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# SOME NEW RESULTS ON ORTHOGONAL POLYNOMIALS FOR LAGUERRE TYPE EXPONENTIAL WEIGHTS 

G. MASTROIANNI ${ }^{1},{ }^{*}$, I. NOTARANGELO ${ }^{1},{ }^{* *}$, L. SZILI $^{2},{ }^{* * *}$ AND P. VÉRTESI ${ }^{3},{ }^{* * *}$<br>${ }^{1}$ DEPARTMENT OF MATHEMATICS, COMPUTER SCIENCES AND ECONOMICS, UNIVERSITY OF BASILICATA, VIA DELL'ATENEO LUCANO 10, 85100 POTENZA, ITALY<br>${ }^{2}$ DEPARTMENT OF NUMERICAL ANALYSIS, LORÁND EÖTVÖS UNIVERSITY, H-1117 BUDAPEST, PÁZMÁNY P. SÉTÁNY I/C, HUNGARY<br>DEPARTMENT OF DIFFERENTIAL EQUATIONS, BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, H-1111 BUDAPEST, EGRY JÓZSEF U. 1., HUNGARY<br>${ }^{3}$ ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, H-1364 BUDAPEST, P.O.B. 127, HUNGARY

Abstract. In this paper we prove some results on the root-distances and the weighted Lebesgue function corresponding to orthogonal polynomials for Laguerre type exponential weights.

## 1. Introduction. Notations. Preliminaries

1.1. In the paper [1] and [2] Eli Levin and Doron Lubinsky investigated certain Laguerre type orthogonal polynomials. Using their results and our papers [3], [4], [5] we state further relations for the root-distances. In addition, we obtain a lower estimation for the weighted Lebesgue function of the weighted Lagrange interpolation with respect to arbitrary point systems.

## 1.2. (For Sections 1.1-1.4 cf. [1] and [2].)

Let

$$
\begin{equation*}
I=[0, d), \tag{1.1}
\end{equation*}
$$

where $0<d \leq \infty$. Let $Q: I \rightarrow[0, \infty)$ be continuous, and

$$
\begin{equation*}
W=\exp (-Q) \tag{1.2}
\end{equation*}
$$

[^0]be such that all moments $\int_{I} x^{n} W(x) d x, n \geq 0$, exists. We call $W$ an exponential weight on $I$. For (a fixed) $\rho>-\frac{1}{2}$, we set
$$
W_{\rho}(x):=x^{\rho} W(x), \quad x \in I .
$$

The orthonormal polynomial of degree $n$ for $W^{2}$ is denoted by $p_{n}\left(W^{2}, x\right)$ or $p_{n}(x)$. That for $W_{\rho}^{2}$ is denoted by $p_{n}\left(W_{\rho}^{2}, x\right)$ or $p_{n, \rho}(x)$. So

$$
\int_{I} p_{n, \rho}(x) p_{m, \rho}(x) x^{2 \rho} W^{2}(x) d x=\delta_{n, m}
$$

and

$$
p_{n, \rho}(x)=\gamma_{n, \rho} x^{n}+\cdots,
$$

where $\gamma_{n, \rho}=\gamma_{n}\left(W_{\rho}^{2}\right)>0$.
We denote the zeros of $p_{n, \rho}$ by $U_{n}\left(W_{\rho}^{2}\right)=\left\{x_{k n}=x_{k n}\left(W_{\rho}^{2}\right)\right\}$, where

$$
x_{n n}<x_{n-1, n}<\cdots<x_{2 n}<x_{1 n} .
$$

As in [1] and [2] we define an even weight $W^{*}$ corresponding to the one-sided weight $W$. Given $I$ and $W$ as in (1.1) and (1.2), let

$$
I^{*}:=(-\sqrt{d}, \sqrt{d})
$$

and for $x \in I^{*}$

$$
\begin{align*}
Q^{*}(x) & :=Q\left(x^{2}\right) \\
W^{*}(x) & :=\exp \left(-Q^{*}(x)\right) \tag{1.3}
\end{align*}
$$

We say that $f: I \rightarrow(0, \infty)$ is quasi-increasing, if there exists $C>0$ such that

$$
f(x) \leq C f(y), \quad 0<x<y<d
$$

The notation

$$
f(x) \sim g(x)
$$

means that there are positive contants $C_{1}, C_{2}$ such that for the relevant range of $x$

$$
C_{1} \leq f(x) / g(x) \leq C_{2}
$$

Similar notation is used for sequences and sequences of functions.
Throughout, $C, C_{1}, C_{2}, c, c_{1}, c_{2}, \ldots$ denotes positive constants independent of $n, x, t$ and polynomials $P$ of degree at most $n$. We write $C=C(\lambda), C \neq C(\lambda)$ to indicate dependence on or independence of, a parameter $\lambda$. The same symbol does not necessarily denote the same constant in different occurrences. We denote the set of polynomials of degree $\leq n$ by $\mathcal{P}_{n}$.
1.3. Following is our class of weights:

