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BILINEAR PSEUDO-DIFFERENTIAL OPERATORS WITH GEVREY-HÖRMANDER SYMBOLS

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Abstract. We consider bilinear pseudo-differential operators whose symbols may have a sub-exponential growth at infinity, together with all their derivatives. It is proved that those symbol classes can be described by the means of the short-time Fourier transform and modulation spaces. Our first main result is the invariance property of the corresponding bilinear operators. Furthermore we prove the continuity of such operators when acting on modulation spaces. As a consequence, we derive their continuity on anisotropic Gelfand-Shilov type spaces.

Introduction

The study of multilinear operators has been influenced by the Calderón-Zygmund theory. Indeed, one of the achievements of Coifman-Meyer's pioneer work [12] is the realization of pseudo-differential operators in terms of singular integrals of Calderón-Zygmund type. Their approach is based on a multilinear point of view and have had a far reaching impact in operator theory and partial differential equations. For example, boundedness of a class of translation invariant bilinear operators on Lebesgue spaces is proved in [12]. Furthermore, the bilinear Calderón-Zygmund theory developed by Grafakos and Torres [23] paved the way to the extension of those results to bilinear pseudo-differential operators which are non-translation invariant, i.e. whose symbols may depend on the space variable as well. We refer to [5] for a brief survey and discussion of applications to partial differential equations, and to [9] for a systematic study of bilinear pseudo-differential operators with symbols in bilinear Hörmander classes. See also [29] for a recent contribution in the context of Triebel-Lizorkin and local Hardy spaces.

Another type of results concerns bilinear (and multilinear) operators whose symbols are not necessarily smooth. Their continuity properties on modulation spaces were first observed in [6]. In contrast to classical bilinear pseudo-differential operators considered in e.g. [12], these operators are treated by the techniques of time-frequency analysis, see also [7–9, 14, 32].

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In this paper, we employ the techniques of time-frequency analysis and modulation spaces, and consider bilinear pseudo-differential operators of Gevrey-Hörmander type whose symbols are of infinite order and may have a (super-)exponential growth at infinity, together with all their derivatives. The linear counterpart of such operators is considered in [10], within the environment of isotropic Gelfand-Shilov spaces of functions and distributions, see also [2, 50], and extended to the anisotropic setting in [1, 4]. The main purpose of this paper is to extend some boundedness results given there to the bilinear case.

More precisely, we consider Gevrey-Hörmander type symbols $a \in \Gamma^{\sigma,\mathbf{s}}_{(\omega)}(\mathbf{R}^{3d})$ (or $a \in \Gamma^{\sigma,\mathbf{s},0}_{(\omega)}(\mathbf{R}^{3d})$), see Definition 1.8, and the corresponding pseudo-differential operators, denoted by $\mathrm{Op}_{r,t}(a)$, see (1.14) below. When $r = t = 0$ we recover the Kohn-Nirenberg correspondence considered, e.g., in [6, 8], while for $r = t = 1/2$ we obtain the Weyl correspondence, considered in [40, 42]. We remark that, in view of this choice of symbol classes, we cannot rely on arguments based on standard (e.g., Littlewood-Paley) localization techniques. The substitute for this is, from the very beginning, a "global approach", aimed at obtaining and employing appropriate characterizations of the involved objects, in terms of suitable estimates which hold true on the whole Euclidean spaces.

The paper is organized as follows. In Section 1 we collect necessary definitions, background material and basic facts on Gelfand-Shilov spaces, weight functions, modulation spaces, symbol classes and the corresponding bilinear pseudo-differential operators. In Section 2, we first study exponential-type operators on Gelfand-Shilov space and prove the corresponding invariance properties. We proceed with a characterization of the symbol spaces in terms of their regularity and decay properties, and suitable estimates related to modulation spaces. Finally, we prove our main results about the continuity of bilinear operators in Section 3.

1. Preliminaries

In this section we provide notation and background material which will be used throughout the paper. Proofs and details are in general omitted, since they can be found, e.g., in [13, 15–20, 25, 40, 43–45].

We use the standard notation for Euclidean spaces and multiindeces, cf. [28]. For example, if $x = (x_1, ..., x_d) \in \mathbf{R}^d$ and $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbf{N}^d$, then $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $\alpha! = \alpha_1! \dots \alpha_d!$. Here $\mathbf{N}$ denotes the set of non-negative integers. If $\alpha \in \mathbf{N}^d$, then $\alpha > 0$ means that $\alpha_j > 0$ for every $j = 1, \dots, d$, and similarly for $\alpha \geq 0$. We write $A(\theta) \lesssim B(\theta)$, $\theta \in \Omega$, if there is a constant $c > 0$ such that $|A(\theta)| \leq c|B(\theta)|$ for all $\theta \in \Omega$. We write $A(\theta) \asymp B(\theta)$ if $A(\theta) \lesssim B(\theta)$ and $B(\theta) \lesssim A(\theta)$ for all $\theta \in \Omega$. Here Ω is an open subset of $\mathbf{R}^N$. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are topological vector spaces, then $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ means that $\mathcal{B}_1$ is continuously embedded into $\mathcal{B}_2$. By $\mathscr{S}(\mathbf{R}^d)$ we denote the Schwartz space of rapidly decreasing functions, and $\mathscr{S}'(\mathbf{R}^d)$ denotes its dual space of tempered distributions.

1.1. **Gelfand-Shilov spaces.** Let $h, s, \sigma > 0$ be fixed. Then $\mathcal{S}_{s;h}^{\sigma}(\mathbf{R}^d)$ is the Banach space of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s;h}^{\sigma}} \equiv \sup_{\alpha,\beta \in \mathbf{N}^d} \sup_{x \in \mathbf{R}^d} \frac{|x^\alpha \partial^\beta f(x)|}{h^{|\alpha+\beta|} \alpha!^s \, \beta!^\sigma} < \infty, \tag{1.1}$$

endowed with the norm (1.1). Obviously, $\mathcal{S}_{s;h}^{\sigma}(\mathbf{R}^d)$ increases as h, s and σ increase, and it is contained in $\mathscr{S}(\mathbf{R}^d)$ for every $h, s, \sigma > 0$.

The Gelfand-Shilov spaces $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$ are defined as the inductive and projective limits respectively of $\mathcal{S}_{s;h}^{\sigma}(\mathbf{R}^d)$, i.e.

$$\mathcal{S}_s^\sigma(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{s;h}^{\sigma}(\mathbf{R}^d) \quad \text{and} \quad \Sigma_s^\sigma(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s;h}^{\sigma}(\mathbf{R}^d), \tag{1.2}$$

with the usual inductive and projective limit topologies. Note that $\Sigma_s^\sigma(\mathbf{R}^d) \neq \{0\}$, if and only if $s + \sigma \geq 1$ and $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, and $\mathcal{S}_s^\sigma(\mathbf{R}^d) \neq \{0\}$, if and only if $s + \sigma \geq 1$, see [22, 33]. For every $s, \sigma > 0$ we have

$$\Sigma_s^\sigma(\mathbf{R}^d) \hookrightarrow \mathcal{S}_s^\sigma(\mathbf{R}^d) \hookrightarrow \Sigma_{s+\varepsilon}^{\sigma+\varepsilon}(\mathbf{R}^d) \hookrightarrow \mathscr{S}(\mathbf{R}^d) \tag{1.3}$$

for every $\varepsilon > 0$. If $s + \sigma \geq 1$, then the last two inclusions in (1.3) are dense, and if in addition $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$ then the first inclusion in (1.3) is dense. Moreover, for $\sigma < 1$ the elements of $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ admit entire extensions to $\mathbf{C}^d$ satisfying suitable exponential bounds, [22].

The spaces $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$ combine global regularity with suitable decay properties at infinity, thus offering an abstract functional analytic framework for some problems in mathematical physics, [24, 31]. The following result is a well-known characterization of $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$ in terms of the exponential decay of derivatives of their elements. The proof is standard, see e.g. [3, Appendix A].

Lemma 1.1. *Let f be a smooth function on $\mathbf{R}^d$, $f \in C^\infty(\mathbf{R}^d)$. Then $f \in \mathcal{S}_s^\sigma(\mathbf{R}^d)$ (respectively $f \in \Sigma_s^\sigma(\mathbf{R}^d)$) if and only if for every $\alpha \in \mathbf{N}^d$*

$$|\partial^\alpha f(x)| \lesssim l^{|\alpha|} \alpha!^\sigma e^{-h|x|^{\frac{1}{s}}}, \quad x \in \mathbf{R}^d, \tag{1.4}$$

for some $l, h > 0$ (respectively for every $l, h > 0$).

The *Gelfand-Shilov distribution spaces* $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ and $(\Sigma_s^\sigma)'(\mathbf{R}^d)$ are the projective and inductive limit respectively of $(\mathcal{S}_{s;h}^{\sigma})'(\mathbf{R}^d)$:

$$(\mathcal{S}_s^\sigma)'(\mathbf{R}^d) = \bigcap_{h>0} (\mathcal{S}_{s;h}^{\sigma})'(\mathbf{R}^d) \quad \text{and} \quad (\Sigma_s^\sigma)'(\mathbf{R}^d) = \bigcup_{h>0} (\mathcal{S}_{s;h}^{\sigma})'(\mathbf{R}^d). \tag{1.2$'$}$$

It follows that $\mathscr{S}'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ when $s+\sigma \ge 1$, and if in addition $(s,\sigma) \neq (\frac{1}{2},\frac{1}{2})$, then $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d) \hookrightarrow (\Sigma_s^\sigma)'(\mathbf{R}^d)$.

Next we rewrite the definition of Gelfand-Shilov spaces in the notation which is convenient for our analysis, see also [10, 22, 27]. We put

$$\mathbf{R}^{d_0+\cdots+d_k} = \mathbf{R}^{d_0} \times \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_k} = \mathbf{R}^{(d_0,\dots,d_k)} = \mathbf{R}^{\mathbf{d}}.$$

Definition 1.2. Let $k \in \mathbf{N}$, $\sigma = (\sigma_0,\dots,\sigma_k) > 0$, $\mathbf{s} = (s_0,\dots,s_k) > 0$, and $\mathbf{d} = d_0 + \cdots + d_k$. The Gelfand-Shilov spaces

$$\mathcal{S}_{\mathbf{s}}^\sigma(\mathbf{R}^{\mathbf{d}}) = \mathcal{S}_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k}(\mathbf{R}^{d_0+\cdots+d_k}) \quad \text{and} \quad \Sigma_{\mathbf{s}}^\sigma(\mathbf{R}^{\mathbf{d}}) = \Sigma_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k}(\mathbf{R}^{d_0+\cdots+d_k}),$$

consist of all $F \in C^\infty(\mathbf{R}^{d_0+\cdots+d_k})$ such that

$$|x_0^{\alpha_0}\dots x_k^{\alpha_k}\partial_{x_0}^{\beta_0}\dots\partial_{x_k}^{\beta_k}F(x_0,\dots,x_k)| \lesssim h^{|\alpha_0+\beta_0+\cdots+\alpha_k+\beta_k|}\prod_{j=0}^k \alpha_j!^{s_j}\beta_j!^{\sigma_j}$$

for some $h>0$ and for every $h>0$ respectively, where $x_j \in \mathbf{R}^{d_j}$, $\alpha_j,\beta_j \in \mathbf{N}^{d_j}$, $j=0,\dots,k$. The dual spaces of $\mathcal{S}_{\mathbf{s}}^\sigma(\mathbf{R}^{\mathbf{d}})$ and $\Sigma_{\mathbf{s}}^\sigma(\mathbf{R}^{\mathbf{d}})$ are respectively denoted by

$$(\mathcal{S}_{\mathbf{s}}^\sigma)'(\mathbf{R}^{\mathbf{d}}) = (\mathcal{S}_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k})'(\mathbf{R}^{d_0+\cdots+d_k})$$

and

$$(\Sigma_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k})'(\mathbf{R}^{\mathbf{d}}) = (\Sigma_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k})'(\mathbf{R}^{d_0+\cdots+d_k}).$$

The space $\mathcal{S}_{\mathbf{s}}^\sigma(\mathbf{R}^{\mathbf{d}})$ is nontrivial if and only if $s_j+\sigma_j \ge 1$, for each $j=0,\dots,k$ and $\Sigma_{\mathbf{s}}^\sigma(\mathbf{R}^{\mathbf{d}})$ is nontrivial if and only if $s_j+\sigma_j \ge 1$, and $(s_j,\sigma_j) \neq (\frac{1}{2},\frac{1}{2})$ for each $j=0,\dots,k$.

