Intertemporal discrete choice

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Daniele Pennesi[1]

Abstract

Random utility models are widely used to estimate preference parameters. In the case of intertemporal choice, the two most common models are the discounted logit and one we call the discounted Luce. Due to their apparent similarity, the choice to use one model or the other seems irrelevant. In this paper, we argue that the discounted Luce is superior to the discounted logit in two significant aspects. First, in relevant applications, the discounted Luce is monotone in the sense of Apesteguia and Ballester (2018), while the discounted logit is not. Second, we show that the discounted logit is incompatible with a property of choice probabilities we call "weak stationarity". The latter is compatible with common assumptions on the random nature of choices and is often not falsifiable with commonly available data. Fitting a logit model to weakly stationary choice probabilities biases the time-preference estimates. On the contrary, the discounted Luce can be safely used when choice probabilities are weakly stationary. An application to an existing dataset supports the theoretical results.

Keywords: Random Choice, Intertemporal Choice, Logit
JEL CLASSIFICATION: D9, D8

Random utility models (RUMs) represent a standard approach to estimate preference parameters, in both field and laboratory experiments. In RUMs, each option is associated with a numerical value that depends on a deterministic component called structural utility and a random shock. The recent wave of empirical research on intertemporal choice has mostly employed two models: the discounted logit (henceforth, Dlogit), and one we call the discounted

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Luce (henceforth, DLuce).[2] The two models are qualitatively similar: both satisfy the independence of irrelevant alternatives property and both determine choice probabilities by a "relative weight" functional form.[3] The choice to use one model or the other thus seems irrelevant and it is often not adequately justified in applied works. In this paper, we argue that this similarity is only superficial, and that the DLuce is superior to the Dlogit in two significant aspects. First, when choices are binary[4] and under fairly general conditions, the DLuce is *monotone* (see Section 3.1) in the sense of Apesteguia and Ballester (2018). Monotonicity allows us to meaningfully interpret the estimated parameters as measures of patience, an interpretation that is invalid in non-monotone models such as the Dlogit.

Second, we introduce a property of choice probabilities, called "weak stationarity" (see Section 3.1), which is compatible with common interpretations of the random nature of choice, for example, from unobserved heterogeneity in a population of individuals. We show that choice data generated by a Dlogit cannot be weakly stationary. Consequently, fitting the Dlogit to weakly stationary data biases the estimates of the patience parameters. The result is relevant because weak stationarity is often *not falsifiable* with the data typically available to the analyst (see the empirical Section 4). In this case, an analyst whose interpretation of random choice is consistent with weak stationarity *should not use the Dlogit.* On the contrary, the DLuce can be safely applied to weakly stationary choice probabilities. Indeed, the DLuce is the unique model that satisfies weak stationarity and the independence of irrelevant alternatives (IIA), plus two technical axioms. Therefore, our results can inform the analyst's choice of which model to use in the estimation exercise.

In the empirical Section 4, we corroborate our theoretical results using an existing dataset, in which the weak stationarity property cannot be falsified. We exploit the experimental data of Tanaka et al. (2010) and we estimate time preference parameters with both the DLuce and the Dlogit. The results show

[2] The Dlogit is used, among others, in Chabris et al. (2008); Louie and Glimcher (2010); Tanaka et al. (2010). The DLuce is used in Harrison et al. (2002); Andersen et al. (2008); Meier and Sprenger (2010); Andreoni et al. (2015); Meier and Sprenger (2015).

[3] They differ in how the shocks are distributed and how the shocks affect the discounted utility: additively in the Dlogit and multiplicatively in the DLuce (see Section 2).

[4] Binary choices are the type of data collected with the popular Multiple Price List method employed among the others in Harrison et al. (2002) Andersen et al. (2008), Tanaka et al. (2010), Meier and Sprenger (2010), Bauer et al. (2012), Halevy (2015).

that the Dlogit systematically estimates a "higher patience" with respect to the DLuce under all the specifications of the discounting regime. For example, when fitting quasi-hyperbolic discounting, the Dlogit estimates a $\delta = 0.99$ and a present bias factor $\beta = 0.644$, while the estimates of the DLuce are $\delta = 0.98$ and $\beta = 0.529$.

While the main results are provided for the simplified setting of choice among dated rewards, we show that the differences between Dlogit and DLuce partly extend to the case of choice among consumption streams. As in the case of dated outcomes, the DLuce satisfies a notion of weak stationarity (suitably adapted to consumption streams) while the Dlogit violates it. Strengthening weak stationarity allows identifying geometric and quasi-hyperbolic discounting in the DLuce. The latter results are also relevant for the application of dynamic random utility models common in the IO literature (see, e.g., Rust, 1987; Aguirregabiria and Mira, 2010).

The paper is structured as follows: after a literature review, in Section 2 we introduce the setting and provide a preview of the results. In Section 3.1, we illustrate the consequences of applying the Dlogit to weak stationary choice probability. In Section 3.2, we prove that the DLuce is essentially the only model that satisfies weak stationarity and the IIA axiom. In Section 3.4, we characterize geometric and quasi-hyperbolic discounting in the DLuce model. In Section 3.5, we study the "weak stationarity-like" properties of the Dlogit. Section 4 contains the empirical assessment of the Dlogit and the DLuce. Lastly, Section 5 provides the extension to consumption streams. Appendix A contains all the proofs of the results in the paper, and Appendix B contains an extension of the characterization of the DLuce dealing with zero-probability choices.

1. Related literature

There is growing interest in the relationship between stochastic choice and time preferences. Blavatskyy (2017) extended Fechner's model to intertemporal choice, hence focusing on binary choices. Proposition 6 in Section 5 is comparable to Proposition 3 in Blavatskyy (2017) even if the setting and axioms are rather different. Blavatskyy (2018) extends Fechner's model to account for multiple alternatives. Recently, Lu and Saito (2018) introduced a model of intertemporal stochastic choice in the spirit of Gul and Pesendorfer (2006). In their theory, random choice follows from the unobservable heterogeneity of the discount factors. They characterize geometric and quasi-hyperbolic discounting using discrete choice among lotteries over consumption streams. In their model, random choice arises from random time preferences, and the utility over payoffs is deterministic. There are some differences with the DLuce: first, their model is always monotone in the sense of Apesteguia and Ballester (2018). Second, in the DLuce, choices are random even in the absence of intertemporal trade-offs, while in their model, choices among immediate options are deterministic. Third, our setting is more general, since we do employ lotteries over consumption streams.

