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The Benefit-Cost Rate Spread for Adjustable-Rate Mortgage with Embedded Options

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Abstract

Adjustable-Rate Mortgages are often embedded with protective derivate instruments to hedge against mortgage rates volatility risk. First, we explore how writing collar options on adjustable-rates affect the total cost of debt. Then, we achieve a measure for the Benefit-Cost Rate Spread with the intent to monitor the Adjustable-Rate Mortgage global cost. Our findings are useful to provide financial practitioners with better awareness about the benefit-cost rate spread in writing protective options.

Keywords: Adjustable-Rate Mortgage (*ARM*), Mortgage effective cost rate, Benefit-Cost Rate Spread, Modified Duration.

1 Introduction

An Adjustable-Rate Mortgage (*ARM*) is a loan embedded with an index-linked adjustable-rate, see for example Briys et al. (1991). The mortgage payments are not set at the beginning of the contract but are modulated over time with the changing of the index-linked rate. To hedge borrower's exposure to interest rate fluctuations, it is common practice to implement risk hedging strategies, such as writing interest rate collars that lock the adjustable-rates to float within a range.

The paper provides an intuitive test to quantify the effective cost rate of an *ARM* embedded with a collar. Then, a measure for the Benefit-Cost Rate Spread is achieved.

An outline of our work is as follows. In Section 2, we provide the notation. In Section 3 we discuss the effective cost of an *ARM* embedded with an interest rate collar. A measure of Benefit-Cost Rate Spread is introduced. Section 4 concludes the note. The proofs are gathered in the Appendix.

2 Notation and layout of the model

Unlike a fixed-rate mortgage, an adjustable-rate mortgage has interest rates varying during the entire lifetime of the mortgage being pegged to an index rate. Lenders should anchor the adjustable-rates on a variety of indices, the most common being rates on one-, three-, or five-year Treasury securities. Another common index is the national or regional average cost of funds to savings and mortgage associations.

The index-linked rate is adjusted at the end of each adjustment period and consequently, the periodic interest payment is reset as well. The rate adjusting procedure offers the borrower the opportunity to have benefits if the index-rate falls but makes losses if the index-rate increases. The advantage for the lender is that of transferring a part of the interest rate volatility risk to the borrower. To hedge against rate fluctuations, the mortgage is often embedded with derivate instruments. In this note, we consider the use of a European[1] collar option, called also hedge wrapper, to bind floating index-linked rates between two limits.

A European collar option written on index-linked rates consists of a series of sequential long positions in European caplets together with a short position in European floorlets on the same floating index-linked rate for the same maturity and notional principal amount. A collar consists of caplets and floorlets involved. Collar-Adjusted mortgage index-linked rates are calculated at the beginning time and take the forward index-rates[2] as substitutes for the future realized index-linked rates.

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[1] The approach can be easily recast for any other protective derivatives.

[2] Clearly any yield curve of forward index-rates can be used as input in the model.

Basic notation follows. Let

- F be the initial principal at time 0;
- T be the total number of years of the mortgage life;
- n be the number of mortgage payments; for the ease of explanation, we assume that the payments are annually paid and so $T = n$. We assume that mortgage index-linked rate adjustment periods coincide with contract's payments periods;
- $j_t^{(0)}$ be the annual compound benchmark *forward index-rate*[3] on the adjustment period $(t, t+1)$ with $t = 0,1,..,n-1$; in the following, we assume $j_t^{(0)}$, with $t = 0,1,..,n-1$, as substitutes for the future realized forward index-rates;
- $i_t^{(0)} = j_t^{(0)} + \Delta$ be the mortgage annual compound *forward index-linked rate* on the adjustment period $(t, t+1)$ with $t = 0,1,..,n-1$, as reference lending rate, where Δ is the *ARM* margin;
- i_{floor} and i_{cap} be the *floor rate* and the *cap rate*, respectively, for the forward index-linked interest rates $i_t^{(0)}$ with $t = 0,1,..,n-1$;
- $r_t^{(0)}$ be the mortgage annual compound *forward Collar-Adjusted interest rate* on the adjustment period $(t, t+1)$ with $t = 0,1,..,n-1$. If we take the forward index-linked rates $i_t^{(0)}$, with $t = 0,1,..,n-1$, as substitutes for the future realized index-linked rates, the Collar-Adjusted interest rate becomes

$$r_t^{(0)} = \begin{cases} i_{floor} & if \quad i_t^{(0)} < i_{floor} \\ i_t^{(0)} & if \quad i_{floor} \le i_t^{(0)} \le i_{cap} \\ i_{cap} & if \quad i_t^{(0)} > i_{cap} \end{cases} \qquad \text{with } t = 0,1,..,n-1;$$

