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EVERY ZERO-DIMENSIONAL HOMOGENEOUS SPACE IS STRONGLY HOMOGENEOUS UNDER DETERMINACY

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Abstract. All spaces are assumed to be separable and metrizable. We show that, assuming the Axiom of Determinacy, every zero-dimensional homogeneous space is strongly homogeneous (that is, all its non-empty clopen subspaces are homeomorphic), with the trivial exception of locally compact spaces. In fact, we obtain a more general result on the uniqueness of zero-dimensional homogeneous spaces which generate a given Wadge class. This extends work of van Engelen (who obtained the corresponding results for Borel spaces) and complements a result of van Douwen.

1. Introduction

Throughout this article, unless we specify otherwise, we will be working in the theory $\mathsf{ZF}+\mathsf{DC}$, that is, the usual axioms of Zermelo-Fraenkel (without the Axiom of Choice) plus the Principle of Dependent Choices (see Section 2 for more details). By space we always mean separable metrizable topological space. A space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \longrightarrow X$ such that $h(x) = y$. For example, using translations, it is easy to see that every topological group is homogeneous (as [vE3, Corollary 3.6.6] shows, the converse is not true, not even for zero-dimensional Borel spaces). Homogeneity is a classical notion in topology, which has been studied in depth (see for example [AvM]). In particular, in his remarkable doctoral thesis [vE3] (see also [vE1] and [vE2]), Fons van Engelen gave a complete classification of the homogeneous zero-dimensional Borel spaces. In fact, as we will make more precise, this article is inspired by his work and relies heavily on some of his techniques.

A space X is *strongly homogeneous* (or *h-homogeneous*) if every non-empty clopen subspace of X is homeomorphic to X. This notion has been studied by several authors, both "instrumentally" and for its own sake (see the list of references in [Me1]). It is well-known that every zero-dimensional strongly homogeneous space is homogeneous (see for example [vE3, 1.9.1] or [Me2, Proposition 3.32]). Our main result shows that, under the Axiom of Determinacy (briefly, AD) the converse also holds (with the trivial exception of locally compact spaces, see Proposition 2.5). For the proof, see Corollary 15.3.

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Theorem 1.1. *Assume* AD*. If X is a zero-dimensional homogeneous space that is not locally compact then X is strongly homogeneous.*

The above theorem follows from a uniqueness result about zero-dimensional homogeneous spaces, namely Theorem 15.2, which is of independent interest. This theorem essentially states that, for every sufficiently high level of complexity $\boldsymbol{\Gamma}$, there are at most two homogeneous zero-dimensional spaces of complexity exactly $\boldsymbol{\Gamma}$ (one meager and one Baire).

Our fundamental tool will be Wadge theory, which was founded by William Wadge in his doctoral thesis [Wa1] (see also [Wa2]), and has become a classical topic in descriptive set theory. In fact, most of this article (Sections 3 to 13) is purely Wadge-theoretic in character. The ultimate goal of the Wadge-theoretic portion of the paper is to show that good Wadge classes are closed under intersection with $\boldsymbol{\Pi}^0_2$ sets (see Section 12), hence they are reasonably closed (see Section 13). Homogeneity comes into play in Section 14, where we show that $[X]$ is a good Wadge class whenever X is a homogeneous space of sufficiently high complexity. This will allow us to use a theorem of Steel from [St2], which will in turn yield the uniqueness result mentioned above (see Section 15). In the preceding sections, the necessary tools are developed. More specifically, Section 4 is devoted to the analysis of the selfdual Wadge classes, Sections 5 to 8 develop the machinery of relativization through Hausdorff operations, and Sections 9 to 11 develop the notions of level and expansion.

The application of Wadge theory to the study of homogeneous spaces was pioneered by van Engelen in [vE3], who obtained the classification mentioned above. As a corollary (namely, [vE3, Corollary 4.4.6]), he obtained the Borel version of Theorem 1.1. The reason why his results are limited to Borel spaces is that they are all based on the fine analysis of the Borel Wadge classes given by Louveau in [Lo1]. Fully extending this analysis beyond the Borel realm appears to be a very hard problem (although partial results have been obtained in [Fo]). Here, we will follow a different strategy, and we will "substitute" facts from [Lo1] about Borel Wadge classes with more general results about arbitrary Wadge classes (under AD). Furthermore, since most of the literature on Wadge theory only deals with ω^ω as the ambient space, while Steel's theorem is stated for 2^ω, we decided to work in the context of arbitrary zero-dimensional uncountable Polish spaces. With regard to these issues, Louveau's book [Lo2] and van Wesep's results on Hausdorff operations from [VW1] were crucial.

At this point, it is natural to ask whether assuming AD is really necessary in the above results. As the following theorem shows, the answer is "yes". This result was essentially proved in [vD], but our exposition is based on [vM, Theorem 5.1]. Following [vM], we will say that $X \subseteq \mathbb{R}$ is a *bi-Bernstein set* if $K \cap X \neq \varnothing$ and $K \cap (\mathbb{R} \setminus X) \neq \varnothing$ for every $K \subseteq \mathbb{R}$ that is homeomorphic to 2^ω.

Theorem 1.2 (van Douwen)**.** *There exists a* ZFC *example X of a homogeneous zero-dimensional space that is not locally compact and not strongly homogeneous.*

Proof. Let X be the space given by [vM, proof of Theorem 5.1]. Notice that X is homogeneous because X is a subgroup of $\mathbb{R}$. Furthermore, X is a bi-Bernstein set by [vM, Proposition 4.5]. It follows that both X and $\mathbb{R} \setminus X$ are dense in $\mathbb{R}$. In particular, X is zero-dimensional and not locally compact.

Given any Borel subset B of X, pick a Borel subset A of $\mathbb{R}$ such that $A \cap X = B$, then define $\overline{\mu}(B) = \mu(A)$, where μ denotes the Lebesgue measure on $\mathbb{R}$. Using the fact that X is bi-Bernstein, it is easy to check that $\overline{\mu}$ is a well-defined measure on the Borel subsets of X. The crucial property of $\overline{\mu}$, as given by the statement of [vM, Theorem 5.1], is that if B and C are homeomorphic Borel subspaces of X, then $\overline{\mu}(B) = \overline{\mu}(C)$.

Now pick $a, b, c \in \mathbb{R}\backslash X$ such that $a < b < c$. Observe that $U = (a, b) \cap X$ and $V = (a, c) \cap X$ are non-empty clopen subsets of X. Furthermore, it is clear from the definition of $\overline{\mu}$ that $\overline{\mu}(U) = b - a \neq c - a = \overline{\mu}(V)$. Therefore U and V are not homeomorphic, which concludes the proof. $\square$

However, we do not know the answer to the following question. Recall that, when $\mathbf{\Gamma} = \mathbf{\Sigma}^1_n$ or $\mathbf{\Gamma} = \mathbf{\Pi}^1_n$ for some $n \geqslant 1$, a space is $\mathbf{\Gamma}$ if it is homeomorphic to a $\mathbf{\Gamma}$ subspace of some Polish space (see [MZ, Section 4] for a more detailed treatment).

Question 1.3. Assuming $\mathsf{V} = \mathsf{L}$, is it possible to construct a zero-dimensional $\mathbf{\Pi}^1_1$ or $\mathbf{\Sigma}^1_1$ space that is homogeneous, not locally compact, and not strongly homogeneous?

The above question is natural because there are many examples of properties (such as the perfect set property[1]) that are known to hold for all spaces under AD, for which definable counterexamples can be constructed under $\mathsf{V} = \mathsf{L}$. Notice that $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_1$ are optimal by [vE3, Corollary 4.4.6]. For other relevant examples, see [vEMS, Theorem 2.6], [Mi], and [Vi].

2. Preliminaries and notation

Let Z be a set, and let $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$. Define $\check{\mathbf{\Gamma}} = \{Z\backslash A : A \in \mathbf{\Gamma}\}$. We will say that $\mathbf{\Gamma}$ is *selfdual* if $\mathbf{\Gamma} = \check{\mathbf{\Gamma}}$. Also define $\Delta(\mathbf{\Gamma}) = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$.

Definition 2.1 (Wadge)**.** Let Z be a space, and let $A, B \subseteq Z$. We will write $A \leqslant B$ if there exists a continuous function $f : Z \longrightarrow Z$ such that $A = f^{-1}[B]$.[2] In this case, we will say that A is *Wadge-reducible* to B, and that f *witnesses* the reduction. We will write $A < B$ if $A \leqslant B$ and $B \nleqslant A$. We will write $A \equiv B$ if $A \leqslant B$ and $B \leqslant A$.

Definition 2.2 (Wadge)**.** Let Z be a space. Given $A \subseteq Z$, define

$$[A] = \{B \subseteq Z : B \leqslant A\}.$$

We will say that $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$ is a *Wadge class* if there exists $A \subseteq Z$ such that $\mathbf{\Gamma} = [A]$. We will say that $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$ is *continuously closed* if $[A] \subseteq \mathbf{\Gamma}$ for every $A \in \mathbf{\Gamma}$.

Both of the above definitions depend of course on the space Z. Often, for the sake of clarity, we will specify what the ambient space is by saying, for example, that "$A \leqslant B$ in Z" or "$\mathbf{\Gamma}$ is a Wadge class in Z". We will say that $A \subseteq Z$ is *selfdual* if $A \leqslant Z\backslash A$ in Z. It is easy to check that A is selfdual iff $[A]$ is selfdual. Given a space Z, we will also use the following shorthand notation:

- $\mathsf{SD}(Z) = \{\mathbf{\Gamma} : \mathbf{\Gamma}$ is a selfdual Wadge class in $Z\}$,

[1] To see that every space has the perfect set property under AD, proceed as in [Ke, Section 21.A]. For the counterexample under $\mathsf{V} = \mathsf{L}$, see [Ka, Theorem 13.12].

[2] Wadge-reduction is usually denoted by $\leqslant_{\mathsf{W}}$, which allows to distinguish it from other types of reduction (such as Lipschitz-reduction). Since we will not consider any other type of reduction in this article, we decided to simplify the notation.

- $\mathsf{NSD}(Z) = \{\mathbf{\Gamma} : \mathbf{\Gamma}$ is a non-selfdual Wadge class in $Z\}$.

Our reference for descriptive set theory is [Ke]. In particular, we assume familiarity with the basic theory of Borel sets and Polish spaces, and use the same notation as in [Ke, Section 11.B]. For example, given a space Z, we use $\mathbf{\Sigma}^0_1(Z)$, $\mathbf{\Pi}^0_1(Z)$, and $\mathbf{\Delta}^0_1(Z)$ to denote the collection of all open, closed, and clopen subsets of Z respectively. Our reference for other set-theoretic notions is [Je].

The classes defined below constitute the so-called *difference hierarchy* (or *small Borel sets*). For a detailed treatment, see [Ke, Section 22.E] or [vE3, Chapter 3]. Here, we will only mention that the $\mathsf{D}_\eta(\mathbf{\Sigma}^0_\xi(Z))$ are among the simplest concrete examples of Wadge classes (see Propositions 9.3 and 9.6).

Definition 2.3 (Kuratowski)**.** Let Z be a space, let $1 \leqslant \eta < \omega_1$ and $1 \leqslant \xi < \omega_1$. Given a sequence of sets $(A_\mu : \mu < \eta)$, define

$$\mathsf{D}_\eta(A_\mu : \mu < \eta) = \begin{cases} \bigcup\{A_\mu \setminus \bigcup_{\zeta<\mu} A_\zeta : \mu < \eta \text{ and } \mu \text{ is odd}\} & \text{if } \eta \text{ is even,} \\ \bigcup\{A_\mu \setminus \bigcup_{\zeta<\mu} A_\zeta : \mu < \eta \text{ and } \mu \text{ is even}\} & \text{if } \eta \text{ is odd.} \end{cases}$$

Define $A \in \mathsf{D}_\eta(\mathbf{\Sigma}^0_\xi(Z))$ if there exists a $\subseteq$-increasing sequence $(A_\mu : \mu < \eta)$ such that $A_\mu \in \mathbf{\Sigma}^0_\xi(Z)$ for each μ and $A = \mathsf{D}_\eta(A_\mu : \mu < \eta)$.

For an introduction to the topic of games, we refer the reader to [Ke, Section 20]. Here, we only want to give the precise definition of determinacy. A *play* of the game $G(\omega, X)$ is decribed by the diagram

$$\begin{array}{cccccc} \text{I} & a_0 & & a_2 & & \cdots \\ \text{II} & & a_1 & & a_3 & \cdots, \end{array}$$

in which $a_n \in \omega$ for every $n \in \omega$ and $X \subseteq \omega^\omega$ is called the *payoff set*. We will say that Player I *wins* this play of the game $G(\omega, X)$ if $(a_0, a_1, \ldots) \in X$. Player II *wins* if Player I does not win.

A *strategy* for a player is a function $\sigma : \omega^{<\omega} \longrightarrow \omega$. We will say that σ is a *winning strategy* for Player I if setting $a_{2n} = \sigma(a_1, a_3, \ldots, a_{2n-1})$ for each n makes Player I win for every $(a_1, a_3, \ldots) \in \omega^\omega$. A winning strategy for Player II is defined similarly. We will say that the game $G(\omega, X)$ (or simply the set X) is *determined* if (exactly) one of the players has a winning strategy. The *Axiom of Determinacy* (briefly, AD) states that every $X \subseteq \omega^\omega$ is determined.[3]

It is well-known that AD is incompatible with the Axiom of Choice (see [Je, Lemma 33.1]). This is the reason why, throughout this article, we will be working in $\mathsf{ZF} + \mathsf{DC}$. [4] Recall that the *Principle of Dependent Choices* (briefly, DC) states that if R is a binary relation on a non-empty set A such that for every $a \in A$ there exists $b \in A$ such that $(b, a) \in R$, then there exists a sequence $(a_0, a_1, \ldots) \in A^\omega$ such that $(a_{n+1}, a_n) \in R$ for every $n \in \omega$. This principle is what is needed to carry out recursive constructions of length ω. Another consequence (in fact, an equivalent formulation) of DC is that a relation R on a set A is well-founded iff there exists no sequence $(a_0, a_1, \ldots) \in A^\omega$ such that $(a_{n+1}, a_n) \in R$ for every $n \in \omega$ (see [Je, Lemma 5.5.ii]). Furthermore, DC implies the Countable Axiom of Choice (see [Je, Exercise 5.7]). To the reader who is unsettled by the lack of the full Axiom of Choice, we recommend [HR].

