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## THREE DIMENSIONAL FRACTAL ATTRACTORS IN A GREEN TRANSITION ECONOMIC GROWTH MODEL

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### Abstract

We analyze a two-sector stochastic economic growth model of green transition with pollution externalities and foreign capital. The final good is produced by combining dirty and clean inputs, with different implications on pollution accumulation. Pollution negatively affects production capabilities and can be reduced by switching to the clean input. The clean input is produced by using the dirty input and the foreign capital received (in the form of dirty input). Random shocks make the effective economy's ability to transition to green activities highly uncertain, eventually undermining its economic development. Such a setting gives rise to a dynamic system represented by a three dimensional affine iterated function system. We show that the economy's steady state is represented by an invariant measure supported on a compact set, characterizing its fractal nature and showing that its attractor may be a distorted Sierpiński tetrahedron.

**Keywords**: Development Aids, Economic Growth, Fractal Attractors, Green Transition, Pollution
**JEL Classification**: C61, O41, Q56

## 1 Introduction

Since Boldrin and Montrucchio's seminal work (1986), the analysis of complexity and chaos in macroeconomic dynamics has received ever growing interest (Montrucchio, 1994; Nishimura and Yano, 1995; Brock and Hommes, 1997). A large share of the literature has focused on stochastic economic growth models showing that they may give rise to nontrivial dynamics eventually converging to invariant measures supported on fractal sets, characterizing the properties of such invariant measures in terms of singularity and absolute continuity along with those of the fractal features of the steady states (Montrucchio and Privileggi, 1999; Mitra et al., 2003; Mitra and Privileggi, 2004, 2006, 2009). Most works focus on traditional one- or two-sector growth models in which economic dynamics are entirely described by (physical and human) capital accumulation and can be represented by a one or two-dimensional affine iterated function systems (IFS) converging to singular measures supported on either the Cantor set or the Sierpiński gasket (La Torre et al., 2011; Marsiglio, 2012; La Torre et al., 2015, 2018b). Very few works extend the traditional growth framework to analyze also environmental and

sustainability problems by accounting for pollution accumulation, showing that the support of the invariant measure could even resemble the Barnsley's fern (Privileggi and Marsiglio, 2013; La Torre et al., 2018a). Due to the growing importance of environmental considerations for the future prospects of economic development, in this paper we wish to contribute to this latter branch of the literature by analyzing a stochastic growth model of green transition in a developing country to explore the fractal properties of the steady state of more sophisticated economic growth models with environmental feedback effects.

Since polluting emissions in developing countries are expected to increase by more than 50% in the next decades (Clarke et al., 2009; IEA, 2010), in order to ensure the sustainability of the world economy it is essential that industrialized countries and international organizations promote and support a process of green transition in poor countries. For example, in recent climate change negotiations industrialized countries have committed to mobilize a substantial amount of resources to address the needs of developing countries and favor mitigation and adaptation policies through the means of development aid and foreign direct investments. However, despite the rhetoric, reality is much different. The development aid effectively in place in developing countries are still to a large extent devoted to projects financing traditional economic activities, such as transportation, building infrastructure and mining, and only a very limited part is employed in green activities, such as favoring access to clean water, biodiversity, $CO_2$ reduction and soil conservation (Hicks et al., 2008). The effects of foreign direct investment are even more problematic: as extensively discussed in the pollution haven literature, regulation and distorted incentives in industrialized countries may push local firms to relocate their activities in poor economies exporting dirty technologies and increasing emissions in such countries (Mani and Wheeler, 1998; Copeland and Taylor, 2004). Therefore, future development prospects in developing countries are highly uncertain, and as a consequence so are those of the entire world economy. In order to consider the effects of development aid and foreign direct investments on the economic and environmental performance of developing countries we extend traditional growth models to account for foreign capital.

Specifically, we focus on a developing country and its transition towards green activities, and in particular on the effects of the foreign capital it receives, inclusive of development aid and foreign direct investments, on such a transitional process. We consider a two-sector economy in which one sector produces the final consumption good while the other a clean production factor. The final good is produced by combining two perfectly substitutable inputs, a dirty and a clean input, which differ only in their environmental implications: while the dirty input generates polluting emissions deteriorating environmental quality, the clean input is completely emissions-free. In particular, the use of the dirty input in the production process contributes to accumulate pollution, which in turn negatively affects the economy's ability to produce the final good through a production externality. However, the dirty input can also be used to produce the clean input which by being totally environmental friendly does not lead to undesirable side-effects on production capabilities. Therefore, a share of the dirty input, along with a share of the foreign capital received (in the form of dirty input), is allocated to produce the clean input lowering thus environmental footprints. The amount of foreign capital received, its productivity in the production of the clean input, and its environmental efficiency are subject to random shocks due to political, technological and environmental issues. The existence of such shocks make the effective economy's ability to transition to green activities highly uncertain, eventually undermining its entire economic development. Such a setting gives rise to a dynamic system which can be described by the means of a three dimensional affine IFS, which by borrowing from the mathematics literature (see, among others, Hutchinson, 1981; Barnsley and Demko, 1985; Barnsley et al., 1986; Vrscay, 1991; Barnsley, 1993; Diaconis and Freedman, 1999; Mendivil

and Vrscay, 2002a, 2002b; Ngai and Wang, 2005; La Torre et al., 2006; Kunze et al. 2007; Niu and Xi, 2007; Barnsley et al., 2008; La Torre and Vrscay, 2009; La Torre et al., 2009; Kunze et al., 2012), can be analyzed to characterize its steady state outcome along with its fractal nature. We show that, under a suitable parameter configuration, the attractor of such a three dimensional IFS is a distorted copy of the Sierpiński tetrahedron and we establish a novel mathematical result that simplifies the process of establishing whether the associated invariant measure is singular. To the best of our knowledge, no other paper has thus far explored the possibility that the fractal attractor of economic growth models may have a three dimensional representation.

The paper proceeds as follows. Section 2 introduces our framework which consists of a two-sector stochastic growth model of green transition with pollution externalities and foreign capital, showing that the economy's steady state is represented by an invariant measure supported on a compact set. Section 3 briefly reviews the theory on IFS which we need in our analysis. Section 4 recalls the characteristics of the classical Sierpiński tetrahedron. Section 5 shows that, under some conditions on parameters, the fractal attractor of our dynamic system may resemble a (possibly highly) distorted of copy of the classical Sierpiński tetrahedron. Section 6 presents concluding remarks and proposes directions for future research. Technicalities are postponed to the Appendix.

# 2   The Model

We analyze a two-sector model of economic growth with environmental feedback effects, and for the sake of simplicity we consider a purely dynamic setting abstracting completely from agents' optimization. The unique final consumption good, $y_t$, is produced through a linear production function combining two perfectly substitutable inputs, a dirty input (i.e., capital or fossil fuels), $k_t$, and a clean input (i.e., renewable energy), $g_t$. A certain share $0 < u < 1$ of the dirty input is invested to produce the clean input, thus only the remaining share $1 - u$ is devoted to the production of the consumable good. Therefore, the final output is produced according to the following technology: $y_t = a(1 - u)k_t + bg_t$, where $a > 0$ and $b > 0$ measure the productivity of the two factors. The use of the dirty input in production activities generates emissions which increase pollution, $p_t$, which in turn reduces production by a random factor $\beta_t \geq 0$ via an additive productivity shock. Net (of pollution externality) output is thus given by: $\tilde{y}_t = y_t - \beta_t p_t$. The clean input is produced according to a linear production technology employing only a given share of the dirty input as follows: $x_t = duk_t$, where $x_t$ is the newly produced quantity and $d > 0$ quantifies the productivity in the clean input sector. Foreign capital, $f_t$, inclusive of development aid and foreign direct investments, is received through a transfer of the dirty input from industrialized countries. A share $0 < v < 1$ of such foreign (dirty) input is devoted to the accumulation of the dirty input and the remaining share $1 - v$ to produce the clean input with random productivity $e_t \geq 0$.