Obviously, if $\sigma_j=\sigma$, $s_j=s$ and $d_j=d$, $j=0,\dots,k$, then

$$\mathcal{S}_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k}(\mathbf{R}^{d_0+\cdots+d_k}) \equiv \mathcal{S}_s^\sigma(\mathbf{R}^{(k+1)d}), \quad \Sigma_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k}(\mathbf{R}^{d_0+\cdots+d_k}) \equiv \Sigma_s^\sigma(\mathbf{R}^{(k+1)d}),$$

$$(\mathcal{S}_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k})'(\mathbf{R}^{d_0+\cdots+d_k}) \equiv (\mathcal{S}_s^\sigma)'(\mathbf{R}^{(k+1)d}),$$

and

$$(\Sigma_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k})'(\mathbf{R}^{d_0+\cdots+d_k}) \equiv (\Sigma_s^\sigma)'(\mathbf{R}^{(k+1)d}).$$

The Fourier transform $\mathscr{F}$ is the linear and continuous map on $\mathscr{S}(\mathbf{R}^{\mathbf{d}})$, given by the formula

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}}\int_{\mathbf{R}^d} f(x)e^{-i\langle x,\xi\rangle}\,dx, \quad \xi \in \mathbf{R}^{\mathbf{d}},$$

when $f \in \mathscr{S}(\mathbf{R}^{\mathbf{d}})$. Here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the usual scalar product on $\mathbf{R}^{\mathbf{d}}$. The Fourier transform extends uniquely to homeomorphisms from $(\mathcal{S}_s^\sigma)'(\mathbf{R}^{\mathbf{d}})$ to $(\mathcal{S}_\sigma^s)'(\mathbf{R}^{\mathbf{d}})$, and from $(\Sigma_s^\sigma)'(\mathbf{R}^{\mathbf{d}})$ to $(\Sigma_\sigma^s)'(\mathbf{R}^{\mathbf{d}})$. Furthermore, it restricts to homeomorphisms from $\mathcal{S}_s^\sigma(\mathbf{R}^{\mathbf{d}})$ to $\mathcal{S}_\sigma^s(\mathbf{R}^{\mathbf{d}})$, and from $\Sigma_s^\sigma(\mathbf{R}^{\mathbf{d}})$ to $\Sigma_\sigma^s(\mathbf{R}^{\mathbf{d}})$, cf. [22] and Proposition 1.4 here below. This, together with the kernel theorem for Gelfand-Shilov spaces (see [30, 35, 39]) implies the following mapping properties of partial Fourier transforms on Gelfand-Shilov spaces. Here, $\mathscr{F}_jF$ is the partial Fourier transforms of $F(x_0,x_1,\dots,x_k)$ with respect to $x_j \in \mathbf{R}^{d_j}$, $j=0,\dots,k$.

Proposition 1.3. *Let $k \in \mathbf{N}$, $s_j, \sigma_j > 0$, $j = 0, \dots, k$. Then the following is true:*

(1) *the mapping $\mathscr{F}_j$ on $\mathscr{S}(\mathbf{R}^{d_0+\cdots+d_k})$ restrict to homeomorphism*

$$\mathscr{F}_j : \mathcal{S}_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k}(\mathbf{R}^{d_0+\cdots+d_k}) \to \mathcal{S}_{s_0,\dots,s_{j-1},\sigma_j,s_{j+1},\dots,s_k}^{\sigma_0,\dots,\sigma_{j-1},s_j,\sigma_{j+1},\dots,\sigma_k}(\mathbf{R}^{d_0+\cdots+d_k});$$

(2) *the mapping $\mathscr{F}_j$ on $\mathscr{S}(\mathbf{R}^{d_0+\cdots+d_k})$ is uniquely extendable to homeomorphism*

$$\mathscr{F}_j : (\mathcal{S}_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k})'(\mathbf{R}^{d_0+\cdots+d_k}) \to (\mathcal{S}_{s_0,\dots,s_{j-1},\sigma_j,s_{j+1},\dots,s_k}^{\sigma_0,\dots,\sigma_{j-1},s_j,\sigma_{j+1},\dots,\sigma_k})'(\mathbf{R}^{d_0+\cdots+d_k}).$$

The same holds true if the $\mathcal{S}_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k}$-spaces and their duals are replaced by corresponding $\Sigma_{s_0,\dots,s_k}^{\sigma_0,\dots,\sigma_k}$ -spaces and their duals in each occurrence.

The result analogous to Proposition 1.3 holds for partial Fourier transforms with respect to some choice of variables.

Proposition 1.4. *Let $k \in \mathbf{N}$, $\sigma = (\sigma_0, \dots, \sigma_k) > 0$, $\mathbf{s} = (s_0, \dots, s_k) > 0$, and $\mathbf{d} = d_1 + \cdots + d_k$. Then the following conditions are equivalent.*

(1) $F \in \mathcal{S}_{\mathbf{s}}^{\sigma}(\mathbf{R}^{\mathbf{d}})$ $\quad$ *(*$F \in \Sigma_{\mathbf{s}}^{\sigma}(\mathbf{R}^{\mathbf{d}})$*);*

(2) *for some $r > 0$ (for every $r > 0$) it holds*

$$|F(x_0, \dots, x_k)| \lesssim e^{-r\left(|x_0|^{\frac{1}{s_0}} + \cdots + |x_k|^{\frac{1}{s_k}}\right)}$$

and

$$|\widehat{F}(\xi_0, \dots, \xi_k)| \lesssim e^{-r\left(|\xi_0|^{\frac{1}{\sigma_0}} + \cdots + |\xi_k|^{\frac{1}{\sigma_k}}\right)}.$$

(3) *for every $\alpha = (\alpha_0, \dots, \alpha_k) \in \mathbf{N}^{\mathbf{d}}$ and for some $h, r > 0$ (for every $h, r > 0$) it holds*

$$|\partial^\alpha F(x_0, \dots, x_k)| \lesssim h^{|\alpha|} \prod_{j=0}^{k} \alpha_j^{\sigma_j} e^{-r\left(|x_0|^{\frac{1}{s_0}} + \cdots + |x_k|^{\frac{1}{s_k}}\right)}.$$

Proof. The equivalence between (1) and (2) follows from [11], and (1) $\Leftrightarrow$ (2) can be proved by a slight modification of the proof of Lemma 1.1 (cf. [3, Appendix A]) and we therefore leave it for the reader.

□

1.2. **Weight functions.** A function ω is called a *weight* or *weight function* on $\mathbf{R}^d$, if $\omega, 1/\omega \in L^\infty_{loc}(\mathbf{R}^d)$ are positive everywhere. Without loss of generality we may assume that the weight functions are continuous on $\mathbf{R}^d$, cf. [25]. Let ω and v be weights on $\mathbf{R}^d$. Then ω is called *v-moderate* or *moderate*, if

$$\omega(x_1 + x_2) \lesssim \omega(x_1) v(x_2), \quad x_1, x_2 \in \mathbf{R}^d. \tag{1.5}$$

If v can be chosen as polynomial, then ω is called a weight of polynomial type. A weight function v is *submultiplicative*, if it is symmetric in each coordinate and

$$v(x_1+x_2) \lesssim v(x_1)v(x_2), \quad x_1,x_2 \in \mathbf{R}^d.$$

From now on, v always denote a submultiplicative weight if nothing else is stated. In particular, if (1.5) holds and v is submultiplicative, then

$$\frac{\omega(x_1)}{v(x_2)} \lesssim \omega(x_1+x_2) \lesssim \omega(x_1)v(x_2), \quad x_1,x_2 \in \mathbf{R}^d. \tag{1.6}$$

If ω is a moderate weight on $\mathbf{R}^d$, then there exists a submultiplicative weight v on $\mathbf{R}^d$ such that (1.5) and (1.6) hold, cf. [45,46,48]. Moreover if v is submultiplicative on $\mathbf{R}^d$, then

$$1 \lesssim v(x) \lesssim e^{c|x|} \tag{1.7}$$

for some constant $c>0$ (cf. [26, Lemma 4.2]).

In particular, if ω is moderate, then there exists $c>0$ such that

$$\omega(x+y) \lesssim \omega(x)e^{c|y|} \quad \text{and} \quad e^{-c|x|} \lesssim \omega(x) \lesssim e^{c|x|}, \quad x,y \in \mathbf{R}^d.$$

For a given $k \in \mathbf{N}$, we let $\mathscr{P}_E(\mathbf{R}^{d_0+\dots+d_k})$ be the set of all moderate weights on $\mathbf{R}^{d_0+\dots+d_k}$, and $\mathscr{P}(\mathbf{R}^{d_0+\dots+d_k})$ be the subset of $\mathscr{P}_E(\mathbf{R}^{d_0+\dots+d_k})$ which consists of weights of polynomial type.

If $\omega \in \mathscr{P}_E(\mathbf{R}^{d_0+\dots+d_k})$ then there exists a submultiplicative weight v on $\mathbf{R}^{d_0+\dots+d_k}$, such that

$$\frac{\omega(x_0,\dots,x_k)}{v(y_0,\dots,y_k)} \lesssim \omega(x_0+y_0,\dots,x_k+y_k) \lesssim \omega(x_0,\dots,x_k)v(y_0,\dots,y_k), \tag{1.8}$$

where $x_j, y_j \in \mathbf{R}^{d_j}$, $j=0,\dots,k$. Note that (1.7) for a submultiplicative weight v on $\mathbf{R}^{d_0+\dots+d_k}$ becomes

$$1 \lesssim v(x_0,\dots,x_k) \lesssim e^{r(|x_0|+\dots+|x_k|)}, \quad x_j \in \mathbf{R}^{d_j},\ j=0,\dots,k, \tag{1.9}$$

for some $r>0$.