In the dynamic setting, Fudenberg and Strzalecki (2015) axiomatized a general version of the recursive logit in which larger menus may be disliked due to choice aversion. The dynamic logit, widely used in applied works (e.g, Rust, 1987), is a particular case of their model. The aim of their paper is different from ours since we are interested in studying the properties of discounted logit and discounted Luce models. Frick et al. (2019) study a general dynamic extension of the random expected utility model of Gul and Pesendorfer (2006). They provide observable restrictions that characterize gradual learning, consumption persistence, and habit formation. Differently from their paper, we focus on the relationship between time preferences and random choice. Like Lu and Saito (2018), Frick et al. (2019) consider choices over lotteries, whereas our framework is more general. For a different approach to dynamic stochastic choice, Dagsvik (2002) and Cerreia-Vioglio et al. (2017) extended the logit to dynamic choices in continuous time, focusing on the dynamic of choice probabilities.

Concerning a more general critique of random utility models, Apesteguia and Ballester (2018) show that common classes of random utility models have paradoxical properties. The probability of selecting, for example, more immediate options, is not necessarily decreasing if the patience parameter increases. They show that the class of random parameter models is free of their critique. Apesteguia et al. (2017) provide testable restrictions to identify a monotone class of random utility models called single-crossing (SCRUM). SCRUM, however, violate the IIA, so, the DLuce is not a single-crossing random utility model despite being monotone when applied to binary choices over dated outcomes.

2. DLuce and Dlogit: a preview

In this section, we introduce our setting and present a preview of the main results. The first part of the paper focuses on discrete choice over *dated rewards*, the type of choices that experimental subjects face in the common Multiple

Price List (MPL) method.[5] We consider a finite set of rewards X and denote by (x,t) a reward $x \in X$ delivered in t periods, for some $t \in \mathbb{N}$. We denote by Z the set of all the possible dated rewards. A choice set is a non-empty and finite subset of Z and we denote by $\mathscr{A}$ the set of all choice sets. Choice probabilities are functions $P : A \times \mathscr{A} \to [0,1]$, such that $\sum_{(x,t)\in A} P((x,t)|A) = 1$.

To simplify the exposition, this section considers only binary choices, i.e. all the choice sets $A = \{(x,t),(y,s)\}$ for some $x, y \in X$ and $t, s \in \mathbb{N}$ and we denote by $P((x,t)|(y,s))$ the probability of selecting (x,t) over (y,s). The choice probabilities in the Dlogit[6] are given by:

$$P^{\text{Dlog}}((x,t)|(y,s)) = \frac{e^{D(t)w(x)}}{e^{D(t)w(x)} + e^{D(s)w(y)}} \tag{Dlogit}$$

where $w : X \to \mathbb{R}$ and D is a *discount function*: a function $D : \mathbb{N} \to (0,1]$, with $D(0) = 1$ and $D(t) \geq D(t+1)$. The function $D(\cdot)$ virtually includes all the discount functions proposed in the literature. For example, geometric $D(t) = \delta^t$, hyperbolic $D(t) = \frac{1}{1+\rho t}$, quasi-hyperbolic $D(0) = 1$, $D(t) = \beta\delta^t$ (e.g Laibson, 1997), risk-adjusted $D(t) = g((1-r)^t)\delta^t$ (Halevy, 2008), non-hyperbolic (Bleichrodt et al., 2009) and, the generalized discount function of Bisin and Hindman (2013), $D(t) = \beta(1-(1-\theta)\rho t)^{1/(1-\theta)}$.

The choice probabilities in the DLuce model[7] are given by:

$$P^{\text{DLuce}}((x,t)|(y,s)) = \frac{D(t)v(x)}{D(t)v(x) + D(s)v(y)} \tag{DLuce}$$

where D is a discount function and $v : X \to \mathbb{R}_{++}$.

Remark 1. The models used in applications contain an extra parameter $\lambda \in [0,\infty)$ often interpreted as a rationality measure. Choice probabilities of the Dlogit are given by $P^{\text{Dlog}}((x,t)|(y,s)) = \frac{e^{\lambda D(t)w(x)}}{e^{\lambda D(t)w(x)} + e^{\lambda D(s)w(y)}}$ and in the DLuce by

[5] In the MPL design, each subject faces a list of binary choices between earlier/lower and later/larger rewards.

[6] The Dlogit belongs to the class of additive random utility models (RUM) denote by P^{a}: $P^{\text{a}}((x,t)|(y,s)) = \mathbb{P}\left(D(t)w(x) + \epsilon_{x,t} \geq D(s)w(y) + \epsilon_{y,s}\right)$ where the shocks are i.i.d. and distributed according to a type I GEV distribution. The c.d.f. of a type I GEV is $F(z) = e^{-e^{-\sigma(z-\mu)}}$ where $\mu \in \mathbb{R}$ and $\sigma > 0$.

[7] The DLuce belongs to the class of *multiplicative* RUMs denoted by P^{m} (Fosgerau and Bierlaire, 2009): $P^{\text{m}}((x,t)|(y,s)) = \mathbb{P}\left(D(t)v(x)\epsilon_{x,t} \geq D(s)v(y)\epsilon_{y,s}\right)$ where the shocks are i.i.d. and distributed according to a random variable ϵ such that $\ln(\epsilon)$ is a type I GEV.

$P^{\mathrm{DLuce}}((x,t)|(y,s)) = \frac{(D(t)w(x))^{\lambda}}{(D(t)w(x))^{\lambda}+(D(s)w(y))^{\lambda}}$. The parameter λ is immaterial for the analysis of our paper, hence we set it equal to one.