Definition 1.1. Let $W=e^{-Q}$ where $Q: I \rightarrow[0, \infty)$ satisfies the following properties:
(a) $\sqrt{x} Q^{\prime}(x)$ is continuous in $I$, with limit 0 at 0 and $Q(0)=0$;
(b) $Q^{\prime \prime}$ exists in $(0, d)$, while $Q^{* \prime \prime}$ is positive in $(0, \sqrt{d})$;
(c)

$$
\lim _{x \rightarrow d-} Q(x)=\infty
$$

(d) The function

$$
\begin{equation*}
T(x):=\frac{x Q^{\prime}(x)}{Q(x)}, \quad x \in(0, d) \tag{1.4}
\end{equation*}
$$

is quasi-increasing in $(0, d)$, with

$$
\begin{equation*}
T(x) \geq \Lambda>\frac{1}{2}, \quad x \in(0, d) \tag{1.5}
\end{equation*}
$$

(e) There exists $C_{1}>0$ such that

$$
\begin{equation*}
\frac{\left|Q^{\prime \prime}(x)\right|}{Q^{\prime}(x)} \leq C_{1} \frac{Q^{\prime}(x)}{Q(x)}, \quad \text { a.e } \quad x \in(0, d) \tag{1.6}
\end{equation*}
$$

There exists a compact subinterval $J$ of $I^{*}$, and $C_{2}>0$ such that

$$
\begin{equation*}
\frac{Q^{* \prime \prime}(x)}{\left|Q^{* \prime}(x)\right|} \geq C_{2} \frac{\left|Q^{* \prime}(x)\right|}{Q^{*}(x)}, \quad \text { a.e. } x \in I^{*} \backslash J \tag{1.7}
\end{equation*}
$$

If the weight $W$ satisfies (a)-(e), then we write $W \in \mathcal{L}\left(C^{2}+\right)$.
Examples (cf. [1] and [2]):

$$
\begin{gather*}
Q(x)=x^{\alpha}, \quad x \in[0,+\infty), \quad \alpha>\frac{1}{2}  \tag{1.8}\\
Q(x)=\exp _{k}\left(x^{\alpha}\right)-\exp _{k}(0), \quad x \in[0,+\infty), \quad \alpha>\frac{1}{2}, \quad k \geq 0 \tag{1.9}
\end{gather*}
$$

where $\exp _{0}(x):=x$ and for $k \geq 1$

$$
\exp _{k}(x)=\underbrace{\exp (\exp (\exp \cdots \exp (x)))}_{k \text { times }}
$$

is the $k$ th iterated exponential.
An example on the finite interval $I=[0,1)$ is

$$
\begin{equation*}
Q(x)=\exp _{k}\left((1-x)^{-\alpha}\right)-\exp _{k}(1), \quad x \in[0,1) \tag{1.10}
\end{equation*}
$$

where $\alpha>0$ and $k \geq 0$.
1.4. One of the important quantities we need is the Mhaskar-Rakhmanov-Saff number for the weight $W_{\rho}$ denoted by $a_{t}=a_{t}(Q)$, defined for $t>0$ as the positive root of the equation

$$
t=\frac{1}{\pi} \int_{0}^{1} \frac{a_{t} u Q^{\prime}\left(a_{t} u\right)}{\sqrt{u(1-u)}} d u
$$

If $x Q^{\prime}(x)$ is strictly increasing and continuous, with limits 0 and $+\infty$ at 0 and $d$ respectively, $a_{t}$ is uniquely defined. Moreover, $a_{t}$ is an increasing function of $t \in(0,+\infty)$, with

$$
\lim _{t \rightarrow+\infty} a_{t}=d
$$

Let us introduce the notation

$$
\begin{equation*}
T_{n}:=T\left(a_{n}\right) \quad(n \in \mathbf{N}) \tag{1.11}
\end{equation*}
$$

Since $T(x) \geq \Lambda>1 / 2, x \in(0, d)$ (see (1.5)), thus we have

$$
\begin{equation*}
\lim _{n \rightarrow+\infty}\left(n T_{n}\right)=+\infty \tag{1.12}
\end{equation*}
$$

From the important relation

$$
Q\left(a_{t}\right) \sim \frac{t}{\sqrt{T\left(a_{t}\right)}},
$$

which holds uniformly for $t>0$ (see [1, (1.27) and Lemma 3.1]), using the condition $\lim _{x \rightarrow d-} Q(x)=+\infty$ (see Definition (c)) it follows that

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \frac{n}{\sqrt{T_{n}}}=+\infty \tag{1.13}
\end{equation*}
$$

