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A CLASS OF ETERNAL SOLUTIONS TO THE G_2-LAPLACIAN FLOW

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Abstract. We explicitly describe the solution of the G_2-Laplacian flow starting from an extremally Ricci-pinched closed G_2-structure on a compact 7-manifold and we investigate its properties. In particular, we show that the solution exists for all real times and that it remains extremally Ricci-pinched. This result holds more generally on any 7-manifold whenever the intrinsic torsion of the extremally Ricci-pinched G_2-structure has constant norm. We also discuss various examples.

1. Introduction

A G_2-structure on a seven-dimensional smooth manifold M is characterized by the existence of a 3-form $\varphi \in \Omega^3(M)$ satisfying a suitable non-degeneracy condition. Such a 3-form gives rise to a Riemannian metric g_φ and to a volume form dV_φ on M.

By [10], the holonomy of g_φ is contained in G_2 if both $d\varphi$ and $d*_\varphi\varphi$ vanish, $*_\varphi$ being the Hodge operator of g_φ. On the other hand, when a Riemannian metric g has $\mathrm{Hol}(g) \subseteq G_2$, then there exists a unique G_2-structure φ satisfying $d\varphi = 0$, $d*_\varphi\varphi = 0$ and such that $g_\varphi = g$. A G_2-structure defined by a non-degenerate 3-form φ which is both closed and co-closed is said to be *torsion-free* and the corresponding Riemannian metric g_φ is Ricci-flat. A G_2-structure φ satisfying the less restrictive condition $d\varphi = 0$ is called *closed*. In such a case, the intrinsic torsion can be identified with a unique 2-form τ such that $d*_\varphi\varphi = \tau\wedge\varphi$, and the scalar curvature of g_φ is given by $-\frac{1}{2}|\tau|^2_\varphi$, where $|\cdot|_\varphi$ denotes the norm induced by g_φ (cf. [3]).

Closed G_2-structures with small torsion constitute the starting point in Joyce's construction of compact 7-manifolds with holonomy G_2 [19]. Besides this and the glueing constructions [7, 21, 24], in recent years a lot of effort has been made in order to understand whether it is possible to obtain metrics with holonomy G_2 using a geometric flow approach. So far, the main results in this direction have been obtained for the G_2-*Laplacian flow* introduced by Bryant in [3]:

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi. \end{cases}$$

Here, φ is a given closed G_2-structure and $\Delta_{\varphi(t)}$ denotes the Hodge Laplacian of $g_{\varphi(t)}$.

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Short-time existence and uniqueness of the solution of the Laplacian flow on a compact manifold were proved by Bryant and Xu in [4], while the geometric and analytic properties of the flow were deeply investigated by Lotay and Wei in [30, 31, 32]. In particular, they proved that the solution $\varphi(t)$ exists as long as the velocity of the flow $|\Delta_{\varphi(t)}\varphi(t)|_{\varphi(t)}$ remains bounded. It is still an open problem whether a bound on the scalar curvature is sufficient to obtain a long-time existence result (cf. [30]). Further aspects of the Laplacian flow were studied in [9, 11, 12, 18, 25, 26, 29].

By [3], on a compact 7-manifold M the Ricci tensor and the scalar curvature of the metric induced by a closed G_2-structure φ must satisfy the following inequality

$$\int_M [\mathrm{Scal}(g_\varphi)]^2\, dV_\varphi \leq 3 \int_M |\mathrm{Ric}(g_\varphi)|^2\, dV_\varphi.$$

In particular, the metric g_φ is Einstein if and only if the G_2-structure is torsion-free (see also [5]). The above inequality reduces to an equality if and only if the intrinsic torsion form τ fulfills the equation $d\tau = \frac{1}{6}\,|\tau|^2_\varphi\, \varphi + \frac{1}{6}\, *_\varphi (\tau\wedge\tau)$. When this happens, the closed G_2-structure is called *extremally Ricci-pinched* (*ERP* for short).

Two examples of manifolds endowed with an ERP closed G_2-structure were obtained by Bryant [3] and Lauret [26]. Both can be described as simply connected solvable Lie groups endowed with a left-invariant ERP closed G_2-structure. In the first case, the Lie group is not unimodular. Nevertheless, it admits a compact quotient by a torsion-free discrete subgroup of the full automorphism group of the G_2-structure. In the second case, the Lie group is unimodular and the existence of a compact quotient has been recently proved in [23]. In [26], Lauret proved that in both examples the ERP closed G_2-structure φ is a *steady Laplacian soliton*, i.e., it satisfies the equation

$$\Delta_\varphi\varphi = \mathcal{L}_X\varphi + \lambda\varphi,$$

for $\lambda = 0$ and for some vector field X. General results on Laplacian solitons (cf. e.g. [30, Sect. 9]) allow one to conclude that the solution of the Laplacian flow starting from one of these ERP closed G_2-structures is self-similar and *eternal*, i.e., it exists for all real times. By [29], compact Laplacian solitons with $\lambda = 0$ are necessarily torsion-free. Thus, none of the above examples can descend to a steady Laplacian soliton on any compact quotient of the corresponding Lie group and, more generally, ERP closed G_2-structures on compact manifolds cannot be steady Laplacian solitons.

In the present paper, we study the behaviour of the Laplacian flow starting from an ERP closed G_2-structure in greater generality. Our main results are contained in Section 4. In Theorem 4.1, we show that the solution of the Laplacian flow starting from an ERP closed G_2-structure φ whose intrinsic torsion form τ has constant norm is given by

$$\varphi(t) = \varphi + f(t)\, d\tau,$$

where $f(t) = \frac{6}{|\tau|^2_\varphi}\left(\exp\left(\frac{|\tau|^2_\varphi}{6}\,t\right) - 1\right)$. From this expression, we easily see that the solution exists for all real times. This result holds, in particular, when the closed G_2-structure is ERP and the manifold M is compact. To prove Theorem 4.1, we first show some useful results on ERP closed G_2-structures in Proposition 4.2, and then we use them to show that the Laplacian flow we are considering is equivalent to a Cauchy problem for the function $f(t)$. The properties obtained in Proposition 4.2 allow us to conclude that the solution $\varphi(t)$

is ERP with constant velocity for all $t \in \mathbb{R}$, and that the Ricci tensor of $g_{\varphi(t)}$ is constant along the flow. Finally, by backward uniqueness and real analyticity of the solution of the Laplacian flow on compact 7-manifolds [30, 32], we conclude that a solution cannot become ERP in finite time unless it starts from an ERP closed G_2-structure.

In Section 5, we study the asymptotic behaviour of the ERP solution $\varphi(t)$ in the compact case. In particular, we show that the volume of the manifold with respect to the Riemannian metric $g_{\varphi(t)}$ increases without bound as $t \to +\infty$, while it shrinks as $t \to -\infty$.

In Section 6, we review the two examples of ERP closed G_2-structures mentioned above, and we discuss some related results. In Example 6.4, we show that Bryant's example belongs to a one-parameter family of inequivalent solvable Lie groups admitting a left-invariant ERP closed G_2-structure, while in Theorem 6.8 we prove that a unimodular Lie group endowed with a left-invariant ERP closed G_2-structure is isomorphic to Lauret's example.

2. Preliminaries

2.1. **Stable forms in dimension seven.** According to [17], a k-form on a real n-dimensional vector space V is said to be *stable* if its $\mathrm{GL}(V)$-orbit is open in $\Lambda^k(V^*)$.

In the present paper, we shall mainly deal with stable 3-forms in dimension seven. They can be characterized as follows.

