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Publisher:

Springer International Publishing

Published version:

DOI:10.1007/978-3-319-72456-0_29

Peter Junghanns, Giuseppe Mastroianni and Incoronata Notarangelo On Nyström and product integration methods for Fredholm integral equations. *Contemporary Computational Mathematics - A Celebration of the 80th Birthday of Ian Sloan* (Dick J., Kuo F., Woniakowski H. eds), Springer International Publishing (2018), 645–673. DOI: 10.1007/978-3-319-72456-0_29

ON NYSTRÖM AND PRODUCT INTEGRATION METHODS FOR FREDHOLM INTEGRAL EQUATIONS

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Abstract. The aim of this paper is to combine classical ideas for the theoretical investigation of the Nyström method for second kind Fredholm integral equations with recent results on polynomial approximation in weighted spaces of continuous functions on bounded and unbounded intervals, where also zeros of polynomials w.r.t. exponential weights are used.

1. Introduction

There exists a huge literature on numerical methods for Fredholm integral equations of second kind,

$$f(x) - \int_I K(x,y) f(y)\, dy = g(x)\,, \quad x \in I\,, \tag{1.1}$$

where I is a bounded or unbounded interval. A very famous method is the Nyström method which is based on an appropriate quadrature rule applied to the integral and on considering (1.1) in the space of (bounded) continuous functions on I. Such quadrature rules can be of different type. In the present paper we will focus on Gaussian rules and product integration rules based on zeros of orthogonal polynomials. The aim of this paper is to combine classical ideas for the theoretical investigation of the Nyström method, in particular the results of Sloan [35, 34], with recent results on polynomial approximation in weighted spaces of continuous functions on bounded and unbounded intervals, where also zeros of polynomials w.r.t. exponential weights come into the play (cf. [13, 29]). Note that the Nyström method, in general, is based on the application of a quadrature rule to the integral part of the operator. Here we focus on quadrature rules of interpolatory type, which are constructed with the help of zeros of orthogonal polynomials, i.e., which are of Gaussian type. Of course, there exists a lot of other possibilities. As an example, let us only mention the paper [12], where quasi-Monte Carlo rules are applied to the case of kernel functions of the form $K(x,y) = h(x-y)$.

Considering weighted spaces of continuous functions is motivated by the fact, that in many practical examples for the unknown function

Dedicated to Ian H. Sloan on the occasion of his 80th birthday.

it is known that it has some kind of singularities at the endpoints of the integration interval. Moreover, the kernel function of the integral operator can have endpoint singularities in both variables. For recent attempts to combine the idea of the Nyström method with weighted polynomial approximation, we refer the reader to [11, 21, 25].

The present paper is organized as follows. In Section 2 we present the notion of collectively compact and strongly convergent operator sequences and the classical result on the application of this concept for proving stability and convergence of approximation methods for operator equations. After formulating the results of Sloan from the 80's on the application of quadrature methods to Fredholm integral equations of the second kind, we show how these results can be generalized by using weighted spaces of continuous functions, where we prefer a unified approach for both bounded and unbounded integration intervals (see Definition 4 and Lemma 5). In Section 3 we prove a general convergence result for the classical Nyström method (see Corollary 9), where "classical" means that usual quadrature rules are used for the discretization of the integral operator, not product integration rules. In Subsections 3.1 and 3.2, this result is applied to the interval $(-1,1)$ involving Jacobi weights and to the half line $(0,\infty)$ involving exponential weights, respectively. Finally, Section 4 contains the most important results of the paper and is devoted to the application of product integration rules in the Nyström method, where again the Jacobi weight case and the exponential weight case are considered separately. In particular, in both cases we show how one can use the respective $\mathbf{L}\log^+\mathbf{L}$ function classes, in order to weaken the conditions on the kernel function of the integral operator (see Propositions 19 and 22).

2. Basic facts

In the sequel, by c we will denote real positive constants, which can assume different values at different places, and by $c \neq c(a,b,\ldots)$ we will explain, that c does not depend on $a,b,\ldots$ If α and β are positive real numbers depending on certain parameters $a,b,\ldots$, then by $\alpha \sim_{a,b,\ldots} \beta$ is meant that there is a positive constant $c \neq c(a,b,\ldots)$ such that $c^{-1}\alpha \le \beta \le c\alpha$.

We say, that a sequence $(\mathcal{K}_n)_{n=1}^\infty$ of linear operators $\mathcal{K}_n : \mathbf{X} \longrightarrow \mathbf{X}$ in the Banach space $\mathbf{X}$ is collectively compact, if the set $\{\mathcal{K}_n f : f \in \mathbf{X}, \|f\| \le 1, n \in \mathbb{N}\}$ is relatively compact in $\mathbf{X}$, i.e., the closure of this set is compact. The concept of collectively compact sets of operators goes back to Anselone and Palmer [1, 4, 2, 5, 6].

For the following proposition, see, for example, [3], or Sections 10.3 and 10.4 in [15], [16], or [17], or Section 4.1.in [7].

Proposition 1. *Let $\mathbf{X}$ be a Banach space and $\mathcal{K}:\mathbf{X}\longrightarrow\mathbf{X}$, $\mathcal{K}_n:\mathbf{X}\longrightarrow\mathbf{X}$, $n\in\mathbb{N}$ be given linear operators with $\lim\limits_{n\to\infty}\|\mathcal{K}_nf-\mathcal{K}f\|=0$ for all $f\in\mathbf{X}$ (i.e., the operators $\mathcal{K}_n$ converge strongly to $\mathcal{K}$ in $\mathbf{X}$). For $g\in\mathbf{X}$, consider the operator equations*

$$(\mathcal{I}-\mathcal{K})f=g \tag{2.1}$$

where $\mathcal{I}$ is the identity operator in $\mathbf{X}$, and

$$(\mathcal{I}-\mathcal{K}_n)f_n=g\,. \tag{2.2}$$

If the sequence $(\mathcal{K}_n)_{n=1}^{\infty}$ is collectively compact and if $\dim\ker(\mathcal{I}-\mathcal{K})=0$, then, for all sufficiently large n equation (2.2) has a unique solution $f_n^*\in\mathbf{X}$, where

$$\|f_n^*-f^*\|\le c\,\|\mathcal{K}_nf^*-\mathcal{K}f^*\|\,,\quad c\ne c(n,g,f^*)\,, \tag{2.3}$$

and $f^*\in\mathbf{X}$ is the unique solution of (2.1).

Let us consider the situation that $\mathbf{X}$ is equal to the space of continuous functions $\mathbf{C}(I)$, where $I=(I,d)$ is one of the compact metric spaces $I=[-1,1]$, $I=[0,\infty]$, or $I=[-\infty,\infty]$, the distance function of which can be given, for example, by $d(x,y)=|a(x)-a(y)|$ or $d(x,y)=\dfrac{|a(x)-a(y)|}{1+|a(x)-a(y)|}$ with $a(x)=\arctan(x)$. As usual, the norm in $\mathbf{C}(I)$ is defined by $\|f\|_\infty:=\max\{|f(x)|:x\in I\}$. As operators $\mathcal{K}$ and $\mathcal{K}_n$ we take

$$(\mathcal{K}f)(x)=\int_I K(x,y)f(y)\,dy\quad\text{as well as}\quad(\mathcal{K}_nf)(x)=\sum_{k=1}^{k_n}\Lambda_{nk}(x)f(x_{nk})\,, \tag{2.4}$$

where the Λ_{nk}'s are certain quadrature weights and we assume $x_{nk}\in I$ $(k=1,\dots,k_n)$, $x_{n1}<x_{n2}<\dots<x_{n\,k_n}$, as well as

(K1) $\displaystyle\int_I|K(x,y)|\,dy<\infty$, i.e., $K(x,.)\in\mathbf{L}^1(I)$ for all $x\in I$,

(K2) $\displaystyle\lim_{x\to x_0}\|K(x,.)-K(x_0,.)\|_{\mathbf{L}^1(I)}=\lim_{x\to x_0}\int_I|K(x,y)-K(x_0,y)|\,dy=0$ for all $x_0\in I$,

(K3) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{k_n}\Lambda_{nk}(x)f(x_{nk})=\int_I K(x,y)f(y)\,dy$ for all $x\in I$ and all $f\in\mathbf{C}(I)$, (K4) $\lim_{x\to x_0} \sup \left\{ \sum_{k=1}^{k_n} |\Lambda_{nk}(x) - \Lambda_{nk}(x_0)| : n \in \mathbb{N} \right\} = 0$ for all $x_0 \in I$.

Note that conditions (K1) and (K2) are necessary and sufficient for the operator $\mathcal{K} : \mathbf{C}(I) \longrightarrow \mathbf{C}(I)$ being a compact one, which is a consequence of the Arzela-Ascoli Theorem characterizing the relatively compact subsets of $\mathbf{C}(I)$. Moreover, the following lemma ist true and crucial for our further considerations (see [35, Section 2, Lemma] and [34, Section 3, Theorem 1]).

Lemma 2. *Suppose that conditions* (K1) *and* (K2) *are fulfilled. The operators $\mathcal{K}_n : \mathbf{C}(I) \longrightarrow \mathbf{C}(I)$, $n \in \mathbb{N}$, defined in* (2.4)*, form a collectively compact sequence, which converges strongly to $\mathcal{K}$, if and only if* (K3) *and* (K4) *are satisfied.*

Remark 3. *For example, in case $I = [0, \infty]$, conditions* (K1) - (K4) *can be written equivalently as (cf.* [34, (3.1)-(3.3)]*)*

(K1') $K(x, .) \in \mathbf{L}^1(0, \infty)\ \forall\, x \in [0, \infty)$,

(K2') $\lim_{x\to x_0} \|K(x, .) - K(x_0, .)\|_{\mathbf{L}^1(0,\infty)} = 0\ \forall\, x_0 \in [0, \infty)$,

(K3') $\lim_{x\to\infty} \sup \left\{ \int_0^\infty |K(x', y) - K(x, y)|\, dy : x' > x \right\} = 0$,

(K4') $\lim_{n\to\infty} \sum_{k=1}^{k_n} \Lambda_{nk}(x) f(x_{nk}) = \int_0^\infty K(x, y) f(y)\, dy\ \forall\, x \in [0, \infty)$ *and* $\forall\, f \in \mathbf{C}[0, \infty]$,

(K5') $\lim_{x\to x_0} \sup \left\{ \sum_{k=1}^{k_n} |\Lambda_{nk}(x) - \Lambda_{nk}(x_0)| : n \in \mathbb{N} \right\} = 0$ *for all* $x_0 \in [0, \infty)$,

(K6') $\lim_{x\to\infty} \sup_{x'>x} \sup \left\{ \sum_{k=1}^{k_n} |\Lambda_{nk}(x) - \Lambda_{nk}(x_0)| : n \in \mathbb{N} \right\} = 0$.

Now, we assume that the kernel function $K(x, y)$ and the quadrature weights $\Lambda_{nk}(x)$ in (2.4) are represented in the form

(2.5)
$$K(x, y) = H(x, y)S(x, y) \quad \text{and} \quad \Lambda_{nk}(x) = \lambda_{nk}^F(H(x, .))S(x, x_{nk}) ,$$

respectively, and consider the conditions (H1) – (H3) below. For this, we need the following notions.

Definition 4. *Let $I_0 = (-1, 1)$, $I_0 = (0, \infty)$, or $I_0 = (-\infty, \infty)$, and let v be a positive weight function on I_0 , where $v : I \longrightarrow [0, \infty)$ is assumed to be continuous and having the property that $p(x)v(x)$ is continuous in I for all polynomials $p(x)$. By $\widetilde{\mathbf{C}}_v = \widetilde{\mathbf{C}}_v(I_0)$ we denote the*

Banach space of all functions $f : I_0 \longrightarrow \mathbb{C}$, for which $vf : I_0 \longrightarrow \mathbb{C}$ can be extended to a continuous function on the whole interval I, where the norm on $\widetilde{\mathbf{C}}_v$ is given by $\|g\|_{\widetilde{\mathbf{C}}_v} = \|g\|_{v,\infty} := \max\{|v(x)g(x)| : x \in I\}$. Moreover, let $\mathbf{C}_v \subset \widetilde{\mathbf{C}}_v$ be the closure (w.r.t. the $\widetilde{\mathbf{C}}_v$-norm) of the set $\mathbf{P}$ of all algebraic polynomials.