In the sequel, we shall denote the positive Mhaskar-Rakhmanov-Saff number for the weight $W^{*}$ by $a_{t}^{*}=a_{t}^{*}\left(Q^{*}\right), t>0$. Thus $a_{t}^{*}$ is defined as the positive root of the equation

$$
t=\frac{1}{\pi} \int_{-a_{t}^{*}}^{a_{t}^{*}} \frac{x Q^{* \prime}(x)}{\sqrt{a_{t}^{* 2}-x^{2}}} d x=\frac{2}{\pi} \int_{0}^{1} \frac{a_{t}^{*} u Q^{* \prime}\left(a_{t}^{*} u\right)}{\sqrt{1-u^{2}}} d u=\frac{2}{\pi} \int_{0}^{1} \frac{a_{t}^{* 2} v Q^{\prime}\left(a_{t}^{* 2} v\right)}{\sqrt{v(1-v)}} d v
$$

whence ([1, p. 211])

$$
a_{t}=a_{2 t}^{* 2} \quad(t>0) .
$$

1.5. Let $W \in \mathcal{L}\left(C^{2}+\right), W^{*}$ is given by (1.3), $m=2 n$ and $y_{k+1, m}<y_{k m}(n \in \mathbf{N})$, where

$$
y_{k m}=y_{k m}\left(W^{* 2}\right)=a_{m}^{*} t_{k m}=a_{m}^{*} \cos \vartheta_{k m}, \quad k=1,2, \ldots, n
$$

are the positive roots of the orthonormal polynomial $p_{m}\left(W^{* 2}\right)$. We define

$$
y_{0 m}=a_{m}^{*}=a_{m}^{*} t_{0 m}, \quad \vartheta_{0 m}=0 \quad \text { and } \quad y_{n+1, m}=t_{n+1, m}=0, \quad \vartheta_{n+1, m}=\pi / 2
$$

By reformulating [5, Theorem 2.1] and using [2, §2], [1, (2.7)-(2.9)], the Hermite-type roots $y_{k m}$ 's satisfy

Theorem 1.1. We have if $W \in \mathcal{L}\left(C^{2}+\right)$ then with $m=2 n \in \mathbf{N}$
(i) $\vartheta_{k m}-\vartheta_{k-1, m} \sim \frac{1}{\left(n T_{n}\right)^{1 / 3} k^{2 / 3}}, \quad 1 \leq k \leq \frac{c_{1} n}{\sqrt{T_{n}}}$,
(i1) $t_{k-1, m}-t_{k m} \sim \frac{1}{\left(n T_{n}\right)^{2 / 3} k^{1 / 3}}, \quad 1 \leq k \leq \frac{c_{1} n}{\sqrt{T_{n}}}$,
(ii) $\vartheta_{k+1, m}-\vartheta_{k m} \sim \frac{1}{n}, \quad \frac{c_{1} n}{\sqrt{T_{n}}} \leq k \leq n$,
(ii1) $t_{k m}-t_{k+1, m} \sim \frac{k}{n^{2}}, \quad \frac{c_{1} n}{\sqrt{T_{n}}} \leq k \leq n$.

## 2. New relations

2.1. Our first relations on the root distances of the Laguerre type roots

$$
x_{k n}\left(W_{\rho}^{2}\right)=x_{k n}=a_{n} u_{k n}=a_{n} \cos \gamma_{k n} \quad(k=1,2, \ldots, n ; n \in \mathbf{N})
$$

are as follows
Theorem 2.1. If $W \in \mathcal{L}\left(C^{2}+\right)$ and $x_{0 n}=a_{n}, x_{n+1, n}=0$, then with $n \in \mathbf{N}$
(a) $\gamma_{k n}-\gamma_{k-1, n} \sim \frac{1}{\left(n T_{n}\right)^{1 / 3} k^{2 / 3}}, \quad 1 \leq k \leq \frac{c_{2} n}{\sqrt{T_{n}}}$,
(a1) $u_{k-1, n}-u_{k n} \sim \frac{1}{\left(n T_{n}\right)^{2 / 3} k^{1 / 3}}, \quad 1 \leq k \leq \frac{c_{2} n}{\sqrt{T_{n}}}$,
(b) $\gamma_{k+1, n}-\gamma_{k n} \sim \frac{n-k+1}{n^{2}}, \quad \frac{c_{2} n}{\sqrt{T_{n}}} \leq k \leq n$,
(b1) $u_{k n}-u_{k+1, n} \sim \frac{(n-k+1) k}{n^{3}}, \quad \frac{c_{2} n}{\sqrt{T_{n}}} \leq k \leq n$.
Remark. Theorem 2.1 shows that the order of the distances does not depend on $\rho$ (cf. (3.1)).
2.2. As an application of the above theorem we prove a lower estimation on the weighted Lebesgue function of the weighted Lagrange interpolation on arbitrary point systems.

We need the following definitions.
If $Z=\left\{z_{k n}\right\}$ is an interpolatory matrix on $I$ that is

$$
0 \leq z_{n n}<z_{n-1, n}<\cdots<z_{2 n}<z_{1 n}<d, \quad n \in \mathbf{N}
$$

then for $f \in C\left(W_{\rho}, I\right)$, $W \in \mathcal{L}\left(C^{2}+\right)$, where

$$
C\left(W_{\rho}, I\right):=\left\{f ; f \text { is continuous on }(0, d) \text { and } \lim _{\substack{x \rightarrow 0+\\ x \rightarrow d-}} f(x) W_{\rho}(x)=0\right\},
$$

we investigate the weighted Lagrange interpolation defined by

$$
\begin{equation*}
L_{n}\left(f, W_{\rho}, Z, x\right)=\sum_{k=1}^{n} f\left(z_{k n}\right) W_{\rho}\left(z_{k n}\right) g_{k n}\left(W_{\rho}, Z, x\right), \quad n \in \mathbf{N} \tag{2.1}
\end{equation*}
$$

