Sharp Weyl Estimates for Tensor Products of Pseudodifferential Operators

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SHARP WEYL ESTIMATES FOR TENSOR PRODUCTS OF PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. We study the asymptotic behavior of the counting function of tensor products of operators, in the cases where the factors are either pseudodifferential operators on closed manifolds, or pseudodifferential operators of Shubin type on $\mathbb{R}^n$, respectively. We obtain, in particular, the sharpness of the remainder term in the corresponding Weyl formulae, which we prove by means of the analysis of some explicit examples.

CONTENTS

INTRODUCTION

Let P be a positive self-adjoint operator of order $m > 0$ with domain $H^m(M) \hookrightarrow L^2(M)$, M a Riemannian, n-dimensional smooth closed manifold. Assume that the resolvent of P is compact, so that the spectrum is discrete and given by a sequence of eigenvalues with finite multiplicities. Let $\{\lambda_j\}_{j \in \mathbb{N}} = \sigma(P)$ be the set of the eigenvalues of P, repeated according to their multiplicity. The counting function $N_P(\tau)$ is defined as

$$(1) \qquad N_P(\tau) = \sum_{\lambda_j \in \sigma(P) \cap [0, \tau)} 1 = \sum_{\lambda_j < \tau} 1.$$

The Weyl law, see, e.g., [Hör68, Hör07], describes the asymptotic expansion of the counting function $N_P(\tau)$, as τ goes to infinity. It is well known that that the leading term of the asymptotic expansion of (1) depends on the dimension of the manifold, on the order of the operator and on its principal symbol, see, e.g., [Hör07]. Similar formulae can be obtained in many other different settings, see [SV97] and [ANPS09] for a detailed analysis and several developments. To mention a few specific situations, see [Shu87, HR81] for the case of the Shubin calculus on $\mathbb{R}^n$, [BN03] for the anisotropic Shubin calculus, [BC11, CM13, Nic03] for the SG-operators on $\mathbb{R}^n$ and the manifolds with ends, [GL02] for operators on conic manifolds, [Mor08] for operators

on cusp manifolds, [DD13] for operators on asymptotic hyperbolic manifolds, [Bat12, BGRP13] for bisingular operators.

In this paper we study the counting function of the tensor product of r pseudodifferential operators. We consider the cases of Hörmander operators on closed manifolds and of the Shubin calculus on $\mathbb{R}^n$. In the case $r = 2$, for classical Hörmander operators on closed manifolds, the operators we consider are a subclass of the so-called *bisingular operators*, studied by L. Rodino in [Rod75] (see also [NR06]) in connection with the multiplicative property of the Atiyah-Singer index [AS68]. An asymptotic expansion of the counting function of bisingular operators was obtained by the first author in [Bat12]. The basic tool was the spectral ζ-function, in the spirit of Guillemin's so-called *soft proof* of the Weyl law [Gui85]. This method allows to determine the leading term of the asymptotic expansion in the non-symmetric case (corresponding to a simple first pole of the spectral ζ-function). In the symmetric case the spectral ζ-function has a first pole of order 2. Using a theorem due to Aramaki [Ara88], it has been possible to determine the leading term, which has a behavior of type $\tau^p \log \tau$, as well as the second term, which has a behavior of type τ^p, p being the first pole of the spectral ζ-function. However, it was not possible, through the aforementioned method, to give a good estimate of the remainder term. We notice that the asymptotic behavior of the counting function in the bisingular case has some similarities with the Weyl law in the setting of SG-classical operators on manifolds with ends [BC11, CM13].

A version of bisingular operators, based on Shubin pseudodifferential calculus on $\mathbb{R}^n$, was introduced in [BGRP13]. The counting function was studied also in this setting, obtaining results analogous to those which hold for the "standard" bisingular calculus.

In this paper we consider the same class of operators studied in [GPRVar], namely, tensor products of r pseudodifferential operators, that is

$$A = A_1 \otimes \ldots \otimes A_r.$$

In the sequel we will assume either that each A_j is a classical Hörmander pseudodifferential operator on a n_j-dimensional closed manifolds M_j, that is $A_j \in L_{\mathrm{cl}}^{m_j}(M_j)$, $j = 1, \ldots, r$, or that each A_j belongs to a classical global Shubin class on $\mathbb{R}^{n_j}$, that is, $A_j \in G_{\mathrm{cl}}^{m_j}(\mathbb{R}^{n_j})$, $j = 1, \ldots, r$. We also assume that A is positive, self-adjoint and Fredholm. It is straightforward to check that the Fredholm property of A implies that A_j is invertible for any $j = 1, \ldots, r$. We illustrate here our results in the case $r = 2$, see Section 3 below for the statements which hold for an arbitrary number of factors.

Denoting by $\sigma(A_1) = \{\lambda_j\}_{j \in \mathbb{N}}$ and $\sigma(A_2) = \{\mu_k\}_{k \in \mathbb{N}}$ the spectra of A_1 and A_2, with eigenvalues repeated according with their multiplicities, we easily obtain that the spectrum of A is given by

$$\sigma(A) = \{\lambda_j \cdot \mu_k\}_{(j,k) \in \mathbb{N}^2}.$$

Therefore

$$(2) \qquad N_A(\tau) = \sum_{\rho \in \sigma(A) \cap [0,\tau)} 1 = \sum_{\lambda_j \cdot \mu_k < \tau} 1.$$

Assume that $A = A_1 \otimes A_2$ is positive, self-adjoint and Fredholm, with $A_1 \in L_{\mathrm{cl}}^{m_1}(M_1)$, $A_2 \in L_{\mathrm{cl}}^{m_2}(M_2)$, $m_1, m_2 > 0$, $\dim M_1 = n_1$, $\dim M_2 = n_2$, and $\frac{n_1}{m_1} > \frac{n_2}{m_2}$. Our first main result, proved in Theorem 2.3, states that, under such assumptions,

(3)
$$
N_A(\tau) = \begin{cases}
\dfrac{C_1}{n_1}\zeta\left(A_2, \dfrac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_1-1}{m_1}}\right) & \text{if } \dfrac{n_2}{m_2} < \dfrac{n_1-1}{m_1}, \\[2.5ex]
\dfrac{C_1}{n_1}\zeta\left(A_2, \dfrac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_1-1}{m_1}}\log\tau\right) & \text{if } \dfrac{n_2}{m_2} = \dfrac{n_1-1}{m_1}, \\[2.5ex]
\dfrac{C_1}{n_1}\zeta\left(A_2, \dfrac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_2}{m_2}}\right) & \text{if } \dfrac{n_2}{m_2} > \dfrac{n_1-1}{m_1}
\end{cases}
$$
, ,

for $\tau \to +\infty$. In (3), ζ denotes the spectral ζ-function and

$$
C_1 = \frac{1}{(2\pi)^{n_1}}\int_{M_1}\int_{\mathbb{S}^{n_1-1}}\frac{d\theta_1 dx_1}{[a_{m_1}(x_1,\theta_1)]^{\frac{n_1}{m_1}}}.
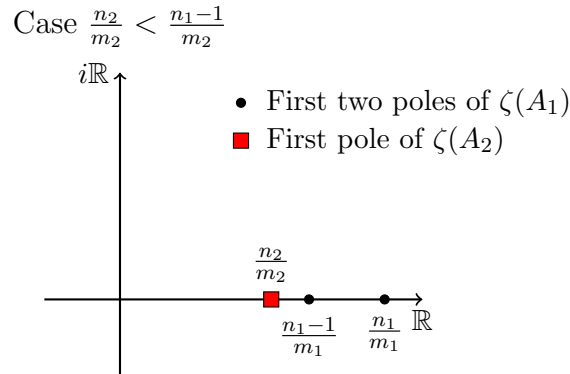$$

A similar statement holds for the tensor product of two Shubin operators with positive order. Moreover, using spherical harmonics, we show that the estimate (3) is sharp.

In [GPRVar], Gramtchev, Pilipović, Rodino and Vindas considered the same class of operators, finding a slightly weaker estimate for the remainder term of the Weyl formula. Explicitely, they prove that, under the assumptions stated above,

$$
N_A(\tau) = \frac{C_1}{n_1}\zeta\left(A_2, \frac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}(\tau^\delta)
$$

where $\max\left\{\frac{n_1-1}{m_1}, \frac{n_2}{m_2}\right\} < \delta < \frac{n_1}{m_1}$.

The asymptotic expansion in (3) is related with the position of the first poles of the spectral ζ-function associated with A_1 and A_2, as sketched in the following pictures.

Case $\frac{n_2}{m_2} < \frac{n_1-1}{m_2}$

Case $\frac{n_2}{m_2} = \frac{n_1-1}{m_2}$

$i\mathbb{R}$

● First two poles of $\zeta(A_1)$

■ First pole of $\zeta(A_2)$

$\frac{n_2}{m_2}$

$\frac{n_1-1}{m_1}$ $\frac{n_1}{m_1}$ $\mathbb{R}$

Case $\frac{n_2}{m_2} > \frac{n_1-1}{m_2}$

$i\mathbb{R}$

● First two poles of $\zeta(A_1)$

■ First pole of $\zeta(A_2)$

$\frac{n_2}{m_2}$

$\frac{n_1-1}{m_1}$ $\frac{n_1}{m_1}$ $\mathbb{R}$
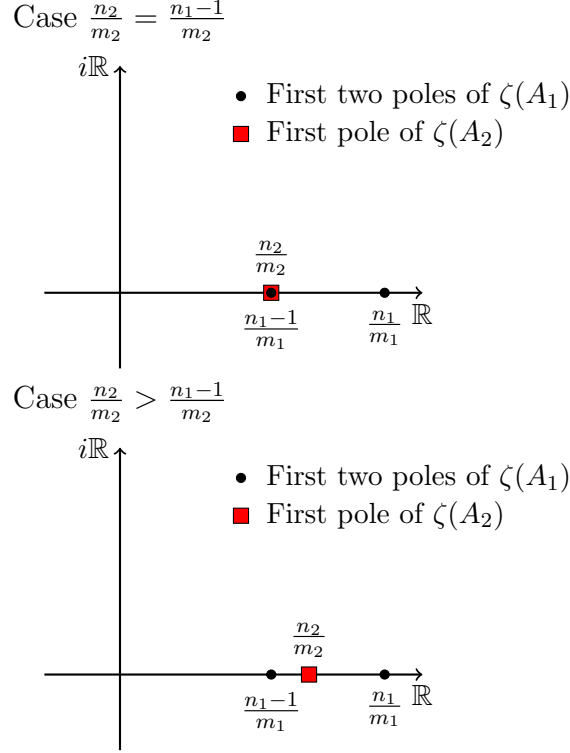
The key point in the proof of our results is the following equivalence, explained in (13):

$$N_A\left(\tau\right) = \sum_{\lambda_j \cdot \mu_k < \tau} 1 = \sum_{\mu_k < \tau} N_{A_2}\left(\frac{\tau}{\mu_k}\right).$$

The argument is then a careful application of the well known sharp Weyl law. A main aspect is the possibility to estimate the reminder term, in the Weyl law of A_2 evaluated in $\frac{\tau}{\mu_k}$, uniformly with respect to μ_k.