Now, we formulate the above mentioned conditions.

(H1) The λ_{nk}^F's, $k = 1, \dots, k_n$, $n \in \mathbb{N}$, are linear and bounded functionals on a Banach space $\mathbf{X}_0$ continuously imbedded in $\mathbf{L}^1_{v^{-1}}(I)$, where $\mathbf{L}_{v^{-1}}(I) = \{f : v^{-1}f \in \mathbf{L}^1\}$ with $\|f\|_{\mathbf{L}^1_{v^{-1}}} = \|v^{-1}f\|_1 := \|v^{-1}f\|_{\mathbf{L}^1}$.

(H2) For all $x \in I$, $H(x,.) \in \mathbf{X}_0$ and $S(x,.) \in \mathbf{C}_v$, and for all $x_0 \in I$,

$$\lim_{x \to x_0} \|H(x,.) - H(x_0,.)\|_{\mathbf{X}_0} = 0 .$$

(H3) It holds $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{k_n} \lambda_{nk}^F(f)\, g(x_{nk}) = \int_I f(y) g(y)\, dy$ for all $f \in \mathbf{X}_0$ and all $g \in \mathbf{C}_v(I_0)$.

In case of $v(x) \equiv 1$ and $I = [-1,1]$, the following lemma is proved in [35, Section 3, Theorem 2].

Lemma 5. *Assume that $K(x,y)$ and $\Lambda_{nk}(x)$ in (2.4) are of the form (2.5), where the conditions* (H1) – (H3) *are fulfilled and where $S(x,y)v(y)$ is continuous on I^2. Then, conditions* (K1) – (K4) *are satisfied.*

Proof. Condition (K1) follows from

$$\int_I |K(x,y)|\, dy \le \|H(x,.)\|_{\mathbf{L}^1_{v^{-1}}} \|S(x,.)\|_{v,\infty} \le c\, \|H(x,.)\|_{\mathbf{X}_0} \|S(x,.)\|_{v,\infty}$$

and condition (H2). Moreover,

$$\begin{aligned} &\|K(x,.) - K(x_0,.)\|_{\mathbf{L}^1} \\ &\quad \le \|H(x,.) - H(x_0,.)\|_{\mathbf{L}^1_{v^{-1}}} \|S(x,.)v\|_\infty + \|H(x_0,.)\|_{\mathbf{L}^1_{v^{-1}}} \|S(x,.)v - S(x_0,.)v\|_\infty \\ &\quad \le \|H(x,.) - H(x_0,.)\|_{\mathbf{X}_0} \|S(x,.)v\|_\infty + \|H(x_0,.)\|_{\mathbf{X}_0} \|S(x,.)v - S(x_0,.)v\|_\infty \longrightarrow 0 \end{aligned}$$

if $x \to x_0 \in [-1,1]$ because of (H2) and the (uniform) continuity of $S(x,y)v(y)$ on I^2. Hence, (K2) is also satisfied. Using (2.5), (H2), and (H3), we get, for $f \in \mathbf{C}(I)$,

$$\sum_{k=1}^{k_n} \Lambda_{nk}(x) f(x_{nk}) = \sum_{k=1}^{k_n} \lambda_{nk}^F(H(x,.)) S(x, x_{nk}) f(x_{nk}) \longrightarrow \int_I H(x,y) S(x,y) f(y)\, dy ,$$

since together with $S(x,.) \in \mathbf{C}_v$ also $S(x,.)f$ belongs to $\mathbf{C}_v$. This shows the validity of (K3). It remains to consider (K4). For this, define $\mathcal{G}_n : \mathbf{X}_0 \longrightarrow \mathbf{C}_v^*$, $f \mapsto \mathcal{G}_n f$ with

$$(\mathcal{G}_n f)(g) = \sum_{k=1}^{k_n} \lambda_{nk}^F(f) g(x_{nk}) \quad \text{for all} \quad g \in \mathbf{C}_v \,.$$

Indeed, $\mathcal{G}_n f \in \mathbf{C}_v^*$, since $|(\mathcal{G}_n f)(g)| \le \sum_{k=1}^{k_n} \frac{|\lambda_{nk}^F(f)|}{v(x_{nk})} \|g\|_{v,\infty}$. Moreover, it is easily seen that

$$\|\mathcal{G}_n f\|_{\mathbf{C}_v^*} = \sum_{k=1}^{k_n} \frac{|\lambda_{nk}^F(f)|}{v(x_{nk})} \,.$$

If we fix $f \in \mathbf{X}_0$, then $\sup\{|(\mathcal{G}_n f)(g)| : n \in \mathbb{N}\} < \infty$ for every $g \in \mathbf{C}_v$, due to (H3). Consequently, in virtue of the principle of uniform boundedness,

$$\sup\left\{\|\mathcal{G}_n f\|_{\mathbf{C}_v^*} : n \in \mathbb{N}\right\} < \infty \quad \text{for every} \quad f \in \mathbf{X}_0 \,. \tag{2.6}$$

Taking into account $\lambda_{nk}^F \in \mathbf{X}_0^*$ and

$$\|\mathcal{G}_n f\|_{\mathbf{C}_v^*} = \sum_{k=1}^{k_n} \frac{|\lambda_{nk}^F(f)|}{v(x_{nk})} \le \sum_{k=1}^{k_n} \frac{\left\|\lambda_{nk}^F\right\|_{\mathbf{X}_0^*}}{v(x_{nk})} \|f\|_{\mathbf{X}_0} \,, \tag{2.7}$$

we see that $\mathcal{G}_n$ belongs to $\mathcal{L}(\mathbf{X}_0, \mathbf{C}_v^*)$. Again by the principle of uniform boundedness and by (2.6), we obtain $c_0 := \sup\left\{\|\mathcal{G}_n\|_{\mathbf{X}_0 \to \mathbf{C}_v^*} : n \in \mathbb{N}\right\} < \infty$. This implies, together with (2.7),

$$\sum_{k=1}^{k_n} \frac{|\lambda_{nk}^F(f)|}{v(x_{nk})} \le c_0 \|f\|_{\mathbf{X}_0} \quad \forall\, f \in \mathbf{X}_0 \,.$$

Hence,

$$\begin{aligned}
&\sum_{k=1}^{k_n} \left|\Lambda_{nk}(x) - \Lambda_{nk}(x_0)\right| \\
&\quad= \sum_{k=1}^{k_n} \Big| \left[\lambda_{nk}^F(H(x,.)) - \lambda_{nk}^F(H(x_0,.))\right] S(x,x_{nk}) \\
&\qquad\qquad\qquad + \lambda_{nk}^F(H(x_0,.)) \left[S(x,x_{nk}) - S(x_0,x_{nk})\right] \Big| \\
&\quad\le \sum_{k=1}^{k_n} \frac{\left|\lambda_{nk}^F(H(x,.) - H(x_0,.))\right|}{v(x_{nk})} \, \|S(x,.)v\|_\infty \\
&\qquad\qquad\qquad + \sum_{k=1}^{k_n} \frac{\left|\lambda_{nk}^F(H(x_0,.))\right|}{v(x_{nk})} \, \|S(x,.)v - S(x_0,.)v\|_\infty \\
&\quad\le c_0 \left[\|H(x,.) - H(x_0,.)\|_{\mathbf{X}_0} \|S(x,.)v\|_\infty + \|H(x_0,.)\|_{\mathbf{X}_0} \|S(x,.)v - S(x_0,v)\|_\infty\right] ,
\end{aligned}$$

and (K4) follows by (H2) and the continuity of $S(x,y)v(y)$ on I^2. $\square$

3. The classical Nyström method

Let u be a positive weight function and w , w_1 be weight functions on I_0 , where $u : I \longrightarrow [0,\infty)$ is assumed to be continuous. For example, all these three weight functions can be Jacobi weights (see Section 3.1) or weights of exponential type (see Section 3.2). Consider a Fredholm integral equation of the second kind

$$\widetilde{f}(x) - \int_I \widetilde{K}(x,y)w(y)\widetilde{f}(y)\,dy = \widetilde{g}(x)\,, \quad x \in I_0\,, \tag{3.1}$$

where $\widetilde{g} \in \widetilde{\mathbf{C}}_u$ and $\widetilde{K} : I^2 \longrightarrow \mathbb{C}$ are given functions and $\widetilde{f} \in \widetilde{\mathbf{C}}_u$ is looked for. Using a set of nodes $x_{nk} \in I_0$ satisfying

$$x_{n1} < x_{n2} < \ldots < x_{n,k_n} \tag{3.2}$$

and a quadrature rule

$$\int_I \widetilde{f}(x)w(x)\,dx \sim \sum_{k=1}^{k_n} \lambda_{nk}\widetilde{f}(x_{nk})\,, \tag{3.3}$$

we look for an approximate solution $\widetilde{f}_n(x)$ for equation (3.1) by solving

$$\widetilde{f}_n(x) - \sum_{k=1}^{k_n} \lambda_{nk}\widetilde{K}(x,x_{nk})\widetilde{f}_n(x_{nk}) = \widetilde{g}(x)\,. \tag{3.4}$$

If we define $f(x) := u(x)\widetilde{f}(x)$, $g(x) := u(x)\widetilde{g}(x)$,

$$K(x,y) = \frac{u(x)\widetilde{K}(x,y)w(y)}{u(y)}\,, \tag{3.5}$$

and

$$\Lambda_{nk}(x) = \frac{\lambda_{nk}u(x)\widetilde{K}(x,x_{nk})}{u(x_{nk})} =: \lambda_{nk}K_1(x,x_{nk})\,, \tag{3.6}$$

then (3.1) considered in $\widetilde{\mathbf{C}}_u(I_0)$ together with (3.4) is equivalent to (2.1) considered in $\mathbf{C}(I)$ together with (2.2), where $\mathcal{K}$ and $\mathcal{K}_n$ are given by (2.4).

Recall, that the function (cf. (3.4))

$$\widetilde{f}_n(x) = \sum_{k=1}^{k_n} \lambda_{nk}\widetilde{K}(x,x_{nk})\widetilde{f}_n(x_{nk}) + \widetilde{g}(x)$$

is called Nyström interpolant at the nodes x_{nk} . For its construction, one needs the values $\xi_{nk} = \widetilde{f}_n(x_{nk})$, which can be computed by considering (3.4) for $x = x_{nj}$, $j = 1, \ldots, k_n$ and solving the system of linear equations

$$\xi_{nj} - \sum_{k=1}^{k_n} \lambda_{nk}\widetilde{K}(x_{nj},x_{nk})\xi_{nk} = \widetilde{g}(x_{nj})\,, \quad j = 1, \ldots, k_n\,.$$

Note, that the convergence of the Nyström interpolant to the solution of the original integral equation is the main feature of the Nyström method. For that reason, the natural spaces, in which the Nyström method together with the integral equation should be considered, are spaces of continous functions. Moreover, the natural class of integral equations, to which the Nyström method together with the concept of collectively compact and strongly convergent operator sequences can be applied, is the class of second kind Fredholm integral equations, since collective compactness and strong convergence imply the compactness of the limit operator.

Nevertheless, there were developed modifications of the Nyström method applicable to integral equations with noncompact integral operators (see, for example, [9, 10, 22]).

