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# Dual Giant Gravitons in Sasaki–Einstein Backgrounds

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## Abstract

We study the dynamics of a BPS D3–brane wrapped on a three–sphere in  $\text{AdS}_5 \times L$ , a so–called dual giant graviton, where  $L$  is a Sasakian five–manifold. The phase space of these configurations is the symplectic cone  $X$  over  $L$ , and geometric quantisation naturally produces a Hilbert space of  $L^2$ –normalisable holomorphic functions on  $X$ , whose states are dual to scalar chiral BPS operators in the dual superconformal field theory. We define classical and quantum partition functions and relate them to earlier mathematical constructions by the authors and S.–T. Yau, hep-th/0603021. In particular, a Sasaki–Einstein metric then minimises an entropy function associated with the D3–brane. Finally, we introduce a grand canonical partition function that counts multiple dual giant gravitons. This is related simply to the index–character of the above reference, and provides a method for counting multi–trace scalar BPS operators in the dual superconformal field theory.

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## 1 Introduction and summary

Recently there has been some interest in counting certain BPS states in type IIB string theory on  $\text{AdS}_5 \times S^5$  [1, 2, 3, 4]. In particular, there are two classes of classical BPS configurations known as giant gravitons and dual giant gravitons, respectively. The former consist of D3-branes wrapping three-dimensional supersymmetric submanifolds of  $S^5$ , whereas the latter consist of D3-branes wrapping a three-sphere in  $\text{AdS}_5$ , and are effectively described by BPS point particles in  $S^5$ . These two sets of classical configurations have recently been quantised in [3] and [4], respectively. Interestingly, the result is the same in each case, with the quantum system being effectively described by a three-dimensional harmonic oscillator. The AdS/CFT correspondence [5] in partic-

ular relates BPS configurations of string theory/supergravity to BPS operators of the dual superconformal field theory. One may introduce appropriate partition functions that count the quantum states of the (dual) giant gravitons above, and compare to the counting of 1/8–BPS scalar chiral primary operators of  $\mathcal{N} = 4$  super Yang–Mills theory. In [3] and [4] these two calculations were successfully matched, both for the giant gravitons and the dual giant gravitons, respectively.

As is well–known, the AdS/CFT correspondence extends to more general type IIB string theory backgrounds of the form  $\text{AdS}_5 \times L$ , where  $L$  is a Sasaki–Einstein 5–manifold [6, 7, 8, 9]. Significant progress has been made in understanding these geometries, and their  $\mathcal{N} = 1$  superconformal field theory duals, over the last two years, starting with the construction of an infinite family of Sasaki–Einstein 5–manifolds [10, 11], together their dual field theories [12, 13].

In this paper we shall extend the work of [4] to an arbitrary Sasakian manifold  $L$ , in particular quantising the space of BPS dual giant gravitons. We show that the phase space of a single BPS dual giant graviton is precisely the cone  $X$  over the Sasakian manifold  $(L, g_L)$ , equipped with the natural symplectic form  $\omega$  induced from the contact structure on  $L$ :

$$\omega = \frac{1}{2}d(r^2\eta) . \tag{1.1}$$

Here  $r$  is the conical direction and  $\eta$  is the contact one–form on  $L$ . Thus the cone  $X$ , minus its singular point, is  $X_0 = \mathbb{R}_+ \times L$  where  $r$  may be thought of as a coordinate on  $\mathbb{R}_+$ . An interesting point about this calculation is that the coordinate  $r$  was initially, up to a proportionality factor, the radial coordinate  $R$  in  $\text{AdS}_5$ . Specifically, the two are related by

$$r^2 = \frac{2NR^2}{l^2} \tag{1.2}$$

where  $N$  is the number of background D3–branes and  $l$  is the  $\text{AdS}_5$  radius. Recall that the latter is given by the AdS/CFT formula

$$l^4 = 4\pi g N \alpha'^2 \tag{1.3}$$

with  $g$  the string coupling constant. The apex  $r = 0$  of the cone  $X$  corresponds here to a D3–brane wrapping a zero–volume three–sphere in  $\text{AdS}_5$ .

The Hamiltonian for the BPS D3–brane is given by

$$H_{\text{BPS}} = \frac{r^2}{2l} . \tag{1.4}$$

Recall that every Kähler cone is equipped with a holomorphic Killing vector field

$$\xi = \mathcal{J} \left( r \frac{\partial}{\partial r} \right) \quad (1.5)$$

where  $\mathcal{J}$  is the complex structure tensor on  $X$ . The BPS D3-brane Hamiltonian is then precisely the Hamiltonian function for the vector field  $l^{-1}\xi$ .

When the Sasakian manifold  $(L, g_L)$  is toric, the isometry group contains  $U(1)^3$  by definition, and there are correspondingly three conserved momenta  $P_{\phi_i}$ ,  $i = 1, 2, 3$ , in the D3-brane dynamics. These are the momenta conjugate to the cyclic coordinates  $\phi_i$  parametrising the three-torus  $\mathbb{T}^3 = U(1)^3$ . The Hamiltonian for BPS configurations may then be written as

$$H_{\text{BPS}} = \frac{1}{l} b_i P_{\phi_i} , \quad (1.6)$$

where  $b_i$  are the components of the Reeb vector field in the above basis

$$\xi = b_i \frac{\partial}{\partial \phi_i} . \quad (1.7)$$

Equipped with a classical phase space  $(X, \omega)$ , which is also Kähler, together with a Hamiltonian  $H = r^2/2l$ , it is straightforward to quantise the system using geometric quantisation [14], which in the present set-up is also similar to Berezin's quantisation [15]. Using the complex polarisation induced by the complex structure on  $X$ , one finds that the Hilbert space  $\mathcal{H}$  is the  $L^2$ -completion of the space of  $L^2$ -normalisable holomorphic functions on  $X$  with inner product

$$\langle f_1, f_2 \rangle = \int_X f_1 \bar{f}_2 e^{-r^2/2} \frac{\omega^3}{3!} . \quad (1.8)$$

Applying the standard rules of geometric quantisation, in particular the Hamiltonian becomes

$$l\hat{H} = -i\mathcal{L}_\xi \quad (1.9)$$

acting on  $\mathcal{H}$ . Thus quantum states of definite energy are precisely holomorphic functions of definite charge under the Reeb vector field. In field theory language, the Hamiltonian governing the dynamics of BPS states is precisely given by the R-charge, or equivalently the dilatation, operator.

One may then define, in the usual way, the classical and quantum partition functions:

$$Z_{\text{classical}}(\beta) = \int_X e^{-\beta H} \frac{\omega^3}{3!} = \frac{8l^3}{\beta^3} \text{vol}[L] \quad (1.10)$$

$$Z_{\text{quantum}}(\beta) = \text{Tr}_{\mathcal{H}} e^{-\beta \hat{H}} . \quad (1.11)$$

The second equality in (1.10), which is straightforward to derive, was essential in [16]. One of the central results of the latter reference was that one can localise this expression for the volume, with respect to  $\xi$ , by appropriately resolving  $X$  and using the formula of Duistermaat and Heckman [17]. This leads to a formula for the volume  $\text{vol}[L]$  which is a rational function of  $\xi$  with rational coefficients. These coefficients are given by certain Chern classes and weights. On the other hand, the quantum partition function (1.11) was called the holomorphic partition function in [18] and is closely related to the character defined in [16]. Indeed, the latter reference implies that  $Z_{\text{quantum}}(\beta)$  has a pole of order 3 as  $\beta \rightarrow 0$ , with

$$Z_{\text{quantum}}(\beta) \sim \frac{1}{(2\pi)^3} Z_{\text{classical}}(\beta), \quad \beta \rightarrow 0. \quad (1.12)$$

For much of this paper we formally consider an arbitrary Sasakian metric on  $L$ . The above partition functions then also become functions of the Reeb vector field  $\xi$  as one varies the background metric [16]. However, in order to satisfy the type IIB equations of motion, the metric  $g_L$  on  $L$  must be Einstein of positive curvature. One can formally define an entropy function for a BPS dual giant graviton from its classical partition function (1.10). A Sasaki–Einstein background then minimises this entropy function with respect to the Reeb vector field  $\xi$ . This is rather analogous to Sen’s entropy function for black holes [19].

Finally, we consider a grand canonical partition function that counts multiple BPS dual giant gravitons. These may effectively be described as  $n$  indistinguishable particles, with  $n$  bounded from above by the number of background D3–branes  $N$  [20, 21, 22]. Suppose that the Kähler cone  $(X, \omega)$  admits a holomorphic  $U(1)^s$  isometry, generated by vector fields  $J_i$ ,  $i = 1, \dots, s$ . As discussed in section 3, these act as Hermitian operators  $\hat{P}_i = -i\mathcal{L}_{J_i}$  on  $\mathcal{H}$  and we may thus define the grand canonical partition function

$$\mathcal{Z}(\zeta, \mathbf{q}, X) = \text{Tr}_{\mathcal{H}_{\text{multi}}} \zeta^{\hat{N}} \mathbf{q}^{\hat{\mathbf{P}}} \quad (1.13)$$

where  $\mathcal{H}_{\text{multi}}$  is the multi BPS dual giant Hilbert space and  $\hat{N}$  is the operator that counts the number of dual giant gravitons in a given state. In particular, the coefficient of  $\zeta$  is precisely the character

$$C(\mathbf{q}, X) = \text{Tr}_{\mathcal{H}} \mathbf{q}^{\hat{\mathbf{P}}} \quad (1.14)$$

of [16]. Moreover, setting

$$\xi = \sum_{i=1}^s b_i J_i \quad (1.15)$$

and  $q_i = \exp(-\beta b_i/l)$ , the character is precisely the quantum partition function (1.11).

It is straightforward to write the partition function (1.13) entirely in terms of the character:

$$\mathcal{Z}(\zeta, \mathbf{q}, X) = \exp \left[ \sum_{n=1}^{\infty} \frac{\zeta^n}{n} C(\mathbf{q}^n, X) \right], \quad (1.16)$$

and thus the partition function  $Z_N(\mathbf{q}, X)$  for  $N$  BPS dual giant gravitons may be extracted rather simply as the coefficient of  $\zeta^N$  in this expression.  $Z_N(\mathbf{q}, X)$  may be interpreted as the trace of the action of  $\mathbf{q} \in (\mathbb{C}^*)^s$  on the space of holomorphic functions on the symmetric product space  $\text{Sym}^N X$ .

In the AdS/CFT dual superconformal field theory, this is precisely the generating function that counts mesonic scalar chiral primary operators according to their  $U(1)^s$  flavour charges. Indeed, if a SCFT arises from the IR limit of  $N$  D3-branes at an isolated singularity  $X$ , then the classical vacuum moduli space should be the symmetric product  $\text{Sym}^N X$ . The coordinate ring of holomorphic functions on this variety is the symmetric product of the coordinate ring of  $X$ . The Hilbert space of  $N$  BPS dual giants above is then spanned by the same set of generators (as a  $\mathbb{C}$ -algebra) as this ring. Thus our counting of chiral primaries in the CFT, obtained via counting  $N$  dual giant gravitons in the geometry, agrees with the results of [2], based on group-theoretic techniques.