Next we extend the weight functions considered in [1,2] to the case when $\mathbf{R}^d = \mathbf{R}^{d_0+\dots+d_k}$.

Definition 1.5. Let $k \in \mathbf{N}$ and $s_j > 0$, $j=0,\dots,k$. Then, the set $\mathscr{P}_{s_0,\dots,s_k}(\mathbf{R}^{d_0+\dots+d_k})$ ($\mathscr{P}^0_{s_0,\dots,s_k}(\mathbf{R}^{d_0+\dots+d_k})$) consists of all weights $\omega \in \mathscr{P}_E(\mathbf{R}^{d_0+\dots+d_k})$ such that

$$\omega(x_0+y_0,\dots,x_k+y_k) \lesssim \omega(x_0,\dots,x_k)e^{r(|y_0|^{\frac{1}{s_0}}+\dots+|y_k|^{\frac{1}{s_k}})},\ x_j,y_j \in \mathbf{R}^{d_j} \tag{1.10}$$

holds for some (for every) $r>0$.

In particular, if $\omega \in \mathscr{P}_{s_0,\dots,s_k}(\mathbf{R}^{d_0+\dots+d_k})$ ($\mathscr{P}^0_{s_0,\dots,s_k}(\mathbf{R}^{d_0+\dots+d_k})$), then

$$e^{-r(|x_0|^{\frac{1}{s_0}}+\dots+|x_k|^{\frac{1}{s_k}})} \lesssim \omega(x_0,\dots,x_k) \lesssim e^{r(|x_0|^{\frac{1}{s_0}}+\dots+|x_k|^{\frac{1}{s_k}})}$$

for some $r>0$ (for every $r>0$).

By (1.8) and (1.9) it follows that

$$\mathscr{P}^0_{s_0,\dots,s_k}(\mathbf{R}^{d_0+\cdots+d_k}) = \mathscr{P}_{\tilde{s}_0,\dots,\tilde{s}_k}(\mathbf{R}^{d_0+\cdots+d_k}) = \mathscr{P}_E(\mathbf{R}^{d_0+\cdots+d_k})$$

when $s_j < 1$ and $\tilde{s}_j \le 1$, $j = 0,\dots,k$. For convenience we set

$$\mathscr{P}^0_E(\mathbf{R}^{d_0+\cdots+d_k}) = \mathscr{P}^0_{E,1}(\mathbf{R}^{d_0+\cdots+d_k}).$$

The following extension of [2, Proposition 1.6], shows that for any weight in $\mathscr{P}_E(\mathbf{R}^{d_0+\cdots+d_k})$, there are equivalent weights that satisfy the anisotropic Gevrey regularity.

Proposition 1.6. *Let there be given $\omega \in \mathscr{P}_E(\mathbf{R}^{d_0+d_1+\cdots+d_k})$. Then there exists a weight $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{d_0+d_1+\cdots+d_k}) \cap C^\infty(\mathbf{R}^{d_0+d_1+\cdots+d_k})$ such that the following is true:*

(1) $\omega_0 \asymp \omega$*;*

(2) *for every (multiindex) $\alpha_j \ge 0$, $j = 0,\dots,k$, we have*

$$|\partial_x^{\alpha_0}\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_k}^{\alpha_k}\omega_0(x,\xi_1,\dots,\xi_k)| \lesssim h^{|\alpha_0+\alpha_1+\cdots+\alpha_k|}\prod_{j=0}^k \alpha_j!^{s_j}\omega(x,\xi_1,\dots,\xi_k)$$

$$\asymp h^{|\alpha_0+\alpha_1+\cdots+\alpha_k|}\prod_{j=0}^k \alpha_j!^{s_j}\omega_0(x,\xi_1,\dots,\xi_k), \quad x \in \mathbf{R}^d_0,\ \xi_j \in \mathbf{R}^{d_j},\ j = 1,\dots,k,$$

for every $h > 0$ and $s_j > 0$, $j = 0,\dots,k$.

The proof is appropriate modification of the proof of [2, Proposition 1.5] and can be found in [3, Appendix].

1.3. **Modulation spaces.** Modulation spaces, originally introduced by Feichtinger in [17], are recognized as appropriate family of spaces when dealing with problems of time-frequency analysis, see [17–21, 25, 36, 38], to mention just a few references. A broader family of modulation spaces is recently studied in [2, 34, 50].

Let $s,\sigma > 0$, such that $s+\sigma \ge 1$, and let $\phi \in \mathcal{S}_s^\sigma(\mathbf{R}^d)$ be fixed. Then the *short-time Fourier transform* $V_\phi f$ of $f \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ with respect to the *window function* ϕ is defined by

$$V_\phi f(x,\xi) \equiv \mathscr{F}(f \cdot \overline{\phi(\cdot - x)})(\xi), \quad x,\xi \in \mathbf{R}^d.$$

This definition makes sense as a Gelfand-Shilov distribution [1, Remark 1.5].

If $f,\phi \in \mathcal{S}_s^\sigma(\mathbf{R}^d)$, then

$$V_\phi f(x,\xi) = (2\pi)^{-\frac{d}{2}}\int f(y)\overline{\phi(y-x)}e^{-i\langle y,\xi\rangle}\,dy.$$

Let $s,\sigma > 0$, such that $s+\sigma \ge 1$. Let there be given $\phi \in \mathcal{S}_s^\sigma(\mathbf{R}^d)\setminus 0$, $p,q \in [1,\infty]$ and $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$. Then the *modulation space* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$

consists of all Gelfand-Shilov distributions f on $\mathbf{R}^d$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \Big(\int \Big(\int |V_\phi f(x,\xi)\omega(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q} < \infty \qquad (1.11)$$

(with the obvious changes if $p=\infty$ and/or $q=\infty$). If $p=q$ we simply write $M^p_{(\omega)}$ instead of $M^{p,p}_{(\omega)}$, and if $\omega=1$, then we set $M^{p,q}=M^{p,q}_{(\omega)}$ and $M^p=M^p_{(\omega)}$.

The spaces $M^{p,q}_{(\omega)}$ are Banach spaces and every $\phi \in M^r_{(v)} \setminus 0$ yields an equivalent norm in (1.11) and so $M^{p,q}_{(\omega)}$ is independent on the choice of $\phi \in M^r_{(v)}$ [50, Proposition 1.1].

Gelfand-Shilov spaces and their dual spaces can be described as projective or inductive limits of modulation spaces [48, Theorem 3.9]. In particular, we have the following characterization of Gelfand-Shilov spaces by the means of the short-time Fourier transform. We refer to [27] for the proof, see also [37, 41, 48].

Proposition 1.7. *Let $k \in \mathbf{N}$, $\sigma=(\sigma_0,\dots,\sigma_k)>0$, $\mathbf{s}=(s_0,\dots,s_k)>0$, and $\mathbf{d}=d_1+\cdots+d_k$. Also let $\phi \in \mathcal{S}^\sigma_{\mathbf{s}}(\mathbf{R}^{\mathbf{d}}) \setminus 0$. Then the following is true:*

(1) *$F \in \mathcal{S}^\sigma_{\mathbf{s}}(\mathbf{R}^{\mathbf{d}})$ if and only if*

$$|V_\phi F(x_0,\dots,x_k,\xi_0,\dots,\xi_k)| \lesssim e^{-r\left(|x_0|^{\frac{1}{s_0}}+\cdots+|x_k|^{\frac{1}{s_k}}+|\xi_0|^{\frac{1}{\sigma_0}}+\cdots+|\xi_k|^{\frac{1}{\sigma_k}}\right)} \qquad (1.12)$$

holds for some $r>0$;

(2) *if, in addition, $\phi \in \Sigma^\sigma_{\mathbf{s}}(\mathbf{R}^{\mathbf{d}}) \setminus 0$, then $F \in \Sigma^\sigma_{\mathbf{s}}(\mathbf{R}^{\mathbf{d}})$ if and only if* (1.12) *holds for every $r>0$.*

1.4. **Symbol classes and Pseudo-differential operators.** First we introduce function spaces related to symbol classes of the multilinear pseudo-differential operators. We consider $a \in C^\infty(\mathbf{R}^{d_0+\cdots+d_k})$ which obey various conditions of the form

$$|\partial^\alpha_x \partial^{\beta_1}_{\xi_1} \dots, \partial^{\beta_k}_{\xi_k} a(x,\xi_1,\dots,\xi_k)| \lesssim h^{|\alpha+\beta_1+\cdots+\beta_k|} \alpha!^\sigma \prod_{j=1}^k \beta_j!^{s_j} \cdot \omega(x,\xi_1,\dots,\xi_k),$$

$w \in \mathscr{P}_E(\mathbf{R}^{d_0+d_1+\cdots+d_k})$, $\alpha \in \mathbf{N}^{d_0}$, $\beta_j \in \mathbf{R}^{d_j}$, $s_j,\sigma,h>0$, $j=1,\dots,k$.

When $k=1$ we recover the condition (1.14) from [4]. Similarly to [4], for a given $w \in \mathscr{P}_E(\mathbf{R}^{d_0+d_1+\cdots+d_k})$ and $s_j,\sigma,h>0$, $j=1,\dots,k$, we consider norms of the form

$$\|a\|_{\Gamma^{\sigma,\mathbf{s};h}_{(\omega)}} \equiv \sup_{\substack{\alpha\in\mathbf{N}^{d_0}\\ \beta_j\in\mathbf{N}^{d_j}}} \left(\sup_{\substack{x\in\mathbf{R}^{d_0}\\ \xi_j\in\mathbf{R}^{d_j}}} \left(\frac{|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \dots, \partial_{\xi_k}^{\beta_k} a(x,\xi_1,\dots,\xi_k)|}{h^{|\alpha+\beta_1+\dots+\beta_k|}\alpha!^\sigma \prod_{j=1}^k \beta_j!^{s_j}\cdot \omega(x,\xi_1,\dots,\xi_k)} \right)\right). \tag{1.13}$$

More precisely, we are interested in invariance and continuity for bilinear pseudo-differential operators when symbols belong to the following symbol classes.

Definition 1.8. Let there be given $\sigma, s_j, h > 0$, $j = 1,\dots,k$ and $\omega \in \mathscr{P}_E(\mathbf{R}^{d_0+d_1+\dots+d_k})$, and set $\mathbf{s} = (s_1,\dots,s_k)$.

(1) The set $\Gamma^{\sigma,\mathbf{s};h}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k})$ consists of all $a \in C^\infty(\mathbf{R}^{d_0+\dots+d_k})$ such that

$$\|a\|_{\Gamma^{\sigma,\mathbf{s};h}_{(\omega)}} < \infty,$$

where the norm $\|\cdot\|_{\Gamma^{\sigma,\mathbf{s};h}_{(\omega)}}$ is given by (1.13).