The dirty input accumulates due to saving and foreign capital while is reduced by depreciation as follows: $k_{t+1} = s\tilde{y}_t + vf_t + (1 - \delta_k)k_t$, where $0 < s < 1$ is the saving rate, $\tilde{y}_t$ net output, and $\delta_k > 0$ the dirty input depreciation rate. Similarly, the clean input accumulates due to the devoted investment and foreign capital while it is reduced by depreciation as follows: $g_{t+1} = x_t + (1 - v)e_t f_t + (1 - \delta_g)g_t$, where $\delta_g > 0$ is the clean input depreciation rate. Pollution increases with the emissions associated with the use of dirty input and foreign capital in the production process, and decreases due to the ability of the natural ecosystem to absorb pollution as follows: $p_{t+1} = \gamma_t k_t + \mu_t f_t + (1 - \eta)p_t$, where $\gamma_t \geq 0$ and $\mu_t \geq 0$ quantify the random environmental inefficiency associated with the use of the dirty input and foreign capital respec-

tively, and $\eta > 0$ is the natural decay rate of pollution. Therefore, given the initial conditions $k_0$, $g_0$ and $p_0$, our model economy can be described by the following system of three difference equations:

$$\begin{cases} k_{t+1} = s\left[a\left(1-u\right)k_t + bg_t - \beta_t p_t\right] + \left(1-\delta_k\right)k_t + vf_t \\ g_{t+1} = duk_t + \left(1-\delta_g\right)g_t + \left(1-v\right)e_t f_t \\ p_{t+1} = \gamma_t k_t + \mu_t f_t + \left(1-\eta\right)p_t \end{cases} \quad (1)$$

Some comments are needed in order to clarify our setting and assumptions. (i) The linearity of the output production function in the dirty and clean inputs implies that the two production factors are perfectly substitutable, and thus the economy is able to produce a positive amount of the consumable good by employing only one of two inputs. We may think of this as a scenario in which the production of the final good requires electricity, which may be alternatively generated through fossil fuels or renewable sources. (ii) The possibility to produce the clean input by employing only the dirty input builds on this type of interpretation allowing thus our model to describe the transition from fossil fuels to renewable energy. For example, polluting machines can be used to set up photovoltaic plans allowing to produce clean energy which can be used to produce consumable goods reducing polluting emissions. (iii) The assumption that foreign capital takes the form of the dirty input is consistent with empirical evidence regarding the functioning of developing aid and foreign direct investments in the real world (Hicks et al., 2008; Mani and Wheeler, 1998). (iv) The assumptions that pollution negatively affects only the production of the final good and that the clean input is entirely emissions-free are mere simplifying hypotheses. Relaxing them to allow for a detrimental effect of pollution also in the clean sector and for the generation of emissions also with the use of the clean input in the production process will not modify our main conclusions.

To emphasize the affine features of system (1) it is convenient to isolate all three variables, $k_t$, $g_t$ and $p_t$, from their coefficients and from the additive constants and rewrite it as follows:

$$\begin{cases} k_{t+1} = s\left[a\left(1-u\right)+1-\delta_k\right]k_t + sbg_t - s\beta_t p_t + vf_t \\ g_{t+1} = duk_t + \left(1-\delta_g\right)g_t + \left(1-v\right)e_t f_t \\ p_{t+1} = \gamma_t k_t + \left(1-\eta\right)p_t + \mu_t f_t. \end{cases} \quad (2)$$

We assume that some of the parameters in (2) represent random exogenous shocks, so that they can have different values according to different realizations of such shocks. Specifically, we shall assume that the economy is affected by $N$ exogenous shocks determining the values of some of the parameters $f_t$, $e_t$, $\mu_t$, $\beta_t$ and $\gamma_t$. The stochastic process generating the sequence of shocks will be assumed to be i.i.d. so that each realization will occur with a constant probability $p_i$ satisfying $0 < p_i < 1$, for $i = 1, \ldots, N$, and such that $\sum_{i=1}^{N} p_i = 1$ (see La Torre et al., 2019, for a generalization in which probabilities are state dependent). Hence, the parameter vector $(f_t, e_t, \mu_t, \beta_t, \gamma_t)$ will take on $N$ values $(f_i, e_i, \mu_i, \beta_i, \gamma_i)$, for $i = 1, \ldots, N$, independently on time $t$. Under these assumptions the dynamics described by (2) represent what is often referred to as an iterated function system with probabilities (IFSP) and turn out to be more conveniently handled in matrix form according to:

$$\begin{bmatrix} k_{t+1} \\ g_{t+1} \\ p_{t+1} \end{bmatrix} = \begin{bmatrix} a_i^{kk} & a_i^{kg} & a_i^{kp} \\ a_i^{gk} & a_i^{gg} & a_i^{gp} \\ a_i^{pk} & a_i^{pg} & a_i^{pp} \end{bmatrix} \begin{bmatrix} k_t \\ g_t \\ p_t \end{bmatrix} + \begin{bmatrix} z_i^k \\ z_i^g \\ z_i^p \end{bmatrix} \quad \text{for } i = 1, \ldots, N, \quad (3)$$

where the matrix's coefficients are $a_i^{kk} \equiv a^{kk} = s\left[a\left(1-u\right)+1-\delta_k\right]$, $a_i^{kg} \equiv a^{kg} = sb$, $a_i^{kp} = -s\beta_i$, $a_i^{gk} \equiv a^{gk} = du$, $a_i^{gg} \equiv a^{gg} = \left(1-\delta_g\right)$, $a_i^{gp} \equiv a^{gp} = 0$, $a_i^{pk} = \gamma_i$, $a_i^{pg} \equiv a^{pg} = 0$ and $a_i^{pp} \equiv a^{pp} = \left(1-\eta\right)$, while the additive constants are $z_i^k = vf_i$, $z_i^g = \left(1-v\right)e_i f_i$ and $z_i^p = \mu_i f_i$.

When the dynamics is stochastic and evolve according to a law of the type defined in (3) the long-run equilibrium steady state ceases to be a fixed point in the common sense and becomes an invariant measure supported on some compact set called attractor (Mitra et al., 2003; La Torre et al., 2015).

# 3 Mathematical Preliminaries

We now briefly review some well known results in the IFSP literature, and specifically we discuss the main definitions and results on IFSP, their attractor, the invariant measure supported on it and its singularity vs. absolute continuity. A huge literature is available on such topics; the reader is referred to, among others, Hutchinson (1981), Barnsley and Demko (1985), Barnsley et al. (1986), Vrscay (1991), Barnsley (1993), Diaconis and Freedman (1999), Mendivil and Vrscay (2002a, 2002b), Ngai and Wang (2005), La Torre et al. (2006), Kunze et al. (2007), Niu and Xi (2007), Barnsley et al. (2008), La Torre and Vrscay (2009), La Torre et al. (2009). For a recent comprehensive and detailed treatment see Kunze et al. (2012). We also present an original extension of these basic results (Corollary 3) that will be useful for our subsequent analysis.

## 3.1 Iterated Function Systems

Let $(X, d)$ denote a compact metric space. An $N$-map *Iterated Function System* (IFS) on $X$, $\mathbf{w} = \{w_1, \ldots, w_N\}$, consists of $N$ contraction mappings on $X$, i.e., $w_i : X \to X$, $i = 1, \cdots, N$, with contraction factors $c_i \in [0, 1)$ (see Barnsley, 1993; Hutchinson, 1981; Barnsley et al., 1986; Kunze et al., 2012). Associated with an $N$-map IFS one can define a set-valued mapping $\hat{\mathbf{w}}$ on the space $\mathcal{H}([a, b])$ of nonempty compact subsets of $X$ as follows:

$$\hat{\mathbf{w}}(S) := \bigcup_{i=1}^{N} w_i(S), \qquad S \in \mathcal{H}([a, b]). \tag{4}$$

The following two results state a convergence property of an $N$-map IFS towards its attractor. More properties and results can be found in Barnsley (1993), Hutchinson (1981), and Kunze et al. (2012).

**Theorem 1 (Hutchinson, 1981)** *For $A, B \in \mathcal{H}(X)$,*

$$h(\hat{\mathbf{w}}(A), \hat{\mathbf{w}}(B)) \leq ch(A, B) \qquad where \qquad c = \max_{1 \leq i \leq N} c_i < 1$$

*and $h$ denotes the Hausdorff metric on $\mathcal{H}(X)$.*

**Corollary 1 (Hutchinson, 1981)** *There exists a unique set $A \in \mathcal{H}([a, b])$, the* attractor *of the IFS $\mathbf{w}$, such that*

$$A = \hat{\mathbf{w}}(A) = \bigcup_{i=1}^{N} w_i(A).$$

*Moreover, for any $B \in \mathcal{H}([a, b])$, $h(A, \hat{\mathbf{w}}^t(B)) \to 0$ as $t \to \infty$.*

## 3.2 Iterated Function Systems with Probabilities and their Invariant Measure

An $N$-map *Iterated Function System with Probabilities* (IFSP) $(\mathbf{w},\mathbf{p})$ is an $N$-map IFS $\mathbf{w}$ with associated probabilities $\mathbf{p} = \{p_1, \ldots, p_N\}$ satisfying $0 < p_i < 1$, for $i = 1, \ldots, N$, and such that $\sum_{i=1}^{N} p_i = 1$.

