Solution theory to semilinear hyperbolic stochastic partial differential equations with polynomially bounded coefficients

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Abstract

We study mild solutions of a class of stochastic partial differential equations, involving operators with polynomially bounded coefficients. We consider semilinear equations under suitable hyperbolicity hypotheses on the linear part. We provide conditions on the initial data and on the stochastic terms, namely, on the associated spectral measure, so that mild solutions exist and are unique in suitably chosen functional classes. More precisely, function-valued solutions are obtained, as well as a regularity result.

Keywords: Semilinear stochastic hyperbolic partial differential equations, Variable coefficients, Fourier integral operators
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1. Introduction

The stochastic partial differential equations (SPDEs in the sequel) that we consider in the present paper are of the general form

$$L(t, x, \partial_t, \partial_x)u(t, x) = \gamma(t, x, u(t, x)) + \sigma(t, x, u(t, x))\dot{\Xi}(t, x), \qquad (1.1)$$

where L is a linear partial differential operator that contains derivatives with respect to time ($t \in \mathbb{R}$) and space ($x \in \mathbb{R}^d$, $d \geq 1$) variables, γ and σ, respectively the drift term and the diffusion coefficient, are real-valued functions, subject to certain regularity conditions, Ξ is a random noise term white in time and colored in space, and u is an unknown stochastic process called *solution* of the SPDE. The equations (1.1) are semilinear: the only possible non-linearities are

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on the right-hand side, and not in the operator L. In Subsection 1.1 below we will describe in more detail the conditions we impose on the operator L, the most important one being (a notion of) hyperbolicity; in Subsection 1.2 we will describe in detail the noise we consider.

Since the sample paths of the solution u are in general not in the domain of the operator L, in view of the singularity of the random noise, we rewrite (1.1) in its corresponding integral (i.e., *weak*) form and look for *mild solutions of* (1.1), that is, stochastic processes $u(t,x)$ satisfying

$$\begin{aligned} u(t,x) = v_0(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y,u(s,y))dyds \\ &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y,u(s,y))\dot{\Xi}(s,y)dyds, \end{aligned} \tag{1.2}$$

where:

- v_0 is a deterministic term, taking into account the initial conditions;
- Λ is a suitable kernel, associated with the fundamental solution of the linear partial differential equation (linear PDE in the sequel) $Lu = 0$;
- the first integral in (1.2) is of deterministic type, while the second is a stochastic integral.

Note that both integrals in (1.2) contain a slight abuse of notation, since $\Lambda(t,s,x,y)$ is, in general, a distribution with respect to the variables $(x,y) \in \mathbb{R}^{2d}$. Given the commonly wide usage of such so-called *distributional integrals*, we will also often adopt here this notation in the representation of our class of mild solutions to (1.1).

The kind of solution u we can construct for equation (1.1) depends on the approach we employ to make sense of the stochastic integral appearing in (1.2). In the present paper we follow the Da Prato-Zabczyk approach (see [19]), which consists in associating an Hilbert space valued Brownian motion with the random noise. One can then define the stochastic integral as an infinite sum of Itô integrals with respect to one-dimensional Brownian motions. This leads to solutions involving random functions taking values in suitable functional spaces. To our best knowledge, the most general result of existence and uniqueness of a function-valued solution to hyperbolic SPDEs is given in [28], where the author considers a semilinear stochastic wave equation having a uniformly elliptic second order operator A in place of the Laplacian, with uniformly bounded coefficients depending on $x \in \mathbb{R}^d$, $d \geq 1$. There, sufficient conditions on the stochastic term $\dot{\Xi}$ and on the coefficients of A are given, in order to find a unique function-valued solution using semigroup theory. In the present paper we show existence and uniqueness of a function-valued solution to a wider class of *semilinear weakly hyperbolic* SPDEs, with *possibly unbounded coefficients* depending on $(t,x) \in [0,T] \times \mathbb{R}^d$, $d \geq 1$, see Subsection 1.1 below.

We recall that an alternative approach to give meaning to (1.1) is the one by Walsh and Dalang (see [10, 17, 34]), where the stochastic integral in (1.2) is defined as a stochastic integral with respect to a martingale measure derived from the random noise $\dot{\Xi}$. With this alternative approach one obtains a so-called *random-field solution*, that is, a solution u defined as a map associating a random variable to each $(t,x) \in [0,T_0] \times \mathbb{R}^d$, where $T_0 > 0$ is the time horizon of the equation. It is well known that in many cases the two approaches lead to the same solution u (in some sense) of an SPDE, see [18] for a precise comparison.

In [2, 7] we have constructed random-field solutions for arbitrary order, *linear* weakly hyperbolic SPDEs with possibly unbounded coefficients, smoothly depending on $(t,x) \in [0,T] \times \mathbb{R}^d$. That construction cannot work for non-linear equations of the form (1.1). Indeed, the stationarity condition $\Lambda = \Lambda(t-s, x-y)$ would be needed, but such condition (fulfilled by SPDEs with constant coefficients) cannot be assumed if we want to deal with general linear operators L in (1.1), that is, admitting variable coefficients. We conclude comparing the function-valued solutions to (1.1) obtained in the present paper, in the special case of the linear equations, with the random-field solutions of the same equation found in [2].

We remark that in the present paper, as well as in [2, 7], the main tools used to construct and study the solutions, namely, pseudodifferential and Fourier integral operators, come from microlocal analysis, within the so-called *SG* (or *scattering*) calculus (see [12, 21, 27]). To our best knowledge, in [7] their full potential has been rigorously applied for the first time within the solution theory of hyperbolic SPDEs. Other applications of these operators in the context of S(P)DEs can be found in [33], where S(P)DEs are investigated in the framework of function-valued solutions by means of pseudodifferential operators, and in [25], where a program for employing Fourier integral operators in stochastic structural analysis is described. We are not aware of any other systematic application of microlocal and Fourier integral operators techniques. In particular, concerning the analysis of weakly semilinear hyperbolic SPDEs with unbounded coefficients, we provide it here. As it is customary for the classes of the associated deterministic PDEs, we are interested in both the smoothness, as well as the decay at spatial infinity, of the solutions. Here we prove an analog of such *global regularity* properties, employing suitable *weighted Sobolev spaces*, namely, the so-called Sobolev-Kato spaces.

1.1. The equations we consider

As mentioned above, we study semilinear SPDEs (1.1) whose partial differential operators L have coefficients in $(t,x) \in [0,T] \times \mathbb{R}^d$ that may admit a polynomial growth as $|x| \to \infty$. Namely, we treat *hyperbolic equations* of arbitrary order $m \in \mathbb{N}$ of the form (1.1), whose coefficients are defined on the whole space $\mathbb{R}^d$, with

$$L = D_t^m - \sum_{j=1}^{m} A_j(t,x,D_x) D_t^{m-j}, \qquad A_j(t,x,D) = \sum_{|\alpha| \le j} a_{\alpha j}(t,x) D_x^\alpha, \quad (1.3)$$

where $m \geq 1$, $a_{\alpha j} \in C^\infty([0,T], C^\infty(\mathbb{R}^d))$ for $|\alpha| \leq j$, $j = 0, \dots, m$, and, for all $k \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^d$, there exists a constant $C_{jk\alpha\beta} > 0$ such that

$$|\partial_t^k \partial_x^\beta a_{\alpha j}(t,x)| \leq C_{jk\alpha\beta} \langle x \rangle^{|\alpha|-|\beta|}, \tag{1.4}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $0 \leq |\alpha| \leq j$, $1 \leq j \leq m$, where $\langle x \rangle := \sqrt{1+|x|^2}$. The hyperbolicity of L means that the symbol $\mathcal{L}_m(t,x,\tau,\xi)$ of the SG-principal part of L, defined here below, satisfies

$$\mathcal{L}_m(t,x,\tau,\xi) := \tau^m - \sum_{j=1}^m \sum_{|\alpha|=j} a_{\alpha j}(t,x) \xi^\alpha \tau^{m-j} = \prod_{j=1}^m \left(\tau - \tau_j(t,x,\xi)\right), \tag{1.5}$$

with $\tau_j(t,x,\xi)$ real-valued, $\tau_j \in C^\infty([0,T]; S^{1,1}(\mathbb{R}^d))$, $j = 1, \dots, m$. The latter means that, for any $\alpha, \beta \in \mathbb{N}_0^d$, $k \in \mathbb{N}_0$, there exists a constant $C_{jk\alpha\beta} > 0$ such that

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta \tau_j(t,x,\xi)| \leq C_{jk\alpha\beta} \langle x \rangle^{1-|\alpha|} \langle \xi \rangle^{1-|\beta|}, \tag{1.6}$$

for $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2d}$, $j = 1, \dots, m$; we shall refer to (1.6) saying that $\tau_j(t)$ is a symbol of class $S^{1,1}(\mathbb{R}^{2d})$, see Section 3 below for the precise definition of the so-called SG-classes of symbols $S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu) \in \mathbb{R}^2$, and the corresponding class of pseudodifferential operators. The real solutions $\tau_j = \tau_j(t,x,\xi)$, $j = 1, \dots, m$, of the equation $\mathcal{L}_m(t,x,\tau,\xi) = 0$ with respect to τ are usually called *characteristic roots* of the operator L.

Definition 1.1. We say that (1.3) is *weakly hyperbolic with roots of constant multiplicities* if the real-valued characteristic roots in (1.5) can be divided into n groups ($1 \leq n \leq m$) of distinct and separated roots, in the sense that, possibly after a reordering of the τ_j, $j = 1, \dots, m$, there exist $l_1, \dots l_n \in \mathbb{N}$ with $l_1 + \ldots + l_n = m$ and n sets

$$G_1 = \{\tau_1 = \cdots = \tau_{l_1}\},\ G_2 = \{\tau_{l_1+1} = \cdots = \tau_{l_1+l_2}\},\ \ldots\ G_n = \{\tau_{m-l_n+1} = \cdots = \tau_m\},$$

satisfying, for a constant $C > 0$,

$$\tau_j \in G_p, \tau_k \in G_q,\ p \neq q,\ 1 \leq p,q \leq n \Rightarrow |\tau_j(t,x,\xi) - \tau_k(t,x,\xi)| \geq C \langle x \rangle \langle \xi \rangle \tag{1.7}$$

for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2d}$. The number $l = \max_{j=1,\dots,n} l_j$ is the *maximum multiplicity of the roots of* $\mathcal{L}_m$.

Notice that, in the case $n = 1$, we have only one group of m coinciding roots, that is, $\mathcal{L}_m$ admits a single real root of multiplicity m, while for $n = m$ we say that the operator is strictly hyperbolic; the most famous example of a strictly hyperbolic operator is given by the wave operator.

Example 1.2. An example of a weakly hyperbolic operator L with roots of constant multiplicities is given by

$$L = (D_t^2 - \langle x \rangle^2 \langle D \rangle^2)^2 = D_t^4 - 2\langle x \rangle^2 \langle D \rangle^2 D_t^2 + \langle x \rangle^4 \langle D \rangle^4 + \mathrm{Op}(p), \qquad x \in \mathbb{R}^d,$$

$p \in S^{3,3}(\mathbb{R}^d)$, where, for $c \in S^{m,\mu}(\mathbb{R}^d)$, $\mathrm{Op}(c)$ denotes the pseudodifferential operator with symbol c, see Section 3. The SG-principal symbol of L is here $L_4(x,\tau,\xi) = (\tau^2 - \langle x\rangle^2\langle\xi\rangle^2)^2$, with *separated* roots $\tau_\pm(x,\xi) = \pm\langle x\rangle\langle\xi\rangle$, both of *multiplicity 2.*

Definition 1.3. We say that (1.3) is *weakly hyperbolic with involutive roots* if the real-valued characteristic roots in (1.5) satisfy

$$\begin{aligned}[D_t - \mathrm{Op}(\tau_j(t)), D_t - \mathrm{Op}(\tau_k(t))] = \quad & \mathrm{Op}(a_{jk}(t))\,(D_t - \mathrm{Op}(\tau_j(t)) \\ + \quad & \mathrm{Op}(b_{jk}(t))\,(D_t - \mathrm{Op}(\tau_k(t))) + \mathrm{Op}(c_{jk}(t)),\end{aligned} \tag{1.8}$$

for some $a_{jk}, b_{jk}, c_{jk} \in C^\infty([0,T], S^{0,0}(\mathbb{R}^d))$, $j,k = 1,\ldots,m$.

Remark 1.4. Recall that roots of constant multiplicities are always involutive, see, e.g., [2] for a proof. The converse statement is not true in general, as shown in [24]: the operator

$$L = (D_t + tD_{x_1} + D_{x_2})(D_t - (t - 2x_2)D_{x_1}), \qquad x \in \mathbb{R}^2,$$

is a weakly hyperbolic operator with involutive roots of non-constant multiplicities.

1.2. The stochastic noise

Here we describe the class of stochastic noises that we allow in our framework. Consider a distribution-valued Gaussian process $\{\Xi(\phi);\ \phi \in \mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with mean zero and covariance functional given by

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} \big(\phi(t) * \tilde{\psi}(t)\big)(x)\,\Gamma(dx)dt, \tag{1.9}$$

where $\widetilde{\psi}(t,x) := \psi(t,-x)$, $*$ is the convolution operator and Γ is a nonnegative, nonnegative definite, tempered measure on $\mathbb{R}^d$. Then, Théorème XVIII in [31, Chapter VII] implies that there exists a nonnegative tempered measure μ on $\mathbb{R}^d$ such that $\mathcal{F}\mu = \widehat{\mu} = \Gamma$. $\mathcal{F}$ and $\widehat{\ }$ denote the Fourier transform given, for functions $f \in L^1(\mathbb{R}^d)$, by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)dx. \tag{1.10}$$

In (1.10), $x \cdot \xi$ denotes the inner product in $\mathbb{R}^d$, and the Fourier transform is extended to tempered distributions $T \in \mathcal{S}'(\mathbb{R}^d)$ by the relation $\langle \mathcal{F}T, \phi\rangle = \langle T, \mathcal{F}\phi\rangle$, for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. By Parseval's identity, the right-hand side of (1.9) can be rewritten as

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} [\mathcal{F}\phi(t)](\xi) \ \cdot \overline{[\mathcal{F}\psi(t)](\xi)}\,\mu(d\xi)dt.$$

The tempered measure Γ is usually called *correlation measure*. The tempered measure μ such that $\Gamma = \widehat{\mu}$ is usually called *spectral measure*.

1.3. The results we get

We consider the SPDE (1.1) with L as in (1.3), (1.5),(1.7) and Ξ an $\mathcal{S}'(\mathbb{R}^d)$-valued Gaussian process with correlation measure Γ and spectral measure μ ad described here above. We derive conditions on the coefficients of L, on the right-hand side terms γ and σ, and on the spectral measure μ (hence, on Ξ), such that there exists a unique function-valued (mild) solution to the corresponding Cauchy problem. The Cauchy data are going to be taken in Sobolev-Kato spaces

$$H^{z,\zeta}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^n) \colon \|u\|_{z,\zeta} = \|\langle\cdot\rangle^z \langle D\rangle^\zeta u\|_{L^2} < \infty\}, \quad (z,\zeta) \in \mathbb{R}^2. \tag{1.11}$$

The coefficients γ, σ will be chosen in suitable classes of Lipschitz functions, denoted by $\mathrm{Lip}_{\mathrm{loc}}(z,\zeta,r,\rho)$. Namely, for suitable $z,\zeta,r,\rho \in \mathbb{R}$, $r,\rho \geq 0$, we say that a function g belongs to $\mathrm{Lip}(z,\zeta,r,\rho)$ if it is measurable and satisfies, for every $t \in [0,T]$,

$$\begin{aligned} \|g(t,\cdot,w)\|_{z,\zeta} &\leq C(t)(1 + \|w\|_{z+r,\zeta+\rho}) \quad \forall w \in H^{z+r,\zeta+\rho}(\mathbb{R}^d), \\ \|g(t,\cdot,w) - g(t,\cdot,v)\|_{z,\zeta} &\leq C(t)\|w - v\|_{z+r,\zeta+\rho} \quad \forall w,v \in H^{z+r,\zeta+\rho}(R^d). \end{aligned}$$

More generally, we say that $g \in \mathrm{Lip}_{\mathrm{loc}}(z,\zeta,r,\rho)$ if the stated properties hold true for $w,v \in U$, with U a suitable open subset of $H^{z+r,\zeta+\rho}(\mathbb{R}^d)$. The precise description of the assumptions on σ and γ are postponed to Section 4, while we immediately give two examples of diffusion coefficients σ which fulfill the requested hypotheses.

Example 1.5. Let $\sigma(t,x,u) = u^2$. Then, σ is an admissible non-linearity for the equations we consider. More generally, we allow $\sigma(t,x,u) = u^n$, $n \in \mathbb{N}$, $n > 2$.

Example 1.6. A right-hand side explicitly depending on $(t,x) \in [0,T] \times \mathbb{R}^d$ and u, which is admissible for the equations we consider, is

$$\sigma(t,x,u) = \langle x\rangle^{l-m} \cdot \widetilde{\sigma}(t,u), \tag{1.12}$$

where l is the maximum multiplicity of the roots and $\widetilde{\sigma}$ is regular in time, satisfies suitable mapping properties with respect to the Sobolev-Kato spaces, and is (uniformly, locally) Lipschitz-continuous with respect to the second variable, see Definition 4.2 and Example 4.13 below for the precise conditions.

To our best knowledge, a diffusion coefficient of the rather general form (1.12) has never been sistematically treated in the literature, except in [30], where, for $m = 2$, it has been incorporated in a certain model equation by means of ad-hoc techniques.