(2) The sets $\Gamma^{\sigma,\mathbf{s}}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k})$ and $\Gamma^{\sigma,\mathbf{s};0}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k})$ are given by

$$\Gamma^{\sigma,\mathbf{s}}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k}) \equiv \bigcup_{h>0} \Gamma^{\sigma,\mathbf{s};h}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k})$$

and

$$\Gamma^{\sigma,\mathbf{s};0}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k}) \equiv \bigcap_{h>0} \Gamma^{\sigma,\mathbf{s};h}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k}),$$

and their topologies are, respectively, the inductive and the projective limit topologies of $\Gamma^{\sigma,\mathbf{s};h}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k})$ with respect to $h > 0$.

As it is common in the theory of ultradifferentiable functions, we say that (the inductive limit) $\Gamma^{\sigma,\mathbf{s}}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k})$ is a Roumieu class, and (the projective limit) $\Gamma^{\sigma,\mathbf{s};0}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k})$ is a Beurling class of test functions.

Notice that $\Gamma^{\sigma,\mathbf{s}}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k})$ and $\Gamma^{\sigma,\mathbf{s};0}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k})$ are nontrivial for any $\sigma, s_j, h > 0$, $j = 1,\dots,k$. For instance by Proposition 1.6 for any $\omega \in \mathscr{P}_E(\mathbf{R}^{d_0+d_1+\dots+d_k})$ there exist a smooth function $\omega_0 \in \mathscr{P}_E(\mathbf{R}^{d_0+d_1+\dots+d_k})$ such that $\omega_0 \in \Gamma^{\sigma,\mathbf{s};0}_{(\omega)}(\mathbf{R}^{d_0+\dots+d_k})$.

When $k = 1$ we put $\sigma_1 = \sigma$ and recover the symbol classes $\Gamma^{\sigma,s}_{(\omega)}(\mathbf{R}^{d_0+d_1})$ and $\Gamma^{\sigma,s;0}_{(\omega)}(\mathbf{R}^{d_0+d_1})$ considered in [4].

Next we recall some facts on pseudo-differential operators. The pseudo-differential operator $\operatorname{Op}_t(a)$ is the linear and continuous operator on

$\mathcal{S}(\mathbf{R}^d)$, defined by the formula

$$\operatorname{Op}_t(a)f(x) = \frac{1}{(2\pi)^d} \iint a(x - t(x-y), \xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi, \quad x \in \mathbf{R}^d.$$

More generally, the definition of $\operatorname{Op}_t(a)$ extends uniquely to $a \in \mathcal{S}'(\mathbf{R}^{2d})$, and then $\operatorname{Op}_t(a)$ is continuous from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$.

Let $\mathbf{t} = (t_1, t_2, \dots, t_m) \in [0,1]^m$, be such that $\sum_{j=1}^m t_j \le 1$, and put $\vec{f} = (f_1, f_2, \dots, f_m) \in \mathscr{S}(\mathbf{R}^{md})$. The multilinear pseudo-differential operator $\operatorname{Op}_{\mathbf{t}}(a)$ from $\mathscr{S}(\mathbf{R}^{md})$ to $\mathscr{S}'(\mathbf{R}^d)$ is defined by the formula

$$\operatorname{Op}_{\mathbf{t}}(a)\vec{f}(x) = \frac{1}{(2\pi)^{2md}} \iint e^{-i\psi(x,\mathbf{y},\xi)} a_{\mathbf{t}}(x, \mathbf{y}, \xi) \prod_{j=1}^m f_j(y_j) d\mathbf{y} d\xi,$$

where

$$a_{\mathbf{t}}(x, \mathbf{y}, \xi) = a(x + \sum_{j=1}^m (t_j y_j - x), \xi, \eta), \quad x, y_j, \xi_j \in \mathbf{R}^d,$$

and the phase function ψ is defined by

$$\psi(x, \mathbf{y}, \xi) = \sum_{j=1}^m \langle y_j - x, \xi_j\rangle, \quad x, y_j, \xi_j \in \mathbf{R}^d.$$

When $m = 2$ we obtain bilinear pseudo-differential operators $\operatorname{Op}_{r,t}(a)$. That is, $\operatorname{Op}_{r,t}(a)$ is the bilinear and continuous operator from $\mathscr{S}(\mathbf{R}^d) \otimes \mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$, defined by the formula

$$\left(\operatorname{Op}_{r,t}(a)(f,g)\right)(x) = (2\pi)^{-2d} \iiiint e^{-i\psi(x,y,z,\xi,\eta)} a_{r,t}(x,y,z,\xi,\eta) f(y) g(z)\, dy dz d\xi d\eta,\ x \in \mathbf{R}^d, \tag{1.14}$$

where $(r,t) \in [0,1] \times [0,1]$, $r + t \le 1$,

$$a_{r,t}(x,y,z,\xi,\eta) = a(x + r(y-x) + t(z-x), \xi, \eta), \quad x,y,z,\xi,\eta \in \mathbf{R}^d,$$

and the phase function ψ is defined by

$$\psi(x,y,z,\xi,\eta) = \langle y - x, \xi\rangle + \langle z - x, \eta\rangle, \quad x,y,z,\xi,\eta \in \mathbf{R}^d.$$

If $r = t = 0$, then the definition of $\operatorname{Op}_0(a)$ coincides with the definition of bilinear pseudo-differential operators

$$T_a(f,g)(x) = (2\pi)^{-d} \iint e^{i\langle x, \xi+\eta\rangle} a(x,\xi,\eta) \widehat{f}(\xi) \widehat{g}(\eta)\, d\xi d\eta, \quad x \in \mathbf{R}^d,$$

considered in e.g [8], and the corresponding multilinear extension is studied in [32].

In fact, in Sections 2 and 3 we will consider the action of $\operatorname{Op}_{r,t}(a)$ when restricted to different Gelfand-Shilov spaces, and related unique extension of such operators to Gelfand-Shilov distributions.

We will use the following results about the continuity of linear pseudo-differential operators with symbols in Gevrey-Hörmander classes, and we refer to [4, Theorem 2.1] and [1, Theorem 3.7] respectively, for the proofs.

Proposition 1.9. *Let $s,\sigma \ge 1$, $p,q \in [1,\infty]$, $\omega,\omega_0 \in \mathscr{P}^0_{s,\sigma}(\mathbf{R}^{2d})$, and $a \in \Gamma^{\sigma,s}_{(\omega_0)}(\mathbf{R}^{2d})$. Then $\operatorname{Op}_t(a)$ is a continuous operators from $M^{p,q}_{(\omega_0\omega)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ for any $t \in [0,1]$.*

Note that, in the notation of Definition 1.5 we have $\mathscr{P}^0_{s,\sigma}(\mathbf{R}^{2d}) = \mathscr{P}^0_{s_0,s_1}(\mathbf{R}^{d_0+d_1})$ when $s_0 = s$, $s_1 = \sigma$ and $d_1 = d_2 = d$.

Proposition 1.10. *Let $A \in \mathbf{M}(d,R)$, $s,\sigma > 0$ be such that $s+\sigma \ge 1$, $\omega \in \mathscr{P}^0_{s,\sigma}(\mathbf{R}^{2d})$ and let $a \in \Gamma^{\sigma,s;h}_0(\mathbf{R}^{2d})$ for some $h > 0$. Then $\operatorname{Op}_A(a)$ is continuous on $\mathcal{S}^\sigma_s(\mathbf{R}^d)$ and on $(\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$.*

2. Characterization and invariance property for bilinear pseudo-differential operators

Our aim in this section is to show that $\Gamma^{\sigma,s_1,s_2}_{(\omega)}(\mathbf{R}^{3d})$ and $\Gamma^{\sigma,s_1,s_2;0}_{(\omega)}(\mathbf{R}^{3d})$ can be characterized in terms of estimates of short-time Fourier transforms and modulation spaces. This is done in subsection 2.2. We refer to [1,49] for similar results related to "standard" pseudo-differential operators. As a preparation, we show that $\operatorname{Op}_{r,t}(a)$ is independent of the choice of r and t, which gives the invariance property for bilinear operators, Theorem 2.2. The counterpart of Theorem 2.2 for "standard" pseudo-differential is proved in e.g. [1,4,49]. The key tools we employ to achieve the desired characterizations and invariance properties are mapping results for exponentials of certain linear operators, similar to the typical ones often appearing in the "usual" Weyl-Hörmander calculus, whose description is given here below.

2.1. Mapping properties of exponential-type operators on Gelfand-Shilov spaces.

For the study of mapping properties of the operator $e^{-i\langle rD_\xi + tD_\eta, D_x\rangle}$ we need the following auxiliary result. By $\mathbf{M}(d,\mathbf{R})$ we denote the set of all $d\times d$-matrices with entries in $\mathbf{R}$.

Lemma 2.1. *Let $A,B \in \mathbf{M}(d,\mathbf{R})$ and $a \in \mathscr{S}(\mathbf{R}^{3d})$. Then*

$$\big(\mathscr{F}^{-1}_{2,3}(e^{i\langle AD_\xi + BD_\eta, D_x\rangle}a)\big)(x+Ay+Bz,y,z) = (\mathscr{F}^{-1}_{2,3}a)(x,y,z), \quad (2.1)$$

$x,y,z \in \mathbf{R}^d$.

Proof. Throughout the proof, the integrals are observed as either Fourier transforms or inverse Fourier transforms of appropriate distributions.

The left-hand side of (2.1) is given by

$$
\begin{aligned}
&\big(\mathscr{F}_{2,3}^{-1}(e^{i\langle AD_\xi+BD_\eta,D_x\rangle}a)\big)(x+Ay+Bz,y,z)\\
&\qquad=\iint e^{i(\langle y,\xi\rangle+\langle z,\eta\rangle)}\left(e^{i\langle AD_\xi+BD_\eta,D_x\rangle}a)\right)(x+Ay+Bz,\xi,\eta)\,d\xi d\eta,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(e^{i\langle AD_\xi+BD_\eta,D_x\rangle}a)\right)(x+Ay+Bz,\xi,\eta)\\
&=\iiint e^{i(\langle x+Ay+Bz,\zeta\rangle+\langle y_1,\xi\rangle+\langle z_1,\eta\rangle)}e^{i(\langle Ay_1+Bz_1,\zeta\rangle)}\widehat{a}(\zeta,y_1,z_1)\,d\zeta dy_1dz_1,
\end{aligned}
$$

where

$$
\widehat{a}(\zeta,y_1,z_1)=\iiint e^{-i(\langle x_1,\zeta\rangle+\langle y_1,\xi_1\rangle+\langle z_1,\eta_1\rangle)}a(x_1,\xi_1,\eta_1)\,dx_1d\xi_1d\eta_1,
$$

$x_1,\xi_1,\eta_1\in\mathbf{R}^d$. Let $\Psi\equiv\Psi(x,x_1,y,y_1,z,z_1,\zeta,\xi,\eta,\xi_1,\eta_1)$ be given by

$$
\begin{aligned}
\Psi=\langle y+y_1,\xi\rangle&+\langle z+z_1,\eta\rangle+\langle x+Ay+Bz,\zeta\rangle\\
&+(\langle Ay_1+Bz_1,\zeta\rangle)-(\langle x_1,\zeta\rangle+\langle y_1,\xi_1\rangle+\langle z_1,\eta_1\rangle),
\end{aligned}
$$