We formulate the conditions

(A) $K_0(x,y) := u(x)\widetilde{K}(x,y)w_1(y)$ is continuous on I^2 ,

(B) $(w_1u)^{-1}\,w \in \mathbf{L}^1(I)$,

(C) there exists a positive weight function $u_1 : I_0 \longrightarrow [0,\infty)$ continuous on I , such that $K_1(x,.) = u(x)\widetilde{K}(x,.)u^{-1}(.) \in \mathbf{C}_{u_1}(I_0)$ for all $x \in I$,

(D) $u_1^{-1}w \in \mathbf{L}^1(I)$,
(E) for the quadrature rule (3.3), we have

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \lambda_{nk} f(x_{nk}) = \int_I f(x)w(x)\,dx$$

for all $f \in \mathbf{C}_{u_1}(I_0)$,
(F) the inequalities

$$\sum_{k=1}^{k_n} \frac{\lambda_{nk}}{u(x_{nk})w_1(x_{nk})} \le c \tag{3.7}$$

hold true for all $n \in \mathbb{N}$, where $c \ne c(n)$.

The following corollary is concerned with condition (E).

Corollary 6. *Let* (D) *be satisfied. If the quadrature rule* (3.3) *is exact for polynomials of degree less than* $\kappa(n)$ *, where* $\kappa(n)$ *tends to infinity if* $n \longrightarrow \infty$ *, and if*

$$\sum_{k=1}^{k_n} \frac{\lambda_{nk}}{u_1(x_{nk})} \le c \tag{3.8}$$

for all $n \in \mathbb{N}$ *, where* $c \ne c(n)$ *, then*

(a) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{k_n} \lambda_{nk} f(x_{nk}) = \int_I f(x)w(x)\,dx \quad \forall\, f \in \mathbf{C}_{u_1}(I_0)$,

(b) $\displaystyle\left|\int_I f(x)w(x)\,dx - \sum_{k=1}^{k_n} \lambda_{nk} f(x_{nk})\right| \le c\, E_{\kappa(n)-1}(f)_{u_1,\infty} \,, \quad c \ne c(n,f)$,

where $E_m(f)_{u_1,\infty} = \inf\{\|f-p\|_{u_1,\infty} : p \in \mathbf{P}_m\}$ *is the best weighted uniform approximation of the function* f *by polynomials of degree less or equal to* m *. Moreover, if* (E) *is satisfied then* (3.8) *and* (b) *hold.*

Proof. Define the linear functionals $\mathcal{F}_n : \mathbf{C}_{u_1}(I_0) \longrightarrow \mathbb{C}$ by

$$\mathcal{F}_n f = \sum_{k=1}^{k_n} \lambda_{nk} f(x_{nk}) \,.$$

Then, in virtue of (3.8),

$$|\mathcal{F}_n f| \le \sum_{k=1}^{k_n} \frac{\lambda_{nk}}{u_1(x_{nk})} \|f\|_{u_1,\infty} \le c\|f\|_{u_1,\infty} \quad \forall\, f \in \mathbf{C}_{u_1} \,,\ c \ne c(n,f) \,.$$

Hence, the linear functionals $\mathcal{F}_n : \mathbf{C}_{u_1}(I_0) \longrightarrow \mathbb{C}$ are uniformly bounded. Moreover, due to our assumptions,

$$\lim_{n\to\infty} \mathcal{F}_n f = \int_I f(x)w(x)\,dx \quad \forall\, f \in \mathbf{P}\,,$$

and the Banach-Steinhaus Theorem gives the assertion (a). For all $p \in \mathbf{P}_{\kappa(n)-1}$, we get

$$\begin{aligned}
&\left|\int_I f(x)w(x)\,dx - \sum_{k=1}^{k_n} \lambda_{nk} f(x_{nk})\right| \\
&\qquad \le \int_I |f(x)-p(x)|\,w(x)\,dx + \sum_{k=1}^{k_n} \lambda_{nk}\,|f(x_{nk})-p(x_{nk})| \\
&\qquad \le \left[\int_I \frac{w(x)\,dx}{u_1(x)} + \sum_{k=1}^{k_n} \frac{\lambda_{nk}}{u_1(x_{nk})}\right] \|f-p\|_{u_1,\infty}\,.
\end{aligned}$$

It remains to take into account (D) and (3.8), and also (b) is proved.

Finally, we make the following observation. The norm of the functionals $\mathcal{F}_n : \mathbf{C}_{u_1}(I_0) \longrightarrow \mathbb{C}$ is equal to $\displaystyle\sum_{k=1}^{k_n} \frac{\lambda_{nk}}{u_1(x_{nk})}$. Hence, due to the uniform boundedness principle, condition (3.8) is also necessary for assertion (a) to be fulfilled. □

Proposition 7. *If the conditions* (A) – (F) *are fulfilled, then the operators* $\mathcal{K}_n \in \mathcal{L}(\mathbf{C}(I))$, *defined in* (2.4) *and* (3.6), *form a collectively compact sequence of strongly convergent to* $\mathcal{K}$ *(cf.* (2.4) *and* (3.5)*) operators in* $\mathbf{C}(I)$.

Proof. We check if conditions (K1) – (K4) are fulfilled. Condition (K1) is a consequence of

$$\int_I |K(x,y)|\,dy \stackrel{(3.5)}{=} \int_I |K_1(x,y)|\,w(y)\,dy \stackrel{\text{(C),(D)}}{\le} \|K_1(x,.)\|_{u_1,\infty} \left\|(u_1)^{-1}w\right\|_{\mathbf{L}^1(I)}\,.$$

Analogously, (K2) follows from

$$\int_I |K(x,y)-K(x_0,y)|\,dy = \int_I |K_0(x,y)-K_0(x_0,y)|\,\frac{w(y)}{u(y)w_1(y)}\,dy$$

by applying the continuity of $K_0(x,y)$ and condition (B). In view of (3.6), condition (C), and condition (E),

$$\sum_{k=1}^{k_n} \Lambda_{nk}(x) f(x_{nk}) = \sum_{k=1}^{k_n} \lambda_{nk} K_1(x, x_{nk}) f(x_{nk}) \longrightarrow \int_I K_1(x,y) f(y) w(y)\, dy = \int_I K(x,y) f(y)\, dy$$

if $n \longrightarrow \infty$ for all $f \in \mathbf{C}(I)$ and all $x \in I$, i.e., $K(x,y)$ satisfies also (K3). Finally, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|K_0(x,y) - K_0(x_0,y)| < \varepsilon$ for all $(x,y) \in U_\delta(x_0) \times I$, where $U_\delta(x_0) = \{x \in I : d(x,x_0) < \delta\}$. Consequently, according to (3.7),

$$\sum_{k=1}^{k_n} |\Lambda_{nk}(x) - \Lambda_{nk}(x_0)| = \sum_{k=1}^{k_n} \lambda_{nk} |K_1(x,x_{nk}) - K_1(x_0,x_{nk})| = \sum_{k=1}^{k_n} \frac{\lambda_{nk}}{u(x_{nk}) w_1(x_{nk})} |K_0(x,x_{nk}) - K_0(x_0,x_{nk})| < c\,\varepsilon$$

for all $x \in U_\delta(x_0)$, which shows the validity of (K4). The application of Lemma 2 completes the proof. $\square$

Remark 8. *In case of $u^{-1}u_1 = w_1$, for the proof of Proposition 7, one can also use Lemma 5. Indeed, if we set $v = u_1$ and define $H(x,y) = w(y)$, $S(x,y) = K_1(x,y)$, $\mathbf{X}_0 = \mathrm{span}\{w\}$ with $\|.\|_{\mathbf{X}_0} = \|.\|_{\mathbf{L}^1_{v^{-1}}(I)}$, $\lambda^F_{nk}(\gamma w) = \gamma \lambda_{nk}$ for $\gamma \in \mathbb{C}$, then, we have $\mathbf{X}_0 \subset \mathbf{L}^1_{v^{-1}}(I)$ continuously (see (D) which now coincides with (B)), $K(x,y) = H(x,y)S(x,y)$ with the continuous function $S(x,y)v(y)$ (see (A)), and $\Lambda_{nk}(x) = \lambda^F_{nk}(w)S(x,x_{nk})$ (cf. (3.6)). Moreover, for all $f = \gamma w \in \mathbf{X}_0$ and all $g \in \mathbf{C}_v(I_0)$,*

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \lambda^F_{nk}(f) g(x_{nk}) = \lim_{n\to\infty} \gamma \sum_{k=1}^{k_n} \lambda_{nk} g(x_{nk}) = \int_I f(y) g(y)\, dy$$

in view of condition (E). Consequently, conditions (H1) – (H3) are fulfilled and Lemma 5 can be applied.

Corollary 9. *Assume (A) – (F). Consider the equations (3.1) and (3.4) with $\widetilde{g} \in \widetilde{\mathbf{C}}_u(I_0)$. Assume further, that the homogeneous equation (3.1) (i.e., $\widetilde{g} \equiv 0$) has in $\widetilde{\mathbf{C}}_u(I_0)$ only the trivial solution. Then, for all sufficiently large n, equation (3.4) possesses a unique solution $\widetilde{f}^*_n \in \widetilde{\mathbf{C}}_u(I_0)$ converging to $\widetilde{f}^*$, where $\widetilde{f}^* \in \widetilde{\mathbf{C}}_u$ is the unique solution of (3.1).*

If the assumptions of Corollary 6 are satisfied, then

$$\left\| \widetilde{f}^* - \widetilde{f}_n^* \right\|_{u,\infty} \le c \sup \left\{ E_{2n-1}\big(u(x)\widetilde{K}(x,.)\widetilde{f}^*\big)_{u_1,\infty} : x \in I \right\} , \tag{3.9}$$

where $c \neq c(n,g)$. *(Note that, due to condition* (C), $u(x)\widetilde{K}(x,.)\widetilde{f}^* \in \mathbf{C}_{u_1}(I_0)$ *for all* $x \in I$.*)*

Proof. In virtue of Proposition 7, we can apply Proposition 1 with $\mathbf{X} = \mathbf{C}(I)$ to the equations (2.1) and (2.2) with the above definitions (3.5) and (3.6). Estimate (2.3) gives

$$\|\widetilde{f}_n^* - \widetilde{f}^*\|_{u,\infty} = \|f_n^* - f^*\|_\infty \le c\, \|\mathcal{K}_n f^* - \mathcal{K} f^*\|_\infty ,$$

where $f^* \in \mathbf{C}(I)$ and $f_n^* \in \mathbf{C}(I)$ are the solutions of (2.1) and (2.2), respectively, and where

$$\begin{aligned} &\|\mathcal{K}_n f^* - \mathcal{K} f^*\|_\infty \\ &\quad = \sup \left\{ \left| \sum_{k=1}^{k_n} \Lambda_{nk}(x) f^*(x_{nk}) - \int_I K(x,y) f^*(y)\, dy \right| : x \in I \right\} \\ &\quad = \sup \left\{ \left| \sum_{k=1}^{k_n} \lambda_{nk} u(x) \widetilde{K}(x,x_{nk}) \widetilde{f}^*(x_{nk}) - \int_I u(x)\widetilde{K}(x,y)\widetilde{f}^*(y) w(y)\, dy \right| : x \in I \right\} . \end{aligned}$$

It remains to use $u(x)\widetilde{K}(x,.)\widetilde{f}^* \in \mathbf{C}_{u_1}(I_0)$ (cf. (C)) and Corollary 6,(b). □

3.1. **The case of Jacobi weights.** Let us apply the above described Nyström method in case of

$$\widetilde{f}(x) - \int_{-1}^1 \widetilde{K}(x,y) v^{\alpha,\beta}(y) \widetilde{f}(y)\, dy = \widetilde{g}(x) , \quad -1 < x < 1 , \tag{3.10}$$

where $\widetilde{g} \in \widetilde{\mathbf{C}}_u = \widetilde{\mathbf{C}}_u(-1,1)$ and $\widetilde{K} : (-1,1)^2 \longrightarrow \mathbb{C}$ are given continuous functions and where $v^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$, $\alpha,\beta > -1$, and $u(x) = v^{\gamma,\delta}(x)$, $\gamma,\delta \ge 0$, are Jacobi weights, and $\widetilde{\mathbf{C}}_u = \widetilde{\mathbf{C}}_{v^{\gamma,\delta}}$. We set $u_1(x) = v^{\gamma_1,\delta_1}(x)$, $w_1(x) = v^{\alpha_1,\beta_1}(x)$ and assume that

- (A1) $K_0 : [-1,1]^2 \longrightarrow \mathbb{C}$ is continuous, where $K_0(x,y) = v^{\gamma,\delta}(x)\widetilde{K}(x,y)v^{\alpha_1,\beta_1}(y)$,
- (B1) $\displaystyle\int_{-1}^1 \frac{v^{\alpha,\beta}(x)\, dx}{v^{\gamma,\delta}(x) v^{\alpha_1,\beta_1}(x)} < \infty$, i.e., $\gamma + \alpha_1 < \alpha + 1$ and $\delta + \beta_1 < \beta + 1$,
- (C1) $0 \le \gamma_1$, $0 \le \delta_1$, and $\gamma + \alpha_1 < \gamma_1 < \alpha + 1$, $\delta + \beta_1 < \delta_1 < \beta + 1$.