When applied to intertemporal choice, the different structure of the Dlogit and the DLuce models becomes relevant. Consider a standard interpretation of random choice: the probability $P((x,t)|(y,s))$ represents the portion of experimental subjects preferring (x,t) to (y,s) in the reference population. The subjects have deterministic preferences that are unobservable to the analyst. Namely, each subject i has a utility function u_i and a discount function D_i and she chooses by maximizing her discounted utility $D_i(t)u_i(x)$. Consider the probability of selecting $(x,0)$ over $(y,0)$ and the probability of selecting (x,t) over (y,t). It is clear that *under the previous unobserved heterogeneity interpretation* of random choice:

$$P((x,t)|(y,t)) = P((x,0)|(y,0)) \tag{1}$$

for all $x,y \in X$ and $t \geq 0$. The subjects preferring x immediately to y immediately (or y to x) will prefer (x,t) to (y,t) ((y,t) to (x,t) respectively). Indeed, $u_i(x) \geq u_i(y)$ if and only if $D_i(t)u_i(x) \geq D_i(t)u_i(y)$ and the equality (1) is satisfied regardless of the individual utility and discount functions. It is not difficult to observe that the DLuce[8] satisfies the equality (1), whereas the Dlogit, generally, does not. For instance, if we apply the Dlogit with geometric discounting $D(t) = \delta^t$ to choice data satisfying the equality (1), we will force δ to be equal to 1 when $w(x) \neq w(y)$:

$$P^{\mathrm{Dlog}}((x,0)|(y,0)) = \frac{1}{1+e^{w(y)-w(x)}} = \frac{1}{1+e^{\delta^t(w(y)-w(x))}} = P^{\mathrm{Dlog}}((x,t)|(y,t))$$

holds only if $\delta^t = 1$ or $w(x) = w(y)$. Therefore, the Dlogit applied to choice data satisfying Eq. (1) mechanically forces the estimated parameter δ to be unitary (see Theorem 1), *even if Eq.* (1) *is consistent with a non-unitary discount function.* Consequently, the Dlogit cannot be used when data satisfy Eq. (1). However, equality (1) is often not falsifiable with the data collected in field or laboratory experiments. In this case, we suggest that *the Dlogit should not be used when random choice arises from unobservable heterogeneity in the pool of subjects.* Similarly, it should not be used when individual random choice arises

[8] More generally, any multiplicative RUM satisfies equality (1): $P^{\mathrm{m}}((x,0)|(y,0)) = \mathbb{P}\left(v(x)\epsilon_{x,0} \geq v(y)\epsilon_{y,0}\right) = \mathbb{P}\left(D(t)v(x)\epsilon_{x,t} \geq D(t)v(y)\epsilon_{y,t}\right) = P^{\mathrm{m}}((x,t)|(y,t))$.

from noise in the evaluation of payoffs alone (see Section 3.1). For example, an individual with deterministic time preferences but a stochastic utility over rewards, will display choice probabilities that satisfy Eq. (1).[9]

A second advantage of the DLuce with respect to the Dlogit is that, when applied to binary choices, the DLuce is free of the critique advanced in Apesteguia and Ballester (2018). They show that a broad class of stochastic choice models, including the Dlogit, is not monotone with respect to parameters measuring impatience (or risk aversion). In these models, an increase in the impatience parameter is not necessarily followed by a larger probability of selecting a more immediate option (see Fig. 1). Failures of monotonicity induce biases in the estimation of preference parameters and prevent the interpretation of $D(t)$ as a measure of time preferences. Differently from the Dlogit, the DLuce is monotone when choices are binary and when the ratio $D(t)/D(s)$ is monotone in the preference parameters determining D (see Proposition 2). Figure 1 illustrates monotonicity and its failure.

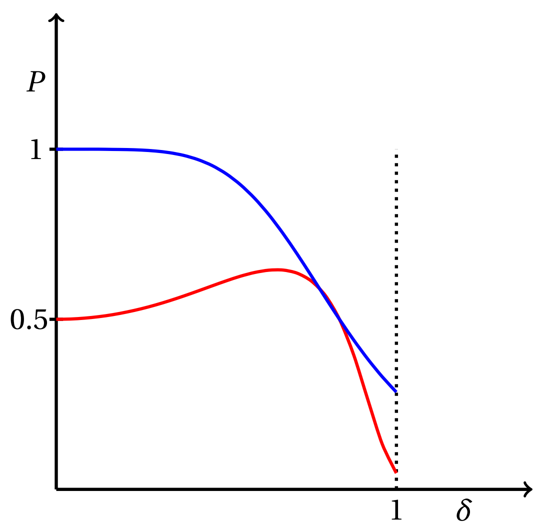

Figure 1: Choice probabilities generated by a Dlogit (red line) and a DLuce (blue line) for $(2,2)$ vs $(5,7)$ with $w(x) = v(x) = x$ and $D(t) = \delta^t$.

[9]In section 3.5, we introduce a form of weak stationarity satisfied by the Dlogit. Although this restricts certain ratios of log-odds, it is sufficient to characterize a more general version of the Dlogit. The testable restrictions of the Dlogit are still unknown in the literature.

3. Weak Stationarity, discounted logit and discounted Luce

3.1. Weak stationarity and the discounted logit

Motivated by the discussion in the previous section, we now formalize the notion of weak stationarity for choice probabilities. For a given $A \in \mathscr{A}$ and $r \geq 0$, we define $A_r = \{(x, t+r) : (x, t) \in A\}$, the set of all the dated rewards in A postponed by r periods. The following definition generalizes equality (1):

$((x, y|t, r)$**-Weak Stationarity).** For all $A \in \mathscr{A}$ such that $(x, t), (y, t) \in A$:

$$\frac{P((x,t)|A)}{P((y,t)|A)} = \frac{P((x,t+r)|A_r)}{P((y,t+r)|A_r)}.$$

$(x, y|t, r)$-Weak Stationarity imposes invariance of the relative probability of choosing between two rewards $x, y \in X$, when they are delivered at *the same dates*, but only for two dates t and $t+r$. Intuitively, these ratios are unaffected by intertemporal trade-offs. When the previous property holds for any pair of payoffs and for any delay, we say that choice probabilities are weakly stationary:

(Weak Stationarity). Choice probabilities satisfy Weak Stationarity if they are $(x, y|t, r)$-weakly stationary for all $x, y \in X$ and all $t, r \geq 0$.