Above

$$
\begin{gather*}
g_{k n}\left(W_{\rho}, Z, x\right)=\frac{W_{\rho}(x)}{W_{\rho}\left(z_{k n}\right)} \ell_{k n}(Z, x), \quad 1 \leq k \leq n,  \tag{2.2}\\
\ell_{k}(x)=\ell_{k n}(Z, x)=\frac{\omega_{n}(Z, x)}{\omega_{n}^{\prime}\left(Z, z_{k n}\right)\left(x-z_{k n}\right)}, \quad 1 \leq k \leq n, \tag{2.3}
\end{gather*}
$$

and

$$
\begin{equation*}
\omega_{n}(Z, x)=c_{n} \prod_{i=1}^{n}\left(x-z_{i n}\right), \quad n \in \mathbf{N} \tag{2.4}
\end{equation*}
$$

The polynomials $\ell_{k}$ of degree exactly $n-1$ (that is $\ell_{k} \in \mathcal{P}_{n-1} \backslash \mathcal{P}_{n-2}$ ) are the fundamental functions of the (usual) Lagrange interpolation while functions $g_{k}$ are the fundamental functions of the weighted Lagrange interpolation.

The classical Lebesgue estimation now has the form

$$
\left|L_{n}\left(f, W_{\rho}, Z, x\right)-f(x) W_{\rho}(x)\right| \leq\left\{\lambda_{n}\left(W_{\rho}, Z, x\right)+1\right\} E_{n-1}\left(f, W_{\rho}\right),
$$

where the weighted Lebesgue function is

$$
\begin{equation*}
\lambda_{n}\left(W_{\rho}, Z, x\right):=\sum_{k=1}^{n}\left|g_{k n}\left(W_{\rho}, Z, x\right)\right|, \quad x \in \mathbf{R}, n \in \mathbf{N} \tag{2.5}
\end{equation*}
$$

and

$$
E_{n-1}\left(f, W_{\rho}\right):=\inf _{p \in \mathcal{P}_{n-1}}\left\|(f-p) W_{\rho}\right\|, \quad n \in \mathbf{N}
$$

where $\|\cdot\|$ is the maximum norm on $I$.

Moreover, the weighted Lebesgue constant is

$$
\Lambda_{n}\left(W_{\rho}, Z\right):=\left\|\lambda_{n}\left(W_{\rho}, Z, x\right)\right\|
$$

We state (cf. [5] and its references).
Theorem 2.2. Let $W \in \mathcal{L}\left(C^{2}+\right)$ and let $0<\varepsilon<1$ be fixed. Then for any fixed interpolatory matrix $Z \subset I$ there exists set $H_{n}=H_{n}\left(W_{\rho}, \varepsilon, Z\right)$ with $\left|H_{n}\right| \leq \varepsilon$ such that

$$
\begin{equation*}
\lambda_{n}\left(W_{\rho}, Z, x\right)>\frac{1}{3840} \varepsilon \log n \quad \text { if } \quad x \in\left[0, a_{n}\left(W_{\rho}\right)\right] \backslash H_{n} \tag{2.6}
\end{equation*}
$$

whenever $n \geq n_{1}$.
2.3. We give another application of Theorem 2.1.

Let $\left\{z_{k n}\right\}=U_{n}\left(W_{\rho}^{2}\right)=\left\{x_{k n}\left(W_{\rho}^{2}\right)\right\}$ with

$$
\begin{equation*}
g_{k n}\left(W_{\rho}, U_{n}\left(W_{\rho}^{2}\right) ; x\right)=\frac{W_{\rho}(x)}{W_{\rho}\left(x_{k n}\left(W_{\rho}^{2}\right)\right)} \ell_{k n}\left(U_{n}\left(W_{\rho}^{2}\right), x\right), \quad 1 \leq k \leq n . \tag{2.7}
\end{equation*}
$$

Then
Corollary 2.3. The weighted Lebesgue constants satisfy

$$
\begin{equation*}
\Lambda_{n}\left(W_{\rho}, U_{n}\left(W_{\rho}^{2}\right)\right) \sim n^{1 / 2}, \quad n \in \mathbf{N} \tag{2.8}
\end{equation*}
$$

Now let

$$
\left\{z_{k n}\right\}=U_{n}\left(W_{\rho}^{2}\right) \cup\left\{x_{0 n}\left(W_{\rho}^{2}\right), x_{n+1, n}\left(W_{\rho}^{2}\right)\right\}=V_{n}\left(W_{\rho}^{2}\right)
$$

Then (cf. [8, Theorem 1])
Corollary 2.4. We have

$$
\begin{equation*}
\Lambda_{n}\left(W_{\rho}, V_{n}\left(W_{\rho}^{2}\right)\right) \sim \log n, \quad n \in \mathbf{N} \tag{2.9}
\end{equation*}
$$

Above

$$
\Lambda_{n}\left(W_{\rho}, V_{n}\left(W_{\rho}^{2}\right)\right)=\left\|\sum_{k=0}^{n+1}\left|g_{k n}\left(W_{\rho}, V_{n}\left(W_{\rho}^{2}\right) ; x\right)\right|\right\|
$$

where $\ell_{k n}\left(V_{n}, x\right)$ (in $\left.g_{k n}\left(V_{n}, x\right)\right)$ is based on the $n+2$ nodes $\left\{x_{k n}\left(W_{\rho}^{2}\right), 0 \leq k \leq n+1\right\}$.