[3] Quite amusingly, Van Wesep referred to AD as a "frankly heretical postulate" (see [VW1, page 64]), and Steel deemed it "probably false" (see [St1, page 63]).

[4] The consistency of $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$ can be obtained under suitable large cardinal assumptions (see [Ka, Proposition 11.13] and [Ne]).

We conclude this section with some miscellaneous topological definitions and results. We will write $X \approx Y$ to mean that the spaces X and Y are homeomorphic. Given a function $s : F \longrightarrow 2$, where $F \subseteq \omega$ is finite, we will use the notation $[s] = \{z \in 2^\omega : s \subseteq z\}$. A space is *crowded* if it is non-empty and it has no isolated points. A space X is *Baire* if every non-empty open subset of X is non-meager in X. A space X is *meager* if X is a meager subset of X. Proposition 2.4 is a particular case of [FZ, Lemma 3.1] (see also [vE3, 1.12.1]). Proposition 2.5 is the reason why we refer to locally compact spaces as the "trivial exceptions". Theorem 2.8 is a special case of [Te, Theorem 2.4] (see also [Me1, Theorem 2 and Appendix A] or [Me2, Theorem 3.2 and Appendix B]).

Proposition 2.4 (Fitzpatrick, Zhou)**.** *Let X be a homogeneous space. Then X is either a meager space or a Baire space.*

Proposition 2.5. *Let X be a zero-dimensional locally compact space. Then X is homogeneous iff X is discrete, $X \approx 2^\omega$, or $X \approx \omega \times 2^\omega$.*

Proof. The right-to-left implication is trivial. For the left-to-right implication, use the well-known characterization of 2^ω as the unique zero-dimensional crowded compact space (see [Ke, Theorem 7.4]). □

Proposition 2.6. *Let X be a zero-dimensional homogeneous space. If there exists a non-empty Polish $U \in \mathbf{\Sigma}^0_1(X)$ then X is Polish.*

Proof. Let $U \in \mathbf{\Sigma}^0_1(X)$ be non-empty and Polish. Since X is zero-dimensional, we can assume without loss of generality that $U \in \mathbf{\Delta}^0_1(X)$. Let $\mathcal{U} = \{h[U] : h \text{ is a homeomorphism of } X\}$. Notice that $\mathcal{U}$ is a cover of X because X is homogeneous and U is non-empty. Let $\{U_n : n \in \omega\}$ be a countable subcover of $\mathcal{U}$. Define $V_n = U_n \setminus \bigcup_{k<n} U_k$ for $n \in \omega$, and observe that each V_n is Polish. Since $V_n \cap V_m$ whenever $m \neq n$, it follows from [Ke, Proposition 3.3.iii] that $X = \bigcup_{n \in \omega} V_n$ is Polish. □

Proposition 2.7. *Assume* AD. *Let Z be a Polish space, and let X be a dense Baire subspace of Z. Then X is comeager in Z.*

Proof. Notice that X has the Baire property because we are assuming AD (see [Ke, Section 21.C]). So, by [Ke, Proposition 8.23.ii], we can write $X = G \cup M$, where $G \in \mathbf{\Pi}^0_2(Z)$ and M is meager in Z. It will be enough to show that G is dense in Z. Assume, in order to get a contradiction, that there exists a non-empty open subset U of Z such that $U \cap G = \varnothing$. Observe that $U \cap X$ is a non-empty open subset of X because X is dense in Z. Furthermore, using the density of X again, it is easy to see that $M = M \cap X$ is meager in X. Since $U \cap X \subseteq M$, this contradicts the fact that X is a Baire space. □

Theorem 2.8 (Terada)**.** *Let X be a non-compact space. Assume that X has a base $\mathcal{B} \subseteq \mathbf{\Delta}^0_1(X)$ such that $U \approx X$ for every $U \in \mathcal{B}$. Then X is strongly homogeneous.*

3. The basics of Wadge theory

The following simple lemma will allow us to generalize many Wadge-theoretic results from ω^ω to an arbitrary zero-dimensional Polish space. Recall that, given a space Z and $W \subseteq Z$, a *retraction* is a continuous function $\rho : Z \longrightarrow W$ such that $\rho \upharpoonright W = \mathsf{id}_W$. By [Ke, Theorem 7.8], every zero-dimensional Polish space is homeomorphic to a closed subspace Z of ω^ω, and by [Ke, Proposition 2.8] there exists a retraction $\rho : \omega^\omega \longrightarrow Z$.

Lemma 3.1. *Let $Z \subseteq \omega^\omega$, and let $\rho : \omega^\omega \longrightarrow Z$ be a retraction. Fix $A, B \subseteq Z$. Then $A \leqslant B$ in Z iff $\rho^{-1}[A] \leqslant \rho^{-1}[B]$ in ω^ω.*

Proof. If $f : Z \longrightarrow Z$ witnesses that $A \leqslant B$ in Z, then $f \circ \rho : \omega^\omega \longrightarrow \omega^\omega$ will witness that $\rho^{-1}[A] \leqslant \rho^{-1}[B]$ in ω^ω. On the other hand, if $f : \omega^\omega \longrightarrow \omega^\omega$ witnesses that $\rho^{-1}[A] \leqslant \rho^{-1}[B]$ in ω^ω, then $\rho \circ (f \upharpoonright Z) : Z \longrightarrow Z$ will witness that $A \leqslant B$ in Z. □

The following result (commonly known as "Wadge's Lemma") shows that antichains with respect to $\leqslant$ have size at most 2.

Lemma 3.2 (Wadge)**.** *Assume* AD*. Let Z be a zero-dimensional Polish space, and let $A, B \subseteq Z$. Then either $A \leqslant B$ or $Z \setminus B \leqslant A$.*

Proof. For the case $Z = \omega^\omega$, see [Ke, proof of Theorem 21.14]. To obtain the full result from this particular case, use Lemma 3.1 and the remarks preceding it. □

Theorem 3.3 (Martin, Monk)**.** *Assume* AD*. Let Z be a zero-dimensional Polish space. Then the relation $\leqslant$ on $\mathcal{P}(Z)$ is well-founded.*

Proof. For the case $Z = \omega^\omega$, see [Ke, proof of Theorem 21.15]. To obtain the full result from this particular case, use Lemma 3.1 and the remarks preceding it. □

Given a zero-dimensional Polish space Z, define

$$\mathsf{W}(Z) = \{\{\mathbf{\Gamma}, \check{\mathbf{\Gamma}}\} : \mathbf{\Gamma} \text{ is a Wadge class in } Z\}.$$

Notice that, by the two previous results, the ordering induced by $\subseteq$ on $\mathsf{W}(Z)$ is a well-order. Therefore, there exists an order-isomorphism $\phi : \mathsf{W}(Z) \longrightarrow \Theta$ for some ordinal Θ.[5] The reason for the "1+" in the definition below is simply a matter of technical convenience (see [AHN, page 45]).

Definition 3.4. Let Z be a zero-dimensional Polish space, and let $\mathbf{\Gamma}$ be a Wadge class in Z. Define

$$||\mathbf{\Gamma}|| = 1 + \phi(\{\mathbf{\Gamma}, \check{\mathbf{\Gamma}}\}).$$

We will say that $||\mathbf{\Gamma}||$ is the *Wadge-rank* of $\mathbf{\Gamma}$.

It is easy to check that $\{\{\varnothing\}, \{Z\}\}$ is the minimal element of $\mathsf{W}(Z)$. Furthermore, elements of the form $\{\mathbf{\Gamma}, \check{\mathbf{\Gamma}}\}$ for $\mathbf{\Gamma} \in \mathsf{NSD}(Z)$ are always followed by $\{\mathbf{\Delta}\}$ for some $\mathbf{\Delta} \in \mathsf{SD}(Z)$, while elements of the form $\{\mathbf{\Delta}\}$ for $\mathbf{\Delta} \in \mathsf{SD}(Z)$ are always followed by $\{\mathbf{\Gamma}, \check{\mathbf{\Gamma}}\}$ for some $\mathbf{\Gamma} \in \mathsf{NSD}(Z)$. This was proved by Van Wesep for $Z = \omega^\omega$ (see [VW1, Corollary to Theorem 2.1]), and it can be generalized to arbitrary uncountable zero-dimensional Polish spaces using Corollary 4.4 and the machinery of relativization that we will develop in Sections 6 to 8. Since these facts will not be needed in the remainder of the paper, we omit the proof.

In fact, as Proposition 6.6 (together with Theorem 6.5) will show, the ordering of the non-selfdual classes is independent of the space Z. However, the situation is more delicate for selfdual classes. For example, it follows easily from Corollary 4.4 that if $\mathbf{\Gamma}$ is a Wadge class in 2^ω such that $||\mathbf{\Gamma}||$ is a limit ordinal of countable cofinality, then $\mathbf{\Gamma}$ is non-selfdual. On the other hand, if $\mathbf{\Gamma}$ is a Wadge class in ω^ω

[5] For a characterization of Θ, see [So, Definition 0.1 and Lemma 0.2].

such that $||\mathbf{\Gamma}||$ is a limit ordinal of countable cofinality, then $\mathbf{\Gamma}$ is selfdual (see [VW1, Corollary to Theorem 2.1] again).

We conclude this section with an elementary result, which shows that clopen sets are "neutral sets" for Wadge-reduction. By this we mean that, apart from trivial exceptions, intersections or unions with these sets do not change the Wadge class. In Section 12, we will prove more sophisticated closure properties.

Proposition 3.5. *Let Z be a space, let $\mathbf{\Gamma}$ be a Wadge class in Z, and let $A \in \mathbf{\Gamma}$.*

- *Assume that $\mathbf{\Gamma} \neq \{Z\}$. Then $A \cap V \in \mathbf{\Gamma}$ for every $V \in \mathbf{\Delta}^0_1(Z)$.*
- *Assume that $\mathbf{\Gamma} \neq \{\varnothing\}$. Then $A \cup V \in \mathbf{\Gamma}$ for every $V \in \mathbf{\Delta}^0_1(Z)$.*

Proof. We will only prove the first statement, since the second one can be obtained by applying it to $\check{\mathbf{\Gamma}}$. So pick $V \in \mathbf{\Delta}^0_1(Z)$, and assume that $\mathbf{\Gamma} = [B]$. Choose $f : Z \longrightarrow Z$ witnessing that $A \leqslant B$. Since $\mathbf{\Gamma} \neq \{Z\}$, we can fix $z \in Z \setminus B$. Define $g : Z \longrightarrow Z$ by setting

$$g(x) = \begin{cases} f(x) & \text{if } x \in V, \\ z & \text{if } x \in Z \setminus V. \end{cases}$$

Since $V \in \mathbf{\Delta}^0_1(Z)$ and f is continuous, the function g is continuous as well. It is clear that g witnesses that $A \cap V \leqslant B$. □

4. The analysis of selfdual sets

The aim of this section is to show that every selfdual set can be built using non-selfdual sets of lower complexity (apply Corollary 4.4 with $U = Z$). This is a well-known result (see for example [Lo2, Lemma 7.3.4]). Our approach is essentially the same as the one used in the proof of [AM, Theorem 16] or in [MR, Theorem 5.3]. However, since the proof becomes slightly simpler in our context, we give all the details below.

Given a space Z and $A \subseteq Z$, define

$$\mathcal{I}(A) = \{V \in \mathbf{\Delta}^0_1(Z) : \text{ there exists a partition } \mathcal{U} \subseteq \mathbf{\Delta}^0_1(V) \text{ of } V \text{ such that } U \cap A < A \text{ for every } U \in \mathcal{U}\}.$$

Notice that $\mathcal{I}(A)$ is *σ-additive*, in the sense that if $V_n \in \mathcal{I}(A)$ for $n \in \omega$ and $V = \bigcup_{n \in \omega} V_n \in \mathbf{\Delta}^0_1(Z)$, then $V \in \mathcal{I}(A)$.

We begin with two simple preliminary results. Recall that $F \subseteq 2^\omega$ is a *flip-set* if whenever $z \in F$ and $w \in 2^\omega$ are such that $|\{n \in \omega : z(n) \neq w(n)\}| = 1$ then $w \notin F$.