The paper is organized as follows. In Section 1, we shortly recall the Weyl laws in the case of the Hörmander calculus on closed manifolds and of the Shubin calculus on $\mathbb{R}^n$. We also study the asymptotic behavior of the sum

$$\sum_{\mu_k < \tau} \frac{1}{\mu_k^c}$$

for different ranges of $c \in \mathbb{R}$, where $\{\mu_k\}_{k \in \mathbb{N}}$ is the spectrum of an operator in the calculus we consider. In Section 2, we prove our main results in the case of tensor products of two factors. In Section 3, we extend the results to the case of tensor products of $r > 2$ factors. In Section 4, we show that our estimates of the remainder term of the Weyl law are sharp, focusing again on the case of tensor products of two factors. Finally, we collect in the Appendix some remarks concerning the connection of this analysis with lattice problems, in particular with the Dirichlet divisor problem in the classic setting and in the anisotropic case.

1. PRELIMINARY RESULTS

We recall well known results on the sharp Weyl law in the case of operators on closed manifolds and of operators of Shubin type on $\mathbb{R}^n$, see, e.g., Hormander [Hör68], Hellfer, Robert [HR81], see also [He84].

Theorem 1.1 (Sharp Weyl law). Let A be a positive self-adjoint elliptic classical pseudodifferential operator in $L_{\mathrm{cl}}^m(M)$, with M a closed manifold of dimension n, and let $\sigma(A) = \{\lambda_j\}_{j \in \mathbb{N}}$ be its spectrum. Then,

$$(4) \qquad N_A(\lambda) = \sum_{\lambda_j < \lambda} 1 = \frac{C_A}{n} \lambda^{\frac{n}{m}} + R_A(\lambda),$$

where

$$C_A = \frac{1}{(2\pi)^n} \int_M \int_{\mathbb{S}^{n-1}} \frac{d\theta dx}{[a_m(x,\theta)]^{\frac{n}{m}}},$$

with a_m the principal homogeneous symbol of A, and

$$\limsup_{\lambda \to +\infty} \frac{\left| N_A(\lambda) - \frac{C_A}{n} \lambda^{\frac{n}{m}} \right|}{\lambda^{\frac{n-1}{m}}}$$

$$(5) \qquad = \limsup_{\lambda \to +\infty} \frac{|R_A(\lambda)|}{\lambda^{\frac{n-1}{m}}} < +\infty.$$

Analogously, let $P \in G_{\mathrm{cl}}^m(\mathbb{R}^n)$ be a positive self-adjoint elliptic classical pseudodifferential operator of Shubin type on $\mathbb{R}^n$ with $m > 0$, and let $\sigma(P) = \{\mu_k\}_{k \in \mathbb{N}}$ be its spectrum. Then,

$$(6) \qquad N_P(\lambda) = \sum_{\mu_k < \lambda} 1 = \frac{K_P}{2n} \lambda^{\frac{2n}{m}} + R_P(\lambda),$$

where

$$K_P = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{S}^{2n-1}} \frac{d\theta}{[p_m(\theta)]^{\frac{2n}{m}}},$$

with p_m the principal homogeneous symbol of P, and

$$\limsup_{\lambda \to +\infty} \frac{\left| N_P(\lambda) - \frac{K_P}{2n} \lambda^{\frac{2n}{m}} \right|}{\lambda^{\frac{2n-1}{m}}} = \limsup_{\lambda \to +\infty} \frac{|R_P(\lambda)|}{\lambda^{\frac{2n-1}{m}}} < +\infty.$$

The next Propositions 1.2 and 1.3 will be crucial in our proof of the Weyl law with sharp remainder for tensor products. They follow as consequence of well known properties of the spectra of positive self-adjoint operators. We examine in detail only the case of Hörmander pseudodifferential operators

on closed manifold, since the argument for the case of Shubin operators is similar.

Proposition 1.2. Let M be a closed manifold of dimension n, and $A \in L^m_{\mathrm{cl}}(M)$, $m > 0$, be elliptic, positive and self-adjoint, with spectrum $\sigma(A) = \{\mu_k\}_{k \in \mathbb{N}}$. Define

$$(7) \qquad F_A(\tau, c) = \sum_{\mu_k < \tau} \frac{1}{\mu_k^c} = \begin{cases} F_1(\tau) & \text{if } c > \dfrac{n}{m}, \\[2mm] F_2(\tau) & \text{if } c = \dfrac{n}{m}, \\[2mm] F_3(\tau) & \text{if } c < \dfrac{n}{m}. \end{cases}$$

Then,

$$\limsup_{\tau \to +\infty} \frac{\zeta(A, c) - F_1(\tau)}{\tau^{\frac{n}{m} - c}} = \kappa_1, \ \limsup_{\tau \to +\infty} \frac{F_2(\tau)}{\log \tau} = \kappa_2, \ \limsup_{\tau \to +\infty} \frac{F_3(\tau)}{\tau^{\frac{n}{m} - c}} = \kappa_3,$$

for suitable positive constants κ_1, κ_2, κ_3. That is, for $\tau \to +\infty$,

$$\zeta(A, c) - F_1(\tau) = \mathcal{O}\left(\tau^{\frac{n}{m} - c}\right), \ F_2(\tau) = \mathcal{O}\left(\log \tau\right), \ F_3(\tau) = \mathcal{O}\left(\tau^{\frac{n}{m} - c}\right).$$

Proof. If $c > \frac{n}{m}$ it is immediate that the series $\sum_{k=0}^{\infty} \frac{1}{\mu_k^c}$ is convergent, in view of the holomorphic properties of the spectral ζ-function associated with A. To prove the asymptotic properties of $\zeta(A, c) - F_1(\tau)$, we switch to $B = A^{1/m}$, so that the order of B is one and $\sigma(B) = \mu_k^{1/m}$. We have

$$\zeta(A, c) - F_1(\tau) = \sum_{\mu_k \geq \tau} \frac{1}{\mu_k^c} = \sum_{\mu_k^{1/m} \geq \tau^{1/m}} \frac{1}{\left(\mu_k^{1/m}\right)^{cm}}$$

$$(8) \qquad\qquad = \int_{\tau^{1/m}}^{+\infty} \frac{1}{\mu^{cm}} dN_B(\mu).$$

Since B is of order one, it is well known that

$$(9) \qquad N_B(\lambda + 1) - N_B(\lambda) \leq \sharp \{\sigma(B) \cap [\lambda, \lambda + 1]\} = \mathcal{O}(\lambda^{n-1}), \lambda \to +\infty$$

(see, e.g., [GS94, § 12]). Using (9) and the properties of Stieltjes integral, we obtain, for $\tau \to +\infty$,

$$\zeta(A, c) - F_1(\tau) = \int_{\tau^{1/m}}^{+\infty} \frac{1}{\mu^{cm}} dN_B(\mu)$$

$$\leq \sum_{j = [\tau^{1/m}] - 1}^{\infty} \sup_{\mu \in [j, j+1]} \left(\frac{1}{\mu^{cm}}\right) (N_B(j+1) - N_B(j))$$

$$\leq \kappa \sum_{j = [\tau^{1/m}] - 1}^{\infty} \frac{1}{j^{cm-n+1}}$$

$$\leq \kappa \int_{[\tau^{1/m}] - 1}^{+\infty} \frac{1}{(t-1)^{cm-n+1}} dt$$

$$= \kappa \frac{1}{cm - n} [\tau^{1/m} - 2]^{n - mc} \in \mathcal{O}\left(\tau^{\frac{n}{m} - c}\right).$$

where $[a]$ denotes the minimum integer such that $[a] \geq a$.

To prove the results for F_2 and F_3 we can assume, without loss of generality, that $\mu_0 = \tilde{\mu}_0 = 1$. Using again the properties of the Stieltjes integral, we write

$$F_A(\tau, c) = \int_1^{\tau^{1/m}} \frac{1}{\mu^{cm}} dN_B(\mu) \leq \sum_{j=1}^{[\tau^{1/m}]} \sup_{\mu \in [j, j+1]} \left(\frac{1}{\mu^{cm}} \right) (N_B(j+1) - N_B(j)).$$

Let us initially suppose that $c > 0$, so that $\frac{1}{x^c}$ is a decreasing function on $[1, +\infty)$. In view of (9), we have

$$\int_1^{\tau^{1/m}} \frac{1}{\mu^{cm}} dN_B(\mu) \leq \sum_{j=1}^{[\tau^{1/m}]} \frac{1}{j^{cm}} \mathcal{O}(j^{n-1}) \leq \tilde{\kappa} \sum_{j=1}^{[\tau^{1/m}]} \frac{1}{j^{cm-n+1}}$$

$$(10) \qquad \leq \tilde{\kappa} \left(\int_1^{[\tau^{1/m}]} t^{n-cm-1} dt + 1 \right).$$

By integration, we find

$$F_A(\tau, c) = \int_1^{\tau^{1/m}} \frac{1}{\mu^{cm}} dN_B(\mu) \leq \begin{cases} \dfrac{\tilde{\kappa}_1}{n - cm} \tau^{\frac{n}{m} - c} & \text{if } \; 0 < c < \dfrac{n}{m}, \\[2ex] \tilde{\kappa}_2 m \log \tau & \text{if } \; c = \dfrac{n}{m}, \end{cases}$$

as claimed. Finally, if $c \leq 0$, then $\frac{1}{\mu^c}$ is a non-decreasing function and also in this case, similarly to (10), we obtain

$$F_A(\tau, c) \leq \kappa \int_1^{[\tau^{1/m}]} (x+1)^{n-cm-1} \, dx \leq \frac{\tilde{\kappa}_3}{n - cm} \tau^{\frac{n}{m} - c}.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proposition 1.3. Let $P \in G_{\mathrm{cl}}^m(\mathbb{R}^n)$ be an elliptic, positive and self-adjoint Shubin operator of order $m > 0$, with spectrum given by $\sigma(P) = \{\lambda_j\}_{j \in \mathbb{N}}$. Define

$$(11) \qquad F_P(\tau, c) = \sum_{\lambda_j < \tau} \frac{1}{\lambda_j^c} = \begin{cases} F_1(\tau) & \text{if } \; c > \dfrac{2n}{m}, \\[2ex] F_2(\tau) & \text{if } \; c = \dfrac{2n}{m}, \\[2ex] F_3(\tau) & \text{if } \; c < \dfrac{2n}{m}. \end{cases}$$

Then,

$$\limsup_{\tau \to +\infty} \frac{\zeta(P, c) - F_1(\tau)}{\tau^{\frac{2n}{m} - c}} = \kappa_1, \; \limsup_{\tau \to +\infty} \frac{F_2(\tau)}{\log \tau} = \kappa_2, \; \limsup_{\tau \to +\infty} \frac{F_3(\tau)}{\tau^{\frac{2n}{m} - c}} = \kappa_3,$$

for suitable positive constants κ_1, κ_2, κ_3. That is, for $\tau \to +\infty$,

$$\zeta(P, c) - F_1(\tau) = \mathcal{O}\left(\tau^{\frac{2n}{m} - c} \right), \; F_2(\tau) = \mathcal{O}\left(\log \tau \right), \; F_3(\tau) = \mathcal{O}\left(\tau^{\frac{2n}{m} - c} \right).$$