Proposition 2.1 ([17]). *Let V be a seven-dimensional real vector space. Consider a 3-form $\phi \in \Lambda^3(V^*)$ and the symmetric bilinear map*

$$b_\phi : V \times V \to \Lambda^7(V^*), \quad b_\phi(v,w) = \frac{1}{6}\, \iota_v\phi \wedge \iota_w\phi \wedge \phi.$$

Then, ϕ is stable if and only if $\det(b_\phi)^{1/9} \in \Lambda^7(V^*)$ is not zero.

Given a stable 3-form ϕ, the symmetric bilinear map

$$g_\phi \coloneqq \det(b_\phi)^{-1/9}\, b_\phi : V \times V \to \mathbb{R} \tag{2.1}$$

is either positive definite or it has signature $(3,4)$. These conditions characterize the only two open $\mathrm{GL}(V)$-orbits contained in $\Lambda^3(V^*)$.

We denote the open orbit of stable 3-forms for which (2.1) is positive definite by $\Lambda^3_+(V^*)$. It is well-known that the $\mathrm{GL}^+(V)$-stabilizer of a 3-form $\phi \in \Lambda^3_+(V^*)$ is isomorphic to the exceptional Lie group G_2, and that there exists a basis $(e^1, \ldots, e^7)$ of V^* for which

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}, \tag{2.2}$$

e^{ijk} being a shorthand for $e^i \wedge e^j \wedge e^k$.

2.2. **Closed $\mathbf{G_2}$-structures.** Let M be a seven-dimensional smooth manifold and let $\Lambda^3_+(T^*M)$ denote the open subbundle of $\Lambda^3(T^*M)$ whose fibre over each point $x \in M$ is given by $\Lambda^3_+(T^*_xM)$.

A G_2-structure on M, namely a G_2-reduction of the frame bundle $FM \to M$, is characterized by the existence of a stable 3-form $\varphi \in \Omega^3_+(M) \coloneqq \Gamma(\Lambda^3_+(T^*M))$. This 3-form gives rise to a Riemannian metric g_φ with volume form dV_φ via the identity

$$g_\varphi(X,Y)\, dV_\varphi = \frac{1}{6}\, \iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi,$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. We denote by $*_\varphi$ the Hodge operator determined by g_φ, and by $|\cdot|_\varphi$ the pointwise norm induced by g_φ.

It follows from the discussion in Section 2.1 that at each point x of M there exists a basis $\mathcal{B}^* = (e^1, \ldots, e^7)$ of the cotangent space T_x^*M for which $\varphi|_x$ can be written as in (2.2). We shall call $\mathcal{B}^*$ an *adapted basis* for the G_2-structure φ.

The intrinsic torsion of a G_2-structure φ can be identified with the covariant derivative of φ with respect to the Levi Civita connection ∇^φ of g_φ. By [10], a G_2-structure φ is *torsion-free*, i.e., $\nabla^\varphi\varphi \equiv 0$, if and only if the 3-form φ is both closed and coclosed.

A G_2-structure is said to be *closed* if the defining 3-form φ satisfies the equation $d\varphi = 0$. When this happens, the intrinsic torsion can be identified with a unique 2-form $\tau \in \Omega^2_{14}(M) \coloneqq \left\{\kappa \in \Omega^2(M) \mid \kappa \wedge \varphi = - *_\varphi \kappa\right\} = \left\{\kappa \in \Omega^2(M) \mid \kappa \wedge *_\varphi\varphi = 0\right\}$ such that

$$d *_\varphi \varphi = \tau \wedge \varphi.$$

Clearly, the *intrinsic torsion form* τ vanishes identically if and only if the G_2-structure is torsion-free. Notice that $\tau = d^*\varphi = - *_\varphi d *_\varphi \varphi$, thus it is coclosed and its exterior derivative coincides with the Hodge Laplacian $\Delta_\varphi\varphi = (dd^* + d^*d)\varphi = -d *_\varphi d *_\varphi \varphi$ of φ. Properties of closed G_2-structures were investigated in [3, sect. 4.6] and in [5].

By [15], the closed 3-form φ defines a calibration on M. An oriented three-dimensional submanifold of M is called *associative* if it is calibrated by φ, while an oriented four-dimensional submanifold N is called *coassociative* if $\varphi|_N \equiv 0$ (see [15, Sect. IV] and [20, Ch. 12] for more details).

By [3], the Ricci tensor and the scalar curvature of the Riemannian metric g_φ induced by a G_2-structure φ can be expressed in terms of the intrinsic torsion. In particular, when φ is closed the Ricci tensor has the following expression,

$$\mathrm{Ric}(g_\varphi) = \frac{1}{4}|\tau|^2_\varphi\, g_\varphi - \frac{1}{4}\,\mathrm{j}_\varphi\left(d\tau - \frac{1}{2} *_\varphi (\tau \wedge \tau)\right),$$

where the map $\mathrm{j}_\varphi : \Omega^3(M) \to \mathcal{S}^2(M)$ is defined as follows

$$\mathrm{j}_\varphi(\beta)(X,Y) = *_\varphi\left(\iota_X\varphi \wedge \iota_Y\varphi \wedge \beta\right),$$

and the scalar curvature is given by

$$\mathrm{Scal}(g_\varphi) = -\frac{1}{2}|\tau|^2_\varphi.$$

2.3. **The $\mathbf{G_2}$-Laplacian flow.** Consider a 7-manifold M endowed with a closed G_2-structure φ. The *Laplacian flow* starting from φ is the initial value problem

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi. \end{cases} \tag{2.3}$$

This flow was introduced by Bryant in [3] to study seven-dimensional manifolds admitting closed G_2-structures. Short-time existence and uniqueness of the solution of (2.3) when M is compact were proved in [4].

Theorem 2.2 ([4])**.** *Assume that M is compact. Then, the Laplacian flow* (2.3) *has a unique solution defined for a short time $t \in [0, \varepsilon)$, with ε depending on φ.*

Remark 2.3. The condition $d\varphi(t) = 0$ implies that the solution of (2.3) must belong to the open set

$$[\varphi]_+ \coloneqq [\varphi] \cap \Omega^3_+(M)$$

in the de Rham cohomology class of φ as long as it exists.

By [30, Thm. 1.6], the solution $\varphi(t)$ exists as long as the velocity of the flow $\left|\Delta_{\varphi(t)}\varphi(t)\right|_{\varphi(t)}$ remains bounded. Moreover, if $\varphi(t)$ is defined on some interval $[0, T]$, then for each fixed time $t \in (0, T]$, $(M, \varphi(t), g_{\varphi(t)})$ is real analytic [32].

Using general results on flows of G_2-structures (see e.g. [3, 22]), it is possible to check that the evolution equation of the Riemannian metric $g_{\varphi(t)}$ induced by a G_2-structure $\varphi(t)$ evolving under the Laplacian flow is given by

$$\frac{\partial}{\partial t} g_{\varphi(t)} = -2\,\mathrm{Ric}(g_{\varphi(t)}) + \frac{|\tau(t)|^2_{\varphi(t)}}{6}\, g_{\varphi(t)} + \frac{1}{4}\,\mathrm{j}_{\varphi(t)}\left(*_{\varphi(t)}(\tau(t) \wedge \tau(t))\right), \tag{2.4}$$

(see also [30]), and the corresponding volume form $dV_{\varphi(t)}$ evolves as follows

$$\frac{\partial}{\partial t} dV_{\varphi(t)} = \frac{|\tau(t)|^2_{\varphi(t)}}{3}\, dV_{\varphi(t)}.$$

In particular, $dV_{\varphi(t)}$ is pointwise non-decreasing.