Lemma 4.1. *Let $F \subseteq 2^\omega$ be a flip-set. Then F does not have the Baire property.*

Proof. Assume, in order to get a contradiction, that F has the Baire property. Since $2^\omega \setminus F$ is also a flip-set, we can assume without loss of generality that F is non-meager in 2^ω. By [Ke, Proposition 8.26], we can fix $n \in \omega$ and $s \in 2^n$ such that $F \cap [s]$ is comeager in $[s]$. Fix $k \in \omega \setminus n$ and let $h : [s] \longrightarrow [s]$ be the homeomorphism defined by

$$h(x)(i) = \begin{cases} x(i) & \text{if } i \neq k, \\ 1 - x(i) & \text{if } i = k \end{cases}$$

for $x \in [s]$ and $i \in \omega$. Observe that $([s] \cap F) \cap h[[s] \cap F]$ is comeager in $[s]$, hence it is non-empty. It is easy to realize that this contradicts the definition of flip-set. □

Lemma 4.2. *Let Z be a space, and let $A \subseteq Z$ be a selfdual set such that $A \notin \mathbf{\Delta}^0_1(Z)$. Assume that $V \in \mathbf{\Delta}^0_1(Z)$ and $V \notin \mathcal{I}(A)$. Then $V \cap A \leqslant V \setminus A$ in V.*

Proof. Using Proposition 3.5, one sees that $V \cap A \leqslant A$ and $V \backslash A \leqslant Z \backslash A$, where both reductions are in Z. On the other hand, since $V \cap A < A$ would contradict the assumption that $V \notin \mathcal{I}(A)$, we see that $V \cap A \equiv A$. It follows that $V \backslash A \leqslant Z \backslash A \equiv A \equiv V \cap A$. Let $f : Z \longrightarrow Z$ be a function witnessing that $V \backslash A \leqslant V \cap A$. Notice that $V \backslash A \neq \varnothing$, otherwise we would have $V = V \cap A \equiv A$, contradicting the assumption that $A \notin \boldsymbol{\Delta}^0_1(Z)$. So we can fix $z \in V \backslash A$, and define $g : Z \longrightarrow V$ by setting

$$g(x) = \begin{cases} x & \text{if } x \in V, \\ z & \text{if } x \in Z \backslash V. \end{cases}$$

Since $V \in \boldsymbol{\Delta}^0_1(Z)$, the function g is continuous. Finally, it is straightforward to verify that $g \circ (f \upharpoonright V) : V \longrightarrow V$ witnesses that $V \cap A \leqslant V \backslash A$ in V. □

Theorem 4.3. *Assume* AD. *Let Z be a zero-dimensional Polish space, and let A be a selfdual subset of Z. Assume that $A \notin \boldsymbol{\Delta}^0_1(Z)$. Then $\boldsymbol{\Delta}^0_1(Z) = \mathcal{I}(A)$.*

Proof. Assume in order to get a contradiction, that $V \in \boldsymbol{\Delta}^0_1(Z) \backslash \mathcal{I}(A)$. Fix a complete metric on Z that induces the given Polish topology. We will recursively construct sets V_n and functions $f_n : V_n \longrightarrow V_n$ for $n \in \omega$. Before specifying which properties we require from them, we introduce some more notation. Given a set X and a function $f : X \longrightarrow X$, set $f^0 = \mathsf{id}_X$ and $f^1 = f$. Furthermore, given $m, n \in \omega$ such that $m \leqslant n$ and $z \in 2^\omega$ (or just $z \in 2^{[m,n]}$), define

$$f^z_{[m,n]} = f^{z(m)}_m \circ \cdots \circ f^{z(n)}_n.$$

We will make sure that the following conditions are satisfied for every $n \in \omega$:

(1) $V_n \in \boldsymbol{\Delta}^0_1(Z)$,
(2) $V_n \notin \mathcal{I}(A)$,
(3) $V_m \supseteq V_n$ whenever $m \leqslant n$,
(4) $f_n : V_n \longrightarrow V_n$ witnesses that $V_n \cap A \leqslant V_n \backslash A$ in V_n,
(5) $\mathsf{diam}(f^s_{[m,n]}[V_{n+1}]) \leqslant 2^{-n}$ whenever $m \leqslant n$ and $s \in 2^{[m,n]}$.

Start by setting $V_0 = V$ and let $f_0 : V_0 \longrightarrow V_0$ be given by Lemma 4.2. Now fix $n \in \omega$, and assume that V_m and f_m have already been constructed for every $m \leqslant n$. Fix a partition $\mathcal{U}$ of Z consisting of clopen sets of diameter at most 2^{-n}. Given $m \leqslant n$ and $s \in 2^{[m,n]}$, define

$$\mathcal{V}^s_m = \{(f^s_{[m,n]})^{-1}[U \cap V_m] : U \in \mathcal{U}\}.$$

Observe that each $\mathcal{V}^s_m \subseteq \boldsymbol{\Delta}^0_1(V_n)$ because each $f^s_{[m,n]}$ is continuous. Furthermore, it is clear that each $\mathcal{V}^s_m$ consists of pairwise disjoint sets, and that $\bigcup \mathcal{V}^s_m = V_n$. Since there are only finitely many $m \leqslant n$ and $s \in 2^{[m,n]}$, it is possible to obtain a partition $\mathcal{V} \subseteq \boldsymbol{\Delta}^0_1(V_n)$ of V_n that simultaneously refines each $\mathcal{V}^s_m$. This clearly implies that any choice of $V_{n+1} \in \mathcal{V}$ will satisfy condition (5). On the other hand since $\mathcal{I}(A)$ is σ-additive and $V_n \notin \mathcal{I}(A)$, it is possible to choose $V_{n+1} \in \mathcal{V}$ such that $V_{n+1} \notin \mathcal{I}(A)$, thus ensuring that condition (2) is satisfied as well. To obtain $f_{n+1} : V_{n+1} \longrightarrow V_{n+1}$ that satisfies condition (4), simply apply Lemma 4.2. This concludes the construction.

Fix an arbitrary $y_{n+1} \in V_{n+1}$ for $n \in \omega$. Given $m \in \omega$ and $z \in 2^\omega$, observe that the sequence $(f^z_{[m,n]}(y_{n+1}) : m \leqslant n)$ is Cauchy by condition (5), hence it makes sense to define

$$x^z_m = \lim_{n \to \infty} f^z_{[m,n]}(y_{n+1}).$$

To conclude the proof, we will show that $F = \{z \in 2^\omega : x_0^z \in A\}$ is a flip-set. This will contradict Lemma 4.1, since AD implies that every subset of 2^ω has the Baire property (see [Ke, Section 21.C]).

Define $A^0 = A$ and $A^1 = Z \backslash A$. Given any $m \in \omega$ and $\varepsilon \in 2$, it is clear from the definition of f_m^ε and condition (4) that

$$x \in A \text{ iff } f_m^\varepsilon(x) \in A^\varepsilon$$

for every $x \in V_m$. Furthermore, using the continuity of f_m^ε and the definition of x_m^z, it is easy to see that

$$f_m^{z(m)}(x_{m+1}^z) = x_m^z$$

for every $z \in 2^\omega$ and $m \in \omega$.

Fix $z \in 2^\omega$ and notice that, by the observations in the previous paragraph,

$$x_0^z \in A \text{ iff } x_1^z \in A^{z(0)} \text{ iff } \cdots \text{ iff } x_{m+1}^z \in (\cdots(A^{z(0)})^{z(1)}\cdots)^{z(m)}$$

for every $m \in \omega$. Now fix $w \in 2^\omega$ and $m \in \omega$ such that $z \upharpoonright \omega\backslash\{m\} = w \upharpoonright \omega\backslash\{m\}$ and $z(m) \neq w(m)$. We need to show that $x_0^z \in A$ iff $x_0^w \notin A$. For exactly the same reason as above, we have

$$x_0^w \in A \text{ iff } x_1^w \in A^{w(0)} \text{ iff } \cdots \text{ iff } x_{m+1}^w \in (\cdots(A^{w(0)})^{w(1)}\cdots)^{w(m)}.$$

Since $z \upharpoonright m = w \upharpoonright m$ and $z(m) \neq w(m)$, in order to finish the proof, it will be enough to show that $x_{m+1}^z = x_{m+1}^w$. To see this, observe that

$$x_{m+1}^z = \lim_{n\to\infty} f_{[m+1,n]}^z(y_{n+1}) = \lim_{n\to\infty} f_{[m+1,n]}^w(y_{n+1}) = x_{m+1}^w,$$

where the middle equality uses the assumption $z \upharpoonright \omega\backslash(m+1) = w \upharpoonright \omega\backslash(m+1)$. □

Corollary 4.4. *Assume* AD. *Let Z be a zero-dimensional Polish space, let A be a selfdual subset of Z, and let $U \in \mathbf{\Delta}_1^0(Z)$. Then there exist pairwise disjoint $V_n \in \mathbf{\Delta}_1^0(U)$ and non-selfdual $A_n < A$ in Z for $n \in \omega$ such that $\bigcup_{n\in\omega} V_n = U$ and $\bigcup_{n\in\omega}(A_n \cap V_n) = A \cap U$.*

Proof. As one can easily check, it will be enough to show that there exists a partition $\mathcal{V} \subseteq \mathbf{\Delta}_1^0(U)$ of U such that for every $V \in \mathcal{V}$ either $A \cap V \in \mathbf{\Delta}_1^0(Z)$ or $A \cap V$ is non-selfdual in Z. If this were not the case, then, using Theorem 4.3, one could recursively construct a strictly $\leqslant$-decreasing sequence of subsets of Z, which would contradict Theorem 3.3. □

5. Basic facts on Hausdorff operations

For a history of the following important notion, see [Ha, page 583]. For a modern survey, we recommend [Za]. Most of the proofs in this section are straightforward, hence we leave them to the reader.

Definition 5.1. Given a set Z and $D \subseteq 2^\omega$, define

$$\mathcal{H}_D(A_0, A_1, \ldots) = \{x \in Z : \{n \in \omega : x \in A_n\} \in D\}$$

whenever $A_0, A_1, \ldots \subseteq Z$, where we identify $\{n \in \omega : x \in A_n\}$ with its characteristic function. Functions of this form are called *Hausdorff operations* (or *ω-ary Boolean operations*).

Of course, the function $\mathcal{H}_D$ depends on the set Z, but what Z is will usually be clear from the context. In case there might be uncertainty about the ambient space, we will use the notation $\mathcal{H}_D^Z$. Notice that, once D is specified, the corresponding Hausdorff operation simultaneously defines functions $\mathcal{P}(Z)^\omega \longrightarrow \mathcal{P}(Z)$ for every Z.

The following proposition lists the most basic properties of Hausdorff operations. Given $n \in \omega$, define $s_n : \{n\} \longrightarrow 2$ by setting $s_n(n) = 1$.

Proposition 5.2. *Let I be a set, and let $D_i \subseteq 2^\omega$ for every $i \in I$. Fix an ambient set Z and $A_0, A_1, \ldots \subseteq Z$.*

- $\mathcal{H}_{[s_n]}(A_0, A_1, \ldots) = A_n$ *for all* $n \in \omega$.
- $\bigcap_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \ldots) = \mathcal{H}_D(A_0, A_1, \ldots)$, *where* $D = \bigcap_{i \in I} D_i$.
- $\bigcup_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \ldots) = \mathcal{H}_D(A_0, A_1, \ldots)$, *where* $D = \bigcup_{i \in I} D_i$.
- $Z \setminus \mathcal{H}_D(A_0, A_1, \ldots) = \mathcal{H}_{2^\omega \setminus D}(A_0, A_1, \ldots)$ *for all* $D \subseteq 2^\omega$.

The point of the above proposition is that any operation obtained by combining unions, intersections and complements can be expressed as a Hausdorff operation. For example, if $D = \bigcup_{n \in \omega}([s_{2n+1}] \setminus [s_{2n}])$, then $\mathcal{H}_D(A_0, A_1, \ldots) = \bigcup_{n \in \omega}(A_{2n+1} \setminus A_{2n})$.

The following proposition shows that the composition of Hausdorff operations is again a Hausdorff operation. We will assume that a bijection $\pi : \omega \times \omega \longrightarrow \omega$ has been fixed, and use the notation $\langle m, n \rangle = \pi(m, n)$.

Proposition 5.3. *Let Z be a set, let $D \subseteq 2^\omega$ and $E_m \subseteq 2^\omega$ for $m \in \omega$. Then there exists a set $F \subseteq 2^\omega$ such that*

$$\mathcal{H}_D(B_0, B_1, \ldots) = \mathcal{H}_F(A_0, A_1, \ldots)$$

for all $A_0, A_1, \ldots \subseteq Z$, where $B_m = \mathcal{H}_{E_m}(A_{\langle m,0 \rangle}, A_{\langle m,1 \rangle}, \ldots)$.

Proof. Define $z \in F$ if $\{m \in \omega : \{n \in \omega : \langle m, n \rangle \in z\} \in E_m\} \in D$, where we identify 2^ω with $\mathcal{P}(\omega)$ through characteristic functions. The rest of the proof is a straightforward verification. □

We conclude this section with a result that will easily imply the fundamental Lemma 6.4.

Proposition 5.4. *Let Z and W be sets, let $A_0, A_1, \ldots \subseteq Z$ and $B_0, B_1, \ldots \subseteq W$.*

1. $W \cap \mathcal{H}_D^Z(A_0, A_1, \ldots) = \mathcal{H}_D^W(A_0 \cap W, A_1 \cap W, \ldots)$ *whenever* $W \subseteq Z$.
2. $f^{-1}[\mathcal{H}_D(B_0, B_1, \ldots)] = \mathcal{H}_D(f^{-1}[B_0], f^{-1}[B_1], \ldots)]$ *for all* $f : Z \longrightarrow W$.
3. $f[\mathcal{H}_D(A_0, A_1, \ldots)] = \mathcal{H}_D(f[A_0], f[A_1], \ldots)]$ *for all bijections* $f : Z \longrightarrow W$.

6. Wadge classes and Hausdorff operations

When one tries to give a systematic exposition of Wadge theory, it soon becomes apparent that it would be very useful to be able to talk about "abstract" Wadge classes, as opposed to Wadge classes in a particular space. More precisely, given a Wadge class $\mathbf{\Gamma}$ in some space Z, one would like to find a way to define what a "$\mathbf{\Gamma}$ subset of W" is, for every other space W, while of course preserving suitable coherence properties. It turns out that Hausdorff operations allow us to do exactly that in a rather elegant way, provided that $\mathbf{\Gamma}$ is a non-selfdual Wadge class, Z and W are uncountable zero-dimensional Polish spaces, and AD holds (see also the discussion in Section 3). For an early instance of this idea, see [LSR2, Theorem 4.2].[6] The following is the crucial definition.