$x,x_1,y,y_1,z,z_1,\zeta,\xi,\eta,\xi_1,\eta_1\in\mathbf{R}^d$. It follows that

$$
\begin{aligned}
&\big(\mathscr{F}_{2,3}^{-1}(e^{i\langle AD_\xi+BD_\eta,D_x\rangle}a)\big)(x+Ay+Bz,y,z)\\
&\qquad\qquad=\idotsint e^{i\Psi(x,x_1,y,y_1,z,z_1,\zeta,\xi,\eta,\xi_1,\eta_1)}\\
&\qquad\qquad\quad\times a(x_1,\xi_1,\eta_1)\,dx_1d\xi_1d\eta_1\,d\zeta dy_1dz_1\,d\xi d\eta,\quad x,y,z\in\mathbf{R}^d. \qquad (2.2)
\end{aligned}
$$

Since

$$
\int e^{i\langle y+y_1,\xi\rangle}\,d\xi=\delta(y+y_1),\qquad\text{and}\quad\int e^{i\langle z+z_1,\eta\rangle}\,d\eta=\delta(z+z_1),
$$

where δ is the Dirac delta distribution, it follows that (2.2) reduces to

$$
\begin{aligned}
&\iiiint e^{i(\langle x,\zeta\rangle-\langle x_1,\zeta\rangle+\langle y,\xi_1\rangle+\langle z,\eta_1\rangle)}a(x_1,\xi_1,\eta_1)\,d\zeta dx_1d\xi_1d\eta_1\\
&\qquad=\iint e^{i(\langle y,\xi_1\rangle+\langle z,\eta_1\rangle)}a(x,\xi_1,\eta_1)\,d\xi_1d\eta_1=\left(\mathscr{F}_{2,3}^{-1}a\right)(x,y,z),
\end{aligned}
$$

and the claim follows. □

Next we show some mapping properties of the operator $e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}$ which are an important ingredient in our analysis.

Theorem 2.2. *Let s_j,σ_j, $j=1,2,3$, be such that*

$$
s_j+\sigma_j\ge 1,\qquad 0<s_2,\,s_3\le s_1,\quad\text{and}\quad 0<\sigma_1\le\sigma_2,\,\sigma_3 \qquad (2.3)
$$

and let $r,t\in[0,1]$ be such that $r+t\le 1$. Then the following is true:

(1) $e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}$ *on* $\mathscr{S}(\mathbf{R}^{3d})$ *restricts to a homeomorphism on* $\mathcal{S}_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3}(\mathbf{R}^{3d})$*, and extends uniquely to a homeomorphism on* $(\mathcal{S}_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3})'(\mathbf{R}^{3d})$*;*

(2) *if in addition* $(s_j,\sigma_j)\neq(\frac{1}{2},\frac{1}{2})$*,* $j=1,2,3$*, then* $e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}$ *on* $\mathscr{S}(\mathbf{R}^{3d})$ *restricts to a homeomorphism on* $\Sigma_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3}(\mathbf{R}^{3d})$*, and extends uniquely to a homeomorphism on* $(\Sigma_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3})'(\mathbf{R}^{3d})$*.*

Proof. We only prove (1) and leave (2) for the reader.

Let $a\in\mathscr{S}(\mathbf{R}^{3d})$ and let $U_{r,t}$ be the map given by

$$(U_{r,t}F)(x,y)=F(x-ry-tz,y,z),\quad x,y\in\mathbf{R}^d.$$

By Lemma 2.1, we have

$$\big(\mathscr{F}_{2,3}^{-1}(e^{i\langle rD_\xi+tD_\eta,D_x\rangle}a)\big)(x+ry+tz,y,z)=(\mathscr{F}_{2,3}^{-1}a)(x,y,z),\quad x,y,z\in\mathbf{R}^d,$$

wherefrom $e^{i\langle rD_\xi+tD_\eta,D_x\rangle}=\mathscr{F}_{2,3}\circ U_{r,t}\circ\mathscr{F}_{2,3}^{-1}$. Therefore it only remains to show that the mapping $U_{r,t}$ is continuous on $\mathcal{S}_{s_1,s_2,s_3}^{\sigma_1,\sigma_2,\sigma_3}$.

Since the Fourier transform with respect to the 2^{nd} and 3^{rd} variables switches between the corresponding decay and regularity properties on Gelfand-Shilov spaces we consider $G=U_{r,t}F$, where $F\in\mathcal{S}_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3}$. Then

$$G(x,y,z)=F(x-ry-tz,y,z)\quad\text{and}\quad\widehat{G}(\zeta,\xi,\eta)=\widehat{F}(\zeta,\zeta+r\xi,\zeta+t\eta),$$

$x,y,z,\zeta,\xi,\eta\in\mathbf{R}^d$. In view of Proposition 1.3 and from the assumptions on s_j and σ_j, it follows that there exist constants $c,r_0>0$, where c depends on r,t, s_j and σ_j only, such that

$$|G(x,y,z)|=|F(x-ry-tz,y,z)|$$
$$\lesssim e^{-r_0(|x-ry-tz|^{\frac{1}{s_1}}+|y|^{\frac{1}{s_2}}+|z|^{\frac{1}{s_3}})}\lesssim e^{-cr_0(|x|^{\frac{1}{s_1}}+|y|^{\frac{1}{s_2}}+|z|^{\frac{1}{s_3}})},$$

$x,y,z\in\mathbf{R}^d$, and

$$|\widehat{G}(\zeta,\xi,\eta)|=|\widehat{F}(\zeta,\zeta+r\xi,\zeta+t\eta)|$$
$$\lesssim e^{-r(|\zeta|^{\frac{1}{\sigma_1}}+|\zeta+r\xi|^{\frac{1}{\sigma_2}}|\zeta+t\eta|^{\frac{1}{\sigma_3}})}\lesssim e^{-cr(|\zeta|^{\frac{1}{\sigma_1}}+|\xi|^{\frac{1}{\sigma_2}}+|\eta|^{\frac{1}{\sigma_3}})},$$

$\zeta,\xi,\eta\in\mathbf{R}^d$. The result follows since by Proposition 1.4 the topology in $\mathcal{S}_{s_1,s_2,s_3}^{\sigma_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d})$ can be defined by the above estimates.

□

Corollary 2.3. *Let* $s,\sigma>0$ *be such that* $s+\sigma\geq1$ *and* $\sigma\leq s$*. Then* $e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}$ *is a homeomorphism on* $\mathcal{S}_s^\sigma(\mathbf{R}^{3d})$*,* $\Sigma_s^\sigma(\mathbf{R}^{3d})$*,* $(\mathcal{S}_s^\sigma)'(\mathbf{R}^{3d})$ *and on* $(\Sigma_s^\sigma)'(\mathbf{R}^{3d})$*.*

Next we study an invariance property of bilinear pseudo-differential operators $\operatorname{Op}_{r,t}(a)$ given by (1.14). More precisely, it can be shown that for every Gelfand-Shilov distribution a there is a unique distribution b in the same Gelfand-Shilov class such that $\operatorname{Op}_{r_1,t_1}(a)=\operatorname{Op}_{r_2,t_2}(b)$, when $r_j,t_j\in[0,1]$ and $r_j+t_j\leq1$. The following result, which explains the relation between such a and b, follows from Theorem 2.2 when the conditions in (2.3) are fulfilled. We refer to [3, Appendix A] for an independent proof.

Proposition 2.4. *Let $r_j, t_j \in [0,1]$ be such that $r_j + t_j \le 1$, and let $a, b \in (\mathcal{S}_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3})'(\mathbf{R}^{3d})$, where $s_j, \sigma_j > 0$, and $s_j + \sigma_j \ge 1$, $j = 1,2,3$. Then*

$$\begin{gathered}\operatorname{Op}_{r_1,t_1}(a) = \operatorname{Op}_{r_2,t_2}(b)\\ \Leftrightarrow \\ e^{-i\langle r_1 D_\xi + t_1 D_\eta, D_x\rangle} a(x,\xi,\eta) = e^{-i\langle r_2 D_\xi + t_2 D_\eta, D_x\rangle} b(x,\xi,\eta), \quad x,\xi,\eta \in \mathbf{R}^d.\end{gathered} \tag{2.4}$$

Note that the latter equality in (2.4) makes sense since it is equivalent to

$$e^{-i\langle r_1 y + t_1 z, \zeta\rangle} \widehat{a}(\zeta, y, z) = e^{-i\langle r_2 y + t_2 z, \zeta\rangle} \widehat{b}(\zeta, y, z), \quad \zeta, y, z \in \mathbf{R}^d.$$

Moreover, by using the similar arguments as in e. g. [1,10,51], it can be shown that the map $a \mapsto e^{-i\langle ry+tz,\zeta\rangle} a$ is continuous on $(\mathcal{S}_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3})'(\mathbf{R}^{3d})$.

The following corollary is a consequence of [10, Theorem 4.6] and Proposition 2.4.

Corollary 2.5. *Let $s_j, \sigma_j > 0$ be such that $s_j + \sigma_j \ge 1$, $(s_j, \sigma_j) \ne (\frac{1}{2}, \frac{1}{2})$, $j = 1,2,3$ and $r_j, t_j \in [0,1]$ be such that $r_j + t_j \le 1$, $j = 1,2$. If $a, b \in (\Sigma_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3})'(\mathbf{R}^{3d})$, are such that $\operatorname{Op}_{r_1,t_1}(a) = \operatorname{Op}_{r_2,t_2}(b)$ Then*

$$a \in \Gamma_{(\omega)}^{\sigma_1,s_2,s_3;0}(\mathbf{R}^{3d}) \qquad \Leftrightarrow \qquad b \in \Gamma_{(\omega)}^{\sigma_1,s_2,s_3;0}(\mathbf{R}^{3d})$$

and

$$a \in \Gamma_{(\omega)}^{\sigma_1,s_2,s_3}(\mathbf{R}^{3d}) \qquad \Leftrightarrow \qquad b \in \Gamma_{(\omega)}^{\sigma_1,s_2,s_3}(\mathbf{R}^{3d}),$$

for any given $\omega \in \mathscr{P}_E(\mathbf{R}^{3d})$.

Passages between different kinds of pseudo-differential calculi have been considered before [28, 51]. On the other hand, for the bilinear pseudo-differential calculi, it seems that the representation $a \mapsto \operatorname{Op}_{r,t}(a)$ for $(r,t) \in [0,1] \times [0,1]$ such that $r + t \le 1$, has not been considered so far.

2.2. **Gevrey-type symbol classes characterizations.** Our first result concerns the Roumieu case of symbols in $\Gamma_{(\omega)}^{\sigma,s,s}(\mathbf{R}^{3d})$. It can be deduced from [10, Proposition 4.3], see also [1, Proposition 2.4]. For the sake of completeness, we give the proof.