A solution of the Laplacian flow is said to be *self-similar* if it is of the form

$$\varphi(t) = \varrho(t)\, F_t^* \varphi,$$

where $F_t \in \mathrm{Diff}(M)$ and $\varrho(t) \in \mathbb{R} \smallsetminus \{0\}$ is a scaling factor. A standard argument allows one to show that the solution of the Laplacian flow is self-similar if and only if the initial datum φ satisfies the equation

$$\Delta_\varphi \varphi = \mathcal{L}_X \varphi + \lambda \varphi, \tag{2.5}$$

for some vector field X on M and some $\lambda \in \mathbb{R}$ (see e.g. [29, 30]). In such a case, $\varrho(t) = \left(1 + \frac{2}{3}\lambda t\right)^{3/2}$. A closed G_2-structure for which (2.5) holds is called a *Laplacian soliton*. Depending on the sign of λ, a Laplacian soliton is said to be *shrinking* ($\lambda < 0$), *steady* ($\lambda = 0$), or *expanding* ($\lambda > 0$), and the corresponding self-similar solution exists on the maximal time interval $\left(-\infty, -\frac{3}{2\lambda}\right)$, $(-\infty, +\infty)$, $\left(-\frac{3}{2\lambda}, +\infty\right)$, respectively.

3. Extremally Ricci-pinched closed G_2-structures

Let M be a compact 7-manifold endowed with a closed G_2-structure φ. It was proved independently in [3] and [5] that the Riemannian metric g_φ cannot be Einstein unless φ is torsion-free. Moreover, by [3] the Ricci tensor $\mathrm{Ric}(g_\varphi)$ and the scalar curvature $\mathrm{Scal}(g_\varphi)$ of g_φ must satisfy the integral inequality

$$\int_M [\mathrm{Scal}(g_\varphi)]^2\, dV_\varphi \le 3 \int_M |\mathrm{Ric}(g_\varphi)|^2\, dV_\varphi, \tag{3.1}$$

and (3.1) reduces to an equality if and only if the intrinsic torsion form τ fulfills

$$d\tau = \frac{1}{6}\, |\tau|^2_\varphi\, \varphi + \frac{1}{6}\, *_\varphi (\tau \wedge \tau). \tag{3.2}$$

This motivates the following.

Definition 3.1 ([3])**.** A closed G_2-structure φ whose intrinsic torsion form τ satisfies (3.2) is said to be *extremally Ricci-pinched* (*ERP* for short).

Useful properties of ERP closed G_2-structures can be derived starting from (3.2) (cf. [3, Sect. 4.6]). We summarize some of them in the next proposition.

Proposition 3.2 ([3])**.** *Let M be a 7-manifold endowed with an ERP closed G_2-structure φ with intrinsic torsion form $\tau \in \Omega^2_{14}(M)$ not identically vanishing. If M is compact, then τ has constant (non-zero) norm. More generally, if $|\tau|_\varphi$ is constant, then the following results hold*

i) $\tau \wedge \tau \wedge \tau = 0$;

ii) $\tau \wedge \tau$ *is a non-zero closed simple 4-form of constant norm;*

iii) $*_\varphi(\tau \wedge \tau)$ *is a non-zero closed simple 3-form of constant norm;*

iv) by points ii) and iii), the tangent bundle of M splits into the orthogonal direct sum of two integrable subbundles $TM = P \oplus Q$ with

$$P \coloneqq \{X \in TM \mid \iota_X(\tau \wedge \tau) = 0\}, \quad Q \coloneqq \{X \in TM \mid \iota_X *_\varphi (\tau \wedge \tau) = 0\}.$$

Moreover, the P-leaves are associative submanifolds calibrated by $-|\tau|_\varphi^{-2} *_\varphi(\tau\wedge\tau)$, while the Q-leaves are coassociative submanifolds calibrated by $-|\tau|_\varphi^{-2}(\tau \wedge \tau)$;

v) the Ricci tensor of g_φ is given by $\mathrm{Ric}(g_\varphi) = \frac{1}{12}\,\mathrm{j}_\varphi(*_\varphi(\tau \wedge \tau)) = -\frac{1}{6}\,|\tau|^2_\varphi\, g_\varphi|_P$. Hence, it is non-positive with eigenvalues $-\frac{1}{6}\,|\tau|^2_\varphi$ of multiplicity three and 0 of multiplicity four.

Further results on ERP closed G_2-structures were obtained by Cleyton and Ivanov in [6]. We shall recall one of them in Section 6.

Remark 3.3. It follows from (3.2) and the identity $|\tau \wedge \tau|^2_\varphi = |\tau|^4_\varphi$ (cf. [3, (2.21)]) that $d\tau$ has constant norm whenever $|\tau|_\varphi$ is constant, indeed

$$|d\tau|^2_\varphi = \frac{1}{6}|\tau|^4_\varphi.$$

Moreover, as τ is coclosed and $d(\tau \wedge \tau) = 0$, we have

$$\Delta_\varphi \tau = - *_\varphi d *_\varphi d\tau = \frac{1}{6}|\tau|^2_\varphi\, \tau.$$

4. The Laplacian flow starting from an ERP closed G_2-structure

In this section, we prove the following result.

Theorem 4.1. *Let M be a seven-dimensional manifold endowed with an ERP closed G_2-structure φ whose intrinsic torsion form τ has constant non-zero norm. Then, the solution of the Laplacian flow starting from φ at $t = 0$ is*

$$\varphi(t) = \varphi + f(t)\, d\tau, \tag{4.1}$$

where

$$f(t) = \frac{6}{|\tau|^2_\varphi}\left(\exp\left(\frac{|\tau|^2_\varphi}{6}\,t\right) - 1\right).$$

In particular, $\varphi(t)$ is defined for all real times, it is ERP, and the corresponding intrinsic torsion form is given by

$$\tau(t) = \exp\left(\frac{|\tau|^2_\varphi}{6}\,t\right)\tau.$$

Finally, the Ricci tensor of the Riemannian metric $g_{\varphi(t)}$ induced by $\varphi(t)$ is constant along the flow, i.e., $\mathrm{Ric}(g_{\varphi(t)}) = \mathrm{Ric}(g_\varphi)$, and $g_{\varphi(t)}$ evolves as follows

$$\frac{\partial}{\partial t} g_{\varphi(t)} = \left.\frac{|\tau|^2_\varphi}{6}\, g_{\varphi(t)}\right|_Q .$$

The proof of the first assertion in Theorem 4.1 consists in showing that the closed 3-form $\varphi(t)$ given by (4.1) defines a G_2-structure and that the G_2-Laplacian flow (2.3) for $\varphi(t)$ is equivalent to a Cauchy problem for the function $f(t)$. To this aim, it is useful to investigate the properties of the 3-form

$$\widetilde{\varphi} \coloneqq \varphi + a\, d\tau = \left(1 + \frac{1}{6}\,|\tau|^2_\varphi\, a\right)\varphi + \frac{a}{6}\, *_\varphi (\tau\wedge\tau), \tag{4.2}$$

where φ is an ERP closed G_2-structure with intrinsic torsion form τ of constant norm, and a is a real number. We collect them in the next result.