Let $\mathcal{M}(X)$ denote the set of probability measures on (Borel subsets of) $X$ and $d_{MK}$ the Monge-Kantorovich distance on this space: For $\mu, \nu \in \mathcal{M}(X)$, with Monge-Kantorovich metric,

$$d_{MK}(\mu, \nu) = \sup_{f \in Lip_1(X)} \left[ \int f \, d\mu - \int f \, d\nu \right].$$

where $Lip_1(X) = \{f : X \to \mathbb{R} \mid |f(x) - f(y)| \le d(x,y)\}$. The metric space $(\mathcal{M}(X), d_{MK})$ is complete (Barnsley, 1993; Hutchinson, 1981; Kunze et al., 2012).

The *Markov operator* associated with an $N$-map IFSP is a mapping $M : \mathcal{M} \to \mathcal{M}$, defined as follows: For any $\mu \in \mathcal{M}(X)$, and for any measurable set $S \subset X$, define a measure $\nu = M\mu$ as:

$$\nu(S) = (M\mu)(S) = \sum_{i=1}^{N} p_i \mu \left[ w_i^{-1}(S) \right].$$

The following results show that the Markov operator has a unique invariant measure $\bar{\mu}$ and it is globally attracting.

**Theorem 2 (Hutchinson, 1981)** *For $\mu, \nu \in \mathcal{M}(X)$,*

$$d_{MK}(M\mu, M\nu) \le c \, d_{MK}(\mu, \nu).$$

**Corollary 2 (Hutchinson, 1981)** *There exists a unique measure $\bar{\mu} \in \mathcal{M}(X)$, the* invariant measure *of the IFSP $(\mathbf{w},\mathbf{p})$, such that*

$$\bar{\mu}(S) = (M\bar{\mu})(S) = \sum_{i=1}^{N} p_i \bar{\mu} \left( w_i^{-1}(S) \right).$$

*Moreover, for any $\nu \in \mathcal{M}(X)$, $d_{MK}(\bar{\mu}, M^t \nu) \to 0$ as $t \to \infty$.*

**Theorem 3 (Hutchinson, 1981)** *The support of the invariant measure $\bar{\mu}$ of an $N$-map IFSP $(\mathbf{w},\mathbf{p})$ is the attractor $A$ of the IFS $\mathbf{w}$, i.e.,*

$$\operatorname{supp} \bar{\mu} = A.$$

## 3.3 Singular Invariant Measures

When a IFSP $(\mathbf{w},\mathbf{p})$ describes the dynamics of an economy the invariant measure $\bar{\mu}$ associated to $(\mathbf{w},\mathbf{p})$ can be interpreted as the stochastic long-run equilibrium (steady state) of such an economy. If $X = \mathbb{R}^n$ the invariant measure $\bar{\mu}$ can be either *absolutely continuous* or *singular* with respect to the $n$-dimensional Lebesgue measure, according to the following general definitions.

**Definition 1** *Two positive measures $\mu$ and $\nu$ defined on a measurable space $(\Omega, \Sigma)$ are called* singular *if there exist two disjoint sets $A$ and $B$ in $\Sigma$ whose union is $\Omega$ such that $\mu$ is zero on all measurable subsets of $B$ while $\nu$ is zero on all measurable subsets of $A$. This is denoted by $\mu \perp \nu$.*

**Definition 2** *If $\mu$ and $\nu$ are two measures defined on a measurable space $(\Omega, \Sigma)$, we say that $\mu$ is* absolutely continuous *with respect to $\nu$ if $\mu(A) = 0$ for any $A \in \Sigma$ such that $\nu(A) = 0$. The absolute continuity of $\mu$ with respect to $\nu$ is denoted by $\mu \ll \nu$.*

This distinction is crucial as in the latter case, whenever $\Omega = X = \mathbb{R}^n$, $\Sigma$ consists of the Borel subsets of $\mathbb{R}^n$ and $\nu$ is the $n$-dimensional Lebesgue measure, $\bar{\mu}$ can be represented by a density depending on some parameters, while in the former case there is no simple way to represent it— one actually has to list all its values on every point in its support. The mathematical literature so far has dealt with this issue by trying to characterize absolute continuity vs. singularity of $\bar{\mu}$ in terms of the parameters characterizing the IFS $(\mathbf{w}, \mathbf{p})$ (see, *e.g.*, Mitra et al., 2003; La Torre et al., 2015, 2018b).

We now focus our attention on the case of affine IFSP on $X = \mathbb{R}^n$. La Torre et al. (2018b) has proved the following theorem that will allow us to study possible singularity properties of the invariant measure in the examples of the following section 5 built by means of IFSP in $X = \mathbb{R}^3$. In what follows let $w_i(x) = A_i x + b_i$, where $A_i$ are $(n \times n)$ matrices and $b_i \in \mathbb{R}^n$, for $i = 1, 2, \ldots, N$, in a fashion similar to the expression in (3), and $p_i$ be the associated probabilities. The following result states a sufficient condition for the singularity of the invariant measure of an affine IFSP.

**Theorem 4 (La Torre et al., 2018b)** *Let $(\mathbf{w}, \mathbf{p}) = \{w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$ be an affine IFSP on $\mathbb{R}^n$ having maps $w_i : \mathbb{R}^n \to \mathbb{R}^n$ defined by $w_i(x) = A_i x + b_i$, for $i = 1, 2, \ldots, N$, and let $p = (p_1, p_2, \ldots, p_N)$ be the associated probability weights. If*

$$|\det(A_1)|^{p_1} |\det(A_2)|^{p_2} \cdots |\det(A_N)|^{p_N} < p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N} \tag{5}$$

*then the invariant measure $\bar{\mu}$ defined by $(\mathbf{w}, \mathbf{p})$ is singular.*

The following corollary provides a lower bound for the RHS in (5) that will be useful in the subsequent sections, where we will be dealing with three-dimensional attractors and invariant measures supported on them.

**Corollary 3** *The IFSP $(\mathbf{w}, \mathbf{p})$ considered in Theorem 4 has a singular invariant measure $\bar{\mu}$ whenever*

$$|\det(A_1)|^{p_1} |\det(A_2)|^{p_2} \cdots |\det(A_N)|^{p_N} \leq e^{-\frac{N}{e}} \simeq (0.6922)^N. \tag{6}$$

**Proof.** See the Appendix. ∎

The lower bound in the RHS of condition (6) is a rough estimate; however it will be enough to establish singularity of all the invariant measures we will obtain in our simulations in the next sections.

# 4 The Standard Sierpiński Tetrahedron

As our goal is to build three-dimensional attractors that resemble the well known Sierpiński tetrahedron by means of the affine IFS (3), before tackling such a construction we first recall how the tetrahedron can be obtained from an affine IFS. Abstracting from probabilities, the IFS that generates the *Sierpiński tetrahedron* in $\mathbb{R}^3$ consists of $N = 4$ affine maps $w_i : \mathbb{R}^3 \to \mathbb{R}^3$ that have the following matrix representation:

$$\begin{bmatrix} k_{t+1} \\ g_{t+1} \\ p_{t+1} \end{bmatrix} = w_i \left( \begin{bmatrix} k_t \\ g_t \\ p_t \end{bmatrix} \right) = \begin{bmatrix} a_i^{kk} & a_i^{kg} & a_i^{kp} \\ a_i^{gk} & a_i^{gg} & a_i^{gp} \\ a_i^{pk} & a_i^{pg} & a_i^{pp} \end{bmatrix} \begin{bmatrix} k_t \\ g_t \\ p_t \end{bmatrix} + \begin{bmatrix} z_i^k \\ z_i^g \\ z_i^p \end{bmatrix} \quad \text{for } i = 1, \ldots, 4, \tag{7}$$

where we use the $(k_t, g_t, p_t)$ notation for the variables consistent with our economic model defined by (3). The *standard* Sierpiński tetrahedron is characterized by six edges of size 1 and has its base on the Sierpiński triangle of vertices $(0,0)$, $(1,0)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so that its vertices in $\mathbb{R}^3$ are $(0,0,0)$, $(1,0,0)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$ and $\left(\frac{1}{2}, \frac{1}{\sqrt{12}}, \sqrt{\frac{2}{3}}\right)$. Table 1 reports the values for the matrix coefficients and the vector of additive constants $z_i^k$, $z_i^g$, $z_i^p$ required by the IFS (7) to generate it recursively. Specifically, the matrix is diagonal having all terms in its diagonal equal to $\frac{1}{2}$ and 0 elsewhere, while the crucial parameters turn out to be the additive terms $z_i^k$, $z_i^g$ and $z_i^p$. Clearly, under the parameterization of Table 1 the IFS (7) is a contraction.

| $i$ | $a_i^{kk}$ | $a_i^{gg}$ | $a_i^{pp}$ | $a_i^{kg}$ | $a_i^{kp}$ | $a_i^{gk}$ | $a_i^{gp}$ | $a_i^{pk}$ | $a_i^{pg}$ | $z_i^k$ | $z_i^g$ | $z_i^p$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | 0 | 0 |
| 3 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{4}$ | $\frac{\sqrt{3}}{4}$ | 0 |
| 4 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{4}$ | $\frac{1}{4\sqrt{3}}$ | $\frac{1}{\sqrt{6}}$ |

TABLE 1: parameters' values for the $N = 4$ maps $w_i$ characterizing the IFS generating the standard Sierpiński tetrahedron.

