**Lagrange interpolation at Pollaczek--Laguerre zeros on the real semiaxis**



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# LAGRANGE INTERPOLATION AT POLLACZEK–LAGUERRE ZEROS ON THE REAL SEMIAXIS

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Abstract. In order to approximate functions defined on the real semiaxis, which can grow exponentially both at 0 and at $+\infty$, we introduce a suitable Lagrange operator based on the zeros of orthogonal polynomials with respect to the weight $w(x) = x^\gamma e^{-x^{-\alpha}-x^\beta}$. We prove that this interpolation process has Lebesgue constant with order $\log m$ in weighted uniform metric and converges with the order of the best approximation in a large subset of weighted $L^p-$spaces, $1 < p < \infty$, with proper assumptions.

**Keywords:** Lagrange interpolation, weighted polynomial approximation, orthogonal polynomials, Pollaczek–Laguerre exponential weights, real semiaxis.

**AMS subject classification:** 41A05, 41A10.

## 1. Introduction

The polynomial approximation of functions defined on the real semiaxis, which can grow exponentially both at 0 and at $+\infty$, has received attention in the literature only recently (see [3, 7, 8, 9, 10, 11, 12]). In these papers, with the contribution of further authors, we have introduced a weight of the form $w(x) = x^\gamma e^{-x^{-\alpha}-x^\beta}$, with $x > 0$, $\alpha > 0$, $\beta > 1$ and $\gamma \geq 0$, and developed the related theory of polynomial approximation in proper function spaces. The properties of the orthonormal system $\{p_m(w)\}_{m\in\mathbb{N}}$ have been studied in [9] also from the computational point of view. To this aim the results proved by Levin and Lubinsky in their book [4] are crucial.

In the present paper, using the zeros of $p_m(w)$, we introduce a new interpolation process, which will be denoted by $\mathcal{L}^*_{m+2}(w)$, in order to approximate the above mentioned class of functions. As main results, we are going to prove that, under suitable necessary and sufficient conditions, this interpolation process has Lebesgue constant with order $\log m$ in weighted uniform metric (cfr. [13]) and behaves like the best approximation in a wide subspace of weighted $L^p-$spaces, $1 < p < \infty$.

## 2. Preliminary results

In the sequel $c, \mathcal{C}$ will stand for positive constants which can assume different values in each formula. We shall write $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ when $\mathcal{C}$ is independent of $a, b, \dots$, and, on the other hand, we will write $\mathcal{C}_a$ or $\mathcal{C}(a)$ when $\mathcal{C}$ depends on $a$. Furthermore $A \sim B$ will mean that if $A$ and $B$ are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$. Finally, we

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will denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most $m$. As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, will stand for the sets of all natural, integer, real numbers, while $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote the sets of positive integers and positive real numbers, respectively.

For the sake of completeness and for the reader's convenience, we recall some basic facts, recently proved in [8].

### 2.1. **Classes of functions.** Letting

$$u(x) = x^\delta \sqrt{\sigma(x)}\,, \quad \sigma(x) = \mathrm{e}^{-x^{-\alpha}-x^\beta} \tag{2.1}$$

with $\alpha > 0$, $\beta > 1$, $\delta \geq 0$, $x > 0$, we introduce the following spaces of functions.

If $1 \leq p < \infty$ we will write $f \in L^p_u$ if and only if

$$\|f\|_{L^p_u} := \|fu\|_p = \left(\int_0^{+\infty} |fu|^p(x)\,\mathrm{d}x\right)^{1/p} < \infty\,.$$

For $p = \infty$, by a slight abuse of notation, we set

$$L^\infty_u = C_u = \left\{ f \in C^0(\mathbb{R}^+) : \lim_{x\to 0^+} f(x)u(x) = 0 = \lim_{x\to+\infty} f(x)u(x) \right\}$$

and

$$\|f\|_{L^\infty_u} := \|fu\|_\infty = \sup_{x\in(0,+\infty)} |f(x)u(x)|\,,$$

where $C^0(\mathbb{R}^+)$ is the collection of all continuous functions on $(0, +\infty)$.

For smoother functions we introduce the Sobolev-type spaces

$$W^p_r(u) = \left\{ f \in L^p_u : \ f^{(r-1)} \in AC(0,+\infty),\ \|f^{(r)}\varphi^r u\|_p < \infty \right\},$$

with

$$\|f\|_{W^p_r(u)} = \|fu\|_p + \|f^{(r)}\varphi^r u\|_p\,,$$

where $1 \leq p \leq \infty$, $1 \leq r \in \mathbb{Z}^+$, $\varphi(x) := \sqrt{x}$ and $AC(0, +\infty)$ denotes the set of all absolutely continuous functions on $(0, +\infty)$.

To characterize further subspaces of $L^p_u$, we introduce the following moduli of smoothness. Let us consider the intervals

$$\mathcal{I}_h(c) = \left[ h^{1/(\alpha+1/2)}, \frac{c}{h^{1/(\beta-1/2)}} \right],$$

with $\alpha$ and $\beta$ in (2.1), $h > 0$ sufficiently small, and $c > 1$ an arbitrary but fixed constant. For any $f \in L^p_u$, $1 \leq p \leq \infty$, $r \geq 1$ and $t > 0$ sufficiently small (say $t < t_0$), we set

$$\Omega^r_\varphi(f,t)_{u,p} = \sup_{0<h\leq t} \left\| \Delta^r_{h\varphi}(f)\,u \right\|_{L^p(\mathcal{I}_h(c))}\,,$$

where

$$\Delta^r_{h\varphi} f(x) = \sum_{i=0}^r (-1)^i \binom{r}{i} f\left(x + (r-i)h\varphi(x)\right)\,, \quad \varphi(x) = \sqrt{x}\,.$$

Moreover, we introduce the following $K$-functional

$$K(f,t^r)_{u,p}=\inf_{g\in W_r^p(u)}\left\{\|(f-g)\,u\|_p+t^r\|g^{(r)}\varphi^r u\|_p\right\}$$

and its main part

$$\widetilde{K}(f,t^r)_{u,p}=\sup_{0<h\le t}\inf_{g\in W_r^p(u)}\left\{\|(f-g)\,u\|_{L^p(\mathcal{I}_h(c))}+h^r\|g^{(r)}\varphi^r u\|_{L^p(\mathcal{I}_h(c))}\right\}\,.$$

Then we define the complete $r$th modulus of smoothness by

$$\begin{aligned}\omega_\varphi^r(f,t)_{u,p} &= \Omega_\varphi^r(f,t)_{u,p}+\inf_{q\in\mathbb{P}_{r-1}}\|(f-q)\,u\|_{L^p(0,t^{1/\left(\alpha+\frac{1}{2}\right)}]}\\ &\quad+\inf_{q\in\mathbb{P}_{r-1}}\|(f-q)\,u\|_{L^p[c\,t^{-1/\left(\beta-\frac{1}{2}\right)},+\infty)}\end{aligned}$$

with $c>1$ a fixed constant. We emphasize that the behaviour of $\omega_\varphi^r(f,t)_{u,p}$ is independent of the constant $c$. Moreover, this modulus of smoothness is equivalent to the $K$-functional. To be more precise (see [8, pp. 171–172, Lemmas 2.3 and 2.4])

$$\omega_\varphi^r(f,t)_{u,p}\sim K(f,t^r)_{u,p}$$

and

$$\Omega_\varphi^r(f,t)_{u,p}\sim\widetilde{K}(f,t^r)_{u,p}\,,$$

where the constants in "$\sim$" are independent of $f$ and $t$, in both cases.

By means of the main part of the modulus of smoothness, for $1\le p\le\infty$, we can define the Zygmund-type spaces

$$Z_s^p(u)=\left\{f\in L_u^p:\ \sup_{t>0}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s}<\infty,\ r>s\right\},$$

$s\in\mathbb{R}^+$, with the norm

$$\|f\|_{Z_s^p(u)}=\|f\|_{L_u^p}+\sup_{t>0}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s}\,.$$

In the sequel, we will use the notation $\Omega_\varphi(f,t)_{u,p}=\Omega_\varphi^1(f,t)_{u,p}$. We remark that, in the definition of $Z_s^p(u)$, the main part of the $r$th modulus of smoothness $\Omega_\varphi^r(f,t)_{u,p}$ can be replaced by the complete modulus $\omega_\varphi^r(f,t)_{u,p}$ (see [8, p. 171]).