Proposition 4.2. *Let φ be an ERP closed G_2-structure, assume that its intrinsic torsion form τ has constant norm, and consider the closed 3-form $\widetilde{\varphi}$ given by (4.2). Then, $\widetilde{\varphi}$ defines a closed G_2-structure for all $a > -6|\tau|^{-2}_\varphi$. Whenever this happens, the following hold*

1) *the g_φ-orthogonal decomposition $TM = P \oplus Q$ given in point iv) of Proposition 3.2 is also $g_{\widetilde{\varphi}}$-orthogonal. Moreover, $g_{\widetilde{\varphi}}\big|_P = g_\varphi|_P$, and $g_{\widetilde{\varphi}}\big|_Q = \left(1 + \frac{1}{6}|\tau|^2_\varphi a\right) g_\varphi|_Q$;*
2) *the volume form induced by $\widetilde{\varphi}$ is $dV_{\widetilde{\varphi}} = \left(1 + \frac{1}{6}|\tau|^2_\varphi a\right)^2 dV_\varphi$;*
3) *the Hodge dual of $\widetilde{\varphi}$ is $*_{\widetilde{\varphi}}\widetilde{\varphi} = \left(1 + \frac{1}{6}|\tau|^2_\varphi a\right)\left(*_\varphi\varphi - \frac{a}{6}\,\tau\wedge\tau\right)$;*
4) *the intrinsic torsion form of $\widetilde{\varphi}$ is given by $\widetilde{\tau} = \left(1 + \frac{1}{6}|\tau|^2_\varphi a\right)\tau$, it satisfies the identity $*_{\widetilde{\varphi}}(\widetilde{\tau}\wedge\widetilde{\tau}) = *_\varphi(\tau\wedge\tau)$, and its $g_{\widetilde{\varphi}}$-norm coincides with the g_φ-norm of τ, i.e., $|\widetilde{\tau}|_{\widetilde{\varphi}} = |\tau|_\varphi$. Consequently, $\mathrm{Scal}(g_{\widetilde{\varphi}}) = \mathrm{Scal}(g_\varphi)$;*
5) *the closed G_2-structure $\widetilde{\varphi}$ is ERP and $\mathrm{Ric}(g_{\widetilde{\varphi}}) = \mathrm{Ric}(g_\varphi)$.*

Proof. We begin showing that $\widetilde{\varphi}$ is stable for all $a \neq -6|\tau|^{-2}_\varphi$ and that it defines a G_2-structure for all $a > -6|\tau|^{-2}_\varphi$. What we need to prove is that $\widetilde{\varphi}|_x \in \Lambda^3_+(T_xM)$ for all $x \in M$. Let us fix a basis $(e_1, \ldots, e_7)$ of T_xM whose dual basis $(e^1, \ldots, e^7)$ of T^*_xM is an adapted basis for the closed G_2-structure φ. Then, we have

$$\varphi|_x = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

and $g_\varphi|_x = \sum_{i=1}^7 (e^i)^2$. By point iv) of Proposition 3.2, we know that $T_xM = P_x \oplus Q_x$. Without loss of generality, we may assume that $P_x = \langle e_1, e_2, e_3\rangle$ and $Q_x = \langle e_4, e_5, e_6, e_7\rangle$. Since $P = \ker(\tau\wedge\tau)$, the simple 4-form $\tau\wedge\tau|_x$ must be proportional to e^{4567}. This implies that $\tau|_x \in \Lambda^2(Q^*_x)$. Consequently, as $\tau\wedge *_\varphi\varphi = 0$, there exist some real numbers c_1, c_2, c_3 for which

$$\tau|_x = c_1\left(e^{45} - e^{67}\right) + c_2\left(e^{46} + e^{57}\right) + c_3\left(e^{47} - e^{56}\right),$$

In particular, $(\tau\wedge\tau)|_x = -|\tau|^2_\varphi\, e^{4567}$ and $*_\varphi(\tau\wedge\tau)|_x = -|\tau|^2_\varphi\, e^{123}$.

We can now compute the symmetric bilinear map $b_{\widetilde{\varphi}}|_x : T_xM\times T_xM \to \Lambda^7(T^*_xM)$ and see that

$$\det(b_{\widetilde{\varphi}}|_x)^{\frac{1}{9}} = \left(1+\frac{1}{6}|\tau|^2_\varphi a\right)^2 \det(b_\varphi|_x)^{\frac{1}{9}}.$$

Thus, the 3-form $\widetilde{\varphi}|_x$ is stable if and only if $a\neq -6|\tau|^{-2}_\varphi$. Moreover, we have that

$$g_{\widetilde{\varphi}}|_x = \det(b_{\widetilde{\varphi}}|_x)^{-\frac{1}{9}}\, b_{\widetilde{\varphi}}|_x = \sum_{i=1}^{3}(e^i)^2 + \left(1+\frac{1}{6}|\tau|^2_\varphi a\right)\sum_{i=4}^{7}(e^i)^2.$$

Hence, $\widetilde{\varphi}|_x\in\Lambda^3_+(T^*_xM)$ if and only if $a > -6|\tau|^{-2}_\varphi$, as we claimed.

Now, assertions 1) and 2) follow immediately from the above discussion, while assertion 3) can be checked pointwise using the adapted basis for φ we are considering.

As for 4), we have

$$d*_{\widetilde{\varphi}}\widetilde{\varphi} = \left(1+\frac{1}{6}|\tau|^2_\varphi a\right)\left(d*_\varphi\varphi - \frac{a}{6}\,d(\tau\wedge\tau)\right) = \left(1+\frac{1}{6}|\tau|^2_\varphi a\right)\tau\wedge\widetilde{\varphi},$$

since $\tau\wedge\tau$ is closed. Thus, by the uniqueness of the 2-form $\widetilde{\tau}$ for which $d*_{\widetilde{\varphi}}\widetilde{\varphi} = \widetilde{\tau}\wedge\widetilde{\varphi}$, we obtain $\widetilde{\tau} = \left(1+\frac{1}{6}|\tau|^2_\varphi a\right)\tau$. We can compute the Hodge dual of $\widetilde{\tau}$ as follows

$$*_{\widetilde{\varphi}}\widetilde{\tau} = -\widetilde{\tau}\wedge\widetilde{\varphi} = -\left(1+\frac{1}{6}|\tau|^2_\varphi a\right)\tau\wedge\varphi = \left(1+\frac{1}{6}|\tau|^2_\varphi a\right)*_\varphi\tau.$$

Consequently, we get

$$|\widetilde{\tau}|^2_{\widetilde{\varphi}}\, dV_{\widetilde{\varphi}} = \widetilde{\tau}\wedge *_{\widetilde{\varphi}}\widetilde{\tau} = \left(1+\frac{1}{6}|\tau|^2_\varphi a\right)^2 |\tau|^2_\varphi\, dV_\varphi = |\tau|^2_\varphi\, dV_{\widetilde{\varphi}}.$$

Finally, a pointwise computation as in 3) allows us to show that $*_{\widetilde{\varphi}}(\widetilde{\tau}\wedge\widetilde{\tau}) = *_\varphi(\tau\wedge\tau)$. Using this, it is straightforward to check that $\widetilde{\varphi}$ is ERP. Now, the integrable subbundles $\widetilde{P}$ and $\widetilde{Q}$ determined by the ERP closed G_2-structure $\widetilde{\varphi}$ coincide with those determined by φ. Consequently, by point v) of Proposition 3.2 and point 1) above, we have

$$\mathrm{Ric}(g_{\widetilde{\varphi}}) = -\frac{1}{6}|\tau|^2_\varphi\, g_{\widetilde{\varphi}}|_{\widetilde{P}} = -\frac{1}{6}|\tau|^2_\varphi\, g_{\widetilde{\varphi}}|_P = -\frac{1}{6}|\tau|^2_\varphi\, g_\varphi|_P = \mathrm{Ric}(g_\varphi).$$

□

We are now ready to prove the main theorem of this section.

