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# EXACT $G_2$ -STRUCTURES ON UNIMODULAR LIE ALGEBRAS

MARISA FERNÁNDEZ, ANNA FINO, AND ALBERTO RAFFERO

ABSTRACT. We consider seven-dimensional unimodular Lie algebras  $\mathfrak{g}$  admitting exact  $G_2$ -structures, focusing our attention on those with vanishing third Betti number  $b_3(\mathfrak{g})$ . We discuss some examples, both in the case when  $b_2(\mathfrak{g}) \neq 0$  and in the case when the Lie algebra  $\mathfrak{g}$  is (2,3)-trivial, i.e., when both  $b_2(\mathfrak{g})$  and  $b_3(\mathfrak{g})$  vanish. These examples are solvable, as  $b_3(\mathfrak{g}) = 0$ , but they are not strongly unimodular, a necessary condition for the existence of lattices on the simply connected Lie group corresponding to  $\mathfrak{g}$ . More generally, we prove that any seven-dimensional (2,3)-trivial strongly unimodular Lie algebra does not admit any exact  $G_2$ -structure. From this, it follows that there are no compact examples of the form  $(\Gamma \backslash G, \varphi)$ , where  $G$  is a seven-dimensional simply connected Lie group with (2,3)-trivial Lie algebra,  $\Gamma \subset G$  is a co-compact discrete subgroup, and  $\varphi$  is an exact  $G_2$ -structure on  $\Gamma \backslash G$  induced by a left-invariant one on  $G$ .

## 1. INTRODUCTION

Let  $M$  be a seven-dimensional smooth manifold. A  $G_2$ -structure on  $M$  is a reduction of the structure group of its frame bundle from  $GL(7, \mathbb{R})$  to the compact exceptional Lie group  $G_2$ . In [12], Gray proved that a smooth 7-manifold carries  $G_2$ -structures if and only if it is orientable and spin.

The existence of a  $G_2$ -structure on  $M$  is characterized by the existence of a globally defined 3-form  $\varphi \in \Omega^3(M)$  satisfying a certain nondegeneracy condition. The  $G_2$ -form  $\varphi$  gives rise to a Riemannian metric  $g_\varphi$  with volume form  $dV_\varphi$  via the identity

$$g_\varphi(X, Y) dV_\varphi = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi, \quad (1.1)$$

for any pair of vector fields  $X, Y$  on  $M$ , where  $\iota_X$  denotes the contraction by  $X$ . Moreover, at each point  $x \in M$  there exists a basis  $\{e^1, \dots, e^7\}$  of the cotangent space  $T_x^*M$  for which

$$\varphi|_x = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

Here,  $e^{127}$  stands for  $e^1 \wedge e^2 \wedge e^7$ , and so on. Such a basis is called an *adapted  $G_2$ -basis*.

A  $G_2$ -structure is said to be *closed* (or *calibrated*) if the defining 3-form  $\varphi$  satisfies the equation  $d\varphi = 0$ , while it is called *co-closed* (or *co-calibrated*) if  $d*_\varphi \varphi = 0$ ,  $*_\varphi$  being the Hodge operator defined by  $g_\varphi$  and  $dV_\varphi$ . When both of these conditions hold, the intrinsic torsion of the  $G_2$ -structure vanishes identically, the Riemannian metric  $g_\varphi$  is Ricci-flat, and  $\text{Hol}(g_\varphi) \subseteq G_2$  (cf. [2, 9]). In this case, the  $G_2$ -structure is said to be *torsion-free*.

The existence of a torsion-free  $G_2$ -structure on a compact 7-manifold  $M$  imposes various constraints on the topology. For instance, the third Betti number must satisfy  $b_3(M) \geq 1$

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[1], and the first Betti number  $b_1(M) \in \{0, 1, 3, 7\}$  [14]. Moreover, if  $\text{Hol}(g_\varphi) = G_2$ , then the fundamental group  $\pi_1(M)$  is finite [14], and so  $b_1(M) = 0$ . However, a non-compact manifold with a torsion-free  $G_2$ -structure may have vanishing third Betti number. An example is given by the metric cone of the nearly Kähler flag manifold  $F_{1,2}$  [2].

In the literature, all known examples of compact 7-manifolds  $M$  admitting a closed  $G_2$ -structure, but not torsion-free  $G_2$ -structures, have  $b_1(M) > 0$  and  $b_3(M) > 0$  (see [4, 6, 7, 8]). A longstanding open question concerns the existence of closed  $G_2$ -structures on compact 7-manifolds with  $b_3(M) = 0$ , such as the 7-sphere. Notice that, in this case, any closed  $G_2$ -structure would be defined by an exact 3-form.

Motivated by this problem, one may investigate the existence of such type of examples when  $M = \Gamma \backslash G$ , where  $G$  is a seven-dimensional simply connected Lie group,  $\Gamma \subset G$  is a co-compact discrete subgroup (lattice), and  $M$  is endowed with an *exact*  $G_2$ -structure induced by a left-invariant one on  $G$ .

Recall that the de Rham cohomology groups of  $\Gamma \backslash G$  are isomorphic to those of the Chevalley-Eilenberg complex of the Lie algebra  $\mathfrak{g}$  of  $G$  when, for instance, the latter is nilpotent or completely solvable (cf. [18, 13]). Thus, a possible strategy to study the problem consists in looking for suitable Lie algebras  $\mathfrak{g}$  with third Betti number  $b_3(\mathfrak{g}) = 0$  and admitting closed  $G_2$ -structures, and see whether the corresponding simply connected Lie group admits a lattice. Since seven-dimensional nilpotent Lie algebras cannot admit exact  $G_2$ -structures (cf. [4]), and since any Lie algebra satisfying  $b_3(\mathfrak{g}) = 0$  is solvable and not nilpotent (see e.g. [15]), one may firstly focus on the latter case. By a result of Garland [11], every simply connected solvable Lie group containing a lattice must be strongly unimodular (see Definition 2.2 for details). This provides a further restriction on  $\mathfrak{g}$ .

In Section 2, we focus on Lie algebras  $\mathfrak{g}$  satisfying the conditions  $b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$ . They are known as *(2,3)-trivial* in the literature, and they are always solvable with codimension one derived algebra  $[\mathfrak{g}, \mathfrak{g}]$  [15].

