow \mathfrak{M}_f$ , be a quasi-monoidal functor. Then there is a coquasi-bialgebra  $H$ , unique up to isomorphism, universal with the property that  $\omega$  factorizes as a monoidal functor  $\omega^H : \mathcal{C} \rightarrow {}^H\mathfrak{M}$  followed by the forgetful functor. Universal means that if  $H'$  is another such coquasi-bialgebra then there is a unique map of coquasi-bialgebras  $\epsilon : H \rightarrow H'$  inducing a functor  ${}^{\epsilon}\mathfrak{M} : {}^H\mathfrak{M} \rightarrow {}^{H'}\mathfrak{M}$  such that  ${}^{\epsilon}\mathfrak{M} \omega^H = \omega^{H'} : \mathcal{C} \rightarrow {}^{H'}\mathfrak{M}$ .*

In [M2] there's no explicit reference to the unitality of the multiplication or of the reassociator. Nevertheless, it can be checked that the above constructed maps satisfy all the conditions defining a coquasi-bialgebra.

**Remark 2.3.** Assume that  $\mathbb{k}$  is a field,  $\mathcal{C}$  is already the category  ${}^B\mathfrak{M}_f$  of finite-dimensional comodules over a coquasi-bialgebra  $B$  and  $\omega$  is already the forgetful functor  $\mathcal{U} : {}^B\mathfrak{M}_f \rightarrow \mathfrak{M}_f$ . Then  $B$  itself is a representing object for  $\text{Nat}(\mathcal{U}, - \otimes \mathcal{U})$  (cf. e.g. [Sc3, Lemma 2.2.1]). In this case, the (co)multiplication, the (co)unit and the reassociator of  $B$  already satisfy the defining relations for  $\Delta$ ,  $\varepsilon$ ,  $m$ ,  $u$  and  $\omega$  stated in §2.1, whence they are the unique coquasi-bialgebra structure maps induced on the  $\mathbb{k}$ -vector space  $B$  by the isomorphisms (9) in view of Theorem 2.2.

<sup>(2)</sup>see e.g. [ML, §IX.6] for details about the coend construction

**2.2. The rigid case.** We recall briefly some facts about rigid objects in a monoidal category.

**Definition 2.4.** A *right dual object*  $X^*$  of  $X$  in  $\mathcal{C}$  is a triple  $(X^*, \text{ev}_X, \text{db}_X)$  in which  $X^*$  is an object in  $\mathcal{C}$  and  $\text{ev}_X : X \boxtimes X^* \rightarrow \mathbb{I}$  and  $\text{db}_X : \mathbb{I} \rightarrow X^* \boxtimes X$  are morphisms in  $\mathcal{C}$ , called *evaluation* and *dual basis* respectively, that satisfy

$$(12) \quad (\text{ev}_X \boxtimes X) \alpha_{X, X^*, X}^{-1} (X \boxtimes \text{db}_X) = \text{id}_X,$$

$$(13) \quad (X^* \boxtimes \text{ev}_X) \alpha_{X^*, X, X^*} (\text{db}_X \boxtimes X^*) = \text{id}_{X^*}.$$

An object which admits a right dual object is said to be *right rigid* (or *dualizable*). If every object in  $\mathcal{C}$  is right rigid, then we say that  $\mathcal{C}$  is *right rigid*.

We will often refer to right dual objects simply as *right duals* or just *duals*.

**Remark 2.5.** Once chosen a right dual object  $X^*$  for every object  $X$  in a right rigid monoidal category  $\mathcal{C}$ , we have that the assignment  $(-)^* : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  defines a functor and  $\text{ev} : (-) \boxtimes (-)^* \rightarrow \mathbb{I}$  and  $\text{db} : \mathbb{I} \rightarrow (-)^* \boxtimes (-)$  define dinatural transformations<sup>(3)</sup>, i.e., for every  $X, Y$  and  $f : X \rightarrow Y$  in  $\mathcal{C}$  we have  $(f^* \boxtimes Y) \text{db}_Y = (X^* \boxtimes f) \text{db}_X$  and  $\text{ev}_X (X \boxtimes f^*) = \text{ev}_Y (f \boxtimes Y^*)$ .

From now on, let us assume that  $\mathcal{C}$  is right rigid. If we have a different choice  $(-)^{\vee} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  of right dual objects, then we write  $\text{ev}^{(*)}$  and  $\text{db}^{(*)}$  to mean the evaluation and dual basis maps associated with the dual  $(-)^*$  and  $\text{ev}^{(\vee)}$  and  $\text{db}^{(\vee)}$  to mean those associated with  $(-)^{\vee}$ . We know (see e.g. [M1, §9.3]) that for every  $X$  in  $\mathcal{C}$ , its right dual is unique up to isomorphism whenever it exists, i.e. we have an isomorphism  $\kappa_X : X^* \rightarrow X^{\vee}$  in  $\mathcal{C}$  given by the composition

$$(14) \quad \kappa_X := \tau_{X^{\vee}} \left( X^{\vee} \boxtimes \text{ev}_X^{(*)} \right) \alpha_{X^{\vee}, X, X^*} \left( \text{db}_X^{(\vee)} \boxtimes X^* \right) \Gamma_{X^*}^{-1}.$$

**Lemma 2.6.** *The isomorphism  $\kappa_X : X^* \rightarrow X^{\vee}$  is natural in  $X$  and the dinatural transformations  $\text{ev}^{(*)}$ ,  $\text{db}^{(*)}$ ,  $\text{ev}^{(\vee)}$  and  $\text{db}^{(\vee)}$  satisfy*

$$(15) \quad (\kappa \boxtimes \text{id}) \text{db}^{(*)} = \text{db}^{(\vee)} \quad \text{and} \quad \text{ev}^{(\vee)} (\text{id} \boxtimes \kappa) = \text{ev}^{(*)}.$$

**Notation 2.7.** In what follows we will retrieve some computations in terms of braided diagrams in the category of  $\mathbb{k}$ -modules. To this aim, let us agree on the following notation

$$\Delta = \begin{array}{c} H \\ \curvearrowright \\ H \quad H \end{array}, \quad \varepsilon = \begin{array}{c} H \\ \downarrow \\ \bullet \end{array}, \quad u = \begin{array}{c} \bullet \\ \uparrow \\ H \end{array}, \quad m = \begin{array}{c} H \quad H \\ \cup \\ H \end{array}, \quad \tau_{V, W} = \begin{array}{c} V \quad W \\ \curvearrowright \\ W \quad V \end{array}, \quad \delta_X = \begin{array}{c} X \\ \diagdown \quad \diagup \\ H \quad X \end{array}.$$

We will also omit to write the functor  $\omega$  in braided diagrams.

Henceforth and unless stated otherwise, we assume also that a choice  $(-)^*$  of dual objects has been performed. Let us consider the following maps

$$(16) \quad \text{ev}_{\omega(X)} := \varphi_0^{-1} \omega(\text{ev}_X) \varphi_{X, X^*} \quad \text{and} \quad \text{db}_{\omega(X)} := \varphi_{X^*, X}^{-1} \omega(\text{db}_X) \varphi_0,$$

which we will represent simply as  $\text{ev}_{\omega(X)} = \bigcup_{X^*}^{X \quad X^*}$  and  $\text{db}_{\omega(X)} = \bigcap_{X^*}^{X \quad X^*}$ .

These do not endow  $\omega(X^*)$  with a structure of right dual object of  $\omega(X)$  in the category  $\mathfrak{M}$  because the functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}$  does not satisfy the associativity

<sup>(3)</sup>More precisely, these should be referred to as *wedges*, since they are dinatural transformations to a constant functor. However, we avoided this in order to spare the proliferation of terminology. For the definition of dinatural transformations and wedges we refer to [ML, §9.4].

condition (2). Nevertheless, we have the following result, whose proof follows easily from the definitions and the dinaturality of  $\mathbf{ev}$  and  $\mathbf{db}$ .

**Lemma 2.8.** *The assignments  $\mathbf{ev}_{\omega(X)}$  and  $\mathbf{db}_{\omega(X)}$  defined in (16) give rise to dinatural transformations  $\mathbf{ev}_{\omega(-)} : \omega \otimes \omega^* \rightarrow \mathbb{k}$  and  $\mathbf{db}_{\omega(-)} : \mathbb{k} \rightarrow \omega^* \otimes \omega$ .*

**Remark 2.9.** Recall that if  $(\mathcal{F}, \phi, \phi_0) : (\mathcal{C}, \boxtimes, \mathbb{I}) \rightarrow (\mathcal{D}, \otimes, \mathbb{J})$  is a monoidal functor between monoidal categories and if  $X$  in  $\mathcal{C}$  has a right dual  $(X^*, \mathbf{ev}_X, \mathbf{db}_X)$ , then  $\mathcal{F}(X)$  is right rigid with dual object  $\mathcal{F}(X^*)$  and structure maps

$$\mathbf{ev}_{\mathcal{F}(X)} = \phi_0^{-1} \mathcal{F}(\mathbf{ev}_X) \phi_{X, X^*} \quad \text{and} \quad \mathbf{db}_{\mathcal{F}(X)} = \phi_{X^*, X}^{-1} \mathcal{F}(\mathbf{db}_X) \phi_0$$