Example 1.7. More generally, a routine extension of the theory developed in the present paper allows for a stochastic term of the very general form

$$\sigma(t,x,u,D_x u,\dots,D_x^\alpha u), \qquad |\alpha| \leq m - 1$$

in the right-hand side of (1.1). The only difference consists in the form of the lipschitzianity assumptions and the corresponding mapping properties, see again Section 4.

We state here below the main result of the paper, whose precise formulation is given in Theorem 4.8. As customary for weakly hyperbolic operators, to achieve well-posedness we need to assume that the lower order terms of L satisfy (an adapted form of) a Levi condition (see (A.24) and Corollary A.13). This allows to give an explicit expression for the distribution $\Lambda(t,s)$ in terms of kernels of suitable Fourier integral operators, see (A.26). We work under an hypothesis of Lipschitz continuity for the nonlinearities in the right-hand side (see Definition 4.2 and Remark 4.3).

Main Theorem. *Consider the Cauchy problem for the SPDE* (1.1) *with L a weakly hyperbolic operator with roots of constant multiplicity, that is, L satisfies* (1.3)*,* (1.5)*,* (1.7)*. Assume, for the spectral measure associated with Ξ, that*

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi)<\infty, \tag{1.13}$$

where l is the maximum multiplicity of the roots of $\mathcal{L}_m$, $1\leq l\leq m$. Moreover, assume that L is of Levi type and that $\gamma,\sigma\in\mathrm{Lip}_{\mathrm{loc}}(z,\zeta,m-l,0)$, $z,\zeta\in\mathbb{R}$. Then, there exists a time horizon $0<T_0\leq T$ such that, for any choice of $u_j\in H^{z+m-1-j,\zeta+m-1-j}(\mathbb{R}^d)$, $0\leq j\leq m-1$, the Cauchy problem admits a unique solution $u\in L^2([0,T_0]\times\Omega,H^{z+m-l,\zeta}(\mathbb{R}^d))$ satisfying (1.2)*, where the first integral is a Bochner integral, and the second integral is understood as the stochastic integral of a suitable $H^{z+m-l,\zeta}(\mathbb{R}^d)$-valued stochastic process with respect to the stochastic noise Ξ.*

Notice that the more general are the assumptions on L (i.e., the larger is l), the smallest is the class of the stochastic noises that we can allow to get a function-valued solution. Our main Theorem extends the results of [28] to the case of general higher order hyperbolic equations with coefficients in (t,x), not uniformly bounded with respect to x and with roots that may coincide.

Remark 1.8. In Corollary 4.10 we explicitly write the result we get in the limit case $l=1$, corresponding to strictly hyperbolic equations. We remark that in this case L automatically satisfies the Levi condition. Moreover, when $m=2$, $l=1$, and Γ is absolutely continuous, condition (1.13) reduces to the well-known condition $\int_{\mathbb{R}^d}\frac{1}{1+|\xi|^2}\mu(d\xi)<\infty$, needed for existence and uniqueness of a solution to the stochastic wave equation.

We conclude the paper with a result concerning operators with involutive characteristics. We show that

> *if L is weakly hyperbolic with involutive roots and $\displaystyle\int_{\mathbb{R}^d}\mu(d\xi)<\infty$, then, under suitable assumptions on γ,σ and the Cauchy data, there exists a unique function-valued solution to the Cauchy problem associated with the SPDE* (1.1)*,*

see Theorem 4.14 for the precise statement. Notice that the condition on the spectral measure for the latter case coincides with (1.13) in the case $l=m$, and that all such conditions coincide when $m=1$.

1.4. *Tools we employ*

The main tools for proving existence and uniqueness of solutions to (1.1) will be the calculus of Fourier integral operators with symbols in the so-called *SG* classes. Such symbols classes have been introduced in the '70s by H.O. Cordes (see, e.g. [12]) and C. Parenti [27] (see also the *scattering calculus* by R. Melrose, e.g. [21]).

Applications of the *SG* FIOs theory to *SG*-hyperbolic Cauchy problems were initially given in [14, 16]. Many authors have, since then, expanded the *SG* FIOs theory and its applications to the solution of hyperbolic problems in various directions. To mention a few, see, e.g., M. Ruzhansky, M. Sugimoto [29], E. Cordero, F. Nicola, L Rodino [11], and the references quoted there and in [5].

In [5], Cauchy problems for general *SG*-hyperbolic first order systems have been studied, constructing their fundamental solution $\{E(t,s)\}_{0\leq s\leq t\leq T}$. The existence of the fundamental solution provides, via Duhamel's formula, existence and uniqueness of the solution to the system, for any given Cauchy data in the weighted Sobolev spaces $H^{z,\zeta}(\mathbb{R}^d)$, $(z,\zeta)\in\mathbb{R}^2$. A remarkable feature, typical for these classes of hyperbolic problems, is the *well-posedness with loss of decay/increase of growth at infinity*, see [3, 4, 16].

There are various techniques to switch from a Cauchy problem for an *SG*-hyperbolic operator L of order $m \geq 2$ to a Cauchy problem for a first order system, see, e.g., [1, 12, 14, 24]. In the approach we follow here, which is the same used in [1, 16], one of the key results for this aim is an adapted version of the so-called Mizohata Lemma of Perfect Factorization, see Proposition A.12 and Lemma A.15 in the Appendix[1]. To construct the fundamental solution of the operator L involved in (1.1), through the fundamental solution of the associated first order system, we need, on one hand, to perform compositions between pseudo-differential operators and Fourier integral operators of *SG* type, using the theory developed in [13], and, on the other hand, compositions between Fourier integral operators of *SG* type with possibly different phase functions. The latter can be achieved using the composition results obtained in [5]. The proof of the main theorems of the paper employs such fundamental solution, together with the application of a fixed point scheme in suitable functional spaces.

1.5. *Organization of the paper*

To provide a presentation of our results as self-contained as possible, for the convenience of the reader, we provide (at different levels of detail) various preliminaries from the existing literature, as described below.

In Section 2 we recall some notions about stochastic integration with respect to Hilbert space-valued processes and the corresponding concept of function-valued solution, following [19].

[1] See also [20, 22, 23], for the original version of such results.

In Section 3 we give a description of the tools coming from microlocal analysis that we use for the construction of the fundamental solution of weakly hyperbolic with polynomially bounded coefficients.

In Section 4 we focus on the semilinear hyperbolic SPDE (1.1), (1.3), (1.5), and in Theorem 4.8 we study existence and uniqueness of a function-valued solution under the assumption of weak hyperbolicity with roots of constant multiplicity (1.7). Notice again that the case of strict hyperbolicity (the one of the waves) reduces to the special case $l = 1$ of Theorem 4.8, and needs no Levi condition. We give sufficient conditions on the coefficients, on the noise and on the right-hand side of (1.1) such that there exists a unique mild function-valued solution of the corresponding Cauchy problem. The key result to achieve existence and uniqueness of the solution is Lemma 4.6, which is a further main result in the present paper. We also prove, in Theorem 4.14, a similar result under the assumption of weak hyperbolicity with involutive roots (1.8). Finally, we make a comparison between the function-valued solutions obtained here, in the special case of linear equations, with the random-field solutions found in [2].

Some additional details about the tools we employ, coming from the microlocal approach to the solution of hyperbolic Cauchy problems for PDEs and systems associated with operators with polynomially bounded coefficients, see [2, 5, 12, 13, 14, 16], are summarized in the Appendix.

1.6. Notation

Throughout this article, we let $\langle a\rangle := (1+|a|^2)^{1/2}$ for all $a \in \mathbb{R}^d$, and we denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}^d_* := \mathbb{R}^d \backslash \{0\}$. Also, α and β will generally denote multiindeces, with their standard arithmetic operations. As usual, we will denote partial derivatives with ∂, and set $D = -\mathrm{i}\partial$, i being the imaginary unit, which is convenient when dealing with Fourier transformations. We will denote by $C^m(X)$, $C^m_0(X)$, $\mathcal{S}(X)$, $\mathcal{D}(X)$, $\mathcal{S}'(X)$ and $\mathcal{D}'(X)$, the m-times continuously differentiable functions, the m-times continuously differentiable functions with compact support, the Schwartz functions, the test functions space $C^\infty_0(X)$, the tempered distributions and the distributions on some finite or infinite-dimensional space X, respectively. Usually, $C > 0$ will denote a generic constant, whose value can change from line to line without further notice. When operator composition is considered, we will usually insert the symbol $\circ$ when the notation $\mathrm{Op}(b)$ and/or $\mathrm{Op}_\varphi(a)$, for pseudodifferential and Fourier integral operators, respectively, are adopted for both factors, as well as in some situations where parameter-dependent operators occurs, for the sake of clarity. When at least one of the operators involved in the product of composition is denoted by a single capital letter, and when no confusion can occur, we will, as customary, omit the symbol $\circ$ completely, and just write, e.g., PQ, RD_t, etc. Finally, $A \asymp B$ means that the estimates $A \lesssim B$ and $B \lesssim A$ hold true, where $A \lesssim B$ means that $|A| \leq c \cdot |B|$, for a suitable constant $c > 0$.

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2. Stochastic integration.

The mild formulation (1.2) is the way in which we understand the SPDE (1.1). In fact, we call *(mild) function-valued solution to* (1.1) an $L^2(\Omega)$-family of random variables $u(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}^d$, jointly measurable, satisfying the stochastic integral equation (1.2) where the last term in the right-hand side is understood within the theory of stochastic integrals taking value in Hilbert spaces.

In this section we recall some of the main results of the theory of stochastic integration with respect to cylindrical Wiener processes. Also, we recall the definition of the Hilbert space $\mathcal{H}$ which will be suitable for our purposes of function-valued solutions to SPDEs. For the latter, we follow the exposition in [18].

Definition 2.1. Let Q be a self-adjoint, nonnegative definite and bounded linear operator on a separable Hilbert space H. An H-valued stochastic process $W = \{W_t(h); h \in H, t \geq 0\}$ is called a *cylindrical Wiener process on H* on the complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ if the following conditions are fulfilled:

1. for any $h \in H$, $\{W_t(h); t \geq 0\}$ is a one-dimensional Brownian motion with variance $t\langle Qh, h\rangle_H$;
2. for all $s, t \geq 0$ and $g, h \in H$,

$$\mathbb{E}[W_s(g)W_t(h)] = (s \wedge t)\langle Qg, h\rangle_H.$$

If $Q = Id_H$, then W is called a standard cylindrical Wiener process.

Let $\mathscr{F}_t$ be the σ-field generated by the random variables $\{W_t(h); 0 \leq s \leq t, h \in H\}$ and the $\mathbb{P}$-null sets. The predictable σ-field is then the σ-field in $[0,T] \times \Omega$ generated by the sets $\{(s,t] \times A, A \in \mathscr{F}_t, 0 \leq s < t \leq T\}$.

We define H_Q to be the completion of the Hilbert space H endowed with the inner product

$$\langle g, h\rangle_{H_Q} := \langle Qg, h\rangle_H,$$

for $g, h \in H$. In the sequel, we let $\{v_k\}_{k\in\mathbb{N}}$ be a complete orthonormal basis of H_Q. Then, the stochastic integral of a predictable, square-integrable stochastic process with values in H_Q, $u \in L^2([0,T]\times\Omega; H_Q)$, is defined as

$$\int_0^t u(s)dW_s := \sum_{k\in\mathbb{N}} \langle u, v_k\rangle_{H_Q} dW_s(v_k).$$

In fact, the series in the right-hand side converges in $L^2(\Omega, \mathscr{F}, \mathbb{P})$ and its sum does not depend on the chosen orthonormal system $\{v_k\}_{k\in\mathbb{N}}$. Moreover, the Itô isometry

$$\mathbb{E}\Big[\Big(\int_0^t u(s)dW_s\Big)^2\Big] = \mathbb{E}\Big[\int_0^t \|u(s)\|^2_{H_Q} ds\Big]$$

holds true for any $u \in L^2([0,T]\times\Omega; H_Q)$. For more on one-dimensional integration, see, e.g., [26].

This notion of stochastic integral can also be extended to operator-valued integrands. Let U be a separable Hilbert space and define $L_2^0 := L_2(H_Q, U)$ the set of Hilbert-Schmidt operators from H_Q to U. With this we can define the space of integrable processes (with respect to W) as the set of $\mathscr{F}$-measureable processes in $L^2([0,T]\times\Omega; L_2^0)$. Since one can identify the Hilbert-Schmidt operators $L_2(H_Q, U)$ with $U \otimes H_Q^*$, one can define the stochastic integral for any $u \in L^2([0,T]\times\Omega; L_2^0)$ coordinatewise in U. Moreover, it is possible to establish an Itô isometry, namely,

$$\mathbb{E}\Big[\Big\|\int_0^t u(s)dW_s\Big\|_U^2\Big] := \int_0^t \mathbb{E}\big[\|u(s)\|^2_{L_2^0}\big]ds. \tag{2.1}$$

The stochastic noise introduced in Subsection 1.2 can be rewritten in terms of a cylindrical Wiener process. The space $\mathcal{C}_0^\infty(\mathbb{R}^d)$, with pre-inner product

$$\langle \phi, \psi\rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi}(\xi)\mu(d\xi),$$

can be completed to

$$\mathcal{H} := \overline{\mathcal{C}_0^\infty(\mathbb{R}^d)}^{\langle\cdot,\cdot\rangle_{\mathcal{H}}},$$

see [18, Lemma 2.4]. Then, $(\mathcal{H}; \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a real separable Hilbert space. We also set

$$\mathcal{H}_T := L^2([0,T]; \mathcal{H}).$$

Then, [18, Proposition 2.5] states the following result.

Proposition 2.2. *For $t \geq 0$ and $\phi \in \mathcal{H}$, set $W_t(\phi) = W(1_{[0,t]}(\cdot)\phi(\cdot))$. Then, the process $W = \{W_t(\phi), t \geq 0, \phi \in \mathcal{H}\}$ is a standard cylindrical Wiener process on $\mathcal{H}$ (where we recall that "standard" here means assuming $Q = Id_{\mathcal{H}}$).*

3. Microlocal analysis for linear operators with polynomially bounded coefficients

We first recall some basic definitions and facts about the so-called SG-calculus of pseudodifferential and Fourier integral operators, through standard material appeared, e.g., in [5] and elsewhere (sometimes with slightly different notational choices). We include in the Appendix some additional details about the theory of hyperbolic linear operators in this context, to give a presentation as self-contained as possible.

The class $S^{m,\mu} = S^{m,\mu}(\mathbb{R}^d)$ of SG symbols of order $(m,\mu) \in \mathbb{R}^2$ is given by all the functions $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with the property that, for any multiindices $\alpha, \beta \in \mathbb{N}_0^d$, there exist constants $C_{\alpha\beta} > 0$ such that the conditions

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle x\rangle^{m-|\alpha|} \langle \xi\rangle^{\mu-|\beta|}, \qquad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{3.1}$$

hold true, see, e.g., [12, 21, 27] for details. For $m, \mu \in \mathbb{R}$, $\ell \in \mathbb{N}_0$, $a \in S^{m,\mu}$, the quantities

$$\|a\|_\ell^{m,\mu} = \max_{|\alpha+\beta|\le \ell} \sup_{x,\xi\in\mathbb{R}^d} \langle x\rangle^{-m+|\alpha|} \langle \xi\rangle^{-\mu+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \tag{3.2}$$

are a family of seminorms, defining the Fréchet topology of $S^{m,\mu}$.

The corresponding classes of pseudodifferential operators $\mathrm{Op}(S^{m,\mu}) = \mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))$ are given by

$$(\mathrm{Op}(a)u)(x) = (a(.,D)u)(x) = (2\pi)^{-d} \int e^{\mathrm{i}x\xi} a(x,\xi) \hat{u}(\xi) d\xi, \quad a \in S^{m,\mu}(\mathbb{R}^d), u \in \mathcal{S}(\mathbb{R}^d), \tag{3.3}$$

extended by duality to $\mathcal{S}'(\mathbb{R}^d)$. The operators in (3.3) form a graded algebra with respect to composition, i.e.,

$$\mathrm{Op}(S^{m_1,\mu_1}) \circ \mathrm{Op}(S^{m_2,\mu_2}) \subseteq \mathrm{Op}(S^{m_1+m_2,\mu_1+\mu_2}).$$

The symbol $c \in S^{m_1+m_2,\mu_1+\mu_2}$ of the composed operator $\mathrm{Op}(a) \circ \mathrm{Op}(b)$, $a \in S^{m_1,\mu_1}$, $b \in S^{m_2,\mu_2}$, admits the asymptotic expansion

$$c(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi), \tag{3.4}$$

which implies that the symbol c equals $a \cdot b$ modulo $S^{m_1+m_2-1,\mu_1+\mu_2-1}$.

The residual elements of the calculus are operators with symbols in

$$S^{-\infty,-\infty} = S^{-\infty,-\infty}(\mathbb{R}^d) = \bigcap_{(m,\mu)\in\mathbb{R}^2} S^{m,\mu}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^{2d}),$$

that is, those having kernel in $\mathcal{S}(\mathbb{R}^{2d})$, continuously mapping $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$. For any $a \in S^{m,\mu}$, $(m,\mu) \in \mathbb{R}^2$, $\mathrm{Op}(a)$ is a linear continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to itself, extending to a linear continuous operator from $\mathcal{S}'(\mathbb{R}^d)$ to itself, and from $H^{z,\zeta}(\mathbb{R}^d)$ to $H^{z-m,\zeta-\mu}(\mathbb{R}^d)$, where $H^{z,\zeta}(\mathbb{R}^d)$, $(z,\zeta)\in\mathbb{R}^2$, denotes the Sobolev-Kato (or *weighted Sobolev*) space defined in (1.11), with the naturally induced Hilbert norm. When $z\geq z'$ and $\zeta\geq\zeta'$, the continuous embedding $H^{z,\zeta}\hookrightarrow H^{z',\zeta'}$ holds true. It is compact when $z>z'$ and $\zeta>\zeta'$. Since $H^{z,\zeta}=\langle\cdot\rangle^z H^{0,\zeta}=\langle\cdot\rangle^z H^{\zeta}$, with H^ζ the usual Sobolev space of order $\zeta\in\mathbb{R}$, we find $\zeta>k+\dfrac{d}{2}\Rightarrow H^{z,\zeta}\hookrightarrow C^k$, $k\in\mathbb{N}_0$.