In Example 2.1, we show the existence of a unimodular *(2,3)-trivial* Lie algebra  $\mathfrak{s}$  admitting closed  $G_2$ -structures. Since  $\mathfrak{s}$  is unimodular, by Hodge duality it also satisfies  $b_4(\mathfrak{s}) = 0 = b_5(\mathfrak{s})$ . However, we prove that it is not strongly unimodular, thus the corresponding simply connected solvable Lie group does not admit lattices. Notice that this example reveals an interesting difference between closed  $G_2$ -structures and symplectic structures (see [10] for further results in this direction). Indeed, both structures are defined by closed differential forms satisfying the same non-degeneracy condition, namely their orbit under the action of the general linear group is open, but a unimodular Lie algebra cannot admit exact symplectic forms (cf. [5]).

More generally, we prove that it is not possible to obtain compact examples of the form  $M = \Gamma \backslash G$ , when the Lie algebra of the simply connected Lie group  $G$  is *(2,3)-trivial* and strongly unimodular. In detail, using a characterisation of *(2,3)-trivial* Lie algebras by Madsen and Swann [15], we first obtain the classification of all seven-dimensional *(2,3)-trivial* strongly unimodular Lie algebras up to isomorphism (Theorem 2.3). Then, we show that none of them admits exact  $G_2$ -structures (Theorem 2.4).

In Section 3, we discuss an example of a seven-dimensional unimodular Lie algebra  $\mathfrak{h}$  satisfying  $b_3(\mathfrak{h}) = 0$ ,  $b_2(\mathfrak{h}) \neq 0$  and admitting closed  $G_2$ -structures. Also in this case, it turns out that the solvable Lie algebra  $\mathfrak{h}$  is not strongly unimodular. This leads to the

following open problem: *is there any seven-dimensional Lie algebra  $\mathfrak{g}$  satisfying  $b_3(\mathfrak{g}) = 0$  which is strongly unimodular and admits closed  $G_2$ -structures?*

## 2. (2,3)-TRIVIAL STRONGLY UNIMODULAR LIE ALGEBRAS

In this section, we focus on seven-dimensional (2,3)-trivial Lie algebras. First of all, we recall that being (2,3)-trivial imposes strong constraints on the structure of a Lie algebra. Indeed, as shown in [15], any finite-dimensional Lie algebra  $\mathfrak{g}$  different from  $\mathbb{R}, \mathbb{R}^2$  and satisfying  $b_3(\mathfrak{g}) = 0$  is solvable and not nilpotent. Moreover, if also  $b_2(\mathfrak{g}) = 0$ , then the derived algebra  $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$  is a codimension one ideal, and  $\mathfrak{g}$  is the semidirect product  $\mathfrak{g} = \mathfrak{n} \rtimes \mathbb{R}$ . We emphasise that  $\mathfrak{n}$  is nilpotent, as  $\mathfrak{g}$  is solvable.

In the next example, we describe a seven-dimensional (2,3)-trivial unimodular Lie algebra admitting closed  $G_2$ -structures. This shows in particular that exact  $G_2$ -structures behave differently from exact symplectic structures, as the latter do not exist on unimodular Lie algebras (cf. [5]).

**Example 2.1.** Let  $\mathfrak{s} = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle$  be the seven-dimensional solvable Lie algebra with the following nonzero Lie brackets

$$\begin{aligned} [e_1, e_3] &= e_4 - 3e_6, & [e_1, e_4] &= e_5, & [e_1, e_5] &= -e_6, & [e_2, e_3] &= e_5, \\ [e_2, e_4] &= -e_6, & [e_1, e_7] &= 2e_1, & [e_2, e_7] &= 4e_2, & [e_3, e_7] &= -\frac{9}{2}e_3 + 6e_5, \\ [e_4, e_7] &= -\frac{5}{2}e_4 + 6e_6, & [e_5, e_7] &= -\frac{1}{2}e_5, & [e_6, e_7] &= \frac{3}{2}e_6. \end{aligned}$$

Notice that  $\mathfrak{s}$  is the semidirect product  $\mathfrak{s} = \mathfrak{n} \rtimes \mathbb{R}$ , where  $\mathbb{R} = \langle e_7 \rangle$  and  $\mathfrak{n} = \langle e_1, \dots, e_6 \rangle$  is isomorphic to the unique six-dimensional 4-step nilpotent Lie algebra with  $b_1(\mathfrak{n}) = 3$  and  $b_2(\mathfrak{n}) = 4$  (cf. [19, Table A.1]). Moreover,  $\mathfrak{s}$  is completely solvable, i.e., for each  $X \in \mathfrak{s}$  the map  $\text{ad}_X \in \text{End}(\mathfrak{s})$  has only real eigenvalues.

Let  $(e^1, \dots, e^7)$  be the dual basis of  $(e_1, \dots, e_7)$ . Then, the structure equations of  $\mathfrak{s}$  can also be written by means of the Chevalley-Eilenberg differentials of the covectors  $e^i$ . In detail:

$$\begin{aligned} de^1 &= -2e^{17}, & de^2 &= -4e^{27}, & de^3 &= \frac{9}{2}e^{37}, & de^4 &= \frac{5}{2}e^{47} - e^{13}, \\ de^5 &= \frac{1}{2}e^{57} - 6e^{37} - e^{14} - e^{23}, & de^6 &= -\frac{3}{2}e^{67} - 6e^{47} + 3e^{13} + e^{15} + e^{24}, & de^7 &= 0. \end{aligned}$$

Using these equations, it is possible to compute the cohomology groups of the Chevalley-Eilenberg complex  $(\Lambda^\bullet(\mathfrak{s}^*), d)$  of  $\mathfrak{s}$ , obtaining

$$H^1(\mathfrak{s}) = \langle [e^7] \rangle, \quad H^6(\mathfrak{s}) = \langle [e^{123456}] \rangle, \quad H^7(\mathfrak{s}) = \langle [e^{1234567}] \rangle, \quad H^i(\mathfrak{s}) = 0, \quad i = 2, 3, 4, 5.$$

We immediately see that  $b_2(\mathfrak{s}) = 0 = b_3(\mathfrak{s})$ , hence  $\mathfrak{s}$  is (2,3)-trivial. Moreover, since  $H^7(\mathfrak{s}) \cong \mathbb{R}$ , the Lie algebra  $\mathfrak{s}$  is unimodular. Recall that this is equivalent to having  $\text{tr}(\text{ad}_X) = 0$  for each  $X \in \mathfrak{s}$ .

The Lie algebra  $\mathfrak{s}$  admits closed (hence exact)  $G_2$ -structures. An example is given by the following 3-form

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} = d \left( \frac{1}{6}e^{12} + \frac{23}{7}e^{34} + 2e^{36} - 2e^{45} + e^{56} \right).$$

Consequently, the simply connected solvable Lie group  $S$  with Lie algebra  $\mathfrak{s}$  is endowed with a *left-invariant* exact  $G_2$ -structure obtained from  $\varphi$  via left multiplication.