Setting $w(x) := v^{\alpha,\beta}(x)$, the conditions (A1) and (B1) are equivalent to (A) and (B) in the present situation, respectively. Condition (C1) leads immediately to (C) and (D), since in case $u(x) = v^{\gamma,\delta}(x)$ and $\gamma, \delta \geq 0$, the set $\mathbf{C}_u$ is equal to the set of all $f \in \widetilde{\mathbf{C}}_u$ satisfying

$$\lim_{x\to 1-0} u(x)f(x) = 0 \text{ if } \gamma > 0 \quad \text{and} \quad \lim_{x\to -1+0} u(x)f(x) = 0 \text{ if } \delta > 0\,.$$

As quadrature rule (3.3) we take the Gaussian rule w.r.t. the Jacobi weight $w(x) = v^{\alpha,\beta}(x)$, i.e., $k_n = n$, the $x_{nk} = x_{nk}^{\alpha,\beta}$'s are the zeros of the nth (normalized) Jacobi polynomial $p_n^{\alpha,\beta}(x)$ w.r.t. $w(x) = v^{\alpha,\beta}(x)$ and the $\lambda_{nk} = \lambda_{nk}^{\alpha,\beta}$'s are the respective Christoffel numbers. Then, for Corollary 6 we have $\kappa(n) = 2n-1$. Moreover, condition (C1) guarantees that (3.7) and (3.8) are also fulfilled, which is due to the following lemma.

Lemma 10 ([33], Theorem 9.25)**.** *For $v^{\alpha,\beta}(x)$ and $v^{\alpha_1,\beta_1}(x)$, assume that $\alpha+\alpha_1 > -1$ and $\beta+\beta_1 > -1$, and let $j \in \mathbb{N}$ be fixed. Then, for each polynomial $q(x)$ with $\deg q \leq jn$,*

$$\sum_{k=1}^{n} \lambda_{nk}^{\alpha,\beta} \left| q(x_{nk}^{\alpha,\beta}) \right| v^{\alpha_1,\beta_1}\left(x_{nk}^{\alpha,\beta}\right) \leq c \int_{-1}^{1} |q(x)| v^{\alpha,\beta}(x) v^{\alpha_1,\beta_1}(x)\, dx\,,$$

where $c \neq c(n,q)$.

Hence, all conditions (A) - (F) are in force and we can apply Corollary 9 together with the estimate (b) of Corollary 6 to equation (3.10) and the Nyström method

$$\widetilde{f}_n(x) - \sum_{k=1}^{n} \lambda_{nk}^{\alpha,\beta} \widetilde{K}(x, x_{nk}^{\alpha,\beta}) \widetilde{f}_n(x_{nk}^{\alpha,\beta}) = \widetilde{g}(x)\,, \quad -1 < x < 1\,, \tag{3.11}$$

to get the following proposition.

Proposition 11. *Assume that* (A1)*,* (B1)*, and* (C1) *are fulfilled and that equation* (3.10) *has only the trivial solution in $\widetilde{\mathbf{C}}_{v^{\gamma,\delta}}$ in case of $\widetilde{g}(x) \equiv 0$. Then, for $\widetilde{g} \in \widetilde{\mathbf{C}}_{v^{\gamma,\delta}}$ and all sufficiently large n, equation* (3.11) *has a unique solution $\widetilde{f}_n^* \in \widetilde{\mathbf{C}}_{v^{\gamma,\delta}}$ and*

$$\left\|\widetilde{f}^* - \widetilde{f}_n^*\right\|_{\gamma,\delta,\infty} \leq c \sup\left\{ E_{2n-1}\big(v^{\gamma,\delta}(x)\widetilde{K}(x,.)\widetilde{f}^*\big)_{v^{\gamma_1,\delta_1},\infty} : -1 \leq x \leq 1 \right\}\,,$$

where $\widetilde{f}^* \in \widetilde{\mathbf{C}}_{v^{\gamma,\delta}}$ is the unique solution of (3.10) *and $c \neq c(n,g)$. (Again we note that the assumptions of the proposition guarantee that $v^{\gamma,\delta}(x)\widetilde{K}(x,.)\widetilde{f}^* \in \mathbf{C}_{v^{\gamma_1,\delta_1}}$ for all $x \in [-1,1]$, (cf. Corollary 9)*

For checking (3.7) and (3.8), we used Lemma 10. The following Lemma will allow us to prove these assumptions also in other cases.

Lemma 12. *Let $w : I_0 \longrightarrow [0,\infty)$ and $v : I_0 \longrightarrow [0,\infty)$ be weight functions and $\lambda_{nk} > 0$, $x_{nk} \in I_0$, $k = 1,\ldots,n$, be given numbers satisfying the conditions $x_{n1} < x_{n2} < \ldots < x_{nn}$ and*

(a) $v^{-1}w \in \mathbf{L}^1(I)$,

(b) $\lambda_{nk} \sim_{n,k} \Delta x_{nk} w(x_{nk})$, $k = 1,\ldots,n$, *where* $\Delta x_{nk} = x_{nk} - x_{n,k-1}$ *and* $x_{n0} < x_{n1}$ *is appropriately chosen,*

(c) $\Delta x_{nk} \sim_{n,k} \Delta x_{n,k-1}$, $k = 2,\ldots,n$,

(d) *for each closed subinterval* $[a,b] \subset I_0$, $v^{-1}w : [a,b] \longrightarrow \mathbb{R}$ *is continuous and*

$$\lim_{n\to\infty} \max\{\Delta x_{nk} : x_{nk} \in [a,b]\} = 0\,, \tag{3.12}$$

(e) *there exists a subinterval* $[A,B] \subset I_0$ *such that* $v^{-1}w : \{x \in I_0 : x \le A\} \longrightarrow \mathbb{R}$ *and* $v^{-1}w : \{x \in I_0 : x \ge B\} \longrightarrow \mathbb{R}$ *are monotone.*

Then, there is a constant $c \ne c(n)$ such that

$$\sum_{k=1}^{n} \frac{\lambda_{nk}}{v(x_{nk})} \le c \int_I \frac{w(x)}{v(x)}\,dx\,. \tag{3.13}$$

Proof. By assumption (b) we have $\displaystyle\sum_{k=1}^{n} \frac{\lambda_{nk}}{v(x_{nk})} \sim_n \sum_{k=1}^{n} \frac{w(x_{nk})}{v(x_{nk})}\,\Delta x_{nk}$. Moreover,

$$\lim_{n\to\infty} \sup\left\{\left|\frac{w(x)}{v(x)} - \frac{w(x_{nk})}{v(x_{nk})}\right| : x \in [x_{n,k-1}, x_{nk}],\ x_{nk} \in [A,B]\right\} = 0\,,$$

due to assumption (d). Hence,

$$\frac{w(x_{nk})}{v(x_{nk})}\,\Delta x_{nk} \le c \int_{x_{n,k-1}}^{x_{nk}} \frac{w(x)}{v(x)}\,dx \quad \forall\, x_{nk} \in [A,B] \quad \text{with} \quad c \ne c(n,k)\,.$$

If $v^{-1}w : \{x \in I_0 : x \le A\} \longrightarrow \mathbb{R}$ is non-increasing, then

$$\frac{w(x_{nk})}{v(x_{nk})}\,\Delta x_{nk} \le \int_{x_{n,k-1}}^{x_{nk}} \frac{w(x)}{v(x)}\,dx \quad \forall\, x_{nk} < A,\ k \ge 1\,.$$

If $v^{-1}w : \{x \in I_0 : x \le A\} \longrightarrow \mathbb{R}$ is non-decreasing, then we use assumption (c) and get

$$\frac{w(x_{nk})}{v(x_{nk})}\,\Delta x_{nk} \sim_{n,k} \frac{w(x_{nk})}{v(x_{nk})}\,\Delta x_{n,k+1} \le \int_{x_{nk}}^{x_{n,k+1}} \frac{w(x)}{v(x)}\,dx \quad \forall\, x_{nk} < A,\ k \ge 1\,,$$

with (if necessary) an appropriately chosen $x_{n,n+1} > x_{nn}$. For $x_{nk} > B$ we can proceed analogously (noting that B can be chosen sufficiently large such that, for all $n \ge n_0$, $v^{-1}w$ is monotone on the interval $[x_{n,k_0-1}, x_{n,k_0})$ containing B). Summarizing we obtain (3.13). $\square$

It is obvious how we have to formulate Lemma 12 in case $\lambda_{nk} > 0$ and $x_{nk} \in I_0$ are given for $k = k_1(n), \ldots, k_2(n)$.

3.2. **The case of an exponential weight on** $(0, \infty)$**.** Consider the integral equation

$$\widetilde{f}(x) - \int_0^\infty \widetilde{K}(x,y) w(y) \widetilde{f}(y)\, dy = \widetilde{g}(x)\,, \quad 0 < x < \infty\,,, \tag{3.14}$$

where $\widetilde{g} \in \widetilde{\mathbf{C}}_u(0,\infty)$ and $\widetilde{K} : (0,\infty)^2 \longrightarrow \mathbb{C}$ are given functions and where $w(x) = w^{\alpha,\beta}(x) = e^{-x^{-\alpha} - x^\beta}$, $\alpha > 0$, $\beta > 1$, $u(x) = u^{a,\delta}(x) = (1+x)^\delta [w(x)]^a$, $a \geq 0$, $\delta \geq 0$. Here we use the Gaussian rule w.r.t. the weight $w(x) = w^{\alpha,\beta}(x)$ and study the Nyström method

$$\widetilde{f}_n(x) - \sum_{k=1}^n \lambda_{nk}^w \widetilde{K}(x, x_{nk}^w) \widetilde{f}_n(x_{nk}^w) = \widetilde{g}(x)\,, \quad 0 < x < \infty\,. \tag{3.15}$$

Let us check conditions (A) - (F), for which we choose

$$w_1(x) = u^{a_0,\delta_0}(x) := (1+x)^{\delta_0} [w(x)]^{a_0}\,, \quad \delta_0, a_0 \in \mathbb{R}\,,$$

and

$$u_1(x) = u^{a_1,\delta_1}(x) = (1+x)^{\delta_1} [w(x)]^{a_1}\,, \quad \delta_1 \geq 0\,,\ 0 < a_1 \leq 1\,,$$

and assume that

(A2) $K_0(x,y) := u(x) \widetilde{K}(x,y) w_1(y)$ is continuous on $[0,\infty]^2$,
(B2) $0 < a + a_0 < 1$, $\delta + \delta_0 \geq 0$ or $a + a_0 = 1$, $\delta + \delta_0 > 1$,
(C2) $0 < a_1 < 1$, $\delta_1 \geq 0$ or $a_1 = 1$, $\delta_1 > 1$,
(D2) $a_1 > a_0 + a$.