The standard Sierpiński tetrahedron, however, does not provide an appropriate reference point for our economic dynamics described by (3) because for values of the variables $(k_t, g_t, p_t)$ in a neighborhood of the vertex $(0,0,0)$—i.e., the origin—a value sufficiently large for parameters $s$ and $\beta_i$ might push the capital into negative territory. Therefore, we shall consider the same tetrahedron defined by the parameterization in Table 1 but shifted inside the positive orthant by the constant $\frac{1}{2}$ with respect to all three variables $k_t$, $g_t$ and $p_t$: specifically, we will focus on the tetrahedron with vertices $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(1, \frac{\sqrt{3}+1}{2}, \frac{1}{2}\right)$ and $\left(1, \frac{\sqrt{3}+1}{2\sqrt{3}}, \frac{\sqrt{3}+2\sqrt{2}}{2\sqrt{3}}\right)$ having parameter values listed in Table 2, where the slopes—the contraction factors—of the maps $w_i$ are the same as in Table 1 and only the additive constants $z_i^k$, $z_i^g$ and $z_i^p$ change their values.

| $i$ | $a_i^{kk}$ | $a_i^{gg}$ | $a_i^{pp}$ | $a_i^{kg}$ | $a_i^{kp}$ | $a_i^{gk}$ | $a_i^{gp}$ | $a_i^{pk}$ | $a_i^{pg}$ | $z_i^k$ | $z_i^g$ | $z_i^p$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| 2 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{3}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| 3 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{\sqrt{3}+1}{4}$ | $\frac{1}{4}$ |
| 4 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{\sqrt{3}+1}{4\sqrt{3}}$ | $\frac{\sqrt{3}+2\sqrt{2}}{4\sqrt{3}}$ |

TABLE 2: parameters' values for the $N = 4$ maps $w_i$ characterizing the IFS generating a Sierpiński tetrahedron shifted inside the positive orthant.

It is well known that the standard Sierpiński tetrahedron has Hausdorff dimension given by $\frac{\ln 4}{\ln 2} = 2$; that is, it is a three-dimensional object having the same consistency of a surface. In fact, if all points are projected onto a plane that is parallel to two of the outer edges, they exactly fill a square of side length $\frac{1}{\sqrt{2}}$ without overlap. Moreover, without making any assumption on the probability vector $(p_1, p_2, ..., p_4)$ to be associated to the maps $w_i$ defined in (7) for the parameters' values provided in both Tables 1 and 2 so to get a full IFSP, we can directly apply Corollary 3 to establish that the invariant measure $\bar{\mu}$ generated by any IFSP as in (7) *is always singular*, for any choice on the probabilities $p_1, p_2, ..., p_4$. To see this, note

that the determinant of the matrices $A_i$ in (7) are the same for all $i = 1, ..., 4$ and equal to $\det(A_i) \equiv \det(A) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ so that condition (6) boils down to

$$|\det(A_1)|^{p_1} |\det(A_2)|^{p_2} \cdots |\det(A_4)|^{p_4} = \left(\frac{1}{8}\right)^{p_1+p_2+\cdots+p_4} = \frac{1}{8} = 0.125 < e^{-\frac{4}{e}} \simeq 0.2296 \quad (8)$$

and the singularity property is established for any probability vector $(p_1, p_2, ..., p_4)$.

By exploiting the "transform" routine in Maple, which is capable of transforming three-dimensional graphic objects in $\mathbb{R}^3$, we built a simple procedure to approximate the standard Sierpiński tetrahedron by iterating the set-valued map defined in (4). Our procedure produces 4 modified copies of any geometric object in $\mathbb{R}^3$ recognizable by Maple by applying the transformation through the $N = 4$ maps defined in (7) according to operator (4). Figure 1 plots the first 6 iterations of operator (4) according to such a procedure starting from the full tetrahedron with vertices $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(1, \frac{\sqrt{3}+1}{2}, \frac{1}{2}\right)$ and $\left(1, \frac{\sqrt{3}+1}{2\sqrt{3}}, \frac{\sqrt{3}+2\sqrt{2}}{2\sqrt{3}}\right)$.
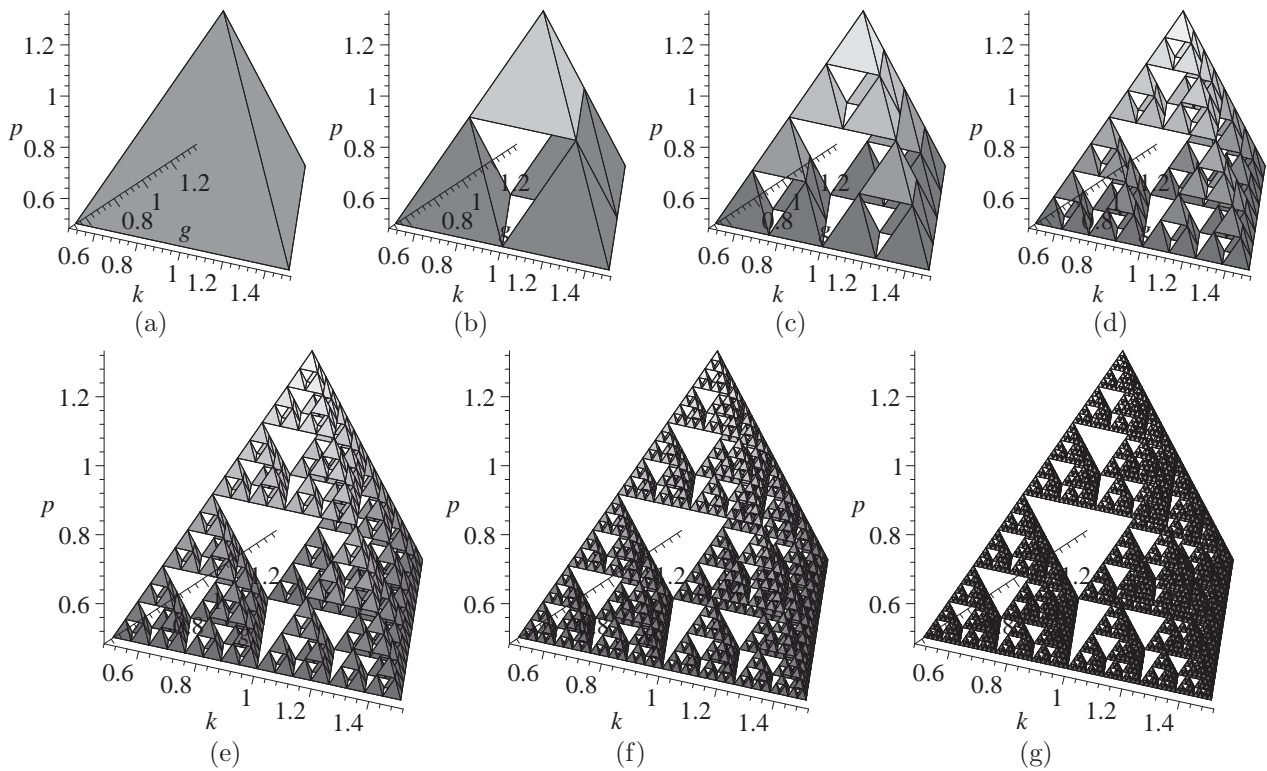


FIGURE 1: first 6 iterations of our algorithm to approximate the Sierpiński tetrahedron starting from the full tetrahedron with vertices $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(1, \frac{\sqrt{3}+1}{2}, \frac{1}{2}\right)$ and $\left(1, \frac{\sqrt{3}+1}{2\sqrt{3}}, \frac{\sqrt{3}+2\sqrt{2}}{2\sqrt{3}}\right)$.

# 5   Sierpiński-Tetrahedron-Like Attractors

We now exploit the construction behind the standard Sierpiński tetrahedron presented in the previous section to build an IFS that embeds the assumptions of the economic model described in Section 2 and turns out to be a version of the IFS (7)—with parameter values slightly different from those in Table 2—capable of generating a distorted version of the Sierpiński tetrahedron as the asymptotic attractor of our model economy. The striking difference between

the IFS (2)—or (3)—and the IFS (7) with parameters defined in Table 2 is the presence of some mixed nonzero multiplicative coefficients in the matrix in place of the zeros featured by Table 2. We will keep such nonzero values small in order to build affine IFS that generate dynamics somewhat resembling those required to obtain the Sierpiński tetrahedron as an attractor.