It is well-known that a Lie group admits a lattice only if it is unimodular (cf. [17]). When the Lie group is solvable, there is a stronger necessary condition for the existence of lattices, namely the group must be *strongly unimodular* (cf. [11, Prop. 3.3]). We recall the definition here.

**Definition 2.2.** Let  $G$  be a simply connected solvable Lie group with Lie algebra  $\mathfrak{g}$ . Denote by  $\mathfrak{n}^0 := \mathfrak{n}$  the nilradical of  $\mathfrak{g}$  and, for each positive integer  $i \geq 1$ , let  $\mathfrak{n}^i := [\mathfrak{n}, \mathfrak{n}^{i-1}]$  denote the  $i^{\text{th}}$  term in the descending central series of  $\mathfrak{n}$ . The Lie algebra  $\mathfrak{g}$  is *strongly unimodular* if for all  $X \in \mathfrak{g}$  the restriction of  $\text{ad}_X$  to each space  $\mathfrak{n}^i/\mathfrak{n}^{i+1}$  is traceless. In this case, the Lie group  $G$  is said to be *strongly unimodular*.

Notice that any solvable strongly unimodular Lie algebra  $\mathfrak{g}$  is unimodular. Indeed, if the nilradical  $\mathfrak{n}$  of  $\mathfrak{g}$  is  $s$ -step nilpotent, then we have the vector space decomposition

$$\mathfrak{g} = \mathfrak{g}/\mathfrak{n} \oplus \mathfrak{n}/\mathfrak{n}^1 \oplus \mathfrak{n}^1/\mathfrak{n}^2 \oplus \cdots \oplus \mathfrak{n}^{s-2}/\mathfrak{n}^{s-1} \oplus \mathfrak{n}^{s-1}, \quad (2.1)$$

where  $\mathfrak{g}/\mathfrak{n}$  is abelian since  $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}$ . Now, for every  $X \in \mathfrak{g}$ , the endomorphism  $\text{ad}_X \in \text{End}(\mathfrak{g})$  preserves every summand of the splitting (2.1). From this and the Definition 2.2, we immediately get that  $\text{ad}_X$  is traceless, i.e.,  $\mathfrak{g}$  is unimodular. On the other hand, a unimodular solvable Lie algebra may not be strongly unimodular. This is the case, for instance, of the Lie algebra  $\mathfrak{s}$  considered in Example 2.1. Indeed, its nilradical  $\mathfrak{n}$  has the following descending central series

$$\mathfrak{n}^0 = \mathfrak{n}, \quad \mathfrak{n}^1 = \langle e_4 - 3e_6, e_5, e_6 \rangle, \quad \mathfrak{n}^2 = \langle e_5, e_6 \rangle, \quad \mathfrak{n}^3 = \langle e_6 \rangle, \quad \mathfrak{n}^4 = \{0\},$$

and the restriction of  $\text{ad}_{e_7}$  to  $\mathfrak{n}^3$  is not traceless. This shows in particular that the simply connected solvable Lie group  $S$  does not contain any lattice.

It is then natural to ask the following.

**Question 1.** Is there any solvable strongly unimodular Lie algebra admitting exact  $G_2$ -structures?

As we are interested in examples having as many zero Betti numbers as possible, we investigate this problem under the assumption that the Lie algebra is (2,3)-trivial. Our strategy is the following: we first determine all possible (2,3)-trivial, strongly unimodular Lie algebras of dimension seven up to isomorphism, and then we investigate whether any of them admits exact  $G_2$ -structures. In the next theorem, we deal with the first problem.

**Theorem 2.3.** *A (2,3)-trivial, strongly unimodular seven-dimensional Lie algebra is solvable and its six-dimensional nilradical is either abelian or isomorphic to one of the following nilpotent Lie algebras*

$$(0, 0, 0, 0, e^{12}, e^{34}), \quad (0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23}).$$

*Proof.* By [15, Thm. 4.3], a Lie algebra  $\mathfrak{g}$  with derived algebra  $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$  is (2,3)-trivial if and only if  $\mathfrak{g}$  is solvable,  $\mathfrak{n}$  is nilpotent of codimension one in  $\mathfrak{g}$  and  $H^1(\mathfrak{n})^{\mathfrak{g}} = \{0\} = H^2(\mathfrak{n})^{\mathfrak{g}} = H^3(\mathfrak{n})^{\mathfrak{g}}$ . Here,  $H^k(\mathfrak{n})$  denotes the  $k^{\text{th}}$  cohomology group of the Chevalley-Eilenberg complex

$(\Lambda^\bullet(\mathfrak{n}^*), \hat{d})$  of  $\mathfrak{n}$ , and  $H^k(\mathfrak{n})^{\mathfrak{g}}$  denotes the fixed point set of the  $\mathfrak{g}$ -action on  $H^k(\mathfrak{n})$  for which  $\mathfrak{n} \subset \mathfrak{g}$  acts trivially and every  $X \in \mathfrak{g}/\mathfrak{n} \cong \mathbb{R}$  acts as follows:

$$X \cdot [\alpha] = [\iota_X d\alpha], \quad \forall [\alpha] \in H^k(\mathfrak{n}),$$

where  $d$  is the Chevalley-Eilenberg differential on  $\mathfrak{g}$ .

This characterisation leads us to considering seven-dimensional solvable Lie algebras of the form  $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$ , where  $\mathfrak{n}$  is a six-dimensional nilpotent Lie algebra and  $\mathbb{R} = \langle X \rangle$  with  $\text{ad}_X = D \in \text{Der}(\mathfrak{n})$ . To determine all possible Lie algebras of this type, up to isomorphism, we proceed as follows. It is known that there exist 34 six-dimensional nilpotent Lie algebras up to isomorphism. For each of them, we consider the basis  $\{e_1, \dots, e_6\}$  of  $\mathfrak{n}$  for which its structure equations are those given in [19, Table A.1]. We can then compute the generic derivation  $D \in \text{Der}(\mathfrak{n})$  and consider the  $6 \times 6$  matrix associated to it with respect to the basis  $\{e_1, \dots, e_6\}$ . This matrix may have up to 36 unknown entries, which we denote by  $a_i$ . Finally, we let  $\mathbb{R} = \langle e_7 \rangle$  with  $\text{ad}_{e_7} = D$ .