2.2. **Best weighted approximation.** Let us denote by

$$E_m(f)_{u,p}=\inf_{P\in\mathbb{P}_m}\|(f-P)\,u\|_p$$

the error of best polynomial approximation of a function $f\in L_u^p$, $1\le p\le\infty$.

Then, for any $f\in L_u^p$, $1\le p\le\infty$, the following Jackson, Stechkin and weak Jackson inequalities hold (see [8, p. 173, Theorems 3.2, 3.3 and 3.4])

$$E_m(f)_{u,p}\le\mathcal{C}\,\omega_\varphi^r\left(f,\frac{\sqrt{a_m}}{m}\right)_{u,p},$$

$$\omega_\varphi^r\left(f,\frac{\sqrt{a_m}}{m}\right)_{u,p}\le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r\sum_{i=0}^m\left(\frac{i}{\sqrt{a_i}}\right)^r\frac{E_i(f)_{u,p}}{i}$$

and, assuming $\Omega_\varphi^r(f,t)_{u,p}\,t^{-1}\in L^1[0,1]$,

$$E_m(f)_{u,p}\le \mathcal{C}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t}\,\mathrm{d}t\,, \tag{2.2}$$

where $m>r\ge 1$, $a_m=a_m(u)\sim m^{1/\beta}$ is the Mhaskar–Rahmanov–Saff number related to $u$ and $\mathcal{C}$ is independent of $m$ and $f$.

In particular, for any $f\in W_r^p(u)$, $1\le p\le +\infty$, we obtain

$$E_m\left(f\right)_{u,p}\le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r\left\|f^{(r)}\varphi^r u\right\|_p\,,\qquad \mathcal{C}\ne\mathcal{C}(m,f)\,. \tag{2.3}$$

Whereas, for any $f\in Z_s^p(u)$, $1\le p\le +\infty$, we get

$$E_m\left(f\right)_{u,p}\le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s\sup_{t>0}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^s}\,,\quad r>s\,,\quad \mathcal{C}\ne\mathcal{C}(m,f)\,.$$

2.3. **The interpolation process.** Let us now consider the weight

$$w(x)=x^\gamma\sigma(x)=x^\gamma\mathrm{e}^{-x^{-\alpha}-x^\beta}\,,$$

with $\alpha>0$, $\beta>1$, $\gamma\ge 0$ and $x\in(0,+\infty)$, and denote by $\{p_m(w)\}_{m\in\mathbb{N}}$ the sequence of orthonormal polynomials with positive leading coefficients.

The zeros of $p_m(w)$ lie in the MRS interval associated with $\sqrt{w}$ (see [4, p. 361] and [11, p. 38, Proposition 2.3], taking into account that $w$ belongs to the Levin–Lubinsky class $\mathcal{F}(C^2+)$), namely

$$\varepsilon_m<x_1<x_2<\cdots<x_m<a_m\,,$$

with $m$ sufficiently large and

$$\varepsilon_\tau=\varepsilon_\tau(\sqrt{w})\sim\left(\frac{\sqrt{a_\tau}}{\tau}\right)^{\frac{1}{\alpha+1/2}}\qquad\text{and}\qquad a_\tau=a_\tau(\sqrt{w})\sim\tau^{1/\beta}\,,\quad \tau>0\,. \tag{2.4}$$

Since $\varepsilon_\tau(\sqrt{w})\sim\varepsilon_\tau(u)$ and $a_\tau(\sqrt{w})\sim a_\tau(u)$, where the constants in "$\sim$" depend only on $\gamma$ and $\delta$, in the sequel with a slight abuse of notation we will simply write $\varepsilon_\tau$ and $a_\tau$ for both weights.

Then, setting $x_0=\varepsilon_m$ and $x_{m+1}=a_m$, for any function $f\in C^0(\mathbb{R}^+)$, we denote by $\mathcal{L}_{m+2}(w,f)$ the Lagrange polynomial, interpolating the function $f$ at the points $x_k$, $k=0,1,\dots,m+1$, namely

$$\mathcal{L}_{m+2}(w,f)=\sum_{k=0}^{m+1}\ell_k(x)f(x_k)\,,$$

where

$$\ell_k(x)=\frac{v(x)p_m(w,x)}{v(x_k)p_m'(w,x_k)(x-x_k)}\,,\qquad 1\le k\le m\,,$$

$v(x) = (x-\varepsilon_m)(a_m - x)$,

$$\ell_0(x) = \frac{a_m - x}{a_m - \varepsilon_m} \frac{p_m(w,x)}{p_m(w,\varepsilon_m)},$$

and

$$\ell_{m+1}(x) = \frac{x-\varepsilon_m}{a_m - \varepsilon_m} \frac{p_m(w,x)}{p_m(w,a_m)}.$$

Let us now fix $\theta \in (0,1)$. With that we introduce a new interpolation process $\mathcal{L}^*_{m+2}(w)$, defined as

$$\mathcal{L}^*_{m+2}(w,f,x) = \sum_{\varepsilon_{\theta m} \le x_k \le a_{\theta m}} \ell_k(x) f(x_k).$$

So, the operator $\mathcal{L}^*_{m+2}(w)$ has the advantage of requiring a smaller number of evaluations of the function. To ensure that this definition is not an unmotivated (though fortunate) "truncation", we need some further observation. To this aim we recall the following inequalities (see [4, p. 97] and [12, 11], taking into account that $u$ belongs to the Levin–Lubinsky class $\mathcal{F}(C^2+)$). For any $P_m \in \mathbb{P}_m$ and $s > 1$, with $u$ given by (2.1) and $1 \le p \le \infty$, we have

$$\|P_m u\|_p \le \mathcal{C} \|P_m u\|_{L^p[\varepsilon_m, a_m]} \tag{2.5}$$

and

$$\|P_m u\|_{L^p(\mathbb{R}^+ \setminus [\varepsilon_{sm}, a_{sm}])} \le \mathcal{C} \mathrm{e}^{-cm^\nu} \|P_m u\|_p, \tag{2.6}$$

where $\mathcal{C}$ and $c$ are independent of $m$ and $P_m$ and

$$\nu = \left(1 - \frac{1}{2\beta}\right) \frac{2\alpha}{2\alpha+1} \in (0,1),$$

and $\varepsilon_m$, $a_m$ are the Mhaskar–Rahmanov–Saff numbers related to $u$.

For any fixed $\theta \in (0,1)$ and for every $f \in L^p_u$, $1 \le p \le \infty$, we get, by using (2.6) and denoting by $P_M$, $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right) m \right\rfloor$, the polynomial of best approximation of $f$ in $L^p_u$,

$$\begin{aligned}
\|fu\|_{L^p(\mathbb{R}^+ \setminus [\varepsilon_{\theta m}, a_{\theta m}])} &\le \|(f - P_M) u\|_p + \|P_M u\|_{L^p(\mathbb{R}^+ \setminus [\varepsilon_{\theta m}, a_{\theta m}])} \\
&\le E_M(f)_{u,p} + \mathcal{C} \mathrm{e}^{-cM^\nu} \|P_M u\|_p \\
&\le E_M(f)_{u,p} + 2\mathcal{C} \mathrm{e}^{-cM^\nu} \|f u\|_p .
\end{aligned}$$

Now, for a sufficiently large $M$ (say $M > M_0$), $2\mathcal{C}\mathrm{e}^{-cM^\nu} < 1$ and we get, using

$$\|fu\|_p \le \|fu\|_{L^p[\varepsilon_{\theta m}, a_{\theta m}]} + \|fu\|_{L^p(\mathbb{R}^+ \setminus [\varepsilon_{\theta m}, a_{\theta m}])},$$

the estimate

$$\|fu\|_p \le \mathcal{C} \left[ \|fu\|_{L^p[\varepsilon_{\theta m}, a_{\theta m}]} + E_M(f)_{u,p} \right]$$

for all sufficiently large $M = \lfloor \frac{\theta m}{\theta+1} \rfloor$ and with $\mathcal{C} = \mathcal{C}_\theta \neq \mathcal{C}(m,f)$. Hence the dominant part of $\|fu\|_p$ is the norm of a finite section of $f$, namely $\chi_\theta f$, where $\chi_\theta$ is the characteristic function of $[\varepsilon_{\theta m}, a_{\theta m}]$. For this reason we apply the operator $\mathcal{L}_{m+2}(w)$ to this finite section.