We pursue such a goal through two examples characterized by opposite approaches. In the first one, after setting parameters $\beta_i$, $\gamma_i$, $s$, $a$, $u$, $\delta_g$, $\eta$, $\delta_k$ and $v$ arbitrarily, we will compute the random additive constants $z_i^k$, $z_i^g$ and $z_i^p$ in order to obtain exactly the vertices of the (shifted) standard Sierpiński tetrahedron as the four fixed points of the four maps $w_i$ in the IFS (7), and, finally, we will determine the unique values of the remaining parameters $(f_i, e_i, \mu_i)$ compatible with these $z_i^k$, $z_i^g$ and $z_i^p$ values. In the second example, on the contrary, we first set all parameters $\beta_i$, $\gamma_i$, $s$, $a$, $u$, $\delta_g$, $\eta$, $\delta_k$, $v$, $f_i$, $e_i$ and $\mu_i$ arbitrarily, and next use such values to evaluate the four (deterministic) fixed points associated to each map $w_i$ in (7), for $i = 1, \ldots, 4$; the latter will then be used as the vertices of the non-standard full tetrahedron to be employed as initial geometric object for the iterations of operator (4) according to our Maple procedure.

## 5.1   Using the Standard Sierpiński Tetrahedron as Reference Point

In our first exercise we assume that the shocks on the final good production and on the environmental inefficiency associated with the use of the dirty input are deterministic and constant over time, $i.e.$, we set

$$\beta_i \equiv \beta = \frac{1}{5} \qquad \text{and} \qquad \gamma_i \equiv \gamma = \frac{1}{10}.$$

Moreover, we assume

$$s = a = u = \delta_g = \eta = \frac{1}{2} \qquad \text{and} \qquad b = d = \frac{1}{5},$$

which imply that the diagonal coefficients $a_i^{gg} \equiv (1 - \delta_g)$ and $a_i^{pp} \equiv (1 - \eta)$ are both $\frac{1}{2}$, as desired, while $a_i^{kk} \equiv s\left[a\left(1 - u\right) + 1 - \delta_k\right] = \frac{1}{2}$ as well whenever

$$\delta_k = \frac{5}{8}.$$

Note that under such parameterization the mixed coefficients are kept small in absolute value, as $a_i^{kg} \equiv sb = \frac{1}{10}$, $a_i^{kp} \equiv -s\beta = -\frac{1}{10}$, $a_i^{gk} \equiv du = \frac{1}{10}$ and $a_i^{pk} \equiv \gamma = \frac{1}{10}$, while still $a_i^{gp} = a_i^{pg} = 0$.

In order to choose the values for the random additive constants $z_i^k$, $z_i^g$ and $z_i^p$ in this first example we set the steady state values $(k_i^*, g_i^*, p_i^*)$ of the $N = 4$ maps $w_i$ in the IFS to be the vertices $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(1, \frac{\sqrt{3}+1}{2}, \frac{1}{2}\right)$ and $\left(1, \frac{\sqrt{3}+1}{2\sqrt{3}}, \frac{\sqrt{3}+2\sqrt{2}}{2\sqrt{3}}\right)$ of the full tetrahedron used as initial set in the approximation of the fractal provided by Figure 1(g), which is generated by the IFS defined by the parameters' values in Table 2. In other words, we solve

$$\begin{cases} k_i^* = a_i^{kk}k_i^* + a_i^{kg}g_i^* + a_i^{kp}p_i^* + z_i^k \\ g_i^* = a_i^{gk}k_i^* + a_i^{gg}g_i^* + z_i^g \\ p_i^* = a_i^{pk}k_i^* + a_i^{pp}p_i^* + z_i^p \end{cases} \iff \begin{cases} z_i^k = \frac{1}{2}k_i^* - \frac{1}{10}g_i^* + \frac{1}{10}p_i^* \\ z_i^g = -\frac{1}{10}k_i^* + \frac{1}{2}g_i^* \\ z_i^p = -\frac{1}{10}k_i^* + \frac{1}{2}p_i^* \end{cases}$$

with respect to $z_i^k$, $z_i^g$ and $z_i^p$ for all steady state (the 4 tetrahedron vertices) values $(k_i^*, g_i^*, p_i^*)$ for $i = 1, \ldots, 4$. The whole set of parameter values is reported in Table 3.

To conclude, we can choose any (constant) value for the deterministic parameter $0 < v < 1$, the share of foreign (dirty) input devoted to the accumulation of the dirty input, $k_t$, and solve

| $i$ | $a_i^{kk}$ | $a_i^{gg}$ | $a_i^{pp}$ | $a_i^{kg}$ | $a_i^{kp}$ | $a_i^{gk}$ | $a_i^{gp}$ | $a_i^{pk}$ | $a_i^{pg}$ | $z_i^k$ | $z_i^g$ | $z_i^p$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $0$ | $\frac{1}{10}$ | $0$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{5}$ |
| 2 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $0$ | $\frac{1}{10}$ | $0$ | $\frac{3}{4}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |
| 3 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $0$ | $\frac{1}{10}$ | $0$ | $\frac{10-\sqrt{3}}{20}$ | $\frac{3+5\sqrt{3}}{20}$ | $\frac{3}{20}$ |
| 4 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $0$ | $\frac{1}{10}$ | $0$ | $0.5528$ | $0.2943$ | $0.5582$ |

TABLE 3: parameters' values for the $N = 4$ maps $w_i$ characterizing the IFS generating a distorted Sierpiński tetrahedron starting from the standard tetrahedron shifted inside the positive orthant.

the system

$$\begin{cases} v f_i = z_i^k \\ (1-v)\, e_t f_t = z_i^g \\ \mu_t f_t = z_i^p \end{cases} \tag{9}$$

with respect to foreign capital, $f_i$, its random productivity to produce the clean input, $e_i$, and the random environmental inefficiency associated with the use of foreign capital, $\mu_i$, in each random realization $i = 1, \ldots, 4$ by using the $\left(z_i^k, z_i^g, z_i^p\right)$ values in the last three columns of Table 3. Following this approach we find 4 triples $(f_i, e_i, \mu_i)$, each corresponding to a realization of the random shocks, for $i = 1, \ldots, 4$. For example, by setting $v_i \equiv v = \frac{1}{3}$, Table 4 lists the corresponding $(f_i, e_i, \mu_i)$ values solving system (9) in each random realization $i = 1, \ldots, 4$.

| $i$ | $f_i$ | $e_i$ | $\mu_i$ |
|---|---|---|---|
| 1 | $0.75$ | $0.4$ | $0.2667$ |
| 2 | $2.25$ | $0.0667$ | $0.0444$ |
| 3 | $1.242$ | $0.7051$ | $0.1209$ |
| 4 | $1.6583$ | $0.2662$ | $0.3366$ |

TABLE 4: values of the foreign capital, $f_i$, its random productivity for the clean input, $e_i$, and the random environmental inefficiency associated with the use of foreign capital, $\mu_i$, for $v = \frac{1}{3}$ and corresponding to each random shock realization $i = 1, \ldots, 4$.

In this example the matrix of the affine maps $w_i$ in the IFS (7) is still constant through all shocks configurations, although it is not diagonal anymore; specifically, according to the values in Table 3:

$$A_i \equiv A = \begin{bmatrix} \frac{1}{2} & \frac{1}{10} & -\frac{1}{10} \\ \frac{1}{10} & \frac{1}{2} & 0 \\ \frac{1}{10} & 0 & \frac{1}{2} \end{bmatrix}.$$

Therefore, again without making any assumption on the probability vector $(p_1, p_2, ..., p_4)$ to be associated with the maps $w_i$ to get a full IFSP, we can directly apply Corollary 3 to establish that the invariant measure $\bar{\mu}$ generated by any IFSP (7) *is always singular*, for any choice on the probabilities $p_1, p_2, ..., p_4$. In fact, the determinant of $A$ above turns out to be the same as that of the diagonal matrix that generates the standard Sierpiński tetrahedron, $\det (A) = \frac{1}{8}$, so that condition (6) still holds and coincides with (8).