Remark 3.1. Notice that in [28] the author uses the space

$$L^2_\omega := \{u\in\mathcal{S}'(\mathbb{R}^d)|\ \sqrt{\omega}u\in L^2(\mathbb{R}^d)\},$$

where $\omega(x)\in\mathcal{S}(\mathbb{R}^d)$ is a strictly positive even function such that for $|x|\geq 1$ we have $\omega(x)=e^{-|x|}$. The weight ω can be substituted by $\omega(x)=\langle x\rangle^{-2z}$, $z>0$, with corresponding space

$$L^2_\omega := \{u\in\mathcal{S}'(\mathbb{R}^d)|\ \langle x\rangle^{-z}u\in L^2(\mathbb{R}^d)\},$$

coinciding with $H^{-z,0}(\mathbb{R}^d)$ in the notation above. In Section 4 we shall use the $H^{z,\zeta}(\mathbb{R}^d)$ spaces to get a function-valued solution to (1.1).

One actually finds

$$\bigcap_{z,\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d)=H^{\infty,\infty}(\mathbb{R}^d)=\mathcal{S}(\mathbb{R}^d),\qquad \bigcup_{z,\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d)=H^{-\infty,-\infty}(\mathbb{R}^d)=\mathcal{S}'(\mathbb{R}^d), \tag{3.5}$$

as well as, for the space of *rapidly decreasing distributions*, see [6, 31],

$$\mathcal{S}'(\mathbb{R}^d)_\infty=\bigcap_{z\in\mathbb{R}}\bigcup_{\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d). \tag{3.6}$$

Cordes introduced the class $\mathcal{O}(m,\mu)$ of the *operators of order* (m,μ) as follows, see, e.g., [12].

Definition 3.2. A linear continuous operator $A\colon\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$ belongs to the class $\mathcal{O}(m,\mu)$, $(m,\mu)\in\mathbb{R}^2$, of the operators of order (m,μ) if, for any $(z,\zeta)\in\mathbb{R}^2$, it extends to a linear continuous operator $A_{z,\zeta}\colon H^{z,\zeta}(\mathbb{R}^d)\to H^{z-m,\zeta-\mu}(\mathbb{R}^d)$. We also define

$$\mathcal{O}(\infty,\infty)=\bigcup_{(m,\mu)\in\mathbb{R}^2}\mathcal{O}(m,\mu),\quad \mathcal{O}(-\infty,-\infty)=\bigcap_{(m,\mu)\in\mathbb{R}^2}\mathcal{O}(m,\mu).$$

Remark 3.3.
1. Trivially, any $A\in\mathcal{O}(m,\mu)$ admits a linear continuous extension $A_{\infty,\infty}\colon\mathcal{S}'(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$. In fact, in view of (3.5), it is enough to set $A_{\infty,\infty}|_{H^{z,\zeta}(\mathbb{R}^d)}=A_{z,\zeta}$.
2. Theorem A.1 implies $\mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))\subset\mathcal{O}(m,\mu)$, $(m,\mu)\in\mathbb{R}^2$.
3. $\mathcal{O}(\infty,\infty)$ and $\mathcal{O}(0,0)$ are algebras under operator multiplication, $\mathcal{O}(-\infty,-\infty)$ is an ideal of both $\mathcal{O}(\infty,\infty)$ and $\mathcal{O}(0,0)$, and $\mathcal{O}(m_1,\mu_1)\circ\mathcal{O}(m_2,\mu_2)\subset\mathcal{O}(m_1+m_2,\mu_1+\mu_2)$.

We now introduce the class of SG-phase functions.

Definition 3.4 (SG-phase function)**.** A real valued function $\varphi \in C^\infty(\mathbb{R}^{2d})$ belongs to the class $\mathfrak{P}$ of SG-phase functions if it satisfies the following conditions:

1. $\varphi \in S^{1,1}(\mathbb{R}^d)$;
2. $\langle \varphi'_x(x,\xi)\rangle \asymp \langle \xi \rangle$ as $|(x,\xi)| \to \infty$;
3. $\langle \varphi'_\xi(x,\xi)\rangle \asymp \langle x \rangle$ as $|(x,\xi)| \to \infty$.

For any $a \in S^{m,\mu}$, $(m,\mu) \in \mathbb{R}^2$, $\varphi \in \mathfrak{P}$, the SG FIOs are defined, for $u \in \mathcal{S}(\mathbb{R}^n)$, as

$$(\mathrm{Op}_\varphi(a)u)(x) = (2\pi)^{-d} \int e^{\mathrm{i}\varphi(x,\xi)} a(x,\xi)\widehat{u}(\xi)\, d\xi, \tag{3.7}$$

and

$$(\mathrm{Op}^*_\varphi(a)u)(x) = (2\pi)^{-d} \iint e^{\mathrm{i}(x\cdot\xi - \varphi(y,\xi))} \overline{a(y,\xi)} u(y)\, dy d\xi. \tag{3.8}$$

Here the operators $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(a)$ are sometimes called SG FIOs of type I and type II, respectively, with symbol a and (SG-)phase function φ. Note that a type II operator satisfies $\mathrm{Op}^*_\varphi(a) = \mathrm{Op}_\varphi(a)^*$, that is, it is the formal L^2-adjoint of the type I operator $\mathrm{Op}_\varphi(a)$.

The analysis of SG FIOs started in [13], where composition results with the classes of SG pseudodifferential operators, and of SG FIOs of type I and type II with regular phase functions, have been proved. Also the basic continuity properties in $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ of operators in the class have been proved there, as well as a version of the Asada-Fujiwara $L^2(\mathbb{R}^d)$-continuity, for operators $\mathrm{Op}_\varphi(a)$ with symbol $a \in S^{0,0}$ and regular SG-phase function $\varphi \in \mathfrak{P}_\delta$, see Definition 3.6. The following theorem summarizes composition results between SG pseudodifferential operators and SG FIOs of type I that we are going to use in the present paper, see [13] for proofs and composition results with SG FIOs of type II.

Theorem 3.5. *Let $\varphi \in \mathfrak{P}$ and assume $b \in S^{m_1,\mu_1}(\mathbb{R}^d)$, $a \in S^{m_2,\mu_2}(\mathbb{R}^d)$, $(m_j,\mu_j) \in \mathbb{R}^2$, $j = 1,2$. Then,*

$$\mathrm{Op}(b) \circ \mathrm{Op}_\varphi(a) = \mathrm{Op}_\varphi(c_1 + r_1) = \mathrm{Op}_\varphi(c_1) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d)),$$

$$\mathrm{Op}_\varphi(a) \circ \mathrm{Op}(b) = \mathrm{Op}_\varphi(c_2 + r_2) = \mathrm{Op}_\varphi(c_2) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d)),$$

for some $c_j \in S^{m_1+m_2,\mu_1+\mu_2}(\mathbb{R}^d)$, $r_j \in S^{-\infty,-\infty}(\mathbb{R}^d)$, $j = 1,2$.

To consider the composition of SG FIOs of type I and type II some more hypotheses are needed, leading to the definition of the classes $\mathfrak{P}_\delta$ and $\mathfrak{P}_\delta(\lambda)$ of regular SG-phase functions.

Definition 3.6 (Regular SG-phase function)**.** Let $\lambda \in [0,1)$ and $\delta > 0$. A function $\varphi \in \mathfrak{P}$ belongs to the class $\mathfrak{P}_\delta(\lambda)$ if it satisfies the following conditions:

1. $|\det(\varphi''_{x\xi})(x,\xi)| \geq \delta$, $\forall (x,\xi)$;
2. the function $J(x,\xi) := \varphi(x,\xi) - x\cdot\xi$ is such that

$$\sup_{\substack{x,\xi\in\mathbb{R}^d \\ |\alpha+\beta|\leq 2}} \frac{|D_\xi^\alpha D_x^\beta J(x,\xi)|}{\langle x\rangle^{1-|\beta|}\langle\xi\rangle^{1-|\alpha|}} \leq \lambda. \tag{3.9}$$

If only condition (1) holds, we write $\varphi \in \mathfrak{P}_\delta$.

The result of a composition of SG FIOs of type I and type II with the same regular SG-phase functions is a SG pseudodifferential operator, see again [13]. The continuity properties of regular SG FIOs on the Sobolev-Kato spaces can be expressed as follows, using the operators of order $(m,\mu)\in\mathbb{R}^2$ introduced above.

Theorem 3.7. *Let φ be a regular SG phase function and $a \in S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu)\in\mathbb{R}^2$. Then, $\mathrm{Op}_\varphi(a)\in\mathcal{O}(m,\mu)$.*

4. Function-valued solutions for semilinear SPDEs.

In this section we state and prove our main result of existence and uniqueness of a function-valued solution of the SPDE (1.1), under suitable assumptions of hyperbolicity for the operator L, see (1.3), (1.5). We work here with a class of operators with more general symbols than the (polynomial) ones appearing in (1.3). Namely, we consider operators of the form

$$L = D_t^m - \sum_{j=1}^m A_j(t,x,D_x) D_t^{m-j}, \tag{4.1}$$

where $A_j(t) = \mathrm{Op}(a_j(t))$ are SG pseudo-differential operators with symbols $a_j \in C^\infty([0,T], S^{j,j})$, $1\leq j\leq m$. Notice that, of course, (1.3) is a particular case of (4.1). The hyperbolicity condition on L becomes

$$\mathcal{L}_m(t,x,\tau,\xi) = \tau^m - \sum_{j=1}^m \tilde{A}_j(t,x,\xi)\tau^{m-j} = \prod_{j=1}^m (\tau - \tau_j(t,x,\xi)), \tag{4.2}$$

where $\tilde{A}_j$ stands for the principal part of A_j, with characteristic roots $\tau_j(t,x,\xi)\in\mathbb{R}$, $\tau_j\in C^\infty([0,T];S^{1,1})$. Let us then consider the Cauchy problem

$$\begin{cases} Lu(t,x) = \gamma(t,x,u(t,x)) + \sigma(t,x,u(t,x))\dot{\Xi}(t,x), & (t,x)\in(0,T]\times\mathbb{R}^d \\ D_t^j u(0,x) = u_j(x), \quad x\in\mathbb{R}^d,\ 0\leq j\leq m-1, \end{cases} \tag{4.3}$$

where L has the form (4.1), under conditions (4.2) and either (1.7) or (1.8). We also assume that $\gamma,\sigma : [0,+\infty)\times\mathbb{R}^d\times\mathbb{R}\longrightarrow\mathbb{R}$ are measurable functions, (at least locally-)Lipschitz-continuous, in our functional setting, with respect to the third variable, see Definition 4.2 and Theorem 4.8 below for the precise hypotheses. Such assumptions are typical in semilinear problems. $\dot{\Xi}$ is the stochastic noise described in Subsection 1.2.

We are interested in finding conditions on L, on the stochastic noise $\dot{\Xi}$, and on $\sigma, \gamma, u_j,\ j = 0, \dots, m-1$, such that (4.3) admits a unique function-valued solution of the form (1.2), following the stochastic integration theory presented in Section 2.

To this aim, we need first the distribution kernel Λ. Its construction for the weakly hyperbolic operators with roots of constant multiplicities is recalled, for the reader's convenience, in the Appendix (see also [2]), and consists of the following steps:

- reduction of the (formal) Cauchy problem

$$\begin{cases} Lu(t) = g(t) & t \in (0,T] \\ D_t^j u(0) = u_j, & 0 \le j \le m-1, \end{cases} \tag{4.4}$$

  where L is the operator in (4.3) and g is a short notation for the right-hand side, to an equivalent first order system;

- construction of the fundamental solution $E(t,s)$ for the system by Theorem A.6, and then of its (formal) solution, following Section 3 and the Appendix;

- construction of the distribution kernel Λ and of the (formal) solution to (4.4), in view of the equivalence of (4.4) and the corresponding first order system.

Notice that all the results on SG-hyperbolic differential operators recalled in Section 3 and the Appendix, in particular, Proposition A.12 and Lemma A.15, still hold true for SG-hyperbolic operators of the form (4.1). We adopt the same terminology and definitions also for this more general operators, with straightforward modifications, where needed. In particular, the mentioned results imply that the distribution Λ is a finite sum of Schwartz kernels of Fourier integral operators with amplitudes of order $(l-m, l-m)$, see (A.26), (A.27).

Next, we need to understand the noise Ξ in terms of a canonically associated Hilbert space $\mathcal{H}_\Xi$, so that we can define the stochastic integral with respect to a cylindrical Wiener process on $\mathcal{H}_\Xi$. This is done in Subsection 4.1 here below. The conditions on the stochastic noise will be given on the spectral measure μ corresponding to the correlation measure Γ related to $\dot{\Xi}$.

Finally, in Subsection 4.2 we state and prove the first main result of this paper, namely Theorem 4.8. We will also prove in Theorem 4.14 a further result, for the involutive roots case, relying on the construction of the kernel Λ performed in [1]. In both situations, we can apply a fixed point technique, in view of the fundamental Lemma 4.6, which is the crucial step to achieve our claims.

Remark 4.1. With respect to the existing literature, in particular [28], we allow here for general hyperbolic equations of higher orders, coefficients depending both on time and space, and possibly with a polynomial growth with respect to x. We observe that in the strictly hyperbolic case, that is, for $l = 1$, the compatibility condition (4.11) exactly corresponds, for $m = 2$, to the one obtained in [28].

4.1. Admissible spectral measures for Hilbert space valued stochastic integrals.

In this subsection we want to make sense of the stochastic integral appearing in (1.2) as a stochastic integral with respect to a cylindrical Wiener process on a Hilbert space, as described in Section 2. We know from (A.27) that, in the stochastic integral appearing in (1.2), Λ is the kernel of (a linear combination of) FIOs Z_{l-m}, with amplitudes of order $(l-m, l-m)$, where l stands for the maximum multiplicity of the characteristic roots ($l = 1$ in the case of a strictly hyperbolic operator, $1 < l \leq m$ in the constant multiplicities case). To give meaning to

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y,u(s,y))\dot{\Xi}(s,y)dyds = \int_0^t Z_{l-m}(t,s)\sigma(s,u(s))d\Xi(s), \quad (4.5)$$

we first introduce the so-called Cameron-Martin space associated with Ξ. Given the Gaussian process Ξ described in Section 1.2, let us define

$$\mathcal{H}_\Xi = \{\widehat{\varphi\mu} \colon \varphi \in L^2_{\mu,s}(\mathbb{R}^d)\}, \quad (4.6)$$

where μ is the spectral measure associated with the noise Ξ, and $L^2_{\mu,s}$ is the space of symmetric functions in L^2_μ, i.e. $\check{\varphi}(x) = \varphi(-x) = \varphi(x)$, $x \in \mathbb{R}^d$, and $\int_{\mathbb{R}^d} |\varphi(x)|^2 \, \mu(dx) < \infty$. Clearly, $\mathcal{H}_\Xi \subset \mathcal{S}'(\mathbb{R}^d)$. The space $\mathcal{H}_\Xi$, endowed with the inner product

$$\langle \widehat{\varphi\mu}, \widehat{\psi\mu} \rangle_{\mathcal{H}_\Xi} := \langle \varphi, \psi \rangle_{L^2_\mu}, \quad \forall \varphi, \psi \in L^2_{\mu,s}(\mathbb{R}^d)$$

with corresponding norm

$$||\widehat{\varphi\mu}||^2_{\mathcal{H}_\Xi} = ||\varphi||^2_{L^2_\mu}$$

turns out to be a real separable Hilbert space, and it is the so-called "Cameron-Martin space" of Ξ, see [28, Propostitition 2.1]. Thus, Ξ is a cylindrical Wiener process on $(\mathcal{H}_\Xi, \langle \cdot, \cdot \rangle_{\mathcal{H}_\Xi})$ which takes values in any Hilbert space $\mathcal{U}$ such that the embedding $\mathcal{H}_\Xi \hookrightarrow \mathcal{U}$ is an Hilbert-Schmidt map.

The following Lemma 4.6 shows that the multiplication operator $\mathcal{H}_\Xi \ni \psi \mapsto Z_{l-m}(t,s)\sigma(s,u) \cdot \psi$ is Hilbert-Schmidt from $\mathcal{H}_\Xi$ to $H^{z+m-l,\zeta}$, under suitable assumptions on σ. Therefore, (4.5) is well-defined as stochastic integral with respect to a cylindrical Wiener process on $(\mathcal{H}_\Xi, \langle \cdot, \cdot \rangle_{\mathcal{H}_\Xi})$ which takes values in $H^{z+m-l,\zeta}$.

Definition 4.2. The class $\mathrm{Lip}(z, \zeta, r, \rho)$, for given $z, \zeta, r, \rho \in \mathbb{R}$, $r, \rho \geq 0$, consists of all measurable functions $g : [0,T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{C}$ such that there exists a real-valued, non negative, $C_t = C(t) \in C[0,T]$, fulfilling the following:

- for every $w \in H^{z+r,\zeta+\rho}(\mathbb{R}^d)$, $t \in [0,T]$, we have $\|g(t,\cdot,w)\|_{z,\zeta} \leq C(t)(1 + \|w\|_{z+r,\zeta+\rho})$;

- for every $w, v \in H^{z+r,\zeta+\rho}(R^d)$, $t \in [0,T]$, we have $\|g(t,\cdot,w)-g(t,\cdot,v)\|_{z,\zeta} \leq C(t)\|w-v\|_{z+r,\zeta+\rho}$.