Now, by definition, it follows that

$$\mathcal{L}^*_{m+2}(w,f,x_k) = f(x_k), \qquad x_k \in [\varepsilon_{\theta m}, a_{\theta m}],$$

and

$$\mathcal{L}^*_{m+2}(w,f,x_k)=0\,,\qquad x_k\notin[\varepsilon_{\theta m},a_{\theta m}]\,.$$

So, $\mathcal{L}^*_{m+2}(w)$ does not preserve all polynomials of degree at most $m+1$. Nevertheless, if we denote by $\mathcal{P}^*_{m+1}$ the set of the polynomials of degree at most $m+1$ vanishing at $x_k\notin[\varepsilon_{\theta m},a_{\theta m}]$, i.e.,

$$\mathcal{P}^*_{m+1}=\{Q\in\mathbb{P}_{m+1}:\ Q(x_k)=0\,,\ x_k\notin[\varepsilon_{\theta m},a_{\theta m}]\,,\ 0\le k\le m+1\}$$

then $\mathcal{L}^*_{m+2}(w,f)\in\mathcal{P}^*_{m+1}$ and, for any $Q\in\mathcal{P}^*_{m+1}$, $\mathcal{L}^*_{m+2}(w,Q)=Q$. Finally, if we denote by

$$E^*_{m+1}(f)_{u,p}=\inf_{Q\in\mathcal{P}^*_{m+1}}\|(f-Q)\,u\|_p$$

the error of best weighted approximation by polynomials of $\mathcal{P}^*_{m+1}$, we can show that (see [9, p. 1664, Lemma 4.1] or [5, 6] for different weight functions)

$$E^*_{m+1}(f)_{u,p}\le\mathcal{C}\left[E_M(f)_{u,p}+\mathrm{e}^{-c\,m^\nu}\|fu\|_p\right] \tag{2.7}$$

with $1\le p\le\infty$, $M=\lfloor\frac{\theta m}{\theta+1}\rfloor$, $\nu=\left(1-\frac{1}{2\beta}\right)\frac{2\alpha}{2\alpha+1}$, $\mathcal{C}$ and $c$ independent of $m$ and $f$. So, $\bigcup_m\mathcal{P}^*_{m+1}$ is dense in $L^p_u$, $1\le p\le\infty$.

## 3. Main results

Let us now state some convergence results for the operator $\mathcal{L}^*_{m+2}(w)$. To this aim we recall that $w(x)=x^\gamma\sigma(x)$ is the weight of the orthonormal system $\{p_m(w)\}_m$ and $u(x)=x^\delta\sqrt{\sigma(x)}$ is the weight of the previously introduced function classes.

**Theorem 3.1.** *For any $f\in C_u$ we have*

$$\|\mathcal{L}^*_{m+2}\,(w,f)\,u\|_\infty\le\mathcal{C}(\log m)\|fu\|_\infty \tag{3.1}$$

*and*

$$\|\left[f-\mathcal{L}^*_{m+2}\,(w,f)\right]u\|_\infty\le\mathcal{C}\left[(\log m)E_M(f)_{u,\infty}+\mathrm{e}^{-c\,m^\nu}\|fu\|_\infty\right] \tag{3.2}$$

*with $M=\lfloor\frac{\theta m}{\theta+1}\rfloor$, $\nu=\left(1-\frac{1}{2\beta}\right)\frac{2\alpha}{2\alpha+1}$, $\mathcal{C}$ and $c$ independent of $m$ and $f$, if and only if*

$$-\frac{3}{4}\le\delta-\frac{\gamma}{2}\le\frac{1}{4}\,. \tag{3.3}$$

Before stating an analogous theorem in $L^p_u$−norm, we recall that (see [7, p. 160 Corollary 3.3]) if $f\in L^p_u$, $1\le p<\infty$, and $\frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\in L^1[0,1]$, then $f$ is continuous on $(0,+\infty)$. So, $\mathcal{L}^*_{m+2}\,(w,f)$ is well defined for this kind of functions.

**Lemma 3.2.** *Let $1<p<\infty$ and $f\in L^p_u$ satisfies*

$$\int_0^1\frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t<\infty\,.$$

*Then we have*

$$\|\mathcal{L}^*_{m+2}(w,f)\,u\|_p \le \mathcal{C}_\theta \left( \sum_{\varepsilon_{\theta m} \le x_k \le a_{\theta m}} \Delta x_k |fu|^p(x_k) \right)^{1/p} \tag{3.4}$$

*with $\mathcal{C}_\theta$ independent of $m$ and $f$, if and only if*

$$-\frac{3}{4} - \frac{1}{p} < \delta - \frac{\gamma}{2} < \frac{1}{4} - \frac{1}{p}. \tag{3.5}$$

We remark that the constant $\mathcal{C}_\theta$ in (3.4) depends on the parameter $\theta$. To be more precise

$$\mathcal{C}_\theta = \mathcal{O}\left(\frac{1}{\theta(1-\theta)}\right),$$

hence the "truncation" does play a crucial role in this case and Lemma 3.2 does not hold if we replace $\mathcal{L}^*_{m+2}(w,x)$ by $\mathcal{L}_{m+2}(w,x)$.

**Remark 3.3.** *We observe that, as a consequence of Lemma 3.2 and Remark 4.2, for any $Q \in \mathcal{P}^*_{m+1}$, with $1 \le p < \infty$ and $0 < \theta < 1$, if conditions (3.5) hold, then we obtain*

$$\|Qu\|_p \sim \left( \sum_{\varepsilon_{\theta m} \le x_k \le a_{\theta m}} \Delta x_k \, |Q(x_k)u(x_k)|^p \right)^{1/p}$$

*where the constants in "$\sim$" are independent of $m$ and $Q$.*

*We note that this Marcinkiewicz-type equivalence is true only for polynomials of the subspace $\mathcal{P}^*_{m+1}$ and not for ordinary polynomials of $\mathbb{P}_{m+1}$.*

**Theorem 3.4.** *Let $1 < p < \infty$. Under the assumptions of Lemma 3.2, if (3.5) holds, we get*

$$\|\mathcal{L}^*_{m+2}(w,f)\,u\|_p \le \mathcal{C}_\theta \left[ \|fu\|_p + \left(\frac{\sqrt{a_m}}{m}\right)^{1/p} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}} \, \mathrm{d}t \right] \tag{3.6}$$

*and*

$$\|\left[f - \mathcal{L}^*_{m+2}(w,f)\right] u\|_p \le \mathcal{C} \left[ \left(\frac{\sqrt{a_m}}{m}\right)^{1/p} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}} + \mathrm{e}^{-cm^\nu} \|fu\|_p \right] \tag{3.7}$$

*with $\nu = \left(1 - \frac{1}{2\beta}\right)\frac{2\alpha}{2\alpha+1}$, $\mathcal{C}$ and $c$ independent of $m$ and $f$.*

We remark that, under the assumption (3.5), if $f$ fulfills only $\frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}} \in L^1[0,1]$, then

$$\|\left[f - \mathcal{L}^*_{m+2}(w,f)\right] u\|_p = o\left(\left(\frac{\sqrt{a_m}}{m}\right)^{1/p}\right),$$

while if $f \in W^p_r(u)$ then

$$\|\left[f - \mathcal{L}^*_{m+2}(w,f)\right] u\|_p \le \mathcal{C} \left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W^p_r(u)}$$

and, by (2.3) this order of convergence is the same of the best weighted polynomial approximation.