Definition 6.1. Given a space Z and $D \subseteq 2^\omega$, define

$$\mathbf{\Gamma}_D(Z) = \{\mathcal{H}_D(A_0, A_1, \ldots) : A_n \in \mathbf{\Sigma}^0_1(Z) \text{ for every } n \in \omega\}.$$

As examples (that will be useful later), consider the following two simple propositions.

Proposition 6.2. *Let $1 \leqslant \eta < \omega_1$. Then there exists $D \subseteq 2^\omega$ such that $\mathbf{\Gamma}_D(Z) = \mathsf{D}_\eta(\mathbf{\Sigma}^0_1(Z))$ for every space Z.*

Proof. This follows from Propositions 5.2 and 5.3 (in case $\eta > \omega$, use a bijection $\pi : \eta \longrightarrow \omega$). The only confusion might result from the fact that the $\mathbf{\Sigma}^0_1(Z)$ sets in the definition of $\mathsf{D}_\eta(\mathbf{\Sigma}^0_1(Z))$ are required to form an increasing sequence, while the $\mathbf{\Sigma}^0_1(Z)$ sets in the definition of $\mathbf{\Gamma}_D(Z)$ have no restrictions. This issue can be resolved by considering $\bigcup_{\mu \leqslant \xi} A_\mu$ instead of A_ξ. □

Proposition 6.3. *Let $1 \leqslant \xi < \omega_1$. Then there exists $D \subseteq 2^\omega$ such that $\mathbf{\Gamma}_D(Z) = \mathbf{\Sigma}^0_\xi(Z)$ for every space Z.*

Proof. This can be proved by induction on ξ, using Propositions 5.2 and 5.3. □

Next, we prove a very useful lemma, which shows that this notion behaves well with respect to subspaces and continuous functions. This lemma is essentially what we refer to when we speak about the "machinery of relativization". It extends (and is inspired by) [vE4, Lemma 2.3].

Lemma 6.4. *Let Z and W be spaces, and let $D \subseteq 2^\omega$.*

(1) *Assume that $W \subseteq Z$. Then $B \in \mathbf{\Gamma}_D(W)$ iff there exists $A \in \mathbf{\Gamma}_D(Z)$ such that $B = A \cap W$.*
(2) *If $f : Z \longrightarrow W$ is continuous and $B \in \mathbf{\Gamma}_D(W)$ then $f^{-1}[B] \in \mathbf{\Gamma}_D(Z)$.*
(3) *If $h : Z \longrightarrow W$ is a homeomorphism then $A \in \mathbf{\Gamma}_D(Z)$ iff $h[A] \in \mathbf{\Gamma}_D(W)$.*

Proof. This is a straightforward consequence of Proposition 5.4. □

The following result (together with the above lemma) is the reason why Hausdorff operations are such an indispensable tool in our treatment of Wadge theory. Its proof is the content of the next two sections.

Theorem 6.5. *Assume AD. Let Z be an uncountable zero-dimensional Polish space. Then*

$$\mathsf{NSD}(Z) = \{\mathbf{\Gamma}_D(Z) : D \subseteq 2^\omega\}$$

Proof. This will follow immediately from Theorem 7.5 and Corollary 8.2. □

Notice that the following simple result, together with the above theorem, shows that the ordering of the non-selfdual Wadge classes is independent of the ambient space Z (provided that AD holds).

Proposition 6.6. *Let Z and W be zero-dimensional spaces that contain a copy of 2^ω, and let $D, E \subseteq 2^\omega$. Then $\mathbf{\Gamma}_D(Z) \subseteq \mathbf{\Gamma}_E(Z)$ iff $\mathbf{\Gamma}_D(W) \subseteq \mathbf{\Gamma}_E(W)$.*

[6] This result is limited to the Borel context. On the other hand, the ambient space is allowed to be analytic, as opposed to Polish.

Proof. Assume that $\mathbf{\Gamma}_D(Z) \subseteq \mathbf{\Gamma}_E(Z)$. Since Z contains a copy of 2^ω and W is zero-dimensional, we see that Z contains a copy of W. Using Lemma 6.4.3, we can assume without loss of generality that $W \subseteq Z$. Then

$$\mathbf{\Gamma}_D(W) = \{A \cap W : A \in \mathbf{\Gamma}_D(Z)\} \subseteq \{A \cap W : A \in \mathbf{\Gamma}_E(Z)\} = \mathbf{\Gamma}_E(W),$$

where the first and last equalities hold by Lemma 6.4.1. The proof of the other implication is similar. □

7. Universal sets

The aim of this section is to prove the easier half of Theorem 6.5 (namely, Theorem 7.5). Our approach is inspired by [Ke, Section 22.A].

Definition 7.1. Let Z and W be spaces, and let $D \subseteq 2^\omega$. Given $U \subseteq W \times Z$ and $x \in W$, let $U_x = \{y \in Z : (x, y) \in U\}$ denote the vertical section of U above x. We will say that $U \subseteq W \times Z$ is a *W-universal set* for $\mathbf{\Gamma}_D(Z)$ if the following two conditions hold:

- $U \in \mathbf{\Gamma}_D(W \times Z)$,
- $\{U_x : x \in W\} = \mathbf{\Gamma}_D(Z)$.

Notice that, by Proposition 6.3, the above yields the definition of a W-universal set for $\mathbf{\Sigma}^0_\xi(Z)$ whenever $1 \leqslant \xi < \omega_1$. Furthermore, this definition agrees with [Ke, Definition 22.2].

Proposition 7.2. *Let Z be a space, and let $D \subseteq 2^\omega$. Then there exists a 2^ω-universal set for $\mathbf{\Gamma}_D(Z)$.*

Proof. By [Ke, Theorem 22.3], we can fix a 2^ω-universal set U for $\mathbf{\Sigma}^0_1(Z)$. Let $h : 2^\omega \longrightarrow (2^\omega)^\omega$ be a homeomorphism, and let $\pi_n : (2^\omega)^\omega \longrightarrow 2^\omega$ be the projection on the n-th coordinate for $n \in \omega$. Notice that, given any $n \in \omega$, the function $f_n : 2^\omega \times Z \longrightarrow 2^\omega \times Z$ defined by $f_n(x, y) = (\pi_n(h(x)), y)$ is continuous. Let $V_n = f_n^{-1}[U]$ for each n, and observe that each $V_n \in \mathbf{\Sigma}^0_1(2^\omega \times Z)$. Set $V = \mathcal{H}_D(V_0, V_1, \ldots)$.

We claim that V is a 2^ω-universal set for $\mathbf{\Gamma}_D(Z)$. It is clear that $V \in \mathbf{\Gamma}_D(2^\omega \times Z)$. Furthermore, using Lemma 6.4, one can easily check that $V_x \in \mathbf{\Gamma}_D(Z)$ for every $x \in 2^\omega$. To complete the proof, fix $A \in \mathbf{\Gamma}_D(Z)$. Let $A_0, A_1, \ldots \in \mathbf{\Sigma}^0_1(Z)$ be such that $A = \mathcal{H}_D(A_0, A_1, \ldots)$. Since U is 2^ω-universal, we can fix $z_n \in 2^\omega$ such that $U_{z_n} = A_n$ for every $n \in \omega$. Set $z = h^{-1}(z_0, z_1, \ldots)$. It is straightforward to verify that $V_z = A$. □

Corollary 7.3. *Let Z be a space that contains a copy of 2^ω, and let $D \subseteq 2^\omega$. Then there exists a Z-universal set for $\mathbf{\Gamma}_D(Z)$.*

Proof. By Proposition 7.2, we can fix a 2^ω-universal set U for $\mathbf{\Gamma}_D(Z)$. Let $W \subseteq Z$ be such that $W \approx 2^\omega$, and fix a homeomorphism $h : 2^\omega \longrightarrow W$. Notice that $(h \times \mathsf{id}_Z)[U] \in \mathbf{\Gamma}_D(W \times Z)$ by Lemma 6.4.3. Therefore, by Lemma 6.4.1, there exists $V \in \mathbf{\Gamma}_D(Z \times Z)$ such that $V \cap (W \times Z) = (h \times \mathsf{id}_Z)[U]$. Using Lemma 6.4 again, one can easily check that V is a Z-universal set for $\mathbf{\Gamma}_D(Z)$. □

Lemma 7.4. *Let Z be a space, and let $D \subseteq 2^\omega$. Assume that there exists a Z-universal set for $\mathbf{\Gamma}_D(Z)$. Then $\mathbf{\Gamma}_D(Z)$ is non-selfdual.*

Proof. Fix a Z-universal set $U \subseteq Z \times Z$ for $\mathbf{\Gamma}_D(Z)$. Assume, in order to get a contradiction, that $\mathbf{\Gamma}_D(Z)$ is selfdual. Let $f : Z \longrightarrow Z \times Z$ be the function defined by $f(x) = (x, x)$, and observe that f is continuous. Since $f^{-1}[U] \in \mathbf{\Gamma}_D(Z) = \check{\mathbf{\Gamma}}_D(Z)$, we see that $Z \backslash f^{-1}[U] \in \mathbf{\Gamma}_D(Z)$. Therefore, since U is Z-universal, we can fix $z \in Z$ such that $U_z = Z \backslash f^{-1}[U]$. If $z \in U_z$ then $f(z) = (z, z) \in U$ by the definition of U_z, contradicting the fact that $U_z = Z \backslash f^{-1}[U]$. On the other hand, If $z \notin U_z$ then $f(z) = (z, z) \notin U$ by the definition of U_z, contradicting the fact that $Z \backslash U_z = f^{-1}[U]$. □

The case $Z = \omega^\omega$ of the following result is [VW1, Proposition 5.0.3], and it is credited to Addison by Van Wesep.

Theorem 7.5. *Let Z be a zero-dimensional space that contains a copy of 2^ω, and let $D \subseteq 2^\omega$. Then $\mathbf{\Gamma}_D(Z) \in \mathsf{NSD}(Z)$.*

Proof. The fact that $\mathbf{\Gamma}_D(Z)$ is non-selfdual follows from Corollary 7.3 and Lemma 7.4. Therefore, it will be enough to show that $\mathbf{\Gamma}_D(Z)$ is a Wadge class. By Proposition 7.2, we can fix a 2^ω-universal set $U \subseteq 2^\omega \times Z$ for $\mathbf{\Gamma}_D(Z)$. Let $W \subseteq Z$ be such that $W \approx 2^\omega \times Z$, and fix a homeomorphism $h : 2^\omega \times Z \longrightarrow W$. By Lemma 6.4, we can fix $A \in \mathbf{\Gamma}_D(Z)$ such that $A \cap W = h[U]$. We claim that $\mathbf{\Gamma}_D(Z) = [A]$. The inclusion $\supseteq$ follows from Lemma 6.4.2. In order to prove the other inclusion, pick $B \in \mathbf{\Gamma}_D(Z)$. Since U is 2^ω-universal, we can fix $z \in 2^\omega$ such that $B = U_z$. Consider the function $f : Z \longrightarrow 2^\omega \times Z$ defined by $f(x) = (z, x)$, and observe that f is continuous. It is straightforward to check that $h \circ f : Z \longrightarrow Z$ witnesses that $B \leqslant A$ in Z. □

8. Van Wesep's theorem

The following is one of the main results of Van Wesep's doctoral thesis (see [VW1, Theorem 5.3.1], whose proof also employs results from [St1]), and it will allow us to obtain the harder half of Theorem 6.5.

Theorem 8.1 (Van Wesep)**.** *Assume* AD*. For every $\mathbf{\Gamma} \in \mathsf{NSD}(\omega^\omega)$ there exists $D \subseteq 2^\omega$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(\omega^\omega)$.*

Corollary 8.2. *Assume* AD*. Let Z be a zero-dimensional Polish space, and let $\mathbf{\Gamma} \in \mathsf{NSD}(Z)$. Then there exists $D \subseteq 2^\omega$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(Z)$.*

Proof. By [Ke, Theorem 7.8], there exists a closed $W \subseteq \omega^\omega$ such that $Z \approx W$. Therefore, using Lemma 6.4.3, we can assume without loss of generality that Z is a closed subspace of ω^ω. Hence, by [Ke, Proposition 2.8], we can fix a retraction $\rho : \omega^\omega \longrightarrow Z$. Let $A \subseteq Z$ be such that $\mathbf{\Gamma} = [A]$. Set $B = \rho^{-1}[A]$, and let $\mathbf{\Lambda} = [B]$ be the Wadge class generated by B in ω^ω.

Using Lemma 3.1, it is easy to see that $\mathbf{\Lambda} \in \mathsf{NSD}(\omega^\omega)$. Therefore, by Theorem 8.1, we can fix $D \subseteq 2^\omega$ such that $\mathbf{\Lambda} = \mathbf{\Gamma}_D(\omega^\omega)$. We claim that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(Z)$. Notice that $A = B \cap Z \in \mathbf{\Gamma}_D(Z)$ by Lemma 6.4.1, hence $\mathbf{\Gamma} \subseteq \mathbf{\Gamma}_D(Z)$ by Lemma 6.4.2. Finally, to see that $\mathbf{\Gamma}_D(Z) \subseteq \mathbf{\Gamma}$, pick $C \in \mathbf{\Gamma}_D(Z)$. Observe that $\rho^{-1}[C] \in \mathbf{\Gamma}_D(\omega^\omega) = \mathbf{\Lambda}$ by Lemma 6.4.2. This means that $\rho^{-1}[C] \leqslant B = \rho^{-1}[A]$ in ω^ω, hence $C \leqslant A$ in Z by Lemma 3.1. So $C \in [A] = \mathbf{\Gamma}$, which concludes the proof. □

9. Basic facts on expansions

The following notion is due to Wadge (see [Wa1, Chapter IV]), and it is inspired by work of Kuratowski. Recall that, given $1 \leqslant \xi < \omega_1$ and spaces Z and W, a function $f : Z \longrightarrow W$ is $\mathbf{\Sigma}^0_\xi$-*measurable* if $f^{-1}[U] \in \mathbf{\Sigma}^0_\xi(Z)$ for every $U \in \mathbf{\Sigma}^0_1(W)$.