Now, for any six-dimensional nilpotent Lie algebra  $\mathfrak{n}$ , we have to check whether the Lie algebra  $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R} = \langle e_1, \dots, e_7 \rangle$  can both be strongly unimodular and satisfy  $H^1(\mathfrak{n})^{\mathfrak{g}} = \{0\} = H^2(\mathfrak{n})^{\mathfrak{g}} = H^3(\mathfrak{n})^{\mathfrak{g}}$ . Clearly,  $\mathfrak{g}$  is strongly unimodular if and only if the restriction of  $\text{ad}_{e_7} = D$  to each space  $\mathfrak{n}^i/\mathfrak{n}^{i+1}$  is traceless. This gives a set of linear equations in the real variables  $a_i$ , say  $\mathcal{S}$ . As for the second condition, we first determine a basis of the cohomology groups  $H^k(\mathfrak{n})$ ,  $k = 1, 2, 3$ , and then we compute the square matrix  $A_k$  associated to the linear endomorphism  $H^k(\mathfrak{n}) \ni [\alpha] \mapsto e_7 \cdot [\alpha]$  with respect to that basis. The entries of these matrices are polynomials in the variables  $a_i$ , and the condition  $H^k(\mathfrak{n})^{\mathfrak{g}} = \{0\}$  is equivalent to  $A_k$  being non-singular.

Summing up, for each six-dimensional nilpotent Lie algebra  $\mathfrak{n}$ , we have to solve the set of linear equations  $\mathcal{S}$  under the constraints  $\det(A_k) \neq 0$ ,  $k = 1, 2, 3$ . For 31 out of 34 non-isomorphic six-dimensional nilpotent Lie algebras this is not possible.

As an example, let us consider the first nilpotent Lie algebra appearing in [19, Table A.1]. Its structure equations with respect to the dual basis  $\{e^1, \dots, e^6\}$  of  $\{e_1, \dots, e_6\}$  are the following

$$\mathfrak{n} = (0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{34} - e^{25}).$$

The descending central series of  $\mathfrak{n}$  is

$$\mathfrak{n}^1 = \langle e_3, e_4, e_5, e_6 \rangle, \quad \mathfrak{n}^2 = \langle e_4, e_5, e_6 \rangle, \quad \mathfrak{n}^3 = \langle e_5, e_6 \rangle, \quad \mathfrak{n}^4 = \langle e_6 \rangle, \quad \mathfrak{n}^5 = \{0\},$$

and the matrix associated to a generic derivation  $D \in \text{Der}(\mathfrak{n})$  with respect to the basis  $\{e_1, \dots, e_6\}$  is the following

$$D = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & 3a_1 & 0 & 0 & 0 \\ a_4 & 0 & a_3 & 4a_1 & 0 & 0 \\ a_5 & a_6 & -a_2 & a_3 & 5a_1 & 0 \\ a_7 & a_8 & a_5 & -a_4 & a_2 & 7a_1 \end{pmatrix}.$$

Now, we see that  $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$  is strongly unimodular if and only if  $a_1 = 0$ . On the other hand, we have  $H^1(\mathfrak{n}) = \langle [e^1], [e^2] \rangle$ , and we see that

$$e_7 \cdot [e^1] = [\iota_{e_7} de^1] = -a_1[e^1], \quad e_7 \cdot [e^2] = [\iota_{e_7} de^2] = -2a_1[e^2].$$

Thus,  $\det(A_1) = 2(a_1)^2$ , whence it follows that  $\mathfrak{g}$  cannot be both (2,3)-trivial and strongly unimodular. In the remaining 30 cases, we proceed similarly. We obtain that any solution of  $\mathcal{S}$  implies  $\det(A_1) = 0$  when  $\mathfrak{n}$  is one of the following Lie algebras

$$\begin{aligned} & (0, 0, e^{12}, e^{13}, e^{14}, e^{34} - e^{25}), & (0, 0, e^{12}, e^{13}, e^{14}, e^{15}), \\ & (0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{15} + e^{24}), & (0, 0, e^{12}, e^{13}, e^{14}, e^{15} + e^{23}), \\ & (0, 0, e^{12}, e^{13}, e^{23}, e^{14}), & (0, 0, 0, e^{12}, e^{14} - e^{23}, e^{15} + e^{34}), \\ & (0, 0, 0, e^{12}, e^{14}, e^{15} + e^{23}), & (0, 0, 0, e^{12}, e^{14}, e^{15} + e^{23} + e^{24}), \\ & (0, 0, 0, e^{12}, e^{14}, e^{15} + e^{24}), & (0, 0, 0, e^{12}, e^{14}, e^{15}), \\ & (0, 0, 0, e^{12}, e^{13}, e^{14} + e^{35}), & (0, 0, 0, e^{12}, e^{23}, e^{14} + e^{35}), \\ & (0, 0, 0, e^{12}, e^{23}, e^{14} - e^{35}), & (0, 0, 0, e^{12}, e^{14}, e^{24}), \\ & (0, 0, 0, e^{12}, e^{13} - e^{24}, e^{14} + e^{23}), & (0, 0, 0, e^{12}, e^{14}, e^{13} - e^{24}), \\ & (0, 0, 0, e^{12}, e^{13} + e^{14}, e^{24}), & (0, 0, 0, e^{12}, e^{13}, e^{14} + e^{23}), \\ & (0, 0, 0, e^{12}, e^{13}, e^{24}), & (0, 0, 0, e^{12}, e^{13}, e^{14}), \\ & (0, 0, 0, 0, e^{12}, e^{15} + e^{34}), & (0, 0, 0, 0, e^{12}, e^{15}), \\ & (0, 0, 0, 0, e^{12}, e^{14} + e^{25}), & (0, 0, 0, 0, 0, e^{12} + e^{34}). \end{aligned}$$

Any solution of  $\mathcal{S}$  implies  $\det(A_2) = 0$  in the following cases

$$(0, 0, 0, e^{12}, e^{13}, e^{23}), \quad (0, 0, 0, 0, e^{12}, e^{14} + e^{23}).$$

Finally, any solution of  $\mathcal{S}$  implies  $\det(A_3) = 0$  when  $\mathfrak{n}$  is one of the following

$$\begin{aligned} & (0, 0, e^{12}, e^{13}, e^{23}, e^{14} - e^{25}), & (0, 0, e^{12}, e^{13}, e^{23}, e^{14} + e^{25}), \\ & (0, 0, 0, 0, e^{12}, e^{13}), & (0, 0, 0, 0, 0, e^{12}), \end{aligned}$$

We now claim that there exist (2,3)-trivial and strongly unimodular Lie algebras  $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$  when  $\mathfrak{n}$  is one of the Lie algebras of [19, Table A.1] not appearing above, namely

$$\mathfrak{a} = (0, 0, 0, 0, 0, 0), \quad \mathfrak{n}_1 = (0, 0, 0, 0, e^{12}, e^{34}), \quad \mathfrak{n}_2 = (0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23}).$$

Let us examine each case separately.