Now it is easy to prove that the operator $\mathcal{L}^*_{m+2}(w)$ is uniformly bounded in Sobolev-type spaces. Here we prove a more general theorem. To this aim we denote by $D^p(u)$, $1<p<\infty$, the following wide class of functions

$$D^p(u)=\left\{f\in L^p_u:\ \Omega_\varphi(f,t)_{u,p}\,t^{-1-1/p}\in L^1[0,1]\right\}, \tag{3.8}$$

with the norm

$$\|f\|_{D^p(u)}=\|fu\|_p+\int_0^1\frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\,.$$

We observe that the Besov-type space $D^p(u)$ contains the continuous functions, the functions of $Z^p_s(u)$, $s>1/p$, and those belonging to $W^p_r(u)$, $r\geq 1$.

**Theorem 3.5.** *For any $f\in D^p(u)$, $1<p<\infty$, if*

$$-\frac{3}{4}-\frac{1}{p}<\delta-\frac{\gamma}{2}<\frac{1}{4}-\frac{1}{p}$$

*then*

$$\sup_m\left\|\mathcal{L}^*_{m+2}(w,f)\right\|_{D^p(u)}\leq\mathcal{C}_\theta\,\|f\|_{D^p(u)}$$

*with $\mathcal{C}_\theta=\mathcal{C}(\theta)\neq\mathcal{C}(f)$.*

Therefore the polynomial $\mathcal{L}^*_{m+2}(w,f)$ behaves essentially as the best approximation in the space $D^p(u)$.

## 4. Proofs

First of all we recall the following inequalities (see [4, pp. 1–34, 325 and 360] and also [11, pp.39–40]). With $w(x)=x^\gamma\mathrm{e}^{-x^{-\alpha}-x^\beta}$ and $v(x)=(a_m-x)(x-\varepsilon_m)$, we have

$$|p_m(w,x)|\sqrt{w(x)\sqrt{|v(x)|}}\leq\mathcal{C}\qquad\forall x\in(0,+\infty)\,, \tag{4.1}$$

$$\frac{1}{|p'_m(w,x_k)|\sqrt{w(x_k)}}\sim\Delta x_k\sqrt[4]{v(x_k)} \tag{4.2}$$

$$\Delta x_k\sim\frac{a_m x_k}{m\sqrt{v(x_k)}}\,,\qquad k=1,2,\dots m\,, \tag{4.3}$$

and

$$\Delta x_k\sim\frac{\sqrt{a_m}}{m}\sqrt{x_k}\,,\qquad x_k\in[\varepsilon_{\theta m},a_{\theta m}]\,, \tag{4.4}$$

where $\mathcal{C}$ and the constants in "$\sim$" are independent of $m$ and $k$.

*Proof Theorem 3.1: (3.3) ⇒ (3.1) and (3.2).* By (2.5) we have

$$\left\|\mathcal{L}^*_{m+2}\left(w,f\right)u\right\|_\infty \leq \mathcal{C}\left\|\mathcal{L}^*_{m+2}\left(w,f\right)u\right\|_{L^\infty(I_m)},$$

where $I_m = [\varepsilon_{m+1}, a_{m+1}]$ Hence, for $x \in I_m$, $x_d$ a zero closest to $x$, $x_k \in [\varepsilon_{\theta m}, a_{\theta m}]$ and $k \neq d, d \pm 1$, using (4.1) and (4.2), we get

$$\begin{aligned} |\ell_k(x)|\,\frac{u(x)}{u(x_k)} &= \left|\frac{v(x)p_m(w,x)u(x)}{v(x_k)p_m'(w,x_k)(x-x_k)u(x_k)}\right| \\ &\leq \mathcal{C}\left(\frac{x}{x_k}\right)^{\delta-\gamma/2}\left(\frac{v(x)}{v(x_k)}\right)^{3/4}\frac{\Delta x_k}{|x-x_k|}\,. \end{aligned}$$

Now, by (2.4), taking into account that $x_k \in [\varepsilon_{\theta m}, a_{\theta m}]$, we have

$$\frac{v(x)}{v(x_k)} \leq \mathcal{C}_\theta \frac{x}{x_k}\,,$$

whence

$$|\ell_k(x)|\,\frac{u(x)}{u(x_k)} \leq \mathcal{C}_\theta\left(\frac{x}{x_k}\right)^{\delta-\gamma/2+3/4}\frac{\Delta x_k}{|x-x_k|}\,, \qquad k \neq d, d\pm 1\,.$$

Since [4, p. 361]

$$|\ell_k(x)|\,\frac{u(x)}{u(x_k)} \sim 1\,, \qquad k = d, d\pm 1\,,$$

it follows that

$$\begin{aligned} &\left\|\mathcal{L}^*_{m+2}\left(w,f\right)u\right\|_\infty \leq \\ &\leq \mathcal{C}_\theta \|fu\|_{L^\infty[\varepsilon_{\theta m}, a_{\theta m}]} \max_{x\in I_m}\left[1 + \sum_{\substack{\varepsilon_{\theta m} \leq x_k \leq a_{\theta m} \\ k \neq d, d\pm 1}} \left(\frac{x}{x_k}\right)^{\delta-\gamma/2+3/4}\frac{\Delta x_k}{|x-x_k|}\right]. \end{aligned}$$

By hypothesis $0 \leq \delta - \gamma/2 + 3/4 \leq 1$ and so the sum at the right-hand side is dominated by $\mathcal{C}\log m$. In fact, for $x \in (\varepsilon_{\theta m}, a_{\theta m})$, this sum is dominated by

$$\mathcal{C}\left\{\int_{\varepsilon_{\theta m}}^{x_{d-1}} + \int_{x_{d+1}}^{a_{\theta m}}\right\}\left(\frac{x}{t}\right)^{\delta-\gamma/2+3/4}\frac{\mathrm{d}t}{|x-t|} = I_1 + I_2$$

For the term $I_2$, since $x < t$, we have

$$I_2 \leq \mathcal{C}\int_{x_{d+1}}^{a_{\theta m}} \frac{\mathrm{d}t}{t-x} \sim \log m\,.$$

While, for the term $I_1$, setting $y=\frac{x}{t}$ and $\lambda=-\delta+\gamma/2-3/4$, by (3.3) we have $-1\le\lambda\le 0$ and then

$$\begin{aligned} I_1 &\le \int_0^{1-\frac{\Delta x_d}{x}} \frac{y^{-\delta+\gamma/2-3/4}}{1-y}\,\mathrm{d}y \\ &\le \int_0^{1-\frac{\Delta x_d}{x}} \frac{y^{\lambda}(1-y^{-\lambda})}{1-y}\,\mathrm{d}y + \int_0^{1-\frac{\Delta x_d}{x}} \frac{\mathrm{d}y}{1-y} \\ &\le \int_0^{1-\frac{\Delta x_d}{x}} y^{\lambda}(1-y)^{-\lambda-1}\,\mathrm{d}y + \int_0^{1-\frac{\Delta x_d}{x}} \frac{\mathrm{d}y}{1-y} \\ &\le \mathcal{C}+\mathcal{C}\log m\,, \end{aligned}$$

which completes the proof, since the other cases are simpler. In the sequel we are going to show that inequality (3.1) implies the assumptions (3.3).

Now, let us prove the error estimate (3.2). Since $\mathcal{L}^*_{m+2}(w)$ preserves the polynomials of $\mathcal{P}^*_{m+1}$, using (3.1) and (2.7), we have

$$\begin{aligned} \left\|\left[f-\mathcal{L}^*_{m+2}(w,f)\right]u\right\|_\infty &\le \mathcal{C}_\theta\,(\log m)\,E^*_{m+1}(f)_{u,\infty} \\ &\le \mathcal{C}_\theta\left[(\log m)E_M(f)_{u,\infty}+\mathrm{e}^{-cm^\nu}\|fu\|_\infty\right] \end{aligned}$$

□

The following proposition will be useful in order to prove Theorem 3.4.