Figure 2 plots the first 6 iterations of operator (4) according to our Maple procedure starting from the full tetrahedron with vertices $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(1, \frac{\sqrt{3}+1}{2}, \frac{1}{2}\right)$ and $\left(1, \frac{\sqrt{3}+1}{2\sqrt{3}}, \frac{\sqrt{3}+2\sqrt{2}}{2\sqrt{3}}\right)$. The distorting effect of the nonzero mixed coefficients $a_i^{kg}$, $a_i^{kp}$, $a_i^{gk}$ and $a_i^{pk}$ is immediately seen in Figure 2(b) already after the first iteration: the first four smaller tetrahedra generated by

the transformation appear to wander around the vertices of the initial tetrahedron and are being rotated counterclockwise. Moreover, by taking a perspective of the same prefractals from a different angle, Figure 3 shows that the smaller tetrahedra arising after each iteration of operator (4) become more flattened along the direction crossing the positive orthant of variables $(k, g)$ from north-west to south-east and stretched along the direction entering the same orthant from the origin.
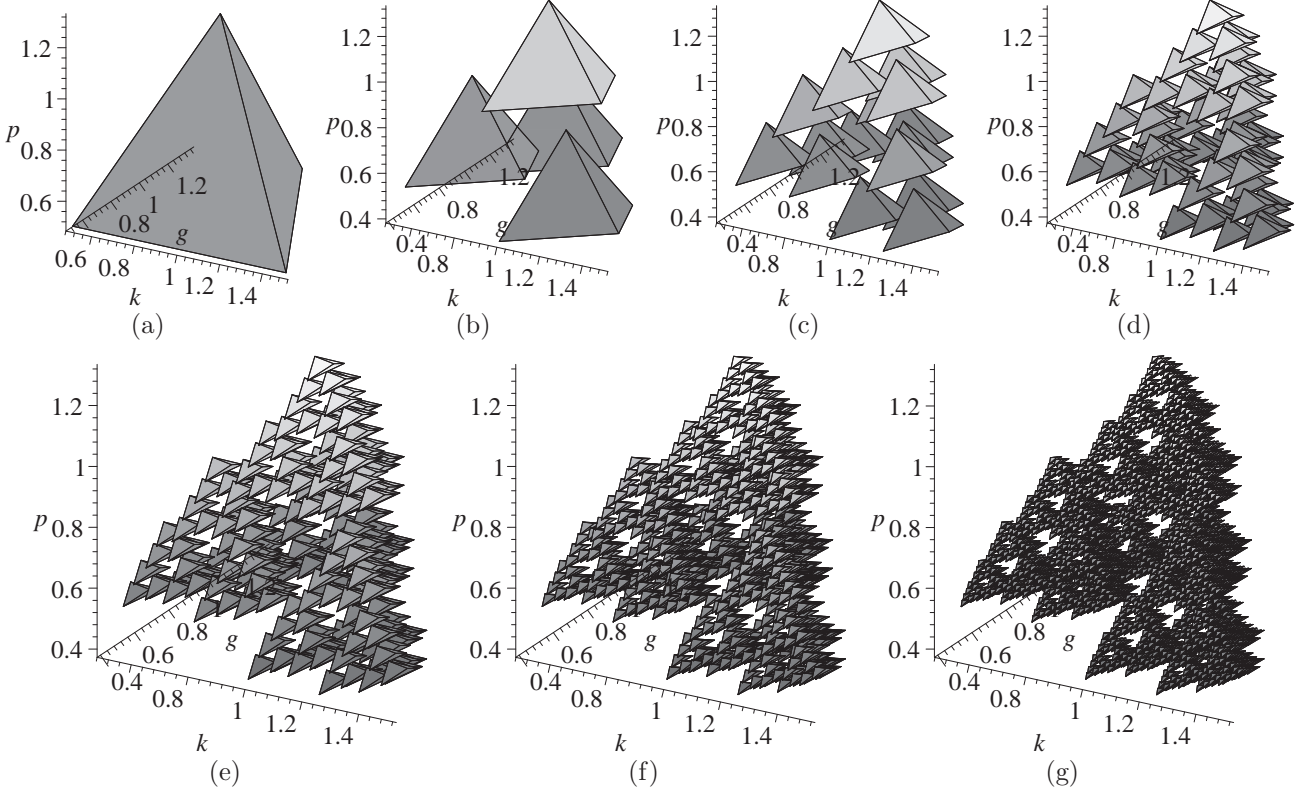


FIGURE 2: first 6 iterations of our algorithm to approximate the distorted Sierpiński tetrahedron generated by the IFS (7) with the coefficients' values provided by Table 3 starting from the full tetrahedron with vertices $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\left(1, \frac{\sqrt{3}+1}{2}, \frac{1}{2}\right)$ and $\left(1, \frac{\sqrt{3}+1}{2\sqrt{3}}, \frac{\sqrt{3}+2\sqrt{2}}{2\sqrt{3}}\right)$.

## 5.2 A More General Example

In the second example we assume that parameters $\beta_i$ and $\gamma_i$ are random, while parameter $f_i$, the amount of foreign capital, will be kept constant. Moreover, now we follow a different approach as we first choose the values of the four random exogenous shocks configurations, that is, the values of parameters $(e_i, \mu_i, \beta_i, \gamma_i)$, and next evaluate the four (deterministic) fixed points associated with each map $w_i$ in (7), for $i = 1, \ldots, 4$; the latter will be the vertices of the non-standard full tetrahedron that will be employed as initial geometric object for the iterations of operator (4) according to our Maple procedure. Specifically, we keep the same constant values $s = a = u = \delta_g = \eta = \frac{1}{2}$, $b = d = \frac{1}{5}$, $\delta_k = \frac{5}{8}$, and $v_i \equiv v = \frac{1}{3}$ as in the first example, while we now assume that $f_i$ is constant (deterministic) as well by setting $f_i \equiv f = 1$. For the random parameters representing the exogenous shocks realizations $(e_i, \mu_i, \beta_i, \gamma_i)$ we consider four possible scenarios:
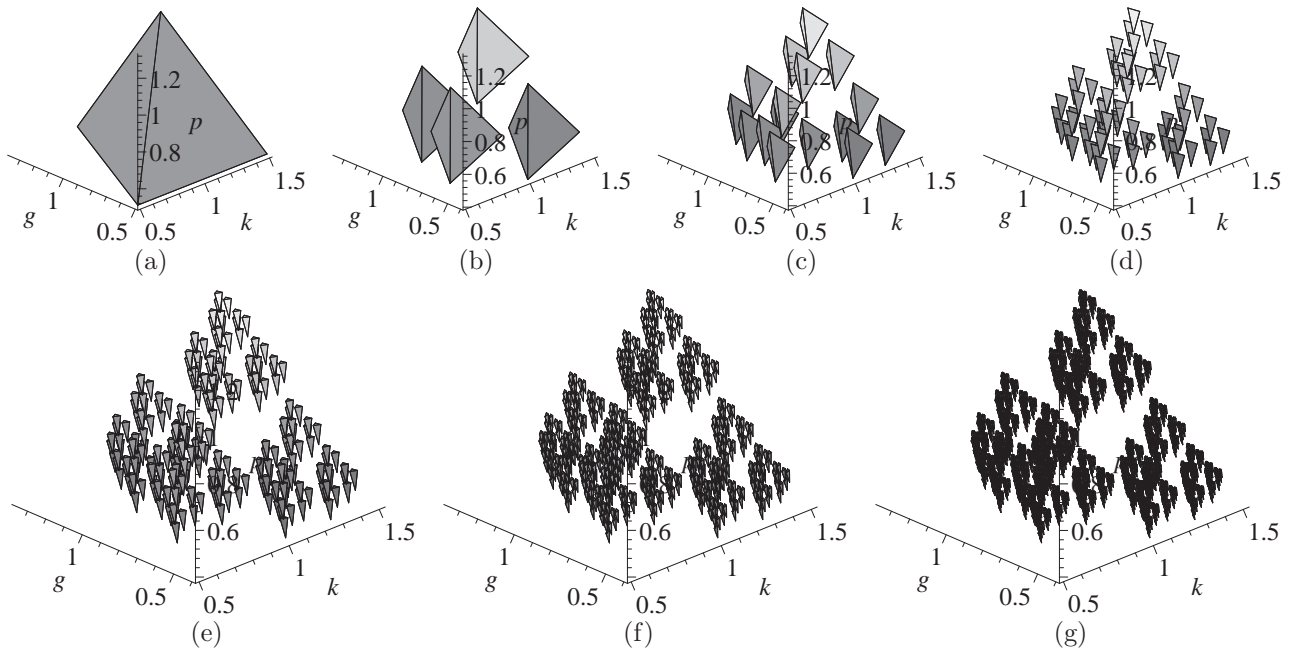
FIGURE 3: same construction as in Figure 2 but with a view of the same prefractals from a different perspective.