For the abelian Lie algebra  $\mathfrak{a}$ , we have  $\text{Der}(\mathfrak{a}) \cong \text{End}(\mathfrak{a})$  and  $H^k(\mathfrak{a}) \cong \Lambda^k(\mathfrak{a}^*)$ . To obtain a (2,3)-trivial Lie algebra  $\mathfrak{g} = \mathfrak{a} \rtimes_D \mathbb{R}$ , it is sufficient to consider a diagonal derivation

$$D = \text{diag}(a_1, a_2, a_3, a_4, a_5, a_6),$$

with  $a_i \neq 0$ , and  $a_i \neq -a_j, -a_j - a_k$ , whenever  $i, j, k$  are distinct (see also [16, Ex. 5.1] for a similar case). Moreover, the Lie algebra  $\mathfrak{g}$  is strongly unimodular if and only if  $\text{tr}(D) = \sum_{i=1}^6 a_i = 0$ . This last equation clearly admits solutions under the above constraints.

The Lie algebra  $\mathfrak{n}_1$  has the following derived series

$$(\mathfrak{n}_1)^0 = \mathfrak{n}_1, \quad (\mathfrak{n}_1)^1 = \langle e_5, e_6 \rangle, \quad (\mathfrak{n}_1)^i = \{0\}, \quad i \geq 2,$$

and the generic derivation  $D_1 \in \text{Der}(\mathfrak{n}_1)$  has the following expression

$$D_1 = \begin{pmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_5 & a_6 & 0 & 0 \\ 0 & 0 & a_7 & a_8 & 0 & 0 \\ a_9 & a_{10} & a_{11} & a_{12} & a_1 + a_4 & 0 \\ a_{13} & a_{14} & a_{15} & a_{16} & 0 & a_5 + a_8 \end{pmatrix}. \quad (2.2)$$

Thus, the seven-dimensional Lie algebra  $\mathfrak{g} = \mathfrak{n}_1 \rtimes_{D_1} \mathbb{R}$  is strongly unimodular if and only if

$$a_1 + a_4 + a_5 + a_8 = 0.$$

We can now compute the determinant of the matrices  $A_1, A_2, A_3$  (see the Appendix for their expression), and notice that there exist solutions of the equation above for which  $\det(A_k) \neq 0$ , for  $k = 1, 2, 3$ . An example is given by

$$a_1 = 1, \quad a_4 = 3, \quad a_5 = 2, \quad a_8 = -6, \quad a_i = 0 \text{ otherwise.}$$

Finally, let us consider the Lie algebra  $\mathfrak{n}_2$ . Its derived series is

$$(\mathfrak{n}_2)^0 = \mathfrak{n}_2, \quad (\mathfrak{n}_2)^1 = \langle e_5, e_6 \rangle, \quad (\mathfrak{n}_2)^i = \{0\}, \quad i \geq 2,$$

and the generic derivation  $D_2 \in \text{Der}(\mathfrak{n}_2)$  is

$$D_2 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ -a_2 & a_1 & -a_4 & a_3 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 & 0 & 0 \\ -a_6 & a_5 & -a_8 & a_7 & 0 & 0 \\ a_9 & a_{10} & a_{11} & a_{12} & a_1 + a_7 & a_2 + a_8 \\ a_{13} & a_{14} & a_{15} & a_{16} & -a_2 - a_8 & a_1 + a_7 \end{pmatrix}. \quad (2.3)$$

From this, we see that  $\mathfrak{g} = \mathfrak{n}_2 \rtimes_{D_2} \mathbb{R}$  is strongly unimodular if and only if

$$a_1 + a_7 = 0.$$

There exist solutions of this equation such that  $\det(A_k) \neq 0$ , for  $k = 1, 2, 3$ . For instance

$$a_1 = 1, \quad a_4 = 2, \quad a_5 = 2, \quad a_7 = -1, \quad a_8 = 1, \quad a_i = 0 \text{ otherwise.}$$

Also in this case, the explicit expression of the matrices  $A_1, A_2, A_3$  can be found in the Appendix.  $\square$

We now show that there are no exact  $G_2$ -structures on any seven-dimensional solvable Lie algebra of the form  $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$ , when  $\mathfrak{n} = \mathfrak{a}, \mathfrak{n}_1$ , and when  $\mathfrak{n} = \mathfrak{n}_2$  and  $D \in \text{Der}(\mathfrak{n}_2)$  is such that  $\mathfrak{g}$  is strongly unimodular. Combining this result with Theorem 2.3, we get that there are no (2,3)-trivial, strongly unimodular seven-dimensional Lie algebras admitting exact  $G_2$ -structures.

**Theorem 2.4.** *Any seven-dimensional solvable Lie algebra  $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$ , with  $\mathfrak{n} = \mathfrak{a}, \mathfrak{n}_1$ , does not admit exact  $G_2$ -structures. Moreover, the same conclusion holds true if  $\mathfrak{n} = \mathfrak{n}_2$  and the derivation  $D \in \text{Der}(\mathfrak{n}_2)$  is such that  $\mathfrak{g}$  is strongly unimodular.*

*Proof.* Recall that a 3-form  $\phi$  on a seven-dimensional Lie algebra  $\mathfrak{g}$  defines a  $G_2$ -structure if and only if

$$\iota_v \phi \wedge \iota_v \phi \wedge \phi \neq 0 \in \Lambda^7(\mathfrak{g}^*), \quad \forall v \in \mathfrak{g} \setminus \{0\}. \quad (2.4)$$

We have to consider the seven-dimensional solvable Lie algebra  $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$  in the case when the nilradical  $\mathfrak{n} = \langle e_1, \dots, e_6 \rangle$  is one of  $\mathfrak{a}, \mathfrak{n}_1, \mathfrak{n}_2$ , with the structure equations given in the proof of Theorem 2.3, and  $\mathbb{R}$  is generated by a vector  $e_7$  such that  $\text{ad}_{e_7}$  is a generic derivation  $D \in \text{Der}(\mathfrak{n})$ .

Under these assumptions, we need to show that there are no exact 3-forms satisfying the condition (2.4) when  $\mathfrak{n} = \mathfrak{a}, \mathfrak{n}_1$ , and when  $\mathfrak{n} = \mathfrak{n}_2$  with  $D \in \text{Der}(\mathfrak{n}_2)$  such that  $\mathfrak{g}$  is strongly unimodular. In what follows, we denote the Chevalley-Eilenberg differential on  $\mathfrak{g}$  and on  $\mathfrak{n}$  by  $d$  and  $\hat{d}$ , respectively. Recall that for any 2-form  $\gamma \in \Lambda^k(\mathfrak{n}^*)$  the following identity holds

$$d\gamma = \hat{d}\gamma + (-1)^{k+1} D^* \gamma \wedge e^7,$$

where  $D^* \gamma(v_1, \dots, v_k) = \gamma(Dv_1, \dots, v_k) + \dots + \gamma(v_1, \dots, Dv_k)$ , for all  $v_1, \dots, v_k \in \mathfrak{n}$ .