(cf. e.g. [St, page 86]). Therefore, even if  $\omega(X^*)$  is not a right dual of  $\omega(X)$  in  $\mathfrak{M}$ ,  $(\omega(X^*), \delta_{X^*})$  is a right dual of  $(\omega(X), \delta_X)$  in  ${}^H\mathfrak{M}$  because  $\omega^H : \mathcal{C} \rightarrow {}^H\mathfrak{M}$  is monoidal. Evaluation and coevaluation maps are the same given in (16) and they are morphisms of comodules. In particular,

$$(17) \quad \begin{array}{c} \begin{array}{c} x \\ | \\ \text{---} \\ | \\ \omega \\ | \\ x \end{array} \\ \text{---} \\ | \\ x \end{array} = \begin{array}{c} x \\ | \\ x \end{array}, \quad \begin{array}{c} \begin{array}{c} x^* \\ | \\ \text{---} \\ | \\ \omega^{-1} \\ | \\ x^* \end{array} \\ \text{---} \\ | \\ x^* \end{array} = \begin{array}{c} x^* \\ | \\ x^* \end{array},$$

$$(18) \quad \begin{array}{c} \begin{array}{c} \text{---} \\ | \\ H \end{array} \\ \text{---} \\ | \\ x^* \end{array} \begin{array}{c} \text{---} \\ | \\ x \end{array} = \begin{array}{c} \bullet \\ | \\ H \end{array} \begin{array}{c} \text{---} \\ | \\ x^* \end{array} \begin{array}{c} \text{---} \\ | \\ x \end{array}, \quad \begin{array}{c} \begin{array}{c} x \\ | \\ \text{---} \\ | \\ H \end{array} \\ \text{---} \\ | \\ x^* \end{array} \begin{array}{c} \text{---} \\ | \\ x^* \end{array} = \begin{array}{c} \bullet \\ | \\ H \end{array} \begin{array}{c} \text{---} \\ | \\ x \end{array} \begin{array}{c} \text{---} \\ | \\ x^* \end{array},$$

where (17) encodes relations (12) and (13).

**2.3. The natural transformation  $\nabla$ .** Consider the distinguished natural transformation  $\nabla^\omega : \mathbf{Nat}(\omega, - \otimes \omega) \rightarrow \mathbf{Nat}(\omega, - \otimes \omega)$  given by

$$(19) \quad \nabla_V^\omega(\xi)_X = (V \otimes \mathbf{ev}_{\omega(X)} \otimes \omega(X)) \tau_{\omega(X), V} \xi_{X^*} (\omega(X) \otimes \mathbf{db}_{\omega(X)})$$

for all  $V$  in  $\mathfrak{M}$ ,  $\xi \in \mathbf{Nat}(\omega, V \otimes \omega)$  and  $X$  in  $\mathcal{C}$  (when it would be clear from the context where to apply a morphism, we will omit to tensor by the identity maps). Graphically,

$$(20) \quad \nabla_V^\omega(\xi)_X = \begin{array}{c} \begin{array}{c} x \\ | \\ \text{---} \\ | \\ \xi_{X^*} \\ | \\ x \end{array} \\ \text{---} \\ | \\ v \end{array}.$$

**Proposition 2.10.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be essentially small right rigid monoidal categories. Let  $(\mathcal{V}, \psi, \psi_0), \mathcal{V} : \mathcal{D} \rightarrow \mathfrak{M}_f$ , be a quasi-monoidal functor and let  $(\mathcal{G}, \zeta, \zeta_0), \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ , be a monoidal one. For all  $V \in \mathfrak{M}$  and  $\xi \in \mathbf{Nat}(\mathcal{V}, V \otimes \mathcal{V})$  we have*

$$(21) \quad \nabla_V^\mathcal{V}(\xi)\mathcal{G} = \nabla_V^{\mathcal{V}\mathcal{G}}(\xi\mathcal{G}).$$

*Proof.* Assume that we are given a choice of right duals  $(-)^*$  in  $\mathcal{C}$  and  $(-)^{\vee}$  in  $\mathcal{D}$ . Since  $\mathcal{G}$  is monoidal we have a natural isomorphism  $\kappa_X : \mathcal{G}(X^*) \rightarrow \mathcal{G}(X)^{\vee}$  as in (14). Note that the composition  $\mathcal{V}\mathcal{G}$  is still a quasi-monoidal functor with structure isomorphisms  $\phi = (\mathcal{V}\zeta) \circ \psi(\mathcal{G} \times \mathcal{G})$  and  $\phi_0 = \mathcal{V}(\zeta_0)\psi_0$ . We will need the following relations, which descend from (15),

$$(22) \quad (\mathcal{V}\kappa \otimes \mathcal{V}\mathcal{G}) \circ \text{db}(\mathcal{V}\mathcal{G}) = (\text{db}\mathcal{V})\mathcal{G} \quad \text{and} \quad \text{ev}(\mathcal{V}\mathcal{G}) = (\text{ev}\mathcal{V})\mathcal{G} \circ (\mathcal{V}\mathcal{G} \otimes \mathcal{V}\kappa).$$

That is, for every object  $X$  in  $\mathcal{C}$  we have

$$(\mathcal{V}(\kappa_X) \otimes \mathcal{V}\mathcal{G}(X)) \text{db}_{\mathcal{V}\mathcal{G}(X)} = \text{db}_{\mathcal{V}(\mathcal{G}(X))}, \quad \text{ev}_{\mathcal{V}\mathcal{G}(X)} = \text{ev}_{\mathcal{V}(\mathcal{G}(X))} (\mathcal{V}\mathcal{G}(X) \otimes \mathcal{V}(\kappa_X)).$$

As a consequence, for every  $\xi \in \text{Nat}(\mathcal{V}, V \otimes \mathcal{V})$  we can compute directly

$$\begin{aligned} \nabla_V^{\mathcal{V}}(\xi)_{\mathcal{G}(X)} &\stackrel{(19)}{=} (V \otimes \text{ev}_{\mathcal{V}(\mathcal{G}(X))} \otimes \mathcal{V}\mathcal{G}(X)) \tau_{\mathcal{V}\mathcal{G}(X), V} \xi_{\mathcal{G}(X)^{\vee}} (\mathcal{V}\mathcal{G}(X) \otimes \text{db}_{\mathcal{V}(\mathcal{G}(X))}) \\ &\stackrel{(22)}{=} (V \otimes \text{ev}_{\mathcal{V}(\mathcal{G}(X))} \otimes \mathcal{V}\mathcal{G}(X)) \tau_{\mathcal{V}\mathcal{G}(X), V} \xi_{\mathcal{G}(X)^{\vee}} \mathcal{V}(\kappa_X) (\mathcal{V}\mathcal{G}(X) \otimes \text{db}_{\mathcal{V}\mathcal{G}(X)}) \\ &\stackrel{(*)}{=} (V \otimes \text{ev}_{\mathcal{V}(\mathcal{G}(X))} \otimes \mathcal{V}\mathcal{G}(X)) \tau_{\mathcal{V}\mathcal{G}(X), V} \mathcal{V}(\kappa_X) \xi_{\mathcal{G}(X^*)} (\mathcal{V}\mathcal{G}(X) \otimes \text{db}_{\mathcal{V}\mathcal{G}(X)}) \\ &\stackrel{(22)}{=} (V \otimes \text{ev}_{\mathcal{V}\mathcal{G}(X)} \otimes \mathcal{V}\mathcal{G}(X)) \tau_{\mathcal{V}\mathcal{G}(X), V} \xi_{\mathcal{G}(X^*)} (\mathcal{V}\mathcal{G}(X) \otimes \text{db}_{\mathcal{V}\mathcal{G}(X)}) \stackrel{(19)}{=} \nabla_V^{\mathcal{V}\mathcal{G}}(\xi\mathcal{G})_X \end{aligned}$$

where in  $(*)$  we used the naturality of  $\xi$ .  $\square$

**Corollary 2.11.** *Let  $\mathcal{C}$  be an essentially small right rigid monoidal category and let  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  be a quasi-monoidal functor. The natural transformation  $\nabla^{\omega}$  does not depend on the choice of the dual objects.*

*Proof.* It is enough to take  $\mathcal{D} = \mathcal{C}$  and  $\mathcal{G} = \text{Id}_{\mathcal{C}}$  in the proof of Proposition 2.10.  $\square$

**Remark 2.12.** Mimicking [Sc3] we may consider a category  $\mathfrak{C}$  whose objects are pairs  $(\mathcal{C}, \mathcal{U})$  where  $\mathcal{C}$  is an essentially small right rigid monoidal category and  $\mathcal{U} : \mathcal{C} \rightarrow \mathfrak{M}_f$  is a quasi-monoidal functor. Morphisms in  $\mathfrak{C}$  between two objects  $(\mathcal{C}, \mathcal{U})$  and  $(\mathcal{D}, \mathcal{V})$  are given by monoidal functors  $\mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$  such that  $\mathcal{V}\mathcal{G} = \mathcal{U}$  as quasi-monoidal functors. It follows from Proposition 2.10 that the transformation  $\nabla^{\sim}$  introduced in the foregoing is a natural transformation between the functor  $\text{Nat}(\sim, - \otimes \sim) : \mathfrak{C} \rightarrow \text{Funct}(\mathfrak{M}, \text{Set})$  sending  $(\mathcal{C}, \mathcal{U})$  to  $\text{Nat}(\mathcal{U}, - \otimes \mathcal{U})$  and itself.