There are at least two relevant interpretations of stochastic choice that are plausibly consistent with Weak Stationarity:

a. Random choice generated by a population of individuals with deterministic preferences that are unobservable to the analyst (as discussed in Section 2).

b. Random choice of an individual with deterministic time preferences but a stochastic evaluation of payoffs: for example, $P((x,t)|A) = \mathbb{P}(D(t)(u(x) + \epsilon_x) \geq D(s)(u(y) + \epsilon_y) \ \ \forall (y, s) \in A)$.

In both cases, relative choice probabilities are not affected by identical shifts along the time axes.

The following result generalizes the intuition provided in Section 2: applying the Dlogit to $(x, y|t, r)$-weakly stationary choice probabilities mechanically biases the estimated discount function.

Theorem 1. *The equality* $\frac{P^{\text{Dlog}}((x,t+r)|A_r)}{P^{\text{Dlog}}((y,t+r)|A_r)} = \frac{P^{\text{Dlog}}((x,t)|A)}{P^{\text{Dlog}}((y,t)|A)}$ *holds for some* $x, y \in X$ *with* $w(x) \neq w(y)$ *and some* $t, r \in \mathbb{N}$ *if and only if* $D(s) = k$ *for all* $s \in [t, t+r]$.

Theorem 1 shows that, if the choice data are $(x, y|t, r)$-weakly stationary for some x, y with $w(x) \neq w(y)$, the structure of the Dlogit forces the discount function to be constant in an interval, even if $(x, y|t, r)$-Weak Stationarity is consistent with arbitrary discounting. Notice that, if $t = 0$ and $w(x) \neq w(y)$ in Theorem 1, then $1 = D(0) = D(s)$ for all $s \in [0, r]$. The estimated patience parameter will thus be unitary between 0 and r. Such a restriction depends on the structure of the Dlogit and not on the property of being $(x, y|t, r)$-weakly stationary, indeed any discount function is compatible with the latter property under the above interpretations. Therefore, applying the Dlogit to $(x, y|t, r)$-weakly stationary data gives potentially biased estimates, because they have to satisfy the condition of Theorem 1. Moreover, most discount functions have the following property: they are either unitary or *strictly* decreasing, e.g. $D(t) = \delta^t$ is either strictly decreasing in t when $\delta \in (0, 1)$ or unitary when $\delta = 1$. A similar property is satisfied by hyperbolic discounting $D(t) = (1 + \rho t)^{-1}$.[10] For these functions, the result of Theorem 1 is stronger:

Corollary 1. *Under the conditions of Theorem 1, if $D(t)$ is either strictly decreasing or unitary, $D(t + r) = D(t)$ for some $t \geq 0$ and $r > 0$ implies $D(t) = 1$ for all t.*

Fitting the Dlogit to weakly stationary choice probabilities and specifying a discount function with the property of Corollary 1 will force the discount function to be completely patient. Therefore, if the choice data are $(x, y|t, r)$-weakly stationary for some x, y with $w(x) \neq w(y)$, they are *incompatible* with a non-trivial Dlogit under common specifications of the discounting function D. Meaning that $(x, y|t, r)$-weakly stationary choice data *cannot* be generated by a Dlogit. Theorem 1 and Corollary 1 can be used to inform the choice of a model in applications. If the data satisfy $(x, y|t, r)$-Weak Stationary, the Dlogit will produce biased estimates. More importantly, the two results offer guidance when Weak Stationarity cannot be falsified, a situation that occurs with choice data that are typically collected in field or laboratory experiments. For example, the subjects never choose between two rewards delivered on the same date when time preferences are elicited with the Multiple Price List method. In this case, the Dlogit should not be used if randomness in the data is generated according to the interpretations a or b above (or any other interpretation consistent with Weak Stationarity).

[10]For an example of a discount function that is weakly decreasing but not constant, take a β-δ model with $\beta \in (0, 1)$ and $\delta = 1$.

The results of Theorem 1 and Corollary 1 are particular cases of a more general feature of the Dlogit. If choice probabilities are "almost"-weakly stationary, fitting the Dlogit potentially generates bounds to the slope of the estimated discount function. "Almost"-weak stationarity may occur, for example, if some subjects in a population have deterministic preferences while the remaining subjects have random preferences. Consider the product $\frac{P((y,t)|A)}{P((x,t)|A)} \cdot \frac{P((x,t+r)|A_r)}{P((y,t+r)|A_r)}$. Its distance from 1 is a measure of how far choice probabilities are from being $(x,y|t,r)$-weakly stationary. For the Dlogit the product $\frac{P^{\text{Dlog}}((y,t)|A)}{P^{\text{Dlog}}((x,t)|A)} \cdot \frac{P^{\text{Dlog}}((x,t+r)|A_r)}{P^{\text{Dlog}}((y,t+r)|A_r)}$ is equal to $e^{(w(y)-w(x))(D(t)-D(t+r))}$, and this number is "close" to 1 (as in the $(x,y|t,r)$-Weak Stationarity condition) when the exponent $(w(y)-w(x))(D(t)-D(t+r))$ is "close" to zero.[11] Hence, when $w(y)-w(x)$ and/or $D(t)-D(t+r)$ are close to zero. Formally:

Theorem 2. *Take some $w(y) \geq w(x)$, some $t, r \in \mathbb{N}$ and consider $\epsilon \geq 0$ such that, $\left|\frac{P((y,t)|A)}{P((x,t)|A)} \cdot \frac{P((x,t+r)|A_r)}{P((y,t+r)|A_r)} - 1\right| \leq \epsilon$, then fitting a Dlogit implies $(w(y)-w(x))(D(t)-D(s)) \leq \epsilon$ for all $s \in [t, t+r]$.*

The result follows directly from the fact that, under the condition $w(y) \geq w(x)$ and the properties of $D(t)$, $(w(y)-w(x))(D(t)-D(t+r)) \geq 0$. Then, the inequality $e^x - 1 \geq x$ for $x \geq 0$ implies

$$(w(y)-w(x))(D(t)-D(t+r)) \leq e^{(w(y)-w(x))(D(t)-D(t+r))} - 1 \leq \epsilon$$

and the monotonicity of D gives the final result.