Let  $\mathfrak{n} = \mathfrak{a}$  be the abelian Lie algebra, and consider the generic 2-form  $\alpha = \beta + \eta \wedge e^7$  on  $\mathfrak{g}$ , where  $\beta \in \Lambda^2(\mathfrak{a}^*)$  and  $\eta \in \Lambda^1(\mathfrak{a}^*)$ . Then, the generic exact 3-form on  $\mathfrak{g}$  is given by

$$\phi = d\alpha = d\beta + d\eta \wedge e^7 = \hat{d}\beta - D^* \beta \wedge e^7 + \hat{d}\eta \wedge e^7 = -D^* \beta \wedge e^7,$$

as  $\hat{d}\gamma = 0$  for any  $\gamma \in \Lambda^k(\mathfrak{a}^*)$ . It is clear that this 3-form cannot define a  $G_2$ -structure, since there exists some nonzero  $v \in \mathfrak{a}$  such that  $\iota_v \phi \wedge \iota_v \phi \wedge \phi = 0$ .

Let us now consider the Lie algebra  $\mathfrak{g} = \mathfrak{n}_1 \rtimes_{D_1} \mathbb{R}$ , where  $D_1 \in \text{Der}(\mathfrak{n}_1)$  is given by (2.2). Let  $\alpha \in \Lambda^2(\mathfrak{g}^*)$ , we write  $\alpha = \sum_{1 \leq i < j \leq 7} c_{ij} e^i \wedge e^j$ , with  $c_{ij} \in \mathbb{R}$ , and see that a generic exact 3-form on  $\mathfrak{g}$  has the following expression

$$\begin{aligned} d\alpha = & (-a_1 c_{12} - a_4 c_{12} + a_9 c_{25} - a_{10} c_{15} + a_{13} c_{26} - a_{14} c_{16} + c_{57}) e^{127} - c_{16} e^{134} \\ & - (a_1 c_{16} + a_3 c_{26} + a_5 c_{16} + a_8 c_{16} + a_9 c_{56}) e^{167} - c_{26} e^{234} - c_{35} e^{123} + c_{56} e^{126} \\ & - (a_2 c_{16} + a_4 c_{26} + a_5 c_{26} + a_8 c_{26} + a_{10} c_{56}) e^{267} - c_{45} e^{124} - c_{56} e^{345} \\ & - (a_1 c_{13} + a_3 c_{23} + a_5 c_{13} + a_7 c_{14} - a_9 c_{35} + a_{11} c_{15} - a_{13} c_{36} + a_{15} c_{16}) e^{137} \\ & - (a_2 c_{13} + a_4 c_{23} + a_5 c_{23} + a_7 c_{24} - a_{10} c_{35} + a_{11} c_{25} - a_{14} c_{36} + a_{15} c_{26}) e^{237} \\ & - (2a_5 c_{36} + a_7 c_{46} + a_8 c_{36} + a_{11} c_{56}) e^{367} - (a_1 c_{25} + a_2 c_{15} + 2a_4 c_{25} - a_{14} c_{56}) e^{257} \\ & - (a_1 c_{14} + a_3 c_{24} + a_6 c_{13} + a_8 c_{14} - a_9 c_{45} + a_{12} c_{15} - a_{13} c_{46} + a_{16} c_{16}) e^{147} \\ & - (a_2 c_{14} + a_4 c_{24} + a_6 c_{23} + a_8 c_{24} - a_{10} c_{45} + a_{12} c_{25} - a_{14} c_{46} + a_{16} c_{26}) e^{247} \\ & - (a_5 c_{34} + a_8 c_{34} - a_{11} c_{45} + a_{12} c_{35} - a_{15} c_{46} + a_{16} c_{36} - c_{67}) e^{347} \\ & - (a_5 c_{46} + a_6 c_{36} + 2a_8 c_{46} + a_{12} c_{56}) e^{467} - (2a_1 c_{15} + a_3 c_{25} + a_4 c_{15} - a_{13} c_{56}) e^{157} \\ & - (a_1 c_{35} + a_4 c_{35} + a_5 c_{35} + a_7 c_{45} - a_{15} c_{56}) e^{357} - c_{56} (a_4 + a_1 + a_8 + a_5) e^{567} \\ & - (a_1 c_{45} + a_4 c_{45} + a_6 c_{35} + a_8 c_{45} - a_{16} c_{56}) e^{457}. \end{aligned}$$

This exact 3-form does not define a  $G_2$ -structure, as we always have  $\iota_{e_6} d\alpha \wedge \iota_{e_6} d\alpha \wedge d\alpha = 0$ , for all  $a_i$  and  $c_{ij}$ .

We are left with the case  $\mathfrak{g} = \mathfrak{n}_2 \rtimes_{D_2} \mathbb{R}$ . Recall from the proof of Theorem 2.3 that  $\mathfrak{g}$  is strongly unimodular if and only if the entries of the derivation  $D_2 \in \text{Der}(\mathfrak{n}_2)$  given in (2.3)

satisfy the condition  $a_1 + a_7 = 0$ . Now, similarly as in the previous case, we can compute the expression of the generic exact 3-form  $d\alpha \in \Lambda^3(\mathfrak{g}^*)$  and see that

$$\iota_{e_6} d\alpha \wedge \iota_{e_6} d\alpha \wedge d\alpha = -12 (c_{56})^3 (a_1 + a_7) e^{1234567}.$$

Thus, if  $\mathfrak{g}$  is strongly unimodular, then  $d\alpha$  cannot define a  $G_2$ -structure.  $\square$

### 3. A UNIMODULAR EXAMPLE WITH $b_2 \neq 0$ AND $b_3 = 0$

Let  $\mathfrak{h} = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle$  be the seven-dimensional solvable Lie algebra with the following nonzero Lie brackets

$$\begin{aligned} [e_1, e_3] &= 4e_3, & [e_1, e_4] &= -e_4, & [e_1, e_5] &= -5e_5, & [e_1, e_6] &= -e_6, & [e_1, e_7] &= 3e_7, \\ [e_2, e_3] &= 3e_3, & [e_2, e_5] &= -4e_5, & [e_2, e_6] &= -e_6, & [e_2, e_7] &= 2e_7, \\ [e_3, e_5] &= -e_6, & [e_3, e_6] &= -e_7. \end{aligned} \quad (3.1)$$

This Lie algebra is the semidirect product  $\mathfrak{h} = \mathbb{R}^2 \ltimes \mathfrak{m}$ , where the abelian Lie algebra  $\mathbb{R}^2 = \langle e_1, e_2 \rangle$  acts on the nilradical  $\mathfrak{m} = \langle e_3, e_4, e_5, e_6, e_7 \rangle$  of  $\mathfrak{h}$  via the representation

$$\rho : \mathbb{R}^2 \rightarrow \text{Der}(\mathfrak{m}), \quad \rho(e_1) = \text{diag}(4, -1, -5, -1, 3), \quad \rho(e_2) = \text{diag}(3, 0, -4, -1, 2).$$

Moreover,  $\mathfrak{h}$  is easily seen to be unimodular and completely solvable.