Remark 4.3. In Definition 4.2 we can actually relax the hypotheses, and ask that the stated properties hold for $w, v \in U$, with U a suitable open subset of $H^{w,\omega}(\mathbb{R}^d)$, for some $w \geq z+r$, $\omega \geq \zeta+\rho$ (typically, a sufficiently small neighbourhood of the initial data of the Cauchy problem). In this case, we indicate the corresponding set by $\mathrm{Lip}_{\mathrm{loc}}(z,\zeta,r,\rho)$.

Remark 4.4. Let $g : [0,T] \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ be measurable and $\zeta = \rho = 0$. Assume that there exists a real-valued, non negative, $C_t = C(t) \in C[0,T]$, satisfying

- for every $w \in \mathbb{R}$, $x \in \mathbb{R}^d$, $t \in [0,T]$, we have $|g(t,x,w)| \leq C(t)(|\kappa(x)|+|w|)$, for some $\kappa \in H^{z,\zeta}(\mathbb{R}^d)$, and
- for every $w, v \in \mathbb{R}$, $x \in \mathbb{R}^d$, $t \in [0,T]$, we have $|g(t,x,w) - g(t,x,v)| \leq C(t)|w-v|$.

Then, $g \in \mathrm{Lip}(z,0,r,0)$. In fact, for some $C > 0$,

$$\begin{aligned}\|g(t,\cdot,w)\|^2_{z,0} &= \|\langle\cdot\rangle^z g(t,\cdot,w)\|^2_{L^2} \leq C_t^2\|\langle\cdot\rangle^z(|\kappa|+|w|)\|^2_{L^2}\\ &\leq 2C_t^2(\|\kappa\|^2_{z,0}+\|w\|^2_{z,0}) \leq C^2C_t^2(1+\|w\|_{z+r,0})^2,\end{aligned}$$

and similarly for the Lipschitz continuity with respect to the third variable, cfr. [28].

Remark 4.5. Let $g(t,x,w) = w^n$, $n \in \mathbb{N}$. Then $g \in \mathrm{Lip}_{\mathrm{loc}}(z,\zeta,r,\rho)$, when $z, r, \rho \geq 0$, $\zeta > \frac{d}{2}$. In fact, when $w \in H^{z+r,\zeta+\rho}(\mathbb{R}^d)$ is such that $\|w\|_{z+r,\zeta+\rho} \leq R$,

$$\|w^n\|_{z,\zeta} \leq C\|w^n\|_{nz,\zeta} \leq C\|w\|^n_{z,\zeta} \leq \widetilde{C}R^{n-1}\|w\|_{z+r,\zeta+\rho},$$

for the algebra properties of the Sobolev-Kato spaces, see e.g. [3, Proposition 2.2].

Lemma 4.6. *Let $Z_{l-m}(t,s)$ be a family of FIOs with amplitudes of order $(l-m, l-m)$, $0 \leq l \leq m$, parametrized by $0 \leq s \leq t \leq T$, and $\sigma \in \mathrm{Lip}(z,\zeta,m-l,0)$. If the spectral measure satisfies*

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi) < \infty, \tag{4.7}$$

(cfr (4.11)), then, for every $w \in H^{z+m-l,\zeta}(\mathbb{R}^d)$, the operator

$$\Phi(t,s) = \Phi_{l,m,\sigma,w}(t,s)\colon \psi \mapsto Z_{l-m}(t,s)\sigma(s,w)\psi$$

belongs to $L^2_0(\mathcal{H}_\Xi, H^{z+m-l,\zeta}(\mathbb{R}^d))$. Moreover, the Hilbert-Schmidt norm of $\Phi(t,s)$ can be estimated by

$$\|\Phi(t,s)\|^2_{L^2_0(\mathcal{H}_\Xi,H^{z+m-l,\zeta})} \leq C^2_{t,s}(1+\|w\|_{z+m-l,\zeta})^2\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi),$$

for some $C_{t,s} > 0$.

Remark 4.7. Lemma 4.6 is the key result to prove Theorems 4.8 and 4.14. It is a generalization, for higher order equations and different functional spaces, of Lemma 2.2 in [28]. There, the author deals with the case $m = 2$ and $l = 1$, related to the wave equation, and works with a multiplication operator by a test function w, obtaining an estimate of the corresponding Hilbert-Schmidt norm involving a weighted L^2 norm of w.

Proof of Lemma 4.6. Let us fix an orthonormal basis $\{e_k\}_{k\in\mathbb{N}} = \{\widehat{f_k\mu}\}_{k\in\mathbb{N}}$ of $\mathcal{H}_\Xi$, where $\{f_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2_{\mu,s}$. We compute

$$\begin{aligned}
||\Phi(t,s)||^2_{L_2^0(\mathcal{H}_\Xi,H^{z+m-l,\zeta})} &= \sum_{k\in\mathbb{N}} ||Z_{l-m}(t,s)\sigma(s,w)\widehat{f_k\mu}||^2_{H^{z+m-l,\zeta}} \\
&= \sum_{k\in\mathbb{N}} ||\langle D\rangle^{l-m}\langle D\rangle^{m-l}\langle\cdot\rangle^{z+m-l}\langle D\rangle^{\zeta} Z_{l-m}(t,s)\sigma(s,w)\widehat{f_k\mu}||^2_{L^2} \\
&= \sum_{k\in\mathbb{N}} ||\langle D\rangle^{l-m}\widetilde{Z}(t,s)\sigma(s,w)\widehat{f_k\mu}||^2_{L^2} \\
&= (2\pi)^{-d}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^d}\langle\xi\rangle^{2(l-m)}\left|\mathcal{F}\left(\widetilde{Z}(t,s)\sigma(s,w)\widehat{f_k\mu}\right)\right|^2(\xi)d\xi \qquad (4.8)
\end{aligned}$$

with $\widetilde{Z}(t,s) = \langle D\rangle^{m-l}\langle\cdot\rangle^{z+m-l}\langle D\rangle^{\zeta}Z_{l-m}(t,s)$ family of FIOs of order (z,ζ). Now, using the well-known fact that the Fourier transform of a product is the ($(2\pi)^{-d}$ multiple of the) convolution of the Fourier transforms, the property $f_k(-x) = f_k(x)$ (by the definition of $L^2_{\mu,s}$), that $\{f_k\}$ is an orthonormal system in L^2_μ, and Bessel's inequality, we get

$$\begin{aligned}
(2\pi)^{-d}\sum_{k\in\mathbb{N}} & \left|\mathcal{F}\left(\widetilde{Z}(t,s)\sigma(s,w)\widehat{f_k\mu}\right)\right|^2(\xi) \\
&= (2\pi)^{-2d}\sum_{k\in\mathbb{N}}|\mathcal{F}\left(\widetilde{Z}(t,s)\sigma(s,w)\right) * \widehat{\widehat{f_k\mu}}|^2(\xi) \\
&= (2\pi)^{-d}\sum_{k\in\mathbb{N}}|\mathcal{F}\left(\widetilde{Z}(t,s)\sigma(s,w)\right) * f_k\mu|^2(\xi) \\
&= (2\pi)^{-d}\sum_{k\in\mathbb{N}}\left|\int_{\mathbb{R}^d}\left[\mathcal{F}\left(\widetilde{Z}(t,s)\sigma(s,w)\right)\right](\xi-\eta)f_k(\eta)\mu(d\eta)\right|^2 \\
&\leq (2\pi)^{-d}\int_{\mathbb{R}^d}\left|\mathcal{F}\left(\widetilde{Z}(t,s)\sigma(s,w)\right)\right|^2(\xi-\eta)\mu(d\eta).
\end{aligned}$$

Inserting this in (4.8), and using the continuity of $\widetilde{Z}$ on Sobolev-Kato spaces we finally get:

$$\begin{aligned}
||\Phi(t,s)||^2_{L_2^0(\mathcal{H}_\Xi,H^{z+m-l,\zeta})} & \\
&\leq (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\langle\xi\rangle^{2(l-m)}\left|\mathcal{F}\left(\widetilde{Z}(t,s)\sigma(s,w)\right)\right|^2(\xi-\eta)\mu(d\eta)d\xi \qquad (4.9)
\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle \eta+\theta\rangle^{2(l-m)} \left| \mathcal{F}\left(\widetilde{Z}(t,s)\sigma(s,w)\right)\right|^2 (\theta)\mu(d\eta)d\theta \\
&\leq (2\pi)^{-d} \left(\sup_{\theta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \langle \theta+\eta\rangle^{2(l-m)}\mu(d\eta)\right) \int_{\mathbb{R}^d} \left| \mathcal{F}\left(\widetilde{Z}(t,s)\sigma(s,w)\right)\right|^2 (\theta)d\theta \\
&= (2\pi)^{-d} \left(\sup_{\theta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \langle \theta+\eta\rangle^{2(l-m)}\mu(d\eta)\right) \|\mathcal{F}(\widetilde{Z}(t,s)\sigma(s,w))\|^2_{L^2} \qquad (4.10)\\
&\leq \left(\sup_{\theta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \langle \theta+\eta\rangle^{2(l-m)}\mu(d\eta)\right) C^2_{t,s}\|\sigma(s,w)\|^2_{z,\zeta} \\
&\leq \left(\sup_{\theta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \langle \theta+\eta\rangle^{2(l-m)}\mu(d\eta)\right) C^2_{t,s}C^2_s\left(1+\|w\|_{z+m-l,\zeta}\right)^2,
\end{aligned}
$$

where $C_{t,s}$ stands for the norm in $\mathscr{L}(H^{z,\zeta}, H^{z,\zeta})$ of the FIO $\widetilde{Z}(t,s)\langle D\rangle^{-\zeta}\langle x\rangle^{-z}$, which, by Theorem 3.5, has amplitude of order $(0,0)$. Since $\sigma \in \mathrm{L}ip(z,\zeta,m-l,0)$, C_s is the constant in Definition 4.2. □

4.2. Function-valued solutions for semilinear hyperbolic equations of arbitrary order.

We are now ready to deal with existence and uniqueness of a function-valued solution for the Cauchy problem (4.3) under conditions (4.2) and either (1.7) or (1.8).

In Theorem 4.8 we study the weakly hyperbolic case with roots of constant multiplicity; in the subsequent Corollary 4.10 we write down the corresponding result in the particular case $l = 1$ of strictly hyperbolic SPDEs. In Theorem 4.14 we state a similar result for the involutive case.

Theorem 4.8. *Let us consider the Cauchy problem* (4.3) *for a hyperbolic SPDE* (1.1)*, where the partial differential operator L of the form* (4.1) *satisfies* (4.2)*. Moreover, assume that L is weakly SG-hyperbolic with constant multiplicities, see Definition 1.1, and let l be the maximum multiplicity of the roots of $\mathcal{L}_m$. Assume also that L is of Levi type, that is, with the notation of Corollary A.13, it satisfies* (A.24)*. Suppose that $\gamma, \sigma \in \mathrm{Lip}_{\mathrm{loc}}(z,\zeta,m-l,0)$, $z,\zeta\in\mathbb{R}$, in some sufficiently small open subset $U \subset H^{z+m-1,\zeta+m-1}(\mathbb{R}^d) \hookrightarrow H^{z+m-l,\zeta}(\mathbb{R}^d)$. Finally, assume for the spectral measure that*

$$
\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi) < \infty. \tag{4.11}
$$

Then, there exists a time horizon $0 < T_0 \leq T$ such that, for any choice of $u_j \in H^{z+m-1-j,\zeta+m-1-j}(\mathbb{R}^d)$, $0 \leq j \leq m-1$, $u_0 \in U$, the Cauchy problem (4.3) *admits a unique solution $u \in L^2([0,T_0]\times\Omega, H^{z+m-l,\zeta}(\mathbb{R}^d))$ satisfying*

$$
\begin{aligned}
u(t,x) = v_0(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y,u(s,y))\,dyds \\
&+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y,u(s,y))\dot{\Xi}(s,y)\,dyds
\end{aligned} \tag{4.12}
$$

where $\Lambda(t,s)$ *is the Schwartz kernel of* $Z_{l-m}(t,s)$*, a sum of FIOs with amplitudes of order* $(l-m,l-m)$*, explicitly obtained in* (A.26)*, the first integral in* (4.12) *is a Bochner integral, and the second integral in* (4.12) *is understood as the stochastic integral of the* $H^{z+m-l,\zeta}(\mathbb{R}^d)$*-valued stochastic process* $Z_{l-m}(t,\cdot)\sigma(\cdot,u(\cdot))$ *with respect to the stochastic noise* Ξ*, in the sense explained in Section 2.*

Remark 4.9. Notice that the noise Ξ defines a cylindrical Wiener process on $(\mathcal{H}_\Xi(\mathbb{R}^d), \langle\cdot,\cdot\rangle_{\mathcal{H}_\Xi(\mathbb{R}^d)})$ with values in $H^{z+m-l,\zeta}(\mathbb{R}^d)$, by Lemma 4.6.

Corollary 4.10. *Let us consider the Cauchy problem* (4.3) *for a hyperbolic SPDE* (1.1)*, where the partial differential operator* L *of the form* (4.1) *satisfies* (4.2)*. Moreover, assume that* L *is strictly SG-hyperbolic, that is,* $\mathcal{L}_m$ *satisfies* (1.5) *and the characteristic roots* τ_j*,* $j=1,\dots,m$*, are distinct, in the sense that for a positive constant* C *we have*

$$|\tau_{j+1}(t,x,\xi)-\tau_j(t,x,\xi)| \geq C\langle x\rangle\langle\xi\rangle \quad \forall (t,x,\xi)\in[0,T]\times\mathbb{R}^{2d}, j=1,\dots,m-1.$$

Suppose that $\gamma,\sigma\in \mathrm{Lip}_{\mathrm{loc}}(z,\zeta,m-1,0)$*,* $z,\zeta\in\mathbb{R}$*, in some sufficiently small open subset* $U\subset H^{z+m-1,\zeta+m-1}(\mathbb{R}^d)$*. Finally, assume for the spectral measure that*

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-1}}\mu(d\xi)<\infty. \tag{4.13}$$

Then, there exists a time horizon $0<T_0\leq T$ *such that, for any choice of* $u_j\in H^{z+m-1-j,\zeta+m-1-j}(\mathbb{R}^d)$*,* $0\leq j\leq m-1$*,* $u_0\in U$*, the Cauchy problem* (4.3) *admits a unique solution* $u\in L^2([0,T_0]\times\Omega, H^{z+m-1,\zeta}(\mathbb{R}^d))$ *satisfying*

$$\begin{aligned} u(t,x) = v_0(t,x) &+ \int_0^t\int_{\mathbb{R}^d}\Lambda(t,s,x,y)\gamma(s,y,u(s,y))\,dyds \\ &+ \int_0^t\int_{\mathbb{R}^d}\Lambda(t,s,x,y)\sigma(s,y,u(s,y))\dot{\Xi}(s,y)\,dyds \end{aligned} \tag{4.14}$$

where $\Lambda(t,s)$ *is the Schwartz kernel of* $Z_{1-m}(t,s)$*, a sum of FIOs with amplitudes of order* $(1-m,1-m)$*, explicitly obtained in* (A.26)*, the first integral in* (4.14) *is a Bochner integral, and the second integral in* (4.12) *is understood as the stochastic integral of the* $H^{z+m-1,\zeta}(\mathbb{R}^d)$*-valued stochastic process* $Z_{1-m}(t,\cdot)\sigma(\cdot,u(\cdot))$ *with respect to the stochastic noise* Ξ*, in the sense explained in Section 2.*

Remark 4.11. Notice that, if the correlation measure Γ is absolutely continuous, then condition (4.13) is equivalent to

$$\int_{\mathbb{R}^d}\frac{1}{(1+|\xi|^2)^{m-1}}\mu(d\xi)<\infty, \tag{4.15}$$

see [28]. Condition (4.15) with $m=2$ on the spectral measure is the one needed for the existence and uniqueness of both a function-valued solution and a random-field solution to a second order SPDE well-known in literature, namely, the stochastic wave equation.

Moreover, the same condition (4.13) has been found in [7], looking for random-field solutions to linear strictly hyperbolic equations with uniformly bounded coefficients. The more general condition (4.11) is exactly the one obtained in [2], looking for random-field solutions to linear hyperbolic SPDEs with possibly unbounded variable coefficients. Thus, the class of the stochastic noises we can deal with if we want to obtain either a function-valued or a random-field solution of the Cauchy problem for an SPDE is described by (4.11) for all *SG*-hyperbolic operators L. Condition (4.11) can be understood as a *compatibility condition* between the noise and the equation: as the order of the equation increases, we can allow for rougher stochastic noises Ξ; as the maximum multiplicity of the roots decreases (i.e., as the regularity of the operator L increases), we can allow for rougher stochastic noises Ξ.

We give here below a couple of examples of right-hand side that we can allow in (4.3).

Example 4.12. Let $\sigma(t,u)=u^2$. Then, σ satisfies all the conditions required in Theorem 4.8. More generally, we can allow also $\sigma(t,u)=u^n$, $n\in\mathbb{N}$, $n>2$, see Remark 4.5.