Definition 9.1. Let Z be a space, and let $\xi < \omega_1$. Given $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$, define

$$\mathbf{\Gamma}^{(\xi)} = \{f^{-1}[A] : A \in \mathbf{\Gamma} \text{ and } f : Z \longrightarrow Z \text{ is } \mathbf{\Sigma}^0_{1+\xi}\text{-measurable}\}.$$

We will refer to $\mathbf{\Gamma}^{(\xi)}$ as an *expansion* of $\mathbf{\Gamma}$.

The following is the corresponding definition in the context of Hausdorff operations. Corollary 10.5 below shows that this is in fact the "right" definition.

Definition 9.2. Let Z be a space, let $D \subseteq 2^\omega$, and let $\xi < \omega_1$. Define

$$\mathbf{\Gamma}^{(\xi)}_D(Z) = \{\mathcal{H}_D(A_0, A_1, \ldots) : A_n \in \mathbf{\Sigma}^0_{1+\xi}(Z) \text{ for every } n \in \omega\}.$$

As an example (that will be useful later), consider the following simple observation.

Proposition 9.3. *Let $1 \leqslant \eta < \omega_1$. Then there exists $D \subseteq 2^\omega$ such that $\mathbf{\Gamma}^{(\xi)}_D(Z) = \mathsf{D}_\eta(\mathbf{\Sigma}^0_{1+\xi}(Z))$ for every space Z and every $\xi < \omega_1$.*

Proof. This is proved like Proposition 6.2 (in fact, the same D will work). □

The following proposition shows that Definition 9.2 actually fits in the context provided by Section 6.

Proposition 9.4. *Let $D \subseteq 2^\omega$, and let $\xi < \omega_1$. Then there exists $E \subseteq 2^\omega$ such that $\mathbf{\Gamma}^{(\xi)}_D(Z) = \mathbf{\Gamma}_E(Z)$ for every space Z.*

Proof. This is proved by combining Propositions 6.3 and 5.3. □

The following useful result is the analogue of Lemma 6.4 in the present context.

Lemma 9.5. *Let Z and W be spaces, let $D \subseteq 2^\omega$, and let $\xi < \omega_1$.*

1. *Assume that $W \subseteq Z$. Then $B \in \mathbf{\Gamma}^{(\xi)}_D(W)$ iff there exists $A \in \mathbf{\Gamma}^{(\xi)}_D(Z)$ such that $B = A \cap W$.*
2. *If $f : Z \longrightarrow W$ is continuous and $B \in \mathbf{\Gamma}^{(\xi)}_D(W)$ then $f^{-1}[B] \in \mathbf{\Gamma}^{(\xi)}_D(Z)$.*
3. *If $f : Z \longrightarrow W$ is $\mathbf{\Sigma}^0_{1+\xi}$-measurable and $B \in \mathbf{\Gamma}_D(W)$ then $f^{-1}[B] \in \mathbf{\Gamma}^{(\xi)}_D(Z)$.*
4. *If $h : Z \longrightarrow W$ is a homeomorphism then $A \in \mathbf{\Gamma}^{(\xi)}_D(Z)$ iff $h[A] \in \mathbf{\Gamma}^{(\xi)}_D(W)$.*

Proof. This is a straightforward consequence of Proposition 5.4. □

Proposition 9.6. *Let Z be an uncountable zero-dimensional Polish space, let $D \subseteq 2^\omega$, and let $\xi < \omega_1$. Then $\mathbf{\Gamma}^{(\xi)}_D(Z) \in \mathsf{NSD}(Z)$.*

Proof. This is proved like Theorem 7.5. More precisely, all results from Section 7 have straightforward adaptations for $\mathbf{\Gamma}^{(\xi)}_D(Z)$, which can be obtained by applying Lemma 9.5 instead of Lemma 6.4, together with the fact that $\mathbf{\Sigma}^0_{1+\xi}(Z)$ has a 2^ω-universal set (see [Ke, Theorem 22.3]). □

10. Kuratowski's transfer theorem

The aim of this section is to develop the tools needed to successfully employ the notion of expansion. For example, Corollary 10.4 will be a crucial ingredient in the proof of Theorem 11.3.

A slightly stronger form of Theorem 10.2 appears as [Lo2, Theorem 7.1.6], where it is called "Kuratowski's transfer theorem". We also point out that Theorem 10.2 can be easily obtained from [Ke, Theorem 22.18], and viceversa. However, our proof seems to be more straightforward.

Lemma 10.1. *Let (Z,τ) be a Polish space, and let $\mathcal{A}\subseteq\tau$ be countable. Then there exists a Polish topology σ on the set Z such that $\tau\subseteq\sigma\subseteq\mathbf{\Sigma}^0_2(Z,\tau)$ and $\mathcal{A}\subseteq\mathbf{\Delta}^0_1(Z,\sigma)$. Furthermore, if τ is zero-dimensional then there exists a zero-dimensional σ as above.*

Proof. By [Ke, Lemma 13.3], it will be enough to consider the case $\mathcal{A}=\{A\}$. Define

$$\sigma=\{(U\cap A)\cup(V\backslash A):U,V\in\tau\}.$$

Notice that (Z,σ) is the topological sum of the subspaces A and $Z\backslash A$ of (Z,τ), which are both Polish by [Ke, Theorem 3.11]. Therefore, (Z,σ) is Polish by [Ke, Proposition 3.3.iii]. All the other claims are straightforward to check. □

Theorem 10.2 (Kuratowski)**.** *Let (Z,τ) be a Polish space, let $1\leqslant\xi<\omega_1$, and let $\mathcal{A}\subseteq\mathbf{\Sigma}^0_\xi(Z,\tau)$ be countable. Then there exists a Polish topology σ on the set Z such that $\tau\subseteq\sigma\subseteq\mathbf{\Sigma}^0_\xi(Z,\tau)$ and $\mathcal{A}\subseteq\sigma$. Furthermore, if τ is zero-dimensional then there exists a zero-dimensional σ as above.*

Proof. By [Ke, Lemma 13.3], it will be enough to consider the case $\mathcal{A}=\{A\}$. We will proceed by induction on ξ. The case $\xi=1$ is proved by setting $\sigma=\tau$. Next, assume that ξ is a limit ordinal and the theorem holds for all $\eta<\xi$. Write $A=\bigcup_{n\in\omega}A_n$, where each $A_n\in\mathbf{\Sigma}^0_{\xi_n}(Z,\tau)$ for suitable $\xi_n<\xi$. By the inductive assumption, for each n, we can fix a Polish topology σ_n on the set Z such that $\tau\subseteq\sigma_n\subseteq\mathbf{\Sigma}^0_{\xi_n}(Z,\tau)$ and $A_n\in\sigma_n$. Using [Ke, Lemma 13.3] again, it is easy to check that the topology σ on Z generated by $\bigcup_{n\in\omega}\sigma_n$ is as desired.

Finally, assume that $\xi=\eta+1$ and the theorem holds for η. Write $A=\bigcup_{n\in\omega}(Z\backslash A_n)$, where each $A_n\in\mathbf{\Sigma}^0_\eta(Z,\tau)$. By the inductive assumption, we can fix a Polish topology σ' on the set Z such that $\tau\subseteq\sigma'\subseteq\mathbf{\Sigma}^0_\eta(Z,\tau)$ and $A_n\in\sigma'$ for each n. Now, applying Lemma 10.1, we can obtain a Polish topology σ on the set Z such that $\sigma'\subseteq\sigma\subseteq\mathbf{\Sigma}^0_2(Z,\sigma')$ and $A_n\in\mathbf{\Delta}^0_1(Z,\sigma)$ for each n. It is clear that $A\in\sigma$ and $\tau\subseteq\sigma$. Furthermore, since $\sigma'\subseteq\mathbf{\Sigma}^0_\eta(Z,\tau)$, we see that $\sigma\subseteq\mathbf{\Sigma}^0_2(Z,\sigma')\subseteq\mathbf{\Sigma}^0_{\eta+1}(Z,\tau)=\mathbf{\Sigma}^0_\xi(Z,\tau)$. □

Corollary 10.3. *Let Z be a Polish space, let $1\leqslant\xi<\omega_1$, and let $\mathcal{A}\subseteq\mathbf{\Sigma}^0_\xi(Z)$ be countable. Then there exists a Polish space W and a $\mathbf{\Sigma}^0_\xi$-measurable bijection $f:Z\longrightarrow W$ such that $f[A]\in\mathbf{\Sigma}^0_1(W)$ for every $A\in\mathcal{A}$. Furthermore, if Z is zero-dimensional then there exists a zero-dimensional W as above.*

Proof. The space W is simply the set Z with the finer topology given by Theorem 10.2, while $f=\mathsf{id}_Z$. □

Corollary 10.4. *Let Z be Polish space, let $D\subseteq 2^\omega$, let $\xi<\omega_1$, and let $\mathcal{A}\subseteq\mathbf{\Gamma}^{(\xi)}_D(Z)$ be countable. Then there exists a Polish space W and a $\mathbf{\Sigma}^0_{1+\xi}$-measurable bijection*

$f : Z \longrightarrow W$ such that $f[A] \in \mathbf{\Gamma}_D(W)$ for every $A \in \mathcal{A}$. Furthermore, if Z is zero-dimensional then there exists a zero-dimensional W as above.

Proof. Let $\mathcal{A} = \{A_m : m \in \omega\}$ be an enumeration. Given $m \in \omega$, fix $B_{m,n} \in \mathbf{\Sigma}^0_{1+\xi}(Z)$ for $n \in \omega$ such that $A_m = \mathcal{H}_D(B_{m,0}, B_{m,1}, \ldots)$. Define $\mathcal{B} = \{B_{m,n} : m, n \in \omega\}$. By Corollary 10.3, we can fix a Polish space W and a $\mathbf{\Sigma}^0_{1+\xi}$-measurable bijection $f : Z \longrightarrow W$ such that $f[B] \in \mathbf{\Sigma}^0_1(W)$ for every $B \in \mathcal{B}$. It remains to observe that

$$f[A_m] = f[\mathcal{H}_D(B_{m,0}, B_{m,1}, \ldots)] = \mathcal{H}_D(f[B_{m,0}], f[B_{m,1}], \ldots) \in \mathbf{\Gamma}_D(W),$$

where the second equality follows from Proposition 5.4.3. □

Corollary 10.5. *Let Z be an uncountable zero-dimensional Polish space, let $D \subseteq 2^\omega$, and let $\xi < \omega_1$. Then $\mathbf{\Gamma}_D(Z)^{(\xi)} = \mathbf{\Gamma}^{(\xi)}_D(Z)$.*

Proof. The inclusion $\mathbf{\Gamma}_D(Z)^{(\xi)} \subseteq \mathbf{\Gamma}^{(\xi)}_D(Z)$ follows from Lemma 9.5.3. In order to prove the other inclusion, pick $A \in \mathbf{\Gamma}^{(\xi)}_D(Z)$. By Corollary 10.4, we can fix a zero-dimensional Polish space W and a $\mathbf{\Sigma}^0_{1+\xi}$-measurable bijection $f : Z \longrightarrow W$ such that $f[A] \in \mathbf{\Gamma}_D(W)$. Since Z contains a copy of 2^ω and W is zero-dimensional, using Lemma 6.4.3 we can assume without loss of generality that W is a subspace of Z, so that $f : Z \longrightarrow Z$. By Lemma 6.4.1, we can fix $B \in \mathbf{\Gamma}_D(Z)$ such that $B \cap W = f[A]$. It is easy to check that $A = f^{-1}[B]$, which concludes the proof. □

Corollary 10.6. *Assume* AD. *Let Z be an uncountable zero-dimensional Polish space, and let $\xi < \omega_1$. Then $\mathbf{\Gamma}^{(\xi)} \in \mathsf{NSD}(Z)$ for every $\mathbf{\Gamma} \in \mathsf{NSD}(Z)$.*

Proof. This follows from Corollary 8.2, Corollary 10.5, and Proposition 9.6. □

Corollary 10.7. *Assume* AD. *Let Z be an uncountable zero-dimensional Polish space, and let $\xi < \omega_1$. Then $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ iff $\mathbf{\Gamma}^{(\xi)} \subseteq \mathbf{\Lambda}^{(\xi)}$ for every $\mathbf{\Gamma}, \mathbf{\Lambda} \in \mathsf{NSD}(Z)$.*

Proof. The fact that $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ implies $\mathbf{\Gamma}^{(\xi)} \subseteq \mathbf{\Lambda}^{(\xi)}$ is a trivial consequence of the definition of expansion. Now fix $\mathbf{\Gamma}, \mathbf{\Lambda} \in \mathsf{NSD}(Z)$ such that $\mathbf{\Gamma}^{(\xi)} \subseteq \mathbf{\Lambda}^{(\xi)}$. Assume, in order to get a contradiction, that $\mathbf{\Gamma} \nsubseteq \mathbf{\Lambda}$. Then $\check{\mathbf{\Lambda}} \subseteq \mathbf{\Gamma}$ by Lemma 3.2, hence

$$\widetilde{\mathbf{\Lambda}^{(\xi)}} = \check{\mathbf{\Lambda}}^{(\xi)} \subseteq \mathbf{\Gamma}^{(\xi)} \subseteq \mathbf{\Lambda}^{(\xi)}.$$

Since $\mathbf{\Lambda}^{(\xi)}$ is non-selfdual by Corollary 10.6, this is a contradiction. □

11. The expansion theorem

The main result of this section is Theorem 11.3, which will be a crucial tool in obtaining the closure properties in the next section, and will be referred to as the expansion theorem. The proof given here is essentially the same as [Lo2, proof of Theorem 7.3.9.ii]. This result can be traced back to [LSR1, Théorème 8], which is however limited to the Borel context. We need to introduce the following notions from [LSR1] (see also [Lo2, Section 7.3.4]).[7]

Definition 11.1 (Louveau, Saint-Raymond)**.** Let Z be a space, let $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$, and let $\xi < \omega_1$. Define $\mathsf{PU}_\xi(\mathbf{\Gamma})$ to be the collection of all sets of the form

$$\bigcup_{n\in\omega} (A_n \cap V_n),$$

[7] In [LSR1], the notation $\mathbf{\Delta}^0_{1+\xi}$-$\mathsf{PU}$ is used instead of PU_ξ, and λ_{C} is used instead of ℓ.

where each $A_n \in \mathbf{\Gamma}$, each $V_n \in \mathbf{\Delta}^0_{1+\xi}(Z)$, the V_n are pairwise disjoint, and $\bigcup_{n\in\omega} V_n = Z$. A set in this form is called a *partitioned union* of sets in $\mathbf{\Gamma}$.