The plan of the rest of the paper is as follows. In section 2 we analyse the classical dynamics of a D3-brane probe wrapping an  $S^3 \subset \text{AdS}_5$ , focusing in particular on BPS configurations. In section 3 we quantise the corresponding phase space using geometric quantisation. In section 4 we write down the classical and quantum partition functions, in particular relating them to the results of [16]. The volume minimisation of the latter reference is related to minimising an entropy function for the D3-brane. Finally, in section 5, we study a grand canonical partition function that counts multiple BPS dual giant gravitons, and discuss the relation to counting BPS chiral primary operators in the AdS/CFT dual superconformal field theory.

## 2 Dual giant gravitons

In this section we study the dynamics of a dual giant graviton in  $\text{AdS}_5 \times L$ , where  $(L, g_L)$  is an arbitrary Sasakian 5-manifold. In order that this background satisfies the type IIB supergravity equations one requires  $g_L$  to be a positively curved Einstein

metric, but for the most part this will be inessential in what follows – the important feature is the Sasakian structure. The dynamics essentially reduces to that of a point particle on  $L$ , and the BPS configurations are described by BPS geodesics on  $L$ . The BPS phase space is precisely the cone  $(X, \omega)$  based on  $L$ , equipped with the standard symplectic form  $\omega$  induced from the contact structure on  $L$ . Moreover, the Hamiltonian restricted to these configurations is proportional to the Hamiltonian function for the Reeb vector field  $\xi$  on  $X$ .

## 2.1 Hamiltonian dynamics and phase space

We begin with the direct product metric on  $\text{AdS}_5 \times L$

$$ds^2 = g_{MN}dX^M dX^N = ds_{\text{AdS}_5}^2 + l^2 ds_L^2 \quad (2.1)$$

where  $X^M$ ,  $M = 0, \dots, 9$ , are local coordinates on  $\text{AdS}_5 \times L$ ,  $ds_L^2$  is the line element of a Sasakian metric on the 5-manifold  $L$ , and  $l$  is the  $\text{AdS}_5$  radius. One may introduce global coordinates on  $\text{AdS}_5$  with line element

$$ds_{\text{AdS}_5}^2 = -V(R)dt^2 + \frac{1}{V(R)}dR^2 + R^2(d\theta^2 + \cos^2 \theta d\phi_1^2 + \sin^2 \theta d\phi_2^2) \quad (2.2)$$

where

$$V(R) = 1 + \frac{R^2}{l^2} . \quad (2.3)$$

$$(2.4)$$

Here  $t$  is the usual global time on  $\text{AdS}_5$ . The coordinate  $R$  is then a radial coordinate on the constant time sections, foliating the latter with round three-spheres.

The simplest way of defining a Sasakian manifold  $(L, g_L)$  is to say that the corresponding metric cone  $(X, g_X)$ , with line element

$$ds_X^2 = dr^2 + r^2 ds_L^2 , \quad (2.5)$$

is Kähler, although there exist other, more intrinsic, definitions. An important fact in what follows is that any Sasakian metric may be written locally as

$$ds_L^2 = h_{\alpha\beta} dx^\alpha dx^\beta + (d\psi + \sigma)^2 . \quad (2.6)$$

Here the Reeb vector field is

$$\xi = \mathcal{J} \left( r \frac{\partial}{\partial r} \right) = \frac{\partial}{\partial \psi} \quad (2.7)$$

which has norm one on the link  $\{r = 1\}$ , which is a copy of  $L$ . The metric transverse to the orbits of  $\xi$  is given locally in components by  $h_{\alpha\beta}(x)$ ,  $\alpha, \beta = 1, \dots, 4$ , and is also a Kähler metric. The contact one-form, metrically dual to  $\xi$ , is, in these local coordinates,

$$\eta = d\psi + \sigma \tag{2.8}$$

and satisfies

$$d\eta = d\sigma = 2\omega_T, \tag{2.9}$$

where  $\omega_T$  is the transverse Kähler form. In particular, the Kähler cone metric on  $X$  then has Kähler form

$$\omega = \frac{1}{2}d(r^2\eta) = \frac{1}{2}i\partial\bar{\partial}r^2. \tag{2.10}$$

For further details on Sasakian geometry we refer the reader to [16] and references therein.

The dynamics of a D3-brane propagating in this background is described by the usual world-volume action, comprising the two terms

$$S_{\text{D3}} = S_{\text{DBI}} + S_{\text{WZ}} = -T_3 \int d^4\zeta \sqrt{-\det G_{\mu\nu}} + T_3 \int C_4 \tag{2.11}$$

where

$$G_{\mu\nu} = \frac{\partial X^M}{\partial \zeta^\mu} \frac{\partial X^N}{\partial \zeta^\nu} g_{MN} \tag{2.12}$$

is the pull-back of the spacetime metric to the D3-brane world-volume, parametrised by coordinates  $\{\zeta^0 = \tau, \zeta^1, \zeta^2, \zeta^3\}$ ,  $C_4$  is the (pull-back of the) Ramond-Ramond four-form potential of type IIB supergravity, and  $T_3$  is the D3-brane tension.

We wish to study configurations in which the D3-brane wraps a round three-sphere in  $\text{AdS}_5$ , a so-called dual giant graviton [23]. We thus choose the following embedding  $X^M(\zeta^\mu)$ :

$$\begin{aligned} t = \tau \quad R = R(\tau) \quad \theta = \zeta^1 \quad \phi_1 = \zeta^2 \quad \phi_2 = \zeta^3 \\ \psi = \psi(\tau) \quad x^\alpha = x^\alpha(\tau). \end{aligned} \tag{2.13}$$

The self-dual Ramond-Ramond five-form for this background is

$$F_5 = -\frac{4}{l} (\text{Vol}[\text{AdS}_5] + l^5 \text{Vol}[L]) \tag{2.14}$$

and it is then easily verified that the pull-back of a choice of  $C_4$  to the world-volume of the D3-brane, under the embedding (2.13), is simply

$$C_4 = \frac{R^4}{l} \sin \zeta^1 \cos \zeta^1 d\tau \wedge d\zeta^1 \wedge d\zeta^2 \wedge d\zeta^3 . \quad (2.15)$$

A short calculation reveals that the determinant factor reads

$$\sqrt{-\det G_{\mu\nu}} = R^3 \cos \zeta^1 \sin \zeta^1 \Delta^{1/2} , \quad (2.16)$$

where we have defined

$$\Delta \equiv V(R) - \frac{\dot{R}^2}{V(R)} - l^2 \left[ h_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta + (\dot{\psi} + \sigma_\alpha \dot{x}^\alpha)^2 \right] \quad (2.17)$$

and a dot denotes differentiation with respect to  $\tau$ . Finally, integrating the action over the  $S^3$  and using the formula (see *e.g.* [24])

$$2\pi^2 T_3 = \frac{N}{l^4} \quad (2.18)$$

for the D3-brane tension, we obtain the effective point-particle Lagrangian

$$L = -\frac{N}{l^4} R^3 \left[ \Delta^{1/2} - \frac{R}{l} \right] . \quad (2.19)$$

To proceed, it is convenient to pass to the Hamiltonian formalism. The canonical momenta are

$$\begin{aligned} P_R &\equiv \frac{\partial L}{\partial \dot{R}} = \frac{NR^3}{l^4 V(R) \Delta^{1/2}} \dot{R} \\ P_\psi &\equiv \frac{\partial L}{\partial \dot{\psi}} = \frac{NR^3}{l^2 \Delta^{1/2}} (\dot{\psi} + \sigma_\alpha \dot{x}^\alpha) \\ P_{x^\alpha} &\equiv \frac{\partial L}{\partial \dot{x}^\alpha} = \frac{NR^3}{l^2 \Delta^{1/2}} \left( h_{\alpha\beta} \dot{x}^\beta + (\dot{\psi} + \sigma_\gamma \dot{x}^\gamma) \sigma_\alpha \right) . \end{aligned} \quad (2.20)$$

The Hamiltonian is then

$$H = \frac{NR^3}{l^4} \left[ V(R)^{1/2} \Omega^{1/2} - \frac{R}{l} \right] , \quad (2.21)$$

where we have defined

$$\Omega = 1 + \frac{l^6}{N^2 R^6} \left[ l^2 V(R) P_R^2 + P_\psi^2 + h^{\alpha\beta} (P_{x^\alpha} - P_\psi \sigma_\alpha) (P_{x^\beta} - P_\psi \sigma_\beta) \right] , \quad (2.22)$$

and  $h^{\alpha\beta}$  denotes the matrix inverse of  $h_{\alpha\beta}$ . It is a standard exercise to verify that the Hamiltonian equations of motion

$$\dot{P}_A = -\frac{\partial H}{\partial Q^A} \quad \dot{Q}^A = \frac{\partial H}{\partial P_A} , \quad (2.23)$$

where  $Q^A$  and  $P_A$  collectively denote the six coordinates and their conjugate momenta, respectively, admit the following solutions

$$\begin{aligned} \dot{R} &= 0 & \dot{\psi} &= \frac{1}{l} & \dot{x}^\alpha &= 0 \\ P_R &= 0 & P_\psi &= \frac{NR^2}{l^2} & P_{x^\alpha} &= \frac{NR^2}{l^2} \sigma_\alpha . \end{aligned} \quad (2.24)$$

In fact, one can show that these solutions are precisely the set of  $\kappa$ -symmetric, or BPS, solutions. The calculation is again standard, although slightly lengthy. The details may be found in appendix A.

The Hamiltonian, restricted to these configurations, is

$$H_{\text{BPS}} = \frac{N}{l^3} R^2 = \frac{1}{l} P_\psi , \quad (2.25)$$

and in particular is proportional to  $R^2$ . We define

$$r^2 \equiv \frac{2NR^2}{l^2} \quad (2.26)$$

so that

$$H_{\text{BPS}} = \frac{r^2}{2l} . \quad (2.27)$$

For  $(X, g_X)$  the Kähler cone over  $(L, g_L)$ , with metric (2.5),  $r^2/2$  is precisely the Hamiltonian function generating the flow along the Reeb vector field; that is,

$$d(r^2/2) = -\xi \lrcorner \omega . \quad (2.28)$$

This Hamiltonian function for  $\xi$  was essential in reference [16] for computing the volume of the link  $(L, g_L)$  using the localisation formula of Duistermaat and Heckman. Here, we recover this Hamiltonian from a different, and more physical, perspective. Indeed, we find that  $r^2/2$  here is precisely the momentum canonically conjugate to the Reeb vector field. Interestingly, the conical coordinate  $r$  on phase space is effectively constructed from the radial coordinate  $R$  in  $\text{AdS}_5$ .