## 3. Proofs

3.1. Proof of Theorem 2.1. First a basic observation: Using [5, Theorem 2.1] and [2, Theorem 1.4] we have

$$
\begin{gather*}
x_{k n}\left(W_{\rho}^{2}\right)-x_{k+1, n}\left(W_{\rho}^{2}\right) \sim x_{k n}\left(W_{-\frac{1}{4}}^{2}\right)-x_{k+1, n}\left(W_{-\frac{1}{4}}^{2}\right),  \tag{3.1}\\
k=1,2, \ldots, n-1,
\end{gather*}
$$

that means it is enough to prove the relations when $\rho=-\frac{1}{4}$, which we suppose from now on.

The relation

$$
p_{n}\left(W_{-\frac{1}{4}}^{2}, x\right)=p_{m}\left(W^{* 2}, y\right), \quad x=y^{2} \in[0, d), \quad m=2 n .
$$

(cf. $[1,(1.7)])$ shows that

$$
\begin{equation*}
x_{k n}=x_{k n}\left(W_{-\frac{1}{4}}^{2}\right)=y_{k m}^{2}, \quad k=1,2, \ldots, n ; m=2 n . \tag{3.2}
\end{equation*}
$$

Our main tool is the formula

$$
\begin{align*}
x_{k n}-x_{k+1, n} & =y_{k m}^{2}-y_{k+1, m}^{2}=\left(y_{k m}+y_{k+1, m}\right)\left(y_{k m}-y_{k+1, m}\right) \sim  \tag{3.3}\\
& \sim y_{k m}\left(y_{k m}-y_{k+1, m}\right), \quad 0 \leq k \leq n,
\end{align*}
$$

using that

$$
y_{k m} \leq y_{k m}+y_{k+1, m} \leq 2 y_{k m}
$$

To get (a1) we write by (i1)

$$
\begin{align*}
1-t_{k m} & =\sum_{s=0}^{k-1}\left(t_{s m}-t_{s+1, m}\right) \sim \frac{1}{\left(n T_{n}\right)^{2 / 3}} \sum_{s=1}^{k-1} s^{-1 / 3} \sim  \tag{3.4}\\
& \sim\left(\frac{k}{n T_{n}}\right)^{2 / 3} \leq \frac{c_{3}}{T_{n}}, \quad 1 \leq k \leq \frac{c_{1} n}{\sqrt{T_{n}}}
\end{align*}
$$

i.e. in (3.4), $t_{k m} \geq \frac{1}{2}$ supposing that $c_{3} / T_{n} \leq 1 / 2$ (say). (This can be attained by a proper $c_{1}>0$ using that $T_{n}>\frac{1}{2}$ (cf. (1.5)). Summarising, we get with a proper $c_{1}>0$

$$
\begin{equation*}
x_{k n}-x_{k+1, n} \sim a_{m}^{*} \frac{a_{m}^{*}}{\left(n T_{n}\right)^{2 / 3} k^{1 / 3}}, \quad 1 \leq k \leq \frac{c_{2} n}{\sqrt{T_{n}}}, \tag{3.5}
\end{equation*}
$$

whence $(a 1)$ is immediate (cf. (3.3), $\left(a_{m}^{*}\right)^{2}=a_{n}($ see $[1,(2.6)])$ and (i1)).

To get (b1) we have by (3.4) and (ii1)

$$
\begin{gather*}
t_{k m}=\sum_{s=1}^{n-k+1}\left(t_{n-s+1, m}-t_{n-s+2, m}\right) \sim  \tag{3.6}\\
\sim \sum_{s=k}^{n} \frac{s}{n^{2}} \sim \frac{(n+1)^{2}-k^{2}}{n^{2}}=\frac{(n+k+1)(n-k+1)}{n^{2}} \sim \frac{n-k+1}{n},
\end{gather*}
$$

whence by (3.3) and (ii1)

$$
\begin{equation*}
x_{k n}-x_{k+1, n} \sim\left(a_{m}^{*}\right)^{2} \frac{n-k+1}{n} \frac{k}{n^{2}}=a_{n} \frac{(n-k+1) k}{n^{3}}, \quad \frac{c_{1} n}{\sqrt{T_{n}}} \leq k \leq n \tag{3.7}
\end{equation*}
$$

whence (b1) is immediate.
Using that $\rho=-\frac{1}{4}$ by (3.2) and

$$
\left(a_{m}^{*} t_{k m}\right)^{2}=y_{k m}^{2}=x_{k n}=a_{n} u_{k n}
$$

we get $1-t_{k m}^{2}=1-u_{k n}$, which means

$$
\begin{equation*}
\sin ^{2} \vartheta_{k m}=2 \sin ^{2} \frac{\gamma_{k n}}{2}, \quad 1 \leq k \leq n \tag{3.8}
\end{equation*}
$$