Note that, due to Lemma 12 (cf. [23, Prop. 3.8], for checking the conditions of Lemma 12 see also [19, 14, 29])

$$\sum_{k=1}^n \frac{\lambda_{nk}^w}{u_1(x_{nk}^w)} \leq c \quad \text{with} \quad c \neq c(n) \tag{3.16}$$

if $u_1^{-1} w \in \mathbf{L}^1(0,\infty)$, which is equivalent to assumption (C2). We also see that (B2) implies $(w_1 u)^{-1} w \in \mathbf{L}^1(0,\infty)$. Condition (A2) together with (D2) guarantess that $u(x)\widetilde{K}(x,.)u^{-1} \in \mathbf{C}_{u_1}(0,\infty)$ for all $x \in [0,\infty]$. Hence, we see that (A2) - (D2) together with Corollary 6,(a) imply (A) – (F), and we can apply Corollary 9 together with Corollary 6,(b) to (3.14) and (3.15) to get the following.

Proposition 13. *Let* $w(x) = e^{-x^{-\alpha} - x^\beta}$, $\alpha > 0$, $\beta > 1$, *and* $u(x) = (1+x)^\delta [w(x)]^a$, $a \geq 0$, $\delta \geq 0$. *Assume that* (A2), (B2), (C2), *and* (D2) *are fulfilled and that equation* (3.14) *has only the trivial solution* *in* $\widetilde{\mathbf{C}}_u(0,\infty)$ *in case of* $\widetilde{g}(x) \equiv 0$. *Then, for* $\widetilde{g} \in \widetilde{\mathbf{C}}_u(0,\infty)$ *and all sufficiently large* n, *equation* (3.11) *has a unique solution* $\widetilde{f}_n^* \in \widetilde{\mathbf{C}}_u(0,\infty)$ *and*

$$\left\|\widetilde{f}^* - \widetilde{f}_n^*\right\|_{u,\infty} \le c \sup\left\{E_{2n-1}\big(u(x)\widetilde{K}(x,.)\widetilde{f}^*\big)_{u_1,\infty} : 0 \le x \le \infty\right\},$$

where $\widetilde{f}^* \in \widetilde{\mathbf{C}}_u(0,\infty)$ *is the unique solution of* (3.14) *and* $c \neq c(n,g)$.

4. The Nyström method based on product integration formulas

Let again I_0 and I be equal to $(-1,1)$, $(0,\infty)$, or $(-\infty,\infty)$ and $[-1,1]$, $[0,\infty]$, or $[-\infty,\infty]$, respectively. Here we discuss the numerical solution of the Fredholm integral equation (3.1) by means of approximating the operator

$$\widetilde{\mathcal{K}} : \widetilde{\mathbf{C}}_u(I_0) \longrightarrow \widetilde{\mathbf{C}}_u(I_0)\,,, \quad \widetilde{f} \mapsto \int_I \widetilde{K}(.,y)w(y)\widetilde{f}(y)\,dy \tag{4.1}$$

by

$$\big(\widetilde{\mathcal{K}}_n\widetilde{f}\big)(x) = \int_I \frac{\widetilde{H}(x,y)}{u(y)}\left[\mathcal{L}_n\widetilde{S}(x,.)u\widetilde{f}\right](y)w(y)\,dy\,, \quad x \in I_0\,, \tag{4.2}$$

where $\widetilde{K}(x,y) = \widetilde{H}(x,y)\widetilde{S}(x,y)$ and $\mathcal{L}_n g$ is the algebraic polynomial of degree less than n with $(\mathcal{L}_n g)(x_{nk}) = g(x_{nk})$, $k = 1,\ldots,n$. Using the formula

$$(\mathcal{L}_n g)(x) = \sum_{k=1}^n g(x_{nk})\ell_{nk}(x) \quad \text{with} \quad \ell_{nk}(x) = \prod_{j=1}^n \frac{x - x_{nj}}{x_{nk} - x_{nj}}\,,$$

we conclude

$$\big(\widetilde{\mathcal{K}}_n\widetilde{f}\big)(x) = \sum_{k=1}^n \int_I \frac{\widetilde{H}(x,y)}{u(y)}\ell_{nk}(y)w(y)\,dy\,\widetilde{S}(x,x_{nk})u(x_{nk})\widetilde{f}(x_{nk})\,.$$

So, here we have $k_n = n$. Furthermore, this means that, for equation (2.1) considered in the space $\mathbf{C}(I)$, the operator $\mathcal{K} : \mathbf{C}(I) \longrightarrow \mathbf{C}(I)$ defined in (2.4) is approximated by $\mathcal{K}_n : \mathbf{C}(I) \longrightarrow \mathbf{C}(I)$ also given by (2.4), where $K(x,y)$ is defined in (3.5) and where (cf. (2.5))

$$\Lambda_{nk}(x) = \int_I H(x,y)\ell_{nk}(y)\,dy\,S(x,x_{nk}) = \lambda_{nk}^F(H(x,.))S(x,x_{nk}) \tag{4.3}$$

with $H(x,y) = \dfrac{u(x)\widetilde{H}(x,y)w(y)}{u(y)}$, $S(x,y) = \widetilde{S}(x,y)$, and

$$\lambda_{nk}^F(f) = \int_I f(y)\ell_{nk}(y)\,dy\,. \tag{4.4}$$

In order to check, under which conditions the assumption (H3) is satisfied, we should use

(4.5)
$$\left|\sum_{k=1}^n \lambda_{nk}^F(f)g(x_{nk}) - \int_I f(y)g(y)\,dy\right| = \left|\int_I f(y)\left[(\mathcal{L}_n g)(y) - g(y)\right]\,dy\right|$$
$$\leq \left(\int_I \left|\frac{f(y)}{u(y)}\right|^p dy\right)^{\frac{1}{p}} \|(\mathcal{L}_n g - g)u\|_{\mathbf{L}^q(I)}\,,$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and u is an appropriate weight function.

4.1. **The case of Jacobi weights.** Consider the case where $w(x) = v^{\alpha,\beta}(x)$, $\alpha,\beta > -1$, and $v(x) = v^{\gamma,\delta}(x)$, $\gamma,\delta \geq 0$.

Lemma 14. *Let $w = v^{\alpha,\beta}$, $\alpha,\beta > -1$, $p > 1$, $\gamma_0,\delta_0 \geq 0$, and $\gamma_0 > \frac{\alpha}{2} + \frac{1}{4} + \frac{1}{p} - 1$, $\delta_0 > \frac{\beta}{2} + \frac{1}{4} + \frac{1}{p} - 1$. Then, condition* (H3) *is fulfilled for $\ell_{nk}(x) = \ell_{nk}^w(x) = \prod_{j=1}^n \frac{x - x_{nj}^{\alpha,\beta}}{x_{nk}^{\alpha,\beta} - x_{nj}^{\alpha,\beta}}$ in* (4.4) *as well as $\mathbf{X}_0 = \mathbf{L}^p_{v^{-\gamma_0,-\delta_0}}$ and $\mathbf{C}_v = \mathbf{C}$, i.e. $v \equiv 1$.*

Proof. First, $\mathbf{X}_0 = \mathbf{L}^p_{v^{-\gamma_0,-\delta_0}}$ is continuously embedded in $\mathbf{L}^1$, since $\gamma_0,\delta_0 \geq 0$. Second, we can use the fact (cf. [31, Theorems 1 and 2]) that there is a constant $c > 0$ such that $\left\|(g - \mathcal{L}_n^w g)v^{\gamma_0,\delta_0}\right\|_q \leq c\,E_{n-1}(g)_\infty$ for all $g \in \mathbf{C}$ if and only if $\frac{v^{\gamma_0,\delta_0}}{\sqrt{w\varphi}} \in \mathbf{L}^q$ with $\varphi(x) = \sqrt{1-x^2}$, i.e.,

$$\gamma_0 - \frac{\alpha}{2} - \frac{1}{4} > -\frac{1}{q} \quad \text{and} \quad \delta_0 - \frac{\beta}{2} - \frac{1}{4} > -\frac{1}{q}\,.$$

Hence, (4.5) can be applied to all $f \in \mathbf{X}_0$, all $g \in \mathbf{C}$, and $u = v^{\gamma_0,\delta_0}$. □

Remark 15. *We remark that Lemma 14 improves the result mentioned in* [34, Section 4.5]*, where γ_0 and δ_0 are chosen as*

$$\max\left\{\frac{\alpha}{2} + \frac{1}{4}, 0\right\} \quad \text{and} \quad \max\left\{\frac{\beta}{2} + \frac{1}{4}, 0\right\}\,,$$

respectively.

As a consequence of Lemma 14 and of Lemma 5, we have to assume that $H(x,.)$ satisfies condition (H2) for $\mathbf{X}_0 = \mathbf{L}^p_{v^{-\gamma_0,-\delta_0}}$ with appropriate γ_0,δ_0 and p as in Lemma 14. The aim of the remaining part of this subsection is to weaken this conditon in a certain way.

By $\mathbf{L}\log^+\mathbf{L}(a,b)$ we denote the set of all measurable functions $f : (a,b) \longrightarrow \mathbb{C}$ for which the integral $\rho_+(f) := \displaystyle\int_a^b |f(x)| \left(1+\log^+|f(x)|\right) dx$ is finite. For $f \in \mathbf{L}^1(a,b)$, by $\mathcal{H}_a^b f$ we denote the Hilbert transform of f,

$$\left(\mathcal{H}_a^b f\right)(x) := \int_a^b \frac{f(y)\,dy}{y-x}\,, \quad a<x<b$$

(as Cauchy principal value integral). From [32, (1),(2)] we infer the following.

Lemma 16. *Let $-\infty < a < b < \infty$. If $f \in \mathbf{L}\log^+\mathbf{L}(a,b)$ and $g \in \mathbf{L}^\infty(a,b)$, then*

$$\left\|g\mathcal{H}_a^b f\right\|_1 + \left\|f\mathcal{H}_a^b g\right\|_1 \le c\|g\|_\infty \rho_+(f) \tag{4.6}$$

with $c \ne c(f,g)$ and

$$\int_a^b g(x)\left(\mathcal{H}_a^b f\right)(x)\,dx = -\int_a^b f(x)\left(\mathcal{H}_a^b g\right)(x)\,dx\,. \tag{4.7}$$

Let us use the abbreviations $w(x) = v^{\alpha,\beta}(x)$, $p_n(x) = p_n^{\alpha,\beta}(x)$, $x_{nk} = x_{nk}^{\alpha,\beta}$, and $\Delta x_{nk} = x_{nk} - x_{n,k-1}$, $k=1,\ldots,n$, $x_{n0} = -1$, $\mathbf{L}^p = \mathbf{L}^p(-1,1)$, and $\mathbf{L}\log^+\mathbf{L} = \mathbf{L}\log^+\mathbf{L}(-1,1)$, as well as $\mathcal{H} = \mathcal{H}_{-1}^1$. The relations

(R1) $|p_n(x)|\sqrt{w(x)\varphi(x)} \le c$ for $x \in A_n := \left[\dfrac{x_{n1}-1}{2}, \dfrac{x_{nn}+1}{2}\right]$, $c \ne c(n)$,

(R2) $\dfrac{1}{|p_n'(x_{nk})|} \sim_{n,k} \Delta x_{nk}\sqrt{w(x_{nk})\varphi(x_{nk})}$ (see [31, (14)]),

(R3) for a fixed summable function $v : [-1,1] \longrightarrow \mathbb{C}$ and a fixed $\ell \in \mathbb{N}$,

$$\sum_{k=1}^n \Delta x_{nk}\,|p(x_{nk})v(x_{nk})| \le c\int_{A_n} |p(x)v(x)|\,dx$$

for all polynomials $p \in \mathbf{P}_{\ell n} := \{P \in \mathbf{P} : \deg P \le \ell n\}$ and with $c \ne c(n,p)$

are well-known. Note that (R1) is a consequence of the estimate (see [8, Theorem 1.1])

$$\left|p_n^{\alpha,\beta}(x)\right|\left(\sqrt{1-x}+\frac{1}{n}\right)^{\alpha+\frac12}\left(\sqrt{1+x}+\frac{1}{n}\right)^{\beta+\frac12} \le c \ne c(n,x)\,, \tag{4.8}$$

$-1<x<1$, and the relation $\theta_{n,k-1} - \theta_{nk} \sim_{n,k} \frac{1}{n}$, $k=1,\ldots,n+1$, $n \in \mathbb{N}$, where $\theta_{nk} \in [0,\pi]$ and $x_{nk} = \cos\theta_{nk}$, $\theta_{n,n+1} = 0$ (cf. [30, (5)]).