Notice that the sets V_n in the above definition are not required to be non-empty. It is easy to check that $\mathsf{PU}_\xi(\mathbf{\Gamma})$ is continuously closed whenever $\mathbf{\Gamma}$ is.

Definition 11.2 (Louveau, Saint-Raymond)**.** Let Z be a space, let $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$, and let $\xi < \omega_1$. Define

- $\ell(\mathbf{\Gamma}) \geqslant \xi$ if $\mathsf{PU}_\xi(\mathbf{\Gamma}) = \mathbf{\Gamma}$,
- $\ell(\mathbf{\Gamma}) = \xi$ if $\ell(\mathbf{\Gamma}) \geqslant \xi$ and $\ell(\mathbf{\Gamma}) \ngeqslant \xi + 1$,
- $\ell(\mathbf{\Gamma}) = \omega_1$ if $\ell(\mathbf{\Gamma}) \geqslant \eta$ for every $\eta < \omega_1$.

We refer to $\ell(\mathbf{\Gamma})$ as the *level* of $\mathbf{\Gamma}$.

As a trivial example, observe that $\ell(\{\varnothing\}) = \ell(\{Z\}) = \omega_1$. Also notice that the inclusion $\mathbf{\Gamma} \subseteq \mathsf{PU}_\xi(\mathbf{\Gamma})$ holds for every $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$. Using the definition of Wadge-reduction, it is a simple exercise to see that $\ell(\mathbf{\Gamma}) \geqslant 0$ for every Wadge class $\mathbf{\Gamma}$.

We remark that it is not clear at this point whether for every Wadge class $\mathbf{\Gamma}$ there exists $\xi \leqslant \omega_1$ such that $\ell(\mathbf{\Gamma}) = \xi$.[8] This happens to be true under AD, and it can be proved by combining techniques from [AM] and [Lo2]. However, since we will not need this fact, we omit the proof.

Theorem 11.3. *Assume* AD. *Let Z be an uncountable zero-dimensional Polish space, let $\mathbf{\Gamma} \in \mathsf{NSD}(Z)$, and let $\xi < \omega_1$. Then the following conditions are equivalent:*

1. $\ell(\mathbf{\Gamma}) \geqslant \xi$,
2. $\mathbf{\Gamma} = \mathbf{\Lambda}^{(\xi)}$ *for some* $\mathbf{\Lambda} \in \mathsf{NSD}(Z)$.

Proof. In order to show that (1) $\rightarrow$ (2), assume that $\ell(\mathbf{\Gamma}) \geqslant \xi$. Let $\mathbf{\Lambda} \in \mathsf{NSD}(Z)$ be minimal with respect to the property that $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}^{(\xi)}$. Assume, in order to get a contradiction, that $\mathbf{\Lambda}^{(\xi)} \nsubseteq \mathbf{\Gamma}$. It follows from Lemma 3.2 that $\mathbf{\Gamma} \subseteq \check{\mathbf{\Lambda}}^{(\xi)}$, hence $\mathbf{\Gamma} \subseteq \Delta(\mathbf{\Lambda}^{(\xi)})$. Fix $A \subseteq Z$ such that $\mathbf{\Gamma} = [A]$. Also fix $D, E \subseteq 2^\omega$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(Z)$ and $\mathbf{\Lambda} = \mathbf{\Gamma}_E(Z)$. Then $\{A, Z \setminus A\} \subseteq \mathbf{\Gamma}_E(Z)^{(\xi)} = \mathbf{\Gamma}^{(\xi)}_E(Z)$, where the equality holds by Corollary 10.5. Then, by Corollary 10.4, we can fix a zero-dimensional Polish space W and a $\mathbf{\Sigma}^0_{1+\xi}$-measurable bijection $f : Z \longrightarrow W$ such that $\{f[A], f[Z \setminus A]\} \subseteq \mathbf{\Gamma}_E(W)$.

Next, we will show that $[f[A]] \in \mathsf{SD}(W)$. Assume, in order to get a contradiction, that this is not the case. By Corollary 8.2, we can fix $F \subseteq 2^\omega$ such that $[f[A]] = \mathbf{\Gamma}_F(W)$. Notice that $\mathbf{\Gamma}_F(W) \subseteq \mathbf{\Gamma}_E(W)$. Furthermore $W \setminus f[A] = f[Z \setminus A] \in \mathbf{\Gamma}_E(W)$, hence $\check{\mathbf{\Gamma}}_F(W) \subseteq \mathbf{\Gamma}_E(W)$. Since $\mathbf{\Gamma}_E(W)$ is non-selfdual by Theorem 7.5, it follows that $\mathbf{\Gamma}_F(W) \subsetneq \mathbf{\Gamma}_E(W)$. Therefore, $\mathbf{\Gamma}_F(Z) \subsetneq \mathbf{\Gamma}_E(Z) = \mathbf{\Lambda}$ by Proposition 6.6. On the other hand, Lemma 9.5.3 and Corollary 10.5 show that $A = f^{-1}[f[A]] \in \mathbf{\Gamma}^{(\xi)}_F(Z) = \mathbf{\Gamma}_F(Z)^{(\xi)}$. Hence $\mathbf{\Gamma} \subseteq \mathbf{\Gamma}_F(Z)^{(\xi)}$, which contradicts the minimality of $\mathbf{\Lambda}$.

Since $[f[A]] \in \mathsf{SD}(W)$, by Corollaries 4.4 and 8.2, we can fix $A_n \subseteq W$, $G_n \subseteq 2^\omega$ and pairwise disjoint $V_n \in \mathbf{\Delta}^0_1(W)$ for $n \in \omega$ such that $f[A] = \bigcup_{n\in\omega}(A_n \cap V_n)$ and $A_n \in \mathbf{\Gamma}_{G_n}(W) \subsetneq \mathbf{\Gamma}_E(W)$ for each n. Notice that $\mathbf{\Gamma}_{G_n}(Z) \subsetneq \mathbf{\Gamma}_E(Z)$ for each n by Proposition 6.6, hence $\mathbf{\Gamma}_D(Z) \nsubseteq \mathbf{\Gamma}_{G_n}(Z)^{(\xi)}$ for each n by the minimality of $\mathbf{\Lambda}$. It follows from Corollary 10.5 and Lemma 3.2 that $\check{\mathbf{\Gamma}}^{(\xi)}_{G_n}(Z) \subseteq \mathbf{\Gamma}_D(Z)$. Then, using Propositions 9.4 and 6.6, one sees that $\check{\mathbf{\Gamma}}^{(\xi)}_{G_n}(W) \subseteq \mathbf{\Gamma}_D(W)$.

[8] For example, it is conceivable that $\mathsf{PU}_\eta(\mathbf{\Gamma}) = \mathbf{\Gamma}$ for all $\eta < \xi$, where ξ is a limit ordinal, while $\mathsf{PU}_\xi(\mathbf{\Gamma}) \neq \mathbf{\Gamma}$.

Set $B_n = W \backslash A_n \in \check{\mathbf{\Gamma}}_{G_n}(W)$ for $n \in \omega$. Observe that $f^{-1}[B_n] \in \check{\mathbf{\Gamma}}^{(\xi)}_{G_n}(Z) \subseteq \mathbf{\Gamma}_D(Z) = \mathbf{\Gamma}$ for each n by Lemma 9.5.3. Furthermore, it is clear that $f^{-1}[V_n] \in \mathbf{\Delta}^0_{1+\xi}(Z)$ for each n. In conclusion, since $W \backslash f[A] = \bigcup_{n\in\omega}(B_n \cap V_n)$, we see that

$$Z \backslash A = \bigcup_{n\in\omega}(f^{-1}[B_n] \cap f^{-1}[V_n]) \in \mathsf{PU}_\xi(\mathbf{\Gamma}) = \mathbf{\Gamma},$$

where the last equality uses the assumption that $\ell(\mathbf{\Gamma}) \geqslant \xi$. This contradicts the fact that $\mathbf{\Gamma}$ is non-selfdual.

In order to show that (2) → (1), let $\mathbf{\Lambda} \in \mathsf{NSD}(Z)$ be such that $\mathbf{\Lambda}^{(\xi)} = \mathbf{\Gamma}$. Pick $A_n \in \mathbf{\Gamma}$ and pairwise disjoint $V_n \in \mathbf{\Delta}^0_{1+\xi}(Z)$ for $n \in \omega$ such that $\bigcup_{n\in\omega} V_n = Z$. We need to show that $\bigcup_{n\in\omega}(A_n \cap V_n) \in \mathbf{\Gamma}$. By Corollary 8.2, we can fix $D \subseteq 2^\omega$ such that $\mathbf{\Lambda} = \mathbf{\Gamma}_D(Z)$. As in the proof of Corollary 10.4, we can fix a Polish space W and a $\mathbf{\Sigma}^0_{1+\xi}$-measurable bijection $f : Z \longrightarrow W$ such that each $f[A_n] \in \mathbf{\Gamma}_D(W)$ and each $f[V_n] \in \mathbf{\Delta}^0_1(W)$. Let $B = \bigcup_{n\in\omega}(f[A_n] \cap f[V_n])$. Since $\mathbf{\Gamma}_D(W)$ is a Wadge class in W by Theorem 7.5, one sees that $B \in \mathsf{PU}_0(\mathbf{\Gamma}_D(W)) = \mathbf{\Gamma}_D(W)$. It follows from Lemma 9.5.3 that

$$\bigcup_{n\in\omega}(A_n \cap V_n) = f^{-1}[B] \in \mathbf{\Gamma}^{(\xi)}_D(Z) = \mathbf{\Lambda}^{(\xi)} = \mathbf{\Gamma},$$

where the second equality holds by Corollary 10.5. □

Corollary 11.4. *Assume* AD*. Let Z and W be uncountable zero-dimensional Polish spaces, let $D \subseteq 2^\omega$, and let $\xi < \omega_1$. Then $\ell(\mathbf{\Gamma}_D(Z)) \geqslant \xi$ iff $\ell(\mathbf{\Gamma}_D(W)) \geqslant \xi$.*

Proof. We will only prove the left-to-right implication, as the other one can be proved similarly. Assume that $\ell(\mathbf{\Gamma}_D(Z)) \geqslant \xi$. Then, by Theorems 7.5 and 11.3, there exists $\mathbf{\Lambda} \in \mathsf{NSD}(Z)$ such that $\mathbf{\Lambda}^{(\xi)} = \mathbf{\Gamma}_D(Z)$. By Corollary 8.2, we can fix $E \subseteq 2^\omega$ such that $\mathbf{\Lambda} = \mathbf{\Gamma}_E(Z)$. By Proposition 9.4, we can fix $F \subseteq 2^\omega$ such that $\mathbf{\Gamma}_F(Z) = \mathbf{\Gamma}^{(\xi)}_E(Z)$ and $\mathbf{\Gamma}_F(W) = \mathbf{\Gamma}^{(\xi)}_E(W)$. Notice that $\mathbf{\Gamma}_F(Z) = \mathbf{\Gamma}_D(Z)$ by Corollary 10.5, hence $\mathbf{\Gamma}_F(W) = \mathbf{\Gamma}_D(W)$ by Proposition 6.6. By applying Corollary 10.5 again, we see that $\mathbf{\Gamma}_D(W) = \mathbf{\Gamma}_E(W)^{(\xi)}$, hence $\ell(\mathbf{\Gamma}_D(W)) \geqslant \xi$ by Theorems 7.5 and 11.3. □

12. Good Wadge classes

The following key notion is essentially due to van Engelen, although he did not give it a name. One important difference is that van Engelen's treatment of this notion is fundamentally tied to Louveau's classification of the Borel Wadge classes from [Lo1], hence it is limited to the Borel context. The notion of level and the expansion theorem allow us to completely bypass [Lo1], and extend this concept to arbitrary Wadge classes.

Definition 12.1. Let Z be a space, and let $\mathbf{\Gamma}$ be a Wadge class in Z. We will say that $\mathbf{\Gamma}$ is *good* if the following conditions are satisfied:

- $\mathbf{\Gamma}$ is non-selfdual,
- $\Delta(\mathsf{D}_\omega(\mathbf{\Sigma}^0_2(Z))) \subseteq \mathbf{\Gamma}$,
- $\ell(\mathbf{\Gamma}) \geqslant 1$.