**Proposition 4.1.** *Let $1\le p<\infty$, $\eta\ge 0$ and $0<\theta<\bar\theta<1$. Then, for any polynomial $P_{lm}\in\mathbb{P}_{lm}$, with $l$ a fixed integer, we have*

$$\sum_{\varepsilon_{\theta m}\le x_k\le a_{\theta m}} \Delta x_k\left|x_k^\eta P_{lm}(x_k)\right|^p \le \mathcal{C}\int_{\varepsilon_{\bar\theta m}}^{a_{\bar\theta m}} \left|x^\eta P_{lm}(x)\right|^p\,\mathrm{d}x\,,$$

*where $x_k$ are the zeros of $p_m(w)$ and $\mathcal{C}$ is independent of $m$ and $P_{lm}$.*

*Proof.* From inequality

$$(b-a)\left|f(a)\right|^p \le 2^p\left[\int_a^b |f(x)|^p\,\mathrm{d}x+(b-a)^p\int_a^b |f'(x)|^p\,\mathrm{d}x\right],$$

setting $a=x_k$, $b=x_{k+1}$ and $f=P_{lm}$, we deduce

$$\Delta x_k\left|P_{lm}(x_k)\right|^p \le 2^p\left[\int_{x_k}^{x_{k+1}} |P_{lm}(x)|^p\,\mathrm{d}x+(\Delta x_k)^p\int_{x_k}^{x_{k+1}} |P'_{lm}(x)|^p\,\mathrm{d}x\right].$$

Using (4.4) and $x_k\sim x$ for $x\in[x_k,x_{k+1}]$, we get

$$\begin{aligned} &\Delta x_k\left|x_k^\eta P_{lm}(x_k)\right|^p \le \\ &\le \mathcal{C}\left[\int_{x_k}^{x_{k+1}} |x^\eta P_{lm}(x)|^p\,\mathrm{d}x+\left(\frac{\sqrt{a_m}}{m}\right)^p\int_{x_k}^{x_{k+1}} \left|x^\eta P'_{lm}(x)\sqrt{x}\right|^p\,\mathrm{d}x\right]. \end{aligned}$$

and

$$\sum_{\varepsilon_{\theta m}\le x_k\le a_{\theta m}} \Delta x_k \left|x_k^\eta P_{lm}(x_k)\right|^p \le$$

$$\le \mathcal{C}\left[\int_{\varepsilon_{\theta m}}^{a_{\theta m}} \left|x^\eta P_{lm}(x)\right|^p \,\mathrm{d}x + \left(\frac{\sqrt{a_m}}{m}\right)^p \int_{\varepsilon_{\theta m}}^{a_{\theta m}} \left|x^\eta P'_{lm}(x)\sqrt{x}\right|^p \,\mathrm{d}x\right].$$

For the second integral at the right-hand side, taking into account that

$$\sqrt{\frac{x}{x-\varepsilon_{\bar\theta m}}}\sqrt{\frac{a_m}{a_{\bar\theta m}-x}} \le C_{\bar\theta}$$

for

$$\varepsilon_m < \varepsilon_{\bar\theta m} < \varepsilon_{\theta m} \le x \le a_{\theta m} < a_{\bar\theta m} < a_m\,,$$

we obtain

$$\begin{aligned}
&\left(\frac{\sqrt{a_m}}{m}\right)^p \int_{\varepsilon_{\theta m}}^{a_{\theta m}} \left|x^\eta P'_{lm}(x)\sqrt{x}\right|^p \,\mathrm{d}x\\
&= \left(\frac{\sqrt{a_m}}{m}\right)^p \int_{\varepsilon_{\theta m}}^{a_{\theta m}} \left|\frac{x^\eta P'_{lm}(x)\sqrt{x}\sqrt{(x-\varepsilon_{\bar\theta m})(a_{\bar\theta m}-x)}}{\sqrt{(x-\varepsilon_{\bar\theta m})(a_{\bar\theta m}-x)}}\right|^p \,\mathrm{d}x\\
&\le \frac{\mathcal{C}}{m^p}\int_{\varepsilon_{\bar\theta m}}^{a_{\bar\theta m}} \left|x^\eta P'_{lm}(x)\sqrt{(x-\varepsilon_{\bar\theta m})(a_{\bar\theta m}-x)}\right|^p \,\mathrm{d}x\\
&\le \mathcal{C}\int_{\varepsilon_{\bar\theta m}}^{a_{\bar\theta m}} \left|x^\eta P_{lm}(x)\right|^p \,\mathrm{d}x\,,
\end{aligned}$$

having used the Bernstein inequality related to the interval $[\varepsilon_{\bar\theta m}, a_{\bar\theta m}]$ in weighted $L^p-$norm. The proposition easily follows. □

**Remark 4.2.** *We note that, proceeding as in the proof of Proposition 4.1 and taking into account that $u(x) \sim u(x_k)$ for $x \in [x_k, x_{k+1}]$ (see [8, p. 170, Proposition 2.1]), for any $P_{lm} \in \mathbb{P}_{lm}$, with $l$ a fixed integer, with $1 \le p < \infty$ and $0 < \theta < \bar\theta < 1$, we obtain*

$$\sum_{\varepsilon_{\theta m}\le x_k\le a_{\theta m}} \Delta x_k \left|P_{lm}(x_k)u(x_k)\right|^p \le \mathcal{C}\int_{\varepsilon_{\bar\theta m}}^{a_{\bar\theta m}} \left|P_{lm}(x)u(x)\right|^p \,\mathrm{d}x\,,$$

*where $x_k$ are the zeros of $p_m(w)$ and $\mathcal{C}$ is independent of $m$ and $P_{lm}$.*

**Proposition 4.3.** *Let $1 \le p < \infty$, $a > 0$ fixed and $g \in L^p$. Then, for $t_m(x) = p_m(w,x)\sqrt{w(x)\sqrt{|(a_m-x)(x-\varepsilon_m)|}}$, the inequality*

$$\int_0^a \left|g(x)t_m(x)\right|^p \,\mathrm{d}x \ge \mathcal{C}\int_0^a \left|g(x)\right|^p \,\mathrm{d}x$$

*holds with $\mathcal{C} \ne \mathcal{C}(m, p_m(w))$.*

*Proof.* Letting $x_k$ be the zeros of $p_m(w)$, we set

$$J_m = \cup_{x_k \le a} \left( x_k - \frac{\mu}{8}\Delta x_k, x_k + \frac{\mu}{8}\Delta x_k \right) ,$$

with $\mu > 0$ small. Taking into account (4.2) and (4.3), since

$$\left| \frac{p_m(w,x)}{p_m'(w,x_d)(x-x_d)} \right| \sim 1 ,$$

where $x_d$ is a zero closest to $x \in (\varepsilon_m, a_m)$, we have

$$|t_m(x)| = \left| p_m(w,x)\sqrt{w(x)\sqrt{|(a_m-x)(x-\varepsilon_m)|}} \right| \sim \frac{|x-x_{d\pm1}|}{\Delta x_{d\pm1}} \tag{4.5}$$

and then

$$|t_m(x)| \ge \mathcal{C}_\mu , \qquad x \in [0,a] \setminus J_m .$$

So we get

$$\begin{aligned} \int_0^a |g(x)t(x)|^p \, \mathrm{d}x &\ge \mathcal{C}_\mu \int_{[0,a]\setminus J_m} |g(x)|^p \, \mathrm{d}x \\ &= \mathcal{C}_\mu \int_0^a |g(x)|^p \, \mathrm{d}x - \mathcal{C}_\mu \int_{J_m} |g(x)|^p \, \mathrm{d}x . \end{aligned}$$

Since the measure of $J_m$ fulfills $|J_m| \le \frac{a\mu}{4}$, using the absolute continuity of the integral, for any fixed $g$ we can choose $\mu$ such that the second integral at the right-hand side is a half of the first one and the proof is complete. □