- F_t be the *principal payment* due at time *t*. By construction, the sum of the principal payments is equal to the initial principal $F = \sum_{t=1}^{n} F_t$;
- $I_t^{\text{No collar}} = D_{t-1} \cdot i_{t-1}^{(0)}$ be the periodic interest payment due at time *t*, where D_{t-1} is the outstanding balance at time $t-1$ and $i_{t-1}^{(0)}$ is the forward index-linked rate at time $t-1$ for the adjustment period $(t-1, t)$, with $t = 1,...,n$ and $D_0 = F$;
- $f_t^{\text{No collar}} = F_t + I_t^{\text{No collar}}$ be the payment due at time *t*, with $t = 1,...,n$;

[3] From now on, notation with exponent *(0)* refers to evalutations made at time *0*.

- $I_t^{\text{With collar}} = D_{t-1} \cdot r_{t-1}^{(0)}$ be the periodic interest payment due at time *t*, where D_{t-1} is the outstanding balance at time $t-1$ and $r_{t-1}^{(0)}$ is the *forward Collar-Adjusted interest rate* for the adjustment period $(t-1, t)$ with $t = 1, ..., n$;
- $f_t^{\text{With collar}} = F_t + I_t^{\text{With collar}}$ be the payment due at time *t*, with $t = 1, ..., n$;
- $\pi^{(0)}$ be the *collar premium intrinsic value* paid at the beginning time 0.

The basic contract conditions are assumed to hold, see for example Brealey et al. (2014) and Magni (2020, and the literature cited therein). The Net Present Value (*NPV*) of an *ARM* is

$$NPV_F^{\text{No (With) collar}}(x) = F - \sum_{t=1}^{n} \frac{f_t^{\text{No (With) collar}}}{(1+x)^t}$$

where *x* is the interest preference rate for valuing the series of cash flows at time 0. The notation No(With) collar stands for distinguishing between a not protected *ARM* from a collar-protected *ARM* whose cash flows series are $f_t^{\text{No collar}}$ or $f_t^{\text{With collar}}$, respectively.

2.1 Effective cost rate of an *ARM*

Let us consider an *ARM*. By definition, the *ARM* effective cost rate r_{ARM} solves the equation $NPV_F^{\text{No collar}}(x) = 0$, i.e.

$$F - \sum_{t=1}^{n} \frac{f_t^{\text{No collar}}}{(1+r_{ARM})^t} = 0$$

and such that

$$F = \sum_{t=1}^{n} \frac{f_t^{\text{No collar}}}{(1+r_{ARM})^t} = \sum_{t=1}^{n} \frac{F_t + I_t^{\text{No collar}}}{(1+r_{ARM})^t} = \sum_{t=1}^{n} \frac{F_t + D_{t-1} \cdot i_{t-1}^{(0)}}{(1+r_{ARM})^t}. \tag{1}$$

2.2 Effective cost rate of an *ARM* embedded with a collar option

Consider a collar option written on the index-linked rates to protect the borrower from rates' fluctuations. An upfront collar premium $\pi^{(0)}$ is laid down in a lump sum at time 0.

The discounted cash flow of the total cash flows of the mortgage including the collar premium $\pi^{(0)}$ is given by:

$$NPV_{Collar\text{-}Adjusted\ ARM}(x) = F - \sum_{t=1}^{n} \frac{f_t^{\text{With collar}}}{(1+x)^t} - \pi^{(0)} \tag{2}$$

where, as before, *x* is the interest rate for discounting the cash flow of each premium payment.