**2.4. Rigidity and the preantipode.** By Yoneda Lemma and the fact that  $H$  represents the functor  $\text{Nat}(\omega, - \otimes \omega)$ , there exists a unique natural transformation in  $\text{Nat}(\omega, H \otimes \omega)$  which corresponds to  $\nabla^{\omega}$  and it is  $\nabla_H^{\omega}(\delta)$ . Its component at  $X$  is

$$(23) \quad \nabla_H^{\omega}(\delta)_X = (H \otimes \text{ev}_{\omega(X)} \otimes \omega(X)) \tau_{\omega(X), H} \delta_{X^*} (\omega(X) \otimes \text{db}_{\omega(X)}).$$

Moreover, there exists a unique linear endomorphism  $S$  of  $H$  such that

$$(24) \quad \vartheta_H(S)_X = \begin{array}{c} \begin{array}{c} x \\ \diagup \\ \textcircled{S} \\ \diagdown \\ H \quad x \end{array} \\ = \\ \begin{array}{c} x \\ | \\ \text{---} \\ | \\ H \quad x \end{array} \\ = \\ \nabla_H^{\omega}(\delta)_X. \end{array}$$

Notice that, by naturality of  $\vartheta$  and  $\nabla^{\omega}$ , for all  $g : H \rightarrow V$  in  $\mathfrak{M}$  we have

$$(25) \quad \vartheta_V(gS) = \nabla_V^{\omega}((g \otimes \omega) \delta).$$

**Proposition 2.13.** *The morphism  $S$  is a preantipode for  $H$ .*

*Proof.* Since  $\text{db}_{\omega(X)}$  is  $H$ -colinear, it follows that

i.e. for every  $h \in H$  we have  $\sum S(h_1)_1 h_2 \otimes S(h_1)_2 = 1_H \otimes S(h)$ . Now, since  $\text{ev}_{\omega(X)}$  is  $H$ -colinear as well, we have also

i.e. for every  $h \in H$  we have  $\sum S(h_2)_1 \otimes h_1 S(h_2)_2 = S(h) \otimes 1_H$ . Finally

so that  $\sum \omega(h_1 \otimes S(h_2) \otimes h_3) = \varepsilon(h)$  for all  $h \in H$ .  $\square$

Summing up, we can state our main theorem, connecting the rigidity of the category  $\mathcal{C}$  with the existence of a preantipode for the coendomorphism coquasi-bialgebra.

**Theorem 2.14.** *Let  $\mathcal{C}$  be an essentially small right rigid monoidal category together with a neutral quasi-monoidal functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$ . Then there exists a preantipode  $S$  for the coendomorphism coquasi-bialgebra  $H$  of  $(\mathcal{C}, \omega)$ .*

**Corollary 2.15** ([U, page 255, Theorem]). *If in addition  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  is monoidal, then the coendomorphism coquasi-bialgebra  $H$  is a Hopf algebra.*

*Proof.* Since  $\omega = \varepsilon \otimes \varepsilon \otimes \varepsilon$ ,  $H$  is a bialgebra and the preantipode provided by Theorem 2.14 satisfies

$$\varepsilon S(h) = \sum \omega(h_1 \otimes S(h_2) \otimes h_3) \stackrel{(5)}{=} \varepsilon(h),$$

i.e. it is an ordinary antipode (see Remark 1.4).  $\square$

**Remark 2.16.** Between the distinguished natural transformations in  $\text{Nat}(\boldsymbol{\omega}, \boldsymbol{\omega})$  that one may consider, there is also  $(\boldsymbol{\omega}(X) \otimes \text{db}_{\boldsymbol{\omega}(X)}) (\text{ev}_{\boldsymbol{\omega}(X)} \otimes \boldsymbol{\omega}(X))$ . This however does not endow  $H$  with a new structure. Instead, it can be checked that

$$\begin{aligned} (\varepsilon S \otimes \boldsymbol{\omega}(X)) \delta_X &= (\boldsymbol{\omega}(X) \otimes \text{db}_{\boldsymbol{\omega}(X)}) (\text{ev}_{\boldsymbol{\omega}(X)} \otimes \boldsymbol{\omega}(X)) \\ &= ((\omega^{-1}(S \otimes H \otimes S)(\Delta \otimes H)\Delta) \otimes \boldsymbol{\omega}(X)) \delta_X \end{aligned}$$

whence  $\omega^{-1}(S(h_1) \otimes h_2 \otimes S(h_3)) = \varepsilon S(h)$  for all  $h \in H$  as in [AP2, Lemma 2.14].

We conclude this subsection by showing that Theorem 2.14 can be refined in order to get a reconstruction theorem for coquasi-Hopf algebras as well (this result already appeared in a sketched form in [M1, §9.4.1, page 476] and it can be considered as a dual version of [H, Lemma 4] and [M1, §9.4.1, page 474]). Henceforth, with  $\text{ev}_V^{(\mathbb{k})} : V \otimes V^* \rightarrow \mathbb{k}$  and  $\text{db}_V^{(\mathbb{k})} : \mathbb{k} \rightarrow V^* \otimes V$  we will denote the ordinary evaluation and dual basis maps of a finitely generated and projective  $\mathbb{k}$ -module  $V$ . Graphically,

$$\begin{array}{c} V \quad V^* \\ \lrcorner \\ \lrcorner \end{array} \quad \text{and} \quad \begin{array}{c} \lrcorner \\ V^* \quad V \end{array}$$

respectively, for every  $V$  in  $\mathfrak{M}_f$ .

Recall that a *coquasi-Hopf algebra* is a coquasi-bialgebra  $H$  endowed with a *coquasi-antipode*, that is to say, a triple  $(s, \alpha, \beta)$  consisting of an anti-coalgebra endomorphism  $s : H \rightarrow H$  and two linear maps  $\alpha, \beta \in H^*$ , such that, for all  $h \in H$

$$\begin{aligned} \sum h_1 \beta(h_2) s(h_3) &= \beta(h) 1_H, & \sum s(h_1) \alpha(h_2) h_3 &= \alpha(h) 1_H, \\ \sum \omega(h_1 \otimes \beta(h_2) s(h_3) \alpha(h_4) \otimes h_5) &= \varepsilon(h), \\ \sum \omega^{-1}(s(h_1) \otimes \alpha(h_2) h_3 \beta(h_4) \otimes s(h_5)) &= \varepsilon(h). \end{aligned}$$

**Remark 2.17.** Let us point out two distinctive features of coquasi-Hopf algebras.

- Differently from preantipodes, coquasi-antipodes are not unique in general. If  $\chi \in H^*$  is convolution invertible and if  $(s, \alpha, \beta)$  is a coquasi-antipode for a coquasi-bialgebra  $H$ , then  $(\chi * s * \chi^{-1}, \chi * \alpha, \beta * \chi^{-1})$  is still a coquasi-antipode for  $H$ .
- If  $H$  is a coquasi-Hopf algebra with coquasi-antipode  $(s, \alpha, \beta)$ , then its category of comodules that are finitely generated and projective over  $\mathbb{k}$  is a rigid monoidal category. In fact, a dual for an  $H$ -comodule  $(V, \rho_V)$  in  ${}^H\mathfrak{M}_f$  is given by its dual  $\mathbb{k}$ -module  $V^*$  with comodule structure, evaluation and dual basis given respectively by

$$(26) \quad \rho_{V^*} := \begin{array}{c} V^* \\ \lrcorner \\ \textcircled{s} \\ \lrcorner \\ H \quad V^* \end{array}, \quad \text{ev}_V := \begin{array}{c} V \quad V^* \\ \textcircled{\beta} \\ \lrcorner \\ \lrcorner \end{array} \quad \text{and} \quad \text{db}_V := \begin{array}{c} \lrcorner \\ \textcircled{\alpha} \\ \lrcorner \\ V^* \quad V \end{array}.$$

**Proposition 2.18** (Reconstruction theorem for coquasi-Hopf algebras). *Let  $\mathcal{C}$  be an essentially small right rigid monoidal category together with a neutral quasi-monoidal functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$ . Then there is a bijective correspondence between natural isomorphisms  $d : \omega(-^*) \rightarrow \omega(-)^*$  in  $\mathfrak{M}_f$  and coquasi-Hopf algebra structures on the coendomorphism coquasi-bialgebra  $H$ . Explicitly, if  $d$  is a natural isomorphism with inverse  $\partial$ , then  $s, \alpha, \beta$  are the unique linear maps such that*

$$(27) \quad \begin{array}{c} x \\ \diagup \\ \textcircled{\alpha} \\ \diagdown \\ x \end{array} = \begin{array}{c} x \\ | \\ \textcircled{d} \\ | \\ x \end{array}, \quad \begin{array}{c} x \\ \diagdown \\ \textcircled{\beta} \\ \diagup \\ x \end{array} = \begin{array}{c} x \\ | \\ \textcircled{\partial} \\ | \\ x \end{array} \quad \text{and} \quad \begin{array}{c} x \\ \diagup \\ \textcircled{s} \\ \diagdown \\ H \quad x \end{array} = \begin{array}{c} x \\ | \\ \textcircled{\partial} \\ | \\ \textcircled{d} \\ | \\ H \quad x \end{array}.$$

Conversely, if  $H$  is a coquasi-Hopf algebra, then the natural isomorphism  $d$  is given by the canonical  $H$ -comodule isomorphism  $\omega(X^*) \cong \omega(X)^*$  from (14).