In order to prove Lemma 3.2, we recall some properties of the Hilbert transform $\mathcal{H}$ extended to an interval $(a,b)$, defined by

$$\mathcal{H}(f,y) = \int_a^b \frac{f(x)}{x-y} \, \mathrm{d}x , \qquad y \in (a,b) ,$$

where the integral is understood in the Cauchy principal value sense. The commutation formula

$$\int_a^b \mathcal{H}(f)g = -\int_a^b \mathcal{H}(g)f$$

holds for any $f \in L^p$ and $g \in L^q$, $1 < p < \infty$, $1/p + 1/q = 1$. We recall that, for any measurable function $f$ such that $fv \in L^p$, $1 < p < \infty$, the inequality

$$\|\mathcal{H}\,(f)\,v\|_p \le \mathcal{C}\|fv\|_p , \qquad \mathcal{C} \ne \mathcal{C}(f) ,$$

holds if and only (see [2]) the weight $v$ belongs to the $A_p$ class, $1 < p < \infty$, i.e.,

$$\left( \frac{1}{|I|} \int_I v^p(x) \, \mathrm{d}x \right)^{1/p} \left( \frac{1}{|I|} \int_I v^{-q}(x) \, \mathrm{d}x \right)^{1/q} \le \mathcal{C} \qquad I \subset (a,b) ,$$

where $|I|$ denotes the measure of $I$ and $\frac{1}{p} + \frac{1}{q} = 1$.

*Proof of Lemma* 3.2. By (2.5) we have

$$\begin{aligned}\left\|\mathcal{L}^*_{m+2}(w,f)\,u\right\|_p &\leq \mathcal{C}\left\|\mathcal{L}^*_{m+2}(w,f)\,u\right\|_{L^p(I_m)}\\ &= \mathcal{C}\sup_{\|g\|_{L^q(I_m)}=1}\int_{I_m}\mathcal{L}^*_{m+2}(w,f,x)\,u(x)g(x)\,\mathrm{d}x\\ &=: \mathcal{C}\sup_{\|g\|_{L^q(I_m)}=1} A(g)\end{aligned}$$

where $I_m := [\varepsilon_{m+1}, a_{m+1}]$ and $q = p/(p-1)$. By using (4.2) and since $v(x_k) \geq \mathcal{C}_\theta a_m x_k$ for $x_k \in [\varepsilon_{\theta m}, a_{\theta m}]$, we obtain

$$\begin{aligned}A(g) &\leq \frac{\mathcal{C}_\theta}{a_m^{3/4}}\sum_{\varepsilon_{\theta m}\leq x_k\leq a_{\theta m}}\frac{\Delta x_k|fu|(x_k)}{x_k^{\delta-\gamma/2+3/4}}\left|\int_{I_m}\frac{v(x)p_m(w,x)u(x)g(x)}{x-x_k}\,\mathrm{d}x\right|\\ &\leq \frac{\mathcal{C}_\theta}{a_m^{3/4}}\sum_{\varepsilon_{\theta m}\leq x_k\leq a_{\theta m}}\frac{\Delta x_k|fu|(x_k)}{x_k^{\delta-\gamma/2+3/4}}\left|\Pi(x_k)\right|,\end{aligned}$$

with

$$\begin{aligned}\Pi(y) &= \int_{I_m}\frac{v(x)p_m(w,x)Q(x)-v(y)p_m(w,y)Q(y)}{x-y}\,\frac{u(x)g(x)}{Q(x)}\,\mathrm{d}x\\ &= \mathcal{H}(vp_m(w)ug,y) - v(y)p_m(w,y)Q(y)\mathcal{H}\left(\frac{ug}{Q},y\right)\end{aligned}$$

where $Q > 0$ is a polynomial of degree $lm$ that will be chosen in the sequel and $\mathcal{H}$ is the Hilbert transform related to the interval $I_m$. Then, using the Hölder inequality, we get

$$A(g) \leq \frac{\mathcal{C}_\theta}{a_m^{3/4}}\left(\sum_{\varepsilon_{\theta m}\leq x_k\leq a_{\theta m}}\Delta x_k|fu|^p(x_k)\right)^{\frac{1}{p}}\left(\sum_{\varepsilon_{\theta m}\leq x_k\leq a_{\theta m}}\Delta x_k\left|x_k^{\gamma/2-\delta-3/4}\Pi(x_k)\right|^q\right)^{\frac{1}{q}}.$$

By Proposition 4.1, the second sum at the right-hand side is dominated by

$$\begin{aligned}&\mathcal{C}\left(\int_{I_m}\left|y^{\gamma/2-\delta-3/4}\Pi(y)\right|^q\,\mathrm{d}y\right)^{1/q}\\ &\quad\leq \mathcal{C}\left[\left(\int_{I_m}\left|y^{\gamma/2-\delta-3/4}\mathcal{H}(vp_m(w)ug,y)\right|^q\,\mathrm{d}y\right)^{1/q}\right.\\ &\quad\left.+\left(\int_{I_m}\left|y^{\gamma/2-\delta-3/4}v(y)p_m(w,y)Q(y)\mathcal{H}\left(\frac{ug}{Q},y\right)\right|^q\,\mathrm{d}y\right)^{1/q}\right]\\ &=: \mathcal{I}_1 + \mathcal{I}_2\,.\end{aligned}$$

Now, by (4.1), we have

$$\begin{aligned}|v(y)p_m(w,y)u(y)| &\leq \mathcal{C}y^{\delta-\gamma/2}(a_m-y)^{3/4}(y-\varepsilon_m)^{3/4}\\ &\leq \mathcal{C}y^{\delta-\gamma/2+3/4}a_m^{3/4}\,.\end{aligned}$$

By virtue of (3.5), $y^{\gamma/2-\delta-3/4}$ is an $A_q$ weight on $I_m$ and we can use the boundedness of the Hilbert transform, obtaining

$$\begin{aligned}\mathcal{I}_1 &\leq \mathcal{C}\left(\int_{I_m}\left|y^{\gamma/2-\delta-3/4}v(y)p_m(w,y)u(y)g(y)\right|^q \mathrm{d}y\right)^{1/q}\\ &\leq \mathcal{C}a_m^{3/4}\|g\|_{L^q(I_m)} = \mathcal{C}a_m^{3/4}.\end{aligned}$$

In order to estimate $\mathcal{I}_2$ we can choose $Q$ such that $Q \sim u$ in $I_m$ (see [12, p. 809, Lemma 3.1]) and, by (4.1), we get

$$\left|y^{\gamma/2-\delta-3/4}v(y)p_m(w,y)Q(y)\right| \leq \mathcal{C}a_m^{3/4},$$

whence, using the boundedness of the Hilbert transform $\mathcal{H}: L^q(I_m) \to L^q(I_m)$, we deduce

$$\begin{aligned}\mathcal{I}_2 &\leq \mathcal{C}a_m^{3/4}\left\|\mathcal{H}\left(\frac{ug}{Q}\right)\right\|_{L^q(I_m)}\\ &\leq \mathcal{C}a_m^{3/4}\|g\|_{L^q(I_m)} = \mathcal{C}a_m^{3/4}.\end{aligned}$$

Collecting the previous inequalities we obtain

$$\left\|\mathcal{L}^*_{m+2}(w,f)\,u\right\|_p \leq \mathcal{C}_\theta\left(\sum_{\varepsilon_{\theta m}\leq x_k\leq a_{\theta m}}\Delta x_k|fu|^p(x_k)\right)^{1/p}.$$