1. a 'best possible world' type scenario characterized by high productivity of foreign capital, $e_i$, and no damage both on production, $\beta_i$, and on the environment, $\gamma_i$ and $\mu_i$, represented by the parameter values reported in the first row of Table 5;

2. an intermediate scenario characterized by average productivity of foreign capital, $e_i$, high damage on production, $\beta_i$, and average damage on the environment, $\gamma_i$ and $\mu_i$, represented by the parameter values reported in the second row of Table 5;

3. another intermediate scenario characterized by average productivity of foreign capital, $e_i$, low damage on production, $\beta_i$, and high damage on the environment, $\gamma_i$ and $\mu_i$, represented by the parameter values reported in the third row of Table 5;

4. a catastrophic scenario characterized by low productivity of foreign capital, $e_i$, together with disruptive damage both on production, $\beta_i$, and on the environment, $\gamma_i$ and $\mu_i$, represented by the parameter values reported in the fourth row of Table 5.

| $i$ | $e_i$ | $\beta_i$ | $\gamma_i$ | $\mu_i$ |
|-----|-------|-----------|------------|---------|
| 1 | $1$ | $0$ | $0$ | $0$ |
| 2 | $\frac{1}{2}$ | $\frac{1}{5}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |
| 3 | $\frac{1}{2}$ | $\frac{1}{10}$ | $\frac{1}{5}$ | $\frac{1}{5}$ |
| 4 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

TABLE 5: values of the foreign capital's productivity, $e_i$, damage on production due to pollution, $\beta_i$, impact of pollution on production, $\gamma_i$, and the environmental inefficiency associated with the use of foreign capital, $\mu_i$, corresponding to each random shock realization $i = 1, \ldots, 4$.

13

The parameters' values listed in Table 5 lead to the values of the matrix coefficients and additive constants for the IFS (7) reported in Table 6.

| $i$ | $a_i^{kk}$ | $a_i^{gg}$ | $a_i^{pp}$ | $a_i^{kg}$ | $a_i^{kp}$ | $a_i^{gk}$ | $a_i^{gp}$ | $a_i^{pk}$ | $a_i^{pg}$ | $z_i^k$ | $z_i^g$ | $z_i^p$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{10}$ | $0$ | $\frac{1}{10}$ | $0$ | $0$ | $0$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $0$ |
| 2 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $0$ | $\frac{1}{10}$ | $0$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{10}$ |
| 3 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{10}$ | $-\frac{1}{20}$ | $\frac{1}{10}$ | $0$ | $\frac{1}{5}$ | $0$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{5}$ |
| 4 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{10}$ | $-\frac{1}{4}$ | $\frac{1}{10}$ | $0$ | $\frac{1}{2}$ | $0$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{2}$ |

TABLE 6: parameters' values for the four maps $w_i$ characterizing the IFS (7) associated to the random shocks' values reported in Table 5.

In this example the matrix of the affine maps $w_i$ in the IFS (7) is not constant but it depends on each shock realization; that is, according to the values in Table 6, there are four different matrices:

$$A_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{10} & 0 \\ \frac{1}{10} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \ A_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{10} & -\frac{1}{10} \\ \frac{1}{10} & \frac{1}{2} & 0 \\ \frac{1}{10} & 0 & \frac{1}{2} \end{bmatrix}, \ A_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{10} & -\frac{1}{20} \\ \frac{1}{10} & \frac{1}{2} & 0 \\ \frac{1}{5} & 0 & \frac{1}{2} \end{bmatrix}, \ A_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{10} & -\frac{1}{4} \\ \frac{1}{10} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

However, Corollary 3 can still be applied to establish that, again without making any assumption on the probability vector $(p_1, p_2, ..., p_4)$ to be associated with the maps $w_i$ to get a full IFSP, the invariant measure $\bar{\mu}$ generated by the IFSP (7) having parameter values as in Table 6 *is singular*, for any choice on the probabilities $p_1, p_2, ..., p_4$. In fact, $\det(A_1) = 0.12$, $\det(A_2) = \det(A_3) = 0.125$ and $\det(A_4) = 0.1825$, so that

$$|\det(A_1)|^{p_1} |\det(A_2)|^{p_2} |\det(A_3)|^{p_3} |\det(A_4)|^{p_4} = (0.12)^{p_1} (0.125)^{p_2} (0.125)^{p_3} (0.1825)^{p_4}$$
$$< (0.1825)^{p_1+p_2+\cdots+p_4} = 0.1825 < e^{-\frac{4}{e}} \simeq 0.2296$$

and condition (6) holds also in this example.

Unlike the construction of the first example in which we have taken the vertices of the standard Sierpiński tetrahedron shifted inside the positive orthant as the fixed points of the four maps $w_i$ for the IFS and then calculated the maps' parameter values which are consistent with them, in this case we follow the opposite route and evaluate the fixed points of the four maps $w_i$ defined by the parameter values already set in Table 6. Specifically, for $i = 1, \ldots, 4$ we solve the four systems

$$\begin{cases} \left(a_i^{kk} - 1\right) k + a_i^{kg} g + a_i^{kp} p + z_i^k = 0 \\ a_i^{gk} k + (a_i^{gg} - 1) g + a_i^{gp} p + z_i^g = 0 \\ a_i^{pk} k + a_i^{pg} g + (a_i^{pp} - 1) p + z_i^p = 0 \end{cases}$$

with respect to $k$, $g$ and $p$ for the parameter values in Table 6 to find the four fixed points $(k_1^*, g_1^*, p_1^*) = (0.9722, 1.5278, 0)$, $(k_2^*, g_2^*, p_2^*) = (0.76, 0.8187, 0.352)$, $(k_3^*, g_3^*, p_3^*) = (0.76, 0.8187, 0.704)$ and $(k_4^*, g_4^*, p_4^*) = (0.1598, 0.3653, 1.1598)$.

Such four vector values are used to define the tetrahedron that will be taken as initial condition in the same Maple recursive procedure already used in previous attractor approximations. The resulting first 6 iterations of operator (4) generated by our Maple procedure in this case are shown in Figures 4 and 5, which report the same prefractals, only observed from different

14

perspectives. Clearly, as the nonzero mixed matrix coefficients $a_i^{kg}$, $a_i^{kp}$, $a_i^{gk}$ and $a_i^{pk}$ now have different values in different shock realizations, and in some scenarios are larger in magnitude than in the previous example, the dynamics generated by this IFS produce a more complex evolution pattern than that reported in Figures 2 and 3, as, after the $t^{th}$ iteration, the initial tetrahedron, already exhibiting a sharp and streamlined shape itself, happens to be recursively sliced into $4^t$ smaller and thinner copies resembling ever sharper blades, some of which are also being rotated along different directions.
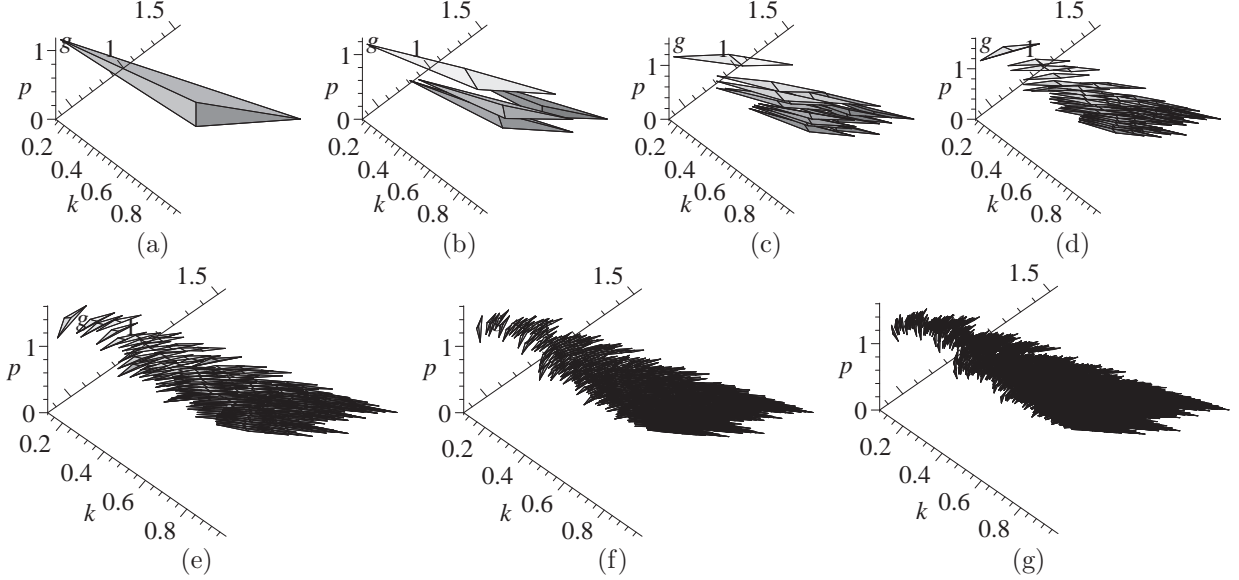


FIGURE 4: first 6 iterations of our algorithm to approximate the attractor of the IFS (7) with the coefficients' values provided by Table 6 starting from the full tetrahedron with vertices $(0.9722, 1.5278, 0)$, $(0.76, 0.8187, 0.352)$, $(0.76, 0.8187, 0.704)$ and $(0.1598, 0.3653, 1.1598)$.