*Proof.* If  $H$  is a coquasi-Hopf algebra then the existence of a natural isomorphism  $d : \omega(-^*) \rightarrow \omega(-)^*$  follows from the fact that both  $(\omega(X)^*, \rho_{\omega(X)^*})$  (Remark 2.17) and  $(\omega(X^*), \delta_{X^*})$  (Remark 2.9) are dual objects of  $(\omega(X), \delta_X)$  in  ${}^H\mathfrak{M}_f$ .

Conversely, let  $(\mathcal{C}, \omega)$  be as in the statement and assume that we have a natural isomorphism  $d : \omega(-^*) \rightarrow \omega(-)^*$  in  $\mathfrak{M}_f$ . Consider the coendomorphism coquasi-bialgebra  $H$  associated with  $(\mathcal{C}, \omega)$ . We may endow  $\omega(X)^*$  with an  $H$ -comodule structure given by  $d_X \delta_{X^*} \partial_X$ . To simplify the exposition, we denote it by  $\delta_{X^*}$ , even if this notation does not strictly make sense. With this coaction,  $\omega(X)^*$  becomes a right dual of  $\omega(X)$  in  ${}^H\mathfrak{M}_f$  with evaluation and dual basis maps given by

$$\text{ev}_{\omega(X)^*}^{(*)} = \begin{array}{c} x \quad x^* \\ | \quad | \\ \textcircled{\partial} \\ | \quad | \\ x^* \quad x \end{array} \quad \text{and} \quad \text{db}_{\omega(X)^*}^{(*)} = \begin{array}{c} \textcircled{d} \\ | \\ x^* \quad x \end{array}.$$

Then, there exist unique linear morphisms  $\alpha, \beta \in H^*$  and  $s : H \rightarrow H$  such that (27) are satisfied. The reader may check that  $(s, \alpha, \beta)$  is a coquasi-antipode for  $H$ .

These two constructions can be showed to be inverses of each other. We verify one composition explicitly and we leave the other one to the reader. Assume that  $H$  admits a coquasi-antipode  $(s, \alpha, \beta)$ . As in the first part of the proof, we define

$$\begin{aligned} \text{ev}_{\omega(X)^*}^{(*)} &:= \text{ev}_{\omega(X)^*}^{(k)}(\beta \otimes \omega(X) \otimes \omega(X)^*)(\delta_X \otimes \omega(X)^*), \\ \text{db}_{\omega(X)^*}^{(*)} &:= (\omega(X)^* \otimes \alpha \otimes \omega(X))(\omega(X)^* \otimes \delta_X) \text{db}_{\omega(X)^*}^{(k)}, \\ \delta_{X^*} &:= \tau_{\omega(X)^*, H} \left( \omega(X)^* \otimes H \otimes \text{ev}_{\omega(X)^*}^{(k)} \right) s \delta_X \left( \text{db}_{\omega(X)^*}^{(k)} \otimes \omega(X)^* \right). \end{aligned}$$

These make of  $\omega(X)^*$  a dual for  $\omega(X)$  and so we have a canonical  $H$ -colinear isomorphism  $d_X : \omega(X^*) \rightarrow \omega(X)^*$  given as in (14). By Lemma 2.6,  $d_X$  satisfies  $(d_X \otimes \omega(X)) \text{db}_{\omega(X)^*}^{(*)} = \text{db}_{\omega(X)^*}^{(*)}$  and  $\text{ev}_{\omega(X)^*}^{(*)}(\omega(X) \otimes d_X) = \text{ev}_{\omega(X)^*}$ , so that

$$\begin{array}{c} x \\ | \\ \textcircled{d} \\ | \\ x \end{array} = \begin{array}{c} x \\ | \\ \textcircled{\alpha} \\ | \\ x \end{array} = \begin{array}{c} x \\ \diagup \\ \textcircled{\alpha} \\ \diagdown \\ x \end{array}$$

and analogously  $(\text{ev}_{\omega(X)} \otimes \omega(X)) \partial_X \left( \omega(X) \otimes \text{db}_{\omega(X)}^{(\mathbb{k})} \right) = (\beta \otimes \omega(X)) \delta_X$ . Moreover, since  $d_X$  is  $H$ -colinear, it follows also that  $d_X \delta_{X^*} \partial_X = \delta_{X^*}$  which implies

Summing up, the unique morphisms  $\alpha, \beta, s$  satisfying (27) are exactly the ones we started with, proving that we have recovered the original coquasi-antipode.  $\square$

**Remark 2.19.** The lack of uniqueness we observed in Remark 2.17 can be clearly perceived in the statement of Proposition 2.18: different choices of the natural isomorphism  $d$  give rise to different coquasi-antipodes and conversely. For the sake of clarity, let us assume that  $\bar{d} : \omega(-^*) \rightarrow \omega(-)^*$  is another natural isomorphism and let us see explicitly which relation connects the new coquasi-antipode  $(\bar{s}, \bar{\alpha}, \bar{\beta})$  with the one coming from  $d$ . The composition  $\bar{d}d^{-1}$  gives a natural automorphism of  $\omega(-)^*$  and there exists a unique natural automorphism  $\zeta$  of  $\omega$  such that  $\zeta^* = \bar{d}d^{-1}$ . By the universal property of  $H$ , there exists a unique  $\mathbb{k}$ -linear map  $\chi : H \rightarrow \mathbb{k}$  such that  $(\chi \otimes \omega(X)) \delta_X = \zeta_X$  for all  $X \in \mathcal{C}$  and since  $\zeta$  is a natural isomorphism,  $\chi$  has to be convolution invertible. Moreover, after recalling that  $\text{ev}_{\omega(X)}^{(\mathbb{k})}$  is a dinatural transformation we may compute

that is to say,  $\bar{\alpha} = \chi * \alpha$ . Similarly, one may check that  $\bar{\beta} = \beta * \chi^{-1}$  and  $\bar{s} = \chi * s * \chi^{-1}$ , as expected.

**2.5. The field case.** Under the additional assumption that  $\mathbb{k}$  is a field, some previous results can be refined and some further conclusions can be drawn. The key point, as we already mentioned in Remark 2.3, is that the reconstruction process applied to  ${}^C\mathfrak{M}_f$  for  $C$  a coalgebra over a field gives back the starting coalgebra.

**Remark 2.20.** Assume that  $B$  is a coquasi-bialgebra with a preantipode  $S_B$ . Denote by  $\mathcal{U} : {}^B\mathfrak{M}_f \rightarrow \mathfrak{M}_f$  the forgetful functor and by  $\rho \in \text{Nat}(\mathcal{U}, B \otimes \mathcal{U})$  the natural coaction of the  $B$ -comodules in  ${}^B\mathfrak{M}_f$ . It can be checked that for  $V$  in  ${}^B\mathfrak{M}_f$ , a right dual of  $V$  is given by  $V^* = (V^* \otimes B)^{\text{co}B}$  with coaction  $\rho_{V^*}(\sum_t f_t \otimes b_t) = \sum_t (b_t)_1 \otimes (f_t \otimes (b_t)_2)$ . Evaluation and dual basis maps are given by

$$\text{ev}_V \left( u \otimes \sum_t (f_t \otimes b_t) \right) = \sum_t f_t(u) \varepsilon(b_t), \quad \text{db}_V(1_{\mathbb{k}}) = \sum_{i=1}^{d_V} (v_0^i \otimes S_B(v_1^i)) \otimes v_i,$$

for all  $\sum_t f_t \otimes b_t \in V^*$ ,  $u \in V$ , where  $\sum_{i=1}^{d_V} v^i \otimes v_i \in V^* \otimes V$  is a dual basis for  $V$  as a finite-dimensional vector space. In particular,  ${}^B\mathfrak{M}_f$  is right rigid.



The following theorem characterizes coquasi-bialgebras with preantipode and coquasi-Hopf algebras via rigidity of their categories of finite-dimensional comodules. Only for this occasion, let us agree that a functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  from a rigid monoidal category  $\mathcal{C}$  to finite-dimensional vector spaces *preserves duals* if there exists a natural isomorphism  $d : \omega(-^*) \rightarrow \omega(-)^*$  (as in Proposition 2.18). Moreover, let us denote by  $\mathfrak{C}$  the category whose objects are  $\mathbb{k}$ -linear abelian rigid monoidal categories  $\mathcal{C}$  together with a  $\mathbb{k}$ -linear exact quasi-monoidal functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  and whose morphisms are monoidal functors  $\mathcal{G} : \mathcal{C} \rightarrow \mathcal{C}'$  between them which are compatible with the  $\omega$ 's, that is to say, such that  $(\omega', \varphi', \varphi'_0) \circ (\mathcal{G}, \psi, \psi_0) = (\omega, \varphi, \varphi_0)$ .