We now show that the simply connected solvable Lie group  $H$  corresponding to  $\mathfrak{h}$  does not contain any lattice. Indeed, the only nonzero terms in the descending central series of the nilradical  $\mathfrak{m}$  of  $\mathfrak{h}$  are

$$\mathfrak{m}^0 = \mathfrak{m}, \quad \mathfrak{m}^1 = \langle e_6, e_7 \rangle, \quad \mathfrak{m}^2 = \langle e_7 \rangle,$$

and from (3.1) it follows that  $\text{ad}_{e_1}|_{\mathfrak{m}^2} = 3\text{Id}_{\mathfrak{m}^2}$ . Hence,  $\mathfrak{h}$  is not strongly unimodular.

Let  $(e^1, \dots, e^7)$  be the dual basis of  $(e_1, \dots, e_7)$ . Then, the Chevalley-Eilenberg differentials of the covectors  $e^i$  are the following:

$$\begin{aligned} de^1 &= 0 = de^2, & de^3 &= -4e^{13} - 3e^{23}, & de^4 &= e^{14}, \\ de^5 &= 5e^{15} + 4e^{25}, & de^6 &= e^{16} + e^{26} + e^{35}, & de^7 &= -3e^{17} - 2e^{27} + e^{36}. \end{aligned}$$

Using these equations, we compute the cohomology groups of the Chevalley-Eilenberg complex  $(\Lambda^\bullet(\mathfrak{h}^*), d)$  of  $\mathfrak{h}$ , obtaining:

$$\begin{aligned} H^1(\mathfrak{g}^*) &= \langle [e^1], [e^2] \rangle, & H^2(\mathfrak{g}^*) &= \langle [e^1 \wedge e^2] \rangle, & H^3(\mathfrak{g}^*) &= H^4(\mathfrak{g}^*) = \{0\}, \\ H^5(\mathfrak{g}^*) &= \langle [e^{34567}] \rangle, & H^6(\mathfrak{g}^*) &= \langle [e^{234567}], [e^{134567}] \rangle, & H^7(\mathfrak{g}^*) &= \langle [e^{1234567}] \rangle. \end{aligned}$$

Therefore, the Betti numbers of the Lie algebra  $\mathfrak{h}$  are the following

$$b_1(\mathfrak{h}) = b_6(\mathfrak{h}) = 2, \quad b_2(\mathfrak{h}) = b_5(\mathfrak{h}) = b_7(\mathfrak{h}) = 1, \quad b_3(\mathfrak{h}) = b_4(\mathfrak{h}) = 0.$$

We now describe an explicit example of a left-invariant exact  $G_2$ -structure  $\varphi$  on  $H$ .

Let us consider the following basis of  $\mathfrak{h}^*$ :

$$\begin{aligned} E^1 &= e^3, & E^2 &= \frac{1}{4\sqrt{3}} e^2, & E^3 &= -\sqrt{3} e^4 + 2\sqrt{3} e^6, \\ E^4 &= e^5, & E^5 &= e^4 + 2e^6, & E^6 &= 8\sqrt{3} e^7, & E^7 &= e^1 + \frac{3}{4} e^2. \end{aligned}$$

The structure equations of  $\mathfrak{h}$  with respect to this new basis are:

$$\begin{cases} dE^1 = 4E^{17}, \\ dE^2 = 0, \\ dE^3 = -E^{37} + 2\sqrt{3}E^{14} - \sqrt{3}E^{23} + 6E^{25}, \\ dE^4 = -5E^{47} + \sqrt{3}E^{24}, \\ dE^5 = -E^{57} + 2E^{14} + 2E^{23} - \sqrt{3}E^{25}, \\ dE^6 = 3E^{67} + 2E^{13} + 2\sqrt{3}E^{15} + \sqrt{3}E^{26}, \\ dE^7 = 0. \end{cases}$$

We consider the  $G_2$ -structure  $\varphi$  on  $\mathfrak{h}$  with adapted  $G_2$ -basis  $(E^1, \dots, E^7)$ , that is,

$$\varphi = E^{127} + E^{347} + E^{567} + E^{135} - E^{146} - E^{236} - E^{245}.$$

This is easily seen to be a closed, hence exact,  $G_2$ -structure. In particular, we have

$$\varphi = d\left(-\frac{1}{4}E^{12} + \frac{1}{6}E^{34} - \frac{1}{2}E^{56}\right).$$

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#### APPENDIX A.

In this appendix, we explicitly write the matrices associated to the endomorphisms  $A_1, A_2, A_3$ , introduced in the proof of Theorem 2.3, in the case when  $\mathfrak{n} = \mathfrak{n}_1$  and  $\mathfrak{n} = \mathfrak{n}_2$ . For each of these Lie algebras, we write an ordered basis of the cohomology group  $H^k(\mathfrak{n}^*)$ ,  $k = 1, 2, 3$ , and the matrix associated to  $A_k$  with respect to that basis. For the sake of convenience, we first impose the conditions on  $D \in \text{Der}(\mathfrak{n})$  for which  $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$  is strongly unimodular.