Now, let us prove that inequality (3.4) implies conditions (3.5). To this aim, letting $m$ fixed and sufficiently large, we consider a piecewise linear function $F_0$ such that

$$F_0(x) = 0 \qquad x \notin [2,3]$$

and

$$F_0(x_k) = \mathrm{sgn}(p_m'(w,x_k)) \qquad x_k \in [2,3],$$

joining two consecutive points with a segment. Since $F_0$ is continuous on $(0,+\infty)$, we can write

$$\left\|\mathcal{L}^*_{m+2}(w,F_0)\,u\right\|_{L^p[\varepsilon_{\theta m},1]} < \left\|\mathcal{L}^*_{m+2}(w,F_0)\,u\right\|_p \leq \mathcal{C}\left(\sum_{2\leq x_k\leq 3}\Delta x_k|F_0u|^p(x_k)\right)^{1/p}$$

and, using (4.2) and $\frac{v(x)}{v(x_k)} \geq \mathcal{C}_\theta\frac{x}{x_k}$ for $x > \varepsilon_{\theta m}$ and $x_k \geq 2$, we get

$$\left|\mathcal{L}^*_{m+2}(w,F_0,x)\,u(x)\right| \geq \mathcal{C}\left|p_m(w,x)v^{1/4}(x)u(x)x^{3/4}\right|\sum_{2\leq x_k\leq 3}\Delta x_k\, x_k^{-\delta+\gamma/2-3/4}u(x_k)$$

and

$$\left\|\mathcal{L}^*_{m+2}(w,F_0)\,u\right\|_{L^p[\varepsilon_{\theta m},1]} \geq \mathcal{C}\left\|p_m(w)v^{1/4}ux^{3/4}\right\|_{L^p[\varepsilon_{\theta m},1]}\sum_{2\leq x_k\leq 3}\Delta x_k\, x_k^{-\delta+\gamma/2-3/4}u(x_k).$$

Using Proposition 4.3 we obtain

$$\sup_m \left(\int_{\varepsilon_{\theta m}}^1 x^{(\delta-\gamma/2+3/4)p}\,\mathrm{d}x\right)^{1/p} \sum_{2\le x_k\le 3} \Delta x_k\, x_k^{-\delta+\gamma/2-3/4} u(x_k) \le \mathcal{C}\left(\sum_{2\le x_k\le 3} \Delta x_k u^p(x_k)\right)^{1/p}$$

Therefore

$$\left(\int_0^1 x^{(\delta-\gamma/2+3/4)p}\,\mathrm{d}x\right)^{1/p} \sum_{2\le x_k\le 3} \Delta x_k\, x_k^{-\delta+\gamma/2-3/4} u(x_k)$$

$$\le \mathcal{C}\left(\sum_{2\le x_k\le 3} \Delta x_k u^p(x_k)\right)^{1/p}. \tag{4.6}$$

Now, let $A = \{A_k\}_{k\in\mathbb{N}}$, where $A_k = (\Delta x_k)^{1/p} u(x_k)$ and $\|A\|_{l^p} = \left(\sum_{2\le x_k\le 3} A_k^p\right)^{1/p}$, the last inequality (4.6) can be rewritten as

$$\left(\int_0^1 x^{(\delta-\gamma/2+3/4)p}\,\mathrm{d}x\right)^{1/p} \sup_A \sum_{2\le x_k\le 3} \frac{A_k}{\|A\|_{l^p}} (\Delta x_k)^{1/q} x_k^{-\delta+\gamma/2-3/4} \le \mathcal{C}\,,$$

with $\frac{1}{p}+\frac{1}{q}=1$, since

$$\sup_A \sum_{2\le x_k\le 3} \frac{A_k}{\|A\|_{l^p}} (\Delta x_k)^{1/q} x_k^{-\delta+\gamma/2-3/4} = \sum_{2\le x_k\le 3} \Delta x_k x_k^{(-\delta+\gamma/2-3/4)q} \sim 1\,.$$

It follows that (3.4) implies $\delta-\gamma/2 > -3/4-1/p$, $1<p<\infty$.

Let us now show that (3.4) implies $\delta-\gamma/2 < 1/4-1/p$. To this aim we consider a piecewise linear function $F_1$ such that

$$F_1(x) = 0 \qquad x \notin [\varepsilon_{\theta m}, 1]$$

and

$$F_1(x_k) = \mathrm{sgn}(p_m'(w,x_k)) \qquad x_k \in [\varepsilon_{\theta m}, 1]\,.$$

Hence we get

$$\left\|\mathcal{L}^*_{m+2}(w,F_1)\,u\right\|_{L^p[2,3]} < \left\|\mathcal{L}^*_{m+2}(w,F_1)\,u\right\|_p \le \mathcal{C}\left(\sum_{\varepsilon_{\theta m}\le x_k\le 1} \Delta x_k |F_1 u|^p(x_k)\right)^{1/p}$$

and, proceeding as before, inequality (4.6) is replaced by

$$\left(\int_2^3 x^{(\delta-\gamma/2+3/4)p}\,\mathrm{d}x\right)^{1/p} \sum_{\varepsilon_{\theta m}\le x_k\le 1} \Delta x_k x_k^{-\delta+\gamma/2-3/4} \le \mathcal{C}\left(\sum_{\varepsilon_{\theta m}\le x_k\le 1} \Delta x_k u^p(x_k)\right)^{1/p}.$$

Now, with $B=\{B_k\}_{k\in\mathbb{N}}$, $B_k=(\Delta x_k)^{1/p}\,u(x_k)$ and $\|B\|_{l^p}=\left(\sum_{0\le x_k\le 1}B_k^p\right)^{1/p}$, the integral at the left-hand side is bounded, while

$$\begin{aligned}\sup_B\sum_{0\le x_k\le 1}\frac{B_k}{\|B\|_{l^p}}\,(\Delta x_k)^{1/q}\,x_k^{-\delta+\gamma/2-3/4} &= \left(\sum_{0\le x_k\le 1}\Delta x_k x_k^{(-\delta+\gamma/2-3/4)q}\right)^{1/q}\\ &\sim \left(\int_0^1 x^{(-\delta+\gamma/2-3/4)q}\,\mathrm{d}x\right)^{1/q}.\end{aligned}$$

From

$$\left(\int_0^1 x^{(-\delta+\gamma/2-3/4)q}\,\mathrm{d}x\right)^{1/q}\le\mathcal{C}$$

condition $\delta-\gamma/2<1/4-\frac{1}{p}$ follows. So the proof is complete. □

*Proof that inequality* (3.1) *of Theorem* 3.1 *implies conditions* (3.3). Let us first prove that (3.1) implies $\delta-\gamma/2-3/4\ge 0$. To this aim, using the same arguments of the previous proof (inequality (3.4) implies $\delta-\gamma/2>-3/4-1/p$), but replacing $p$ with $\infty$ and $q$ with 1, we obtain

$$\begin{aligned}\max_{x\in[\varepsilon_{\theta m},1]}&\left|p_m(w,x)v^{1/4}x^{3/4}u(x)\right|\sum_{2\le x_k\le 3}\Delta x_k u(x_k)x_k^{-\delta+\gamma/2-3/4}\\ &\le\left\|\mathcal{L}^*_{m+2}(w,F_0)\,u\right\|_{L^\infty[\varepsilon_{\theta m},1]}\\ &\le\mathcal{C}\,\|F_0u\|_{L^\infty[\varepsilon_{\theta m},1]}\log m\\ &\le\mathcal{C}\log m\,,\end{aligned}\tag{4.7}$$

where $F_0$ is the above defined function. The sum at the left hand side is bounded from below and using (4.5), it follows that

$$\max_{x\in[\varepsilon_{\theta m},1]}x^{\delta-\gamma/2+3/4}\le\mathcal{C}\log m\,,$$

i.e. $\delta-\gamma/2+3/4\ge 0$.