2. Spectral asymptotics for the tensor product of two operators

We start considering the case of the tensor product of 2 operators. Let M_1, M_2 be two compact manifolds of dimension n_1, n_2, respectively. Let $A = A_1 \otimes A_2$, $A_j \in L_{\mathrm{cl}}^{m_j}(M_j)$, $m_j > 0$, $j = 1, 2$. Assume that the spectra of A_1 and A_2 are sequences of eigenvalues, and set

$$\sigma(A_1) = \{\lambda_j\}_{j \in \mathbb{N}}, \qquad \sigma(A_2) = \{\mu_k\}_{k \in \mathbb{N}},$$

so that

$$\sigma(A) = \{\lambda_j \cdot \mu_k : \lambda_j \in \sigma(A_1), \mu_k \in \sigma(A_2)\}.$$

For simplicity, we start with the case $m_1 = 1$ and $n_1 > \frac{n_2}{m_2}$. Let c be an arbitrary positive constant and B an operator with spectrum $\sigma(B) = \{\mu_k\}_{k \in \mathbb{N}}$, then $\sigma(c\,B) = \{c \cdot \mu_k\}_{k \in \mathbb{N}}$. There is a simple and useful formula relating the counting functions $N_{c\,B}$ and N_B, namely

$$(12) \qquad N_{c\,B}(\tau) = \sum_{c \cdot \mu_k < \tau} 1 = \sum_{\mu_k < \frac{\tau}{c}} 1 = N_B\left(\frac{\tau}{c}\right).$$

In particular, (12) implies that, without loss of generality, we can assume[1] $\lambda_j > 1$ and $\mu_k > 1$ for all j, k. Let us now summarize the hypotheses on the factors A_1, A_2.

Assumptions 1.

M_1, M_2 smooth closed manifolds of dimensions n_1, n_2, respectively;

$A = A_1 \otimes A_2$, $A_1 \in L_{\mathrm{cl}}^1(M_1), A_2 \in L_{\mathrm{cl}}^{m_2}(M_2)$, $\quad m_2 > 0, n_1 > \dfrac{n_2}{m_2}$;

A_1, A_2 positive, self-adjoint, elliptic;

$\sigma(A_1) = \{\lambda_j\}_{j \in \mathbb{N}}$, $\sigma(A_2) = \{\mu_k\}_{k \in \mathbb{N}}$, $\quad \lambda_j > 1, \mu_k > 1$, for all j, k.

Since $\lambda_j, \mu_k > 1$ for all j, k, using (12), we have[2]

$$N_A(\tau) = \sum_{\lambda_j \cdot \mu_k < \tau} 1 = \sum_{\mu_k < \tau} \left(\sum_{\lambda_j \cdot \mu_k < \tau} 1 \right) =$$

$$(13) \qquad = \sum_{\mu_k < \tau} N_{\mu_k A_1}(\tau) = \sum_{\mu_k < \tau} N_{A_1}\left(\frac{\tau}{\mu_k}\right).$$

Proposition 2.1. Let A, A_1 and A_2 be as in Assumptions 1. Then,

$$N_A(\tau) = \sum_{\mu_k < \tau} \left(\frac{C_1}{n_1} \left(\frac{\tau}{\mu_k}\right)^{n_1} + \frac{1}{\mu_k^{n_1 - 1}} r_k(\tau) \right),$$

with

$$(14) \qquad C_1 = \frac{1}{(2\pi)^{n_1}} \int_{M_1} \int_{\mathbb{S}^{n_1-1}} \frac{d\theta_1 dx_1}{[a_{m_1}(x_1, \theta_1)]^{\frac{n_1}{m_1}}},$$

---

[1] In fact, if that condition were not true, we could consider the operator $c^2 A$, with $c = (\min\{\lambda_j, \mu_k\} - \varepsilon)^{-1}$, $\varepsilon > 0$ small enough.

[2] Recall that $\lambda_j > 1$ for all j. In the first term of (13) we can reduce the summation to $\mu_k < \tau$ since, otherwise, we would have $\lambda_k \cdot \mu_k \geq \tau$ for all j, and the second summation would be zero.

and $r_k(\tau)$ is $\mathcal{O}\left(\tau^{n_1-1}\right)$, uniformly with respect to μ_k. That is, there exists a positive constant C such that

(15) $$|r_k(\tau)| \le C\tau^{n_1-1}, \quad \text{for all } k \in \mathbb{N}.$$

Proof. By (13) we have

$$N_A(\tau) = \sum_{\mu_k < \tau} N_{A_1}\left(\frac{\tau}{\mu_k}\right).$$

Using (4), we can write

(16) $$N_A(\tau) = \sum_{\mu_k < \tau} \left(\frac{C_1}{n_1} \frac{\tau^{n_1}}{\mu_k^{n_1}} + R_A\left(\frac{\tau}{\mu_k}\right)\right).$$

Equation (5) implies that

$$|R_A(t)| \le \kappa t^{n_1-1}, \quad t > 1,$$

for a suitable constant κ. Since $\mu_k < \tau \Rightarrow \frac{\tau}{\mu_k} > 1$ in the summation (16), we can write

$$\left|R_A\left(\frac{\tau}{\mu_k}\right)\right| \le C\left(\frac{\tau}{\mu_k}\right)^{n_1-1}.$$

Hence, setting

$$r_k(\tau) = \mu_k^{n_1-1} R_A\left(\frac{\tau}{\mu_k}\right),$$

we have the assertion.

$\square$

Lemma 2.2. Let A, A_1, A_2 be as in Assumptions 1, and assume $n_1 > \dfrac{n_2}{m_2}$. Then we have, for $\tau \to +\infty$,

$$N_A(\tau) = \begin{cases} \dfrac{C_1}{n_1} \zeta(A_2, n_1)\tau^{n_1} + \mathcal{O}(\tau^{n_1-1}) & \text{if } \dfrac{n_2}{m_2} < n_1 - 1, \\[2mm] \dfrac{C_1}{n_1} \zeta(A_2, n_1)\tau^{n_1} + \mathcal{O}\left(\tau^{n_1-1}\log\tau\right) & \text{if } \dfrac{n_2}{m_2} = n_1 - 1, \\[2mm] \dfrac{C_1}{n_1} \zeta(A_2, n_1)\tau^{n_1} + \mathcal{O}\left(\tau^{\frac{n_2}{m_2}}\right) & \text{if } \dfrac{n_2}{m_2} > n_1 - 1, \end{cases}$$

where C_1 is given by (14).

Proof. Using Proposition 2.1 we obtain

$$N_A(\tau) = \sum_{\mu_k < \tau} \left(\frac{C_1}{n_1}\left(\frac{\tau}{\mu_k}\right)^{n_1} + \frac{1}{\mu_k^{n_1-1}} r_k(\tau)\right),$$

where $r_k(\tau)$ is uniformly $\mathcal{O}\left(\tau^{n_1-1}\right)$ for $\tau \to +\infty$, in the sense of (15). We can then write

$$\left| N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2, n_1)\tau^{n_1} \right|$$

$$= \left| \sum_{\mu_k < \tau} \left(\frac{C_1}{n_1}\frac{\tau^{n_1}}{\mu_k^{n_1}} + \frac{1}{\mu_k^{n_1-1}}r_k(\tau^{n_1-1}) \right) - \frac{C_1}{n_1}\zeta(A_2, n_1)\tau^{n_1} \right|$$

$$\leq \frac{C_1}{n_1}\tau^{n_1}\left| F_{A_2}(\tau, n_1) - \zeta(A_2, n_1) \right| + \left| \sum_{\mu_k < \tau} \frac{1}{\mu_k^{n_1-1}}r_k(\tau^{n_1-1}) \right|$$

$$(17) \qquad \leq \frac{C_1}{n_1}\tau^{n_1}\left| F_{A_2}(\tau, n_1) - \zeta(A_2, n_1) \right| + C\tau^{n_1-1}F_{A_2}(\tau, n_1-1).$$

Let us start with the case $n_1 - 1 > \frac{n_2}{m}$. Using (17), we find

$$\limsup_{\tau \to +\infty} \frac{\left| N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2, n_1)\tau^{n_1} \right|}{\tau^{n_1-1}}$$

$$\leq \frac{C_1}{n_1}\limsup_{\tau \to +\infty} \tau\left| \zeta(A_2, n_1) - F_{A_2}(\tau, n_1) \right| + C\limsup_{\tau \to +\infty} F_{A_2}(\tau, n_1-1).$$

Since

$$n_1 > n_1 - 1 > \frac{n_2}{m_2} \Rightarrow \frac{n_2}{m_2} - n_1 < -1,$$

$\zeta(A_2, n_1) - F_1(\tau) = \mathcal{O}\left(\tau^{\frac{n_2}{m_2}-n_1}\right)$ for $\tau \to +\infty$, in view of Proposition 1.2. It follows that

$$\limsup_{\tau \to +\infty} \tau|\zeta(A_2, n_1) - F_1(\tau)| \leq \widetilde{C}\limsup_{\tau \to +\infty} \tau^{\frac{n_2}{m_2}-n_1+1} = 0,$$

which implies

$$\limsup_{\tau \to +\infty} \frac{\left| N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2, n_1)\tau^{n_1} \right|}{\tau^{n_1-1}} \leq C\limsup_{\tau+\infty} F_{A_2}(\tau, n_1-1) = C\zeta(A_2, n_1-1).$$

Since $n_1-1 > \frac{n_2}{m_2}$, $\zeta(A_2, n_1-1)$ is finite, and we have the desidered assertion.

In the case $n_1 - 1 = \frac{n_2}{m_2}$, from (17) we analogously get

$$\limsup_{\tau \to +\infty} \frac{\left| N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2, n_1)\tau^{n_1} \right|}{\tau^{n_1-1}\log\tau}$$

$$\leq \frac{C_1}{n_1}\limsup_{\tau \to +\infty} \frac{\tau}{\log\tau}\left| \zeta(A_2, n_1) - F_{A_2}(\tau, n_1) \right| + C\limsup_{\tau \to +\infty} \frac{1}{\log\tau}F_{A_2}\left(\tau, \frac{n_2}{m_2}\right).$$

Since $n_1 > n_1 - 1 = \frac{n_2}{m_2}$, in view of Proposition 1.2 we find

$$\zeta(A_2, n_1) - F_1(\tau) = \mathcal{O}\left(\tau^{-1}\right), \quad F_{A_2}\left(\tau, \frac{n_2}{m_2}\right) = F_2(\tau) = \mathcal{O}\left(\log\tau\right),$$

so that

$$\limsup_{\tau \to +\infty} \frac{\left| N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2, n_1)\tau^{n_1} \right|}{\tau^{n_1-1}\log\tau} \leq \widetilde{C},$$

as claimed.