Example 4.13. A class of explicitly (t,x)-dependent nonlinear stochastic coefficients which satisfy the requirements of Theorem 4.8 are those of the form

$$\sigma(t,x,u)=\langle x\rangle^{l-m}\cdot\widetilde{\sigma}(t,u), \tag{4.16}$$

where $\widetilde{\sigma}\in\mathrm{Lip}_{\mathrm{loc}}(z+m-l,\zeta,0,0)$. Indeed, the function σ in (4.16) fulfills the assumptions of Theorem 4.8, being an element of $\mathrm{Lip}_{\mathrm{loc}}(z,\zeta,m-l,0)$. In fact, for every w in a sufficiently small subset $U\subset H^{z+m-l,\zeta}(\mathbb{R}^d)$, we have

$$||\sigma(t,\cdot,w)||_{z,\zeta}=||\tilde{\sigma}(t,\cdot,w)||_{z+m-l,\zeta}\leq C(t)\left(1+||w||_{z+m-l,\zeta}\right),$$

and the verification of $||\sigma(t,\cdot,w_1)-\sigma(t,\cdot,w_2)||_{z,\zeta}\leq C(t)||w_1-w_2||_{z+m-l,\zeta}$ follows similarly.

Proof of Theorem 4.8. To start, we follow the computations in the Appendix. First, we perform a change of variable, defining the (nm)-dimensional vector of unknowns W having entries given by (A.21). The equation $Lu(t)=g(t,u)$, where formally $g(t,u):=\gamma(t,u)+\sigma(t,u)\dot{\Xi}(t)$, is then equivalent to the semilinear hyperbolic system of first order (A.23) in the unknown W, with $g(t,u)$ in place of $g(t)$. Such system has the form

$$\begin{cases}(D_t-\mathrm{Op}(\kappa_1(t))-\mathrm{Op}(\kappa_0(t)))W(t)=F(t,W(t))+G(t,W(t))\dot{\Xi}(t), & t\in[0,T],\\ W(0)=W_0,\end{cases} \tag{4.17}$$

with $\kappa_1\in C^\infty([0,T],S^{1,1})$ real-valued and diagonal, $\kappa_0\in C^\infty([0,T],S^{0,0})$, and (nm)-dimensional vectors $F(t,W(t))$, $G(t,W(t))$ given by

$$F(t,W(t))=(\underbrace{\tilde{F}(t,W),\dots,\tilde{F}(t,W(t))}_{n\text{ times}})^t,\quad \tilde{F}(t,W(t))=(\underbrace{0,\dots,0}_{m-1\text{ times}},\gamma(t,W_1^{(1)}))^t,$$

$$G(t,W(t)) = (\underbrace{\tilde{G}(t,W),\dots,\tilde{G}(t,W(t))}_{n \text{ times}})^t, \quad \tilde{G}(t,W(t)) = (\underbrace{0,\dots,0}_{m-1 \text{ times}}, \sigma(t,W_1^{(1)}))^t.$$

We also have that $W_0 = \mathrm{Op}(b)U_0$, with a $(mn \times m)$-dimensional block-matrix symbol b with structure analogous to (A.25) and entries with the same orders, so that, by the assumptions of Theorem 4.8, we get $W_0 \in H^{z,\zeta}$.

By Theorem A.6 we can formally construct, via Duhamel's formula, the "mild solution" to (4.17):

$$W(t) = E(t,0)W_0 + i\int_0^t E(t,s)F(s,W(s))ds + i\int_0^t E(t,s)G(s,W(s))d\Xi(s), \quad t \in [0,T_0],$$

for a suitable $T_0 \in (0,T]$. Now, we go back to the equation (1.1) to get its (formal) solution u. By Lemma A.19, we know that $u(t)$ is the first entry of the vector $\mathrm{Op}(\Upsilon_n(t))W(t)$. Thus, as in (A.26), we obtain (formally)

$$\begin{aligned} u(t,x) &= v_0(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y,u(s,y))\,dyds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y,u(s,y))\dot{\Xi}(s,y)\,dyds \\ &= v_0(t,x) + \int_0^t Z_{l-m}(t,s)\gamma(s,u(s))ds + \int_0^t Z_{l-m}(t,s)\sigma(s,u(s))\dot{\Xi}(s)ds, \end{aligned}$$

where $v_0 \in \bigcap_{j\geq 0} C^j([0,T_0], H^{z+m-l-j,\zeta+m-l-j})$ depends on the Cauchy data, and $\Lambda \in C^\infty(\Delta_{T_0}, \mathcal{S}')$ is, for any $(t,s) \in \Delta_{T_0}$, the Schwartz kernel of the Fourier integral operator family Z_{l-m}, with amplitudes of order $(l-m, l-m)$. We then construct the map $u \to \mathcal{T}u$ on $L^2([0,T_0]\times\Omega, H^{z+m-l,\zeta}(\mathbb{R}^d))$, defined as follows:

$$\begin{aligned} \mathcal{T}u(t) &:= v_0(t) + \int_0^t Z_{l-m}(t,s)\gamma(s,u(s))ds + \int_0^t Z_{l-m}(t,s)\sigma(s,u(s))dB_s \quad (4.18) \\ &:= v_0(t) + \mathcal{T}_1 u(t) + \mathcal{T}_2 u(t), \quad t \in [0,T_0], \end{aligned}$$

where the last integral on the right-hand side is understood as the stochastic integral of the stochastic process $Z_{l-m}(t,\cdot)\sigma(\cdot,u(\cdot)) \in L^2([0,T_0]\times\Omega, H^{z+m-l,\zeta})$ with respect to the cylindrical Wiener process $\{W_t(h)\}_{t\in[0,T], h\in H^{z+m-l,\zeta}}$ associated with the random noise $\Xi(t)$, which is well-defined by Lemma 4.6 and takes values in $H^{z+m-l,\zeta}$.

To prove that the solution (4.12) of the Cauchy problem (4.3) is indeed well-defined, we have to check that

$$\mathcal{T}\colon L^2([0,T_0]\times\Omega, H^{z+m-l,\zeta}(\mathbb{R}^d)) \longrightarrow L^2([0,T_0]\times\Omega, H^{z+m-l,\zeta})$$

is well-defined, it is Lipschitz continuous on $L^2([0,T_0]\times\Omega, H^{z+m-l,\zeta})$, and it becomes a contraction if we take T_0 small enough. Then, an application

of Banach's fixed point Theorem will provide existence of a unique solution $u \in L^2([0,T_0] \times \Omega, H^{z+m-l,\zeta})$ satisfying $u = \mathcal{T}u$, that is (4.12).
To verify that $\mathcal{T}u$ in (4.18) belongs to $L^2([0,T_0] \times \Omega, H^{z+m-l,\zeta})$ for every $u \in L^2([0,T_0] \times \Omega, H^{z+m-l,\zeta})$ we notice that:
- $v_0 \in \bigcap_{j\geq 0} C^j([0,T_0], H^{z+m-l-j,\zeta+m-l-j}) \subset L^2([0,T_0] \times \Omega, H^{z+m-l,\zeta})$;
- $\mathcal{T}_1 u$ is in $L^2([0,T_0]\times\Omega, H^{z+m-l,\zeta})$; indeed, $\mathcal{T}_1 u(t)$ is defined as the Bochner integral on $[0,t]$ of the function $s \to Z_{l-m}(t,s)\gamma(s,u(s))$ with values in $L^2(\Omega, H^{z+m-l,\zeta})$, and, by the properties of Bochner integrals, the continuity of $Z_{l-m}(t,s)$ on Sobolev-Kato spaces, and the fact that $\gamma \in \mathrm{Lip}(z,\zeta,m-l,0)$, we have

$$
\begin{aligned}
\|\mathcal{T}_1 u\|^2_{L^2([0,T_0]\times\Omega,H^{z+m-l,\zeta})} &= \mathbb{E}\left[\int_0^{T_0} \|\mathcal{T}_1 u(t)\|^2_{z+m-l,\zeta} dt\right] \\
&= \int_0^{T_0} \mathbb{E}\left[\left\|\int_0^t Z_{l-m}(t,s)(\gamma(s,u(s))ds\right\|^2_{z+m-l,\zeta}\right] dt \\
&\leq \int_0^{T_0}\int_0^t \mathbb{E}\left[\|Z_{l-m}(t,s)(\gamma(s,u(s))\|^2_{z+m-l,\zeta}\right] dsdt \\
&\leq \int_0^{T_0}\int_0^t C^2_{t,s}\mathbb{E}\left[\|\gamma(s,u(s)))\|^2_{z,\zeta+l-m}\right] dsdt \\
&\leq \int_0^{T_0}\int_0^t C^2_{t,s}C^2_s\mathbb{E}\left[(1+\|u(s)\|_{z+m-l,\zeta+l-m})^2\right] dsdt \\
&\leq 2\left(\max_{0\leq s\leq t\leq T_0} C^2_{t,s}C^2_s\right) T_0 \int_0^{T_0}\left(1+\mathbb{E}\left[\|u(s)\|^2_{z+m-l,\zeta}\right]\right) ds \\
&= 2C_{T_0}T_0(T_0 + \|u\|^2_{L^2([0,T_0]\times\Omega,H^{z+l-m,\zeta})}) < \infty;
\end{aligned}
$$

- $\mathcal{T}_2 u$ is in $L^2([0,T_0] \times \Omega, H^{z+m-l,\zeta})$, in view of the fundamental isometry (2.1), Lemma 4.6 and the fact that the expectation can be moved inside and outside time integrals, by Fubini's Theorem:

$$
\begin{aligned}
\|\mathcal{T}_2 u\|^2_{L^2([0,T_0]\times\Omega,H^{z+m-l,\zeta})} &= \mathbb{E}\left[\int_0^{T_0} \|\mathcal{T}_2 u(t)\|^2_{z+m-l,\zeta} dt\right] \\
&= \int_0^{T_0} \mathbb{E}\left[\left\|\int_0^t Z_{l-m}(t,s)\sigma(s,u(s))dW_s\right\|^2_{z+m-l,\zeta}\right] dt \\
&= \int_0^{T_0}\int_0^t \mathbb{E}\left[\|Z_{l-m}(t,s)\sigma(s,u(s))\|^2_{L^2_0(\mathcal{H}_\Xi,H^{z+m-l,\zeta})}\right] dsdt \\
&\leq \int_0^{T_0}\int_0^t \mathbb{E}\left[C^2_{(t,s)}\left(1+\|u(s)\|_{H^{z+m-l,\zeta}}\right)^2 \sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi)\right] dsdt \\
&= \left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi)\right)\int_0^{T_0}\int_0^t C^2_{(t,s)}\mathbb{E}\left[\left(1+\|u(s)\|_{H^{z+m-l,\zeta}}\right)^2\right] dsdt
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi)\right)\left(\max_{0\leq s\leq t\leq T_0}C^2_{(t,s)}\right)T_0\left(T_0+\int_0^{T_0}\mathbb{E}\left[\|u(s)\|^2_{z+m-l,\zeta}\right]ds\right)\\
&=2C_{T_0,m,l}T_0(T_0+\|u\|^2_{L^2([0,T_0]\times\Omega,H^{z+l-m,\zeta})})<\infty.
\end{aligned}
$$

Now, we take $u_1,u_2\in L^2([0,T_0]\times\Omega,H^{z+m-l,\zeta})$ and compute

$$
\begin{aligned}
&\|\mathcal{T}u_1-\mathcal{T}u_2\|^2_{L^2([0,T_0]\times\Omega,H^{z+m-l,\zeta})}\\
&\quad\leq 2\left(\|\mathcal{T}_1u_1-\mathcal{T}_1u_2\|^2_{L^2([0,T_0]\times\Omega,H^{z+m-l,\zeta})}+\|\mathcal{T}_2u_1-\mathcal{T}_2u_2\|^2_{L^2([0,T_0]\times\Omega,H^{z+m-l,\zeta})}\right)\\
&\quad=2\int_0^{T_0}\mathbb{E}\left[\left\|\int_0^t Z_{l-m}(t,s)(\gamma(s,u_1(s))-\gamma(s,u_2(s)))ds\right\|^2_{z+m-l,\zeta}\right]dt \qquad (4.19)\\
&\quad+2\int_0^{T_0}\mathbb{E}\left[\left\|\int_0^t Z_{l-m}(t,s)(\sigma(s,u_1(s))-\sigma(s,u_2(s)))dB_s\right\|^2_{z+m-l,\zeta}\right]dt. \qquad (4.20)
\end{aligned}
$$

In the term (4.19) here above we can move the expectation and the $(z+m-l,\zeta)-$norm inside the integral with respect to s. Then, by continuity of Z_{l-m} on Sobolev-Kato spaces, Definition 4.2, and the embedding $H^{z+m-l,\zeta}\hookrightarrow H^{z+m-l,\zeta+l-m}$, we obtain

$$
\begin{aligned}
&2\int_0^{T_0}\mathbb{E}\left[\left\|\int_0^t Z_{l-m}(t,s)(\gamma(s,u_1(s))-\gamma(s,u_2(s)))ds\right\|^2_{z+m-l,\zeta}\right]dt\\
&\qquad\leq 2\int_0^{T_0}\int_0^t\mathbb{E}\left[\|Z_{l-m}(t,s)(\gamma(s,u_1(s))-\gamma(s,u_2(s)))\|^2_{z+m-l,\zeta}\right]dsdt\\
&\qquad\leq 2\int_0^{T_0}\int_0^t C^2_{t,s}\mathbb{E}\left[\|\gamma(s,u_1(s))-\gamma(s,u_2(s))\|^2_{z,\zeta+l-m}\right]dsdt\\
&\qquad\leq 2\int_0^{T_0}\int_0^t C^2_{t,s}C^2_s\mathbb{E}\left[\|u_1(s)-u_2(s)\|^2_{z+m-l,\zeta+l-m}\right]dsdt\\
&\qquad\leq 2\left(\max_{0\leq s\leq t\leq T_0}C^2_{t,s}C^2_s\right)T_0\int_0^{T_0}\mathbb{E}\left[\|u_1(s)-u_2(s)\|^2_{z+m-l,\zeta}\right]ds\\
&\qquad=2C_{T_0}T_0\|u_1-u_2\|^2_{L^2([0,T_0]\times\Omega,H^{z+l-m,\zeta})}.
\end{aligned}
$$

To the term (4.20) we apply, here below, the fundamental isometry (2.1) to pass from the first to the second line, formula (4.10) of Lemma 4.6 to pass from the second to the third line, Definition 4.2 to pass from the third to the fourth line, and finally get:

$$
\begin{aligned}
&2\int_0^{T_0}\mathbb{E}\left[\left\|\int_0^t Z_{l-m}(t,s)(\sigma(s,u_1(s))-\sigma(s,u_2(s)))dB_s\right\|^2_{z+m-l,\zeta}\right]dt\\
&\qquad=2\int_0^{T_0}\int_0^t\mathbb{E}\left[\|Z_{l-m}(t,s)(\sigma(s,u_1(s))-\sigma(s,u_2(s)))\|^2_{L^0_2(\mathcal{H}_\Xi,H^{z+m-l,\zeta})}\right]dsdt
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2\int_0^{T_0}\int_0^t \mathbb{E}\left[C_{t,s}^2\|\sigma(s,u_1(s))-\sigma(s,u_2(s))\|^2_{H^{z,\zeta}}\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi)\right]dsdt\\
&\leq 2\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi)\right)\int_0^{T_0}\int_0^t C_{t,s}^2C_s^2\mathbb{E}\left[\|u_1(s)-u_2(s)\|^2_{z+m-l,\zeta}\right]dsdt\\
&\leq 2C_{T_0}T_0\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi)\right)\|u_1-u_2\|^2_{L^2([0,T_0]\times\Omega,H^{z+m-l,\zeta})}.
\end{aligned}
$$

Summing up, we have proved that

$$
\begin{aligned}
&\|\mathcal{T}u_1-\mathcal{T}u_2\|^2_{L^2([0,T_0]\times\Omega,H^{z,\zeta})}\\
&\qquad\leq 2C_{T_0}T_0\left(1+\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi)\right)\|u_1-u_2\|^2_{L^2([0,T_0]\times\Omega,H^{z+m-l,\zeta})},
\end{aligned}
$$

that is, $\mathcal{T}$ is Lipschitz continuous on $L^2([0,T_0]\times\Omega,H^{z+m-l,\zeta})$. Moreover, in view of the assumption (4.11), if we take $T_0>0$ such that

$$
2C_{T_0}T_0\left(1+\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi)\right)<1, \tag{4.21}
$$

then $\mathcal{T}$ becomes a strict contraction on $L^2([0,T_0]\times\Omega,H^{z+m-l,\zeta})$, and so it admits a unique fixed point $u=\mathcal{T}u$. That is, there exists a unique, well-defined solution of (4.3). To prove the estimate (4.21), it is sufficient to take T_0 small enough, since the constant C_{T_0} is continuously dependent on T_0. The proof is complete. □

4.3. The weakly hyperbolic case with involutive roots

We conclude the section with the statement of a result of existence and uniqueness of a solution to the Cauchy problem (4.3) for the SPDE (1.1) in the more general case of involutive roots, cfr. (1.8). With these even weaker hyperbolicity assumption we can still switch from (4.3) to an equivalent first order system (A.5), but at the price, as usual, of some further requirement on the lower order terms of the operator L. Namely, we ask that L admits a factorization (A.13) with symbols h_{jk}, $j=1,\dots,m$, $k=1,\dots,l_j$, such that $h_{jk}\in C^\infty([0,T],S^{0,0})$. Notice that this is automatically true in the case of strict hyperbolicity, and that only the request on the order of the symbols h_{jk} has to be fulfilled in the case of hyperbolicity with constant multiplicities. We say, in the present case, that L satisfies the *strong Levi condition*, or, equivalently, that it is *of strong Levi type*. We state and discuss here below our further result, under the hypothesis (1.8).