**Theorem 2.21.** *A coalgebra  $C$  is a coquasi-bialgebra with preantipode if and only if  ${}^C\mathfrak{M}_f$  is a right rigid monoidal category in such a way that the forgetful functor  $\mathcal{U} : {}^C\mathfrak{M}_f \rightarrow \mathfrak{M}_f$  is a quasi-monoidal functor. It is a coquasi-Hopf algebra if and only if, in addition,  $\mathcal{U}$  preserves duals in the above sense, if and only if for every  $(N, \rho_N) \in {}^C\mathfrak{M}_f$  the vector space  $N^*$  admits a left  $C$ -comodule structure and an evaluation  $\text{ev}_N$  and a dual basis  $\text{db}_N$  maps which are left  $C$ -colinear. A posteriori, the latter ones are going to be of the form (26).*

Moreover, the assignments  $H \mapsto ({}^H\mathfrak{M}_f, \mathcal{U})$  and  $(\mathcal{C}, \omega) \mapsto H_\omega$  provide an equivalence of categories between the category of coquasi-bialgebras with preantipode and coquasi-bialgebra morphisms on the one hand, and  $\mathfrak{C}$  on the other.

*Proof.* The first claim follows from Theorem 2.2, Theorem 2.14 and Remark 2.20. The second one from Theorem 2.2, Remark 2.17 and Proposition 2.18. The last assertion follows from [Sc3, §2.2] and the first claim of the statement. In fact, in [Sc3, §2.2] it has been proven that the assignments  $C \mapsto ({}^C\mathfrak{M}_f, \mathcal{U})$  and  $(\mathcal{C}, \omega) \mapsto C_\omega$  provide an equivalence of categories between the category of coalgebras and coalgebra morphisms on the one hand, and the category of  $\mathbb{k}$ -linear abelian categories  $\mathcal{C}$  together with a  $\mathbb{k}$ -linear exact faithful functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  and functors  $\mathcal{G} : \mathcal{C} \rightarrow \mathcal{C}'$  between them such that  $\omega'\mathcal{G} = \omega$ , on the other. The first claim entails that if we restrict the foregoing equivalence to the subcategory of coquasi-bialgebras with preantipode, then they correspond to  $\mathbb{k}$ -linear abelian rigid monoidal categories  $\mathcal{C}$  together with a  $\mathbb{k}$ -linear exact quasi-monoidal functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  and Theorem 2.14 states the converse. Moreover, if  $f : H \rightarrow H'$  is a morphism of coquasi-bialgebras then the induced functor  ${}^f\mathfrak{M} : {}^H\mathfrak{M}_f \rightarrow {}^{H'}\mathfrak{M}_f$  is a strict monoidal functor such that  $(\mathcal{U}', \text{ld}, \text{ld}) \circ ({}^f\mathfrak{M}, \text{ld}, \text{ld}) = (\mathcal{U}, \text{ld}, \text{ld})$ . Conversely, if  $(\mathcal{G}, \psi, \psi_0) : (\mathcal{C}, \omega) \rightarrow (\mathcal{C}', \omega')$  is a monoidal functor such that  $(\omega', \varphi', \varphi'_0) \circ (\mathcal{G}, \psi, \psi_0) = (\omega, \varphi, \varphi_0)$  then it induces a unique morphism of coquasi-bialgebras  $g : H_\omega \rightarrow H_{\omega'}$  by the universal property of  $H_\omega$ . The reason why faithfulness is no longer required explicitly is that a  $\mathbb{k}$ -linear exact functor between rigid monoidal categories is faithful in view of [DM, Proposition 1.19, page 121].  $\square$

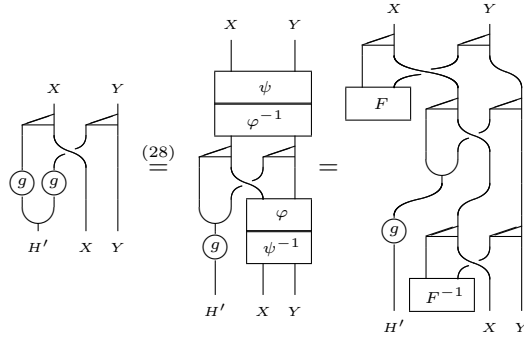
**Remark 2.22.** The second paragraph in Theorem 2.21 tells us that we have a bijective correspondence between coquasi-bialgebras with preantipode (up to isomorphism) and  $\mathbb{k}$ -linear abelian rigid monoidal categories  $\mathcal{C}$  together with a  $\mathbb{k}$ -linear exact quasi-monoidal functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  (up to monoidal equivalence compatible with the  $\omega$ 's). In relation to Remark 1.2, let us point out that there is a (more general) bijective correspondence between coquasi-bialgebras with preantipode (up to twist equivalence) and  $\mathbb{k}$ -linear abelian rigid monoidal categories  $\mathcal{C}$  together with a  $\mathbb{k}$ -linear exact quasi-monoidal functor  $\omega : \mathcal{C} \rightarrow \mathfrak{M}_f$  (up to monoidal

equivalences  $\mathcal{G}$  such that  $\omega'\mathcal{G} = \omega$ ). The difference here is that we are no longer requiring that  $(\omega', \varphi', \varphi'_0) \circ (\mathcal{G}, \psi, \psi_0) = (\omega, \varphi, \varphi_0)$  as quasi-monoidal functors, but only that  $\omega'\mathcal{G} = \omega$  as functors. As a consequence, the natural isomorphism  $\varphi^{-1} \circ \psi : \omega \otimes \omega \rightarrow \omega \otimes \omega$  corresponds to a unique convolution invertible linear map  $F : H \otimes H \rightarrow \mathbb{k}$  and  $\mathcal{G}$  to a unique coquasi-bialgebra isomorphism  $g : H_F \rightarrow H'$ .

For the sake of completeness, assume that we have a monoidal equivalence  $\mathcal{G} : (\mathcal{C}, \omega) \rightarrow (\mathcal{C}', \omega')$ . By universal property of  $H := H_\omega$ , there is a unique coalgebra isomorphism  $g : H \rightarrow H' := H_{\omega'}$  induced by  $\mathcal{G}$ . We retrieve explicitly the argument to show that  $g \circ m_F = m' \circ (g \otimes g)$  and we leave the other verifications to the reader. For every  $X$  in  $\mathcal{C}$  denote by  $\delta'_X : \omega(X) \rightarrow H' \otimes \omega(X)$  the (natural)  $H'$ -coaction on  $\omega(X) = \omega'(\mathcal{G}(X))$ . By definition,  $g$  is the unique coalgebra morphism such that  ${}^g\mathfrak{M}^H = \omega'^{H'}\mathcal{G}$ , whence  $\delta'_X = (g \otimes \omega(X))\delta_X$ . By construction, the multiplication  $m'$  on  $H'$  is the unique map such that

$$(28) \quad (m' \otimes \omega'(W) \otimes \omega'(Z)) \tau_{\omega'(W), H'} (\delta'_W \otimes \delta'_Z) = (H' \otimes \psi_{W,Z}^{-1}) \delta'_{W \otimes Z} \psi_{W,Z}$$

for all  $W, Z$  in  $\mathcal{C}'$ . Therefore, by definition of  $m$  and  $F$ ,

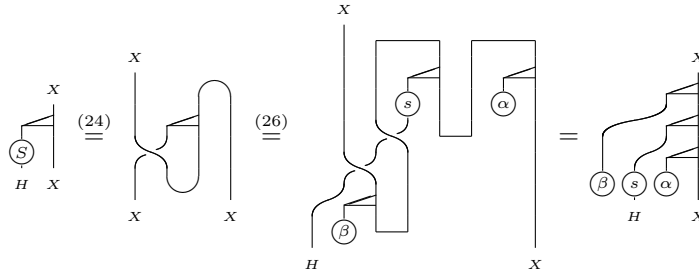


which implies that  $m' \circ (g \otimes g) = g \circ m_F$ .

Conversely, if there is a coquasi-bialgebra isomorphism  $g : H_F \rightarrow H'$  then this induces a strict monoidal isomorphism  $({}^g\mathfrak{M}, \text{ld}, \text{ld}) : {}^{H_F}\mathfrak{M}_f \rightarrow {}^{H'}\mathfrak{M}_f$ , which in turn gives, by composition, a monoidal isomorphism  $({}^g\mathfrak{M}, \psi_F, \text{ld}) : {}^H\mathfrak{M}_f \rightarrow {}^{H'}\mathfrak{M}_f$  where  $\psi_{F_{M,N}} : M \otimes N \rightarrow M \otimes N, m \otimes n \mapsto F(m_{-1} \otimes n_{-1})m_0 \otimes n_0$ . Observe that  $(\mathcal{U}', \text{ld}, \text{ld}) \circ ({}^g\mathfrak{M}, \psi_F, \text{ld}) \neq (\mathcal{U}, \text{ld}, \text{ld})$  as quasi-monoidal functors, even if  $\mathcal{U}' {}^g\mathfrak{M} = \mathcal{U}$ .