Finally, in the case $n_1 - 1 < \frac{n_2}{m_2}$, (17) gives

$$\limsup_{\tau \to +\infty} \frac{\left| N_A(\tau) - \frac{C_1}{n_1} \zeta(A_2, n_1) \tau^{n_1} \right|}{\tau^{\frac{n_2}{m_2}}}$$

$$\leq \frac{C_1}{n_1} \limsup_{\tau \to +\infty} \tau^{n_1 - \frac{n_2}{m_2}} \left| \zeta(A_2, n_1) - F_{A_2}(\tau, n_1) \right| +$$

$$C \limsup_{\tau \to +\infty} \tau^{n_1 - 1 - \frac{n_2}{m_2}} F_{A_2}(\tau, n_1 - 1).$$

Since $n_1 > \frac{n_2}{m_2} > n_1 - 1$, Proposition 1.2 implies

$$\zeta(A_2, n_1) - F_1(\tau) = \mathcal{O}\left(\tau^{\frac{n_2}{m_2} - n_1} \right), \quad F_{A_2}(\tau, n_1 - 1) = F_3(\tau) = \mathcal{O}\left(\tau^{\frac{n_2}{m} - n_1 + 1} \right).$$

Therefore,

$$\limsup_{\tau \to +\infty} \frac{\left| N_A(\tau) - \frac{C_1}{n_1} \zeta(A_2, n_1) \tau^{n_1} \right|}{\tau^{\frac{n_2}{m_2}}} < +\infty.$$

The proof is complete. $\qquad\square$

We can now prove our main result.

Theorem 2.3. Let M_1, M_2 be two closed manifolds of dimension n_1, n_2, respectively. Let $A = A_1 \otimes A_2$, where $A_j \in L^{m_j}_{\mathrm{cl}}(M_j)$, $m_j > 0$, $j = 1, 2$, are positive, self-adjoint, invertible operators, with $\dfrac{n_1}{m_1} > \dfrac{n_2}{m_2}$. Then, for $\tau \to +\infty$,

$$N_A(\tau) = \begin{cases} \dfrac{C_1}{n_1} \zeta\left(A_2, \dfrac{n_1}{m_1} \right) \tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_1 - 1}{m_1}} \right) & \text{if } \dfrac{n_2}{m_2} < \dfrac{n_1 - 1}{m_1}, \\[3mm] \dfrac{C_1}{n_1} \zeta\left(A_2, \dfrac{n_1}{m_1} \right) \tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_1 - 1}{m_1}} \log \tau \right) & \text{if } \dfrac{n_2}{m_2} = \dfrac{n_1 - 1}{m_1}, \\[3mm] \dfrac{C_1}{n_1} \zeta\left(A_2, \dfrac{n_1}{m_1} \right) \tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_2}{m_2}} \right) & \text{if } \dfrac{n_2}{m_2} > \dfrac{n_1 - 1}{m_1}, \end{cases}$$

where C_1 is given by (14).

Proof. Without loss of generality, we can assume $m_1 = 1, m_2 = m > 0$, possibly considering an appropriate power of A, see [Bat12]. Moreover, again without loss of the generality, we can assume that all the eigenvalues are strictly larger than one, so that the Assumptions 1 are fulfilled. Then, the claim follows from Lemma 2.2. $\qquad\square$

The case of the tensor product of two Shubin operators can be treated in a completely similar fashion, using Proposition 1.3 in place of Proposition 1.2, and the Weyl law (6) which holds in this setting.

Theorem 2.4. Let $P = P_1 \otimes P_2$ and $P_j \in G^{m_j}_{\mathrm{cl}}(\mathbb{R}^{n_j})$, $m_j > 0$, $j = 1, 2$, be positive, self-adjoint, invertible operators, with $\dfrac{2n_1}{m_1} > \dfrac{2n_2}{m_2}$. Then, for

$\tau \to +\infty$,

$$
N_P(\tau) = \begin{cases} \dfrac{K_1}{2n_1}\zeta\left(P_2, \dfrac{2n_1}{m_1}\right)\tau^{\frac{2n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{2n_1-1}{m_1}}\right) & \text{if } \dfrac{2n_2}{m_2} < \dfrac{2n_1-1}{m_1}, \\[2em] \dfrac{K_1}{2n_1}\zeta\left(P_2, \dfrac{2n_1}{m_1}\right)\tau^{\frac{2n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{2n_1-1}{m_1}}\log\tau\right) & \text{if } \dfrac{2n_2}{m_2} = \dfrac{2n_1-1}{m_1}, \\[2em] \dfrac{K_1}{2n_1}\zeta\left(P_2, \dfrac{2n_1}{m_1}\right)\tau^{\frac{2n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{2n_2}{m_2}}\right) & \text{if } \dfrac{2n_2}{m_2} > \dfrac{2n_1-1}{m_1}, \end{cases}
$$

where

$$
K_1 = \frac{1}{(2\pi)^{2n_1}}\int_{\mathbb{S}^{2n_1-1}} \frac{d\theta_1}{[p_{m_1}(\theta_1)]^{\frac{2n_1}{m_1}}}.
$$

3. Spectral asymptotics for the tensor product of r operators

As in the previous sections, to avoid redundancy we will prove in detail our results for tensor products of r factors only in the case of operators belonging to the Hörmander calculus on closed manifolds. We will then omit the proof of the analogous Theorem 3.4 for the case of operators belonging to the Shubin calculus, which can be obtained by similar arguments.

The main tool in the study of the extension of Theorem 2.3 to the product of $r \geq 2$ factors is a refined version of Proposition 1.2. Let us first state the hypotheses.

Assumptions 2.

$M_1, \ldots, M_r$ smooth closed manifolds of dimensions $n_1, \ldots, n_r$, respectively;

$A = A_1 \otimes \cdots \otimes A_r, \quad A_j \in L^{m_j}_{\mathrm{cl}}(M_j), \ m_j > 0, \quad j = 1, \ldots, r;$

A_j positive, self-adjoint, elliptic, $\quad j = 1, \ldots, r;$

$\sigma(A_j) = \left\{{}^j\mu_{k_j}\right\}_{k_j \in \mathbb{N}}, \quad {}^j\mu_1 > 1, \quad j = 1, \ldots, r.$

Proposition 3.1. Let $A, A_j, j = 1, \ldots, r$, be as in Assumptions 2. Set

$$
p = \max\left\{\frac{n_1}{m_1}, \ldots, \frac{n_r}{m_r}\right\}, \quad S = \left\{j \in \{1, \ldots, r\}\colon \frac{n_j}{m_j} = p\right\}, \quad s = \sharp S,
$$

and define, for $\tau \to +\infty$,

$$
F_A(\tau, c) = \sum_{{}^1\mu_{k_1}\cdot\ldots\cdot{}^r\mu_{k_r} < \tau} \frac{1}{({}^1\mu_{k_1})^c \cdot \ldots \cdot ({}^r\mu_{k_r})^c} = \begin{cases} F_1(\tau) & \text{if } p < c, \\ F_2(\tau) & \text{if } p = c, \\ F_3(\tau) & \text{if } p > c. \end{cases}
$$

Then,

$$
\limsup_{\tau \to +\infty} \frac{\left|\prod_{j=1}^r \zeta(A_j, c) - F_1(\tau)\right|}{\tau^{p-c}(\log\tau)^{s-1}} = \kappa_1,
$$

$$
\limsup_{\tau \to +\infty} \frac{F_2(\tau)}{(\log\tau)^s} = \kappa_2,
$$

$$
\limsup_{\tau \to +\infty} \frac{F_3(\tau)}{\tau^{p-c}(\log\tau)^{s-1}} = \kappa_3,
$$

that is, for $\tau \to +\infty$,

$$\prod_{j=1}^{r} \zeta(A_j, c) - F_1(\tau) = \mathcal{O}\left(\tau^{p-c} (\log \tau)^{s-1}\right),$$

$$F_2(\tau) = \mathcal{O}\left((\log \tau)^s\right), \ F_3(\tau) = \mathcal{O}\left(\tau^{p-c} (\log \tau)^{s-1}\right).$$

Proof. We will make use of the straightforward inequality

$$(18) \quad F_A(\tau, c) = \sum_{^1\mu_{k_1} \cdots \cdot^r\mu_{k_r} < \tau} \frac{1}{(^1\mu_{k_1})^c \cdot \ldots \cdot (^r\mu_{k_r})^c} \leq \prod_{j=1}^{r} \sum_{^j\mu_{k_j} < \tau} \frac{1}{(^j\mu_{k_j})^c},$$

as well as of the following consequence of the absolute convergence of the involved series,

$$(19) \quad \prod_{j=1}^{r} \zeta(A_j, c) = \lim_{\tau \to +\infty} \prod_{j=1}^{r} \sum_{^j\mu_{k_j} < \tau} \frac{1}{(^j\mu_{k_j})^c} = \sum \frac{1}{(^1\mu_{k_1})^c \cdot \ldots \cdot (^r\mu_{k_r})^c},$$

where c belongs to the holomorphic domain of the functions $\zeta(A_j, z)$, $j = 1, \ldots, r$, and the last summation in (19) is taken on all the r-tuples of eigenvalues $(^1\mu_{k_1}, \ldots, ^r\mu_{k_r}) \in \sigma(A_1) \oplus \cdots \oplus \sigma(A_r)$.

Case $p = c$. Let us split the last term in (18) as

$$\prod_{j=1}^{r} \sum_{^j\mu_{k_j} < \tau} \frac{1}{(^j\mu_{k_j})^c} = \left(\prod_{j \notin S} \sum_{^j\mu_{k_j} < \tau} \frac{1}{(^j\mu_{k_j})^c}\right) \cdot \left(\prod_{t \in S} \sum_{^t\mu_{k_t} < \tau} \frac{1}{(^t\mu_{k_t})^c}\right).$$

Recalling that $\frac{n_j}{m_j} < c$ for all $j \notin S$, that is, c belongs to the holomorphic domain of $\zeta(A_j, \cdot)$ for $j \notin S$, and that $\frac{n_t}{m_t} = c$ for all $t \in S$, using Proposition 1.2 we have

$$\left(\prod_{j \notin S} \sum_{^j\mu_{k_j} < \tau} \frac{1}{(^j\mu_{k_j})^c}\right) \cdot \left(\prod_{t \in S} \sum_{^t\mu_{k_t} < \tau} \frac{1}{(^t\mu_{k_t})^c}\right)$$

$$= \left(\prod_{j \notin S} \zeta(A_j, c)\right) \cdot \mathcal{O}\left((\log \tau)^s\right) = \mathcal{O}\left((\log \tau)^s\right),$$

which implies our claim in this case, in view of (18).