To make this correspondence more precise, we turn to analysing the resulting reduced phase space. Given the constraints (2.24), the reduced phase space may be parametrised by the six local coordinates

$$Q^A = (R, \psi, x^1, \dots, x^4)^A . \quad (2.29)$$

Indeed, it is straightforward to see that this phase space is naturally a copy of the cone  $X$  over  $L$  with conical coordinate  $r$ . The tip of the cone  $r = 0$  corresponds to

the singular configuration in which the D3–brane wraps a zero–volume  $S^3$  in  $\text{AdS}_5$ . A standard way to obtain the symplectic structure on phase space is to compute the Dirac brackets. Let us repackage the six constraints as follows:

$$f_1 = P_R = 0 \quad f_2 = P_\psi - \frac{N}{l^2} R^2 = 0 \quad f_{\alpha+2} = P_{x^\alpha} - \frac{N}{l^2} R^2 \sigma_\alpha = 0 \quad (2.30)$$

and compute the following Poisson brackets

$$\{Q^A, Q^B\}_{PB} = 0 \quad \{Q^A, f_B\}_{PB} = \delta_B^A \quad (2.31)$$

and

$$M_{AB} = \{f_A, f_B\}_{PB} = \frac{N}{l^2} 2R \begin{pmatrix} 0 & 1 & \sigma_\alpha \\ -1 & 0 & 0 \\ -\sigma_\beta & 0 & R(\omega_T)_{\alpha\beta} \end{pmatrix}. \quad (2.32)$$

We thus obtain

$$\{Q^A, Q^B\}_{DB} = \{Q^A, Q^B\}_{PB} - \{Q^A, f_C\}_{PB} M_{CD}^{-1} \{f_D, Q^B\}_{PB} = M^{AB} \quad (2.33)$$

where  $M^{AB}$  is the matrix inverse of  $M_{AB}$ , giving the symplectic structure

$$\omega = \frac{N}{l^2} d(R^2 \eta) = \frac{1}{2} d(r^2 \eta). \quad (2.34)$$

Thus we recover the standard symplectic structure on the Kähler cone  $(X, g_X)$ . Alternatively, the same result may be obtained by computing the symplectic one–form  $\nu$  on phase space:

$$\nu = P_R dR + P_\psi d\psi + P_{x^\alpha} dx^\alpha = \frac{N}{l^2} R^2 \eta = \frac{1}{2} r^2 \eta. \quad (2.35)$$

## 2.2 Toric geometries

In this subsection we discuss the particular case that  $(L, g_L)$  is a toric Sasakian manifold. This is of course a subcase of the general discussion in the previous subsection, but it is nevertheless instructive to analyse explicitly. When  $(L, g_L)$  is toric, by definition there is at least a  $U(1)^3$  isometry group, and there are correspondingly three conserved momenta in the D3–brane dynamics considered in the previous subsection. These are dual to certain global flavour charges in the AdS/CFT dual conformal field theory, with the R–charge arising as a linear combination of these three charges.

The group  $\mathbb{T}^3 = U(1)^3$  acts Hamiltonianly on the Kähler cone  $(X, g_X)$ , which in the present case means that the  $\mathbb{T}^3$  action preserves the Kähler form  $\omega$ . A general toric Kähler cone  $(X, g_X)$  metric may be written as [25]

$$ds_X^2 = G_{ij} dy^i dy^j + G^{ij} d\phi_i d\phi_j . \quad (2.36)$$

Here  $\phi_i \sim \phi_i + 2\pi$  are angular coordinates on  $\mathbb{T}^3$ , with corresponding moment map coordinates  $y^i$ ,  $i, j = 1, 2, 3$ . The  $y^i$  are homogeneous solutions to

$$dy^i = -\frac{\partial}{\partial \phi_i} \lrcorner \omega . \quad (2.37)$$

The symplectic form  $\omega$  is then simply

$$\omega = dy^i \wedge d\phi_i \quad (2.38)$$

while the metric is parametrised by the Hessian matrix  $G_{ij}$  of the symplectic potential  $G(\mathbf{y})$

$$G_{ij} = \frac{\partial^2 G}{\partial y^i \partial y^j} , \quad (2.39)$$

with  $G^{ij}$  denoting the matrix inverse of  $G_{ij}$ . The map

$$\begin{aligned} \mu & : X \rightarrow \mathbb{R}^3 \\ & (y^i, \phi_i) \mapsto y^i \end{aligned} \quad (2.40)$$

is called the moment map, and the image  $\mu(X) = \mathcal{C}^*$  is a strictly convex rational polyhedral cone

$$\mathcal{C}^* = \{ \mathbf{y} \in \mathbb{R}^3 \mid l^a(\mathbf{y}) = v_i^a y^i \geq 0, a = 1, \dots, d \} . \quad (2.41)$$

Thus it is the convex cone formed by  $d$  planes through the origin of  $\mathbb{R}^3$ . The inward-pointing normal vectors to these planes may be taken to be integral primitive vectors  $v^a \in \mathbb{Z}^3$ ,  $a = 1, \dots, d$ . The symplectic potential  $G(\mathbf{y})$  is then a function on  $\mathcal{C}^*$  with a certain prescribed singular behaviour near the bounding planes. Moreover,  $G_{ij}$  is required to be homogeneous degree minus one in the  $y^i$ . For further details, the reader is referred to [25].

The metric  $ds_L^2$  on the link is simply given by the restriction of the metric (2.36) to the hypersurface

$$2b_i y^i = 1 . \quad (2.42)$$

Here the constants  $b_i$  are the components of the Reeb vector field in the above basis:

$$\xi = b_i \frac{\partial}{\partial \phi_i} . \quad (2.43)$$

To study the dynamics of a D3-brane wrapped on an  $S^3$  inside  $\text{AdS}_5$ , one proceeds as before. The Hamiltonian can be written as in (2.21), with

$$\Omega = 1 + \frac{l^6}{N^2 R^6} [l^2 V(R) P_R^2 + G^{ij} P_{y^i} P_{y^j} + G_{ij} P_{\phi_i} P_{\phi_j}] . \quad (2.44)$$

where the canonical momenta are

$$P_{y^i} = G_{ij} \dot{y}^j \quad P_{\phi_i} = G^{ij} \dot{\phi}_j . \quad (2.45)$$

The  $\phi_i$  are cyclic coordinates, so that it immediately follows that the  $P_{\phi_i}$  are constant. The BPS solutions are given by

$$\begin{aligned} \dot{R} = 0 & \quad \dot{y}^i = 0 & \quad \dot{\phi}_i = \frac{1}{l} b_i \\ P_R = 0 & \quad P_{y^i} = 0 & \quad P_{\phi_i} = \frac{2NR^2}{l^2} y^i \end{aligned} \quad (2.46)$$

which also solve the Hamiltonian equations. Note that  $y^i$  are not independent variables. Thus one might introduce a Lagrange multiplier and implement the usual Hamiltonian description of constrained systems. Alternatively, one can simply solve for one variable in terms of the other two. For instance, one can choose  $y^2, y^3$  as independent variables, and regard  $y^1$  as a function of these. Taking note of this, and using formulae from [25], one can indeed verify that (2.46) is a solution. We define

$$r^i \equiv P_{\phi_i} = \frac{2NR^2}{l^2} y^i = r^2 y^i , \quad (2.47)$$

so that the  $r^i$  effectively become moment map coordinates on the cone  $X$ , and the reduced phase space is parametrised by the coordinates  $(r^i, \phi_i)$  and has symplectic form

$$\omega = dr^i \wedge d\phi_i . \quad (2.48)$$

Finally, the BPS Hamiltonian is

$$H_{\text{BPS}} = \frac{1}{l} b_i P_{\phi_i} . \quad (2.49)$$

Notice that we could have arrived at the same results by simply implementing the change of coordinates discussed in appendix B.

### Example: 1/8–BPS dual giants in $\text{AdS}_5 \times S^5$

As an example of this formalism, let us recover the results of [4]. Of course, 1/8–BPS configurations in an  $S^5$  background preserve the same supersymmetry as 1/2–BPS configuration in a Sasaki–Einstein background. Equivalently, 1/8–BPS operators of  $\mathcal{N} = 4$  super Yang–Mills preserve the same supersymmetry as 1/2–BPS operators in an  $\mathcal{N} = 1$  SCFT.

We view  $S^5$  as a toric Sasaki–Einstein manifold, with  $U(1)^3 \subset SO(6)$  the Cartan subgroup. Setting  $y^i = (\mu^i)^2/2$  and taking

$$G(\mathbf{y}) = \frac{1}{2} \sum_{i=1}^3 y^i \log y^i \quad (2.50)$$

the toric metric (2.36) reads

$$ds_{\mathbb{C}^3}^2 = \sum_{i=1}^3 ((d\mu^i)^2 + (\mu^i)^2 d\phi_i^2) \quad (2.51)$$

while the constraint is

$$\sum_{i=1}^3 (\mu^i)^2 = 1 . \quad (2.52)$$

If one wishes, one may introduce unconstrained angular variables to solve for (2.52). It is straightforward to check<sup>1</sup>, for example using formulae in [25], that the Reeb vector field is  $b_i = (1, 1, 1)$ . Inserting this into the general toric formula (2.49) for the BPS Hamiltonian one obtains

$$H_{S^5} = \frac{1}{l} (P_{\phi_1} + P_{\phi_2} + P_{\phi_3}) , \quad (2.53)$$

which is the result presented in [4].

## 2.3 Relation to BPS geodesics

In this subsection, we point out that the dynamics of a BPS point–particle<sup>2</sup> in an arbitrary Sasakian manifold is equivalent to that of a BPS dual giant graviton, previously discussed. Geometric quantisation of this dynamics, to be discussed in section 3, will

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<sup>1</sup>Notice that here the toric fan is generated by the standard orthonormal basis for  $\mathbb{R}^3$ ,  $v_a^i = \delta_a^i$ .

<sup>2</sup>BPS geodesics in the  $Y^{p,q}$  [10, 11] and  $L^{a,b,c}$  [28, 29] manifolds were considered in [26, 27], and were argued to be related to chiral primary operators of the dual quiver gauge theories [13, 30, 31, 27].

then lead to a precise relation to chiral primary operators in the dual conformal field theory.