By (3.8) $\sin \vartheta_{k m}=\sqrt{2} \sin \frac{\gamma_{k n}}{2}$ whence

$$
\begin{gathered}
\sin \vartheta_{k+1}-\sin \vartheta_{k}=2 \sin \frac{\vartheta_{k+1}-\vartheta_{k}}{2} \cdot \cos \frac{\vartheta_{k+1}+\vartheta_{k}}{2}= \\
=\sqrt{2}\left(\sin \frac{\gamma_{k+1}}{2}-\sin \frac{\gamma_{k}}{2}\right)=\sqrt{2} \cdot 2 \cdot \sin \frac{\gamma_{k+1}-\gamma_{k}}{4} \cdot \cos \frac{\gamma_{k+1}+\gamma_{k}}{4} .
\end{gathered}
$$

Now by $\cos \frac{\vartheta_{k+1}+\vartheta_{k}}{2} \approx \cos \vartheta_{k}=t_{k}$ and $\cos \frac{\gamma_{k+1}+\gamma_{k}}{4} \approx \cos \frac{\gamma_{k}}{2} \geq \cos \frac{\pi}{4}=\frac{\sqrt{2}}{2}$, we get

$$
\begin{equation*}
\left(\vartheta_{k+1, m}-\vartheta_{k m}\right) t_{k m} \sim \gamma_{k+1}-\gamma_{k}, \quad 0 \leq k \leq n \tag{3.9}
\end{equation*}
$$

(We omitted some obvious details.) By (3.9) we get (a) (or (b)) using Theorem 1.1 (i), (ii); (3.4) and (3.6).
3.2. Proof of Theorem 2.2. First we formulate some other relations applied later. Let $I_{n}(\varepsilon)=\left[\varepsilon a_{n},(1-\varepsilon) a_{n}\right], 0<\varepsilon<1$ fixed. Then

$$
\begin{equation*}
x_{k n}-x_{k+1, n} \sim \frac{a_{n}}{n}, \quad x_{k n}, x_{k+1, n} \in I_{n}(\varepsilon) \tag{3.10}
\end{equation*}
$$

Indeed, by $u_{k n}=t_{k m}^{2}\left(\right.$ again $\left.\rho=-\frac{1}{4}\right)$

$$
u_{k} \in[\varepsilon, 1-\varepsilon] \text { iff } t_{k} \in[\sqrt{\varepsilon}, \sqrt{1-\varepsilon}] \text {, }
$$

whence using that

$$
u_{k n}-u_{k+1, n} \sim t_{k m}\left(t_{k m}-t_{k+1, m}\right), \quad t_{k m} \sim 1 \text { and } t_{k m}-t_{k+1, m} \sim \frac{1}{n}
$$

(cf. $[5,(3.9)])$ we get (3.10).
Moreover, by [2, (1.13) and (1.17)], using (3.10) and the relations $x_{k n} \sim a_{n}-x_{k n} \sim a_{n}$ whenever $x_{k n} \in I_{n}(\varepsilon)$, we get

$$
\begin{equation*}
\left|p_{n \rho}^{\prime}\left(x_{k n}\right) W_{\rho}\left(x_{k n}\right)\right| \sim \frac{n}{a_{n}^{3 / 2}}, \quad x_{k n} \in I_{n}(\varepsilon) . \tag{3.11}
\end{equation*}
$$

Finally, we quote [2, Theorem 1.2] saying that

$$
\begin{equation*}
\sup _{x \in I}\left|p_{n \rho}(x)\right| W(x)\left(x+\frac{a_{n}}{n^{2}}\right)^{\rho} \sim\left(\frac{n}{a_{n}}\right)^{1 / 2}, \quad x \in I \tag{3.12}
\end{equation*}
$$

Using relations (3.10)-(3.12), we can prove Theorem 2.2. as we did in [5, §3.4-3.10]. We can omit the details.
3.3. Proof of Corollary 2.3. By $[1,(1.15)]$ and $[8$, Lemma 1] we can restrict ourselves to the interval $\left[0, a_{n}\right]$.