Case $p > c$. To simplify notation, we can suppose, without loss of generality, $p = \frac{n_1}{m_1}$. Recalling the assumption $^j\mu_{k_j} > 1$, $j = 1, \ldots, r$, we observe that

$$^1\mu_{k_1} \cdot \ldots \cdot ^r\mu_{k_r} < \tau \Leftrightarrow \left[\prod_{j=2}^{r} (^j\mu_{k_j}) < \tau \ \wedge \ 1 < ^1\mu_{k_1} < \frac{\tau}{\prod_{j=2}^{r} (^j\mu_{k_j})}\right].$$

In fact, the $\Leftarrow$ implication is immediate, while

$$^1\mu_{k_1} \cdot \ldots \cdot {}^r\mu_{k_r} < \tau \ \wedge \ {}^1\mu_{k_1} > 1$$

$$\Rightarrow 1 < {}^1\mu_{k_1} < \frac{\tau}{\prod_{j=2}^r ({}^j\mu_{k_j})}$$

$$\Rightarrow 1 < {}^1\mu_{k_1} < \frac{\tau}{\prod_{j=2}^r ({}^j\mu_{k_j})} \ \wedge \ \prod_{j=2}^r ({}^j\mu_{k_j}) < \tau.$$

Then, we can write

$$F_A(\tau, c) = F_3(\tau)$$

$$= \sum_{{}^2\mu_{k_2} \cdot \ldots \cdot {}^r\mu_{k_r} < \tau} \frac{1}{({}^2\mu_{k_2})^c \cdot \ldots \cdot ({}^r\mu_{k_r})^c} \sum_{1 < {}^1\mu_{k_1} < \frac{\tau}{{}^2\mu_{k_2} \cdot \ldots \cdot {}^r\mu_{k_r}}} \frac{1}{({}^1\mu_{k_1})^c}$$

$$= \sum_{\prod_{j=2}^r ({}^j\mu_{k_j}) < \tau} \frac{1}{\prod_{j=2}^r ({}^j\mu_{k_j})^c} \int_1^{\frac{\tau}{\prod_{j=2}^r ({}^j\mu_{k_j})}} \frac{1}{\mu^c} dN_{A_1}(\mu).$$

Switching to $B = (A_1)^{\frac{1}{m_1}}$, recalling that then $\sigma(B) = \{({}^1\mu_{k_1})^{\frac{1}{m_1}}\}$ and[3]

$$\int_1^{\frac{\tau}{\prod_{j=2}^r ({}^j\mu_{k_j})}} \frac{1}{\mu^c} dN_{A_1}(\mu) = \int_1^{\left[\frac{\tau}{\prod_{j=2}^r ({}^j\mu_{k_j})}\right]^{\frac{1}{m_1}}} \frac{1}{\mu^{c\,m_1}} dN_B(\mu),$$

using (9) with n_1 in place of n, it turns out that, for $\tau \to +\infty$,

$$F_3(\tau) = \sum_{\prod_{j=2}^r ({}^j\mu_{k_j}) < \tau} \frac{1}{\prod_{j=2}^r ({}^j\mu_{k_j})^c} \ \mathcal{O}\left(\left(\frac{\tau}{\prod_{j=2}^r ({}^j\mu_{k_j})}\right)^{p-c}\right)$$

$$= \left[\sum_{{}^2\mu_{k_2} \cdot \ldots \cdot {}^r\mu_{k_r} < \tau} \frac{1}{({}^2\mu_{k_2})^p \cdot \ldots \cdot ({}^r\mu_{k_r})^p}\right] \mathcal{O}\left(\tau^{p-c}\right).$$

Using the result of the case $p = c$ above, with $s - 1$ in place of s, we conclude

$$F_3(\tau) = \mathcal{O}\left(\tau^{p-c} (\log \tau)^{s-1}\right),$$

as claimed.

Case $p < c$. Since $c > \frac{n_j}{m_j}$ for all $j = 1, \ldots, r$, c belongs to the holomorphic domain of all the functions $\zeta(A_j, z)$, $j = 1, \ldots, r$. Then, by (19), in

---

[3]That is,

$$\sum_{1 < {}^1\mu_{k_1} < \frac{\tau}{{}^2\mu_{k_2} \cdot \ldots \cdot {}^r\mu_{k_r}}} \frac{1}{({}^1\mu_{k_1})^c} = \sum_{1 < ({}^1\mu_{k_1})^{\frac{1}{m_1}} < \left[\frac{\tau}{{}^2\mu_{k_2} \cdot \ldots \cdot {}^r\mu_{k_r}}\right]^{\frac{1}{m_1}}} \frac{1}{\left[({}^1\mu_{k_1})^{\frac{1}{m_1}}\right]^{c\,m_1}}$$

similarly to the proof of Proposition 1.2.

this case we have, for all τ,

$$\prod_{j=1}^{r} \zeta(A_j, c) - F_A(\tau, c) = \prod_{j=1}^{r} \zeta(A_j, c) - F_1(\tau)$$

$$= \sum \frac{1}{(^1\mu_{k_1})^c \cdot \ldots \cdot (^r\mu_{k_r})^c} - \sum_{^1\mu_{k_1} \cdot \ldots \cdot ^r\mu_{k_r} < \tau} \frac{1}{(^1\mu_{k_1})^c \cdot \ldots \cdot (^r\mu_{k_r})^c}$$

$$= \sum_{^1\mu_{k_1} \cdot \ldots \cdot ^r\mu_{k_r} \geq \tau} \frac{1}{(^1\mu_{k_1})^c \cdot \ldots \cdot (^r\mu_{k_r})^c}$$

We will prove the claim by induction on the number of operators. The case $r = 2$ is proven in Proposition 1.2. Let us then suppose that the desired estimate holds true for a tensor product of $r - 1$ operators, $r \geq 2$, and let us prove that it holds true also for a tensor product of r operators.

We can again suppose, without loss of generality, $p = \frac{n_1}{m_1}$. Since, clearly,

$$^1\mu_{k_1} \cdot \ldots \cdot ^r\mu_{k_r} \geq \tau \Rightarrow$$

$$^1\mu_{k_1} \geq \frac{\tau}{\prod_{j=2}^{r}(^j\mu_{k_j})} \wedge \left[\prod_{j=2}^{r}(^j\mu_{k_j}) < \tau \vee \prod_{j=2}^{r}(^j\mu_{k_j}) \geq \tau \right],$$

we can write

$$\prod_{j=1}^{r} \zeta(A_j, c) - F_1(\tau) =$$

$$(20) \qquad \sum_{\prod_{j=2}^{r}(^j\mu_{k_j}) < \tau} \frac{1}{\prod_{j=2}^{r}(^j\mu_{k_j})^c} \int_{\frac{\tau}{\prod_{j=2}^{r}(^j\mu_{k_j})}}^{+\infty} \frac{1}{\mu^c} dN_{A_1}(\mu)$$

$$(21) \qquad + \sum_{\prod_{j=2}^{r}(^j\mu_{k_j}) \geq \tau} \frac{1}{\prod_{j=2}^{r}(^j\mu_{k_j})^c} \int_{\frac{\tau}{\prod_{j=2}^{r}(^j\mu_{k_j})}}^{+\infty} \frac{1}{\mu^c} dN_{A_1}(\mu).$$

Let us first consider (20). Arguing as in the previous case $p > c$, and using the case $p = c$ with $s - 1$ in place of s, we find, for $\tau \to +\infty$,

$$\sum_{\prod_{j=2}^{r}(^j\mu_{k_j}) < \tau} \frac{1}{\prod_{j=2}^{r}(^j\mu_{k_j})^c} \int_{\frac{\tau}{\prod_{j=2}^{r}(^j\mu_{k_j})}}^{+\infty} \frac{1}{\mu^c} dN_{A_1}(\mu)$$

$$= \sum_{\prod_{j=2}^{r}(^j\mu_{k_j}) < \tau} \frac{1}{\prod_{j=2}^{r}(^j\mu_{k_j})^c} \mathcal{O}\left(\left(\frac{\tau}{\prod_{j=2}^{r} {}^j\mu_{k_j}} \right)^{p-c} \right)$$

$$= \mathcal{O}\left(\tau^{p-c} (\log \tau)^{s-1} \right),$$

which is the desired estimate. We now show that (21) fulfills the same the same holds for. Using the fact that $\zeta(A_1, c)$ is finite, we

can estimate (21) as

$$\sum_{\prod_{j=2}^{r}(^j\mu_{k_j})\geq\tau}\frac{1}{\prod_{j=2}^{r}(^j\mu_{k_j})^c}\int_{\frac{\tau}{\prod_{j=2}^{r}(^j\mu_{k_j})}}^{+\infty}\frac{1}{\mu^c}dN_{A_1}(\mu)$$

$$(22)\qquad\leq\sum_{\prod_{j=2}^{r}(^j\mu_{k_j})\geq\tau}\frac{1}{\prod_{j=2}^{r}(^j\mu_{k_j})^c}\zeta(A_1,c).$$

By the inductive hypothesis, we see that (22) is $\mathcal{O}\left(\tau^{\widetilde{p}-c}(\log\tau)^{\widetilde{s}-1}\right)$ for $\tau\to+\infty$, where $\widetilde{p}=\max\left\{\frac{n_j}{m_j}\right\}_{j=2}^{r}\leq p$ and $\widetilde{s}<s$. Therefore, it is also $\mathcal{O}\left(\tau^{p-c}(\log\tau)^{s-1}\right)$ for $\tau\to+\infty$, and the same of course holds for (21), in view of the above estimate.

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Assumptions 3. Let $A,A_1,\ldots,A_r$ be as in Assumptions 2, and suppose that there exists $l\in\{1,\ldots,r\}$ such that

$$\frac{n_l}{m_l}>\max\left\{\frac{n_j}{m_j}\right\}_{j\in\{1,\ldots,r\}\setminus\{l\}}.$$

For notational simplicity, in the next two statements we also assume, without loss of generality, that $l=1$. As in the previous section, we first consider the case when $m_1=1$. We will denote by $\mu_{\mathbf{j}_{\geq2}}$ the product $\prod_{j=2}^{r}{}^j\mu_{k_j}$, where $\mathbf{j}_{\geq2}$ denotes the multiindex $(k_2,\ldots,k_r)\in\mathbb{N}^{r-1}$. The following proposition is an extension of Proposition 3.2.

Proposition 3.2. Let $A,A_1,\ldots,A_r$ be as in Assumptions 3. Then,

$$N_A(\tau)=\sum_{\mu_{\mathbf{j}_{\geq2}}<\tau}\left(\frac{C_1}{n_1}\left(\frac{\tau}{\mu_{\mathbf{j}_{\geq2}}}\right)^{n_1}+\frac{1}{\mu_{\mathbf{j}_{\geq2}}^{n_1-1}}r_{\mathbf{j}_{\geq2}}(\tau)\right),$$

where C_1 is given by (14) and $r_{\mathbf{j}_{\geq2}}$ is $\mathcal{O}\left(\tau^{n_1-1}\right)$, uniformly with respect to $\mu_{\mathbf{j}_{\geq2}}$, for any $\mathbf{j}_{\geq2}$. That is, there exists a positive constant C such that

$$|r_{\mathbf{j}_{\geq2}}(\tau)|\leq C\tau^{n_1-1},\quad\text{for all }\mathbf{j}_{\geq2}\in\mathbb{N}^{r-1}.$$

Proposition 3.2 implies the next lemma, which is a multidimensional version of Lemma 2.2. We omit the proof, since the argument is analogue to the one used to prove Lemma 2.2, similarly to what has been done in the proof of Proposition 3.1.