We define the *ARM effective Collar-Adjusted interest rate* $r_{Collar\text{-}Adjusted\ ARM}$ as the *Internal Rate of Return (IRR)*[4] of the stream of total payments. By definition $r_{Collar\text{-}Adjusted\ ARM}$ complies the condition[5]

$$NPV_{Collar\text{-}Adjusted\ ARM}\left(r_{Collar\text{-}Adjusted\ ARM}\right)=0\,. \tag{3}$$

In general, no closed formula exists for $r_{Collar\text{-}Adjusted\ ARM}$, to overcome this limitation an analytical approximated measure is achieved in the following Section.

3 The Benefit-Cost Rate Spread

In this Section, we provide a simple test to compare the cost of a collar-protected *ARM* against that of a no-collar protected by using the NPV criterion or, equivalently in this case, by the IRR criterion. Our framework takes the forward index-rates $j_t^{(0)}$, with $t=0,1,..,n-1$, as substitutes for future realized forward index-rates. Consequently, we can calculate the realized Collar-Adjusted interest rates $r_t^{(0)}$. We call the collar indifference price $\pi_{Indifference\ price}$ that collar premium that makes $r_{Collar\text{-}Adjusted\ ARM}=r_{ARM}$, i.e., substituting Equation (1) in Equation (3),

$$\pi_{Indifference\ price}=\sum_{t=1}^{n}\frac{I_t^{\text{No collar}}-I_t^{\text{With collar}}}{\left(1+r_{ARM}\right)^t}=\sum_{t=1}^{n}\frac{D_{t-1}\cdot\left(i_{t-1}^{(0)}-r_{t-1}^{(0)}\right)}{\left(1+r_{ARM}\right)^t}= \tag{4}$$

Formula (4) simply states that $\pi_{Indifference\ price}$ is equal to the present value of the periodic interest payments mismatching between a not-protected *ARM* and a collar-protected one. Following intuitive relations stand:

- $r_{Collar\text{-}Adjusted\ ARM}>r_{ARM}$ if and only if $\pi^{(0)}>\pi_{Indifference\ price}$;
- $r_{Collar\text{-}Adjusted\ ARM}=r_{ARM}$ if and only if $\pi^{(0)}=\pi_{Indifference\ price}$;
- $r_{Collar\text{-}Adjusted\ ARM}<r_{ARM}$ if and only if $\pi^{(0)}<\pi_{Indifference\ price}$.

The proof is an immediate consequence of the definitions of $r_{Collar\text{-}Adjusted\ ARM}$ and r_{ARM}.

Now we investigate the *Benefit-Cost Rate Spread* defined as

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[4] Note that $NPV_{Collar\text{-}Adjusted\ ARM}\left(x\right)$ is a strictly increasing function in *x*.

[5] Due to Norstrøm (1972) condition, the equation $NPV_{Collar\text{-}Adjusted\ ARM}\left(x\right)=0$ admits a unique solution.

$$Benefit - Cost\ Rate\ Spread = r_{Collar\text{-}Adjusted\ ARM} - r_{ARM}\ , \tag{5}$$

So, the collar-protected *ARM* and not-protected *ARM* are indifferent in cost only if the current collar price $\pi^{(0)} = \pi_{Indifference\ price}$.

Proposition 1.

Let F be the initial principal. If $r_{Collar\text{-}Adjusted\ ARM} \neq r_{ARM}$, the Benefit-Cost Rate Spread admits the following approximate measure[6]:

$$Benefit - Cost\ Rate\ Spread \cong \frac{-NPV_F^{\text{With collar}}\left(r_{ARM}\right) + \pi^{(0)}}{F \cdot Duration_F} \tag{6}$$

where

- $NPV_F^{\text{With collar}}\left(r_{ARM}\right) = F - \sum_{t=1}^{n} \frac{f_t^{\text{With collar}}}{\left(1+r_{ARM}\right)^t} = F - \sum_{t=1}^{n} \frac{F_t + D_{t-1} \cdot r_{t-1}^{(0)}}{\left(1+r_{ARM}\right)^t}$;
- $Duration_F = \frac{\sum_{t=1}^{n} t \cdot f_t^{\text{With collar}} \left(1+r_{ARM}\right)^{-t}}{\left(1+r_{ARM}\right) \cdot F}$, with $\frac{1}{1+r_{ARM}} \le Duration_F \le \frac{n}{1+r_{ARM}}$, is the *Modified Duration*[7] *of ARM's Collar-Adjusted total cash flow* discounted at the financing interest rate r_{ARM}.