Fix $n \in \mathbf{N}$ and the point $x \in\left[x_{j+1, n}, x_{j, n}\right]=: \triangle x_{j n}(0 \leq j \leq n$; obviously $j=j(n))$. Even more, by [2, Theorem 1.3], we can suppose that $x$ is "far" from the nodes, namely

$$
\begin{equation*}
\left|x-\left(x_{j n}+x_{j+1, n}\right) / 2\right| \leq\left|\triangle x_{j n}\right| / 4 \tag{3.13}
\end{equation*}
$$

Then by (2.5), (2.7) and [2, Theorem 1.3] we have

$$
\begin{equation*}
\lambda_{n}\left(W_{\rho}, U_{n}\left(W_{\rho}^{2}\right), x\right) \sim \sum_{k=1}^{n} \frac{\left|\triangle u_{k n}\right|}{\left|u_{j n}-u_{k n}\right|}\left(\frac{u_{k n}\left(1-u_{k n}\right)}{u_{j n}\left(1-u_{j n}\right)}\right)^{1 / 4} \tag{3.14}
\end{equation*}
$$

Here and hereafter for a fixed index $j$ we use the notation

$$
\sum_{k=1}^{n} a(k, j)=\sum_{\substack{k=1 \\ k \neq j}}^{n} a(k, j)
$$

Moreover we used the fact that the $j$ th term of the sum (2.5) has the same order as the $l$ th ones, whenever $|l-j| \leq c$ (see [2, Theorem 1.3]).

Let from now

$$
\sum_{k=1}^{n}{ }^{\prime} \cdots=\sum_{k=1}^{c n-1} \cdots+\sum_{k=c n}^{n} \cdots=: S_{2}+S_{1}
$$

In order to estimate $S_{1}$ and $S_{2}$, we distinguish several cases.
A. First we suppose that $0<u_{j n} \leq c<1$. Let

$$
v_{K n}=u_{n-K+1, n}, \quad\left|\triangle v_{K n}\right|=v_{K+1, n}-v_{K n} \quad(0<K \leq n)
$$

and $v_{J n}=u_{n-J+1, n}(0 \leq J \leq n)$. By (b1) of Theorem 2.1 we have $(J>K$, say)

$$
\begin{aligned}
\left|v_{J n}-v_{K n}\right| & \sim \frac{1}{n^{2}} \sum_{s=K}^{J} s \sim \frac{|J-K||J+K|}{n^{2}} \text { and } \\
v_{K n} & \sim \sum_{s=1}^{K} \frac{s}{n^{2}} \sim \frac{K^{2}}{n^{2}} .
\end{aligned}
$$

We can write as follows

$$
\begin{gathered}
S_{1} \sim \sum_{K=1}^{c n} \frac{\Delta v_{K n}}{\left|v_{J n}-v_{K n}\right|}\left(\frac{v_{K n}}{v_{J n}}\right)^{1 / 4} \sim \sum_{K \leq J / 2} \cdots+\sum_{J / 2 \leq K<2 J}^{\prime} \cdots+\sum_{K=2 J}^{c n} \cdots \sim \\
\sim\left(\frac{n^{2}}{J^{2}}\right)^{5 / 4} \sum_{K=1}^{J} \frac{K}{n^{2}}\left(\frac{K^{2}}{n^{2}}\right)^{1 / 4}+\log 2 J+\sum_{K=2 J}^{c n} \frac{K}{n^{2}}\left(\frac{n^{2}}{K^{2}}\right)^{3 / 4}\left(\frac{n^{2}}{J^{2}}\right)^{1 / 4} \sim \\
\sim 1+\log (2 J)+\left(\frac{n}{J}\right)^{1 / 2} \leq c n^{1 / 2}
\end{gathered}
$$

(For example, the second sum can be estimated by Theorem 2.1 as follows

$$
\begin{aligned}
\sum_{J / 2 \leq K<2 J}^{\prime} \cdots & \leq c \sum_{J / 2 \leq K<2 J}^{\prime} \frac{K}{n^{2}} \cdot \frac{n^{2}}{|K+J||K-J|} \cdot\left(\frac{K}{J}\right)^{1 / 4} \sim \\
& \sim \sum_{J / 2 \leq K<2 J}^{\prime} \frac{1}{|K-J|} \sim \log (2 J)
\end{aligned}
$$

using that now $0<v_{J n} \leq c<1$ and $K \sim J$.)
B. If $0 \leq u_{j n} \leq c$ but $\left|u_{k n}-u_{j n}\right| \geq c_{1}$ (where $0<c_{1}<c<1$ ), using that now

$$
S_{1}=\sum_{\substack{k=c n \\\left|u_{k n}-u_{j n}\right| \geq c_{1}}}^{n} \cdots
$$

we get by analogue consideration as before that $S_{1} \leq c$.
Similarly one can obtain:
C. If $0<c_{1} \leq u_{j n} \leq c_{2}<1$, then $S_{1} \sim \log n$.
D. Let $\left|u_{j n}-u_{k n}\right| \geq c_{1}$. Then by arguments as in Part $\mathbf{A}$ we have

$$
\begin{gathered}
S_{1} \sim n^{1 / 2}, \quad c_{2}, \quad\left(n T_{n}\right)^{1 / 6}, \quad \text { if } \\
u_{j n} \sim n^{-2}, \quad c_{3} \leq u_{j n} \leq c_{4}, \quad 1-u_{j n} \sim\left(n T_{n}\right)^{-2 / 3}, \quad \text { respectively }
\end{gathered}
$$

Using that $T_{n}<n^{2-\varepsilon}$ (see [7, (3.38)]), we get

$$
S_{1} \leq c\left(n^{1 / 2}+\left(n \cdot n^{2-\varepsilon}\right)^{1 / 6}\right) \leq c n^{1 / 2}
$$