The following proposition gives some concrete examples of good Wadge classes.

Proposition 12.2. *Let Z be an uncountable zero-dimensional Polish space, let $\omega \leqslant \eta < \omega_1$, and let $2 \leqslant \xi < \omega_1$. Then $\mathsf{D}_\eta(\mathbf{\Sigma}^0_\xi(Z))$ is a good Wadge class in Z.*

Proof. Set $\mathbf{\Gamma} = \mathsf{D}_\eta(\mathbf{\Sigma}^0_\xi(Z))$. The fact that $\mathbf{\Gamma} \in \mathsf{NSD}(Z)$ follows from Propositions 9.3 and 9.6. The inclusion $\Delta(\mathsf{D}_\omega(\mathbf{\Sigma}^0_2(Z))) \subseteq \mathbf{\Gamma}$ holds trivially. Finally, using Corollary 10.5 and [LSR1, Théorème 8],[9] one sees that $\ell(\mathbf{\Gamma}) \geqslant 1$. □

The main result of this section is Corollary 12.4, which will be crucial in showing that good Wadge classes are reasonably closed (see Lemma 13.2). The case $Z = \omega^\omega$ of the following theorem is due to Andretta, Hjorth, and Neeman (see [AHN, Lemma 3.6.a]), and the general case follows easily from this particular case (thanks to the machinery of relativization).

Theorem 12.3. *Assume* AD. *Let Z be an uncountable zero-dimensional Polish space, and let $\mathbf{\Gamma} \in \mathsf{NSD}(Z)$. Assume that $\mathsf{D}_n(\mathbf{\Sigma}^0_1(Z)) \subseteq \mathbf{\Gamma}$ for every $n \in \omega$.*

- *If $A \in \mathbf{\Gamma}$ and $C \in \mathbf{\Pi}^0_1(Z)$ then $A \cap C \in \mathbf{\Gamma}$.*
- *If $A \in \mathbf{\Gamma}$ and $U \in \mathbf{\Sigma}^0_1(Z)$ then $A \cup U \in \mathbf{\Gamma}$.*

Proof. Observe that, since $\check{\mathbf{\Gamma}}$ also satisfies the assumptions of the theorem, it will be enough to prove the first statement. So pick $A \in \mathbf{\Gamma}$ and $C \in \mathbf{\Pi}^0_1(Z)$. By Corollary 8.2, we can fix $D \subseteq 2^\omega$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(Z)$, and let $\mathbf{\Lambda} = \mathbf{\Gamma}_D(\omega^\omega)$. Using Lemma 6.4.3, we can assume without loss of generality that Z is a closed subspace of ω^ω. By [Ke, Proposition 2.8], we can fix a retraction $\rho : \omega^\omega \longrightarrow Z$. Notice that $\rho^{-1}[A] \in \mathbf{\Lambda}$ by Lemma 6.4.2. Furthermore, it is clear that $C \in \mathbf{\Pi}^0_1(\omega^\omega)$.

Next, we claim that $||\mathbf{\Lambda}|| \geqslant \omega$. Since $\mathsf{D}_n(\mathbf{\Sigma}^0_1(Z)) \subseteq \mathbf{\Gamma}$ for every $n \in \omega$, using Propositions 6.2 and 6.6 one sees that $\mathsf{D}_n(\mathbf{\Sigma}^0_1(\omega^\omega)) \subseteq \mathbf{\Lambda}$ for every $n \in \omega$. Since these are Wadge classes by Theorem 7.5, and they form a strictly increasing sequence by [Ke, Exercise 22.26.iv], our claim is proved. Therefore, we can apply [AHN, Lemma 3.6.a], which shows that $\rho^{-1}[A] \cap C \in \mathbf{\Lambda}$. Finally, Lemma 6.4.1 shows that $A \cap C = (\rho^{-1}[A] \cap C) \cap Z \in \mathbf{\Gamma}_D(Z) = \mathbf{\Gamma}$. □

Corollary 12.4. *Assume* AD. *Let Z be an uncountable zero-dimensional Polish space, and let $\mathbf{\Gamma} \in \mathsf{NSD}(Z)$. Assume that $\mathsf{D}_n(\mathbf{\Sigma}^0_2(Z)) \subseteq \mathbf{\Gamma}$ for every $n \in \omega$ and $\ell(\mathbf{\Gamma}) \geqslant 1$.*

- *If $A \in \mathbf{\Gamma}$ and $G \in \mathbf{\Pi}^0_2(Z)$ then $A \cap G \in \mathbf{\Gamma}$.*
- *If $A \in \mathbf{\Gamma}$ and $F \in \mathbf{\Sigma}^0_2(Z)$ then $A \cup F \in \mathbf{\Gamma}$.*

In particular, the above two statements hold for every good Wadge class $\mathbf{\Gamma}$ in Z.

Proof. Observe that, since $\check{\mathbf{\Gamma}}$ also satisfies the assumptions of the theorem, it will be enough to prove the first statement. So pick $A \in \mathbf{\Gamma}$ and $G \in \mathbf{\Pi}^0_2(Z)$. By Theorem 11.3, we can pick $\mathbf{\Lambda} \in \mathsf{NSD}(Z)$ such that $\mathbf{\Lambda}^{(1)} = \mathbf{\Gamma}$. By Corollary 8.2, we can fix $E \subseteq 2^\omega$ such that $\mathbf{\Gamma}_E(Z) = \mathbf{\Lambda}$.

Since $\mathbf{\Lambda}^{(1)} = \mathbf{\Gamma}$, there exists a $\mathbf{\Sigma}^0_2$-measurable function $f : Z \longrightarrow Z$ and $B \in \mathbf{\Lambda}$ such that $A = f^{-1}[B]$. Furthermore, using Corollary 10.5 for a suitable choice of D, it is easy to check that $\mathbf{\Pi}^0_1(Z)^{(1)} = \mathbf{\Pi}^0_2(Z)$. Therefore, there exists a $\mathbf{\Sigma}^0_2$-measurable function $g : Z \longrightarrow Z$ and $C \in \mathbf{\Pi}^0_1(Z)$ such that $G = g^{-1}[C]$. By applying Lemma 6.4.2 to the projection on the first coordinate $\pi : Z \times Z \longrightarrow Z$, one sees that $B \times Z \in \mathbf{\Gamma}_E(Z \times Z)$. Furthermore, it is clear that $Z \times C \in \mathbf{\Pi}^0_1(Z \times Z)$.

We claim that $\mathsf{D}_n(\mathbf{\Sigma}^0_1(Z \times Z)) \subseteq \mathbf{\Gamma}_E(Z \times Z)$ for every $n \in \omega$. So fix $n \in \omega$, and let $D \subseteq 2^\omega$ be the set given by Proposition 9.3 when $\eta = n$. Notice that

$$\mathbf{\Gamma}_D(Z)^{(1)} = \mathbf{\Gamma}^{(1)}_D(Z) = \mathsf{D}_n(\mathbf{\Sigma}^0_2(Z)) \subseteq \mathbf{\Gamma} = \mathbf{\Gamma}_E(Z)^{(1)},$$

[9] Here, we apply [LSR1, Théorème 8] instead of Theorem 11.3 simply because the former does not require AD.

where the first equality holds by Corollary 10.5. Therefore $\mathbf{\Gamma}_D(Z) \subseteq \mathbf{\Gamma}_E(Z)$ by Corollary 10.7. An application of Proposition 6.6 with $W = Z \times Z$ concludes the proof of our claim.

Therefore, we can apply Theorem 12.3, which shows that $B \times C = (B \times Z) \cap (Z \times C) \in \mathbf{\Gamma}_E(Z \times Z)$. Consider the function $(f, g) : Z \longrightarrow Z \times Z$ defined by $(f, g)(x) = (f(x), g(x))$, and observe that (f, g) is $\mathbf{\Sigma}^0_2$-measurable. By Lemma 9.5.3, it follows that

$$A \cap G = (f, g)^{-1}[B \times C] \in \mathbf{\Gamma}^{(1)}_E(Z) = \mathbf{\Lambda}^{(1)} = \mathbf{\Gamma},$$

where the second equality holds by Corollary 10.5. □

13. Reasonably closed Wadge classes

In this section we will define reasonably closed Wadge classes and prove that every good Wadge class is reasonably closed. This notion is an ad hoc definition, and it is the key idea of an ingenious lemma due to Harrington (see [St2, Lemma 3]). This lemma is a crucial ingredient in the proof of Theorem 15.1. Here, we will follow the approach of [vE3, Section 4.1].

Given $i \in 2$, set

$$Q_i = \{x \in 2^\omega : x(n) = i \text{ for all but finitely many } n \in \omega\}.$$

Notice that every element of $2^\omega \setminus (Q_0 \cup Q_1)$ is obtained by alternating finite blocks of zeros and finite blocks of ones. Define the function $\phi : 2^\omega \setminus (Q_0 \cup Q_1) \longrightarrow 2^\omega$ by setting

$$\phi(x)(n) = \begin{cases} 0 & \text{if the } n^{\text{th}} \text{ block of zeros of } x \text{ has even length,} \\ 1 & \text{otherwise,} \end{cases}$$

where we start counting with the 0^{th} block of zeros. It is easy to check that ϕ is continuous.

Definition 13.1. Let $\mathbf{\Gamma}$ be a Wadge class in 2^ω. We will say that $\mathbf{\Gamma}$ is *reasonably closed* if $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$ for every $A \in \mathbf{\Gamma}$.

The following result is essentially the same as [vE3, Lemma 4.2.17], except that it is not limited to the Borel context.

Lemma 13.2. *Assume* AD. *Let* $\mathbf{\Gamma}$ *be a good Wadge class in* 2^ω. *Then* $\mathbf{\Gamma}$ *is reasonably closed.*

Proof. By Corollary 8.2, we can fix $D \subseteq 2^\omega$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(2^\omega)$. Set $Z = 2^\omega \setminus (Q_0 \cup Q_1)$. Pick $A \in \mathbf{\Gamma}$. Notice that $\phi^{-1}[A] \in \mathbf{\Gamma}_D(Z)$ by Lemma 6.4.2. Therefore, by Lemma 6.4.1, there exists $B \in \mathbf{\Gamma}$ such that $B \cap Z = \phi^{-1}[A]$. Since $\mathbf{\Gamma}$ is a good Wadge class and $Z \in \mathbf{\Pi}^0_2(2^\omega)$, it follows from Corollary 12.4 that $\phi^{-1}[A] \in \mathbf{\Gamma}$. Finally, again by Corollary 12.4, we see that $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$, which concludes the proof. □

14. Wadge classes of homogeneous spaces are good

The main result of this section is that $[X]$ is a good Wadge class whenever X is a homogeneous space of sufficiently high complexity (see Theorem 14.4 for the precise statement). Together with Lemma 13.2, this will allow us to apply Theorem 15.1 in the next section.

We will need three preliminary results. Lemmas 14.1, 14.2, and 14.3 correspond to [vE3, Lemma 4.2.16], [vE3, Lemma 4.4.2], and [vE3, Lemma 4.4.1] respectively, while Theorem 14.4 corresponds to [vE3, Lemma 4.4.3]. Once again, the difference is that we work with arbitrary sets instead of just Borel sets. In the case of Lemma 14.3, this yields at the same time a substantially simpler proof, inspired by [Lo2, proof of Theorem 7.3.10.ii].

Lemma 14.1. *Assume* AD*. Let Z be an uncountable zero-dimensional Polish space, and let $\mathbf{\Gamma}$ be a good Wadge class in Z. Assume that A and B are subspaces of Z such that $B \in \mathbf{\Gamma}$ and $A \approx B$. Then $A \in \mathbf{\Gamma}$.*

Proof. Let $h : A \longrightarrow B$ be a homeomorphism. By [Ke, Theorem 3.9], we can fix $G, H \in \mathbf{\Pi}^0_2(Z)$ and a homeomorphism $f : G \longrightarrow H$ such that $h \subseteq f$. By Corollary 8.2, we can fix $D \subseteq 2^\omega$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(Z)$. Notice that $B \in \mathbf{\Gamma}_D(H)$ by Lemma 6.4.1. It follows from Lemma 6.4.2 that $A \in \mathbf{\Gamma}_D(G)$. Therefore, according to Lemma 6.4.1, there exists $C \in \mathbf{\Gamma}_D(Z)$ such that $C \cap G = A$. Since $G \in \mathbf{\Pi}^0_2(Z)$, an application of Corollary 12.4 concludes the proof. □

Lemma 14.2. *Assume* AD*. Let Z be an uncountable zero-dimensional Polish space, let $\mathbf{\Gamma}$ be a good Wadge class in Z, and let X be a homogeneous subspace of Z. Assume that $A \in \mathbf{\Sigma}^0_1(X)$ is non-empty and $A \in \mathbf{\Gamma}$. Then $X \in \mathbf{\Gamma}$.*

Proof. Define $\mathcal{U} = \{h[A] : h \text{ is a homeomorphism of } X\}$. Notice that $\mathcal{U}$ is a cover of X because X is homogeneous and A is non-empty. Let $\{A_n : n \in \omega\}$ be a countable subcover of $\mathcal{U}$. Observe that each $A_n \in \mathbf{\Gamma}$ by Lemma 14.1. Fix $U_n \in \mathbf{\Sigma}^0_1(Z)$ for $n \in \omega$ such that $U_n \cap X = A_n$ for each n. Set $V_n = U_n \setminus \bigcup_{k<n} U_k$ for $n \in \omega$, and observe that $V_n \in \mathbf{\Delta}^0_2(Z)$ for each n. Furthermore, it is easy to check that

$$X = \bigcup_{-1 \leqslant n < \omega} (V_n \cap A_n),$$

where $V_{-1} = Z \setminus \bigcup_{n<\omega} V_n = Z \setminus \bigcup_{n<\omega} U_n$ and $A_{-1} = \varnothing$. In conclusion, we see that $X \in \mathsf{PU}_1(\mathbf{\Gamma})$. Since $\ell(\mathbf{\Gamma}) \geqslant 1$, it follows that $X \in \mathbf{\Gamma}$. □