Lemma 3.3. Let $A,A_1,\ldots,A_r$ be as in Assumptions 3. Let us suppose that $m_1=1$ and $n_1>\frac{n_j}{m_j}$, $j=2,\ldots,r$, and set

$$p=\max\left\{\frac{n_j}{m_j}\right\}_{j=2,\ldots,r},\quad S=\left\{j=2,\ldots,r:\frac{n_j}{m_j}=p\right\},\quad s=\sharp S.$$

Then we have, for $\tau \to +\infty$,

$$N_A(\tau) = \begin{cases} C_A \tau^{n_1} + \mathcal{O}\left(\tau^{n_1-1}\right) & \text{if } p < n_1 - 1, \\[2mm] C_A \tau^{n_1} + \mathcal{O}\left(\tau^{n_1-1} \left(\log \tau\right)^s\right) & \text{if } p = n_1 - 1, \\[2mm] C_A \tau^{n_1} + \mathcal{O}\left(\tau^p \left(\log \tau\right)^{s-1}\right) & \text{if } p > n_1 - 1, \end{cases}$$

where

$$C_A = \frac{C_1}{n_1} \prod_{j=2}^{r} \zeta\left(A_j, n_1\right)$$

and C_1 is given by (14).

Finally, using powers of the operator A, it is possible to extend the result to the case where all the factors have arbitrary positive order, which is, together with Theorem 3.5 below for the tensor product of r factors in the Shubin calculus, our next main result.

Theorem 3.4. Let $M_1, \ldots, M_r$ be closed manifolds of dimension $n_1, \ldots, n_r$, respectively. Let $A = A_1 \otimes \cdots \otimes A_r$, where $A_j \in L_{\mathrm{cl}}^{m_j}(M_j)$, $m_j > 0$, $j = 1, \ldots, r$, are positive, self-adjoint, invertible operators, and assume that there exists $l \in \{1, \ldots, r\}$ such that $\frac{n_l}{m_l} > \max\left\{\frac{n_j}{m_j}\right\}_{j \in \{1,\ldots,r\}\backslash\{l\}}$. Set

$$p = \max\left\{\frac{n_j}{m_j}\right\}_{j \in \{1,\ldots,r\}\backslash\{l\}}, \quad S = \left\{j = 1, \ldots, r, j \neq l : \frac{n_j}{m_j} = p\right\}, \quad s = \sharp S.$$

Then, for $\tau \to +\infty$,

$$N_A(\tau) = \begin{cases} C_A \tau^{\frac{n_l}{m_l}} + \mathcal{O}\left(\tau^{\frac{n_l-1}{m_l}}\right) & \text{if } p < \frac{n_l - 1}{m_l}, \\[2mm] C_A \tau^{\frac{n_l}{m_l}} + \mathcal{O}\left(\tau^{\frac{n_l-1}{m_l}} \left(\log \tau\right)^s\right) & \text{if } p = \frac{n_l - 1}{m_l}, \\[2mm] C_A \tau^{\frac{n_l}{m_l}} + \mathcal{O}\left(\tau^p \left(\log \tau\right)^{s-1}\right) & \text{if } p > \frac{n_l - 1}{m_l}, \end{cases}$$

where

$$C_A = \frac{C_l}{n_l} \prod_{\substack{j=1,\ldots,r \\ j \neq l}} \zeta\left(A_j, \frac{n_l}{m_l}\right), \quad C_l = \frac{1}{(2\pi)^{n_l}} \int_{M_l} \int_{\mathbb{S}^{n_l-1}} \frac{d\theta_l dx_l}{\left[a_{m_l}\left(x_l, \theta_l\right)\right]^{\frac{n_l}{m_l}}}.$$

Theorem 3.5. Let $P = P_1 \otimes \cdots \otimes P_r$ and $P_j \in G_{\mathrm{cl}}^{m_j}(\mathbb{R}^{n_j})$, $m_j > 0$, $j = 1, \ldots, r$, be positive, self-adjoint, invertible operators, and assume that there exists $l \in \{1, \ldots, r\}$ such that $\frac{2n_l}{m_l} > \max\left\{\frac{2n_j}{m_j}\right\}_{j \in \{1,\ldots,r\}\backslash\{l\}}$. Set

$$p = \max\left\{\frac{2n_j}{m_j}\right\}_{j \in \{1,\ldots,r\}\backslash\{l\}}, \quad S = \left\{j = 1, \ldots, r, j \neq l : \frac{2n_j}{m_j} = p\right\}, \quad s = \sharp S.$$

Then, for $\tau \to +\infty$,

$$
N_P(\tau) = \begin{cases} K_P \, \tau^{\frac{2n_l}{m_l}} + \mathcal{O}\left(\tau^{\frac{2n_l-1}{m_l}}\right) & \text{if} \quad p < \dfrac{2n_l-1}{m_l}, \\[2ex] K_P \, \tau^{\frac{2n_l}{m_l}} + \mathcal{O}\left(\tau^{\frac{2n_l-1}{m_l}} (\log \tau)^s\right) & \text{if} \quad p = \dfrac{2n_l-1}{m_l}, \\[2ex] K_P \, \tau^{\frac{2n_l}{m_l}} + \mathcal{O}\left(\tau^p (\log \tau)^{s-1}\right) & \text{if} \quad p > \dfrac{2n_l-1}{m_l}, \end{cases}
$$

where

$$
K_P = \frac{V_l}{2n_l} \prod_{\substack{j=1,\dots,r \\ j \neq l}} \zeta\left(P_j, \frac{2n_l}{m_l}\right), \quad V_l = \frac{1}{(2\pi)^{2n_l}} \int_{\mathbb{S}^{2n_l-1}} \frac{d\theta_l}{[p_{m_l}(\theta_l)]^{\frac{2n_l}{m_l}}}.
$$

4. SHARPNESS OF THE RESULT

In this section we show that the estimates obtained in Theorem 2.3 are sharp. To begin, we choose two pseudodifferential operators on spheres, whose spectrum we can describe explicitly. Namely, we set

$$
A_1 = (-\Delta_{\mathbb{S}^2} + 2) - 2\left(-\Delta_{\mathbb{S}^2} + \frac{1}{4}\right)^{\frac{1}{2}} \in L^2_{\mathrm{cl}}(\mathbb{S}^2), \ A_2 = -\Delta_{\mathbb{S}^1} + 1 \in L^2_{\mathrm{cl}}(\mathbb{S}^1),
$$

where A_1 is considered as an unbounded operator on $L^2(\mathbb{S}^2)$, where $\mathbb{S}^2$ is the 2-dimensional sphere, and A_2 is considered as an unbounded operator on $L^2(\mathbb{S}^1)$, where $\mathbb{S}^1$ is the 1-dimensional sphere. It is well known, see, e.g., [Shu87, §3], that

$$
\sigma\left(-\Delta_{\mathbb{S}^2}\right) = \left\{k^2 + k \mid k \in \mathbb{N}, \ \mathbf{mult}\left(k^2 + k\right) = (2k+1)\right\},
$$
$$
\sigma\left(-\Delta_{\mathbb{S}^1}\right) = \left\{n^2 \mid n \in \mathbb{N}, \ \mathbf{mult}\left(n^2\right) = 2\right\},
$$

where $\mathbf{mult}(\tau)$ is the multiplicity of the eigenvalue τ. Therefore, by the functional calculus of operators,

(23) $\sigma(A_1) = \left\{k^2 - k + 1 \mid k \in \mathbb{N}, \ \mathbf{mult}\left(k^2 - k + 1\right) = (2k+1)\right\},$

(24) $\sigma(A_2) = \left\{n^2 + 1 \mid n \in \mathbb{N}, \ \mathbf{mult}\left(n^2 + 1\right) = 2\right\},$

since the eigenfunction of A_1 and $-\Delta_{\mathbb{S}^2}$ are the same. Notice that all the eigenvalues of A_1 are larger then 1, therefore

(25) $N_{A_1}(\tau) = 0, \quad \tau \leq 1.$

Knowing precisely the eigenvalues of A_1 together with their multiplicities, we can write, for $\tau > 1$,

$$
N_{A_1}(\tau) = \sum_{k^2-k+1 < \tau} \mathbf{mult}\left(k^2 - k + 1\right)
$$

$$
= \sum_{k^2-k+1 < \tau} (2k+1) = \sum_{k=0}^{\bar{k}} (2k+1)
$$

where

$$
\bar{k}^2 - \bar{k} + 1 < \tau \leq \left(\bar{k}+1\right)^2 - \left(\bar{k}+1\right) + 1 = \bar{k}^2 + \bar{k} + 1, \tau > 1.
$$

That is,

(26)
$$N_{A_1}(\tau) = \sum_{k=0}^{\bar{k}} (2k+1) = \sum_{k^2+k \leq \bar{k}^2+\bar{k}} \mathbf{mult}\left(k^2+k\right) = N_{-\Delta_{\mathbb{S}^2}}\left(\bar{k}^2+\bar{k}+\frac{1}{2}\right),$$

provided that

$$\bar{k}^2 - \bar{k} + 1 < \tau \leq \left(\bar{k}+1\right)^2 - \left(\bar{k}+1\right) + 1 = \bar{k}^2 + \bar{k} + 1, \tau > 1.$$

Using a well known result on the counting function of the Laplacian on the spheres (see [Shu87]), we have, for each $\bar{k} \in \mathbb{N}$,

$$N_{-\Delta_{\mathbb{S}^2}}\left(\bar{k}^2+\bar{k}+\frac{1}{2}\right) = \bar{k}^2 + 2\bar{k} + 1.$$

So, in view of (26), supposing $\tau > 1$, we find

$$N_{A_1}(\tau) = \bar{k}^2 + 2\bar{k} + 1,$$
$$\bar{k}^2 - \bar{k} + 1 < \tau \leq \left(\bar{k}+1\right)^2 - \left(\bar{k}+1\right) + 1 = \bar{k}^2 + \bar{k} + 1.$$

The asymptotic expansion (4) implies that

$$N_{A_1}(\tau) = \tau + R(\tau), \quad R = \mathcal{O}\left(\tau^{\frac{1}{2}}\right).$$

We can then obtain a bound for $R(\tau)$:

$$R(\tau) = N_{A_1}(\tau) - \tau$$
$$= \bar{k}^2 + 2\bar{k} + 1 - \tau, \quad \bar{k}^2 - \bar{k} + 1 < \tau \leq \bar{k}^2 + \bar{k} + 1.$$

Therefore, for $\tau > 16$,

$$R(\tau) \geq \bar{k}^2 + 2\bar{k} + 1 - \bar{k}^2 - \bar{k} - 1 = \bar{k} > \frac{3\sqrt{\tau}}{4},$$

which implies, in particular, that the remainder is positive for $\tau > 16$. We also have

$$R(\tau) < \bar{k}^2 + 2\bar{k} + 1 - \bar{k}^2 + \bar{k} - 1 = 2\bar{k} < 4\sqrt{\tau},$$

and we can conclude that

(27)
$$\frac{3\sqrt{\tau}}{4} \leq R(\tau) \leq 4\sqrt{\tau}, \quad \tau > 16.$$

Summing up, we proved that

(28)
$$N_{A_1}(\tau) = \tau + R(\tau),$$

(29)
$$N_{A_2}(\tau) = 2\,\tau^{1/2} + \mathcal{O}(1),$$

where the $R(\tau)$ in (28) satisfies (27). Notice that both A_1 and A_2 are elliptic, invertible and positive, so it is possible to consider powers of these operators of arbitrary exponent. Now, we examine separately the three different situations that can arise.