We consider the motion of a free point-like particle in the metric

$$ds^2 = -dt^2 + ds_L^2 \quad (2.54)$$

where  $(L, g_L)$  is a Sasakian manifold, with Reeb vector field  $\xi = \partial/\partial\psi$ . We therefore consider the following action<sup>3</sup>

$$S = \int d\tau \left[ -\dot{t}^2 + h_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta + (\dot{\psi} + \sigma_\alpha\dot{x}^\alpha)^2 \right] \quad (2.55)$$

where dots denote derivatives with respect to  $\tau$ , and  $\alpha, \beta = 1, \dots, 4$ . Passing to the Hamiltonian formalism, we have

$$H = -p_t^2 + p_\psi^2 + h^{\alpha\beta}(p_\alpha - p_\psi\sigma_\alpha)(p_\beta - p_\psi\sigma_\beta) , \quad (2.56)$$

in terms of the conjugate momenta. As  $t$  and  $\psi$  are cyclic coordinates, their conjugate momenta are constant:  $p_t = E$ ,  $p_\psi = \lambda$ . Setting to zero the total Hamiltonian, as follows from reparametrisation invariance, one obtains an expression for the energy:

$$E^2 = \lambda^2 + h^{\alpha\beta}(p_\alpha - \lambda\sigma_\alpha)(p_\beta - \lambda\sigma_\beta) . \quad (2.57)$$

This is positive definite; in particular we have the bound

$$E \geq \lambda . \quad (2.58)$$

It is then natural to define BPS geodesics as those trajectories for which the inequality (2.58) is saturated. This immediately implies that

$$p_\alpha = \lambda\sigma_\alpha . \quad (2.59)$$

One can check that the full solution to the equations of motion is given by

$$\dot{x}^\alpha = 0 , \quad \dot{\psi} = \lambda . \quad (2.60)$$

Thus a BPS geodesic is precisely an orbit of the Reeb vector field  $\xi$ , with the particle moving at constant speed  $\lambda$  in this direction. Thus the configuration space is a copy of the cone over  $(L, g_L)$ : for each point in  $L$  there is a unique BPS geodesic starting at that point<sup>4</sup>. The conical direction may then identified with the speed as

$$\lambda = r^2/2 . \quad (2.61)$$

---

<sup>3</sup>We suppress overall multiplicative factors in the action.

<sup>4</sup>Assuming  $\lambda > 0$ .

Indeed, one easily checks that the symplectic form on phase space is

$$\nu = p_\psi d\psi + p_\alpha dx^\alpha = \lambda \eta = \frac{1}{2} r^2 \eta \quad (2.62)$$

with this identification. Moreover, the energy is

$$E = \frac{r^2}{2} \quad (2.63)$$

which is the same as the rescaled BPS Hamiltonian  $lH_{\text{BPS}}$  for the BPS dual giant gravitons.

Finally, for toric geometries, using the change of coordinates in appendix B it is straightforward to write down the BPS geodesics in terms of the Reeb vector:

$$\dot{\phi}_i = \lambda b_i \quad \dot{y}^i = 0 \quad (2.64)$$

and obtain the following expression for the energy

$$E = b_i p_{\phi_i} . \quad (2.65)$$

### 3 Geometric quantisation

In this section we quantise the BPS dual giant graviton. This is a fairly routine exercise in applying geometric quantisation to the phase space  $X$  derived in the previous section. The result is rather simple: the Hilbert space  $\mathcal{H}$  is the space of  $L^2$ -normalisable holomorphic functions on  $X$ , with respect to the inner product (1.8). There is then a standard map from quantisable functions on  $X$  to operators on  $\mathcal{H}$ , which in particular maps the Hamiltonian  $H_{\text{BPS}} = r^2/2l$  to  $\hat{H} = -il^{-1}\mathcal{L}_\xi$ . Thus states of definite energy are described by holomorphic functions on  $X$  of definite R-charge under the Reeb vector field  $\xi$ .

#### 3.1 Hilbert space

Given a phase space  $X$ , with symplectic form  $\omega$ , one would like to quantise the classical system  $(X, \omega)$ . Thus, one would like to associate a Hilbert space  $\mathcal{H}$  in a natural way to  $(X, \omega)$ . Moreover, to every classical observable, namely a function  $A$  on  $X$ , one would like to associate a symmetric operator  $\hat{A}$  on  $\mathcal{H}$ . According to Dirac, the map  $A \rightarrow \hat{A}$  should be linear and map Poisson brackets to commutators. Thus, operators should form a Hilbert space representation of the classical observables.

Given any symplectic manifold  $(X, \omega)$ , a natural Hilbert space is simply the  $L^2$ -completion of the space of smooth  $L^2$ -normalisable complex-valued functions on  $X$ , with norm being  $\langle f_1, f_2 \rangle = \int_X f_1 \bar{f}_2 \omega^n / n!$ . One could then map the function  $A \mapsto -iX_A$  where  $X_A$  is the Hamiltonian vector field for  $A$ , which by definition satisfies

$$dA = -X_A \lrcorner \omega . \quad (3.1)$$

In fact this map from functions on  $X$  to vector fields is a Lie algebra homomorphism with respect to the Poisson bracket

$$[A, B] = \omega^{ij} \partial_i A \partial_j B \quad (3.2)$$

and the usual Lie bracket  $[X_A, X_B]$  of vector fields. However, there are various problems with this, not least that the constant function is mapped to zero, so that for example position and momentum then commute. Moreover, since our phase space  $X$  is a non-compact cone, the wavefunctions would have to have very rapid decrease in the conical direction.

Geometric quantisation is an attempt to solve this quantisation problem in general<sup>5</sup>. The first step is to define an Hermitian line bundle  $\mathcal{L}$  over  $X$  with unitary connection for which  $-2\pi i \omega$  is the curvature 2-form. In general, a necessary condition is that the periods of  $\omega$  are integral. However, in our case,  $\omega$  is in fact exact

$$\omega = \frac{1}{2} d(r^2 \eta) = \frac{1}{2} i \partial \bar{\partial} r^2 \quad (3.3)$$

and moreover is a Kähler form with a globally defined Kähler potential  $r^2/2$ . This also happens to be the Hamiltonian for the BPS D3-brane of course, up to a factor of the AdS radius  $l$ . Since  $\omega$  is exact, the line bundle  $\mathcal{L}$  is trivial as a complex line bundle, and we thus take

$$\mathcal{L} = \mathbb{C} \times X . \quad (3.4)$$

In particular, sections of  $\mathcal{L}$  may be identified with complex-valued functions on  $X$ . Given two functions  $f_1, f_2$ , viewed as sections of  $\mathcal{L}$ , their pointwise inner product is

$$(f_1, f_2) = f_1 \bar{f}_2 \exp(-r^2/2) . \quad (3.5)$$

This metric has the property that it is compatible with the connection

$$\nabla = d - i\nu \quad (3.6)$$

---

<sup>5</sup>For a review, see reference [14].

where

$$\nu = -\frac{1}{2}i\partial r^2 \tag{3.7}$$

is a connection 1-form with  $d\nu = \omega$ . Compatability means that

$$d(f_1, f_2) = (\nabla f_1, f_2) + (f_1, \nabla f_2) \tag{3.8}$$

which the reader may easily verify.

Thus, a first attempt at assigning a Hilbert space to  $(X, \omega)$  would be to set  $\mathcal{H}(X)$  equal to the  $L^2$ -completion of the space of smooth complex-valued functions on  $X$  with bounded norm with respect to the with inner product

$$\langle f_1, f_2 \rangle = \int_X (f_1, f_2) \frac{\omega^n}{n!} . \tag{3.9}$$

Notice this inner product includes the Kähler potential factor  $\exp(-r^2/2)$ . This has the added bonus of making the measure more convergent. However, as is well-known, this Hilbert space is too big – roughly, in quantum mechanics, the wave functions should depend on only “half” the phase space variables, for example either position or momentum variables. In forming  $\mathcal{H}(X)$  we have so far made no such distinction.

The most natural way of solving this problem in general, geometrically, is to pick a *polarisation* of  $(X, \omega)$ . The reader is referred to reference [14] for the general set-up. Here we note that, since  $X$  has a natural complex structure, namely that induced by the Sasakian structure on  $L$ , there exists a natural choice of complex polarisation, namely the integral distribution  $F$  in  $T_{\mathbb{C}}X$  spanned by the anti-holomorphic vector fields on  $X$ ,  $\partial/\partial\bar{z}_i$ . The Kähler form  $\omega$  of course vanishes on  $F$ , and by construction so does the Kähler 1-form  $\nu$  in (3.7). One then says that this choice of  $\nu$  is *adapted* to the polarisation. The Hilbert space  $\mathcal{H}$  is then the subspace of polarised elements of  $\mathcal{H}(X)$ :

$$\nabla_X f = 0 \tag{3.10}$$

where  $X \in F$ . In the present set-up, with  $\nu$  adapted to the polarisation, this reduces to

$$\bar{\partial}f = 0 \tag{3.11}$$

so that polarised sections of  $\mathcal{L}$  may be identified with  $L^2$ -normalisable *holomorphic functions* on  $X$ . In fact, this Hilbert space, with norm (1.8), was constructed by Berezin in his quantisation of Kähler manifolds, introduced in [15].

## 3.2 Operators

The main point of the above construction, however, is that there is a natural map from a certain class of functions on  $X$  to symmetric operators acting on  $\mathcal{H}$ , that satisfies Dirac's requirements. Namely,

$$A \mapsto \hat{A} \equiv -i\nabla_{X_A} + A = -i\mathcal{L}_{X_A} - (X_A \lrcorner \nu) + A . \quad (3.12)$$

Clearly this is linear, and for any three functions satisfying the Poisson bracket relation  $A = [B, C]$ , the reader can easily check that indeed  $-i\hat{A} = [\hat{B}, \hat{C}]$ . The space of *quantisable* functions is then the space of  $A$ s such that  $\hat{A} : \mathcal{H} \rightarrow \mathcal{H}$ . In particular,  $X_A$  must preserve the polarisation  $F$ , which means that  $[X_A, X] \in F, \forall X \in F$ .

We may now apply this to some of the observables of interest. The Hamiltonian is  $H_{\text{BPS}} = r^2/2l$ . The corresponding Hamiltonian vector field is, up to a factor of  $1/l$ , just the Reeb vector field  $\xi$ . This vector field is holomorphic, and thus preserves the polarisation  $F$ . Thus the Hamiltonian is indeed quantisable<sup>6</sup>

$$lH_{\text{BPS}} \mapsto l\hat{H} = -i\mathcal{L}_\xi - (\xi \lrcorner \nu) + r^2/2 . \quad (3.13)$$

We must now recall that

$$\nu = -\frac{1}{2}i\partial r^2 = -\frac{1}{4}i(d + id^c)r^2 \quad (3.14)$$

where as usual  $d^c = \mathcal{J} \circ d$  and we have

$$\begin{aligned} \partial &= \frac{1}{2}(d + id^c) \\ \bar{\partial} &= \frac{1}{2}(d - id^c) . \end{aligned} \quad (3.15)$$

Recall also (see *e.g.* [16]) that  $d^c r^2 = 2r^2\eta$ , and  $\xi \lrcorner \eta = 1, \xi \lrcorner dr = 0$ . Putting all this together, we see that the last two terms in (3.13) cancel, giving

$$H_{\text{BPS}} \mapsto \hat{H} = -il^{-1}\mathcal{L}_\xi . \quad (3.16)$$

The energy eigenstates of  $\hat{H}$  acting on  $\mathcal{H}$  are therefore simply holomorphic functions of fixed charge under the Reeb vector field.