The condition $D(t)-D(t+r)$ "close to zero", however, may not be binding. For example, when t is large, any discount function is almost flat. The situation is different when $t=0$, namely when choice probabilities are "almost"-weakly stationary at $t=0$. Indeed, the inequality in Theorem 2 mechanically forces the discount function in the Dlogit toward high patience, as formalized in the following corollary:

Corollary 2. *Take some $w(y) \geq w(x)$, some $r \in \mathbb{N}$ and consider $\epsilon \geq 0$ such that $\left|\frac{P((y,0)|A)}{P((x,0)|A)} \cdot \frac{P((x,r)|A_r)}{P((y,r)|A_r)} - 1\right| \leq \epsilon$, then fitting a Dlogit implies $D(s) \geq 1 - \frac{\epsilon}{w(y)-w(x)}$ for all $s \in [0, r]$.*

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[11] Using a first-order Taylor expansion $e^{(w(y)-w(x))(D(t)-D(t+r))} \approx 1 + (w(y)-w(x))(D(t)-D(t+r))$ which is approximately equal to 1 when $(w(y)-w(x))(D(t)-D(t+r))$ is close to zero.

The lower the ϵ and the larger the difference between $w(y)$ and $w(x)$, the higher the estimated patience. Therefore, the Dlogit estimates of time preferences will be constrained, even if the property of being "almost" $(x, y|t, r)$-weakly stationary should not restrict intertemporal preferences. The next example numerically illustrates Corollary 2:

Example 1. *Consider the inequality in Corollary 2 for* $w(y) = 1$, $w(x) = 0$. *Take* $\epsilon = 0.1$ *and* $r = 1$. *Then fitting the Dlogit, implies* $D(0) - D(1) \leq 0.1$, *hence* $D(1) \geq 0.9$. *For example, if* $D(s) = \delta^s$ *and* $r = 1$, *then* $D(1) = \delta \geq 0.9$ *(see the blue line in the top-left panel of Figure 2). If discounting is quasi-hyperbolic* $D(0) = 1$ *and* $D(s) = \beta\delta^s$ *for* $s \geq 1$, $\epsilon = 0.1$ *implies* $\beta\delta \geq 0.9$ *which also bounds* $\beta \geq 0.9$ *(the green line in the top-left panel of Figure 2 represents* $D(t) = 0.85 \cdot \delta^t$ *as a function of* δ. *In this case, no* $\delta \in (0, 1]$ *can satisfy the inequality of Corollary 2). For hyperbolic discounting* $D(s) = (1+\rho \cdot s)^{-1}$, $1 - D(1) = 1 - (1+\rho)^{-1} \leq 0.1$ *holds if* $\rho \leq 0.111$ *(see the red line in the top-left panel of Figure 2)), which corresponds to high patience. The bottom-left and bottom-right panels in Figure 2 have similar interpretations. For example, with* $r = 3$ *and* $\epsilon = 0.2$, $D(3) = \delta^3 \geq 1 - 0.1$ *implies* $\delta \geq 0.92831$ *(see the blue line in the bottom-right panel of Figure 2).*

In the empirical section (Sec. 4), we use the data of Tanaka et al. (2010) in which the Weak Stationarity axiom is not falsifiable, and we show that the Dlogit always estimates higher patience than the DLuce in any specification of the discounting regime. According to the previous results, such a difference is potentially driven by the very structure of the Dlogit.

3.2. Weak Stationarity and the discounted Luce

It is straightforward to observe that the DLuce necessarily satisfies the Weak Stationarity axiom. In this section, we show the converse implication: jointly with the independence of irrelevant alternatives and two technical axioms, the Weak Stationarity axiom is also *sufficient* to characterize the DLuce model. This means that the DLuce is the only model that is consistent with Weak Stationarity within the class of models satisfying the IIA and two technical axioms. The first technical axiom is

(Positivity). For all $A \in \mathscr{A}$ and $(x, t) \in A$, $P((x, t)|A) > 0$.

A small but positive probability is empirically indistinguishable from a zero probabilit; therefore, positivity is a rather weak requirement. We will relax it in Section Appendix B. The standard Luce's choice axiom or Independence of Irrelevant Alternatives (IIA) assuming Positivity becomes:

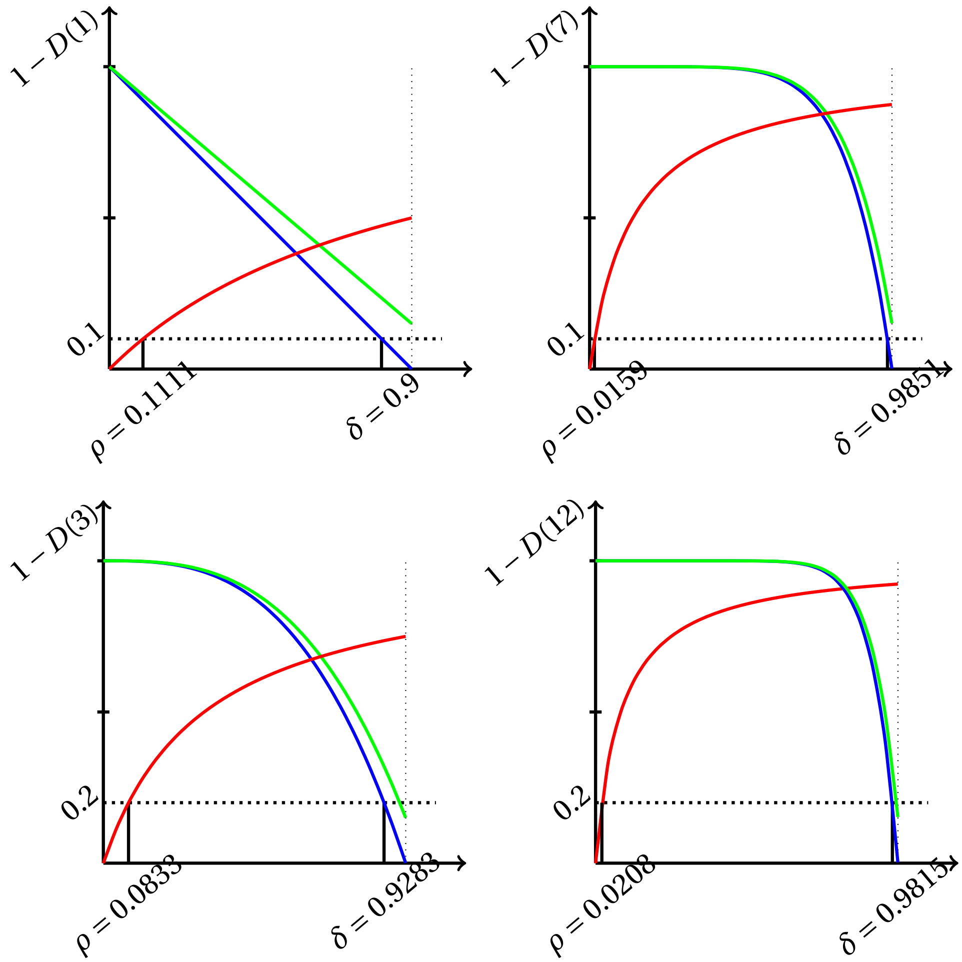

Figure 2: The blue lines correspond to geometric discounting as a function of δ. The red lines correspond to hyperbolic discounting as a function of ρ. The green lines corresponds to quasi-hyperbolic discounting with $\beta = 0.85$ as a function of δ.

(IIA). For all $A, B \in \mathscr{A}$ and $(x,t),(y,s) \in A \cap B$,

$$\frac{P((x,t)|A)}{P((y,s)|A)} = \frac{P((x,t)|B)}{P((y,s)|B)}.$$

The relative probability of choosing a reward x in t periods over a reward y in s periods is constant across menus. Positivity and the IIA axiom are necessary and sufficient for the existence of a positive random scale[12] $u : Z \to R_{++}$, such that

$$P((x,t)|A) = \frac{u(x,t)}{\sum_{(y,s)\in A} u(y,s)}.$$

The last condition imposes a stochastic form of impatience: the probability of choosing a reward decreases when it is delayed.

(Stochastic Impatience). For all $x \in X$, $t, r \geq 0$ and $A \in \mathscr{A}$, with $(x,t),(x,t+r) \notin A$, $P((x,t)|A \cup (x,t)) \geq P((x,t+r)|A \cup (x,t+r))$.

Stochastic Impatience bounds the rewards in X to be goods and not bads. We have the following result:

Proposition 1 (DLuce)**.** *The choice probabilities P satisfy Positivity, the IIA axiom, Stochastic Impatience and Weak Stationarity if and only if there exist a random scale $v : X \to R_{++}$ and a unique discount function $D : \mathbb{N} \to (0,1]$ such that:*

$$P((x,t)|A) = \frac{D(t)\,v(x)}{\sum_{(y,s)\in A} D(s)\,v(y)}$$

for all $A \in \mathscr{A}$.

Positivity and the IIA axiom give the Luce's relative weight form choice probabilities. The Weak Stationarity axiom imposes separability between the discount function and the value function.

Remark 2. The interpretation of the DLuce (or the Dlogit) as arising from a pool of subjects with potentially different time preferences may seem at odds with

[12]A random scale is a value function which is unique up to positive multiplication, i.e. if u and u' define the same probabilities, there exists an $\alpha > 0$ such that $u = \alpha u'$.

the unique discount function D in the DLuce (or Dlogit). However, as is common in applications, the estimated discount function depends on the observable characteristics of the subject (e.g. age, income, sex). For example, to estimate δ of a geometric discount function, one can fit the following model:

$$P^{\text{DLuce}}((x,0)|(y,t)) = \frac{x}{x + (\delta_0 + \sum \delta_i C_i)^t y}$$

where the C_i's are the observable characteristics of the subjects and δ_i their coefficients. The estimate $\hat{\delta}$ is correlated with observables and each subject in the pool can be mapped to a discount factor.

3.3. Apesteguia and Ballester's monotonicity

A second fundamental difference between the DLuce and the Dlogit is that the former may be monotone in the sense of Apesteguia and Ballester (2018), while the latter is not. In the intertemporal choice domain, monotonicity means that the probability of selecting a delayed reward cannot decrease if the patience parameter increases. For models violating the monotonicity property, such as the Dlogit (see Figure 1), the interpretation of D as a measure of intertemporal preference is flawed. For example, if $D(t) = \delta^t$ in the Dlogit, an increase of δ is not necessarily followed by an increase in the probability of selecting a more delayed option. On the contrary, we show that the DLuce is (in general) monotone when restricted to binary choices.