E. We estimate $S_{2}=\sum_{k=1}^{c n} \cdots$. If $u_{j n} \geq c_{0}$ then using Theorems 1.1 and 2.1 further the argument in [4, Part 4.7] we get that $S_{2} \sim\left(n T_{n}\right)^{1 / 6}$. Now let $0<u_{j n}<c_{0}<c_{1} \leq u_{k n}$, $k \leq c n$. Then

$$
S_{2} \leq \sum_{k=1}^{c n} \frac{\left|\triangle u_{k n}\right|}{\left|u_{1 n}-u_{k n}\right|}\left(\frac{u_{k n}\left(1-u_{k n}\right)}{u_{1 n}\left(1-u_{1 n}\right)}\right)^{1 / 4}<n^{1 / 2} \sum_{k=1}^{c n}\left|\triangle u_{k n}\right| \sim n^{1 / 2} \quad\left(\text { by } u_{1 n} \sim 1\right) .
$$

Summarizing the points $\mathbf{A}-\mathbf{E}$, we obtain Corollary 2.3.
3.4. Proof of Corollary 2.4. The argument is similar to the ones in Part 3.3, so we only sketch it.

We suppose that $x \in\left[x_{n n}, x_{1 n}\right]$ and moreover $x$ satisfies (3.13). Then

$$
\begin{gather*}
\lambda_{n}\left(W_{\rho}, V_{n}\left(W_{\rho}^{2}\right) ; x\right) \sim \\
\sim \sum_{k=1}^{n} \frac{\left|x_{j n}-x_{n+1, n}\right|\left|x_{j n}-x_{0 n}\right|}{\left|x_{k n}-x_{n+1, n}\right|\left|x_{k n}-x_{0 n}\right|}\left|\ell_{k n}\left(U_{n}\left(W_{\rho}^{2}\right), x\right)\right|+\sum_{k=0, n+1} \cdots \sim \\
\sim \sum_{k=1}^{\prime} \frac{u_{j n}\left(1-u_{j n}\right)}{u_{k n}\left(1-u_{k n}\right)} \frac{\left|\triangle u_{k n}\right|}{\left|u_{j n}-u_{k n}\right|}\left(\frac{u_{k n}\left(1-u_{k n}\right)}{u_{j n}\left(1-u_{j n}\right)}\right)^{1 / 4}=  \tag{3.15}\\
=\sum_{k=1}^{n}\left(\frac{u_{j n}\left(1-u_{j n}\right)}{u_{k n}\left(1-u_{k n}\right)}\right)^{3 / 4} \frac{\left|\triangle u_{k n}\right|}{\left|u_{j n}-u_{k n}\right|}
\end{gather*}
$$

considering that the second sum can be considered by $\sum_{k=1, n} \cdots$ (cf. [4, Part 4.8], say).
As before we write
a. Suppose $0<u_{j n} \leq c<1$. Then

$$
\begin{gathered}
S_{1} \sim \sum_{K=1}^{c n}\left(\frac{v_{J n}}{v_{K n}}\right)^{3 / 4} \frac{\left|\Delta v_{K n}\right|}{\left|v_{J n}-v_{K n}\right|} \sim \sum_{K \leq J / 2} \cdots+\sum_{J / 2 \leq K \leq 2 J}^{\prime} \cdots+\sum_{K=2 J}^{c n} \cdots \sim \\
\sim\left(\frac{n^{2}}{J^{2}}\right)^{1 / 4} \sum_{K=1}^{J} \frac{K}{n^{2}}\left(\frac{n^{2}}{K^{2}}\right)^{3 / 4}+\log (2 J)+\sum_{K=2 J}^{c n} \frac{K}{n^{2}}\left(\frac{J^{2}}{n^{2}}\right)^{3 / 4}\left(\frac{n^{2}}{K^{2}}\right)^{7 / 4} \sim \\
\sim 1+\log (2 J)+1 .
\end{gathered}
$$

The cases b. and c. correspond to B. and C. and give

$$
S_{1} \leq c \quad \text { and } \quad S_{1} \sim \log n, \quad \text { respectively. }
$$

d. Let $\left|u_{j n}-u_{k n}\right| \geq c_{1}$. Then by (3.15) and using

$$
u_{j n}\left(1-u_{j n}\right) \leq\left(\frac{u_{j n}+1-u_{j n}}{2}\right)^{2}=\frac{1}{4}
$$

we get

$$
S_{1} \leq c \sum_{K=1}^{c n} \frac{\left|\triangle v_{K n}\right|}{v_{K n}^{3 / 4}} \sim \sum_{K=1}^{c n} \frac{K}{n^{2}}\left(\frac{n^{2}}{K^{2}}\right)^{3 / 4} \sim \frac{1}{n^{1 / 2}} \sum_{K=1}^{n} \frac{1}{K^{1 / 2}} \leq c .
$$

e. The estimation of $S_{2}$ is analogous to the Part 4.8 in [4], whence we get that

$$
S_{2} \sim \log n .
$$

So we get (2.9).
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