Proposition 2.6. *Let $s_j, \sigma_j > 0$, $j = 1,2,3$, be such that the conditions in (2.3) hold, let $\omega \in \mathscr{P}_{s_1,\sigma_2,\sigma_3}^0(\mathbf{R}^{3d})$ and let a be a Gelfand-Shilov distribution on $\mathbf{R}^{3d}$.*

Then the following conditions are equivalent:

(1) $a \in \Gamma^{\sigma_1,s_2,s_3}_{(\omega)}(\mathbf{R}^{3d})$, *that is,* $a \in C^\infty(\mathbf{R}^{3d})$ *and*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma a(x,\xi,\eta)| \lesssim h^{|\alpha+\beta+\gamma|}\alpha!^{\sigma_1}\beta!^{s_2}\gamma!^{s_3}\omega(x,\xi,\eta), \quad x,\xi,\eta \in \mathbf{R}^d,$$

for every $\alpha,\beta,\gamma \in \mathbf{N}^d$ *and some* $h>0$*;*

(2) *For every* $\phi \in \mathcal{S}^{\sigma_1,s_2,s_3}_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d}) \setminus 0$*, there exist constants* $h,R>0$ *such that for every* $\alpha,\beta,\gamma \in \mathbf{N}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \left(e^{i(\langle x,\zeta\rangle+\langle y,\xi\rangle+\langle z,\eta\rangle)}V_\phi a(x,\xi,\eta,\zeta,y,z)\right)| \lesssim h^{|\alpha+\beta+\gamma|}\alpha!^{\sigma_1}\beta!^{s_2}\gamma!^{s_3}\omega(x,\xi,\eta)e^{-R(|\zeta|^{\frac{1}{\sigma_1}}+|y|^{\frac{1}{s_2}}+|z|^{\frac{1}{s_3}})}, \tag{2.5}$$

$x,\xi,\eta,\zeta,y,z \in \mathbf{R}^d$.

(3) *For every* $\phi \in \mathcal{S}^{\sigma_1,s_2,s_3}_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d}) \setminus 0$*, there exist a constant* $R>0$ *such that*

$$|V_\phi a(x,\xi,\eta,\zeta,y,z)| \lesssim \omega(x,\xi,\eta)e^{-R(|\zeta|^{\frac{1}{\sigma_1}}+|y|^{\frac{1}{s_2}}+|z|^{\frac{1}{s_3}})}, \quad x,\xi,\eta,\zeta,y,z \in \mathbf{R}^d. \tag{2.6}$$

Proof. That (2) implies (3) is immediate, since (2.5) is equal to (2.6) when $\alpha=\beta=\gamma=0$.

Let $X=(x,\xi,\eta)$, $Y=(x_1,\xi_1,\eta_1)$, $Z=(\zeta,y,z) \in \mathbf{R}^{3d}$, and set

$$F_a(X,Y) = a(X+Y)\phi(Y) = a(x+x_1,\xi+\xi_1,\eta+\eta_1)\phi(x_1,\xi_1,\eta_1).$$

Assume that (1) holds true. By the Leibniz rule, (1.10) and Proposition 1.4 we obtain

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma F_a(x,\xi,\eta,x_1,\xi_1,\eta_1)| \lesssim h^{|\alpha+\beta+\gamma|}\alpha!^{\sigma_1}\beta!^{s_2}\gamma!^{s_3}\omega(x,\xi,\eta)e^{-R(|x_1|^{\frac{1}{s_1}}+|\xi_1|^{\frac{1}{\sigma_2}}+|\eta_1|^{\frac{1}{\sigma_3}})},$$

$x,\xi,\eta,x_1,\xi_1,\eta_1 \in \mathbf{R}^d$, for some constants $h,R>0$.

It follows that the set

$$\left\{ G_{a,h,X}(Y) \;\middle|\; G_{a,h,x,\xi,\eta}(x_1,\xi_1,\eta_1) = \frac{\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma F_a(x,\xi,\eta,x_1,\xi_1,\eta_1)}{h^{|\alpha+\beta+\gamma|}\alpha!^{\sigma_1}\beta!^{s_2}\gamma!^{s_3}\omega(x,\xi,\eta)} \right\}$$

is bounded in $\mathcal{S}^{\sigma_1,s_2,s_3}_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d})$. If $\mathscr{F}_2 F_a$ denotes the partial Fourier transform of $F_a(X,Y)$ with respect to the Y-variable, then we get

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma (\mathscr{F}_2 F_a)(x,\xi,\eta,\zeta,y,z)| \lesssim h^{|\alpha+\beta+\gamma|}\alpha!^{\sigma_1}\beta!^{s_2}\gamma!^{s_3}\omega(x,\xi,\eta)e^{-R(|y|^{\frac{1}{\sigma_1}}+|z|^{\frac{1}{s_2}}+|\zeta|^{\frac{1}{s_3}})},$$

$x,\xi,\eta,x_1,\xi_1,\eta_1 \in \mathbf{R}^d$, for some constants $h,R>0$. This, together with the Leibnitz rule applied to $\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \left(e^{i(\langle x,\zeta\rangle+\langle y,\xi\rangle+\langle z,\eta\rangle)}V_\phi a(x,\xi,\eta,\zeta,y,z)\right)$ gives (2).

Assume now that (3) holds. By the inversion formula we get

$$a(X) = \frac{(2\pi)^{-\frac{3d}{2}}}{\|\phi\|_{L^2}^2} \iint V_\phi a(Y,Z)\phi(X-Y)e^{i\langle X,Z\rangle}\, dY dZ, \quad X \in \mathbf{R}^{3d}, \tag{2.7}$$

in the weak sense. Since $\phi \in \mathcal{S}_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3}(\mathbf{R}^{3d})$ we notice that

$$(X,Y,Z) \mapsto V_\phi a(Y,Z)\phi(X-Y)e^{i\langle X,Z\rangle}$$

is a smooth map, and

$$(Y,Z) \mapsto Z^{\boldsymbol{\alpha}} V_\phi a(Y,Z)\partial^{\boldsymbol{\beta}}\phi(X-Y)e^{i\langle X,Z\rangle}$$

is an integrable function for every $X \in \mathbf{R}^{3d}$, $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\alpha_3) \in \mathbf{N}^{3d}$, and $\boldsymbol{\beta} = (\beta_1,\beta_2,\beta_3) \in \mathbf{N}^{3d}$ in view of (3). Hence the derivatives of a in (2.7) satisfy the following estimates:

$$|\partial^{\boldsymbol{\alpha}} a(X)| \le \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \iint |Z^{\boldsymbol{\beta}} V_\phi a(Y,Z)(\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\phi)(X-Y)|\, dY dZ$$

$$\lesssim \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \iint |Z^{\boldsymbol{\beta}} \omega(Y) e^{-R(|\zeta|^{\frac{1}{\sigma_1}}+|y|^{\frac{1}{s_2}}+|z|^{\frac{1}{s_3}})}(\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\phi)(X-Y)|\, dY dZ$$

$$\lesssim \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} h_2^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|}(\alpha_1-\beta_1)!^{\sigma_1}(\alpha_2-\beta_2)!^{s_2}(\alpha_3-\beta_3)!^{s_3}$$

$$\times \iint |Z^{\boldsymbol{\beta}}| e^{-R(|\zeta|^{\frac{1}{\sigma_1}}+|y|^{\frac{1}{s_2}}+|z|^{\frac{1}{s_3}})}\omega(Y)e^{-h_1(|x-x_1|^{\frac{1}{s_1}}+|\xi-\xi_1|^{\frac{1}{\sigma_2}}+|\eta-\eta_1|^{\frac{1}{\sigma_3}})}\, dY dZ,$$

$X \in \mathbf{R}^{3d}$, for some constants $h_1, h_2 > 0$, and we used Lemma 1.1. For any $\boldsymbol{\beta} \in \mathbf{N}^{3d}$, $\sigma, s_j > 0$ such that $s_j + \sigma \ge 1$, $j = 1,2$, and $h_2, R > 0$, it holds

$$|\zeta^{\beta_1} y^{\beta_2} z^{\beta_3} e^{-R(|\zeta|^{\frac{1}{\sigma}}+|y|^{\frac{1}{s}}+|z|^{\frac{1}{s}})}| \lesssim h_2^{|\boldsymbol{\beta}|}\beta_1!^{\sigma_1}\beta_2!^{s_2}\beta_3!^{s_3} e^{-\frac{R}{2}(|\zeta|^{\frac{1}{\sigma_1}}+|y|^{\frac{1}{s_2}}+|z|^{\frac{1}{s_3}})},$$

$\zeta, y, z \in \mathbf{R}^d$, so that

$$|\partial^{\boldsymbol{\alpha}} a(X)|$$

$$\lesssim h_2^{|\boldsymbol{\alpha}|} \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} (\beta_1!(\alpha_1-\beta_1)!)^{\sigma_1}(\beta_2!(\alpha_2-\beta_2)!)^{s_2}(\beta_3!(\alpha_3-\beta_3)!)^{s_3}$$

$$\times \iint e^{-\frac{R}{2}(|\zeta|^{\frac{1}{\sigma_1}}+|y|^{\frac{1}{s_2}}+|z|^{\frac{1}{s_3}})}\omega(Y)e^{-h_1(|x-x_1|^{\frac{1}{s_1}}+|\xi-\xi_1|^{\frac{1}{\sigma_2}}+|\eta-\eta_1|^{\frac{1}{\sigma_3}})}\, dY dZ$$

$$\lesssim (2h_2)^{|\boldsymbol{\alpha}|}\alpha_1!^{\sigma_1}\alpha_2!^{s_2}\alpha_3!^{s_3}$$

$$\times \int \omega(X+(Y-X))e^{-h_1(|x-x_1|^{\frac{1}{s_1}}+|\xi-\xi_1|^{\frac{1}{\sigma_2}}+|\eta-\eta_1|^{\frac{1}{\sigma_3}})}\, dY, \quad X \in \mathbf{R}^{3d}. \tag{2.8}$$

Since $\omega \in \mathscr{P}^0_{s_1,\sigma_2,\sigma_3}$, that is (1.10) holds for every $r > 0$, by choosing $r \in (0, h_1/2)$, from (2.8) it follows that

$$|\partial^{\boldsymbol{\alpha}} a(X)| \lesssim (2h_2)^{|\boldsymbol{\alpha}|} \alpha_1!^{\sigma_1} \alpha_2!^{s_2} \alpha_3!^{s_3} \omega(X), \quad X \in \mathbf{R}^{3d},$$

for some constant $h_2 > 0$ (and we conclude that (2.7) holds also in the pointwise sense). Therefore (3) implies (1) and the result follows. $\square$

We refer to Proposition 2.7 in the extended version of this paper [3] for the Beurling case which can be proved by following similar arguments as in the proof of Proposition 2.6.

In the next result however, we consider the Beurling case and give a description of the symbol class $\Gamma^{\sigma_1,s_2,s_3;0}_{(\omega)}(\mathbf{R}^{3d})$ in terms of modulation spaces $M^{\infty,q}_{(1/\omega_R)}(\mathbf{R}^{3d})$ for $q \in [1, \infty]$ and ω_R defined in (2.9) below. To prove Proposition 2.7 we follow arguments analogous to those used in the proofs of [1, Proposition 3.5] and [10, Proposition 4.4].