Theorem 4.14. *Let us consider the Cauchy problem* (4.3) *for an SPDE* (1.1)*, where the partial differential operator L of the form* (4.1) *satisfies the hyperbolicity hypothesis* (4.2)*. Assume that L is SG-hyperbolic with involutive roots,* *that is, all the roots of the principal part $\mathcal{L}_m$ of L are real-valued and form an involutive system, in the sense of* (1.8)*. Moreover, assume that L is of strong Levi type. Suppose that $\gamma, \sigma \in \mathrm{Lip}_{\mathrm{loc}}(z, \zeta, 0, 0)$, $z, \zeta \in \mathbb{R}$, in some sufficiently small open subset $U \subset H^{z+m-1,\zeta+m-1}(\mathbb{R}^d) \hookrightarrow H^{z,\zeta}(\mathbb{R}^d)$. Finally, assume that the spectral measure satisfies the compatibility condition*

$$\int_{\mathbb{R}^d} \mu(d\xi) < \infty. \tag{4.22}$$

Then, there exists a time horizon $0 \leq T_0 \leq T$ such that for any choice of $u_j \in H^{z+m-1-j,\zeta+m-1-j}(\mathbb{R}^d)$, $0 \leq j \leq m-1$, $u_0 \in U$, the Cauchy problem (4.3) *admits a unique solution $u \in L^2([0,T_0] \times \Omega, H^{z,\zeta}(\mathbb{R}^d))$ satisfying*

$$\begin{aligned} u(t,x) = v_0(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y,u(s,y))\, dyds \\ &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y,u(s,y))\dot{\Xi}(s,y)\, dyds, \end{aligned}$$

where $\Lambda(t,s)$ is obtained through the Schwartz kernels of Fourier integral operators with amplitudes of order $(0,0)$, the first integral is a Bochner integral, and the second integral is intended to be the stochastic integral of the $H^{z,\zeta}(\mathbb{R}^d)$-valued stochastic process $E_0(t,\cdot)\sigma(\cdot,u(\cdot))$ with respect to the stochastic noise Ξ.

Remark 4.15. Ξ defines a cylindrical Wiener process on $(\mathcal{H}_\Xi(\mathbb{R}^d), \langle\cdot,\cdot\rangle_{\mathcal{H}_\Xi(\mathbb{R}^d)})$ with values in $H^{z,\zeta}$, by Lemma 4.6.

Proof of Theorem 4.14. By the analysis in [1], we know that, also in this case, using (A.26), the Cauchy problem (4.4) can be (formally) written as

$$\begin{aligned} u(t,x) &= v_0(t,x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y,u(s,y))\, dyds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y,u(s,y))\dot{\Xi}(s,y)\, dyds \\ &= v_0(t,x) + \int_0^t Z_0(t,s)\gamma(s,u(s))ds + \int_0^t Z_0(t,s)\sigma(s,u(s))\dot{\Xi}(s)ds, \end{aligned}$$

where $v_0 \in \bigcap_{j\geq 0} C^j([0,T_0], H^{z-j,\zeta-j})$ depends on the Cauchy data, and $\Lambda \in C^\infty(\Delta_{T_0}, \mathcal{S}')$ is, for any $(t,s) \in \Delta_{T_0}$, the Schwartz kernel of the Fourier integral operator family $Z_0(t,s)$, with amplitudes of order $(0,0)$. Given the assumption (4.22), identical to the case $l = m$ in the proof of Theorem 4.8, the result can then be achieved through the same argument. ☐

4.4. Function-valued solutions and random-field solutions in the linear case.

Consider now the special case of (4.3), with a SG-hyperbolic operator L with constant multiplicities, where $\sigma(t,x,u(t,x)) = \sigma(t,x)$ and $\gamma(t,x,u(t,x)) =$

$\gamma(t,x)$, $\gamma,\sigma\in C([0,T],H^{z,\zeta})$, $z\geq 0$, $\zeta>\frac{d}{2}$, $s\mapsto\mathcal{F}(\sigma)(s)=\nu_s\in L^2([0,T],\mathcal{M}_b)$, $\mathcal{M}_b$ the space of complex-valued measures with finite total variation. That is, we look at the Cauchy problem

$$\begin{cases} Lu(t,x)=\gamma(t,x)+\sigma(t,x)\dot{\Xi}(t,x), & (t,x)\in(0,T]\times\mathbb{R}^d \\ D_t^j u(0,x)=u_j(x), & x\in\mathbb{R}^d,\ 0\leq j\leq m-1, \end{cases} \tag{4.23}$$

for the linear SPDEs studied in [2]. Such (more restrictive) hypotheses imply $\gamma,\sigma\in\mathrm{Lip}(z,\zeta,r,\rho)\subset\mathrm{Lip}_{\mathrm{loc}}(z,\zeta,r,\rho)$ for any $r,\rho\geq 0$. In fact, recalling Definition 4.2, trivially:

- for every $w\in H^{z+r,\zeta+\rho}$, $t\in[0,T]$, $\|g(t,\cdot,w)\|_{z,\zeta}=\|g(t,\cdot)\|_{z,\zeta}\leq C(t)(1+\|w\|_{z+r,\zeta+\rho})$, with $C(t)=\|g(t,\cdot)\|_{z,\zeta}$;
- for every $w,v\in H^{z+r,\zeta+\rho}$, $t\in[0,T]$, $\|g(t,\cdot,w)-g(t,\cdot,v)\|_{z,\zeta}\equiv 0\leq C(t)\|w-v\|_{z+r,\zeta+\rho}$.

Applying Theorem 4.8, we obtain the existence and uniqueness of a function-valued solution for the linear Cauchy problem (4.23), which we here denote by u_{fv}. Since in Theorem 4.12 of [2] we proved the existence and uniqueness of a random-field solution of (4.23), which we here denote by u_{rf}, we now wish to compare it with u_{fv}.

Remark 4.16. Notice that, in analogy with (4.12), u_{rf} satisfies

$$\begin{aligned} u_{\mathrm{rf}}(t,x)=v_0(t,x)&+\int_0^t\int_{\mathbb{R}^d}\Lambda(t,s,x,y)\gamma(s,y)\,dyds \\ &+\int_0^t\int_{\mathbb{R}^d}\Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)\,dyds. \end{aligned} \tag{4.24}$$

While the first two terms in the right-hand side of (4.24) clearly coincide with the first two terms in the right-hand side of (4.12), the corresponding third, stochastic terms in (4.12) and (4.24) are defined in different ways.

We now prove that a random-field solution of (4.23) is also a function-valued solution.

Proposition 4.17. *Let u_{rf} and u_{fv} be the random-field solution and the function-valued solution of* (4.23)*, respectively, with L SG-hyperbolic with constant multiplicities, $\gamma,\sigma\in C([0,T],H^{z,\zeta})$, $z\geq 0$, $\zeta>\frac{d}{2}$, $s\mapsto\mathcal{F}(\sigma)(s)=\nu_s\in L^2([0,T],\mathcal{M}_b)$, $\mathcal{M}_b$ the space of complex-valued measures with finite total variation. Then, $u_{\mathrm{rf}}=u_{\mathrm{fv}}=u$.*

Proof. Our analysis in [2] shows that $\Lambda\sigma\in\mathcal{P}_0$, the completion of the class $\mathcal{E}$ of simple processes via the pre-inner product (defined for suitable f,g)

$$\langle f,g\rangle_0=\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d}\big(f(s)*\tilde{g}(s)\big)(x)\,\Gamma(dx)ds\right]$$

$$= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} [\mathcal{F}f(s)](\xi) \cdot \overline{[\mathcal{F}g(s)](\xi)}\, \mu(d\xi) ds \right].$$

By Proposition 3.12 in [18], it follows that the stochastic integrals of $\Lambda\sigma$ with respect to the martingale measure associated with $\dot{\Xi}$ (considered in Section 4 of [2]), and with respect to the cylindrical Wiener process considered in Section 4 are equal. This proves that $u_{\rm rf} = u_{\rm fv} = u$, as claimed. □

Appendix. Microlocal techniques for the solution of *SG*-hyperbolic problems for linear operators with polynomially bounded coefficients.

We collect in this Appendix, for the convenience of the reader, some additional results concerning the *SG*-calculus and its applications to hyperbolic problems, which we mentioned along the main text. This material appeared, sometimes in slightly different form, in [5] and the references quoted therein.

A.1. Boundedness and ellipticity

The continuity property of the elements of $\mathrm{Op}(S^{m,\mu})$ on the scale of spaces $H^{z,\zeta}(\mathbb{R}^d)$, $(m,\mu),(z,\zeta) \in \mathbb{R}^2$, is precisely expressed in the next Theorem A.1 (see [12] and the references quoted therein for the result on more general classes of *SG*-symbols).

Theorem A.1. *Let $a \in S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu) \in \mathbb{R}^2$. Then, for any $(z,\zeta) \in \mathbb{R}^2$, $\mathrm{Op}(a) \in \mathcal{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-m,\zeta-\mu}(\mathbb{R}^d))$, and there exists a constant $C > 0$, depending only on d, m, μ, z, ζ, such that*

$$\|\mathrm{Op}(a)\|_{\mathscr{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-m,\zeta-\mu}(\mathbb{R}^d))} \le C \|a\|^{m,\mu}_{[\frac{d}{2}]+1}, \tag{A.1}$$

where $[t]$ denotes the integer part of $t \in \mathbb{R}$.

The following characterization of the class $\mathcal{O}(-\infty,-\infty)$ is often useful, see [12].

Theorem A.2. *The class $\mathcal{O}(-\infty,-\infty)$ coincides with $\mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d))$ and with the class of smoothing operators, that is, the set of all the linear continuous operators $A\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. All of them coincide with the class of linear continuous operators A admitting a Schwartz kernel k_A belonging to $\mathcal{S}(\mathbb{R}^{2d})$.*

An operator $A = \mathrm{Op}(a)$ and its symbol $a \in S^{m,\mu}$ are called *elliptic* (or $S^{m,\mu}$*-elliptic*) if there exists $R \ge 0$ such that

$$C\langle x\rangle^m \langle \xi\rangle^\mu \le |a(x,\xi)|, \qquad |x| + |\xi| \ge R,$$

for some constant $C > 0$. If $R = 0$, a^{-1} is everywhere well-defined and smooth, and $a^{-1} \in S^{-m,-\mu}$. If $R > 0$, then a^{-1} can be extended to the whole of $\mathbb{R}^{2d}$

so that the extension $\widetilde{a}_{-1}$ satisfies $\widetilde{a}_{-1} \in S^{-m,-\mu}$. An elliptic SG operator $A \in \mathrm{Op}(S^{m,\mu})$ admits a parametrix $A_{-1} \in \mathrm{Op}(S^{-m,-\mu})$ such that

$$A_{-1}A = I + R_1, \quad AA_{-1} = I + R_2,$$

for suitable $R_1, R_2 \in \mathrm{Op}(S^{-\infty,-\infty})$, where I denotes the identity operator. In such a case, A turns out to be a Fredholm operator on the scale of functional spaces $H^{z,\zeta}(\mathbb{R}^d)$, $(z,\zeta) \in \mathbb{R}^2$.

The study of the composition of $M \geq 2$ SG FIOs of type I $\mathrm{Op}_{\varphi_j}(a_j)$ with regular SG-phase functions $\varphi_j \in \mathfrak{P}_\delta(\lambda_j)$ and symbols $a_j \in S^{m_j,\mu_j}(\mathbb{R}^d)$, $j = 1, \dots, M$, has been done in [5]. The result of such composition is still an SG-FIO with a regular SG-phase function φ given by the so-called *multi-product* $\varphi_1 \sharp \cdots \sharp \varphi_M$ of the phase functions φ_j, $j = 1, \dots, M$, and symbol a as in Theorem A.3 here below.

Theorem A.3. *Consider, for $j = 1, 2, \dots, M$, $M \geq 2$, the SG FIOs of type I $\mathrm{Op}_{\varphi_j}(a_j)$ with $a_j \in S^{m_j,\mu_j}(\mathbb{R}^d)$, $(m_j, \mu_j) \in \mathbb{R}^2$, and $\varphi_j \in \mathfrak{P}_\delta(\lambda_j)$ such that $\lambda_1 + \cdots + \lambda_M \leq \lambda \leq \frac{1}{4}$ for some sufficiently small $\lambda > 0$. Then, there exists $a \in S^{m,\mu}(\mathbb{R}^d)$, $m = m_1 + \cdots + m_M$, $\mu = \mu_1 + \cdots + \mu_M$, such that, setting $\phi = \varphi_1 \sharp \cdots \sharp \varphi_M$, we have*

$$\mathrm{Op}_{\varphi_1}(a_1) \circ \cdots \circ \mathrm{Op}_{\varphi_M}(a_M) = \mathrm{Op}_{\phi}(a).$$

Moreover, for any $\ell \in \mathbb{N}_0$ there exist $\ell' \in \mathbb{N}_0$, $C_\ell > 0$ such that

$$\|a\|_\ell^{m,\mu} \leq C_\ell \prod_{j=1}^{M} \|a_j\|_{\ell'}^{m_j,\mu_j}. \tag{A.2}$$

Theorem A.3 is a corollary of the main Theorem in [5]. There, the *multi-product* of regular SG-phase functions is defined and its properties are studied, parametrices and compositions of regular SG FIOs with amplitude identically equal to 1 are considered, leading to the general composition $\mathrm{Op}_{\varphi_1}(a_1) \circ \cdots \circ \mathrm{Op}_{\varphi_M}(a_M)$. It is needed for the determination of the fundamental solutions of the hyperbolic operators (1.3), involved in (1.1), in the case of involutive roots with non-constant multiplicities, see [1].

A.2. First order SG-hyperbolic linear systems

Here we summarize the main results concerning the analysis of Cauchy problems for SG-hyperbolic linear systems with diagonal principal part, by means of the corresponding class of Fourier operators. Given a symbol $\varkappa \in C([0,T]; S^{1,1})$, set $\Delta_{T_0} = \{(s,t) \in [0,T_0]^2 \colon 0 \leq s \leq t \leq T_0\}$, $0 < T_0 \leq T$, and consider the eikonal equation

$$\begin{cases} \partial_t \varphi(t,s,x,\xi) = \varkappa(t,x,\varphi'_x(t,s,x,\xi)), & t \in [s,T_0], \\ \varphi(s,s,x,\xi) = x \cdot \xi, & s \in [0,T_0), \end{cases} \tag{A.3}$$

with $0 < T_0 \leq T$. By an extension of the theory developed in [14], it is possible to prove that the following Proposition A.4 holds true.

Proposition A.4. *For any small enough $T_0 \in (0, T]$, equation* (A.3) *admits a unique solution $\varphi \in C^1(\Delta_{T_0}, S^{1,1}(\mathbb{R}^d))$, satisfying $J \in C^1(\Delta_{T_0}, S^{1,1}(\mathbb{R}^d))$ and*

$$\partial_s \varphi(t, s, x, \xi) = -\varkappa(s, \varphi'_\xi(t, s, x, \xi), \xi), \tag{A.4}$$

for any $(t, s) \in \Delta_{T_0}$. Moreover, for every $\ell \in \mathbb{N}_0$ there exists $\delta > 0$, $c_\ell \geq 1$ and $\widetilde{T}_\ell \in [0, T_0]$ such that $\varphi(t, s, x, \xi) \in \mathfrak{P}_\delta(c_\ell|t-s|)$, with $\|J\|_{2,\ell} \leq c_\ell|t-s|$ for all $(t, s) \in \Delta_{\widetilde{T}_\ell}$.

Remark A.5. Of course, if additional regularity with respect to $t \in [0, T]$ is fulfilled by the symbol $\varkappa$ in the right-hand side of (A.3), this reflects in a corresponding increased regularity of the resulting solution φ with respect to $(t, s) \in \Delta_{T_0}$. Since here we are not dealing with problems concerning the t-regularity of the solution, we assume smooth t-dependence of the coefficients of L. Some of the results below will anyway be formulated in situations of lower regularity with respect to t.

Let us consider the Cauchy problem

$$\begin{cases} (D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\kappa_0(t)))W(t) = Y(t), & t \in [0, T], \\ W(s) = W_0, & s \in [0, T], \end{cases} \tag{A.5}$$

where the $(\nu \times \nu)$-system is hyperbolic with diagonal principal part, that is:

- the matrix κ_1 satisfies $\kappa_1 \in C^\infty([0, T], S^{1,1})$, it is real-valued and diagonal, and each entry on the principal diagonal coincides with the value of one of the roots $\tau_j \in C^\infty([0, T]; S^{1,1})$, possibly repeated a number of times, depending on their multiplicities;

- the matrix κ_0 satisfies $\kappa_0 \in C^\infty([0, T], S^{0,0})$.

In analogy with the terminology introduced above, we will say that the system (A.5) is hyperbolic with constant multiplicities when the elements on the main diagonal of κ_1 are all distinct and satisfy (1.7). Similarly, we will say that the system is hyperbolic with involutive roots when they satisfy (1.8). We will also generally assume $W_0 \in H^{z,\zeta}$, $Y \in C([0, T], H^{z,\zeta})$, $(z, \zeta) \in \mathbb{R}^2$.

The fundamental solution, or *solution operator*, of (A.5) is a family

$$\{E(t, s) \colon (t, s) \in [0, T_0]^2\}, \quad 0 < T_0 \leq T$$

of linear continuous operators in the strong topology of $\mathscr{L}(H^{z,\zeta}, H^{z,\zeta})$, $(z, \zeta) \in \mathbb{R}^2$, see [12]. In the cases of strict SG-hyperbolicity or of SG-hyperbolicity with constant multiplicities, such family can be explicitly expressed in terms of suitable (matrices of) SG FIOs of type I, modulo smoothing terms, see [14, 16] and Subsection A.3 below. In the case of SG-hyperbolicity with variable multiplicities, it is, in general, a limit of a sequence of (matrices of) SG FIOs of type I, see [5]. A remarkable special case is the involutive roots one, where, again, $E(t, s)$ can be expressed as a finite linear combination of (matrices of)

SG FIOs of type I, modulo smoothing terms, see [1]. See, e.g., [20] and [32] for the results in the classical situations, where the variable x belongs to a bounded set.