**Corollary 2.23** ([AP1, Theorem 3.10]). *Every coquasi-Hopf algebra  $H$  with coquasi-antipode  $(s, \alpha, \beta)$  admits a preantipode given by  $S := \beta * s * \alpha$ .*

*Proof.* By Theorem 2.21 there is a coquasi-bialgebra with preantipode structure on  $H$ , where  $\omega$  is the former one and  $S$  is uniquely given by



Therefore,  $S = \beta * s * \alpha$ . □

**Lemma 2.24.** *If a preantipode for a coquasi-bialgebra  $B$  exists, it is unique.*

*Proof.* It can be checked directly that  $S_B$  satisfies condition (24) and hence  $S_B = S$ , the unique linear endomorphism induced on  $B$ .  $\square$

**Lemma 2.25.** *Let  $g : A \rightarrow B$  be a morphism between coquasi-bialgebras  $A$  and  $B$  with preantipodes  $S_A$  and  $S_B$  respectively. Then  $g S_A = S_B g$ .*

*Proof.* Since  $g$  is a coquasi-bialgebra morphism, it induces a strict monoidal functor  ${}^g\mathfrak{M} : {}^A\mathfrak{M} \rightarrow {}^B\mathfrak{M}$ , which in turn restricts to a strict monoidal functor  $\mathcal{G} : {}^A\mathfrak{M}_f \rightarrow {}^B\mathfrak{M}_f$  such that  $\mathcal{V}\mathcal{G} = \mathcal{U}$ , where  $\mathcal{U} : {}^A\mathfrak{M}_f \rightarrow \mathfrak{M}_f$  and  $\mathcal{V} : {}^B\mathfrak{M}_f \rightarrow \mathfrak{M}_f$  are the forgetful functors. Observe that, in particular, this implies that  $(g \otimes \mathcal{U}(X)) \rho_X^A = \rho_{\mathcal{G}(X)}^B$  for every  $X$  in  ${}^A\mathfrak{M}_f$ . Let us denote by  $\vartheta : \text{Hom}(A, -) \rightarrow \text{Nat}(\mathcal{U}, - \otimes \mathcal{U})$  the natural isomorphism such that  $\vartheta_V(f) = (f \otimes \mathcal{U}) \rho^A$  for all  $V$  in  $\mathfrak{M}$  and  $f \in \text{Hom}(A, V)$ . We want to show that  $\vartheta_B(g S_A) = \vartheta_B(S_B g)$ . For all  $X$  in  ${}^A\mathfrak{M}_f$  we may compute

$$\begin{aligned} \vartheta_B(g S_A)_X &\stackrel{(25)}{=} \nabla_B^{\mathcal{U}}((g \otimes \mathcal{U}) \rho^A)_X = \nabla_B^{\mathcal{V}\mathcal{G}}(\rho^B \mathcal{G})_X \stackrel{(21)}{=} \nabla_B^{\mathcal{V}}(\rho^B)_{\mathcal{G}(X)} \\ &= (S_B \otimes \mathcal{V}\mathcal{G}(X)) \rho_{\mathcal{G}(X)}^B = (S_B \otimes \mathcal{U}(X)) (g \otimes \mathcal{U}(X)) \rho_X^A = \vartheta_B(S_B g)_X. \end{aligned}$$

Hence  $g S_A = S_B g$  as claimed.  $\square$

**Remark 2.26.** Let  $B$  be a coquasi-bialgebra over a commutative ring  $\mathbb{k}$ . The natural coaction  $\rho \in \text{Nat}(\mathcal{U}, B \otimes \mathcal{U})$  coming from  ${}^B\mathfrak{M}_f$  induces, via the isomorphism  $\text{Hom}(H_{\mathcal{U}}, B) \cong \text{Nat}(\mathcal{U}, B \otimes \mathcal{U})$ , a canonical morphism  $\text{can}_B : H_{\mathcal{U}} \rightarrow B$ . Then, all the results in §2.5 still remain true if we assume to work with  $\mathbb{k}$ -flat coquasi-bialgebras  $B$  such that  $\text{can}_B$  is an isomorphism (mimicking [EG, GV], these may be referred to as *Galois coalgebras*).

### 3. THE FINITE-DUAL OF A QUASI-BIALGEBRA WITH PREANTIPODE

Assume that  $\mathbb{k}$  is a field. As a final application of the theory we developed, let us show that the finite dual coalgebra of a quasi-bialgebra with preantipode is a coquasi-bialgebra with preantipode (for the definition of the finite dual coalgebra we refer to [Sw, Chapter VI], for the definition of a preantipode for a quasi-bialgebra and its properties we refer to [Sa] and Appendix A). The proof of this fact lies on the following result, which can be deduced from [A, Chapter 3, §1.2].

**Lemma 3.1.** *Let  $A$  be an algebra and  $A^\circ$  be its finite dual coalgebra. We have an isomorphism  $\mathcal{L} : {}^{A^\circ}\mathfrak{M}_f \rightarrow {}_f\mathfrak{M}_A$  between the category of finite-dimensional left  $A^\circ$ -comodules and that of finite-dimensional right  $A$ -modules that satisfies  $\mathcal{V}\mathcal{L} = \mathcal{U}$ , where  $\mathcal{V} : {}_f\mathfrak{M}_A \rightarrow \mathfrak{M}_f$  and  $\mathcal{U} : {}^{A^\circ}\mathfrak{M}_f \rightarrow \mathfrak{M}_f$  are the obvious forgetful functors.*

For the sake of completeness, let us recall that  $\mathcal{L}$  associates every left  $A^\circ$ -comodule  $(N, \rho_N)$  with the right  $A$ -module  $(N, \mu_N^\rho)$  where the action is given by  $\mu_N^\rho(n \otimes a) = \sum n_{-1}(a)n_0$ . Its inverse  $\mathcal{R} : {}_f\mathfrak{M}_A \rightarrow {}^{A^\circ}\mathfrak{M}_f$  assigns to every finite-dimensional right  $A$ -module  $(M, \mu_M)$ , the left  $A^\circ$ -comodule  $(M, \rho_M^\mu)$  with coaction

$$(29) \quad \rho_M^\mu(m) = \sum_{i=1}^{d_M} (e^i \mu_m) \otimes e_i$$

where  $\mu_m(a) := \mu_M(m \otimes a)$  for all  $a \in A$  and  $\sum_{i=1}^{d_M} e^i \otimes e_i \in M^* \otimes M$  is a dual basis for  $M$  as a vector space. Notice that  $\mathcal{U}\mathcal{R} = \mathcal{V}$  as well.

**Lemma 3.2.** *Let  $(A, m, u, \Delta, \varepsilon, \Phi, S)$  be a quasi-bialgebra with preantipode. The category of finite-dimensional right  $A$ -modules  ${}_f\mathfrak{M}_A$  is a right rigid monoidal category with quasi-monoidal forgetful functor  $\mathcal{V} : {}_f\mathfrak{M}_A \rightarrow \mathfrak{M}_f$ .*

*Proof.* As it happens for coquasi-bialgebras, the axioms of a quasi-bialgebra ensures that the category of right  $A$ -modules  $\mathfrak{M}_A$  becomes a monoidal category with tensor product the tensor product over  $\mathbb{k}$ , unit object  $\mathbb{k}$  itself, and associativity constraint

$$\mathfrak{a}_{M,N,P}((m \otimes n) \otimes p) = (m \otimes (n \otimes p)) \cdot \Phi^{-1}$$

for all  $M, N, P$  in  $\mathfrak{M}_A$  and  $m, n, p$  in  $M, N, P$  respectively. The unit constraints are the same of  $\mathfrak{M}$ . In particular, the forgetful functor  $\mathcal{V} : \mathfrak{M}_A \rightarrow \mathfrak{M}$  is a quasi-monoidal functor and the same property holds for its restriction to finite-dimensional modules. One may check directly that a dual object of an  $A$ -module  $M$  is given by

$$M^* := \frac{A \otimes M^*}{A^+(A \otimes M^*)}$$

where  $A^+ := \ker(\varepsilon)$  and  $M^*$  is the  $\mathbb{k}$ -linear dual of  $M$ . The  $A$ -module structure on  $M^*$  is  $\overline{a \otimes f} \cdot x = \overline{ax \otimes f}$  for all  $a, x \in A$  and  $f \in M^*$ . Evaluation and dual basis maps are explicitly given by

$$\text{ev}_M(m \otimes \overline{a \otimes f}) = f(m \cdot S(a)) \quad \text{and} \quad \text{db}_M(1_{\mathbb{k}}) = \sum_{i=1}^{d_M} \overline{1_A \otimes e^i} \otimes e_i$$

for all  $m \in M$ ,  $f \in M^*$  and  $a \in A$  and where  $\sum_{i=1}^{d_M} e^i \otimes e_i$  is a dual basis of  $M$  as a finite-dimensional vector space.  $\square$

**Proposition 3.3.** *Let  $(A, m, u, \Delta, \varepsilon, \Phi, S)$  be a quasi-bialgebra with preantipode. Let  $(A^\circ, \Delta_\circ, \varepsilon_\circ)$  be its finite dual coalgebra. Then  $A^\circ$  can be endowed with a structure of a coquasi-bialgebra with preantipode.*

*Proof.* Denote by  $\mathcal{V} : {}_f\mathfrak{M}_A \rightarrow \mathfrak{M}_f$  and  $\mathcal{U} : {}^{A^\circ}\mathfrak{M}_f \rightarrow \mathfrak{M}_f$  the forgetful functors. As a consequence of Lemma 3.1, we have a chain of natural isomorphism

$$\text{Nat}(\mathcal{V}, - \otimes \mathcal{V}) \cong \text{Nat}(\mathcal{U}, - \otimes \mathcal{U}) \cong \text{Hom}(A^\circ, -)$$

which allows us to consider  $A^\circ$  itself as a representing object for  $\text{Nat}(\mathcal{V}, - \otimes \mathcal{V})$ . If we consider then the category of finite-dimensional right  $A$ -modules  ${}_f\mathfrak{M}_A$  as a right rigid monoidal category together with the quasi-monoidal forgetful functor  $\mathcal{V} : {}_f\mathfrak{M}_A \rightarrow \mathfrak{M}_f$ , then  $A^\circ$  can be endowed with a structure of a coquasi-bialgebra with preantipode in view of Theorem 2.14.  $\square$

**Remark 3.4.** It is worthy to point out that the corestriction  $\mathcal{V}^{A^\circ} : {}_f\mathfrak{M}_A \rightarrow {}^{A^\circ}\mathfrak{M}_f$  of the functor  $\mathcal{V}^{A^\circ} : {}_f\mathfrak{M}_A \rightarrow {}^{A^\circ}\mathfrak{M}$  provided by Theorem 2.14 coincides with the functor  $\mathcal{R}$ , which becomes this way a strict monoidal functor.