Lemma 17. *Let* $w(x) = v^{\alpha,\beta}(x)$ *and* $v(x) = v^{\gamma,\delta}(x)$ *be Jacobi weights satisfying*

$$\frac{\alpha}{2} + \frac{1}{4} > \gamma \geq 0 \quad \textit{and} \quad \frac{\beta}{2} + \frac{1}{4} > \delta \geq 0 \,. \tag{4.9}$$

Then, there is a constant $c \neq c(n, f, g)$ *such that, for all functions* $f : (-1, 1) \longrightarrow \mathbb{C}$ *with* $fv \in \mathbf{L}^\infty$ *and all* g *with* $\dfrac{g}{\sqrt{w\varphi}} \in \mathbf{L}\log^+ \mathbf{L}$,

$$\|g\mathcal{L}_n^w f\|_1 \leq c\,\rho_+\left(\frac{g}{\sqrt{w\varphi}}\right)\|fv\|_\infty \,.$$

Proof. Write $\|g\mathcal{L}_n^w f\|_1 = J_1 + J_2 + J_3$, where

$$J_1 = \|g\mathcal{L}_n^w f\|_{\mathbf{L}^1(A_n)}\,, \quad J_2 = \|g\mathcal{L}_n^w f\|_{\mathbf{L}^1\left(-1,\frac{x_{n1}-1}{2}\right)}\,, \quad J_3 = \|g\mathcal{L}_n^w f\|_{\mathbf{L}^1\left(\frac{x_{nn}+1}{2},1\right)}\,.$$

Define

$$\widetilde{p}_n(y) := \begin{cases} p_n(y) & : \; y \in A_n\,, \\ 0 & : \; y \notin A_n\,, \end{cases} \quad \text{and} \quad \widetilde{g}_n(y) := \begin{cases} g(y) & : \; y \in A_n\,, \\ 0 & : \; y \notin A_n\,, \end{cases}$$

as well as $h_n(y) := \operatorname{sgn}\left[g(y)\left(\mathcal{L}_n^w f\right)(y)\right]$, and consider

$$J_1 = \int_{A_n} h_n(y)g(y)\left(\mathcal{L}_n^w f\right)(y)\,dy = \sum_{k=1}^n \frac{f(x_{nk})}{p_n'(x_{nk})}\int_{A_n}\frac{p_n(y)}{y - x_{nk}}\,g(y)h_n(y)\,dy$$

$$\overset{(R2)}{\leq} c\|fv\|_\infty \sum_{k=1}^n \Delta x_{nk}\frac{\sqrt{w(x_{nk})\varphi(x_{nk})}}{v(x_{nk})}\,|G_n(x_{nk})|\,,$$

where

$$G_n(x) = \int_{A_n}\frac{p_n(y)Q_n(y) - p_n(x)Q_n(x)}{y - x}\,\frac{g(y)h_n(y)}{Q_n(y)}\,dy$$

for some polynomial $Q_n \in \mathbf{P}_{\ell n}$ positive on A_n ($\ell \in \mathbb{N}$ fixed). Then, due to $G_n \in \mathbf{P}_{\ell n+n-1}$ and (R3),

$$J_1 \leq c\|fv\|_\infty \int_{A_n}|G_n(x)|\,\frac{\sqrt{w(x)\varphi(x)}}{v(x)}\,dx$$

$$\leq c\|fv\|_\infty\left[\int_{-1}^1 \frac{\sqrt{w(x)\varphi(x)}}{v(x)}\left(\mathcal{H}\widetilde{p}_n\widetilde{g}_n h_n\right)(x)\,k_n^1(x)\,dx\right.$$

$$\left. + \int_{-1}^1 \frac{\sqrt{w(x)\varphi(x)}}{v(x)}\,|\widetilde{p}_n(x)|\,Q_n(x)\left(\mathcal{H}\frac{gh_n}{Q_n}\right)(x)\,k_n^2(x)\,dx\right]$$

$$=: c\|fv\|_\infty\left[J_1' + J_1''\right],$$

where $k_n^1(x) = \operatorname{sgn}\left[\left(\mathcal{H}\widetilde{p}_n\widetilde{g}_n h_n\right)(x)\right]$ and $k_n^2(x) = \operatorname{sgn}\left[\left(\mathcal{H}\dfrac{gh_n}{Q_n}\right)(x)\right]$. With the help of (4.7), (R1), and (4.6), we get

$$\begin{aligned} J_1' &= -\int_{-1}^1 \widetilde{p}_n(x)\widetilde{g}_n(x)h_n(x)\left(\mathcal{H}\frac{\sqrt{w\varphi}}{v}\,k_n^1\right)(x)\,dx \\ &\le c\left\|\frac{g}{\sqrt{w\varphi}}\,\mathcal{H}\frac{\sqrt{w\varphi}}{v}\,k_n^1\right\|_1 \le c\left\|\frac{\sqrt{w\varphi}}{v}\right\|_\infty \rho_+\left(\frac{g}{\sqrt{w\varphi}}\right) \end{aligned}$$

and, by choosing $Q_n(x) \sim_{n,x} \sqrt{w(x)\varphi(x)}$ for $x \in A_n$ (see [28, Lemma 2.1]),

$$\begin{aligned} J_1'' &\le c\int_{-1}^1 \frac{\sqrt{w(x)\varphi(x)}}{v(x)}\left(\mathcal{H}\frac{gh_n}{Q_n}\right)(x)\,k_n^2(x)\,dx \\ &= -c\int_{-1}^1 \frac{g(x)h_n(x)}{Q_n(x)}\left(\mathcal{H}\frac{\sqrt{w\varphi}}{v}k_n^2\right)(x)\,dx \\ &\le c\left\|\frac{g}{\sqrt{w\varphi}}\,\mathcal{H}\frac{\sqrt{w\varphi}}{v}\,k_n^2\right\|_1 \le c\left\|\frac{\sqrt{w\varphi}}{v}\right\|_\infty \rho_+\left(\frac{g}{\sqrt{w\varphi}}\right). \end{aligned}$$

Now, let us estimate J_3, the term J_2 can be handled analogously. We get

$$\begin{aligned} J_3 &= \int_{\frac{x_{nn}+1}{2}}^1 h_n(y)g(y)\left(\mathcal{L}_n^w f\right)(y)\,dy = \sum_{k=1}^n \frac{f(x_{nk})}{p_n'(x_{nk})}\int_{\frac{x_{nn}+1}{2}}^1 \frac{p_n(y)}{y-x_{nk}}\,g(y)h_n(y)\,dy \\ &\overset{\text{(R2)}}{\le} c\|fv\|_\infty \sum_{k=1}^n \Delta x_{nk}\frac{\sqrt{w(x_{nk})\varphi(x_{nk})}}{v(x_{nk})}\int_{\frac{x_{nn}+1}{2}}^1 \frac{|p_n(y)g(y)|}{y-x_{nk}}\,dy \end{aligned}$$

Note that, due to the assumptions on w and u, $\alpha + \frac{1}{2} > 0$. Hence, in view of (4.8),

$$\frac{|p_n(y)|\sqrt{w(y)\varphi(y)}}{y-x_{nk}} \le \frac{c}{1-x_{nk}}, \quad y \in \left[\frac{x_{nn}+1}{2}, 1\right],$$

since, for $y \in \left[\dfrac{x_{nn}+1}{2}, 1\right]$, we have $y - x_{nk} \ge \dfrac{1-x_{nk}}{2}$. We conclude

$$\begin{aligned} J_3 &\le c\|fv\|_\infty \sum_{k=1}^n \frac{\sqrt{w(x_{nk})\varphi(x_{nk})}}{v(x_{nk})(1-x_{nk})}\int_{\frac{x_{nn}}{2}+1}^1 \frac{|g(y)|\,dy}{\sqrt{w(y)\varphi(y)}} \\ &\overset{\text{(R3)}}{\le} c\|fv\|_\infty \int_{-1}^1 \frac{\sqrt{w(x)\varphi(x)}}{(1-x)v(x)}\,dx\left\|\frac{g}{\sqrt{w\varphi}}\right\|_1 \le c\|fv\|_\infty\,\rho_+\left(\frac{g}{\sqrt{w\varphi}}\right), \end{aligned}$$

since $\dfrac{\alpha}{2}+\dfrac{1}{4}-\gamma-1>-1$. □

Lemma 18. *Let $v: I \longrightarrow [0,\infty)$ be a weight function as in Definition 4 and $R: I^2 \longrightarrow \mathbb{C}$ be a function such that $R_x \in \mathbf{C}_v$ for all $x \in I$, where $R_x(y)=R(x,y)$, and such that $R(x,y)v(y)$ is continuous on I^2. Then, for every $n \in \mathbb{N}$, there is a function $P_n(x,y)$ such that $P_{n,x}(y) = P_n(x,y)$ belongs to $\mathbf{P}_n$ for every $x \in I$ and $\lim_{n\to\infty} \sup\left\{|R(x,y)-P_n(x,y)|v(y) : (x,y)\in I^2\right\} = 0$.*

Proof. Let $\varepsilon_n > 0$ and, for every $x \in I$, choose $P_{n,x} \in \mathbf{P}_n$ such that

$$\|(R_x - P_{n,x})v\|_\infty < E_n(R_x)_{v,\infty} + \varepsilon_n .$$

It remains to prove that $\lim_{n\to\infty} \sup\{E_n(R_x)_{v,\infty} : x \in I\} = 0$. If this is not the case, then there are an $\varepsilon > 0$ and $n_1 < n_2 < \dots$ such that $E_{n_k}(R_{x_k})_{v,\infty} \ge 2\varepsilon$ for certain $x_k \in I$. Due to the compactness of I, we can assume that $x_k \longrightarrow x^*$ for $k \longrightarrow \infty$. In virtue of the continuity of $R(x,y)v(y)$, we can conclude that $\|(R_{x_k} - R_{x^*})v\|_\infty < \varepsilon$ for all $k \ge k_0$. Since $\|(R_{x_k} - p)v\|_\infty \ge 2\varepsilon$ for all $p \in \mathbf{P}_{n_k}$ and $k \in \mathbb{N}$, we obtain, for $p \in \mathbf{P}_{nk}$ and $k \ge k_0$,

$$2\varepsilon \le \|(R_{x_k} - p)v\|_\infty \le \|(R_{x_k} - R_{x^*})v\|_\infty + \|(R_{x^*} - p)v\|_\infty < \varepsilon + \|(R_{x^*} - p)v\|_\infty$$

and, consequently, $\|(R_{x*} - p)v\|_\infty > \varepsilon$ for all $p \in \mathbf{P}_{n_k}$ and $k \in \mathbb{N}$, in contradiction to $R_{x*} \in \mathbf{C}_v$. □