Proof of Theorem 4.1. Consider the closed 3-form $\varphi(t) = \varphi + f(t)d\tau$, where f is a real valued smooth function such that $f(0)=0$. Let t be small enough so that $f(t) > -6|\tau|^{-2}_\varphi$. Then, by Proposition 4.2 the 3-form $\varphi(t)$ defines a closed G_2-structure with intrinsic torsion form $\tau(t) = \left(1+\frac{1}{6}|\tau|^2_\varphi f(t)\right)\tau$. Now, the Laplacian flow equation $\frac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t)$ reads

$$\frac{d}{dt}f(t)\, d\tau = \left(1+\frac{1}{6}|\tau|^2_\varphi f(t)\right)d\tau.$$

Consequently, $\varphi(t)$ is the solution of the Laplacian flow starting from φ at $t=0$ if and only if $f(t)$ solves the Cauchy problem

$$\begin{cases} \frac{d}{dt}f(t) = 1 + \frac{1}{6}|\tau|^2_\varphi f(t), \\ f(0) = 0. \end{cases}$$

Thus, we have

$$f(t) = \frac{6}{|\tau|^2_\varphi}\left(\exp\left(\frac{|\tau|^2_\varphi}{6}\,t\right) - 1\right),$$

whence we see that $f(t)$ is defined for all $t \in \mathbb{R}$ and that it satisfies the condition $f(t) > -6|\tau|^{-2}_\varphi$. The second part of the theorem follows immediately from points 4) and 5) of Proposition 4.2. In particular, the evolution equation of the metric $g_{\varphi(t)}$ can be obtained starting from equation (2.4) and using the identity $\frac{1}{4}\,\mathrm{j}_\varphi(*_\varphi(\tau\wedge\tau)) = 3\,\mathrm{Ric}(g_\varphi)$ (cf. point v) of Proposition 3.2). □

When M is compact, backward uniqueness and real analyticity of the solution of the Laplacian flow (cf. [30, Thm. 1.4] and [32]) together with Theorem 4.1 imply the following.

Corollary 4.3. *Let $\varphi(t)$ be the solution of the Laplacian flow* (2.3) *on a compact 7-manifold M and assume that $\varphi(0)$ is not ERP. Then, $\varphi(t)$ cannot become ERP in finite time.*

Furthermore, from point 4) of Proposition 4.2 and Remark 3.3 we see that the velocity of the flow is constant for all $t \in \mathbb{R}$:

$$\left|\Delta_{\varphi(t)}\varphi(t)\right|_{\varphi(t)} = |d\tau(t)|_{\varphi(t)} = \frac{1}{\sqrt{6}}|\tau|^2_\varphi.$$

Since the solution $\varphi(t)$ of the Laplacian flow starting from an ERP closed G_2-structure φ exists for all real times, and since the Ricci tensor of the corresponding metric $g_{\varphi(t)}$ does not evolve along the flow, it is natural to ask whether $\varphi(t)$ is self-similar or, equivalently, if φ is a steady Laplacian soliton. It follows from [29, Cor. 1] that the answer is negative when the manifold M is compact (see also [30, Prop. 9.5]). In detail:

Theorem 4.4 ([29])**.** *Let M be a compact 7-manifold. Then, the only steady Laplacian solitons on M are given by torsion-free* G_2*-structures.*

On the other hand, it was recently shown in [27] that any left-invariant ERP closed G_2-structure on a non-compact Lie group is a steady Laplacian soliton. The converse of this result does not hold, as there are examples of left-invariant steady Laplacian solitons that are not ERP, see [14].

5. Asymptotic behaviour of the ERP solution

In this section, we assume that the 7-manifold M is compact and we investigate the behaviour of the ERP solution when $t \to \pm\infty$.

Using point 2) of Proposition 4.2, we obtain the following expression for the volume form induced by $\varphi(t)$

$$dV_{\varphi(t)} = \exp\left(\frac{|\tau|^2_\varphi}{3}\,t\right) dV_\varphi.$$

Consequently, the total volume of the compact 7-manifold M with respect to the metric $g_{\varphi(t)}$ is

$$\mathrm{Vol}_{g_{\varphi(t)}}(M) = \int_M dV_{\varphi(t)} = \exp\left(\frac{|\tau|^2_\varphi}{3}\,t\right)\mathrm{Vol}_{g_\varphi}(M),$$

whence we easily see that it increases without bound as t goes to $+\infty$, while it shrinks as t goes to $-\infty$:

$$\lim_{t\to+\infty}\mathrm{Vol}_{g_{\varphi(t)}}(M) = +\infty, \qquad \lim_{t\to-\infty}\mathrm{Vol}_{g_{\varphi(t)}}(M) = 0.$$

We now study the behaviour of the associative P-leaves and coassociative Q-leaves along the flow. As the integrable subbundles $P(t)$ and $Q(t)$ determined by the ERP closed G_2-structure $\varphi(t)$ coincide with the subbundles P and Q determined by $\varphi = \varphi(0)$, at each time $t \in \mathbb{R}$ we can endow the P-leaves and the Q-leaves with the Riemannian metric induced by $g_{\varphi(t)}$. We denote the corresponding Riemannian volume forms by $dV_P(t)$ and $dV_Q(t)$, respectively, and we let $dV_P \coloneqq dV_P(0)$, $dV_Q \coloneqq dV_Q(0)$.

Consider an oriented P-leaf $L_P \hookrightarrow M$. By point iv) of Proposition 3.2, we know that the volume form $dV_P(t)$ on L_P coincides with the restriction of the closed 3-form $-|\tau(t)|^{-2}_{\varphi(t)} *_{\varphi(t)} (\tau(t)\wedge\tau(t))$. Such a volume form is constant along the flow, as

$$dV_P = -|\tau|^{-2}_\varphi *_\varphi (\tau\wedge\tau)\big|_{L_P} = dV_P(t),$$

by point 4) of Proposition 4.2. Using the same result, we see that the volume form $dV_Q(t)$ of an oriented Q-leaf $L_Q \hookrightarrow M$ is given by

$$\begin{aligned} dV_Q(t) &= -|\tau(t)|^{-2}_{\varphi(t)}(\tau(t)\wedge\tau(t))\Big|_{L_Q} = -|\tau|^{-2}_\varphi \exp\left(\frac{|\tau|^2_\varphi}{3}t\right)(\tau\wedge\tau)\Bigg|_{L_Q} \\ &= \exp\left(\frac{|\tau|^2_\varphi}{3}t\right) dV_Q. \end{aligned}$$

It is now immediate to prove the following.

Proposition 5.1. *Let $\varphi(t)$ be the solution of the Laplacian flow starting from an ERP closed G_2-structure on a compact 7-manifold M. Then, the volume of the P-leaves is constant along the flow. Moreover:*

1. *when $t \to +\infty$, the volume of the P-leaves goes to zero relative to the volume of the manifold, while the volume of the Q-leaves and the volume of the manifold M grow at the same rate;*
2. *when $t \to -\infty$, the volume of the Q-leaves and the volume of the manifold M tend to zero at the same rate.*

Remark 5.2. Rescaling the metric $g_{\varphi(t)}$ as $\exp\left(-\frac{|\tau|^2_\varphi}{6}t\right) g_{\varphi(t)}$ shows that the volume of the P-leaves goes to zero as $t \to +\infty$, that is, the P-leaves collapse as $t \to +\infty$.

6. Examples

In this section, we review two examples of 7-manifolds endowed with an ERP closed G_2-structure obtained in [3, 26], and we discuss some related results. Further examples are considered in [2, 27, 28].

We begin with the example obtained by Bryant in [3, Ex. 1]. It consists of an invariant ERP closed G_2-structure on the non-compact homogeneous space $SL(2,\mathbb{C})\ltimes\mathbb{C}^2/SU(2)$. For the sake of convenience, we give the following alternative description (cf. [6, Sect. 6.3] and [26, Ex. 4.13]).