Case $\frac{n_1}{m_1} > \frac{n_2}{m_2}$ **and** $\frac{n_1-1}{m_1} > \frac{n_2}{m_2}$**.** Let us consider the operator

$$B = A_1 \otimes A_2^2.$$

Clearly $\frac{n_1}{m_1} = \frac{2}{2} = 1 > \frac{n_2}{m_2} = \frac{1}{4}$ and $\frac{n_1-1}{m_1} = \frac{1}{2} > \frac{n_2}{m_2} = \frac{1}{4}$, so we are in the first case of Theorem 2.3, which states that

$$N_B(\tau) = \zeta(A_2^2, 1)\tau + \mathcal{O}\left(\tau^{1/2}\right). \tag{30}$$

By equations (23) and (24) we obtain

$$\sigma(B) = \{(k^2 - k + 1)(n^2 + 1)^2 \mid k, n \in \mathbb{N},$$
$$\mathbf{mult}\left((k^2 - k + 1)(n^2 + 1)^2\right) = 2(2k + 1)\}.$$

Therefore,

$$N_B(\tau) = \sum_{\substack{(k^2-k+1)(n^2+1)^2 < \tau}}^{n \in \mathbb{N},\, k \in \mathbb{N}} \mathbf{mult}\left((k^2 - k + 1)(n^2 + 1)^2\right)$$

$$= \sum_{\substack{(k^2-k+1)(n^2+1)^2 < \tau}}^{n \in \mathbb{N},\, k \in \mathbb{N}} 2(2k + 1)$$

$$= 2 \sum_{\substack{(k^2-k+1) < \frac{\tau}{(n^2+1)^2}}}^{n \in \mathbb{N},\, k \in \mathbb{N}} \mathbf{mult}\left(k^2 - k + 1\right)$$

$$= 2 \sum_{(n^2+1)^2 < \tau}^{n \in \mathbb{N}} N_{A_1}\left(\frac{\tau}{(n^2+1)^2}\right) \tag{31}$$

$$= 2\left(\sum_{(n^2+1)^2 < \tau}^{n \in \mathbb{N}} \frac{\tau}{(n^2+1)^2} + R\left(\frac{\tau}{(n^2+1)^2}\right)\right). \tag{32}$$

Notice that in (31) we have made use of (25) to reduce the summation. Let us now show that the estimate (30) is indeed sharp, that is

$$\limsup_{\tau \to +\infty} \frac{\left|N_B(\tau) - \zeta\left(A_2^2, 1\right)\tau\right|}{\tau^{1/2}} > 0,$$

by direct computation. In view of (32), we can write

$$\limsup_{\tau \to +\infty} \frac{\left|N_B(\tau) - \zeta\left(A_2^2, 1\right)\tau\right|}{\tau^{1/2}}$$

$$= \limsup_{\tau \to +\infty} \frac{\left|2\sum_{(n^2+1)^2 < \tau}\left(\frac{\tau}{(n^2+1)^2} + R\left(\frac{\tau}{(n^2+1)^2}\right)\right) - \zeta\left(A_2^2, 1\right)\tau\right|}{\tau^{1/2}}$$

$$\tag{33}$$

$$= \limsup_{\tau \to +\infty} \frac{\left|2\sum_{(n^2+1)^2 < \tau} \frac{\tau}{(n^2+1)^2} - \zeta\left(A_2^2, 1\right)\tau + 2\sum_{(n^2+1)^2 < \tau} R\left(\frac{\tau}{(n^2+1)^2}\right)\right|}{\tau^{1/2}}.$$

We notice that

$$\limsup_{\tau \to +\infty} \frac{\left| 2 \sum_{(n^2+1)^2 < \tau} \frac{\tau}{(n^2+1)^2} - \zeta\left(A_2^2, 1\right) \tau \right|}{\tau^{1/2}}$$

$$= \limsup_{\tau \to +\infty} \tau^{1/2} (F_{A_2^2}(\tau, 1) - \zeta\left(A_2^2, 1\right)),$$

where we have used the notation introduced in Section 1. By Proposition 1.2, $F_{A_2^2}(\tau, 1) - \zeta\left(A_2^2, 1\right) = \mathcal{O}\left(\tau^{-\frac{3}{4}}\right)$, therefore[4],

$$\limsup_{\tau \to +\infty} \frac{\left| 2 \sum_{(n^2+1)^2 < \tau} \frac{\tau}{(n^2+1)^2} - \zeta\left(A_2^2, 1\right) \tau \right|}{\tau^{1/2}} = 0.$$

Since, for all τ,

$$\sum_{(n^2+1)^2 < \tau} 2 \frac{\tau}{(n^2+1)^2} - \zeta\left(A_2^2, 1\right) \tau \leq 0,$$

(33) becomes

$$\limsup_{\tau \to +\infty} \frac{\left| N_B(\tau) - \zeta\left(A_2^2, 1\right) \tau \right|}{\tau^{1/2}}$$

$$\geq -\limsup_{\tau \to +\infty} \frac{\zeta\left(A_2^2, 1\right) - 2 \sum_{(n^2+1)^2 < \tau} \frac{\tau}{(n^2+1)^2}}{\tau^{1/2}} + 2 \limsup_{\tau \to +\infty} \sum_{(n^2+1)^2 < \tau} \frac{\left| R\left(\frac{\tau}{(n^2+1)^2}\right) \right|}{\tau^{1/2}}$$

$$\geq \frac{3}{2} \limsup_{\tau \to +\infty} \sum_{(n^2+1)^2 < \tau} \frac{\tau^{1/2}}{(n^2+1)\tau^{1/2}}$$

$$= \frac{3}{2} \zeta\left(A_2^2, \frac{1}{2}\right).$$

Here, we have used the estimates (27) and that the quantities $\frac{n_1}{m_1} = 1$ and $\frac{n_1-1}{n_2} = \frac{1}{2}$ are larger than $\frac{n_2}{m_2} = \frac{1}{4}$. The latter implies that $\zeta\left(A_2^2, \frac{1}{2}\right)$ is a finite, positive quantity[5], in view of the holomorphic properties of the spectral ζ-function of elliptic positive pseudodifferential operators on closed manifolds, see [See67]. This proves the desired result.

Case $\frac{n_1}{m_1} > \frac{n_2}{m_2}$ **and** $\frac{n_1-1}{m_1} = \frac{n_2}{m_2}$. We consider the operator

$$C = A_1 \otimes A_2.$$

Clearly $\frac{n_1}{m_1} = \frac{2}{2} = 1 > \frac{n_2}{m_2} = \frac{1}{2}$ and $\frac{n_1-1}{m_1} = \frac{1}{2} = \frac{n_2}{m_2}$ so that we are in second case of Theorem 2.3, which now states that

$$N_C(\tau) = \zeta(A_2, 1)\tau + \mathcal{O}\left(\tau^{1/2} \log \tau\right).$$

Using (23) and (24) we obtain explicitly the spectrum of C, namely

$$\sigma(C) = \left\{ \left(k^2 - k + 1\right)\left(n^2 + 1\right) \mid \mathbf{mult}\left(\left(k^2 - k + 1\right)\left(n^2 + 1\right)\right) = 2(2k+1) \right\}.$$

---

[4]Actually, here one could prove directly that $F_1(\tau) - \zeta\left(A_2^2, 1\right)$ is asymptotic to $\tau^{-\frac{3}{4}}$.

[5]The convergence of the involved series is straightforward.

Therefore, using (25),

$$N_C(\tau) = \sum_{\substack{(k^2-k+1)(n^2+1)<\tau}}^{n\in\mathbb{N},k\in\mathbb{N}} 2\,(2k+1)$$

$$= 2 \sum_{\substack{(k^2-k+1)<\frac{\tau}{n^2+1}}}^{n\in\mathbb{N},k\in\mathbb{N}} \mathbf{mult}\,\left(k^2+k+1\right)$$

$$= 2 \sum_{\substack{(n^2+1)<\tau}}^{n\in\mathbb{N}} N_{A_1}\left(\frac{\tau}{n^2+1}\right)$$

$$(34) \qquad\qquad = 2 \sum_{\substack{n^2+1<\tau}}^{n\in\mathbb{N}} \left(\frac{\tau}{n^2+1} + R\left(\frac{\tau}{n^2+1}\right)\right).$$

Let us check directly that

$$(35) \qquad\qquad \limsup_{\tau\to+\infty} \frac{|N_C(\tau) - \zeta(A_2,1)\tau|}{\tau^{1/2}\log\tau} > 0.$$

Using (34) and (27) we can write

$$\limsup_{\tau\to+\infty} \frac{|N_C(\tau) - \zeta(A_2,1)\tau|}{\tau^{1/2}\log\tau}$$

$$= \limsup_{\tau\to+\infty} \frac{\left|2\sum_{n^2+1<\tau}\left(\frac{\tau}{n^2+1} + R\left(\frac{\tau}{n^2+1}\right)\right) - \zeta(A_2,1)\tau\right|}{\tau^{1/2}\log\tau}$$

$$\geq -\limsup_{\tau\to+\infty} \frac{\tau^{1/2}\left(\zeta(A_2,1) - 2\sum_{n^2+1<\tau}\frac{1}{n^2+1}\right)}{\log\tau}$$

$$+ \limsup_{\tau\to+\infty} \frac{3}{4}\tau^{1/2}\frac{2\sum_{n^2+1<\tau}\frac{1}{(n^2+1)^{1/2}}}{\tau^{1/2}\log\tau}$$

$$(36) \qquad \geq -\limsup_{\tau\to+\infty} \tau^{\frac{1}{2}}\frac{2\sum_{n^2+1\geq\tau}\frac{1}{n^2+1}}{\log\tau} + \limsup_{\tau\to+\infty} \frac{3}{2}\frac{\sum_{n^2+1<\tau}\frac{1}{(n^2+1)^{1/2}}}{\log\tau}$$

Finally, using the results of Proposition 1.2 (or directly, by integral inequalities), we obtain that

$$\limsup_{\tau\to+\infty} \tau^{\frac{1}{2}}\frac{2\sum_{n^2+1\geq\tau}\frac{1}{n^2+1}}{\log\tau} = \lim_{\tau\to+\infty} \tau^{\frac{1}{2}}\frac{2\sum_{n^2+1\geq\tau}\frac{1}{n^2+1}}{\log\tau} = 0.$$

Moreover,

$$\limsup_{\tau\to+\infty} \frac{3}{2}\frac{\sum_{n^2+1<\tau}\frac{1}{n^2+1}}{\log\tau} = \frac{3}{4},$$

so that, by means of (36), the desired result is proven also in this second case.