To highlight the dependency of the discount function D from a parameter, we denote it by $D(t,\alpha)$. For instance, we write $D(t,\rho) = \frac{1}{1+\rho t}$. We have the following result, in which the partial derivative of $D(t,\alpha)$ is denoted by $D_\alpha(t,\alpha)$:[13]

Proposition 2. *If the choice probabilities have a DLuce representation* $P^{\text{DLuce}}((x,t)|(y,s)) = \frac{1}{1+\frac{D(s,\alpha)v(y)}{D(t,\alpha)v(x)}}$, *then for all* $t,s \geq 0$ *and all* $x,y \in X$,

$$\frac{\partial P^{\text{DLuce}}((x,t)|(y,s))}{\partial \alpha} \geq (\leq) 0 \iff \frac{D_\alpha(t,\alpha)}{D(t,\alpha)} \geq (\leq) \frac{D_\alpha(s,\alpha)}{D(s,\alpha)}.$$

[13] Indeed, $\frac{\partial P^{\text{DLuce}}((x,t)|(y,s))}{\partial \alpha} = -K\left[\frac{\partial D(s,\alpha)}{\partial \alpha} D(t,\alpha) - D(s,\alpha)\frac{\partial D(t,\alpha)}{\partial \alpha}\right]$ where $K = \left(1 + \frac{D(s,\alpha)v(y)}{D(t,\alpha)v(x)}\right)^{-2} \frac{v(y)}{v(x)} \frac{1}{D(t,\alpha)^2}$.

The condition in Proposition 2 is satisfied, for example, by the following discount functions:[14]

1. $D(t,\delta) = \delta^t$.

2. $D(0,\delta) = 1$ and $D(t,\delta) = \beta\delta^t$.

3. $D(t,\rho) = (1+\rho t)^{-1}$.

Therefore, in the DLuce model, one can correctly interpret D as a measure of intertemporal preference under all the common specifications of the discount function.

Remark 3. The structure of the DLuce model permits identification of the "degree of impatience" directly from choice probabilities. Indeed, the square root of the following ratio,

$$r_{x,y}(t,t+1) = \frac{P^{\text{DLuce}}((x,t)|(y,t+1))}{P^{\text{DLuce}}((x,t+1)|(y,t))}\frac{P^{\text{DLuce}}((y,t)|(x,t+1))}{P^{\text{DLuce}}((y,t+1)|(x,t))}$$

is a number that can be interpreted as a measure of impatience, when the condition of Proposition 2 is satisfied. Indeed, $\sqrt{r_{x,y}(t,t+1)} = \frac{D(t)}{D(t+1)}$ for all $t \geq 0$ and all $x, y \in X$.

3.4. Geometric and quasi-hyperbolic discounting

In this section, we show that strengthening the Weak Stationarity axiom allows us to identify geometric and quasi-hyperbolic discounting within the DLuce model.

Consider again the unobservable heterogeneity interpretation of choice probabilities. Each subject has a deterministic preference that is unobservable to the analyst. Suppose that all subjects discount the future *geometrically*. Namely, each subject i has a utility function u_i and a discount factor δ_i and decides to maximize her discounted utility $\delta_i^t u_i(x)$. Alternatively, consider an individual with noisy perception of the payoffs but geometric discounting of future utilities, e.g. $P((x,t)|(y,s)) = \mathbb{P}(\delta^t(u(x)+\epsilon_x) \geq \delta^s(u(y)+\epsilon_y))$. Take the probability of

[14]For geometric discounting, $\frac{\partial\delta^t}{\partial\delta}\frac{1}{\delta^t} = t\delta^{-1} \geq s\delta^{-1} = \frac{\partial\delta^s}{\partial\delta}\frac{1}{\delta^s}$, if $t \geq s$ (the opposite inequality holds for $s \geq t$ as expected). For quasi-hyperbolic discounting the proof is identical to geometric discounting. For hyperbolic discounting, $\frac{\partial D(t,\rho)}{\partial\rho}D(t,\rho) = -(1+\rho t)^{-1}t$, which is decreasing in t, hence the condition is satisfied.

selecting $(x,0)$ over (y,s) and the probability of selecting (x,t) over $(y,t+s)$. It is clear that, *under both the previous interpretations* of random choice

$$P((x,0)|(y,s)) = P((x,t)|(y,t+s)) \tag{2}$$

for all $x, y \in X$ and $t, s \geq 0$. The subjects preferring x immediately to y in s periods will prefer (x,t) to $(y,t+s)$, indeed, $u_i(x) \geq \delta_i^s u_i(y)$ if and only if $\delta_i^t u_i(x) \geq \delta_i^{t+s} u_i(y)$ and the equality is satisfied regardless of the individual utility and discount factor. A similar consideration is valid for the individual with random tastes over payoffs and geometric discounting. The following axiom formalizes the previous intuition:

(Stationarity). For all $A \in \mathscr{A}$, $x, y \in X$ and all $t, s, r \geq 0$ such that $(x,t),(y,s) \in A$,

$$\frac{P((x,t)|A)}{P((y,s)|A)} = \frac{P((x,t+r)|A_r)}{P((y,s+r)|A_r)}.$$

The relative probability of selecting (x,t) and (y,s) from A is unaffected if *all* the rewards in A are *equally* postponed. Notice that the Dlogit cannot satisfy the Stationarity axiom even when $D(t) = \delta^t$, indeed

$$\frac{P^{\text{Dlog}}((x,t)|A)}{P^{\text{Dlog}}((y,s)|A)} = \frac{e^{\delta^t w(x)}}{e^{\delta^s w(y)}} \neq \left(\frac{e^{\delta^t w(x)}}{e^{\delta^s w(y)}}\right)^{\delta^r} = \frac{e^{\delta^{t+r} w(x)}}{e^{\delta^{s+r} w(y)}} = \frac{P^{\text{Dlog}}((x,t+r)|A_r)}{P^{\text{Dlog}}((y,s+r)|A_r)}$$

only if $\delta = 1$ or $w(x) = w(y)$. The Stationarity axiom characterizes geometric discounting:

Proposition 3 (Geometric DLuce)**.** *The choice probabilities P satisfy Positivity, the IIA axiom, Stochastic Impatience and Stationarity if and only if there exist a random scale $v : X \to R_{++}$ and a unique $\delta \in (0,1]$ such that:*

$$P((x,t)|A) = \frac{\delta^t v(x)}{\sum_{(y,s)\in A} \delta^s v(y)}$$

for all $A \in \mathscr{A}$.