Example 6.1. Let $\mathfrak{r} = \langle e_1, e_2, e_3\rangle$ be the three-dimensional non-unimodular solvable Lie algebra with non-zero Lie brackets

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3.$$

Consider the abelian Lie algebra $\mathbb{R}^4 = \langle e_4, e_5, e_6, e_7\rangle$, and the Lie algebra homomorphism $\mu : \mathfrak{r} \to \mathrm{Der}(\mathbb{R}^4) \cong \mathfrak{gl}(4,\mathbb{R})$ defined as follows

$$\mu(e_1) = \mathrm{diag}\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \ \mu(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \ \mu(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The semidirect product $\mathfrak{s} := \mathfrak{r} \ltimes_\mu \mathbb{R}^4$ is a seven-dimensional non-unimodular completely solvable Lie algebra. Denoted by $(e^1, \dots, e^7)$ the dual basis of $(e_1, \dots, e_7)$, the structure equations $\left(de^i\right)_{i=1,\dots,7}$ of $\mathfrak{s}$ are given by

$$\left(0, -e^{12}, -e^{13}, \frac{1}{2}\,e^{14}, \frac{1}{2}\,e^{15}, -\frac{1}{2}\,e^{16} + e^{25} + e^{34}, -\frac{1}{2}\,e^{17} + e^{24} - e^{35}\right).$$

It is now straightforward to check that the 3-form φ given by (2.2) defines an ERP closed G_2-structure on $\mathfrak{s}$ with intrinsic torsion form

$$\tau = 3\,e^{45} - 3\,e^{67}.$$

From this we see that $|\tau|^2_\varphi = 18$, $P = \mathfrak{r}$, and $Q = \mathbb{R}^4$. Notice that φ is exact:

$$\varphi = d\left(-\frac{1}{2}e^{23} + e^{45} - e^{67}\right).$$

Left-multiplication allows to extend the 3-form φ to a left-invariant one, say $\widetilde{\varphi}$, on the simply connected solvable Lie group S with Lie algebra $\mathfrak{s}$. Bryant's example is then described by the pair $(\mathrm{S}, \widetilde{\varphi})$.

Remark 6.2. Albeit the Lie algebra $\mathfrak{s}$ is not unimodular, the corresponding simply connected solvable Lie group S admits a compact quotient, as it is acted on by a torsion-free discrete subgroup $\Gamma \subset \mathrm{Aut}(\mathrm{S}, \widetilde{\varphi})$ (cf. [3, Ex. 1] and [26, Remark 4.14]). This gives rise to a compact locally homogeneous example of ERP closed G_2-structure on $\Gamma\backslash\mathrm{S}$.

As observed by Cleyton and Ivanov [6], in the above example the intrinsic torsion form τ is parallel with respect to the canonical G_2-connection $\overline{\nabla}$. More generally, the following holds.

Theorem 6.3 ([6]). *Let M be a 7-manifold endowed with a closed* G_2*-structure* φ *and let* $\overline{\nabla}$ *be the corresponding canonical* G_2*-connection. If the intrinsic torsion form* τ *of* φ *is parallel with respect to* $\overline{\nabla}$*, then* φ *is ERP and* (M, g_φ) *is locally isometric to Bryant's example.*

We now describe a one-parameter family of pairwise non-isomorphic solvable Lie algebras admitting an ERP closed G_2-structure and including the above example.

Example 6.4. Let us consider the one-parameter family of three-dimensional non-unimodular solvable Lie algebras $\mathfrak{r}_\eta = \langle e_1, e_2, e_3\rangle$ with non-zero Lie brackets

$$[e_1, e_2] = e_2 + \eta\, e_3, \quad [e_1, e_3] = -\eta\, e_2 + e_3,\ \eta \in \mathbb{R}.$$

Notice that $\mathfrak{r}_0 = \mathfrak{r}$, while the number $1 + \eta^2$ provides a complete isomorphism invariant for $\mathfrak{r}_\eta$ when $\eta \neq 0$ (see [33, Lemma 4.10]). Moreover, it is possible to check that the Ricci endomorphism of the inner product $g = (e^1)^2+(e^2)^2+(e^3)^2$ on $\mathfrak{r}_\eta$ is diagonal with eigenvalue -2 of multiplicity three (cf. [33, Thm. 4.11]).

We let $\mathfrak{s}_\eta \coloneqq \mathfrak{r}_\eta \ltimes_{\mu_\eta} \mathbb{R}^4$, where $\mathbb{R}^4 = \langle e_4, e_5, e_6, e_7\rangle$ is the abelian Lie algebra, and the Lie algebra homomorphism $\mu_\eta : \mathfrak{r}_\eta \to \mathrm{Der}(\mathbb{R}^4) \cong \mathfrak{gl}(4, \mathbb{R})$ is defined as follows

$$\mu_\eta(e_1) = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \eta \\ 0 & 0 & -\eta & \frac{1}{2} \end{pmatrix},\ \mu_\eta(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},\ \mu_\eta(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

When $\eta \neq 0$, the seven-dimensional non-unimodular Lie algebra $\mathfrak{s}_\eta$ is solvable but not completely solvable, as the complex numbers $1 \pm i\eta$, $\frac{1}{2} \pm i\eta$ are eigenvalues of ad_{e_1}. Furthermore, $\mathfrak{s}_0$ coincides with the Lie algebra $\mathfrak{s}$ considered in Example 6.1.

Now, the 3-form φ given by (2.2) is exact, and it defines an ERP closed G_2-structure on $\mathfrak{s}_\eta$ with intrinsic torsion form $\tau = 3\,e^{45} - 3\,e^{67}$. By left-multiplication, we can extend φ to a left-invariant ERP closed G_2-structure $\widetilde{\varphi}_\eta$ on the simply connected solvable Lie group S_η with Lie algebra $\mathfrak{s}_\eta$. Clearly, the Lie groups S_η are pairwise non-isomorphic for all $\eta \geq 0$. However, it is possible to check that the intrinsic torsion form of the ERP closed G_2-structure on S_η is parallel with respect to the canonical G_2-connection. Hence, by the result of Cleyton and Ivanov recalled above, $(\mathrm{S}_\eta, g_{\widetilde{\varphi}_\eta})$ is locally isometric to Bryant's example.

Remark 6.5. Although the simply connected solvable Lie groups S_η are pairwise non-isomorphic for different values of $\eta \geq 0$, it is possible to show that the G_2-structures $(\mathrm{S}_\eta, \widetilde{\varphi}_\eta)$ are pairwise equivalent. This has been proved in the recent work [27].

The next example was obtained by Lauret in [26]. As shown in the same paper, it is not equivalent to Bryant's example (cf. [26, Rem. 4.15]).

Example 6.6. Consider the abelian Lie algebras $\mathbb{R}^3 = \langle e_1, e_2, e_3\rangle$ and $\mathbb{R}^4 = \langle e_4, e_5, e_6, e_7\rangle$, and define the seven-dimensional unimodular solvable Lie algebra $\mathfrak{u}$ as the semidirect product $\mathbb{R}^3 \ltimes_\mu \mathbb{R}^4$, with $\mu : \mathbb{R}^3 \to \mathrm{Der}(\mathbb{R}^4) \cong \mathfrak{gl}(4, \mathbb{R})$ given by

$$\mu(e_1) = \mathrm{diag}(1, 1, -1, -1), \quad \mu(e_2) = \mathrm{diag}(1, -1, 1, -1), \quad \mu(e_3) = \mathrm{diag}(1, -1, -1, 1).$$

The structure equations of $\mathfrak{u}$ can be written with respect to the dual basis $(e^1, \ldots, e^7)$ of $(e_1, \ldots, e_7)$ as follows

$$\left(0, 0, 0, -e^{14} - e^{24} - e^{34}, -e^{15} + e^{25} + e^{35}, e^{16} - e^{26} + e^{36}, e^{17} + e^{27} - e^{37}\right).$$

The 3-form φ given by (2.2) defines an ERP closed G_2-structure on $\mathfrak{u}$ with intrinsic torsion form

$$\tau = -2\,e^{45} + 2\,e^{67} - 2\,e^{46} - 2\,e^{57} + 2\,e^{47} - 2\,e^{56}.$$

Notice that $|\tau|^2_\varphi = 24$, and that P and Q coincide with $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.