We leave for the reader to write down Proposition 2.7 when the (Roumieu case) symbol class $\Gamma^{\sigma_1,s_2,s_3}_{(\omega)}(\mathbf{R}^{3d})$ is considered instead.

Proposition 2.7. *Let $R > 0$, $q \in [1, \infty]$, $s_j, \sigma_j > 0$, $(s_j, \sigma_j) \neq (\frac{1}{2}, \frac{1}{2})$, $j = 1, 2, 3$, and let the conditions in (2.3) hold. Also, let $\phi, \phi_0 \in \Sigma^{\sigma_1,s_2,s_3}_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d}) \setminus 0$, $\omega \in \mathscr{P}_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d})$, and let*

$$\omega_R(x, \xi, \eta, \zeta, y, z) = \omega(x, \xi, \eta) e^{-R(|\zeta|^{\frac{1}{\sigma_1}} + |y|^{\frac{1}{s_2}} + |z|^{\frac{1}{s_3}})}. \tag{2.9}$$

Then

$$\Gamma^{\sigma_1,s_2,s_3;0}_{(\omega)}(\mathbf{R}^{3d}) = \bigcap_{R>0} \{\, a \in (\Sigma^{\sigma_1,s_2,s_3}_{s_1,\sigma_2,\sigma_3})'(\mathbf{R}^{3d})\,;\, \|\omega_R^{-1} V_\phi a\|_{L^{\infty,q}(\mathbf{R}^{3d}\times\mathbf{R}^{3d})} < \infty \,\}. \tag{2.10}$$

Proof. When $q = \infty$, (2.10) becomes

$$\Gamma^{\sigma_1,s_2,s_3;0}_{(\omega)} = \bigcap_{R>0} M^{\infty}_{(1/\omega_R)}(\mathbf{R}^{3d}),$$

which is a straightforward consequence of Proposition **??**. Therefore it is enough to prove that

$$\bigcap_{R>0} M^{\infty}_{(1/\omega_R)}(\mathbf{R}^{3d}) = \bigcap_{R>0} \{\, a \in (\Sigma^{\sigma_1,s_2,s_3}_{s_1,\sigma_2,\sigma_3})'(\mathbf{R}^{3d})\,;\, \|\omega_R^{-1} V_\phi a\|_{L^{\infty,q}(\mathbf{R}^{3d}\times\mathbf{R}^{3d})} < \infty \,\}.$$

Put

$$V_{0,a}(X, Y) = |(V_{\phi_0} a)(x, \xi, \eta, \zeta, y, z)|, \quad V_a(X, Y) = |(V_\phi a)(x, \xi, \eta, \zeta, y, z)|$$

$$\text{and} \quad G(x, \xi, \eta, \zeta, y, z) = |(V_\phi \phi_0)(x, \xi, \eta, \zeta, y, z)|,$$

where $X=(x,\xi,\eta)\in \mathbf{R}^{3d}$ and $Y=(\zeta,y,z)\in \mathbf{R}^{3d}$. By Proposition 1.7 we have

$$0\le G(x,\xi,\eta,\zeta,y,z)\lesssim e^{-R(|x|^{\frac{1}{s_1}}+|\xi|^{\frac{1}{\sigma_2}}+|\eta|^{\frac{1}{\sigma_3}}+|\zeta|^{\frac{1}{\sigma_1}}+|y|^{\frac{1}{s_2}}+|z|^{\frac{1}{s_3}})}, \qquad (2.11)$$

$x,\xi,\eta,\zeta,y,z\in \mathbf{R}^d$, for every $R>0$.

From [25, Lemma 11.3.3] (when extended to the duality between Gelfand-Shilov spaces and their dual spaces of distributions) it follows that $V_a\lesssim V_{0,a}*G$, so we obtain

$$(\omega_R^{-1}\cdot V_a)(X,Y)\lesssim \big((\omega_{cR}^{-1}\cdot V_{0,a})*G_1\big)(X,Y),,\quad X,Y\in \mathbf{R}^{3d}, \qquad (2.12)$$

for some G_1 which satisfies (2.11), and for a constant $c>0$ independent of R. By applying the L^∞-norm on the both sides of (2.12) we obtain

$$\begin{aligned}\|\omega_R^{-1}V_a\|_{L^\infty(\mathbf{R}^{6d})} &= \sup_Y\sup_X \big|\omega_R^{-1}V_a(X,Y)\big|\\ &\lesssim \sup_Y\sup_X\big|\omega_{cR}^{-1}V_{0,a}*G_1(X,Y)\big|\end{aligned}$$

$$\lesssim \sup_Y\Big(\iint\big(\sup_X(\omega_{cR}^{-1}\cdot V_{0,a})(X,Y-Y_1)\big)G_1(X_1,Y_1)\,dX_1dY_1\Big)$$

$$\le \|\omega_{cR}^{-1}\cdot V_{0,a}\|_{L^{\infty,q}}\|G_1\|_{L^{1,q'}}\asymp \|\omega_{cR}^{-1}\cdot V_{0,a}\|_{L^{\infty,q}},$$

wherefrom

$$\bigcap_{R>0}\{\, a\in (\Sigma_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3})'(\mathbf{R}^{3d})\,;\,\|\omega_R^{-1}V_\phi a\|_{L^{\infty,q}(\mathbf{R}^{3d}\times\mathbf{R}^{3d})}<\infty\,\}$$

$$\subset \bigcap_{R>0}M^\infty_{(1/\omega_R)}(\mathbf{R}^{3d}).$$

For the opposite inclusion we put $K_j=\omega_{jcR}^{-1}\cdot V_{0,a}$, $j=1,2$. By (2.12) and Minkowski's inequality we have

$$\begin{aligned}\|\omega_R^{-1}\cdot V_a\|^q_{L^{\infty,q}} &\lesssim \|\,((K_1*G)\,\|^q_{L^{\infty,q}}\\ &\lesssim \int\sup_X\Big(\iint K_1(X-X_1,Y-Y_1)G(X_1,Y_1)\,dX_1dY_1\Big)^q\,dY\end{aligned}$$

$$\lesssim \int\Big(\iint \sup\,(K_2(\,\cdot\,,Y-Y_1))\,G(X_1,Y_1)$$

$$\times e^{-cR(|\zeta-\zeta_1|^{\frac{1}{\sigma_1}}+|y-y_1|^{\frac{1}{s_2}}+|z-z_1|^{\frac{1}{s_3}})}\,dX_1dY_1\Big)^q\,dY$$

$$\lesssim \|K_2\|^q_{L^\infty}\int\Big(\iint G(X_1,Y_1)e^{-cR(|\zeta-\zeta_1|^{\frac{1}{\sigma_1}}+|y-y_1|^{\frac{1}{s_2}}+|z-z_1|^{\frac{1}{s_3}})}\,dX_1dY_1\Big)^q\,dY$$

$$\lesssim \|K_2\|^q_{L^\infty}\equiv \|\omega_{2cR}^{-1}\cdot V_{0,a}\|^q_{L^\infty}.$$

Finally, by interchanging the roles of ϕ and ϕ_0 we get

$$\|\omega_R^{-1}\cdot V_{0,a}\|_{L^{\infty,q}} \lesssim \|\omega_{2cR}^{-1}\cdot V_a\|_{L^\infty},$$

i.e.

$$\bigcap_{R>0} M^\infty_{(1/\omega_R)}(\mathbf{R}^{3d})$$
$$\subset \bigcap_{R>0}\{\, a\in (\Sigma^{\sigma_1,s_2,s_3}_{s_1,\sigma_2,\sigma_3})'(\mathbf{R}^{3d})\,;\, \|\omega_R^{-1}V_\phi a\|_{L^{\infty,q}(\mathbf{R}^{3d}\times\mathbf{R}^{3d})}<\infty\,\},$$

and the result follows. □

In [10, Theorem 4.1], it is shown that if A is a $d\times d$-matrix with real entries, then the operator $e^{i\langle AD_\xi,D_x\rangle}$ is a homeomorphism between certain classes of symbols. We proceed with an analogous result in the context of the symbol class $\Gamma^{\sigma_1,s_2,s_3;0}_{(\omega)}(\mathbf{R}^{3d})$.

Theorem 2.8. *Let $s_j,\sigma_j>0$ be such that the conditions in (2.3) hold, and let $r,t\in[0,1]$ be such that $r+t\le 1$.*

If $\omega\in\mathscr{P}^0_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d})$, then $a\in\Gamma^{\sigma_1,s_2,s_3}_{(\omega)}(\mathbf{R}^{3d})$ if and only if

$$e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}a\in\Gamma^{\sigma_1,s_2,s_3}_{(\omega)}(\mathbf{R}^{3d}).$$

If $\omega\in\mathscr{P}_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d})$ instead, and if, in addition to (2.3), $(s_j,\sigma_j)\neq(\frac12,\frac12)$, $j=1,2,3$, then $a\in\Gamma^{\sigma_1,s_2,s_3;0}_{(\omega)}(\mathbf{R}^{3d})$ if and only if

$$e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}a\in\Gamma^{\sigma_1,s_2,s_3;0}_{(\omega)}(\mathbf{R}^{3d}).$$

Proof. We give the proof for the Beurling case, and the Roumieu case is left for the reader.

We will use the result of Proposition 2.7. Therefore we fix a window function $\phi\in\Sigma^{\sigma_1,s_2,s_3}_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d})$ and let $\phi_{r,t}=e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}\phi$. Then, in view of Theorem 2.2 (2), $\phi_{r,t}$ belongs to $\Sigma^{\sigma_1,s_2,s_3}_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d})$.

By similar arguments as in the proof of Lemma 2.1, we get

$$\begin{aligned}|(V_{\phi_{r,t}}(e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}a))(x,\xi,\eta,\zeta,y,z)|&\\ =|(V_\phi a)(x+ry+tz,\xi+r\zeta,\eta+t\zeta,\zeta,y,z)|,&\qquad(2.13)\end{aligned}$$

$x,\xi,\eta,\zeta,y,z\in\mathbf{R}^d$. Then using (2.13) and a change of variables argument, we get

$$\|\omega_{0,0;R}^{-1}V_\phi a\|_{L^{p,q}}=\|\omega_{r,t;R}^{-1}V_{\phi_{r,t}}(e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}a)\|_{L^{p,q}},$$

where

$$\omega_{r,t;R}(x,\xi,\eta,\zeta,y,z)=\omega(x+ry+tz,\xi+r\zeta,\eta+t\zeta)e^{-R(|y|^{\frac{1}{s_2}}+|z|^{\frac{1}{s_3}}+|\zeta|^{\frac{1}{\sigma_1}})},$$

and $p,q\in[1,\infty]$.