Case $\frac{n_1}{m_1} > \frac{n_2}{m_2}$ and $\frac{n_1-1}{m_1} < \frac{n_2}{m_2}$. In this situation we consider the operator

$$D = A_1 \otimes A_2^{\frac{3}{4}}.$$

Clearly, $\frac{n_1}{m_1} = \frac{2}{2} = 1 > \frac{n_2}{m_2} = \frac{2}{3}$ and $\frac{n_1-1}{m_1} = \frac{1}{2} < \frac{n_2}{m_2} = \frac{2}{3}$, so we are in the third case of Theorem 2.3, which implies that

$$\text{(37)} \qquad N_D(\tau) = \zeta\left(A_2^{\frac{3}{4}}, 1\right) \tau + \mathcal{O}\left(\tau^{\frac{2}{3}}\right).$$

It is immediate to observe that

$$\text{(38)} \quad \sigma(D) = \{\left(k^2 + k + 1\right)\left(n^2 + 1\right)^{3/4} \mid$$
$$\mathbf{mult}\left((k^2 + k + 1)(n^2 + 1)^{3/4}\right) = 2\left(2k + 1\right)\}.$$

Therefore, using again (25), we obtain

$$N_D(\tau) = \sum_{\substack{(k^2-k+1)(n^2+1)^{3/4}<\tau}}^{n\in\mathbb{N}, k\in\mathbb{N}} 2\left(2k+1\right)$$

$$= 2 \sum_{\substack{(k^2-k+1)<\frac{\tau}{(n^2+1)^{3/4}}}}^{n\in\mathbb{N}, k\in\mathbb{N}} \mathbf{mult}\left(k^2 - k + 1\right)$$

$$= 2 \sum_{\substack{(n^2+1)^{3/4}<\tau}}^{n\in\mathbb{N}} N_{A_1}\left(\frac{\tau}{(n^2+1)^{3/4}}\right)$$

$$\text{(39)} \qquad = 2 \sum_{\substack{(n^2+1)^{3/4}<\tau}}^{n\in\mathbb{N}} \left(\frac{\tau}{(n^2+1)^{3/4}} + R\left(\frac{\tau}{(n^2+1)^{3/4}}\right)\right).$$

Let us now compute directly

$$\limsup_{\tau\to+\infty} \frac{\left|N_D(\tau) - \zeta\left(A_2^{3/4}, 1\right)\tau\right|}{\tau^{2/3}}.$$

By (39), we find

$$\limsup_{\tau\to+\infty} \frac{\left|N_D(\tau) - \zeta(A_2^{3/4}, 1)\tau\right|}{\tau^{2/3}}$$

$$= \limsup_{\tau\to+\infty} \frac{\left|2\sum_{(n^2+1)^{3/4}<\tau}\left(\frac{\tau}{(n^2+1)^{3/4}} + R\left(\frac{\tau}{(n^2+1)^{\frac{3}{4}}}\right)\right) - \zeta\left(A_2^{3/4}, 1\right)\tau\right|}{\tau^{2/3}}$$

$$= \limsup_{\tau\to+\infty} \tau^{-2/3} \cdot \left| 2\sum_{(n^2+1)^{3/4}<\tau} \frac{\tau}{(n^2+1)^{3/4}} - \zeta\left(A_2^{3/4}, 1\right)\tau + \right.$$

$$\left. + 2\sum_{(n^2+1)^{3/4}<\tau} R\left(\frac{\tau}{(n^2+1)^{\frac{3}{4}}}\right)\right|.$$

We also notice that

$$\lim_{\tau \to +\infty} \frac{\left|2\sum_{(n^2+1)^{3/4}<\tau} \frac{\tau}{(n^2+1)^{3/4}} - \zeta\left(A_2^{3/4},1\right)\tau\right|}{\tau^{2/3}}$$

$$= \lim_{\tau \to +\infty} \frac{\zeta\left(A_2^{3/4},1\right)\tau - 2\sum_{(n^2+1)^{3/4}<\tau} \frac{\tau}{(n^2+1)^{3/4}}}{\tau^{2/3}}$$

$$= \lim_{\tau \to +\infty} 2\,\tau^{1/3} \sum_{(n^2+1)^{3/4}\geq\tau} \frac{1}{(n^2+1)^{3/4}},$$

and that

$$\sum_{(n+1)^{3/2}\geq\tau} \frac{1}{(n+1)^{3/2}} \leq \sum_{(n^2+1)^{3/4}\geq\tau} \frac{1}{(n^2+1)^{3/4}} \leq \sum_{n^{3/2}\geq\tau} \frac{1}{n^{3/2}}.$$

Using the standard integral criteria of series convergence, one can easily check that

$$\lim_{\tau \to +\infty} \tau^{1/3} \sum_{(n+1)^{3/2}\geq\tau} \frac{1}{(n+1)^{3/2}} = \lim_{\tau \to +\infty} \tau^{1/3} \sum_{n^{3/2}\geq\tau} \frac{1}{n^{3/2}} = 2.$$

Hence

$$(40) \qquad \lim_{\tau \to +\infty} 2\,\tau^{1/3} \sum_{(n^2+1)^{3/4}\geq\tau} \frac{1}{(n^2+1)^{3/4}} = 4.$$

By a similar argument, we also have that

$$(41) \qquad \lim_{\tau \to +\infty} \tau^{-1/6} \sum_{(n^2+1)^{3/4}<\tau} \frac{1}{(n^2+1)^{3/8}} = 4.$$

In view of (27), (40) and (41) we finally obtain

$$\limsup_{\tau \to +\infty} \frac{\left|N_D(\tau) - \zeta(A_2^{3/4},1)\tau\right|}{\tau^{2/3}}$$

$$\geq \limsup_{\tau \to +\infty} \frac{N_D(\tau) - \zeta(A_2^{3/4},1)\tau}{\tau^{2/3}}$$

$$= -\lim_{\tau \to +\infty} 2\,\tau^{1/3} \sum_{(n^2+1)^{3/4}\geq\tau} \frac{1}{(n^2+1)^{3/4}}$$

$$+ \limsup_{\tau \to +\infty} 2 \frac{\sum_{(n^2+1)^{3/4}<\tau} R\left(\frac{\tau}{(n^2+1)^{3/4}}\right)}{\tau^{2/3}}$$

$$\geq -4 + \frac{3}{2} \limsup_{\tau \to +\infty} \tau^{-1/6} \sum_{(n^2+1)^{3/4}<\tau} \frac{1}{(n^2+1)^{3/8}}$$

$$(42) \qquad \geq -4 + 6 = 2 > 0.$$

Equation (42) proves the desired result also in this last case.

5. Appendix. The Dirichlet divisors problem

Counting functions of the type (2) suggest a spectral approach to a prominent type of lattice problem, the so-called Dirichlet divisors problem. Let us suppose that the spectrum of both A_1 and A_2 in (2) is formed by all strictly positive natural numbers, each with multiplicity one. Then,

$$N_A(\tau) = \sum_{n \cdot m < \tau} 1 = D(\tau).$$

The function $D(\tau)$ is called Dirichlet divisor summatory function and it is straightforward to check that it amounts the number of points with integer coordinates belonging to the first quadrant of the Cartesian plane which lie below the hyperbola $xy = \tau$. In 1849, Dirichlet proved that

$$(43) \qquad D(\tau) = \tau \log \tau + (2\gamma - 1)\tau + \mathcal{O}(\tau^{1/2}),$$

where γ is the Euler-Mascheroni constant, namely

$$\gamma = \lim_{\tau \to +\infty} \left(\sum_{0 < n < \tau} \frac{1}{n} - \int_0^\tau \frac{1}{x} dx \right),$$

or, equivalently,

$$\gamma = \lim_{z \to 1} (z - 1) \zeta_R(z),$$

where $\zeta_R(z)$ is the Riemann ζ-function. Several papers aimed at finding the sharp remainder term in (43), see [IKKN06] for an overview on this type of problems. Hardy, in [Har16], proved that $\mathcal{O}(\tau^{\frac{1}{4}})$ is a lower bound for the remainder in (43). It is conjectured that the sharp estimate in this case is $\mathcal{O}(\tau^{\frac{1}{4}+\epsilon})$ or, more precisely, $\mathcal{O}\left(\tau^{1/4} \log \tau\right)$. The best known result, due to Huxley, is that the remainder is $\mathcal{O}(\tau^\alpha (\log \tau)^{\beta+1})$, where

$$\alpha = \frac{131}{416} \sim 0,3149\ldots \quad \beta = \frac{18627}{8320} \sim 2,2513\ldots.$$

In order to have a spectral interpretation of the Dirichlet divisor problem, a global bisingular calculus based on Shubin calculus has been introduced in [BGRP13]. Then, the following Hermite-type operator

$$H_j = \frac{1}{2} \left(-\partial_{x_j}^2 + x_j^2 \right) + \frac{1}{2}, \quad j = 1, 2,$$

has been examined. Using Hermite polynomials, it turns out that $\sigma(H_j) = \{n\}_{n \in \mathbb{N}^*}$, $j = 1, 2$, and each eigenvalue has multiplicity one. Therefore $\sigma(H_1 \otimes H_2) = \{n \cdot m\}_{(n,m) \in (\mathbb{N}^*)^2}$ and

$$N_{H_1 \otimes H_2}(\tau) = D(\tau).$$

This clear spectral meaning of the Dirichlet divisor problem was one of the main motivation of the papers [BGRP13, GPRVar]. For the connection between Dirichlet divisor problem and standard bisingular operators on the product of closed manifolds see [Bat12]. Actually, since we deal with the non-symmetric case, it is not possible to attack directly the traditional Dirichlet divisor problem through the approach described in the previous sections,

while our techniques are well suited to treat generalized anisotropic Dirichlet divisors problems like, for instance,

$$N_{H_1^\alpha \otimes H_2^\beta}(\tau) = \sum_{n^\alpha \cdot m^\beta < \tau} 1, \quad \alpha \neq \beta.$$

In [GPRVar] it is proven that

$$(44) \qquad N_{H_1^\alpha \otimes H_2^\beta}(\tau) = \zeta\left(\frac{\alpha}{\beta}\right)\tau^{\frac{1}{\beta}} + \zeta\left(\frac{\beta}{\alpha}\right)\tau^{\frac{1}{\alpha}} + \mathcal{O}\left(\tau^{\frac{1}{\alpha+\beta}}\right),$$

where ζ is the meromorphic continuation of the Riemann ζ-function. Notice that (44) proves the sharpness of (3) in the case $\frac{n_2}{m_2} > \frac{n_1 - 1}{m_1}$.

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