**Case  $\mathfrak{n} = \mathfrak{n}_1$ .** For the Lie algebra  $\mathfrak{n}_1 = (0, 0, 0, 0, e^{12}, e^{34})$  with generic derivation  $D_1$  given by (2.2), we have that  $\mathfrak{g} = \mathfrak{n} \rtimes_{D_1} \mathbb{R}$  is strongly unimodular if and only if  $a_8 = -a_1 - a_4 - a_5$ . In this case, we get

- $H^1(\mathfrak{n}_1) = \langle [e^1], [e^2], [e^3], [e^4] \rangle$ ,

$$A_1 = - \begin{pmatrix} a_1 & a_3 & 0 & 0 \\ a_2 & a_4 & 0 & 0 \\ 0 & 0 & a_5 & a_7 \\ 0 & 0 & a_6 & -a_1 - a_4 - a_5 \end{pmatrix};$$

- $H^2(\mathfrak{n}_1) = \langle [e^{13}], [e^{14}], [e^{15}], [e^{23}], [e^{24}], [e^{25}], [e^{36}], [e^{46}] \rangle$ ,

$$A_2 = \begin{pmatrix} -a_1 - a_5 & -a_7 & -a_{11} & -a_3 & 0 & 0 & a_{13} & 0 \\ -a_6 & a_4 + a_5 & -a_{12} & 0 & -a_3 & 0 & 0 & a_{13} \\ 0 & 0 & -2a_1 - a_4 & 0 & 0 & -a_3 & 0 & 0 \\ -a_2 & 0 & 0 & -a_4 - a_5 & -a_7 & -a_{11} & a_{14} & 0 \\ 0 & -a_2 & 0 & -a_6 & a_1 + a_5 & -a_{12} & 0 & a_{14} \\ 0 & 0 & -a_2 & 0 & 0 & -2a_4 - a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 + a_4 - a_5 & -a_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_6 & 2a_4 + 2a_1 + a_5 \end{pmatrix};$$

- $H^3(\mathfrak{n}_1) = \langle [e^{125}], [e^{135}], [e^{136}], [e^{145}], [e^{146}], [e^{235}], [e^{236}], [e^{245}], [e^{246}], [e^{346}] \rangle$ ,

$$A_3 = \begin{pmatrix} 2a_4 + 2a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2a_1 + a_5 + a_4 & 0 & a_6 & 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_5 - a_4 & 0 & a_6 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_7 & 0 & a_1 - a_5 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_7 & 0 & -2a_4 - a_5 - a_1 & 0 & 0 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & 0 & 0 & 2a_4 + a_5 + a_1 & 0 & a_6 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & a_5 - a_1 & 0 & a_6 & 0 \\ 0 & 0 & 0 & a_3 & 0 & a_7 & 0 & a_4 - a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 & a_7 & 0 & -2a_1 - a_5 - a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2a_4 - 2a_1 \end{pmatrix}.$$

**Case  $\mathfrak{n} = \mathfrak{n}_2$ .** Consider the Lie algebra  $\mathfrak{n}_2 = (0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23})$  with generic derivation  $D_2$  given by (2.3). Then,  $\mathfrak{g} = \mathfrak{n}_2 \rtimes_{D_2} \mathbb{R}$  is strongly unimodular if and only if  $a_7 = -a_1$ . In this case, we have

- $H^1(\mathfrak{n}_2) = \langle [e^1], [e^2], [e^3], [e^4] \rangle$ ,

$$A_1 = \begin{pmatrix} -a_1 & a_2 & -a_5 & a_6 \\ -a_2 & -a_1 & -a_6 & -a_5 \\ -a_3 & a_4 & a_1 & a_8 \\ -a_4 & -a_3 & -a_8 & a_1 \end{pmatrix};$$

- $H^2(\mathfrak{n}_2) = \langle [e^{12}], [e^{34}], [e^{13} + e^{24}], [e^{14} - e^{23}], [e^{16} + e^{25}], [-e^{15} + e^{26}], [e^{36} + e^{45}], [-e^{35} + e^{46}] \rangle$ ,

$$A_2 = \begin{pmatrix} -2a_1 & -2a_6 & -2a_5 & a_9 - a_{14} & a_{10} + a_{13} & 0 & 0 & 0 \\ a_4 & 0 & a_8 - a_2 & -\frac{1}{2}a_{15} & \frac{1}{2}a_{11} - \frac{1}{2}a_{16} & -a_6 & \frac{1}{2}a_{13} + \frac{1}{2}a_{10} & -\frac{1}{2}a_9 + \frac{1}{2}a_{14} \\ -a_3 & a_2 - a_8 & 0 & -\frac{1}{2}a_{16} & \frac{1}{2}a_{12} + \frac{1}{2}a_{15} & -a_5 & -\frac{1}{2}a_{14} + \frac{1}{2}a_9 & \frac{1}{2}a_{13} + \frac{1}{2}a_{10} \\ 0 & 0 & 0 & -a_1 & a_8 + 2a_2 & 0 & -a_5 & a_6 \\ 0 & 0 & 0 & -a_8 - 2a_2 & -a_1 & 0 & -a_6 & -a_5 \\ 0 & 2a_4 & -2a_3 & 0 & 0 & 2a_1 & a_{11} - a_{16} & a_{12} + a_{15} \\ 0 & 0 & 0 & -a_3 & a_4 & 0 & a_1 & a_2 + 2a_8 \\ 0 & 0 & 0 & -a_4 & -a_3 & 0 & -a_2 - 2a_8 & a_1 \end{pmatrix};$$

- $H^3(\mathfrak{n}_2) = \langle [e^{125}], [e^{126}], [e^{345}], [e^{346}], [-e^{135} + e^{146}], [-e^{135} + e^{236}], [e^{135} + e^{245}], [e^{136} + e^{246}], [e^{145} - e^{246}], [e^{235} - e^{246}] \rangle$ ,

$$A_3 = \begin{pmatrix} -2a_1 & 0 & 0 & -a_4 & -a_4 & 0 & -a_3 & 0 & -a_4 & a_2 + a_8 \\ 0 & -2a_1 & -a_2 - a_8 & a_5 & -a_6 & -2a_5 & -a_6 & 2a_6 & -a_5 & 0 \\ 0 & a_2 + a_8 & 2a_1 & a_6 & a_5 & 0 & -a_5 & 0 & -a_6 & 0 \\ a_6 & 0 & a_4 & 0 & -a_2 - 2a_8 & 0 & -a_8 - 2a_2 & a_2 + a_8 & 0 & 0 \\ a_5 & 0 & a_3 & a_2 + 2a_8 & 0 & -a_2 - a_8 & 0 & 0 & -a_8 - 2a_2 & 0 \\ 0 & -a_3 & 0 & 0 & -a_2 & 0 & -a_8 - 2a_2 & a_2 - a_8 & 0 & -a_5 \\ -a_5 & 0 & -a_3 & a_2 & 0 & a_2 + a_8 & 0 & 0 & -a_8 & 0 \\ 0 & -a_4 & 0 & a_2 & 0 & -a_2 + a_8 & 0 & 0 & -a_8 - 2a_2 & a_6 \\ -a_6 & 0 & a_4 & 0 & a_2 & 0 & a_8 & a_2 + a_8 & 0 & 0 \\ -a_2 - a_8 & 0 & 0 & a_3 & a_3 & -2a_3 & a_4 & -2a_4 & -a_3 & -2a_1 \end{pmatrix}.$$

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