Lemma 14.3. *Assume* AD*. Let Z be an uncountable zero-dimensional Polish space, let $\mathbf{\Gamma} \in \mathsf{NSD}(Z)$ be such that $\ell(\mathbf{\Gamma}) = 0$, and let $X \in \mathbf{\Gamma}$ be codense in Z. Then there exists a non-empty $U \in \mathbf{\Delta}^0_1(Z)$ and $\mathbf{\Lambda} \in \mathsf{NSD}(Z)$ such that $\mathbf{\Lambda} \subsetneq \mathbf{\Gamma}$ and $X \cap U \in \mathbf{\Lambda}$.*

Proof. First we claim that $\check{\mathbf{\Gamma}} \subseteq \mathsf{PU}_1(\mathbf{\Gamma})$. Since $\ell(\mathbf{\Gamma}) = 0$, we can fix $A \in \mathsf{PU}_1(\mathbf{\Gamma}) \setminus \mathbf{\Gamma}$. Using Lemma 3.2 and the fact that $\mathsf{PU}_1(\mathbf{\Gamma})$ is continuously closed, it is easy to see that $\check{\mathbf{\Gamma}} \subseteq [A] \subseteq \mathsf{PU}_1(\mathbf{\Gamma})$. Therefore, we can fix $A_n \in \mathbf{\Gamma}$ and pairwise disjoint $V_n \in \mathbf{\Delta}^0_2(Z)$ for $n \in \omega$ such that $\bigcup_{n\in\omega}(A_n \cap V_n) = Z \setminus X$. Since Z is a Baire space, we can fix $n \in \omega$ and a non-empty $U \in \mathbf{\Delta}^0_1(Z)$ such that $U \subseteq V_n$.

Notice that $\mathbf{\Gamma} \neq \{Z\}$ and $\mathbf{\Gamma} \neq \{\varnothing\}$ because $\ell(\mathbf{\Gamma}) = 0$, hence it is possible to apply Proposition 3.5. In particular, one sees that $U \setminus X = A_n \cap U \in \mathbf{\Gamma}$, hence $Z \setminus (X \cap U) = (Z \setminus U) \cup (U \setminus X) \in \mathbf{\Gamma}$. So, we have $X \cap U \in \mathbf{\Gamma}$ (again by Proposition 3.5) and $Z \setminus (X \cap U) \in \mathbf{\Gamma}$. This easily yields the desired result if $X \cap U$ is non-selfdual, so assume that $X \cap U$ is selfdual. By Corollary 4.4, we can fix pairwise disjoint $U_n \in \mathbf{\Delta}^0_1(U)$ and non-selfdual $B_n < X \cap U$ in Z for $n \in \omega$ such that $\bigcup_{n\in\omega} U_n = U$ and $\bigcup_{n\in\omega}(B_n \cap U_n) = X \cap U$. If we had $B_n = Z$ for each n such that $U_n \neq \varnothing$ then the assumption that X is codense in Z would be contradicted, so assume that

$n \in \omega$ is such that $B_n \neq Z$ and $U_n \neq \varnothing$. To conclude the proof, set $\mathbf{\Lambda} = [B_n]$ and observe that $X \cap U_n = B_n \cap U_n \leqslant B_n$ by Proposition 3.5. □

Theorem 14.4. *Assume* AD. *Let Z be an uncountable zero-dimensional Polish space, and let X be a homogeneous dense subspace of Z such that $X \notin \Delta(\mathsf{D}_\omega(\mathbf{\Sigma}^0_2(Z)))$. Then $[X]$ is a good Wadge class in Z.*

Proof. Fix $\mathbf{\Gamma} \in \mathsf{NSD}(Z)$ minimal with respect to the property that $X \cap U \in \mathbf{\Gamma} \cup \check{\mathbf{\Gamma}}$ for some non-empty $U \in \mathbf{\Delta}^0_1(Z)$. Fix a non-empty $U \in \mathbf{\Delta}^0_1(Z)$ such that $X \cap U \in \mathbf{\Gamma} \cup \check{\mathbf{\Gamma}}$. Assume without loss of generality that $X \cap U \in \mathbf{\Gamma}$ (the case $X \cap U \in \check{\mathbf{\Gamma}}$ is similar). First we will prove that $\mathbf{\Gamma}$ is a good Wadge class, then we will show that $[X] = \mathbf{\Gamma}$. Observe that $\mathsf{D}_\omega(\mathbf{\Sigma}^0_2(Z))$ and $\check{\mathsf{D}}_\omega(\mathbf{\Sigma}^0_2(Z))$ are good Wadge classes in Z by Proposition 12.2. We claim that $X \cap U \notin \Delta(\mathsf{D}_\omega(\mathbf{\Sigma}^0_2(Z)))$. Assume, in order to get a contradiction, that $X \cap U \in \Delta(\mathsf{D}_\omega(\mathbf{\Sigma}^0_2(Z)))$. Then, by the density of X, it is possible to apply Lemma 14.2 (twice), obtaining that $X \in \Delta(\mathsf{D}_\omega(\mathbf{\Sigma}^0_2(Z)))$. Since this contradicts our assumptions, our claim is proved. By Lemma 3.2, it follows that $\Delta(\mathsf{D}_\omega(\mathbf{\Sigma}^0_2(Z))) \subseteq \mathbf{\Gamma}$.

Next, we claim that $\ell(\mathbf{\Gamma}) \geqslant 1$. Assume, in order to get a contradiction, that $\ell(\mathbf{\Gamma}) = 0$. By Corollary 8.2, we can fix $D \subseteq 2^\omega$ such that $\mathbf{\Gamma}_D(Z) = \mathbf{\Gamma}$. Since X is dense in Z and homogeneous, if U were countable then X would be countable, by the same argument as in the proof of Proposition 2.6. So U is an uncountable zero-dimensional Polish space, and $\ell(\mathbf{\Gamma}_D(U)) = 0$ by Corollary 11.4. Furthermore, X must be codense in Z, otherwise it would follow that X is Polish by Proposition 2.6, hence $X \in \mathbf{\Pi}^0_2(Z)$ by [Ke, Theorem 3.11]. Therefore, by Lemma 14.3, there exists a non-empty $V \in \mathbf{\Delta}^0_1(U)$ and $\mathbf{\Lambda} \in \mathsf{NSD}(U)$ such that $\mathbf{\Lambda} \subsetneq \mathbf{\Gamma}_D(U)$ and $X \cap V \in \mathbf{\Lambda}$. By Corollary 8.2, we can fix $E \subseteq 2^\omega$ such that $\mathbf{\Gamma}_E(U) = \mathbf{\Lambda}$. Observe that $\mathbf{\Gamma}_E(Z) \subsetneq \mathbf{\Gamma}_D(Z)$ by Proposition 6.6. Therefore, in order to contradict the minimality of $\mathbf{\Gamma}$, it remains to show that $X \cap V \in \mathbf{\Gamma}_E(Z)$. By Lemma 6.4.1, there exists $A \in \mathbf{\Gamma}_E(Z)$ such that $A \cap U = X \cap V$. Notice that $\mathbf{\Gamma}_E(Z) \neq \{Z\}$, otherwise it would follow that $X = Z$, which contradicts our assumptions. Therefore $X \cap V = A \cap U \in \mathbf{\Gamma}_E(Z)$ by Proposition 3.5.

At this point, we know that $\mathbf{\Gamma}$ is a good Wadge class, so we can apply Lemma 14.2, obtaining that $X \in \mathbf{\Gamma}$. To conclude the proof, it will be enough to show that X is non-selfdual, as it will follow from the minimality of $\mathbf{\Gamma}$ and Proposition 3.5 that $[X] = \mathbf{\Gamma}$. Assume, in order to get a contradiction, that X is selfdual. Then, by Corollary 4.4, there exist a non-empty $V \in \mathbf{\Delta}^0_1(Z)$ and a non-selfdual $A < X$ in Z such that $A \cap V = X \cap V$. Set $\mathbf{\Lambda} = [A]$, and observe that $\mathbf{\Lambda} \subsetneq \mathbf{\Gamma}$. Notice that $\mathbf{\Lambda} \neq \{Z\}$, otherwise it would follow that $V \subseteq X$, hence X would not be codense in Z. Therefore $X \cap V = A \cap V \in \mathbf{\Lambda}$ by Proposition 3.5. This contradicts the minimality of $\mathbf{\Gamma}$. □

15. The main results

This sections contains our main results. Theorem 15.2 extends (and is inspired by) [vE4, Lemma 2.7]. All the work done so far was aimed at applying the following result, which is a particular case of [St2, Theorem 2]. Given a Wadge class $\mathbf{\Gamma}$ in 2^ω and $X \subseteq 2^\omega$, we will say that X is *everywhere properly* $\mathbf{\Gamma}$ if $X \cap [s] \in \mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$ for every $s \in 2^{<\omega}$.

Theorem 15.1 (Steel)**.** *Assume* AD. *Let $\mathbf{\Gamma}$ be a reasonably closed Wadge class in 2^ω. Assume that X and Y are subsets of 2^ω that satisfy the following conditions:*

- *X and Y are everywhere properly $\mathbf{\Gamma}$,*
- *X and Y are either both meager in 2^ω or both comeager in 2^ω.*

Then there exists a homeomorphism $h : 2^\omega \longrightarrow 2^\omega$ such that $h[X] = Y$.

Theorem 15.2. *Assume* AD. *Let X and Y be homogeneous dense subspaces of 2^ω. Assume that $X \notin \Delta(\mathsf{D}_\omega(\mathbf{\Sigma}^0_2(2^\omega)))$, and that the following conditions are satisfied:*

- *$[X] = [Y]$,*
- *X and Y are either both meager spaces or both Baire spaces.*

Then there exists a homeomorphism $h : 2^\omega \longrightarrow 2^\omega$ such that $h[X] = Y$.

Proof. Let $\mathbf{\Gamma} = [X]$. Notice that $\mathbf{\Gamma}$ is a good Wadge class by Theorem 14.4, hence it is reasonably closed by Lemma 13.2. It is clear that if X and Y are both meager spaces, then they are both meager in 2^ω. On the other hand, if X and Y are both Baire spaces, then they are comeager in 2^ω by Proposition 2.7. Hence, by Theorem 15.1, it will be enough to show that X and Y are everywhere properly $\mathbf{\Gamma}$. We will only prove this for X, since the proof for Y is perfectly analogous. Pick $s \in 2^{<\omega}$. Using Proposition 3.5, one sees that $X \cap [s] \in \mathbf{\Gamma}$. In order to get a contradiction, assume that $X \cap [s] \in \check{\mathbf{\Gamma}}$. Since $\check{\mathbf{\Gamma}}$ is also a good Wadge class, it follows from Lemma 14.2 that $X \in \check{\mathbf{\Gamma}}$, which contradicts the fact that $\mathbf{\Gamma}$ is non-selfdual. □

Corollary 15.3. *Assume* AD. *Let X be a zero-dimensional homogeneous space that is not locally compact. Then X is strongly homogeneous.*

Proof. Notice that X is crowded, otherwise it would be discrete by homogeneity. Therefore, we can assume without loss of generality that X is a dense subspace of 2^ω. If $X \in \Delta(\mathsf{D}_\omega(\mathbf{\Sigma}^0_2(2^\omega)))$, then the desired result follows from [vE3, Corollary 4.4.6]. So assume that $X \notin \Delta(\mathsf{D}_\omega(\mathbf{\Sigma}^0_2(2^\omega)))$.

By Theorem 2.8, it will be enough to show that $X \cap [s] \approx X$ for every $s \in 2^{<\omega}$. Pick $s \in 2^{<\omega}$. Let $h : [s] \longrightarrow 2^\omega$ be a homeomorphism, and let $Y = h[X \cap [s]]$. It is easy to check that Y is a homogeneous dense subspace of 2^ω. Furthermore, it is clear that X and Y are either both meager spaces or both Baire spaces. We claim that $[X] = [Y]$. By Theorem 15.2, this will conclude the proof.

Set $\mathbf{\Gamma} = [X]$, and observe that $\mathbf{\Gamma}$ is a good Wadge class by Theorem 14.4. In particular, $\mathbf{\Gamma}$ is non-selfdual. Hence, by Corollary 8.2, we can fix $D \subseteq 2^\omega$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(2^\omega)$. Notice that $X \cap [s] \in \mathbf{\Gamma}_D([s])$ by Lemma 6.4.1, hence $Y \in \mathbf{\Gamma}_D(2^\omega)$ by Lemma 6.4.3. This shows that $[Y] \subseteq [X]$. In order to prove the other inclusion, by Lemma 3.2, it will be enough to show that $Y \notin \check{\mathbf{\Gamma}}_D(2^\omega)$. Assume, in order to get a contradiction, that $Y \in \check{\mathbf{\Gamma}}_D(2^\omega)$. Then $X \cap [s] \in \check{\mathbf{\Gamma}}_D([s])$ by Lemma 6.4.3. It follows easily from Lemma 6.4.1 and Proposition 3.5 that $X \cap [s] \in \check{\mathbf{\Gamma}}_D(2^\omega) = \check{\mathbf{\Gamma}}$. This implies that $X \in \check{\mathbf{\Gamma}}$ by Lemma 14.2, which contradicts the fact that $\mathbf{\Gamma}$ is non-selfdual. □

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