The above calculation in fact generalises to any holomorphic vector field on  $X$  that acts isometrically on  $(L, g_L)$ . Any such vector field  $V$  is tangent to the link, meaning that

$$\mathcal{L}_V r = 0 . \quad (3.17)$$

---

<sup>6</sup>In general this is not guaranteed to be the case, which is one of the problems with geometric quantisation.

The corresponding Hamiltonian function  $A_V$  by definition satisfies

$$dA_V = -V \lrcorner \omega . \quad (3.18)$$

The unique homogeneous solution to this is

$$A_V = \frac{1}{2} r^2 \eta(V) . \quad (3.19)$$

It follows that

$$-(V \lrcorner \nu) + A_V = 0 \quad (3.20)$$

where we have used the above equations, together with  $dr(V) = \mathcal{L}_V r = 0$  and the fact that  $V$  is holomorphic.

A holomorphic Killing vector field  $V$  gives rise to a conserved quantity in the dynamics of BPS dual giant gravitons. A coordinate along the orbits of  $V$  is then canonically conjugate to the function  $A_V$  on the BPS phase space. On quantisation, this maps to the operator

$$\hat{A}_V = -i\mathcal{L}_V \quad (3.21)$$

acting on the Hilbert space  $\mathcal{H}$  of holomorphic functions.

In particular, we may apply this to the generators of the  $\mathbb{T}^3$  isometry for toric geometries. Here one may take  $V = \partial/\partial\phi_i$  for any  $i = 1, 2, 3$ . These give rise to the conserved quantities  $P_{\phi_i}$ , which may be identified with the symplectic coordinates  $y^i$  for BPS solutions. On quantisation, these map to operators that we shall call  $\hat{P}^i = -i\mathcal{L}_{\partial/\partial\phi_i}$ , acting on holomorphic functions on  $X$ . Applying this to the toric BPS Hamiltonian (2.49), we recover the result (3.16)

$$H_{\text{BPS}} \mapsto l^{-1} b_i \hat{P}^i = -il^{-1} \mathcal{L}_\xi . \quad (3.22)$$

Recall that holomorphic functions on  $X$  are spanned by elements of the abelian semi-group

$$\mathcal{S}_C = \mathbb{Z}^3 \cap \mathcal{C}^* \subset \mathbb{R}^3 \quad (3.23)$$

of integral points inside the polyhedral cone  $\mathcal{C}^*$ . We can give a physical interpretation of this by using the fact that, upon quantisation, the linear functions  $l^a(\mathbf{y})$  map to operators

$$l^a(\mathbf{y}) \mapsto \hat{L}^a = v_i^a \hat{P}^i . \quad (3.24)$$

Thus, the equations defining  $\mathcal{C}^*$  become conditions to be imposed on states  $f$  of the Hilbert space  $\mathcal{H}$ :

$$l^a(\mathbf{y}) \geq 0 \mapsto \langle f, \hat{L}^a f \rangle \geq 0 . \quad (3.25)$$

The latter precisely means that the quantum numbers of a state  $f$ , which are the eigenvalues of  $\hat{P}^i$ , are  $\mathbf{m} \in \mathcal{S}_C$ .

## 4 Partition functions and entropy minimisation

In this section we analyse the classical and quantum partition functions for a BPS dual giant graviton. This gives a physical interpretation to the results in [16]. Moreover, we show that the classical entropy, viewed as a function of the background Sasakian metric, is minimised for Sasaki–Einstein backgrounds. This is rather similar to Sen’s entropy function for black holes [19], which is also defined off-shell and is extremised on solutions. In both cases these extremisation problems allow one to compute the entropy of a solution, without knowing its explicit form, but assuming that the solution in fact exists. For black holes the extremal entropy typically depends only on the conserved electric and magnetic charges, whereas the results of [16] determine the extremal entropy of the BPS dual giant in terms of equivariant holomorphic invariants of the geometry, namely certain Chern classes and weights. For toric geometries, these can be replaced by the toric data defining the Calabi–Yau singularity [25].

### 4.1 Classical and quantum partition functions

Given a classical phase space  $(X, \omega)$  with Hamiltonian  $H$ , together with a quantisation with Hilbert space  $\mathcal{H}$  and quantised Hamiltonian  $\hat{H}$ , one can define classical and quantum partition functions. The classical partition function is obtained by integrating  $\exp(-\beta H)$  over the phase space:

$$Z_{\text{classical}}(\beta) = \int_X e^{-\beta H} \frac{\omega^3}{3!} , \quad (4.1)$$

where  $\beta = 1/T$  is the inverse temperature, in units where  $k_B = 1$ . Since we have  $H_{\text{BPS}} = r^2/2l$ , a change of variable shows that (4.1) is given by

$$Z_{\text{classical}}(\beta) = \frac{l^3}{\beta^3} \int_X e^{-r^2/2} \frac{\omega^3}{3!} . \quad (4.2)$$

The integrand in this last expression is then easily written in polar coordinates. A simple calculation gives

$$Z_{\text{classical}}(\beta) = \frac{8l^3}{\beta^3} \text{vol}[L] \quad (4.3)$$

where  $\text{vol}[L]$  is the volume of the link  $(L, g_L)$  [16]. In fact the formula (4.2) for the volume of  $(L, g_L)$  was crucial in [16]. By writing the volume of the link in terms of a classical partition function, one can make contact with the formula of Duistermaat and Heckman [17]. This localises the integral on the fixed point set of  $\xi$ , being the Hamiltonian vector field for the Hamiltonian  $r^2/2$ . Since  $\xi$  vanishes only at  $r = 0$ , the integral effectively localises at the tip of the cone. One gets a useful formula only by taking an appropriate (partial) resolution of the cone  $X$ . Any such resolution will suffice, and the localisation formula expresses the volume, and hence the classical partition function, in terms of certain equivariant holomorphic (topological) invariants. The reader is referred to [16] for further details, which also contains a number of detailed examples.

The quantum (canonical) partition function is equally simple to define. This time one takes a trace of the operator  $\exp(-\beta\hat{H})$  over the Hilbert space, rather than integrating over the classical phase space:

$$Z_{\text{quantum}}(\beta) = \text{Tr}_{\mathcal{H}} e^{-\beta\hat{H}} . \quad (4.4)$$

Recall that in section 3, we showed that

$$\beta\hat{H} = \frac{\beta}{l} \mathcal{L}_{r\partial/\partial r} \quad (4.5)$$

when acting on holomorphic functions. Thus we see that the quantum partition function is precisely<sup>7</sup> what was defined as the *holomorphic partition function* in [18], and is essentially the character introduced in [16]. The variables are related by  $t = \beta/l$ . Holomorphic functions on  $X$  of charge  $\lambda$  under  $\xi$ , or equivalently degree  $\lambda$  under  $r\partial/\partial r$ , give rise to eigenfunctions of the scalar Laplacian  $\nabla_L^2$  on the link  $L$ . Writing

$$f = r^\lambda \tilde{f} \quad (4.6)$$

---

<sup>7</sup>This is not quite obvious, since in [18] the partition function was defined as a trace over all holomorphic functions on  $X$ , whereas here the Hilbert space  $\mathcal{H}$  is the space of bounded holomorphic functions, with inner product (1.8). The traces are nevertheless equal. To see this, let  $f$  be a holomorphic function with eigenvalue  $\lambda > 0$  under  $l\hat{H}$ . Then  $f = r^\lambda \tilde{f}$  where  $\tilde{f}$  is a function on  $L$ . The  $r$  integral in the square norm  $\|f\|^2$  of  $f$  is then finite, since the exponential dominates any monomial in  $r$ . The remaining integral over  $L$  is then bounded, since any continuous function on a compact space is bounded.

we have

$$-\nabla_L^2 \tilde{f} = \lambda(\lambda + 4)\tilde{f} . \quad (4.7)$$

Thus the quantum partition function is a holomorphic analogue of the usual partition function of a Riemannian manifold  $(L, g_L)$ , arising from the spectrum of the scalar Laplacian.

As discussed in [18], the relation to the character<sup>8</sup> of [16] can be seen as follows. Given a holomorphic  $(\mathbb{C}^*)^s$  action on  $X$ , we may define the character<sup>9</sup>

$$C(\mathbf{q}, X) = \text{Tr } \mathbf{q} \quad (4.8)$$

as the trace of the action of  $\mathbf{q} \in (\mathbb{C}^*)^s$  on the holomorphic functions on  $X$ . Holomorphic functions on  $X$  that are eigenvectors of the induced  $(\mathbb{C}^*)^s$  action with eigenvalue  $\mathbf{q}^{\mathbf{m}} = \prod_{i=1}^s q_i^{m_i}$  form a vector space over  $\mathbb{C}$  of dimension  $d_{\mathbf{m}}$ . Each eigenvalue then contributes<sup>10</sup>  $d_{\mathbf{m}}\mathbf{q}^{\mathbf{m}}$  to the trace (4.8):

$$C(\mathbf{q}, X) = \sum_{\mathbf{m}} d_{\mathbf{m}}\mathbf{q}^{\mathbf{m}} . \quad (4.9)$$

Letting  $\zeta^i$  be a basis for the Lie algebra of  $U(1)^s \subset (\mathbb{C}^*)^s$ , and writing the Reeb vector field as

$$\xi = \sum_{i=1}^s b_i \zeta^i , \quad (4.10)$$

the eigenvalue of  $f$  may be written as

$$\lambda_{\mathbf{m}} = \sum_{i=1}^s b_i m_i , \quad (4.11)$$

thus<sup>11</sup>

$$\text{Tr}_{\mathcal{H}} e^{-\beta \hat{H}} = \sum_{\mathbf{m}} d_{\mathbf{m}} e^{-\beta \lambda_{\mathbf{m}}/l} = C(\exp(-\beta \mathbf{b}/l), X) . \quad (4.12)$$

It is well known that the volume of  $(L, g_L)$  arises as the coefficient of a pole in the partition function, which is also the trace of the heat kernel [33]. One of the results of [16] may then be considered the holomorphic Sasakian analogue of this, namely that

$$Z_{\text{quantum}}(\beta) = \frac{1}{(2\pi)^3} Z_{\text{classical}}(\beta) + \mathcal{O}(1/\beta^2) \quad (4.13)$$

<sup>8</sup>See [32] for computations of the character in a large number of non-toric examples.