Note that, unlike the IFS generating the standard Sierpiński tetrahedron in Figure 1, whose prefractals are all contained in the full initial tetrahedron having as its vertices the four fixed points of the maps $w_i$, in Figures 2 – 5 the prefractals' components wander all around the vertices of the initial tetrahedron because of the rotations induced by the mixed coefficients $a_i^{kg}$, $a_i^{kp}$, $a_i^{gk}$ and $a_i^{pk}$ (either deterministic or stochastic). As a result, after each iteration all prefractal's components are being scattered around inside a set which is larger the initial tetrahedron; such a feature is clearly more evident in Figures 4 and 5.

# 6   Conclusion

Since polluting emissions in developing countries are expected to increase substantially in the future, in order to ensure a sustainable process of economic development at world level it is essential that industrialized economies support the green transition in developing countries. In order to shed some light on this issue, we analyze the implications of foreign capital on the economic development of a developing economy transiting from dirty to clean activities in a two-sector stochastic economic growth model with pollution externalities. Output is produced by combining dirty and clean inputs with the latter being completely emissions-free, while the clean input is produced by using the dirty input. Foreign capital (in the form of dirty input), inclusive of development aid and foreign direct investments, is partly allocated to the production of the clean input. Such a setting gives rise to a dynamic system represented by a three dimensional
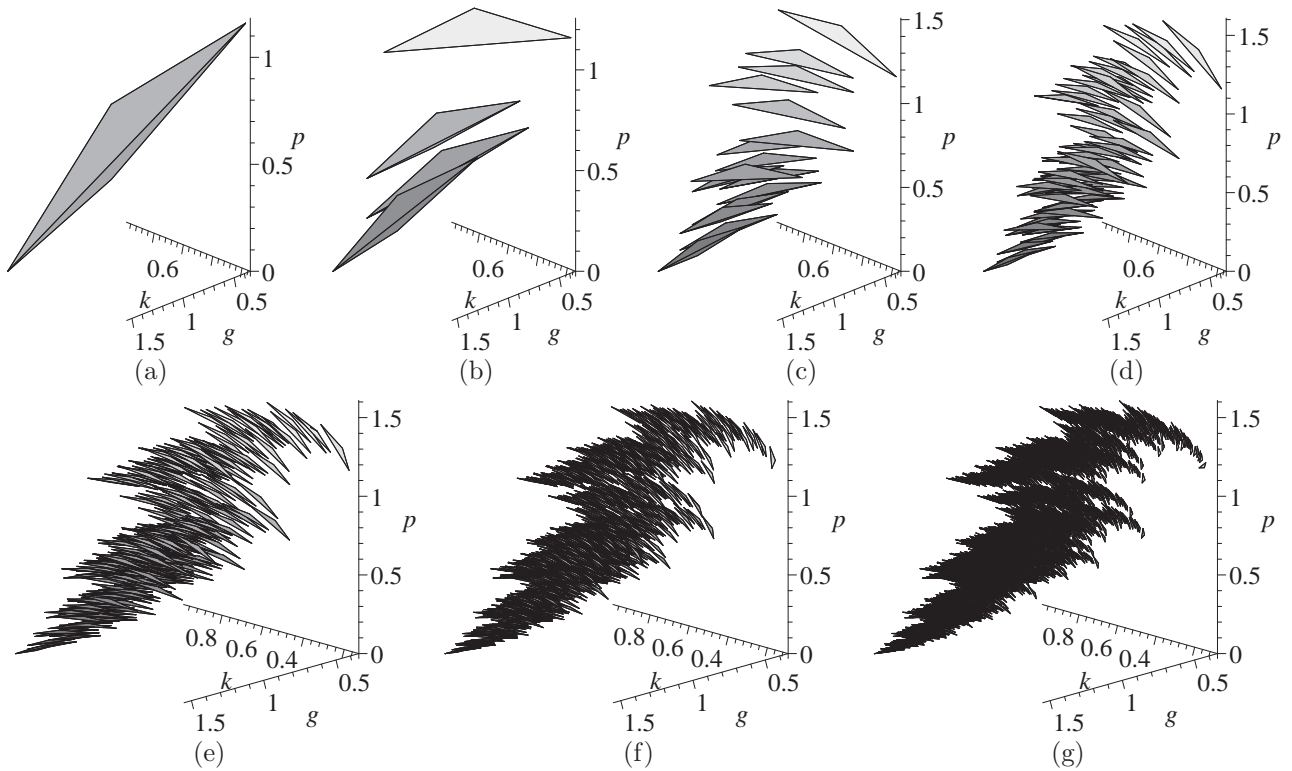
FIGURE 5: same construction as in Figure 4 but with a view of the same prefractals from a different perspective.

affine IFS. We show that the economy's steady state is represented by an invariant measure supported on a compact set, characterizing its fractal nature and showing that (under a specific parametrization) its attractor may look like a distorted Sierpiński tetrahedron.

To the best of our knowledge, our paper is the first work investigating the possibility that the fractal attractor of stochastic economic growth models may be have a three dimensional representation. In order to do so we have kept the model as simple as possible but it would be interesting to extend it along different directions to capture some other aspects of the problem under investigation. The Sierpiński tetrahedron is the natural candidate to start the analysis of three dimensional fractal attractors but it may also be worthwhile to explore whether economic growth models may give rise to other more complicated, but well known, attractors, like the Sierpiński pyramid or the Menger sponge. The analysis has been carried out in a purely dynamic setting abstracting completely from agents' optimization while it may also be interesting to assess whether and how optimal saving and investment decisions may change our conclusions. These further issues are left for future research.

## Technical Appendix

**Proof of Corollary 3.** We just need to show that $e^{-\frac{N}{e}}$ is a lower bound for the term $p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N}$ for any choice of probabilities $p_1, ..., p_N$ satisfying $0 < p_i < 1$, for $i = 1, \ldots, N$, and such that $\sum_{i=1}^{N} p_i = 1$. To this purpose we consider the generalized open cube

$$\Phi = \{(p_2, ..., p_N) : 0 < p_i < 1, i = 1, \ldots, N\}$$

16

and solve $\min_{(p_1,\ldots,p_N)\in\Phi} \ln\left(p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N}\right)$, which is equivalent to $\min_{(p_1,\ldots,p_N)\in\Phi} \left(p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N}\right)$. Note that $\ln\left(p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N}\right) = p_1 \ln p_1 + p_2 \ln p_2 + \cdots + p_N \ln p_N$ is strictly convex as sum of strictly convex functions of each variable $p_i$; therefore $(p_1^*,\ldots,p_N^*) = \arg\min_{(p_1,\ldots,p_N)\in\Phi} \ln\left(p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N}\right)$ is unique provided it exists. FOC on $p_1 \ln p_1 + p_2 \ln p_2 + \cdots + p_N \ln p_N$ yields $(p_1^*,\ldots,p_N^*) = (e^{-1},\ldots,e^{-1})$, which is the unique (interior) solution of $\min_{(p_1,\ldots,p_N)\in\Phi} \ln\left(p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N}\right) = \min_{(p_1,\ldots,p_N)\in\Phi} \left(p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N}\right)$, to which corresponds the minimum value $\left(e^{-1}\right)^{e^{-1}} \left(e^{-1}\right)^{e^{-1}} \cdots \left(e^{-1}\right)^{e^{-1}} = e^{-\frac{N}{e}}$. As $\sum_{i=1}^{N} e^{-1} = \frac{N}{e} \neq 1$ for any $N \in \mathbb{N}$, the open simplex

$$\Psi = \left\{ (p_2,\ldots,p_N) : 0 < p_i < 1, i = 1,\ldots,N, \sum_{i=1}^{N} p_i = 1 \right\}$$

is a proper subset of the open cube $\Phi$ which does not contain the point $(p_1^*,\ldots,p_N^*) = (e^{-1},\ldots,e^{-1})$. Therefore, necessarily $p_1^{p_1} p_2^{p_2} \cdots p_N^{p_N} > e^{-\frac{N}{e}}$ for any $(p_2,\ldots,p_N) \in \Psi$ and the proof is complete. ∎

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