Hence Proposition 2.7, and the fact that there exists a constant $c>0$ such that

$$\omega_{0,0;R+c}\lesssim\omega_{r,t;R}\lesssim\omega_{0,0;R-c},$$

give

$$a\in\Gamma^{\sigma_1,s_2,s_3;0}_{(\omega)}(\mathbf{R}^{3d})\quad\Leftrightarrow\quad\|\omega^{-1}_{0,0;R}V_\phi a\|_{L^\infty}<\infty \text{ for every } R>0$$

$$\Leftrightarrow\quad\|\omega^{-1}_{r,t;R}V_{\phi_{r,t}}(e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}a)\|_{L^\infty}<\infty \text{ for every } R>0$$

$$\Leftrightarrow\ \|\omega^{-1}_{0,0;R}V_{\phi_{r,t}}(e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}a)\|_{L^\infty}<\infty \text{ for every } R>0$$

$$\Leftrightarrow\quad e^{-i\langle rD_\xi+tD_\eta,D_x\rangle}a\in\Gamma^{\sigma_1,s_2,s_3;0}_{(\omega)}(\mathbf{R}^{3d}),$$

and the result follows.

□

3. Continuity of bilinear pseudo-differential operators with symbols of Gevrey-regularity and infinite order

We first discuss the continuity of bilinear operators in $\operatorname{Op}(\Gamma^{\sigma_1,s_2,s_3}_{(\omega)})$ and $\operatorname{Op}(\Gamma^{\sigma_1,s_2,s_3;0}_{(\omega)})$ when acting on products of modulation spaces. In particular, Theorem 3.1 can be considered as an extension of [47, Theorem 3.2] to bilinear operators and a more general class of weights.

Theorem 3.1. *Let $s_j,\sigma_j>0$ be such that the conditions in* (2.3) *hold. Also, let $v_1\in\mathscr{P}^0_{s_1}(\mathbf{R}^d)$, $v_j\in\mathscr{P}^0_{\sigma_j}(\mathbf{R}^d)$, $j=2,3$, $\omega_0,\omega\in\mathscr{P}^0_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d})$, and let ω_0 be $\otimes_{j=1}^3 v_j$-moderate. Furthermore, let $r,t\in[0,1]$ such that $r+t\le 1$, and let $p,q\in[1,\infty]$. If $a\in\Gamma^{\sigma_1,s_2,s_3}_{(\omega_0)}(\mathbf{R}^{3d})$, then there exists $R>0$ such that $\operatorname{Op}_{r,t}(a)$ is continuous from $M^{p,q}_{(\omega_0\omega)}(\mathbf{R}^d)\times M^{\infty,\infty}_{(1/\omega_R)}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, where*

$$\omega_R(x,\xi,\eta)=e^{-R(|x|^{\frac{1}{s_1}}+|\xi|^{\frac{1}{\sigma_2}}+|\eta|^{\frac{1}{\sigma_3}})},\quad x,\xi,\eta\in\mathbf{R}^d.$$

Remark 3.2. We will use estimates similar to those obtained in the proof of [9, Theorem 6.1]. We observe that out arguments are anyway different since, in view of the fact that we employ Gevrey tpye symbols, we cannot rely on standard localization techniques. The idea is that for a fixed function g in appropriate space of test functions, $a\in\Gamma^{\sigma_1,s_2,s_3}_{(\omega)}(\mathbf{R}^{3d})$, and $r=t=0$, the operator $\operatorname{Op}_{0,0}(a)(\,\cdot\,,g)\equiv T_a(\,\cdot\,,g)$ can be regarded as a linear pseudo-differential operator (with symbol depending on g), that is,

$$\operatorname{Op}_{0,0}(a)(f,g)(x)=(2\pi)^{-\frac{d}{2}}\int e^{i\langle x,\xi\rangle}a_g(x,\xi)\widehat{f}(\xi)\,d\xi,$$

where

$$a_g(x,\xi)=(2\pi)^{-\frac{d}{2}}\int e^{i\langle x,\eta\rangle}a(x,\xi,\eta)\widehat{g}(\eta)\,d\eta. \tag{3.1}$$

If $a_g\in\Gamma^{\sigma_1,s_2}_{(\omega_0)}(\mathbf{R}^{2d})$, then the continuity of $\operatorname{Op}_{0,0}(a)(\,\cdot\,,g)$ from $M^{p.q}_{(\omega_0\omega)}$ to $M^{p.q}_{(\omega)}$ follows by Proposition 1.9.

Lemma 3.3. *Let $s_j, \sigma_j > 0$ be such that the conditions in* (2.3) *hold. Also, let $v_1 \in \mathscr{P}^0_{s_1}(\mathbf{R}^d)$, $v_j \in \mathscr{P}^0_{\sigma_j}(\mathbf{R}^d)$, $j = 2, 3$, $\omega_0, \omega \in \mathscr{P}^0_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d})$, and let ω_0 be $\otimes_{j=1}^3 v_j$-moderate.*

If $g \in \mathcal{S}^{\sigma_1}_{s_1}(\mathbf{R}^d)$ and $a \in \Gamma^{\sigma_1,s_2,s_3}_{(\omega_0)}(\mathbf{R}^{3d})$, then the symbol a_g given by (3.1) *belongs to $\Gamma^{\sigma_1,s_2}_{(\omega)}(\mathbf{R}^{2d})$, where $\omega(x,\xi) \equiv \omega_0(x,\xi,0) \in \mathscr{P}^0_{s_1,\sigma_2}(\mathbf{R}^{2d})$.*

Proof. By (3.1) it follows that a_g is a smooth function. Indeed,

$$(x,\xi,\eta) \mapsto e^{i\langle x,\eta\rangle} a(x,\xi,\eta)\widehat{g}(\eta)$$

is a smooth mapping and

$$\eta \mapsto \eta^\gamma e^{i\langle x,\eta\rangle} \partial_x^\alpha \partial_\xi^\beta a(x,\xi,\eta)\widehat{g}(\eta)$$

is an integrable function for every x, ξ, α, β and γ.

Since $\widehat{g} \in \mathcal{S}^{s_1}_{\sigma_1}(\mathbf{R}^d)$ (cf. Proposition 1.3) and since $\omega_0 \in \mathscr{P}^0_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d})$, it follows that

$$\left|\partial_x^\alpha \partial_\xi^\beta a_g(x,\xi)\right| \lesssim \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \int \left|\eta^\gamma \partial_x^{\alpha-\gamma} \partial_\xi^\beta a(x,\xi,\eta)\widehat{g}(\eta)\right| \, d\eta$$

$$\lesssim \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \int h^{|\alpha+\beta-\gamma|} (\alpha-\gamma)!^{\sigma_1} \beta!^{s_2} \left|\omega_0(x,\xi,\eta)\eta^\gamma e^{-r|\eta|^{\frac{1}{\sigma_1}}}\right| \, d\eta$$

$$\lesssim \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \int h^{|\alpha+\beta-\gamma|} (\alpha-\gamma)!^{\sigma_1} \beta!^{s_2} \omega_0(x,\xi,0) \left|e^{r_0|\eta|^{\frac{1}{\sigma_3}}} \eta^\gamma e^{-r|\eta|^{\frac{1}{\sigma_1}}}\right| \, d\eta,$$

for every $r_0 > 0$, and some constants $r, h > 0$. Since r_0 can be chosen such that $r_0 < r$, and since

$$\left|\eta^\gamma e^{-(r-r_0)|\eta|^{\frac{1}{\sigma_1}}}\right| \lesssim h^{|\gamma|} \gamma!^{\sigma_1} e^{-\frac{(r-r_0)}{2}|\eta|^{\frac{1}{\sigma_1}}},$$

we get

$$\left|\partial_x^\alpha \partial_\xi^\beta a_g(x,\xi)\right|$$

$$\lesssim h^{|\alpha+\beta|} \beta!^{s_2} \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} ((\alpha-\gamma)!\gamma!)^{\sigma_1} \int \omega_0(x,\xi,0) e^{-(r-r_0)|\eta|^{\frac{1}{\sigma_1}}} \, d\eta,$$

$$\lesssim (4h)^{|\alpha+\beta|} \alpha!^{\sigma_1} \beta!^{s_2} \omega(x,\xi), \quad x,\xi \in \mathbf{R}^d,$$

for some constant $h > 0$, where $\omega(x,\xi) \equiv \omega_0(x,\xi,0) \in \mathscr{P}^0_{s_1,\sigma_2}(\mathbf{R}^{2d})$. This gives the desired result. □

Proof of Theorem 3.1. In view of the invariance properties for the bilinear pseudo-differential operators given in Theorem 2.8, we may assume $r = t = 0$ without loss of generality.

By Proposition 1.9 and Lemma 3.3, it follows that $\operatorname{Op}(a)(f,g)$ is a continuous mapping from $M^{p,q}_{(\omega_0\omega)}(\mathbf{R}^d) \times \mathcal{S}^{\sigma_1}_{s_1}(\mathbf{R}^d)$ to $M^{p,q}_{(\omega)}(\mathbf{R}^d)$. Now the result follows from Proposition 1.7. □

We refer to [3, Theorem 3.4] for the Beurling case counterpart of Theorem 3.1.

Finally, the characterization of Gelfand-Shilov spaces via modulation spaces gives the following result (cf. [27, 41, 48]), see also Proposition 1.7.

Theorem 3.4. *Let there be given* $s,\sigma > 0$ *such that* $s+\sigma \geq 1$, $v_1 \in \mathscr{P}^0_s(\mathbf{R}^d)$, $v_j \in \mathscr{P}^0_\sigma(\mathbf{R}^d)$, $j = 2,3$, *and* $\omega_0 \in \mathscr{P}^0_{s,\sigma,\sigma}(\mathbf{R}^{3d})$, *such that* ω_0 *is* $\otimes_{j=1}^3 v_j$*-moderate. If* $r,t \in [0,1]$, *such that* $r+t \leq 1$, *and* $a \in \Gamma^{\sigma,s,s}_{(\omega)}(\mathbf{R}^{3d})$ *then* $\operatorname{Op}_{r,t}(a)$ *is continuous from* $\mathcal{S}^\sigma_s(\mathbf{R}^d)\times\mathcal{S}^\sigma_s(\mathbf{R}^d)$ *to* $\mathcal{S}^\sigma_s(\mathbf{R}^d)$, *and from* $(\mathcal{S}^\sigma_s)'(\mathbf{R}^d) \times (\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$ *to* $(\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$.

Proof. In view of Theorem 2.8, it is enough to consider the case when $r = t = 0$, i.e. $\operatorname{Op}_{0,0}(a)$.

By Proposition 1.10 and Remark 3.2, it is enough to show that a_g given by (3.1) belongs to $\Gamma^{\sigma,s}_{(\omega)}(\mathbf{R}^{2d})$ for $\omega(x,\xi) \equiv \omega_0(x,\xi,0) \in \mathscr{P}^0_{s,\sigma}(\mathbf{R}^{2d})$. This follows from Lemma 3.3. Now the the continuity of $\operatorname{Op}_{r,t}(a)$ from $\mathcal{S}^\sigma_s(\mathbf{R}^d) \times \mathcal{S}^\sigma_s(\mathbf{R}^d)$ to $\mathcal{S}^\sigma_s(\mathbf{R}^d)$ follows from Theorem 3.1 and Proposition 1.7.

The continuity of $\operatorname{Op}_{r,t}(a)$ from $(\mathcal{S}^\sigma_s)'(\mathbf{R}^d) \times (\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$ to $(\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$ follows by duality. $\square$

The analogous result hold for the Beurling case, see [3, Theorem 3.6].

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