Remark 6.7. As shown in [26], in both Examples 6.1 and 6.6 the left-invariant ERP closed G_2-structure is a steady Laplacian soliton on the corresponding non-compact simply connected solvable Lie group. By [29, Cor. 1], it cannot descend to an invariant steady Laplacian soliton on any compact quotient.

In the next theorem, we show that the Lie algebra $\mathfrak{u}$ described in the previous example is the unique unimodular Lie algebra admitting ERP closed G_2-structures up to isomorphism.

Theorem 6.8. *Let* G *be a unimodular Lie group endowed with a left-invariant ERP closed* G_2*-structure. Then, its Lie algebra is isomorphic to the Lie algebra* $\mathfrak{u}$ *described in Example 6.6.*

Proof. Consider the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ endowed with the ERP closed G_2-structure φ corresponding to the left-invariant one on G, and let $\mathfrak{g} = P \oplus Q$ be the induced g_φ-orthogonal decomposition of $\mathfrak{g} \cong T_{1_\mathrm{G}}\mathrm{G}$. Since $\mathfrak{g}$ is unimodular and the Ricci tensor of g_φ is non-positive, the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is abelian by [8, Cor. 1]. Moreover, by [8, Lemma 1] we have $\mathrm{Ric}(g_\varphi)|_\mathfrak{n} = 0$. Consequently, point v) of Proposition 3.2 implies that the nilradical $\mathfrak{n}$ is contained in Q.

To prove the assertion, we consider all possible seven-dimensional unimodular Lie algebras with abelian nilradical of dimension at most four, and we show that $\mathfrak{u}$ is the only one admitting an ERP closed G_2-structure up to isomorphism. We shall deal with the cases $\mathfrak{g}$ solvable and $\mathfrak{g}$ non-solvable separately.

If $\mathfrak{g}$ is solvable, by [36, Thm. 1]

$$\dim(\mathfrak{g}) = \dim(\mathfrak{n}) + k,$$

with $k \leq \dim(\mathfrak{n}) - \dim([\mathfrak{n},\mathfrak{n}])$. As $\mathfrak{n}$ is abelian, we have

$$7 = \dim(\mathfrak{n}) + k, \quad k \leq \dim(\mathfrak{n}) \leq \dim(Q) = 4,$$

whence $\mathfrak{n} = Q$. Thus, the nilradical of $\mathfrak{g}$ is four-dimensional and abelian. Moreover, since $\mathfrak{g}$ is solvable, the dimension of its center $\mathfrak{z}(\mathfrak{g})$ must satisfy $\dim(\mathfrak{z}(\mathfrak{g})) \leq 2\dim(\mathfrak{n}) - \dim(\mathfrak{g}) = 1$ (see e.g. [34]). We now claim that $\mathfrak{g}$ is not decomposable. Indeed, otherwise we could write $\mathfrak{g} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ for some solvable unimodular ideals $\mathfrak{s}_1, \mathfrak{s}_2 \subseteq \mathfrak{g}$. If $\dim(\mathfrak{s}_1) = 2$, then necessarily $\mathfrak{s}_1 \cong \mathbb{R}^2$ and $\mathfrak{z}(\mathfrak{g})$ would have dimension at least two, a contradiction. If $\dim(\mathfrak{s}_1) = 3$, then $\mathfrak{s}_1$ is isomorphic either to the Lie algebra $\mathfrak{e}(1,1)$ of the group of rigid motions of the Minkowski 2-space or to the Lie algebra $\mathfrak{e}(2)$ of the group of rigid motions of the Euclidean 2-space (see, for instance, [1, Thm. 1.1]). Since both these Lie algebras have two-dimensional abelian nilradical, it follows that the unimodular solvable Lie algebra $\mathfrak{s}_2$ must be four-dimensional with two-dimensional abelian nilradical. However, there are no four-dimensional Lie algebras satisfying these properties by [1, Thm. 1.5]. Finally, if $\dim(\mathfrak{s}_1) = 1$, then $\mathfrak{s}_1 \cong \mathbb{R}$ and $\mathfrak{s}_2$ must be six-dimensional with three-dimensional nilradical. Any such $\mathfrak{s}_2$ must be decomposable by [35, Thm. 3]. We can then conclude as in the previous cases. Therefore, $\mathfrak{g}$ is indecomposable.

Indecomposable seven-dimensional solvable Lie algebras with an abelian nilradical of dimension four were classified in [16]. A scan of all possibilities allows to conclude that only three unimodular Lie algebras of this type occur up to isomorphism. One is isomorphic to the Lie algebra $\mathfrak{u}$ described in Example 6.6, thus it admits an ERP closed G_2-structure. The remaining ones correspond to the fourth and the fifth pencil in [16, Prop. 5.4]. Their structure equations with respect to a suitable basis $(e^1,\ldots,e^7)$ are the following:

$$\begin{cases} de^i &=& 0, \quad i=1,2,3,\\ de^4 &=& \alpha\, e^{14}-e^{15}+\gamma\, e^{24},\\ de^5 &=& e^{14}+\alpha\, e^{15}+\gamma\, e^{25},\\ de^6 &=& -\alpha\, e^{16}-\beta\, e^{17}-\gamma\, e^{26}-\rho\, e^{27}-\sigma\, e^{37},\\ de^7 &=& \beta\, e^{16}-\alpha\, e^{17}+\rho\, e^{26}-\gamma\, e^{27}+\sigma\, e^{36}, \end{cases} \tag{6.1}$$

where $\alpha,\beta,\rho\in\mathbb{R}$ and $\gamma,\sigma\in\mathbb{R}\smallsetminus\{0\}$, and

$$\begin{cases} de^i &=& 0, \quad i=1,2,3,\\ de^4 &=& \frac{1}{2}\alpha\, e^{14}-e^{15}+\frac{1}{2}\beta\, e^{24},\\ de^5 &=& e^{14}+\frac{1}{2}\alpha\, e^{15}+\frac{1}{2}\beta\, e^{25},\\ de^6 &=& -e^{36},\\ de^7 &=& -\alpha\, e^{17}-\beta\, e^{27}+e^{37}, \end{cases} \tag{6.2}$$

where $\alpha\in\mathbb{R}$ and $\beta\in\mathbb{R}\smallsetminus\{0\}$. We now show that none of these Lie algebras admits closed G_2-structures. The generic closed 3-form ϕ on the Lie algebra with structure equations (6.1) is given by

$$\begin{aligned} \phi \;=\; & \phi_{123}e^{123}+\phi_{124}e^{124}+\phi_{125}e^{125}+\phi_{135}e^{135}+\phi_{136}e^{136}+\phi_{137}e^{137}+\phi_{147}e^{147}+\phi_{157}e^{157}\\ & +\frac{\rho}{\sigma}\phi_{357}e^{257}+\frac{\rho}{\sigma}\phi_{347}e^{247}+\frac{\alpha}{\gamma}\phi_{245}e^{145}+(\sigma\phi_{147}-\beta\phi_{347})\,e^{356}+(\beta\phi_{357}-\sigma\phi_{157})\,e^{346}\\ & +(\alpha\phi_{235}-\gamma\phi_{135})\,e^{234}+\rho\left(\phi_{147}-\frac{\beta}{\sigma}\phi_{347}\right)e^{256}+\rho\left(\frac{\beta}{\sigma}\phi_{357}-\phi_{157}\right)e^{246}+\phi_{245}e^{245}\\ & +\frac{\alpha^2\phi_{235}-\alpha\gamma\phi_{135}+\phi_{235}}{\gamma}e^{134}+\phi_{235}e^{235}+\phi_{236}e^{236}+\phi_{237}e^{237}+\phi_{347}e^{347}+\phi_{357}e^{357}\\ & +\frac{\beta^2\phi_{357}-\beta\sigma\phi_{157}-\phi_{357}}{\sigma}e^{146}-\frac{\beta^2\phi_{347}-\beta\sigma\phi_{147}-\phi_{347}}{\sigma}e^{156}+\frac{\alpha}{\gamma}\phi_{267}e^{167}+\phi_{267}e^{267}\\ & +\frac{\alpha\phi_{236}-\beta\phi_{237}-\gamma\phi_{136}+\rho\phi_{137}}{\sigma}e^{127}-\frac{\alpha\phi_{237}+\beta\phi_{236}-\gamma\phi_{137}-\rho\phi_{136}}{\sigma}e^{126}, \end{aligned}$$