In all the three cases mentioned above, the fundamental solution satisfies

$$\begin{cases} (D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\kappa_0(t)))E(t,s) = 0, & (t,s) \in [0,T_0]^2, \\ E(s,s) = I, & s \in [0,T_0]. \end{cases} \tag{A.6}$$

The fundamental solution of a first order SG-hyperbolic system with diagonal principal part, $E(t,s)$, has the following properties, which actually hold for the broader class of symmetric first order system of the type (A.5), of which systems with real-valued, diagonal principal part are a special case, see [12], Ch. 6, §3, and [14].

Theorem A.6. *Let the system* (A.5) *be hyperbolic with diagonal principal part* $\kappa_1 \in C^1([0,T], S^{1,1}\,(\mathbb{R}^d))$*, and lower order part* $\kappa_0 \in C^1([0,T], S^{0,0}(\mathbb{R}^d))$*. Then, for any choice of* $W_0 \in H^{z,\zeta}(\mathbb{R}^d)$*,* $Y \in C([0,T], H^{z,\zeta}(\mathbb{R}^d))$*, there exists a unique solution* $W \in C([0,T], H^{z,\zeta}(\mathbb{R}^d)) \cap C^1([0,T], H^{z-1,\zeta-1}(\mathbb{R}^d))$ *of* (A.5)*,* $(z,\zeta) \in \mathbb{R}^2$*, given by Duhamel's formula*

$$W(t) = E(t,s)W_0 + i\int_s^t E(t,\vartheta)Y(\vartheta)d\vartheta, \quad t \in [0,T].$$

Moreover, the solution operator $E(t,s)$ *has the following properties:*

1. $E(t,s)\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ *is an operator belonging to* $\mathcal{O}(0,0)$*,* $(t,s) \in [0,T]^2$*; its first order derivatives,* $\partial_t E(t,s)$*,* $\partial_s E(t,s)$*, exist in the strong operator convergence of* $\mathscr{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-1,\zeta-1}(\mathbb{R}^d))$*,* $(z,\zeta) \in \mathbb{R}^2$*, and belong to* $\mathcal{O}(1,1)$*;*
2. $E(t,s)$ *is bounded and strongly continuous from* $[0,T]^2_{ts}$ *to* $\mathscr{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z,\zeta}(\mathbb{R}^d))$*,* $(z,\zeta) \in \mathbb{R}^2$*;* $\partial_t E(t,s)$ *and* $\partial_s E(t,s)$ *are bounded and strongly continuous from* $[0,T]^2_{ts}$ *to* $\mathscr{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-1,\zeta-1}(\mathbb{R}^d))$*,* $(z,\zeta) \in \mathbb{R}^2$*;*
3. *for* $t, s, t_0 \in [0,T]$ *we have*
$$E(t_0,t_0) = I, \quad E(t,s)E(s,t_0) = E(t,t_0), \quad E(t,s)E(s,t) = I;$$
4. $E(t,s)$ *satisfies, for* $(t,s) \in [0,T]^2$*, the differential equations*
$$D_t E(t,s) - (\mathrm{Op}(\kappa_1(t)) + \mathrm{Op}(\kappa_0(t)))E(t,s) = 0, \tag{A.7}$$
$$D_s E(t,s) + E(t,s)(\mathrm{Op}(\kappa_1(s)) + \mathrm{Op}(\kappa_0(s))) = 0; \tag{A.8}$$
5. *the operator family* $E(t,s)$ *is uniquely determined by the properties (1)-(3) here above, and one of the differential equations* (A.7)*,* (A.8)*.*

Corollary A.7. 1. *Under the hypotheses of Theorem A.6,* $E(t,s)$ *is invertible on* $\mathcal{S}(\mathbb{R}^d)$*,* $\mathcal{S}'(\mathbb{R}^d)$*, and* $H^{z,\zeta}(\mathbb{R}^d)$*,* $(z,\zeta) \in \mathbb{R}^2$*, with inverse given by* $E(s,t)$*,* $s,t \in [0,T]$*.*

2. *If, additionally, one assumes* $\kappa_1 \in C^m([0,T], S^{1,1}(\mathbb{R}^d))$, $\kappa_0 \in C^m([0,T], S^{0,0}(\mathbb{R}^d))$, $m \geq 2$, *the partial derivatives* $\partial_t^j \partial_s^k E(t,s)$ *exist in strong operator convergence of* $\mathcal{S}(\mathbb{R}^d)$ *and* $\mathcal{S}'(\mathbb{R}^d)$, *and* $\partial_t^j \partial_s^k E(t,s) \in \mathcal{O}(j+k, j+k)$, $j+k \leq m$. *Moreover,* $\partial_t^j \partial_s^k E(t,s)$ *is strongly continuous from* $[0,T]^2_{ts}$ *to every* $\mathscr{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-j-k,\zeta-j-k}(\mathbb{R}^d))$, $(z,\zeta) \in \mathbb{R}^2$, $j+k \leq m$.

In [5] we have proved the next Theorem A.8, concerning the structure of $E(t,s)$, in the spirit of the approach followed in [20].

Theorem A.8. *Under the same hypotheses of Theorem A.6, if* T_0 *is small enough, for every fixed* $(t,s) \in \Delta_{T_0}$, $E(t,s)$ *is a limit of a sequence of matrices of SG FIOs of type I, with regular phase functions* $\varphi_{jk}(t,s)$ *belonging to* $\mathfrak{P}_\delta(c_h|t-s|)$, $c_h \geq 1$, *of class* C^1 *with respect to* $(t,s) \in \Delta_{T_0}$, *and amplitudes belonging to* $C^1(\Delta_{T_0}, S^{0,0}(\mathbb{R}^d))$.

In the case of strict hyperbolicity, or, more generally, hyperbolicity with constant multiplicities, we can actually "decouple" the equations in (A.5) into n blocks of smaller dimensions, by means of the so-called *perfect diagonalizer*, an element of $C^\infty([0,T], \mathrm{Op}(S^{0,0}))$. Thus, the solution of (A.5) can be reduced to the solution of n independent smaller systems. The principal part of the coefficient matrix of each one of such decoupled subsystems admits then a single distinct eigenvalue of maximum multiplicity, so that it can be treated, essentially, like a scalar SG-hyperbolic equations of first order. Explicitely, see, e.g., [14, 20],

Theorem A.9. *Assume that the system* (A.5) *is hyperbolic with constant multiplicities* ν_j, $j = 1, \dots, N$, $\nu_1 + \dots + \nu_n = \nu$, *with diagonal principal part* $\kappa_1 \in C^\infty([0,T], S^{1,1}(\mathbb{R}^d))$ *and* $\kappa_0 \in C^\infty([0,T], S^{0,0}(\mathbb{R}^d))$, *both of them* $(\nu \times \nu)$*-dimensional matrices. Then, there exist* $(\nu \times \nu)$*-dimensional matrices* $\omega \in C^\infty([0,T], S^{0,0}(\mathbb{R}^d))$ *and* $\widetilde{\kappa}_0 \in C^\infty([0,T], S^{0,0}(\mathbb{R}^d))$ *such that*

$$\det(\omega) \asymp 1 \Rightarrow \omega^{-1} \in C^\infty([0,T], S^{0,0}(\mathbb{R}^d)), \quad \widetilde{\kappa}_0 = \mathrm{diag}(\widetilde{\kappa}_{01}, \dots, \widetilde{\kappa}_{0n}),$$

$\widetilde{\kappa}_{0j} (\nu_j \times \nu_j)$*-dimensional matrix, and*

$$\begin{aligned}&(D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\kappa_0(t)))\mathrm{Op}(\omega(t)) - \mathrm{Op}(\omega(t))(D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\widetilde{\kappa}_0(t)))\\ &\in C^\infty([0,T], \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d)).\end{aligned} \tag{A.9}$$

In this situation, by an extension of the results in [14, 16], we can give an explicit form to the fundamental solution $E(t,s)$ in Theorem A.8, in terms of (smooth families of) SG FIOs of type I, modulo smoothing remainders. With the results of Theorem A.9 at hand, we solve, by means of the so-called *geometrical optics* (or FIOs) method, the system

$$\begin{cases}(D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\widetilde{\kappa}_0(t)))\widetilde{E}(t,s) = 0, & t \in [0,T_0],\\ \widetilde{E}(s,s) = I, & s \in [0,T_0).\end{cases} \tag{A.10}$$

Notice that the *approximate solution operator* $\widetilde{A}(t,s)$, $(t,s) \in \Delta_{T_0}$, in terms of *SG* FIOs solves the corresponding operator problem up to smoothing remainders. Namely, the FIOs family $\widetilde{A}(t,s)$ solves the system

$$\begin{cases} (D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\widetilde{\kappa}_0(t)))\widetilde{A}(t,s) = \widetilde{R}_1(t,s), & (t,s) \in \Delta_{T_0}, \\ \widetilde{A}(s,s) = I + \widetilde{R}_2(s), & s \in [0,T_0), \end{cases} \tag{A.11}$$

where $\widetilde{R}_1$ and $\widetilde{R}_2$ are suitable smooth families of operators in $\mathcal{O}(-\infty,-\infty)$, coming from the solution method, see [12, 13, 14, 16, 20] for more details. It turns out that $\widetilde{A}(t,s)$ belongs to $\mathcal{O}(0,0)$ for any $(t,s) \in \Delta_{T_0}$. Explicitely,

$$\widetilde{A}(t,s) = \mathrm{diag}(\widetilde{A}^{(1)}(t,s), \dots, \widetilde{A}^{(m)}(t,s)),$$
$$\widetilde{A}^{(p)}(t,s) = \mathrm{diag}(\mathrm{Op}_{\varphi_{\varpi_p(1)}(t,s)}(a_1^{(p)}(t,s)), \dots, \mathrm{Op}_{\varphi_{\varpi_p(m)}(t,s)}(a_m^{(p)}(t,s))), p = 1, \dots, m,$$

with phase functions $\varphi_j \in C^\infty(\Delta_{T_0}, \mathfrak{P}_\delta(\lambda))$, $\lambda = \lambda(T_0)$ suitably small, solutions of the eikonal equations (A.3) with τ_j in place of $\varkappa$, and symbols $a_j^{(p)} \in C^\infty(\Delta_{T_0}, S^{0,0})$, $p, j = 1, \dots, m$, see [14]. Solving the equations in (A.10) modulo smoothing terms is enough for our aims. Indeed, we have the following result (see [2] for its proof).

Proposition A.10. *Under the hypotheses* (4.1), (4.2)*, let* $A(t,s) = \mathrm{Op}(\omega(t)) \circ \widetilde{A}(t,s) \circ \mathrm{Op}(\omega_{-1})(s)$*, with* $\widetilde{A}(t,s)$ *solution of* (A.11)*,* $(t,s) \in \Delta_{T_0}$*, and* $\mathrm{Op}(\omega_{-1})(s)$ *parametrix of the perfect diagonalizer* $\mathrm{Op}(\omega(s))$*,* $s \in [0,T]$*. Then, the solution* $E(t,s)$ *of* (A.6) *and the operator family* $A(t,s)$ *satisfy* $E - A \in C^\infty(\Delta_{T_0}, \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d)))$.

Remark A.11. Proposition A.10 means that the Schwartz kernels of E and A differ by a family of elements of $\mathcal{S}(\mathbb{R}^{2d})$, smoothly depending on $(t,s) \in \Delta_{T_0}$.

Using Proposition A.10, by repeated applications of Theorem 3.5, we finally obtain

$$E(t,s) = E_0(t,s) + R(t,s), \quad (t,s) \in \Delta_{T_0}, \tag{A.12}$$

where

- E_0 is a $(nm \times nm)$-dimensional matrix of operators in $\mathcal{O}(0,0)$ given by

  $$E_0(t,s) = \left(\sum_{p=1}^{n} \mathrm{Op}_{\varphi_p(t,s)}(e_{pjk}(t,s)) \right)_{j,k=0,\dots,nm-1},$$

  with the regular phase-functions $\varphi_p(t,s)$, solutions of the eikonal equations associated with τ_p, and symbols $e_{pjk}(t,s) \in S^{0,0}$, $j,k = 0, \dots, nm-1$, $p = 1, \dots, n$, smoothly depending on $(t,s) \in \Delta_{T_0}$;

- R is a $(nm \times nm)$-dimensional matrix of elements in $C^\infty(\Delta_{T_0}, \mathrm{Op}(S^{-\infty,-\infty}))$, operators with kernel in $S(\mathbb{R}^{2d})$, smoothly depending on $(t,s) \in \Delta_{T_0}$, that is,

  $$R = (\mathrm{Op}(r_{jk}(t,s)))_{j,k=0,\dots,nm-1},$$

with symbols $r_{jk} \in C^\infty(\Delta_{T_0}, S^{-\infty,-\infty})$, $j,k = 0,\dots,nm-1$, collecting the remainders of the compositions in $\mathrm{Op}(\omega) \circ \widetilde{A} \circ \mathrm{Op}(\omega_{-1})$ and the difference $E - A$.

Achieving a similar result for systems with involutive roots is not straightforward. In fact, in this case, the system cannot, in general, be diagonalized block by block, and a quite technical analysis is needed, see [1].

A.3. Fundamental solution for SG-hyperbolic linear operators

By the hyperbolicity hypotheses, as it will be explained below, to obtain the term depending on the initial conditions and the kernel Λ, associated with the linear operator in (1.1), it is enough to know the fundamental solution of first order systems with diagonal principal part. The next results are employed to switch from (4.4) to a first order linear system of the form (A.5).

Proposition A.12. *Let L be a hyperbolic operator with constant multiplicities l_j, $j = 1,\dots,n \le m$. Denote by $\theta_j \in G_j$, $j = 1,\dots,n$, the distinct real roots of $\mathcal{L}_m$ in* (1.5)*. Then, it is possible to factor L as*

$$L = L_n \cdots L_1 + \sum_{j=1}^{m} \mathrm{Op}(r_j(t)) D_t^{m-j} \tag{A.13}$$

with

$$L_j = (D_t - \mathrm{Op}(\theta_j(t)))^{l_j} + \sum_{k=1}^{l_j} \mathrm{Op}(h_{jk}(t))\,(D_t - \mathrm{Op}(\theta_j(t)))^{l_j-k}, \tag{A.14}$$

$$h_{jk} \in C^\infty([0,T], S^{k-1,k-1}(\mathbb{R}^d)), \quad r_j \in C^\infty([0,T], S^{-\infty,-\infty}(\mathbb{R}^d)), \quad j = 1,\dots,n, k = 1,\dots,l_j. \tag{A.15}$$

The following corollary is an immediate consequence of Proposition A.12, and is proved by means of a reordering of the distinct roots θ_j, $j = 1,\dots,n$.

Corollary A.13. *Let ϖ_j, $j = 1,\dots,n$, denote the reordering of the n-tuple $(1,\dots,n)$, given, for $k = 1,\dots,n$, by*

$$\varpi_j(k) = \begin{cases} j+k-1 & \text{for } j+k \le n+1, \\ j+k-n-1 & \text{for } j+k > n+1, \end{cases} \tag{A.16}$$

That is, for $n \ge 2$, $\varpi_1 = (1,\dots,n), \varpi_2 = (2,\dots,n,1),\dots,\varpi_n = (n,1,\dots,n-1)$. Then, under the same hypotheses of Proposition A.12, we have, for any $p = 1,\dots,n$,

$$L = L^{(p)}_{\varpi_p(n)} \dots L^{(p)}_{\varpi_p(1)} + \sum_{j=1}^{m} \mathrm{Op}(r^{(p)}_j(t)) D_t^{m-j} \tag{A.17}$$

with

$$L_j^{(p)} = (D_t - \mathrm{Op}(\theta_j(t)))^{l_j} + \sum_{k=1}^{l_j} \mathrm{Op}(h_{jk}^{(p)}(t))\, (D_t - \mathrm{Op}(\theta_j(t)))^{l_j-k}, \tag{A.18}$$

$$h_{jk}^{(p)} \in C^\infty([0,T], S^{k-1,k-1}(\mathbb{R}^d)), j = 1, \ldots, n, k = 1, \ldots, l_j, \tag{A.19}$$

$$r_j^{(p)} \in C^\infty([0,T], S^{-\infty,-\infty}(\mathbb{R}^d)), j = 1, \ldots, m. \tag{A.20}$$

Remark A.14. Of course, for $n = 1$, we only have the single "reordering" $\varpi_1 = (1)$, $l_1 = l = m$, and

$$L = L_1^{(1)} + \sum_{j=1}^{m} \mathrm{Op}(r_j^{(1)}(t)) D_t^{m-j}$$

with

$$L_1^{(1)} = (D_t - \mathrm{Op}(\theta_1(t)))^m + \sum_{k=1}^{m} \mathrm{Op}(h_{1k}^{(1)}(t))\, (D_t - \mathrm{Op}(\theta_1(t)))^{m-k},$$

$$h_{1k}^{(1)} \in C^\infty([0,T], S^{k-1,k-1}(\mathbb{R}^d)), k = 1, \ldots, m, \quad r_j^{(1)} \in C^\infty([0,T], S^{-\infty,-\infty}(\mathbb{R}^d)), j = 1, \ldots, m$$

With inductive procedures similar to those performed in [8, 9] and [23], respectively, it is possible to prove the following Lemma A.15.