Let us come back to the integral operator $\mathcal{K} : \mathbf{C}[-1,1] \longrightarrow \mathbf{C}[-1,1]$,

$$(\mathcal{K}f)(x) = \int_{-1}^{1} K(x,y) f(y)\, dy \tag{4.10}$$

and its product integration approximation $\mathcal{K}_n : \mathbf{C}[-1,1] \longrightarrow \mathbf{C}[-1,1]$,

$$(\mathcal{K}_n f)(x) = \sum_{k=1}^{n} \Lambda_{nk}(x) f(x_{nk}^w) = \int_{-1}^{1} H(x,y) \left(\mathcal{L}_n^w S_x f\right)(x)\, dx , \tag{4.11}$$

where $S_x(y) = S(x,y)$,

(4.12)

$$K(x,y) = H(x,y)S(x,y), \quad \text{and} \quad \Lambda_{nk}(x) = S(x,x_{nk}^w) \int_{-1}^{1} H(x,y)\ell_{nk}^w(y)\, dy .$$

Proposition 19. *Consider (4.10) and (4.11) together with (4.12) in the Banach space $\mathbf{C}[-1,1]$. If the Jacobi weights $w = w^{\alpha,\beta}$ and $v = v^{\gamma,\delta}$ satisfy the conditions of Lemma 17 and if*

(a) $\dfrac{H_x}{\sqrt{w\varphi}} \in \mathbf{L}\log^+\mathbf{L}$ *for all $x \in [-1,1]$, where $H_x(y) = H(x,y)$,*

(b) $\sup\left\{\rho_+\left(\dfrac{H_x}{\sqrt{w\varphi}}\right) : -1 \le x \le 1\right\} < \infty$,

(c) $\lim\limits_{x\to x_0} \rho_+\left(\dfrac{H_x - H_{x_0}}{\sqrt{w\varphi}}\right) = 0$ *for all* $x_0 \in [-1,1]$,

(d) *the map* $[-1,1]^2 \longrightarrow \mathbb{C}$, $(x,y) \mapsto S(x,y)v(y)$ *is continuous with* $S_x \in \mathbf{C}_v$ *for all* $x \in [-1,1]$,

then the operators $\mathcal{K}_n$ *form a collectively compact sequence, which converges strongly to the operator* $\mathcal{K}$.

Proof. At first we show that $\mathcal{K}_n$ converges strongly to $\mathcal{K}$. Indeed, for $f \in \mathbf{C}[-1,1]$, a function $P(x,y)$, which is a polynomial in y of degree less than n , and $P_x(y) = P(x,y)$, we have

$$
\begin{aligned}
&|(\mathcal{K}_n f)(x) - (\mathcal{K} f)(x)| \\
&\qquad \le \int_{-1}^{1} \left| H(x,y)\left[\mathcal{L}_n^w(S_x f - P_x)\right](x)\right| dx + \int_{-1}^{1} \left| H(x,y)\left[S(x,y)f(y) - P(x,y)\right]\right| dx \\
&\qquad \le c\left[\rho_+\left(\frac{H_x}{\sqrt{w\varphi}}\right) + \left\|H_x v^{-1}\right\|_1\right] \|(S_x f - P_x)v\|_\infty ,
\end{aligned}
$$

where we took into account Lemma 17 and that condition (a) together with (4.9) implies $H_x v^{-1} \in \mathbf{L}^1(-1,1)$. Moreover, $\sup\left\{\left\|H_x v^{-1}\right\|_1 : -1 \le x \le 1\right\} < \infty$ due to condition (b). Thus,

$$
\|\mathcal{K}_n f - \mathcal{K} f\|_\infty \le c \sup_{-1\le x\le 1} \|(S_x f - P_x)v\|_\infty ,
$$

which proves the desired strong convergence by referring to Lemma 18. A consequence of this is that the set $\{\|\mathcal{K}_n f\|_\infty : f \in \mathbf{C}[-1,1],\ \|f\|_\infty \le 1\}$ is bounded. Furthermore, for $\|f\|_\infty \le 1$,

$$
\begin{aligned}
&|(\mathcal{K}_n f)(x) - (\mathcal{K}_n f)(x_0)| \\
&\quad \le \int_{-1}^{1} \left| H(x,y)\left[\mathcal{L}_n^w(S_x - S_{x_0})f\right](y)\right| dy \\
&\qquad\qquad + \int_{-1}^{1} \left| \left[H(x,y) - H(x_0,y)\right]\left(\mathcal{L}_n^w S_{x_0} f\right)(y)\right| dy \\
&\quad \overset{\text{Lemma 17}}{\le} c\left[\rho_+\left(\frac{H_x}{\sqrt{w\varphi}}\right)\|(S_x - S_{x_0})v\|_\infty + \rho_+\left(\frac{H_x - H_{x_0}}{\sqrt{w\varphi}}\right)\|S_{x_0} v\|_\infty\right].
\end{aligned}
$$

Hence, due to (b), (c), and (d), the set $\{\mathcal{K}_n f : f \in \mathbf{C}[-1,1],\ \|f\|_\infty \le 1\}$ is equicontinuous in each point $x_0 \in [-1,1]$, and so equicontinuous on $[-1,1]$. $\square$

4.2. **The case of an exponential weight on $(0,\infty)$.** Here, in case $w(x)=w_{\alpha,\beta}(x)=x^\alpha e^{-x^\beta}$, $0<x<\infty$, $\alpha\ge 0$, $\beta>\frac{1}{2}$, we are going to prove results analogous to Lemma 17 and Proposition 19. Note that quadrature rules with such weights were introduced and investigated in [27]. Moreover, we mention that in [20] there are considered numerical methods and presented numerical results for Fredholm integral equations of second kind, basing on interpolation processes w.r.t. the nodes $\{x_{nk}^w\}$.

We again set $p_n(x)=p_n^w(x)$ and $\{x_{nk}\}=\{x_{nk}^w\}$ and, additionally, $x_{n,n+1}=a_n$, where $a_n=a_n(\sqrt{w})\sim_n n^{\frac{1}{\beta}}$ is the Mhaskar-Rahmanov-Saff number associated with the weight $\sqrt{w(x)}$. Let us fix $\theta\in(0,1)$, set $n_\theta=\min k\in 1,\dots,n : x_{nk}\ge\theta a_n$, and define, for a function $f:(0,\infty)\to\mathbb{C}$,

$$\mathcal{L}_n^* f=\sum_{k=1}^{n_\theta} f(x_{nk})\ell_{nk}^*\,,\quad \ell_{nk}^*(x)=\frac{p_n^w(x)(a_n-x)}{p_n'(x_{nk})(x-x_{nk})(a_n-x_{nk})}\,. \tag{4.13}$$

Then, we have $(\mathcal{L}_n^* f)(x_{nk})=f(x_{nk})$ for $k=1,\dots,n_\theta$ and $(\mathcal{L}_n^* f)(x_{nk})=0$ for $k=n_\theta+1,\dots,n+1$, as well as, for $\Delta x_{nk}=x_{nk}-x_{n,k-1}$, $k=1,\dots,n$, $x_{n0}=0$,

- (R4) $\sup\left\{\,|p_n(x)|\sqrt{w(x)\sqrt{|a_n-x|x}} : 0<x<\infty\right\}\le c<\infty$ with $\ne c(n)$ (see [19, 14]),
- (R5) $\dfrac{1}{|p_n'(x_{nk})|}\sim_{n,k}\Delta x_{nk}\sqrt{w(x_{nk})\sqrt{(a_n-x_{nk})x_{nk}}}$, $k=1,\dots,n$ (see [19, 14]),
- (R6) for fixed $\ell\in\mathbb{N}$, there is a constant $c\ne c(n,p)$ such that (see [18])

$$\sum_{k=1}^{n_\theta}\Delta x_{nk}\,|p(x_{nk})|\le c\int_0^{\theta a_n}|p(x)|\,dx\quad\text{for all}\quad p\in\mathbf{P}_{\ell n}\,.$$

Remark 20. *The constant on the right-hand side of (4.6) does not depend on the interval $[a,b]$, i.e., we have, for $-\infty<a<b<\infty$,*

$$\left\|g\mathcal{H}_a^b f\right\|_1+\left\|f\mathcal{H}_a^b g\right\|_1\le c\|g\|_\infty\rho_+(f) \tag{4.14}$$

for all $g\in\mathbf{L}^\infty(a,b)$ and $f\in\mathbf{L}\log^+\mathbf{L}(a,b)$, where $c\ne c(f,g,a,b)$.

Indeed, if c_1 is the constant in (4.6) in case $a=0$ and $b=1$, then, by setting $x=\chi(t)=(b-a)t+a$ and $y=\chi(s)$,

$$\begin{aligned}
&\int_a^b \left| g(x)\int_a^b \frac{f(y)\,dy}{y-x}\right| dx + \int_a^b \left| f(x)\int_a^b \frac{g(y)\,dy}{y-x}\right| dx \\
&\quad = (b-a)\left[\int_0^1 \left| g(\chi(t))\int_0^1 \frac{f(\chi(s))\,ds}{s-t}\right| dt + \int_0^1 \left| f(\chi(t))\int_0^1 \frac{g(\chi(s))\,ds}{s-t}\right| dt\right] \\
&\quad \le c_1\|g\|_{\infty,[a,b]}\int_0^1 |f(\chi(t))|\left(1+\log^+|f(\chi(t))|\right) dt = c_1\|g\|_{\infty,[a,b]}\rho_{+,[a,b]}(f)\,.
\end{aligned}$$

Lemma 21. *Let $\psi(x)=\sqrt{x}$, $x\ge 0$ and $v(x)=(1+x)^\delta\sqrt{w(x)}$, $\delta\ge\frac14$. Then, there is a constant $c\ne c(n,f,g)$ such that, for all functions $f:(0,\infty)\longrightarrow\mathbb{C}$ with $fv\in\mathbf{L}^\infty(0,\infty)$ and all g with $\frac{g}{\sqrt{w\psi}}\in\mathbf{L}\log^+\mathbf{L}(0,\infty)$,*

$$\|g\mathcal{L}_n^*f\|_{\mathbf{L}^1(0,\infty)} \le c\,\rho_+\left(\frac{g}{\sqrt{w\psi}}\right)\|fv\|_\infty\,.$$

Proof. Write $\|g\mathcal{L}_n^*f\|_{\mathbf{L}^1(0,\infty)} = \|g\mathcal{L}_n^*f\|_{\mathbf{L}^1(0,2a_n)} + \|g\mathcal{L}_n^*f\|_{\mathbf{L}^1(2a_n,\infty)} =: J_1+J_2$. Using (R5) we get, with $h_n(y)=\operatorname{sgn}\left[g(y)\left(\mathcal{L}_n^*f\right)(y)\right]$,

$$\begin{aligned}
J_1 &\le c\|fv\|_\infty\sum_{k=1}^{n_\theta}\Delta x_{nk}\frac{\sqrt{w(x_{nk})\psi(x_{nk})}}{v(x_{nk})(a_n-x_{nk})^{\frac34}}\left|\int_0^{2a_n}\frac{p_n(y)(a_n-y)g(y)h_n(y)}{y-x_{nk}}\,dy\right| \\
&= c\|fv\|_\infty\sum_{k=1}^{n_\theta}\Delta x_{nk}\frac{(x_{nk})^{\frac14}}{(1+x_{nk})^\delta(a_n-x_{nk})^{\frac34}}\left|\int_0^{2a_n}\frac{p_n(y)(a_n-y)g(y)h_n(y)}{y-x_{nk}}\,dy\right| \\
&\le c\frac{\|fv\|_\infty}{(a_n)^{\frac34}}\sum_{k=1}^{n_\theta}\Delta x_{nk}\left|G_n(x_{nk})\right|\,,
\end{aligned}$$

where

$$G_n(t)=\int_0^{2a_n}\frac{p_n(y)(a_n-y)Q_n(y)-p_n(t)(a_n-t)Q_n(t)}{y-t}\,\frac{g(y)h_n(y)}{Q_n(y)}\,dy$$

and $Q_n\in\mathbf{P}_{\ell n}$ a polynomial positive on $(0,a_n)$ ($\ell\in\mathbb{N}$ fixed). Since $G_n\in\mathbf{P}_{(\ell+1)n}$, with the help of (R6) we can estimate

$$\begin{aligned}
J_1 \le \frac{c\|fv\|_\infty}{(a_n)^{\frac34}}&\left[\int_0^{2a_n}\left|\left(\mathcal{H}_0^{2a_n}p_n(a_n-\cdot)gh_n\right)(x)\right| dx\right. \\
&\left.+\int_0^{2a_n}\left|p_n(x)(a_n-x)Q_n(x)\left(\mathcal{H}_0^{2a_n}\frac{gh_n}{Q_n}\right)(x)\right| dx\right] =: J_1'+J_1''\,.
\end{aligned}$$