Proof. See the proof in Appendix A.

Therefore

$$r_{Collar\text{-}Adjusted\ ARM} \cong r_{ARM} + Benefit - Cost\ Rate\ Spread$$

where the *Benefit-Cost Rate Spread* is given in (4). If the *Benefit-Cost Rate Spread* is positive (negative), $r_{Collar\text{-}Adjusted\ ARM}$ is higher (smaller) than r_{ARM}.

Formula (4) gets simpler if the principal is paid back with principal payments equal in amount, i.e. $F_t = \frac{F}{n}$ for $t = 1,..,n$. It is known that the Modified Duration of a series of cash flows with equal amounts of payments is independent on amounts' size and it is dependent only on the interest rate and the number n of payments (see Dierkes and Ortmann, 2015 and, for a review of the use of the duration in the literature, see Shah et al. (2020)), then

[6] In the following, the term proxy is also used as shorthand by approximated measure.

[7] The Modified Duration is an adjusted version of the Macaulay Duration, see Macaulay (1938); for a discussion on the use of the Modified Duration for bonds see for example Fabozzi and Frank (1999).

$$Duration_F = \frac{1 + \frac{1}{r_{ARM}} - \frac{n}{r_{ARM} \cdot s_{n|r_{ARM}}}}{1 + r_{ARM}},$$

where $s_{n|r_{ARM}} = \frac{(1+r_{ARM})^n - 1}{r_{ARM}}$ denotes the final value of an annuity of n payments of 1 per period.

Proposition 2. *Repayment of the mortgage via amortization with level principal payments.*
Let F be the initial principal and the repayment of F occurs via amortization with level principal payments of amount $F_t = \frac{F}{n}$ for $t = 1,..,n$. If $r_{Collar\text{-}Adjusted\ ARM} \neq r_{ARM}$, formula (4) can be simplified:

$$Benefit - Cost\ Rate\ Spread \cong \frac{-F + \frac{F}{n} \cdot a_{n|r_{ARM}} + \sum_{t=1}^{n} \frac{I_t^{\text{With collar}}}{(1+r_{ARM})^t} + \pi^{(0)}}{F \cdot \frac{1 + \frac{1}{r_{ARM}} - \frac{n}{r_{ARM} \cdot s_{n|r_{ARM}}}}{1 + r_{ARM}}} \qquad (7)$$

where $a_{n|r_{ARM}} = \frac{1-(1+r_{ARM})^{-n}}{r_{ARM}}$ denotes the present value of an annuity of n payments of 1 per period.

Proof. See the proof in Appendix B.

Formulae (6) and (7) make evidence that under *ceteris paribus* contract conditions, the *Benefit-Cost Rate Spread* is increasing (decreasing) in the collar premium $\pi^{(0)}$ if $\pi^{(0)}$ is positive (negative).

4 Conclusions

In this paper, we analyse the impact on the effective mortgage cost of writing a collar option on the mortgage index-linked rates and introduce a measure of the *Benefit-Cost Rate Spread*.

Appendix A *Proof of Proposition 1*
Given the annual compound financing interest rate r_{ARM}, let us approximate the function $NPV_{Collar\text{-}Adjusted\ ARM}(x)$ in (2) with the Taylor polynomial approximation of the first order

$$NPV_{Collar\text{-}Adjusted\ ARM}(x) \cong NPV_{Collar\text{-}Adjusted\ ARM}(r_{ARM}) + NPV'_{Collar\text{-}Adjusted\ ARM}(r_{ARM}) \cdot (x - r_{ARM}) \tag{A.1}$$

where $NPV'_{Collar\text{-}Adjusted\ ARM}(r_{ARM})$ is the derivative of $NPV_{Collar\text{-}Adjusted\ ARM}$ concerning *x* at $x = r_{ARM}$.