Let us now prove that (3.1) implies $\delta-\gamma/2-3/4\le 1$. As before, we will use a procedure similar to that of previous proof (inequality (3.4) implies $\delta-\gamma/2<1/4-1/p$), with $p=\infty$ and $q=1$ and $F_1$ in place of $F_0$. In this case (4.7) becomes

$$\begin{aligned}\max_{x\in[2,3]}&\left|p_m(w,x)v^{1/4}x^{3/4}u(x)\right|\sum_{\varepsilon_{\theta m}\le x_k\le 1}\Delta x_k u(x_k)x_k^{-\delta+\gamma/2-3/4}\\ &\le\left\|\mathcal{L}^*_{m+2}(w,F_1)\,u\right\|_{L^\infty[2,3]}\\ &\le\mathcal{C}\,\|F_1u\|_{L^\infty[2,3]}\log m\\ &\le\mathcal{C}\,\|u\|_{L^\infty[2,3]}\log m\,.\end{aligned}$$

Since $\left|p_m(w,x)v^{1/4}x^{3/4}u(x)\right|$ is bounded from below in $[2,3]$, we get

$$\sup_u \sum_{\varepsilon_{\theta m}\le x_k\le 1} \Delta x_k \frac{u(x_k)}{\|u\|_{L^\infty[2,3]}} x_k^{-\delta+\gamma/2-3/4} \le \mathcal{C}\log m$$

and

$$\sum_{\varepsilon_{\theta m}\le x_k\le 1} \Delta x_k x_k^{-\delta+\gamma/2-3/4} \le \mathcal{C}\log m\,,$$

whence

$$\int_{\varepsilon_{\theta m}}^1 t^{-\delta+\gamma/2-3/4}\,\mathrm{d}t \le \mathcal{C}\log m\,,$$

i.e. $\delta-\gamma/2+3/4\le 1$. So, (3.1) implies conditions (3.3). □

*Proof of Theorem* 3.4. First of all, inequality (3.6) can be obtained proceeding like in [15, pp. 232–234]), since

$$\begin{aligned}\left\|\mathcal{L}^*_{m+2}(w,f)\,u\right\|_p &\le \mathcal{C}_\theta\left(\sum_{\varepsilon_{\theta m}\le x_k\le a_{\theta m}} \Delta x_k |fu|^p(x_k)\right)^{1/p}\\ &\le \mathcal{C}_\theta\left[\|fu\|_p+\left(\frac{\sqrt{a_m}}{m}\right)^{1/p}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\right]\end{aligned}\tag{4.8}$$

Then, for any $P\in P^*_{m+1}$ of quasi best approximation, we can write

$$f-\mathcal{L}^*_{m+2}(w,f)=(f-P)-\mathcal{L}^*_{m+2}(w,f-P)\,.$$

By (2.7) and (2.2) we get

$$\begin{aligned}\|(f-P)u\|_p &\le \mathcal{C}\left[E_M(f)_{u,p}+\mathrm{e}^{-cm^\nu}\|fu\|_p\right]\\ &\le \mathcal{C}\left[\left(\frac{\sqrt{a_m}}{m}\right)^{1/p}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t+\mathrm{e}^{-cm^\nu}\|fu\|_p\right]\end{aligned}$$

and, using (4.8) and arguments similar to those in [14, Prop. 4.2, pp.280–281],

$$\begin{aligned}\left\|\mathcal{L}^*_{m+2}(w,f-P)\,u\right\|_p &\le \mathcal{C}_\theta\left[\|(f-P)u\|_p+\left(\frac{\sqrt{a_m}}{m}\right)^{1/p}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi(f-P,t)_{u,p}}{t^{1+1/p}}\right]\\ &\le \mathcal{C}\left[\mathrm{e}^{-cm^\nu}\|fu\|_p+\left(\frac{\sqrt{a_m}}{m}\right)^{1/p}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\right]\end{aligned}$$

we obtain (3.7).

□

*Proof of Theorem* 3.5. Let us consider the space $D^p(u)$, defined by (3.8). We have

$$\left\|\mathcal{L}^*_{m+2}(w,f)\right\|_{D^p(u)} \le \|f\|_{D^p(u)} + \left\|f-\mathcal{L}^*_{m+2}(w,f)\right\|_{D^p(u)}$$

and

$$\left\|f-\mathcal{L}^*_{m+2}(w,f)\right\|_{D^p(u)} = \left\|\left[f-\mathcal{L}^*_{m+2}(w,f)\right]u\right\|_p + \int_0^1 \frac{\Omega_\varphi\left(f-\mathcal{L}^*_{m+2}(w,f),t\right)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\,.$$

Let us estimate the second addend, decomposing it as

$$\int_0^1 \frac{\Omega_\varphi\left(f-\mathcal{L}^*_{m+2}(w,f),t\right)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t = \left\{\int_0^{\frac{\sqrt{a_m}}{m}} + \int_{\frac{\sqrt{a_m}}{m}}^1\right\} \frac{\Omega_\varphi\left(f-\mathcal{L}^*_{m+2}(w,f),t\right)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\,.$$

For the second integral we get

$$\begin{aligned}\int_{\frac{\sqrt{a_m}}{m}}^1 \frac{\Omega_\varphi\left(f-\mathcal{L}^*_{m+2}(w,f),t\right)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t &\le \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^{1/p}\left\|\left[f-\mathcal{L}^*_{m+2}(w,f)\right]u\right\|_p \\ &\le \mathcal{C}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t + \mathcal{C}\mathrm{e}^{-cm^\nu}\|fu\|_p\,.\end{aligned}$$

In order to estimate the first integral we can write

$$f-\mathcal{L}^*_{m+2}(w,f) = (f-P) - \mathcal{L}^*_{m+2}(w,f-P) + \mathcal{L}_{m+2}(w,(1-\chi_\theta)P)$$

where $P\in\mathbb{P}_M$, $M=\lfloor\frac{\theta m}{\theta+1}\rfloor$, is a polynomial of quasi best approximation for $f$, $\chi_\theta$ is the characteristic function of $[\varepsilon_{\theta m}, a_{\theta m}]$. Since (using a procedure in [1, pp . 98–100], see also [8, p. 174, Theorem 3.5])

$$\frac{\sqrt{a_m}}{m}\|P'_m\varphi u\|_p \le \mathcal{C}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi(f,t)_{u,p}}{t}\,\mathrm{d}t$$

we obtain

$$\begin{aligned}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi(f-P,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t &\le \mathcal{C}\left[\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t + \left(\frac{\sqrt{a_m}}{m}\right)^{1-\frac{1}{p}}\|P'_m\varphi u\|_p\right] \\ &\le \mathcal{C}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\,.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi\left(\mathcal{L}^*_{m+2}(w, f-P), t\right)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t &\le \left(\frac{\sqrt{a_m}}{m}\right)^{1-1/p} \left\|\left[\mathcal{L}^*_{m+2}(w, f-P)\right]' \varphi u\right\|_p \\ &\le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{-1/p} \left\|\mathcal{L}^*_{m+2}(w, f-P)\, u\right\|_p \\ &\le \mathcal{C}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t1, .\end{aligned}$$

In this last inequality we have used the estimate for $\left\|\mathcal{L}^*_{m+2}(w, f-P)\, u\right\|_p$ given in Theorem 3.4.It follows that

$$\begin{aligned}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi\left(\mathcal{L}^*_{m+2}(w, (1-\chi_\theta)P), t\right)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t &\le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{1-1/p} \left\|\left[\mathcal{L}^*_{m+2}(w, (1-\chi_\theta)P)\right]' \varphi u\right\|_p \\ &\le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{-1/p} \left\|\mathcal{L}^*_{m+2}(w, (1-\chi_\theta)P)\, u\right\|_p .\end{aligned}$$

Since

$$\left|\mathcal{L}^*_{m+2}(w, (1-\chi_\theta)P, x)\, u(x)\right| \le \|Pu\|_{L^\infty(\mathbb{R}^+\setminus[\varepsilon_{\theta m}, a_{\theta m}])} \sum_{x_k\notin[\varepsilon_{\theta m}, a_{\theta m}])} \frac{|\ell_k(x)|\, u(x)}{u(x_k)}$$

where the sum is dominated by $\mathcal{C}m^\tau$, for some $\tau > 0$, using (2.6) and the Nikolskii inequality (see [12, p. 810, Theorem 3.4])

$$\|P\,u\|_\infty \le \mathcal{C}\left(\frac{m}{\sqrt{\varepsilon_m a_m}}\right)^{\frac{1}{p}} \|P\,u\|_p \,,$$

we get

$$\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi\left(\mathcal{L}^*_{m+2}(w, (1-\chi_\theta)P), t\right)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t \le \mathcal{C}\mathrm{e}^{-c\,m^\nu} \|fu\|_p$$

which completes the proof. □

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