Defining $k_n^1(x) = \operatorname{sgn}\left[\left(\mathcal{H}_0^{2a_n} p_n(a_m - \cdot) g h_n\right)(x)\right]$ and using (4.7) and (R4), we obtain

$$J_1' \le \frac{c\|fv\|_\infty}{(a_n)^{\frac{3}{4}}} \int_0^{2a_n} p_n(x)(a_n - x) g(x) h_n(x) \left(\mathcal{H}_0^{2a_n} k_n^1\right)(x)\, dx$$

$$\le c\|fv\|_\infty \int_0^{2a_n} \frac{|g(x)|}{\sqrt{w(x)\psi(x)}} \left|\left(\mathcal{H}_0^{2a_n} k_n^1\right)(x)\right| dx \overset{(4.14)}{\le} c\|fv\|_\infty \rho_+\left(\frac{g}{\sqrt{w\psi}}\right).$$

In order to estimate J_1'', we choose $Q_n \in \mathbf{P}_{\ell n}$ such that $Q_n(x) \sim_{n,x} \sqrt{w(x)\psi(x)}$ for $x \in (0, 2a_n)$ (see [26]). Then, due to (R4) and (4.7),

$$J_1'' \le c\|fv\|_\infty k_n^2(x) \left(\mathcal{H}_0^{2a_n} \frac{g h_n}{Q_n}\right)(x)\, dx$$

$$\le c\|fv\|_\infty \int_0^{2a_n} \left|\frac{g(x)}{\sqrt{w(x)\psi(x)}} \left(\mathcal{H} k_n^2\right)(x)\right| dx \overset{(4.14)}{\le} c\|fv\|_\infty \rho_+\left(\frac{g}{\sqrt{w\psi}}\right),$$

where $k_n^2(x) = \operatorname{sgn}\left[\left(\mathcal{H}_0^{2a_n} \frac{g h_n}{Q_n}\right)(x)\right]$,. Finally, let us consider J_2. Again taking into account (R5), we get

$$J_2 \le c\|fv\|_\infty \sum_{k=1}^{n_\theta} \Delta x_{nk} \frac{\sqrt{w(x_{nk})\psi(x_{nk})}}{v(x_{nk})(a_n - x_{nk})^{\frac{3}{4}}} \left|\int_{2a_n}^\infty \frac{p_n(y)(a_n - y) g(y) h_n(y)}{y - x_{nk}}\, dy\right|$$

$$= c\|fv\|_\infty \sum_{k=1}^{n_\theta} \Delta x_{nk} \frac{(x_{nk})^{\frac{1}{4}}}{(1 + x_{nk})^\delta (a_n - x_{nk})^{\frac{3}{4}}} \left|\int_{2a_n}^\infty \frac{p_n(y)(a_n - y) g(y) h_n(y)}{y - x_{nk}}\, dy\right|$$

$$\le \frac{c\|fv\|_\infty}{(a_n)^{\frac{3}{4}}} \sum_{k=1}^{n_\theta} \Delta x_{nk} \int_{2a_n}^\infty \frac{|p_n(y)| \sqrt{w(y)\sqrt{y(y - a_n)}}}{(a_n)^{\frac{1}{4}}} \left(\frac{y - a_n}{y - x_{nk}}\right)^{\frac{3}{4}} \frac{|g(y)|}{\sqrt{w(y)\psi(y)}}\, dy,$$

where we also used that $y - x_{nk} \ge 2a_n - a_n = a_n$. Hence, in virtue of $\left(\frac{y - a_n}{y - x_{nk}}\right)^{\frac{3}{4}} \le 1$ for $y > 2a_n$, $\sum_{k=1}^{n_\theta} \Delta x_{nk} \le a_n$, and (R1),

$$J_2 \le c\|fv\|_\infty \int_{2a_n}^\infty \frac{|g(y)|}{\sqrt{w(y)\psi(y)}}\, dy \le c\|fv\|_\infty \rho_+\left(\frac{g}{\sqrt{w\psi}}\right).$$

$\square$

Let us apply Lemma 21 to the integral operator $\mathcal{K} : \mathbf{C}[0,\infty] \longrightarrow \mathbf{C}[0,\infty]$,

$$(\mathcal{K}f)(x) = \int_0^\infty K(x,y) f(y)\, dy \tag{4.15}$$

and its product integration approximation $\mathcal{K}_n : \mathbf{C}[0,\infty] \longrightarrow \mathbf{C}[0,\infty]$,

$$(\mathcal{K}_n f)(x) = \sum_{k=1}^{n_\theta} \Lambda_{nk}^*(x) f(x_{nk}^w) = \int_0^\infty H(x,y) \left(\mathcal{L}_n^* S_x f\right)(x)\, dx\,, \tag{4.16}$$

where $w(x) = w_{\alpha,\beta}(x) = x^\alpha e^{-x^\beta}$, $\alpha > -1$, $\beta > \frac{1}{2}$, where $\mathcal{L}_n^*$ is defined in (4.13), and where $S_x(y) = S(x,y)$,

$$K(x,y) = H(x,y)S(x,y)\,, \quad \Lambda_{nk}^*(x) = S(x, x_{nk}^w) \int_0^\infty H(x,y) \ell_{nk}^*(y)\, dy\,. \tag{4.17}$$

Proposition 22. *Consider (4.15) and (4.16) together with (4.17) in the Banach space $\mathbf{C}[0,\infty]$. If $v(x) = (1+x)^\delta \sqrt{w(x)}$ with $\delta \geq \frac{1}{4}$ and if*

(a) $\dfrac{H_x}{\sqrt{w\psi}} \in \mathbf{L}\log^+\mathbf{L}(0,\infty)$ *for all* $x \in [0,\infty]$, *where* $H_x(y) = H(x,y)$,

(b) $\sup\left\{\rho_+\left(\dfrac{H_x}{\sqrt{w\varphi}}\right) : 0 \leq x \leq \infty\right\} < \infty$,

(c) $\displaystyle\lim_{d(x,x_0)\to 0} \rho_+\left(\frac{H_x - H_{x_0}}{\sqrt{w\psi}}\right) = 0$ *for all* $x_0 \in [0,\infty]$,

(d) *the map* $[0,\infty]^2 \longrightarrow \mathbb{C}$, $(x,y) \mapsto S(x,y)v(y)$ *is continuous with* $S_x \in \mathbf{C}_v$ *for all* $x \in [-1,1]$,

then the operators $\mathcal{K}_n$ form a collectively compact sequence, which converges strongly to the operator $\mathcal{K}$.

Proof. We proceed in an analogous way as in the proof of Proposition 19. For $f \in \mathbf{C}[0,\infty]$ and a function $P(x,y) = P_x(y)$, which is a polynomial in y of degree less than n, we have

$$\begin{aligned} &\left|\left(\mathcal{K}_n f\right)(x) - \left(\mathcal{K} f\right)(x)\right| \\ &\quad \leq \int_{-1}^1 \left|H(x,y)\left[\mathcal{L}_n^*(S_x f - P_x)\right](x)\right|\, dx + \int_0^\infty \left|H(x,y) \sum_{k=n_\theta+1}^{n+1} P_x(x_{nk}^w)\ell_{nk}^*(y)\right| dy \\ &\qquad + \int_{-1}^1 \left|H(x,y)\left[S(x,y)f(y) - P(x,y)\right]\right|\, dx =: J_1 + J_1 + J_3\,, \end{aligned}$$

By Lemma 21,

$$J_1 \leq c\,\rho_+\left(\frac{H_x}{\sqrt{w\varphi}}\right) \left\|(S_x f - P_x)v\right\|_\infty\,.$$

Condition (a) together with $\delta \geq \frac{1}{4}$ implies $H_x v^{-1} \in \mathbf{L}^1(-1,1)$, and hence

$$J_3 \leq \left\|H_x v^{-1}\right\|_1 \left\|(S_x f - P_x)v\right\|_\infty\,.$$

Consequently, since we have also $\sup\{\|H_x v^{-1}\|_1 : -1 \le x \le 1\} < \infty$ by condition (b), we get

$$J_1 + J_3 \le c \sup_{-1\le x\le 1} \|(S_x f - P_x)v\|_\infty \,. \tag{4.18}$$

To estimate J_2, we recall that (see [29, (2.3)])

$$\|P_n u\|_{\mathbf{L}^\infty(x_{n_\theta},\infty)} \le c\, e^{-\widetilde{c}\, n} \|P_n u\|_\infty \quad \text{for} \quad P_n \in \mathbf{P}_{m(n)}$$

($m(n) < n$, $\lim_{n\to\infty} m(n) = \infty$) for some positive constants $c \ne c(n, P_n)$ and $\widetilde{c} \ne \widetilde{c}(n, P_n)$ and (cf. [24, pp. 362,373])

$$\sum_{k=n_\theta+1}^{n+1} \frac{v(x)\ell^*_{nk}(x)}{v(x^w_{nk})} \le c\, n^\sigma$$

for some $\sigma > 0$ and $c \ne c(n, x)$. Thus,

$$J_2 \le c\, n^\sigma \left\|H_x v^{-1}\right\|_1 \|P_x v\|_{\mathbf{L}^\infty(x_{n\theta},\infty)} \le c n^\sigma e^{-\widetilde{c}\, n} \left\|H_x v^{-1}\right\|_1 \|P_x v\|_\infty$$

$P_x \in \mathbf{P}_{m(n)}$ can be chosen in such a way that $\sup\{\|P_x v\|_\infty : x \in [0,\infty]\} < \infty$ (in view of Lemma 18). Hence, together with (4.18) we conclude the strong convergence of $\mathcal{K}_n$ to $\mathcal{K}$. Consequently, the set

$$\{\|\mathcal{K}_n f\|_\infty : f \in \mathbf{C}[0,\infty],\ \|f\|_\infty \le 1\}$$

is bounded. Furthermore, for $\|f\|_\infty \le 1$,

$$\begin{aligned} &|(\mathcal{K}_n f)(x) - (\mathcal{K}_n f)(x_0)| \\ &\le \int_{-1}^{1} |H(x,y)\,[\mathcal{L}^*_n (S_x - S_{x_0}) f]\,(y)|\, dy \\ &\qquad\qquad + \int_{-1}^{1} |[H(x,y) - H(x_0,y)]\,(\mathcal{L}^*_n S_{x_0} f)\,(y)|\, dy \\ &\overset{\text{Lemma 21}}{\le} c\left[\rho_+\left(\frac{H_x}{\sqrt{w\varphi}}\right)\|(S_x - S_{x_0})v\|_\infty + \rho_+\left(\frac{H_x - H_{x_0}}{\sqrt{w\varphi}}\right)\|S_{x_0} v\|_\infty\right]. \end{aligned}$$

Hence, due to (b), (c), and (d), the set $\{\mathcal{K}_n f : f \in \mathbf{C}[-1,1],\ \|f\|_\infty \le 1\}$ is equicontinuous in each point $x_0 \in [0,\infty]$, and so equicontinuous on $[0,\infty]$. □

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