Set $x = r_{\text{Collar-Adjusted ARM}}$. By Equation (3), it follows that Equation (A.1) can be solved for $r_{Collar\text{-}Adjusted\ ARM} \neq r_{ARM}$:

$$r_{Collar\text{-}Adjusted\ ARM} - r_{ARM} \cong \frac{-NPV_{Collar\text{-}Adjusted\ ARM}(r_{ARM})}{NPV'_{Collar\text{-}Adjusted\ ARM}(r_{ARM})}, \tag{A.2}$$

since $NPV'_{Collar\text{-}Adjusted\ ARM}(r_{ARM}) \neq 0$.

The derivative of (2) at $x = r_{ARM}$ can be reformulated as

$$NPV'_{Collar\text{-}Adjusted\ ARM}(r_{ARM}) = \left.\frac{\partial NPV_{Collar\text{-}Adjusted\ ARM}(x)}{\partial x}\right|_{x=r_{ARM}} =$$

$$= \sum_{t=1}^{n} t \cdot f_t^{\text{With collar}} (1 + r_{ARM})^{-t-1} = \frac{\sum_{t=1}^{n} t \cdot f_t^{\text{With collar}} (1 + r_{ARM})^{-t}}{(1 + r_{ARM})} = F \cdot D_F \tag{A.3}$$

where

$$Duration_F = \frac{\sum_{t=1}^{n} t \cdot f_t^{\text{With collar}} (1 + r_{ARM})^{-t}}{(1 + r_{ARM}) \cdot F}, \text{ with } \frac{1}{1 + r_{ARM}} \leq Duration_F \leq \frac{n}{1 + r_{ARM}}, \text{ is}$$

the *Modified Duration of the total cash flow* discounted at the financing interest rate r_{ARM}.

Let $r_{Collar\text{-}Adjusted\ ARM} \neq r_{ARM}$ be. Since, by definition (see Equation (5))

$$Benefit - Cost\ Rate\ Spread = r_{Collar\text{-}Adjusted\ ARM} - r_{ARM}$$

substituting the approximation (A.2) in the above Equation, where by Equation (1) we get $NPV_{Collar\text{-}Adjusted\ ARM}(r_{ARM}) = NPV_F^{\text{With collar}}(r_{ARM}) - \pi^{(0)}$ and $NPV'_{Collar\text{-}Adjusted\ ARM}(r_{ARM})$ is approximated by (A.3), we obtain

$$Benefit - Cost\ Rate\ Spread \cong \frac{-NPV_{Collar\text{-}Adjusted\ ARM}(r_{ARM})}{NPV'_{Collar\text{-}Adjusted\ ARM}(r_{ARM})} = \frac{-NPV_F^{\text{With collar}}(r_{ARM}) + \pi^{(0)}}{F \cdot Duration_F}$$

since $NPV'_{Collar\text{-}Adjusted\ ARM}(r_{ARM}) \neq 0$. ■

Appendix B *Proof of Proposition 2*

Let the principal payments be equal in amount, i.e. $F_t = \frac{F}{n}$ for $t = 1,..,n$. It follows that

$$NPV_F^{\text{With collar}}\left(r_{ARM}\right)=F-\sum_{t=1}^{n}\frac{F_t+I_t^{\text{With collar}}}{\left(1+r_{ARM}\right)^t}=F-\frac{F}{n}\cdot a_{n|r_{ARM}}-\sum_{t=1}^{n}\frac{I_t^{\text{With collar}}}{\left(1+r_{ARM}\right)^t} \quad \text{(B.1).}$$

The Modified Duration for a cash flow of payments of equal amounts is independent on the amounts, then

$$Duration_F=\frac{1+\frac{1}{r_{ARM}}-\frac{n}{r_{ARM}\cdot s_{n|r_{ARM}}}}{1+r_{ARM}} \text{ with } \frac{1}{1+r_{ARM}}\leq Duration_F<\frac{1}{r_{ARM}} \quad \text{(B.2)}$$

where $a_{n|r_{ARM}}=\frac{1-\left(1+r_{ARM}\right)^{-n}}{r_{ARM}}$ and $s_{n|r_{ARM}}=\frac{\left(1+r_{ARM}\right)^{n}-1}{r_{ARM}}$ denote the present value and the final value, respectively of an annuity of n payments of 1 per period.

Let us substitute $NPV_F^{\text{With collar}}\left(r_{ARM}\right)$ with Equation (B.1) and $Duration_F$ with Equation (B.2) in the formula (6) and the results come out. ■

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