Lemma A.15. *Under the same hypotheses of Proposition A.12, for all* $k = 0, \ldots, m-1$, *it is possible to find symbols* $\varsigma_{kpq} \in C^\infty([0,T], S^{k-q+l_p-n,k-q+l_p-n}(\mathbb{R}^d))$, $p = 1, \ldots, n$, $q = 0, \ldots, l_p - 1$, *such that, for all* $t \in [0,T]$,

$$\theta^k = \sum_{p=1}^{n} \left[\sum_{q=0}^{l_p-1} \varsigma_{kpq}(t)(\theta - \theta_p(t))^q \right] \cdot \left[\prod_{\substack{1 \le j \le n \\ j \ne p}} (\theta - \theta_j(t))^{l_j} \right].$$

Let us denote by θ_j, $j = 1, \ldots, n$, the distinct values of the roots τ_k, $k = 1, \ldots, m$, and with ϖ_p, $p = 1, \ldots, n$, the reorderings of the n-tuple $(1, \ldots, n)$ defined in (A.16).

The equivalence of the Cauchy problems for the equation $Lu(t) = g(t)$ and a 1×1 system (A.5) is then trivial for $m = 1$. For $m \ge 2$, we will now define a (nm)-dimensional vector of unknown W and construct a corresponding linear first order hyperbolic system, with diagonal principal part and constant multiplicities, equivalent to $Lu(t) = g(t)$.

Let us set, for convenience, with the notation introduced in Corollary A.13,

$$l^{(p,k)} = \begin{cases} 0, & k = 0, \\ \displaystyle\sum_{1 \le j \le k} l_{\varpi_p(j)}, & 1 \le k \le n-1, \text{ if } n \ge 2, \\ m, & k = n, \end{cases}$$

$$L^{(p,k)} = \begin{cases} I, & k=0, \\ L^{(p)}_{\varpi_p(k)} \cdots L^{(p)}_{\varpi_p(1)}, & 1 \le k \le n-1, \text{ if } n \ge 2, \end{cases}$$

$p = 1, \dots, n$, and define

$$W^{(p)}_{l^{(p,k)}+j+1}(t) = (D_t - \mathrm{Op}(\theta_{\varpi_p(k+1)}(t)))^j L^{(p,k)} u(t) \tag{A.21}$$

for $p = 1, \dots, n$, $k = 0, \dots, n-1$, $j = 0, \dots, l_{\varpi_p(k+1)} - 1$. Using Lemma A.15, we can express the t derivatives of u in terms of the components of W from (A.21). In fact:

Lemma A.16. *Under the hypotheses of Lemma A.15, for all $k = 1, \dots, m-1$, $p = 1, \dots, n$, it is possible to find symbols $w^{(p)}_{kj} \in C^\infty([0,T], S^{j,j}(\mathbb{R}^d))$, $j = 1, \dots, k$, such that, with the (nm)-dimensional vector W defined in* (A.21)*,*

$$D^k_t u(t) = \sum_{j=1}^{k} \mathrm{Op}(w^{(p)}_{kj}(t)) W^{(p)}_{k-j+1}(t) + W^{(p)}_{k+1}(t). \tag{A.22}$$

By the definition (A.21), we find the extension of (A.22) to $k = 0$ in the form $u(t) = W^{(p)}_1(t)$, $p = 1, \dots, n$. Using Lemma A.16 we see that (A.17), (A.21) and (A.22) give rise to a block diagonal linear system in the nm unknown $W^{(p)}_{l^{(p,k)}+j+1}(t)$ with blocks labeled by $p = 1, \dots, n$, of the type

$$\begin{cases} \qquad\qquad \dots, \\ (D_t - \mathrm{Op}(\theta_{\varpi_p(1)}(t)))W^{(p)}_{j+1}(t) = \quad W^{(p)}_{j+2}(t), \quad j = 0, \dots, l_{\varpi_p(1)} - 2, \text{ if } l_{\varpi_p(1)} \ge 2, \\ (D_t - \mathrm{Op}(\theta_{\varpi_p(1)}(t)))W^{(p)}_{l^{(p,1)}}(t) = -\displaystyle\sum_{k=1}^{l_{\varpi_p(1)}} \mathrm{Op}(h^{(p)}_{\varpi_p(1)k}(t)) W^{(p)}_{l^{(p,1)}-k+1}(t) + W^{(p)}_{l^{(p,1)}+1}(t), \\ (D_t - \mathrm{Op}(\theta_{\varpi_p(2)}(t)))W^{(p)}_{l^{(p,1)}+j+1}(t) = \quad W^{(p)}_{l^{(p,1)}+j+2}(t), \quad j = 0, \dots, l_{\varpi_p(2)} - 2, \text{ if } l_{\varpi_p(2)} \ge 2, \ n \ge 2, \\ (D_t - \mathrm{Op}(\theta_{\varpi_p(2)}(t)))W^{(p)}_{l^{(p,2)}}(t) = -\displaystyle\sum_{k=1}^{l_{\varpi_p(2)}} \mathrm{Op}(h^{(p)}_{\varpi_p(2)k}(t)) W^{(p)}_{l^{(p,2)}-k+1}(t) + W^{(p)}_{l^{(p,2)}+1}(t), \text{ if } n \ge 2, \\ \qquad\qquad \dots, \\ (D_t - \mathrm{Op}(\tau_{\varpi_p(n)}(t)))W^{(p)}_{m}(t) = -\displaystyle\sum_{k=1}^{l_{\varpi_p(n)}} \mathrm{Op}(h^{(p)}_{\varpi_p(n)k}(t)) W^{(p)}_{m-k+1}(t) \\ \qquad - \displaystyle\sum_{j=1}^{m-1} \left(\sum_{q=1}^{m-j} \mathrm{Op}(r^{(p)}_j(t)) \circ \mathrm{Op}(w^{(p)}_{m-j,q}(t)) W^{(p)}_{m-j-q+1}(t) + \mathrm{Op}(r^{(p)}_j(t)) W^{(p)}_{m-j+1}(t) \right) \\ \qquad - \mathrm{Op}(r^{(p)}_m(t)) W^{(p)}_1(t) + g(t), \\ \qquad\qquad \dots \end{cases} \tag{A.23}$$

and equivalent, block by block, to the equation $Lu(t) = g(t)$.

As it is very well-known in the usual hyperbolic theory, in the case of weak hyperbolicity the principal term does not provide enough information, by itself, to imply well-posedness of the Cauchy problem. In other words, lower order terms are also relevant in this case, and one needs to impose additional conditions on them. We will then assume that L satisfies the SG-Levi condition

$$h_{jk}^{(p)} \in C^\infty([0,T], S^{0,0}(\mathbb{R}^d)), \quad p, j = 1, \dots, n, k = 1, \dots, l_j, \tag{A.24}$$

see Corollary A.13.

Remark A.17. Let us observe that, indeed, (A.24) needs to be fulfilled only for a single value of $p = 1, \dots, n$. Also, (A.24) is automatically fulfilled when L is strictly SG-hyperbolic. If L satisfies (A.24) we will also say that L is of Levi type.

It is clear, in view of the calculus of SG pseudodifferential operators, the fact that $r_j^{(p)} \in C^\infty([0,T], S^{-\infty,-\infty})$, $p = 1, \dots, n$, and the inclusions among the SG symbols, that the system (A.23) is a hyperbolic first order linear system of the form (A.5), where:
- the $(nm \times nm)$-dimensional, block-diagonal matrix $\kappa_1 \in C^\infty([0,T], S^{1,1})$ is given by $\kappa_1 = \operatorname{diag}(\kappa_{11}, \dots, \kappa_{1n})$, with each block defined by

$$\kappa_{1p} = \operatorname{diag}(\underbrace{\theta_{\omega_p(1)}, \dots, \theta_{\omega_p(1)}}_{l_{\omega_{p(1)}} \text{ times}}, \underbrace{\theta_{\omega_p(2)}, \dots, \theta_{\omega_p(2)}}_{l_{\omega_{p(2)}} \text{ times}}, \dots, \underbrace{\theta_{\omega_p(n)}, \dots, \theta_{\omega_p(n)}}_{l_{\omega_{p(n)}} \text{ times}}),\ p = 1, \dots, n;$$

- the $(nm \times nm)$-dimensional, block-diagonal matrix $\kappa_0 \in C^\infty([0,T], S^{0,0})$ is given by $\kappa_0 = \operatorname{diag}(\kappa_{01}, \dots, \kappa_{0m})$ with suitable matrices κ_{0p} having entries in $C^\infty([0,T], S^{0,0})$, $p = 1, \dots, n$;
- the right-hand side is

$$Y(t) = (\underbrace{G(t), \dots, G(t)}_{n \text{ times}})^t, \quad G(t) = (\underbrace{0, \dots, 0}_{m-1 \text{ times}}, g(t))^t.$$

The initial data W_0 is obtained by $W_0 = \operatorname{Op}(b)U_0$, with $U_0 = (u_0, \dots, u_{m-1})^t$ and a $(mn \times m)$-dimensional block-matrix symbol b with the following structure:

$$b = \begin{pmatrix} b^{(1)} \\ \hline \cdots \\ \hline b^{(n)} \end{pmatrix}, \quad b^{(p)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ b_{10}^{(p)} & 1 & 0 & 0 & \dots \\ b_{20}^{(p)} & b_{21}^{(p)} & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, p = 1, \dots, n, \tag{A.25}$$

and the $(m \times m)$-dimensional matrices $b^{(p)}$ satisfying

- if $m \geq 2$, $b_{jk}^{(p)} \in S^{j-k,j-k}$, $j > k$, $j = 1, \dots, m-1$, $k = 0, \dots, j-1$,

- $b_{jj}^{(p)} = 1 \in S^{0,0}$, $j = 0, \dots, m-1$,

- if $m \geq 2$, $b_{jk}^{(p)} = 0$, $j < k$, $j = 0, \dots, m-2$, $k = j+1, \dots, m-1$,

$p = 1, \dots, m$.

Remark A.18. Consider, for instance, the case $n = 1$, that is, $\mathcal{L}_m$ admits a unique real root $\theta_1 = \tau_1$ of maximum multiplicity $l = l_1 = m$. Then, there is a single "reordering" $\varpi_1 = (1)$, the vector W has m components, $W = (W_1^{(1)}, \dots, W_m^{(1)})$, and (A.23) consists of a single block of m equations. Namely, in view of Corollary A.13, assuming $n \geq 2$ and dropping everywhere the $^{(1)}$ label, (A.21) reads, in this case,

$$\begin{aligned}
W_1(t) &= u(t),\\
W_2(t) &= (D_t - \mathrm{Op}(\tau_1(t)))u(t) = (D_t - \mathrm{Op}(\tau_1(t)))W_1(t),\\
&\dots,\\
W_m(t) &= (D_t - \mathrm{Op}(\tau_1(t)))^{m-1}u(t) = (D_t - \mathrm{Op}(\tau_1(t)))W_{m-1}(t),
\end{aligned}$$

while $Lu(t) = g(t)$ is then equivalent to

$$\begin{aligned}
&(D_t - \mathrm{Op}(\tau_1(t)))^m u(t) + \sum_{k=1}^{m} \mathrm{Op}(h_{1k}(t))(D_t - \mathrm{Op}(\tau_1(t)))^{m-k}u(t)\\
&\qquad\qquad + \sum_{j=1}^{m} \mathrm{Op}(r_j(t)) D_t^{m-j} u(t) = g(t)\\
&\qquad\qquad \Leftrightarrow\\
&(D_t - \mathrm{Op}(\tau_1(t)))W_m(t) = -\sum_{k=1}^{m} \mathrm{Op}(h_{1k}(t))W_{m-k+1}(t)\\
&\qquad - \sum_{j=1}^{m-1}\left(\sum_{q=1}^{m-j} \mathrm{Op}(r_j(t)) \circ \mathrm{Op}(w_{m-j,q}(t))W_{m-j-q+1}(t) + \mathrm{Op}(r_j(t))W_{m-j+1}(t)\right)\\
&\qquad - \mathrm{Op}(r_m(t))W_1(t) + g(t),
\end{aligned}$$

that is,

$$\begin{cases}
(D_t - \mathrm{Op}(\tau_1(t)))W_1(t) = \quad W_2(t)\\
\qquad\dots\\
(D_t - \mathrm{Op}(\tau_1(t)))W_{m-1}(t) = \quad W_m(t)\\
(D_t - \mathrm{Op}(\tau_1(t)))W_m(t) = -\displaystyle\sum_{k=1}^{m} \mathrm{Op}(h_{1k}(t))W_{m-k+1}(t)\\
\qquad - \displaystyle\sum_{j=1}^{m-1}\left(\sum_{q=1}^{m-j} \mathrm{Op}(r_j(t)) \circ \mathrm{Op}(w_{m-j,q}(t))W_{m-j-q+1}(t) + \mathrm{Op}(r_j(t))W_{m-j+1}(t)\right)\\
\qquad - \mathrm{Op}(r_m(t))W_1(t) + g(t),
\end{cases}$$

which has the form (A.5) with $Y(t) = (\underbrace{0,\dots,0}_{m-1 \text{ times}}, g(t))^t$, as claimed, since $\kappa_1(t) = \operatorname{diag}(\tau_1(t),\dots,\tau_1(t))$, while the coefficients of the components of W in the right-hand sides of the equations are all symbols of order $(0,0)$, since $S^{-\infty,-\infty} \subset S^{0,0}$.

The next Lemma A.19 from [16], see also [8, 9] and [23], is the key result to achieve, from (A.12) and the expressions of E_0 and R, the correct regularity of u.

Lemma A.19. *There exists a ($m\times mn$)-dimensional matrix $\Upsilon_n \in C^\infty([0,T_0], S^{0,0}(\mathbb{R}^d)$ such that the k-th row consists of symbols of order $(l-m+k, l-m+k)$, $k = 0,\dots,m-1$, and*

$$\begin{pmatrix} u(t) \\ \dots \\ D_t^{m-1}u(t) \end{pmatrix} = \operatorname{Op}(\Upsilon_n(t))W(t), \quad t \in [0,T_0].$$

Assume that $g \in C([0,T], H^{z,\zeta})$, $(z,\zeta) \in \mathbb{R}^2$. Then, the Cauchy problem for the first order system (A.5) with $s = 0$, equivalent to (4.4), fulfills all the assumptions of Theorem A.6. An application of Theorem A.6, together with (A.12) and Lemma A.19 initially gives

$$\begin{aligned} \begin{pmatrix} u(t) \\ \dots \\ D_t^{m-1}u(t) \end{pmatrix} &= [\operatorname{Op}(\Upsilon_n(t)) \circ (E_0(t,0) + R(t,0)) \circ \operatorname{Op}(b)]U_0 \\ &+ i\int_0^t [\operatorname{Op}(\Upsilon_n(t)) \circ (E_0(t,s) + R(t,s))]Y(s)ds, t \in [0,T_0]. \end{aligned}$$

Then, taking into account that the only non-vanishing entries of Y coincide with g, computations with matrices, the structure of the entries of Υ_n and b, and further applications of Theorem 3.5 give

$$\begin{aligned} u(t) &= \sum_{j=0}^{m-1} \left[\sum_{p=1}^{n} \operatorname{Op}_{\varphi_p(t,0)}(z^0_{pj}(t)) + \operatorname{Op}(r^0_j(t)) \right] u_j \\ &+ i\int_0^t \left[\sum_{p=1}^{n} \operatorname{Op}_{\varphi_p(t,s)}(z^1_p(t,s)) + \operatorname{Op}(r^1(t,s)) \right] g(s)ds, \\ &= v_0(t) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,.,y)g(s,y)\,dyds, \end{aligned} \tag{A.26}$$

where

- the phase functions φ_p are solution to the eikonal equations (A.3), with θ_p in place of $\varkappa$, $p = 1,\dots,n$;
- $z^0_{pj} \in C^\infty([0,T_0], S^{l-1-j,l-1-j})$, $p = 1,\dots,n$, $r^0_j \in C^\infty([0,T_0], S^{-\infty,-\infty})$, $j = 0,\dots,m-1$, so that $v_0 \in \bigcap_{j\geq 0} C^j([0,T_0], H^{z+m-l-j,\zeta+m-l-j})$;

- $\Lambda \in C^\infty(\Delta_{T_0}, \mathcal{S}')$ is, for any $(t,s) \in \Delta_{T_0}$, the Schwartz kernel of the operator

$$Z_{l-m}(t,s) = i\left[\sum_{p=1}^{n} \mathrm{Op}_{\varphi_p(t,s)}(z_p^1(t,s)) + \mathrm{Op}(r^1(t,s))\right], \tag{A.27}$$

with $z_p^1 \in C^\infty(\Delta_{T_0}, S^{l-m,l-m})$, $p = 1,\dots,m$, $r^1 \in C^\infty(\Delta_{T_0}, S^{-\infty,-\infty})$, so that also

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,.,y) g(s,y)\, dyds \in \bigcap_{j\geq 0} C^j([0,T_0], H^{z+m-l-j,\zeta+m-l-j}).$$

Notice the usual abuse of notation, using the kernel $\Lambda(t,s)$ in the *distributional integral* in (A.26). By Proposition A.2, $\Lambda(t,s)$ differs by an element of $C^\infty(\Delta_{T_0}, \mathcal{S})$ from the kernel of

$$\widetilde{Z}_{l-m}(t,s) = i\sum_{p=1}^{n} \mathrm{Op}_{\varphi_p(t,s)}(z_p^1(t,s)). \tag{A.28}$$

By the analysis in [1], in the case of involutive roots analogous formulae can be obtained for u and Λ. Namely, the final expression (A.26) for u, $v_0 \in \bigcap_{j\geq 0} C^j([0,T_0],\, H^{z-j,\zeta-j})$, as well as (A.27) and (A.28) with $l = m$, hold true.

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