**Remark 3.5.** If we want to know explicitly the coquasi-bialgebra structure on  $A^\circ$  we may proceed as follows. First of all observe that the quasi-monoidal structure on  $\mathcal{V} : {}_f\mathfrak{M}_A \rightarrow \mathfrak{M}_f$  is the strict one:  $\varphi_{M,N} = \text{id}_{M \otimes N}$  and  $\varphi_0 = \text{id}_{\mathbb{k}}$ . Secondly, for every object  $M$  in  ${}_f\mathfrak{M}_A$  the natural transformation  $\rho_M : \mathcal{V}(M) \rightarrow A^\circ \otimes \mathcal{V}(M)$  is given by the coaction (29). Let us denote by  $\sum_{i=1}^{d_M} e_M^i \otimes e_i^M \in M^* \otimes M$  a dual basis for  $M$  as a vector space, for all  $M$  in  ${}_f\mathfrak{M}_A$ . If we denote by  $\mu^{M \otimes N}$  the  $A$ -action on the tensor product, then

$$\rho_{M \otimes N}(x) = \sum_{i,j} ((e_M^i \otimes e_N^j) \mu_x^{M \otimes N}) \otimes (e_i^M \otimes e_j^N)$$

for all  $x \in M \otimes N$ , where we considered  $M^* \otimes N^*$  injected in  $(M \otimes N)^*$ . Furthermore, it is well-known from the associative case that the convolution product  $*$  restricts to a morphism  $*$  :  $A^\circ \otimes A^\circ \rightarrow A^\circ$ . It is also clear that  $\varepsilon \in A^\circ$ . To show that they are the multiplication and the unit induced on  $A^\circ$ , denote by  $\mu^M$  and  $\mu^N$  the  $A$ -actions on  $M$  and  $N$  respectively and compute for  $\sum_{i=1}^t m_i \otimes n_i \in M \otimes N$

$$(A^\circ \otimes \varphi_{M,N}) \left( \vartheta_{A^\circ}^2(*)_{M,N} \left( \sum_{i=1}^t m_i \otimes n_i \right) \right) = \sum_{i,h,k} ((e_h^M \mu_{m_i}^M) * (e_N^k \mu_{n_i}^N)) \otimes (e_h^M \otimes e_k^N).$$

Since for every  $a \in A$ ,  $f \in M^*$ ,  $g \in N^*$  and  $x = \sum_{i=1}^t m_i \otimes n_i \in M \otimes N$  we have

$$\sum_{i=1}^t ((f \mu_{m_i}^M) * (g \mu_{n_i}^N)) (a) = \sum_{i=1}^t (f \mu_{m_i}^M) (a_1) (g \mu_{n_i}^N) (a_2) = (f \otimes g) \mu_x^{M \otimes N} (a),$$

we conclude that  $(A^\circ \otimes \varphi_{M,N}) \vartheta_{A^\circ}^2(*)_{M,N} = \rho_{M \otimes N} \varphi_{M,N}$  and by uniqueness of the morphism  $A^\circ \otimes A^\circ \rightarrow A^\circ$  satisfying this relation we have that the multiplication induced on  $A^\circ$  is exactly  $*$ . Moreover, if we compute

$$r_{A^\circ} (\rho_{\mathbb{k}} (\mathbf{1}_{\mathbb{k}})) = r_{A^\circ} (\varepsilon \otimes \mathbf{1}_{\mathbb{k}}) = \varepsilon,$$

then we recover that the unit of the multiplication  $*$  is  $\varepsilon$ , in view of (11) and the fact that  $\varphi_0 = \text{id}_{\mathbb{k}}$ . Consider also the assignment

$$\omega : A^\circ \otimes A^\circ \otimes A^\circ \rightarrow \mathbb{k}; \quad \omega (f \otimes g \otimes h) = \sum f (\Phi^1) g (\Phi^2) h (\Phi^3).$$

For every  $M, N, P$  in  ${}_f\mathfrak{M}_A$  and all  $m \in M, n \in N, p \in P$ , it satisfies

$$\begin{aligned} & \varphi_{M \otimes N, P} \left( (\varphi_{M,N} \otimes \mathcal{V}(P)) \left( \vartheta_{\mathbb{k}}^3(\omega)_{M,N,P} (m \otimes n \otimes p) \right) \right) \\ &= \sum_{i,j,k} \omega \left( (e_M^i \mu_m^M) \otimes (e_N^j \mu_n^N) \otimes (e_P^k \mu_p^P) \right) e_i^M \otimes e_j^N \otimes e_k^P \\ &= \sum m \cdot \Phi^1 \otimes n \cdot \Phi^2 \otimes p \cdot \Phi^3, \end{aligned}$$

whence  $\varphi_{M \otimes N, P} (\varphi_{M,N} \otimes \mathcal{V}(P)) \vartheta_{\mathbb{k}}^3(\omega)_{M,N,P} = \mathcal{V}(\mathfrak{a}_{M,N,P}^{-1}) \varphi_{M,N \otimes P} (\mathcal{V}(M) \otimes \varphi_{N,P})$  and so  $\omega$  is in fact the induced reassociator. The antipode can be constructed explicitly as well. Consider the transpose  $S^* : A^* \rightarrow A^*$ . Let us show firstly that  $S^*$  factors through a linear map  $S^\circ : A^\circ \rightarrow A^\circ$ ; the proof relies on formula (36) from Appendix A. Pick  $f \in A^\circ$  and compute

$$\begin{aligned} S^* (f) (ab) &= f (S(ab)) \stackrel{(36)}{=} \sum f (S(\varphi^1 b) \varphi^2 S(\psi^1 \varphi^3) \psi^2 S(a \psi^3)) \\ &= \sum f_1 S(\varphi^1 b) f_2 (\varphi^2 S(\psi^1 \varphi^3) \psi^2) f_3 S(a \psi^3) \\ &= \left( \sum (\psi^3 \leftarrow f_3 S) \otimes f_2 (\varphi^2 S(\psi^1 \varphi^3) \psi^2) (f_1 S \leftarrow \varphi^1) \right) (a \otimes b). \end{aligned}$$

Since this implies that  $m^* (S^* (f)) \in A^* \otimes A^*$ , in view of [Sw, Proposition 6.0.3] we have that  $S^* (f) \in A^\circ$ . Let us prove now that  $S^\circ$  satisfies the relation  $\vartheta_{A^\circ} (S^\circ) = \nabla_{A^\circ}^\vee (\rho)$ . For all  $M$  in  ${}_f\mathfrak{M}_A$  and all  $m \in M$  we need to show that

$$(30) \quad \sum S^\circ (m_{-1}) \otimes m_0 = \sum_{i=1}^{d_M} (\overline{1_A \otimes e^i})_0 (m) (\overline{1_A \otimes e^i})_{-1} \otimes e_i.$$

Since  $M^*$  is finite-dimensional, we may fix a dual basis  $\sum_{j=1}^{d_{M^*}} \gamma^j \otimes \gamma_j$  of  $M^*$  as an object in  $\mathfrak{M}_f$  and then, by (29), the right-hand member of (30) can be rewritten as

$$\sum_{i=1}^{d_M} \sum_{j=1}^{d_{M^*}} \gamma_j(m) \left( \gamma^j \mu_{1_A \otimes e^i}^{M^*} \right) \otimes e_i.$$

Let us focus on  $\sum_{j=1}^{d_{M^*}} \gamma_j(m) \left( \gamma^j \mu_{1_A \otimes e^i}^{M^*} \right) \in A^\circ$ . For all  $a \in A$ ,

$$\sum_{j=1}^{d_{M^*}} \gamma_j(m) \left( \gamma^j \mu_{1_A \otimes e^i}^{M^*} \right) (a) = \sum_{j=1}^{d_{M^*}} \gamma_j(m) \gamma^j (\overline{a \otimes e^i}) = \overline{a \otimes e^i}(a) = e^i(m \cdot S(a))$$

and since  $e^i(m \cdot S(a)) = S^\circ(e^i \mu_m^M)(a)$ , we have

$$\sum_{i=1}^{d_M} \sum_{j=1}^{d_{M^*}} \gamma_j(m) \left( \gamma^j \mu_{1_A \otimes e^i}^{M^*} \right) \otimes e_i = \sum_i S^\circ(e^i \mu_m^M) \otimes e_i.$$

We can conclude then that relation (30) is satisfied, as desired.

**Remark 3.6.** The fact that the finite dual coalgebra of a quasi-bialgebra is a coquasi-bialgebra has already been shown in [AES, §5.2] with a different approach.