<sup>9</sup>In the following, as in [16], we will not be concerned where sums such as this converge. Similar remarks apply later to the grand canonical partition function.

<sup>10</sup>Note that we include the contribution  $\mathbf{m} = \mathbf{0}$ ,  $d_{\mathbf{0}} = 1$ , coming from the constant functions.

<sup>11</sup>cf the previous footnote. Since, by assumption,  $\xi$  is in the Lie algebra of  $U(1)^s$ , any holomorphic function contributing to the trace in the character in [16] will have bounded norm.

as  $\beta \rightarrow 0$ . This means that the classical and quantum partition functions coincide (1.12) to leading order in  $\beta$  as  $\beta \rightarrow 0$ , a familiar result in statistical mechanics.

## 4.2 Entropy minimisation

The classical entropy associated to a single BPS dual giant graviton is given by the standard formula

$$S = \frac{\partial}{\partial T} (T \log Z_{\text{classical}}) \quad (4.14)$$

where recall  $T = 1/\beta$  is the equilibrium temperature. Since  $Z_{\text{classical}}$  is homogeneous degree three in  $T$ , we immediately deduce that

$$\exp(S) = e^3 T^3 Z_{\text{classical}}(T = 1) = (2leT)^3 \text{vol}[L, g_L], \quad (4.15)$$

where we regard  $S$  as a function of the Reeb vector field. The main result of [16] was that Sasaki–Einstein metrics minimise the volume in the space of Sasakian metrics on  $L$  satisfying

$$\mathcal{L}_\xi \Omega = 3i\Omega \quad (4.16)$$

where  $\Omega$  is the (fixed) holomorphic  $(3,0)$ -form on  $X$ . This is dual to fixing the R-charge of the superpotential in the dual superconformal field theory to be 2. Thus the entropy function of BPS dual giant gravitons is minimised on backgrounds that satisfy the type IIB equations of motion.

## 5 Counting BPS states

In this final section we introduce a grand canonical partition function that counts multiple BPS dual giant gravitons. This may be used to count scalar chiral primaries of the dual superconformal field theory, and is related to the character of [16].

### 5.1 Grand canonical partition function

A single BPS dual giant graviton has Hilbert space  $\mathcal{H}$ . A wavefunction is described by a holomorphic function on the Kähler cone  $X$ . It is then straightforward to consider states consisting of  $n$  BPS dual giant gravitons. Since these are mutually BPS, the total energy is just the sum of the individual energies. These multi states are thus

effectively described by  $n$  indistinguishable particles, each with Hilbert space  $\mathcal{H}$ . The  $n$ -particle Hilbert space is hence the symmetric tensor product  $\mathcal{H}_n = \text{Sym}^n \mathcal{H}$ .

Suppose now that  $(X, g_X)$  admits some number  $s$  of commuting holomorphic Killing vector fields, generated by vector fields  $J_i, i = 1, \dots, s$ . Of course, all geometries admit at least the Reeb vector field  $\xi$  as such a symmetry. We may assume these generate a  $U(1)^s$  isometry. As discussed in section 3, these symmetries give rise to an  $s$ -tuple of commuting operators

$$\hat{\mathbf{P}} = -i\mathcal{L}_{\mathbf{J}} \quad (5.1)$$

acting on  $\mathcal{H}$  which quantise the Hamiltonian functions canonically conjugate to the corresponding cyclic coordinates in the D3-brane dynamics. It is then natural to construct a grand canonical partition function that counts multiple BPS dual giant gravitons, weighted by these charges, as the trace of  $\mathbf{q}^{\hat{\mathbf{P}}}$  over the multi-particle Hilbert space<sup>12</sup>

$$\mathcal{H}_{\text{multi}} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad (5.2)$$

namely:

$$\mathcal{Z}(\zeta, \mathbf{q}, X) = \text{Tr}_{\mathcal{H}_{\text{multi}}} \zeta^{\hat{N}} \mathbf{q}^{\hat{\mathbf{P}}}, \quad (5.3)$$

where

$$\mathbf{q}^{\hat{\mathbf{P}}} \equiv \prod_{i=1}^s q_i^{\hat{P}^i}. \quad (5.4)$$

Here  $\hat{N}$  is the operator that counts the number of giant gravitons. Thus if  $\Psi \in \mathcal{H}_n$  one has  $\hat{N}\Psi = n\Psi$ .

The usual expression for the grand canonical partition function of a system of indistinguishable bosonic particles is given by

$$\mathcal{Z}(\zeta, \mathbf{q}, X) = \prod_{\mathbf{m}} \frac{1}{(1 - \zeta \mathbf{q}^{\mathbf{m}})^{d_{\mathbf{m}}}} \quad (5.5)$$

where the product is taken over all states with quantum numbers  $\mathbf{m}$ , and  $d_{\mathbf{m}}$  is the degeneracy of states with equal quantum numbers<sup>13</sup>. It is then easy to express  $\mathcal{Z}(\zeta, \mathbf{q}, X)$

<sup>12</sup>We define  $\mathcal{H}_0 = \{1\}$ .

<sup>13</sup>Notice that the infinite product includes the term with  $\mathbf{m} = \mathbf{0}$ ,  $d_{\mathbf{0}} = 1$ .

in terms of the character  $C(\mathbf{q}, X)$ . By taking the logarithm of (5.5) and expanding the terms  $\log(1 - \zeta \mathbf{q}^{\mathbf{m}})$  in a formal power series, we obtain

$$\mathcal{Z}(\zeta, \mathbf{q}, X) = \exp \left[ \sum_{n=1}^{\infty} \frac{\zeta^n}{n} C(\mathbf{q}^n, X) \right]. \quad (5.6)$$

One may then formally expand

$$\mathcal{Z}(\zeta, \mathbf{q}, X) = \sum_{n=0}^{\infty} \zeta^n Z_n(\mathbf{q}, X) \quad (5.7)$$

where  $Z_n(\mathbf{q}, X)$  counts the  $n$ -particle states. In particular, notice that the single-particle partition function

$$Z_1(\mathbf{q}, X) = C(\mathbf{q}, X) \quad (5.8)$$

is precisely the character.

The argument of [21, 22], showing that the number of BPS dual giants is bounded from above by  $N$  (the number of background D3-branes), applies, since it is entirely based on considerations in  $\text{AdS}_5$ . Thus, the physical quantity of interest is a truncation of (5.7) to order  $N$ .

### Toric geometries

In the case  $(L, g_L)$  is toric, the Kähler cone  $X$  is an affine toric variety. The Hamiltonian  $\mathbb{T}^3$  action fibres  $X$  over a conical convex subspace  $\mathcal{C}^*$  of  $\mathbb{R}^3$ , which is the image of the moment map (2.40). Equivalently, we may specify the abelian semi-group

$$\mathcal{S}_{\mathcal{C}} = \mathbb{Z}^3 \cap \mathcal{C}^* \subset \mathbb{R}^3. \quad (5.9)$$

It is a standard result that holomorphic functions on  $X$  are spanned by elements of the semi-group  $\mathbf{m} \in \mathcal{S}_{\mathcal{C}}$ , as we already discussed earlier. An  $n$ -particle BPS dual giant state, which is an eigenstate under the torus action, is then a vector

$$|\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_n\rangle \in \text{Sym}^n \mathcal{S}_{\mathcal{C}}, \quad (5.10)$$

and the grand canonical partition function is hence

$$\mathcal{Z}(\zeta, \mathbf{q}, X) = \prod_{\mathbf{m} \in \mathcal{S}_{\mathcal{C}}} \frac{1}{1 - \zeta \mathbf{q}^{\mathbf{m}}} = 1 + \zeta \sum_{\mathbf{m} \in \mathcal{S}_{\mathcal{C}}} \mathbf{q}^{\mathbf{m}} + \mathcal{O}(\zeta^2). \quad (5.11)$$

Notice that in this case there is precisely one holomorphic function for each set of quantum numbers  $\mathbf{m} \in \mathcal{S}_C$ , thus  $d_{\mathbf{m}} = 1$ .

The simplest example is that of  $S^5$ , discussed in subsection 2.2. The Kähler cone over the round  $S^5$  is of course  $\mathbb{C}^3$ , with its standard Kähler form  $\omega$ . The Cartan subgroup  $U(1)^3 \subset U(3)$  preserves  $\omega$  and the corresponding moment map (2.40) has image  $\mu(\mathbb{C}^3) = (\mathbb{R}_+)^3$ . The semi-group for this affine toric variety is thus

$$\mathcal{S}_C = \mathbb{Z}^3 \cap (\mathbb{R}_+)^3 = (\mathbb{Z}_+)^3 = \{\mathbf{m} = (m_1, m_2, m_3) \mid m_i \geq 0, i = 1, 2, 3\}. \quad (5.12)$$

Equivalently, recalling that  $v_i^a = \delta_i^a$ , the fact that  $m_i \geq 0$  follows from the argument at the end of section 3. Thus the Hilbert space is isomorphic to that of the three-dimensional harmonic oscillator, with  $l\hat{H}$  being the energy operator<sup>14</sup>, and the partition function (5.11) precisely reduces to that in [4]. Alternatively, it is given by (5.6), with

$$C(\mathbf{q}^n, \mathbb{C}^3) = \frac{1}{(1 - q_1^n)(1 - q_2^n)(1 - q_3^n)}. \quad (5.13)$$

## 5.2 Counting BPS operators in the dual SCFT

According to the AdS/CFT dictionary, BPS states in the geometry are dual to BPS operators in the SCFT, with the same quantum numbers. It is then natural to interpret the dual giant gravitons that we have considered in terms of BPS operators of the dual conformal field theory.

Let us first recall how this correspondence works in the prototypical example of  $\mathcal{N} = 4$  super Yang–Mills. Generic (1/8–BPS) single–trace scalar chiral primary operators of the  $\mathcal{N} = 4$  super Yang–Mills theory are of the type

$$\text{Tr}(X^{m_1} Y^{m_2} Z^{m_3}) \quad \Delta = m_1 + m_2 + m_3 = \sum_{i=1}^3 m_i b_i \quad (5.14)$$

where  $X, Y, Z$  are the three complex scalar fields, in the adjoint of  $U(N)$ . In the abelian theory, with  $N = 1$ , these operators are simply monomials in three complex variables  $x, y, z$ , of the type  $x^{m_1} y^{m_2} z^{m_3}$  with  $m_i \geq 0$ , and span the coordinate ring of  $\mathbb{C}^3$ . When  $N > 1$ , and including also multi–trace operators, one obtains monomials in the eigenvalues of the three operators, which span the coordinate ring of the symmetric product  $\text{Sym}^N \mathbb{C}^3$ . In other words, the scalar sector of the chiral ring of the  $\mathcal{N} = 4$  super Yang–Mills theory, for finite  $N$ , is isomorphic to the ring of holomorphic functions on

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<sup>14</sup>Note that the ground state energy is zero.