where $\phi_{ijk}\in\mathbb{R}$. A simple computation shows that

$$b_\phi(e_6,e_6)=-\phi_{267}\left(\phi_{147}\phi_{357}-\phi_{157}\phi_{347}\right)e^{1234567}=-b_\phi(e_7,e_7),$$

thus ϕ cannot define a G_2-structure.

As for the Lie algebra with structure equations (6.2), the generic closed 3-form has the following expression

$$\begin{aligned}\phi \;=\;& \phi_{123}e^{123} + \phi_{124}e^{124} + \phi_{125}e^{125} + \phi_{135}e^{135} + \phi_{136}e^{136} + \phi_{137}e^{137} + \phi_{235}e^{235} + \phi_{236}e^{236}\\ &+\phi_{237}e^{237} + \phi_{245}e^{245} + \phi_{267}e^{267} + \phi_{346}e^{346} + \phi_{347}e^{347} + \phi_{356}e^{356} + \phi_{357}e^{357}\\ &+\frac{1}{2}\left(\alpha\phi_{235} - \beta\phi_{135}\right)e^{234} + \left(\phi_{346} - \frac{\alpha}{2}\phi_{356}\right)e^{156} - \left(\phi_{347} + \frac{\alpha}{2}\phi_{357}\right)e^{157} + \alpha\phi_{267}e^{167}\\ &-\left(\phi_{356} + \frac{\alpha}{2}\phi_{346}\right)e^{146} + \left(\phi_{357} - \frac{\alpha}{2}\phi_{347}\right)e^{147} + \left(\alpha\phi_{237} - \beta\phi_{137}\right)e^{127} + \alpha\phi_{245}e^{145}\\ &-\frac{\beta}{2}\left(\phi_{347}e^{247} + \phi_{346}e^{246} + \phi_{357}e^{257} + \phi_{356}e^{256}\right) + \left(\frac{\alpha^2}{2}\phi_{235} - \frac{\alpha}{2}\phi_{135} + 2\phi_{235}\right)e^{134},\end{aligned}$$

with $\phi_{ijk} \in \mathbb{R}$, and we immediately see that also in this case $b_\phi(e_6, e_6) = -b_\phi(e_7, e_7)$.

We now focus on the case when $\mathfrak{g}$ is unimodular and non-solvable. By the classification in [13], $\mathfrak{g}$ is isomorphic to one of the following

$$\begin{gathered}\left(-e^{23}, -2e^{12}, 2e^{13}, 0, -e^{45}, \tfrac{1}{2}e^{46} - e^{47}, \tfrac{1}{2}e^{47}\right);\\ \left(-e^{23}, -2e^{12}, 2e^{13}, 0, -e^{45}, -\alpha\, e^{46}, (1+\alpha)\, e^{47}\right),\ -1 < \alpha \leq -\tfrac{1}{2};\\ \left(-e^{23}, -2e^{12}, 2e^{13}, 0, -\alpha\, e^{45}, \tfrac{\alpha}{2}\, e^{46} - e^{47}, e^{46} + \tfrac{\alpha}{2}\, e^{47}\right),\ \alpha > 0;\\ \left(-e^{23}, -2e^{12}, 2e^{13}, -e^{14} - e^{25} - e^{47}, e^{15} - e^{34} - e^{57}, 2e^{67}, 0\right).\end{gathered}$$

All of these Lie algebras have a three-dimensional abelian nilradical $\mathfrak{n}$. Since the 3-form defining an ERP closed G_2-structure vanishes on Q, it must vanish on $\mathfrak{n}$. The first three Lie algebras in the above list do not admit any stable closed 3-form satisfying this condition. Indeed, in each case the abelian nilradical is given by $\mathfrak{n} = \langle e_5, e_6, e_7\rangle$ and imposing that the generic closed 3-form ϕ satisfies $\phi(e_5, e_6, e_7) = 0$ gives $b_\phi(e_i, e_i) = 0$, for $i = 5, 6, 7$. We are then left with the last Lie algebra, whose nilradical is $\mathfrak{n} = \langle e_4, e_5, e_6\rangle$. Let us consider the generic closed 3-form

$$\begin{aligned}\phi \;=\;& \phi_{123}e^{123} - 3\phi_{247}e^{124} - \phi_{234}e^{125} + \phi_{267}e^{126} + \phi_{127}e^{127} - \phi_{235}e^{134} + 3\phi_{357}e^{135}\\ &-\phi_{367}e^{136} + \phi_{137}e^{137} + \phi_{467}e^{146} + \left(\phi_{257} - \phi_{234}\right)e^{147} - \phi_{567}e^{156} - \left(\phi_{235} + \phi_{347}\right)e^{157}\\ &+2\phi_{236}e^{167} + \phi_{234}e^{234} + \phi_{235}e^{235} + \phi_{236}e^{236} + \phi_{237}e^{237} + \phi_{247}e^{247} + \phi_{467}e^{256}\\ &+\phi_{257}e^{257} + \phi_{267}e^{267} + \phi_{567}e^{346} + \phi_{347}e^{347} + \phi_{357}e^{357} + \phi_{367}e^{367} + \phi_{456}e^{456}\\ &+\phi_{457}e^{457} + \phi_{467}e^{467} + \phi_{567}e^{567},\end{aligned}$$

where $\phi_{ijk} \in \mathbb{R}$. The condition $\phi(e_4, e_5, e_6) = 0$ gives $\phi_{456} = 0$. Up to a basis change, we may assume that $Q = \langle e_4, e_5, e_6, v\rangle$ with $v = v_1e_1 + v_2e_2 + v_3e_3 + v_7e_7$ for some real numbers v_1, v_2, v_3, v_7. Now, if we consider the equation $0 = \phi(e_4, e_5, v) = \phi_{457}\, v_7$, we see that necessarily $v_7 = 0$, otherwise $b_\phi(e_4, e_4) = 0$ and ϕ would not be stable. An inspection of the equations $\phi(e_4, e_6, v) = 0 = \phi(e_5, e_6, v)$ when ϕ is stable gives the following possibilities

- $\phi_{467} \neq 0$, $\phi_{567} \neq 0$, and $Q = \left\langle e_4, e_5, e_6, e_1 + \frac{\phi_{567}}{\phi_{467}}e_2 - \frac{\phi_{467}}{\phi_{567}}e_3\right\rangle$;
- $\phi_{467} = 0$, $\phi_{567} \neq 0$, and $Q = \langle e_4, e_5, e_6, e_2\rangle$;
- $\phi_{467} \neq 0$, $\phi_{567} = 0$, and $Q = \langle e_4, e_5, e_6, e_3\rangle$.

Now, if τ is the intrinsic torsion form of an ERP closed G_2-structure, then the 4-form $\tau \wedge \tau \in \Lambda^4(Q^*)$ must be closed and simple. However, in none of the above cases there exist closed simple 4-forms on Q. $\square$

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