#### APPENDIX A. A RELATION FOR THE PREANTIPODE OF A QUASI-BIALGEBRA

Recall from [Sa] that a preantipode for a quasi-bialgebra  $(A, \Delta, \varepsilon, m, u, \Phi)$  is a  $\mathbb{k}$ -linear map  $S: A \rightarrow A$  that satisfies

$$(31) \quad \sum a_1 S(ba_2) = \varepsilon(a)S(b) = \sum S(a_1b)a_2, \quad \sum \Phi^1 S(\Phi^2) \Phi^3 = 1,$$

for all  $a, b \in A$ , where  $\sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3 = \Phi$ . Let us introduce also the following extended notation for the reassociator and its inverse:

$$\begin{aligned} \Phi &= \sum \Phi^1 \otimes \Phi^2 \otimes \Phi^3 = \sum \Psi^1 \otimes \Psi^2 \otimes \Psi^3 = \dots \\ \Phi^{-1} &= \sum \varphi^1 \otimes \varphi^2 \otimes \varphi^3 = \sum \psi^1 \otimes \psi^2 \otimes \psi^3 = \dots \end{aligned}$$

Let  $(A, m, u, \Delta, \varepsilon, \Phi, S)$  be a quasi-bialgebra with preantipode and consider the  $A$ -actions on  $\text{End}(A) = \text{Hom}(A, A)$  defined by  $(f \leftarrow a)(b) = f(ab)$  and  $(a \rightarrow f)(b) = f(ba)$  for all  $a, b \in A$  and for all  $f \in \text{End}(A)$ . Define the elements

$$(32) \quad \begin{aligned} p &:= \sum \varphi^1 \otimes \varphi^2 (\varphi^3 \rightarrow S) \in A \otimes \text{End}(A), \\ q &:= \sum (S \leftarrow \varphi^1) \varphi^2 \otimes \varphi^3 \in \text{End}(A) \otimes A, \end{aligned}$$

where  $(x(y \rightarrow f))(a) = xf(ay)$  and  $((f \leftarrow x)y)(a) = f(ax)y$  for all  $a, x, y \in A$  and for all  $f \in \text{End}(A)$ . Let us introduce the following notation for shortness:

$$p := \sum p^1 \otimes p^2 \quad \text{and} \quad q := \sum q^1 \otimes q^2.$$

**Lemma A.1.** *In the foregoing notation we have that for every  $a \in A$*

$$(33) \quad \begin{aligned} \sum p^1 \otimes p^2(a) &= \sum \varphi_1^1 \psi^1 \otimes \varphi_2^1 \psi^2 \Phi^1 S(a \varphi^2 \psi_1^3 \Phi^2) \varphi^3 \psi_2^3 \Phi^3, \\ \sum q^1(a) \otimes q^2 &= \sum \Phi^1 \varphi_1^1 \psi^1 S(\Phi^2 \varphi_2^1 \psi^2 a) \Phi^3 \varphi^2 \psi_1^3 \otimes \varphi^3 \psi_2^3. \end{aligned}$$

Moreover, the following relations hold for every  $a, b \in A$

$$(34) \quad \sum p^1 a \otimes p^2(b) = \sum a_{11} p^1 \otimes a_{12} p^2(ba_2),$$

$$(35) \quad \sum q^1(a) \otimes bq^2 = \sum q^1(b_1a)b_{21} \otimes q^2b_{22}.$$

*Proof.* The reassociator  $\Phi$  satisfies the dual relation to (3), i.e.

$$(1_A \otimes \Phi) \cdot (A \otimes \Delta \otimes A) (\Phi) \cdot (\Phi \otimes 1_A) = (A \otimes A \otimes \Delta) (\Phi) \cdot (\Delta \otimes A \otimes A) (\Phi).$$

In particular, it satisfies

$$\sum \varphi_1^1 \psi^1 \otimes \varphi_2^1 \psi^2 \Phi^1 \otimes \varphi^2 \psi_1^3 \Phi^2 \otimes \varphi^3 \psi_2^3 \Phi^3 = \sum \varphi^1 \psi^1 \otimes \varphi^2 \psi_1^2 \otimes \varphi^3 \psi_2^2 \otimes \psi^3.$$

Applying  $(A \otimes m)$   $(A \otimes A \otimes m)$   $(A \otimes A \otimes (S \leftarrow a) \otimes A)$  to both sides we get

$$\begin{aligned} & \sum \varphi_1^1 \psi^1 \otimes \varphi_2^1 \psi^2 \Phi^1 S(a \varphi^2 \psi_1^3 \Phi^2) \varphi^3 \psi_2^3 \Phi^3 \\ &= \sum \varphi^1 \psi^1 \otimes \varphi^2 \psi_1^2 S(a \varphi^3 \psi_2^2) \psi^3 \stackrel{(31)}{=} \sum \varphi^1 \otimes \varphi^2 S(a \varphi^3) = \sum p^1 \otimes p^2(a), \end{aligned}$$

which is the first identity in (33). The second one is proved analogously. Let us check that (34) holds as well ((35) is proved similarly). We compute

$$\begin{aligned} \sum p^1 a \otimes p^2(b) & \stackrel{(32)}{=} \sum \varphi^1 a \otimes \varphi^2 S(b \varphi^3) \stackrel{(31)}{=} \sum \varphi^1 a_1 \otimes \varphi^2 a_{21} S(b \varphi^3 a_{22}) \\ & \stackrel{(*)}{=} \sum a_{11} \varphi^1 \otimes a_{12} \varphi^2 S(b a_2 \varphi^3) = \sum a_{11} p^1 \otimes a_{12} p^2(b a_2), \end{aligned}$$

where in (\*) we used the quasi-coassociativity  $\Phi \cdot (\Delta \otimes A) \Delta = (A \otimes \Delta) \Delta \cdot \Phi$ .  $\square$

**Lemma A.2.** *Let  $(A, m, u, \Delta, \varepsilon, \Phi, S)$  be a quasi-bialgebra with preantipode and let  $p, q$  be defined as above. For all  $a \in A$  we have that*

$$S(a) = \sum q^1(1_A) S(p^1 a q^2) p^2(1_A) = \sum S(\varphi^1) \varphi^2 S(\psi^1 a \varphi^3) \psi^2 S(\psi^3).$$

*Proof.* Keeping in mind that  $\Phi^{-1}$  is counital, i.e. that it satisfies

$$(\varepsilon \otimes A \otimes A) (\Phi^{-1}) = 1_A \otimes 1_A = (A \otimes \varepsilon \otimes A) (\Phi^{-1}) = 1_A \otimes 1_A = (A \otimes A \otimes \varepsilon) (\Phi^{-1}),$$

we may compute directly

$$\begin{aligned} & \sum S(\varphi^1) \varphi^2 S(\psi^1 a \varphi^3) \psi^2 S(\psi^3) = \sum q^1(1_A) S(p^1 a q^2) p^2(1_A) \\ & \stackrel{(33)}{=} \sum \Phi^1 \varphi_1^1 \psi^1 S(\Phi^2 \varphi_2^1 \psi^2) \Phi^3 \varphi^2 \psi_1^3 S(\gamma_1^1 \phi^1 a \varphi^3 \psi_2^3) \gamma_2^1 \phi^2 \Psi^1 S(\gamma^2 \phi_1^3 \Psi^2) \gamma^3 \phi_2^3 \Psi^3 \\ & \stackrel{(31)}{=} \sum \Phi^1 \varphi_1^1 S(\Phi^2 \varphi_2^1) \Phi^3 \varphi^2 S(\phi^1 a \varphi^3) \phi^2 \Psi^1 S(\phi_1^3 \Psi^2) \phi_2^3 \Psi^3 \\ & \stackrel{(31)}{=} \sum \Phi^1 S(\Phi^2) \Phi^3 S(a) \Psi^1 S(\Psi^2) \Psi^3 = S(a). \quad \square \end{aligned}$$

**Proposition A.3.** *Let  $(A, m, u, \Delta, \varepsilon, \Phi, S)$  be a quasi-bialgebra with preantipode. For all  $a, b \in A$  we have*

$$(36) \quad S(ab) = \sum S(\varphi^1 b) \varphi^2 S(\psi^1 \varphi^3) \psi^2 S(a \psi^3).$$

*Proof.* We know from Lemma A.2 that  $S(a) = \sum q^1(1_A) S(p^1 a q^2) p^2(1_A)$ . Relation (36) is proved directly by applying it to  $S(ab)$ :

$$\begin{aligned} S(ab) &= \sum q^1(1_A) S(p^1 a b q^2) p^2(1_A) \stackrel{(34)}{=} \sum q^1(1_A) S(a_{11} p^1 b q^2) a_{12} p^2(a_2) \\ & \stackrel{(31)}{=} \sum q^1(1_A) S(p^1 b q^2) p^2(a) \stackrel{(35)}{=} \sum q^1(b_1) b_{21} S(p^1 q^2 b_{22}) p^2(a) \\ & \stackrel{(31)}{=} \sum q^1(b) S(p^1 q^2) p^2(a) = \sum S(\varphi^1 b) \varphi^2 S(\psi^1 \varphi^3) \psi^2 S(a \psi^3). \quad \square \end{aligned}$$

Formula (36) can be viewed as an anti-multiplicativity of the preantipode.

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