$\text{Sym}^N \mathbb{C}^3$  (for more details see, *e.g.* [34]). The results of [4] then show that this space arises from quantising the phase spaces of precisely  $N$  non-spinning BPS dual giant gravitons in  $\text{AdS}_5 \times S^5$ . The reason why one considers the Hilbert space for precisely  $N$  dual giants is that these are viewed as excitations of the background  $N$  D3-branes. The  $N$ -particle Hilbert space is the symmetric tensor product  $\mathcal{H}_N = \text{Sym}^N \mathcal{H}$  of the single-particle Hilbert space  $\mathcal{H}$ , and the partition function for  $N$  BPS dual giant gravitons may be obtained as the coefficient  $Z_N(\mathbf{q}, \mathbb{C}^3)$  of  $\zeta^N$  in (5.3). In fact, this partition function agrees precisely with the counting of mesonic scalar<sup>15</sup> chiral primary operators in the gauge theory, obtained using the index of [2].

More generally, if a SCFT arises from the IR limit of  $N$  D3-branes at an isolated singularity  $X$ , then the classical vacuum moduli space should be the symmetric product  $\text{Sym}^N X$ . The Hilbert space of  $N$  BPS dual giants is then spanned by the same set of generators (as a  $\mathbb{C}$ -algebra) as the ring of holomorphic functions on  $\text{Sym}^N X$ . This is the scalar sector of the chiral ring of the dual superconformal field theory. When the singularity  $X$  admits a crepant resolution<sup>16</sup>, one expects to be able to describe the dual SCFT by a quiver gauge theory. In this case, the mesonic scalar chiral primary operators are constructed from closed loops in the quiver, by matrix-multiplying the corresponding bifundamental fields. These gauge-invariant operators then generate the ring of holomorphic functions on the vacuum moduli space. Thus we see that the partition function  $Z_N(\mathbf{q}, X)$  for  $N$  dual giants also counts the mesonic scalar chiral primary operators in the dual SCFT, weighted by their  $U(1)^s$  flavour charges. This is in full agreement with the results of [2]. We emphasize that this partition function is related simply to the character  $C(\mathbf{q}, X)$ , and the latter may be computed using only a minimal amount of geometric information. In particular, one may apply localisation techniques to compute  $C(\mathbf{q}, X)$ , as described in [16].

**Note:** Just as this paper was about to be submitted to the arXiv, we became aware of [35]. Their results overlap with our section 5. Their conclusions are in agreement with ours.

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<sup>15</sup>The full chiral ring in general contains operators with non-zero spin. These are accounted for in the index of [2]. However, throughout this paper, we restrict our attention to spinless configurations.

<sup>16</sup>We note that there are plenty of examples of  $X$ , admitting Ricci-flat Kähler cone metrics, which have no crepant resolution. In this case, the dual SCFTs might in principle be quite exotic, and in particular not be described by quiver gauge theories.

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## A $\kappa$ –symmetry analysis

In this appendix we demonstrate that the dual giant graviton solutions considered in the main text are precisely the set of BPS solutions. That is, they are the general set of solutions of the D3–brane dynamics respecting  $\kappa$ –symmetry of the world–volume action. One must thus impose that the Killing spinor  $\epsilon$  of type IIB supergravity,

$$\nabla_M \epsilon + \frac{i}{192} F_{MP_1P_2P_3P_4} \Gamma^{P_1P_2P_3P_4} \epsilon = 0 , \quad (\text{A.1})$$

in the background of  $\text{AdS}_5 \times L$ , also satisfies the  $\kappa$ –symmetry projection

$$\Gamma_\kappa \epsilon = i \epsilon , \quad (\text{A.2})$$

where the  $\kappa$ –symmetry projection matrix is defined as

$$\Gamma_\kappa = \frac{1}{4! \sqrt{-\det G_{\mu\nu}}} \epsilon^{\mu\nu\rho\sigma} \gamma_{\mu\nu\rho\sigma} . \quad (\text{A.3})$$

The  $\gamma$ –matrices above are the world–volume gamma matrices, defined as

$$\gamma_\mu = \frac{\partial X^M}{\partial \zeta^\mu} e_M^{\hat{M}} \Gamma_{\hat{M}} , \quad (\text{A.4})$$

where  $e_M^{\hat{M}}$  is a vielbein, that is, a local orthonormal frame, and  $\Gamma_{\hat{M}}$  are ten–dimensional flat spacetime gamma matrices, obeying

$$\{\Gamma_{\hat{M}}, \Gamma_{\hat{N}}\} = 2\eta_{\hat{M}\hat{N}} . \quad (\text{A.5})$$

We find

$$\begin{aligned} \gamma_0 &= V(R)^{1/2} \Gamma_{\hat{0}} + \frac{\dot{R}}{V(R)^{1/2}} \Gamma_{\hat{1}} + l \left( (\dot{\psi} + \sigma_\alpha \dot{x}^\alpha) \Gamma_{\hat{5}} + \dot{x}^\alpha \hat{e}_\alpha^{\hat{a}} \Gamma_{\hat{a}+5} \right) \\ \gamma_1 &= R \Gamma_{\hat{2}} \quad \gamma_2 = R \cos \zeta^1 \Gamma_{\hat{3}} \quad \gamma_3 = R \sin \zeta^1 \Gamma_{\hat{4}} , \end{aligned} \quad (\text{A.6})$$

where  $\hat{e}_\alpha^{\hat{\alpha}} = e_{\alpha+5}^{\hat{\alpha}+5}$ ,  $\alpha = 1, \dots, 4$ , is a vielbein such that  $h_{\alpha\beta} = \hat{e}_\alpha^{\hat{\alpha}} \hat{e}_\beta^{\hat{\beta}} \delta_{\hat{\alpha}\hat{\beta}}$ . The  $\kappa$ -symmetry projector is then

$$\Gamma_\kappa = \Delta^{-1/2} \left[ V(R)^{1/2} \Gamma_{\hat{0}} + \frac{\dot{R}}{V(R)^{1/2}} \Gamma_{\hat{1}} + l \left( (\dot{\psi} + \sigma_\alpha \dot{x}^\alpha) \Gamma_{\hat{5}} + \dot{x}^\alpha \hat{e}_\alpha^{\hat{\alpha}} \Gamma_{\hat{\alpha}+5} \right) \right] \Gamma_{\hat{2}\hat{3}\hat{4}} . \quad (\text{A.7})$$

In the following we adopt the conventions of [36]. Using the spinor ansatz

$$\epsilon = \Psi \otimes \chi \otimes \theta \quad (\text{A.8})$$

where  $\epsilon$  is the complexified type IIB spinorial parameter obeying, in our conventions, the following chirality projection

$$\Gamma_{11} \epsilon = -\epsilon , \quad (\text{A.9})$$

(A.1) reduces to the two equations

$$\nabla_m \Psi + \frac{1}{2l} \rho_m \Psi = 0 , \quad m = 0, \dots, 4 \quad (\text{A.10})$$

$$\nabla_\alpha \chi - \frac{i}{2l} \gamma_\alpha \chi = 0 , \quad \alpha = 0, \dots, 4 , \quad (\text{A.11})$$

where we now take the  $\alpha$  index to run from 0 to 4. These may be recognised as the standard equations obeyed by any Killing spinor  $\Psi$  of  $\text{AdS}_5$  (of radius  $l$ ) and the Killing spinor  $\chi$  of an arbitrary Sasaki–Einstein manifold with line element  $l^2 ds_L^2$  and corresponding Ricci tensor  $\text{Ric} = 4l^2 g_L$ . Notice that, in the notation of [36],  $f = -4/l$  is the overall constant factor multiplying the Ramond–Ramond five–form. The warp factor has been set to zero. We have also chosen the following decomposition of the ten–dimensional Dirac matrices:

$$\begin{aligned} \Gamma_{\hat{m}} &= \rho_{\hat{m}} \otimes \mathbf{1} \otimes \sigma^3 , & \hat{m} &= 0, \dots, 4 \\ \Gamma_{\hat{\alpha}+5} &= \mathbf{1} \otimes \gamma_{\hat{\alpha}} \otimes \sigma^1 , & \hat{\alpha} &= 0, \dots, 4 , \end{aligned} \quad (\text{A.12})$$

where

$$\{\rho_{\hat{m}}, \rho_{\hat{n}}\} = 2\eta_{\hat{m}\hat{n}} , \quad \{\gamma_{\hat{\alpha}}, \gamma_{\hat{\beta}}\} = 2\delta_{\hat{\alpha}\hat{\beta}} , \quad (\text{A.13})$$

are (flat) gamma matrices of  $\text{Cliff}(4, 1)$  and  $\text{Cliff}(5, 0)$ , respectively, and  $\sigma^1, \sigma^2, \sigma^3$  are the Pauli matrices.

We may now return to the  $\kappa$ -symmetry projection. First note that  $\gamma_{\hat{\alpha}} \chi$ ,  $\hat{\alpha} = 0, 1, \dots, 4$  are linearly independent spinors, over the *real* numbers, on  $L$ . For, consider the linear combination

$$\sum_{\hat{\alpha}=0}^4 a_{\hat{\alpha}} \gamma_{\hat{\alpha}} \chi = 0 \quad (\text{A.14})$$

with  $a_{\hat{\alpha}} \in \mathbb{R}$ . One may now apply  $\bar{\chi}\gamma_{\hat{1}}$  on the left to obtain

$$a_{\hat{1}} + a_{\hat{2}}\bar{\chi}\gamma_{\hat{1}}\gamma_{\hat{2}}\chi = 0 . \quad (\text{A.15})$$

Here we have used  $\bar{\chi}\chi = 1$ , together with the fact that

$$-\frac{i}{2}\bar{\chi}\gamma_{\alpha\beta}\chi dx^\alpha \wedge dx^\beta = \omega_T = \hat{e}^{\hat{1}} \wedge \hat{e}^{\hat{2}} + \hat{e}^{\hat{3}} \wedge \hat{e}^{\hat{4}} \quad (\text{A.16})$$

is the transverse Kähler form. Also note that the Killing spinor  $\chi$  obeys the projection (see *e.g.* [37])

$$\gamma_{\hat{0}}\chi = \pm\chi , \quad (\text{A.17})$$

and  $\bar{\chi}\gamma_{\hat{\alpha}}\chi = 0$  for  $\hat{\alpha} = 1, \dots, 4$ . In particular, the second term in (A.15) is pure imaginary, and this immediately gives  $a_{\hat{1}} = a_{\hat{2}} = 0$ . A similar argument gives  $a_{\hat{3}} = a_{\hat{4}} = 0$  and thus we also have  $a_{\hat{0}} = 0$ . Using these facts we see that the  $\kappa$ -symmetry condition (A.2) implies that

$$\dot{x}^\alpha = 0 , \quad \alpha = 1, \dots, 4 . \quad (\text{A.18})$$

Also note that, since  $\psi$  is a cyclic coordinate ( $\partial/\partial\psi$  is a Killing vector field), the conjugate momentum is

$$P_\psi = k , \quad (\text{A.19})$$

a constant. Using the expression for  $P_\psi$  in terms of  $\dot{\psi}$ , we have

$$\dot{\psi} = \frac{l^2 k \Delta^{1/2}}{NR^3} \quad (\text{A.20})$$

with

$$\Delta = \frac{V(R) - \frac{\dot{R}^2}{V(R)}}{1 + \frac{l^6 k^2}{N^2 R^6}} . \quad (\text{A.21})$$

Inserting this into  $\Gamma_\kappa$  we obtain

$$\Gamma_\kappa = \Delta^{-1/2} \left[ V(R)^{1/2} \Gamma_{\hat{0}} + \frac{\dot{R}}{V(R)^{1/2}} \Gamma_{\hat{1}} + \frac{l^3 k \Delta^{1/2}}{NR^3} \Gamma_{\hat{5}} \right] \Gamma_{\hat{2}\hat{3}\hat{4}} . \quad (\text{A.22})$$

It is convenient to define

$$\frac{R}{l} \equiv \sinh \alpha , \quad (\text{A.23})$$

so that the projection (A.2) becomes

$$\begin{aligned} & \left[ \cosh \alpha \rho_0 \otimes \mathbf{1} \otimes \mathbf{1} + \frac{\dot{R}}{\cosh \alpha} \rho_1 \otimes \mathbf{1} \otimes \mathbf{1} + \frac{l^3 k \Delta^{1/2}}{NR^3} \mathbf{1} \otimes \gamma_0 \otimes i\sigma^2 \right] \epsilon = \\ & = \Delta^{1/2} i\rho_{2\hat{3}\hat{4}} \otimes \mathbf{1} \otimes \mathbf{1} \epsilon . \end{aligned} \quad (\text{A.24})$$

Finally, imposing the chirality condition (A.9),  $\sigma^2\theta = -\theta$ , after a little algebra, the projection (A.24) reduces to

$$\Lambda\Psi \equiv \left[ \cosh \alpha \mathbf{1} - \frac{\dot{R}}{\cosh \alpha} \rho_0 \rho_1 + \Delta^{1/2} \rho_1 \pm \frac{l^3 k \Delta^{1/2}}{NR^3} i\rho_0 \right] \Psi = 0 . \quad (\text{A.25})$$

To proceed, we shall need an explicit form of the Killing spinor in AdS space. A useful expression, that may be adapted for instance from [4], reads, in our notation:

$$\Psi = e^{-\frac{\alpha}{2}\rho_1} e^{-\frac{t}{2l}\rho_0} e^{\frac{\theta}{2}\rho_{1\hat{2}}} e^{\frac{\phi_1}{2}\rho_{1\hat{3}}} e^{\frac{\phi_2}{2}\rho_{2\hat{4}}} \Psi_0 \equiv D\Psi_0 , \quad (\text{A.26})$$

where  $\Psi_0$  is a constant spinor. In order to compute  $\Lambda$  acting on  $\Psi_0$  we need to commute this through  $D$ . It is useful to record the following identities:

$$\begin{aligned} \rho_0 D &= D \left[ \cosh \alpha \mathbf{1} + \sinh \alpha e^{\frac{t}{l}\rho_0} \left[ \cos \theta e^{-\phi_1 \rho_{1\hat{3}}} \rho_1 + \sin \theta e^{-\phi_2 \rho_{2\hat{4}}} \rho_2 \right] \right] \rho_0 \equiv DA , \\ \rho_1 D &= D e^{\frac{t}{l}\rho_0} \left[ \cos \theta e^{-\phi_1 \rho_{1\hat{3}}} \rho_1 + \sin \theta e^{-\phi_2 \rho_{2\hat{4}}} \rho_2 \right] \equiv DB . \end{aligned} \quad (\text{A.27})$$

We then find

$$\Lambda D\Psi_0 = D \left[ \cosh \alpha \mathbf{1} - \frac{\dot{R}}{\cosh \alpha} AB + \Delta^{1/2} B \pm i \frac{l^3 k \Delta^{1/2}}{NR^3} A \right] \Psi_0 = 0 . \quad (\text{A.28})$$

First, let us restrict to the solutions (2.24) and check that they are indeed BPS. Thus, let us set

$$\dot{R} = 0 \quad k = \frac{NR^2}{l^2} \quad \Delta^{1/2} = \frac{R_0}{l} = \sinh \alpha_0 . \quad (\text{A.29})$$

Equation (A.28) reduces to

$$D \left[ \cosh \alpha_0 \mathbf{1} + \sinh \alpha_0 e^{\frac{t}{l}\rho_0} \left[ \cos \theta e^{-\phi_1 \rho_{1\hat{3}}} \rho_1 + \sin \theta e^{-\phi_2 \rho_{2\hat{4}}} \rho_2 \right] \right] (\mathbf{1} \pm i\rho_0) \Psi_0 = 0 \quad (\text{A.30})$$

and thus we simply require that

$$i\rho_0 \Psi_0 = \mp \Psi_0 . \quad (\text{A.31})$$

This can always be achieved, as can be seen for instance from an explicit expression for the  $\rho_{\hat{0}}$  matrix. One can choose the following basis of Dirac matrices in  $\text{AdS}_5$ <sup>17</sup>

$$\begin{aligned}\rho_{\hat{0}} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbf{1} & \rho_{\hat{1}} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1} \\ \rho_{\hat{2}} &= \mathbf{1} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \rho_{\hat{3}} &= \mathbf{1} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\end{aligned}\tag{A.32}$$

and  $\rho_{\hat{4}} = -i\rho_{\hat{0}\hat{1}\hat{2}\hat{3}}$ . We may then take

$$\Psi_0 = \begin{pmatrix} i \\ \mp 1 \end{pmatrix} \otimes \tilde{\Psi}_0\tag{A.33}$$

The condition (A.31) is the AdS analogue of (A.17) for Sasaki–Einstein manifolds.

Finally, let us show that in fact the solutions we considered are the set of all BPS solutions. The projections must hold at any point of the world–volume of the D3–brane, thus we may simplify the calculation by conveniently setting  $t = \theta = \phi_1 = \phi_2 = 0$ . Then

$$\begin{aligned}A &= (\cosh \alpha \mathbf{1} + \sinh \alpha \rho_{\hat{1}}) \rho_{\hat{0}} \\ B &= \rho_{\hat{1}}.\end{aligned}\tag{A.34}$$

Next, we choose a spinor that obeys the projection (A.31). Thus, in our particular basis, we may choose (A.33). It follows that

$$\bar{\Psi}_0 \Psi_0 = 2 \|\tilde{\Psi}_0\|^2 \quad \bar{\Psi}_0 \rho_{\hat{1}} \Psi_0 = 0\tag{A.35}$$

and in particular, applying  $\bar{\Psi}_0 D^{-1}$  to the left of equation (A.28), we obtain

$$\cosh \alpha \pm i \dot{R} \tanh \alpha - \frac{l^3 k \Delta^{1/2}}{NR^3} \cosh \alpha = 0.\tag{A.36}$$

Thus we conclude that necessarily

$$\dot{R} = 0 \quad k = \frac{R^2 N}{l^2},\tag{A.37}$$

while the remaining components of (A.28) proportional to  $\rho_{\hat{1}}$  are automatically satisfied. This concludes our proof that (2.24) are all the  $\kappa$ –symmetric solutions to the D3–brane equations of motion.

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<sup>17</sup>The construction is standard, see *e.g.* [38].

## B A change of coordinates

In this appendix we give an explicit change of coordinates between Sasakian coordinates and symplectic coordinates, in the case that the Kähler cone  $X$  is toric.

When the cone is toric, the metric may be written as either

$$ds_X^2 = G_{ij}dy^i dy^j + G^{ij}d\phi_i d\phi_j \quad i, j = 1, 2, 3 \quad (\text{B.1})$$

or as

$$ds_X^2 = dr^2 + r^2(d\psi + \sigma)^2 + r^2 h_{\alpha\beta} dx^\alpha dx^\beta \quad \alpha, \beta = 1, 2, 3, 4 \quad (\text{B.2})$$

where

$$\begin{aligned} r^2 &= 2b_i y^i \\ h_{\alpha\beta} dx^\alpha dx^\beta &= H_{pq} d\eta^p d\eta^q + H^{pq} d\varphi_p d\varphi_q \quad p, q = 1, 2 \\ \sigma &= 2\eta^p d\varphi_p \end{aligned} \quad (\text{B.3})$$

and we have used that, locally, the transverse Kähler metric is also toric. Now define

$$\tilde{\varphi}_i = (\psi, \varphi_1, \varphi_2)_i \quad (\text{B.4})$$

and suppose that the angular variables in the above metrics are linearly related

$$\phi_i = A_i^j \tilde{\varphi}_j . \quad (\text{B.5})$$

This allows one to determine the matrix  $G^{ij}$  by direct comparison. Inverting this, we see that

$$G = AMA^T \quad (\text{B.6})$$

with

$$M = \frac{1}{r^2} \begin{pmatrix} 1 + 4H_{rs}\eta^r\eta^s & -2H_{rq}\eta^r \\ -2H_{pr}\eta^r & H_{pq} \end{pmatrix} . \quad (\text{B.7})$$

It may then be verified that the following is the change of coordinates for the non-angular part of the metric

$$y^i = r^2 (A^{-T})^i_j w^j \quad w^j = \left( \frac{1}{2}, \eta^1, \eta^2 \right)^j . \quad (\text{B.8})$$

There is a consistency condition that the matrix  $A$  must satisfy. In particular,

$$A_i^{-1} y^i = \frac{1}{2} r^2 \tag{B.9}$$

implies that  $A_i^{-1} = b_i$ . Thus we must set

$$A = \begin{pmatrix} b_1 & * & * \\ b_2 & * & * \\ b_3 & * & * \end{pmatrix} \tag{B.10}$$

with the entries  $*$  arbitrary, provided that  $A$  is invertible. Note that the Reeb vector transforms as

$$\frac{\partial}{\partial \psi} = \frac{\partial \phi_i}{\partial \psi} \frac{\partial}{\partial \phi_i} = b_i \frac{\partial}{\partial \phi_i} . \tag{B.11}$$

Finally, using the fact that all the transverse Kähler coordinates are constants for BPS motion (*cf* (2.24)), we see from (B.5) that

$$\dot{\phi}_i = b_i \dot{\psi} . \tag{B.12}$$

This may be used to obtain a quick derivation of the BPS solutions for toric geometries, discussed in the main text.

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