A weakening of the Stationarity axiom permits characterization of the popular quasi-hyperbolic discounting model in the DLuce:

(Quasi-hyperbolic Stationarity). For all $A \in \mathscr{A}$ and all $x, y \in X$:

1. (Quasi-stationarity): $\frac{P((x,t)|A)}{P((y,s)|A)} = \frac{P((x,t+r)|A_r)}{P((y,s+r)|A_r)}$, for all $t, s > 0, r \geq 0$
2. (Present Bias): $\frac{P((x,0)|A)}{P((y,s)|A)} \geq \frac{P((x,r)|A_r)}{P((y,s+r)|A_r)}$, for all $s > 0, r \geq 0$.

All the intuitive features of quasi-hyperbolic discounting affect relative choice probabilities. For non-immediate outcomes, the relative choice probabilities are constant when outcomes are equally delayed. However, the relative probability of choosing an immediate payment over a delayed one is strictly greater than the same proportion when both payments are equally delayed, this is the present bias. We have the following result:

Proposition 4 (Quasi-hyperbolic DLuce)**.** *The choice probabilities P satisfy Positivity, the IIA axiom, Stochastic Impatience, Weak Stationarity and the Quasi-hyperbolic Stationarity axiom, if and only if, there exist a random scale* $v : X \to \mathbb{R}_{++}$ *and unique* $\beta, \delta \in (0,1]$ *such that:*

$$P((x,t)|A) = \frac{D(t)\,v(x)}{\sum_{(y,s)\in A} D(s)\,v(y)}$$

and $D(t) = \beta\delta^t$ *if* $t > 0$ *and* $D(0) = 1$.

Alternative restrictions, in the same spirit of Stationarity and Quasi-hyperbolic Stationarity, can be devised to characterize diminishing impatience (i.e. $D(t)/D(t+1) \geq D(t+1)/D(t+2)$ for all t) or variations of quasi-hyperbolic discounting, such as $D(t) = 1$ for all $t \leq t^*$ and $D(t) = \beta\delta^t$, for all $t > t^*$ and some $t^* \in \mathbb{N}$.

Notice that the Present Bias (point 2. of the Quasi-hyperbolic Stationarity axiom) is consistent with the Dlogit even when discounting is geometric. Indeed,

$$\frac{P^{\text{Dlog}}((x,0)|A)}{P^{\text{Dlog}}((y,s)|A)} = \frac{e^{w(x)}}{e^{\delta^s w(y)}} \geq \left(\frac{e^{w(x)}}{e^{\delta^s w(y)}}\right)^{\delta^r} = \frac{e^{\delta^r w(x)}}{e^{\delta^{s+r} w(y)}} = \frac{P^{\text{Dlog}}((x,r)|A_r)}{P^{\text{Dlog}}((y,s+r)|A_r)}$$

whenever $\delta^s w(y) \leq w(x)$. Therefore, the Dlogit may fail to detect non-stationary structural preferences.

3.5. Weak Stationarity in the Dlogit

A natural question arises at this point, what are the "weak stationary-like" properties satisfied by the Dlogit? Before answering, we introduce the log-odds of choosing between x and y when both are delivered in t periods:

$$l_{x,y}(t) = \ln \frac{P((x,t)|A)}{P((y,t)|A)}$$

for some $A \in \mathscr{A}$. Note that by assuming the IIA log-odds are independent of the set A. The following property of log-odds is (necessarily) satisfied by the Dlogit:

(Log-odds Stationarity). For all $t, r \geq 0$:

$$\frac{l_{x,y}(t)}{l_{x,y}(t+r)} = \frac{l_{x',y'}(t)}{l_{x',y'}(t+r)}. \tag{3}$$

for all $x, x', y, y' \in X$ such that either ratio is well defined.

Indeed, if $P^{\text{Dlog}}((x,t)|A) = \frac{e^{D(t)w(x)}}{\sum_{(y,s)\in A} e^{D(s)w(y)}}$, $l_{x,y}(t) = D(t)(w(x) - w(y))$. Since the ratio is well-defined, it must be that $w(x) \neq w(y)$, hence:

$$\frac{l_{x,y}(t)}{l_{x,y}(t+r)} = \frac{D(t)(w(x)-w(y))}{D(t+r)(w(x)-w(y))} = \frac{D(t)(w(x')-w(y'))}{D(t+r)(w(x')-w(y'))} = \frac{l_{x',y'}(t)}{l_{x',y'}(t+r)}.$$

As for the interpretation of Log-odds Stationarity, the log-odds are delay-dependent utility differences, hence the axiom means that relative differences in utility are delay-independent in the Dlogit. While necessary, the Log-odds Stationarity axiom is sufficient to characterize a generalized version of the Dlogit. We say that $x, y \in X$ are 0-distinguishable if $l_{x,y}(0) \neq 0$.[15]

Proposition 5 (Generalized logit)**.** *There are $x, y \in X$ that are 0-distinguishable and he choice probabilities P satisfy Positivity, the IIA axiom and Log-odds Stationarity if and only if there exist a function $w : X \to \mathbb{R}$ not identically zero and two functions $\lambda, D : \mathbb{N} \to \mathbb{R}_+$ such that:*

$$P((x,t)|A) = \frac{\lambda(t)e^{D(t)w(x)}}{\sum_{(y,s)\in A} \lambda(s)e^{D(s)w(y)}}$$

for all $A \in \mathscr{A}$.

4. DLuce vs Dlogit: an application

In this section, we apply the DLuce and the Dlogit to a dataset in which the Weak Stationarity property cannot be falsified, and compare their estimates. We use the data of Tanaka et al. (2010), where time preferences are elicited

[15] Equivalently, $P((x,0)|A) \neq P((y,0)|A)$.

through the multiple price list method. The authors ask experimental subjects to choose between immediate/smaller versus later/larger monetary rewards (i.e. $(x,0)$ versus (y,t) where x, y are dong, the Vietnamese currency), letting the rewards x, y and the delay t of the rewards vary (from 30,000 to 300,000 for the rewards, from three days to three months for t). Each subject made 75 choices, for a total of 5340 observations. The Weak Stationarity axiom is not falsifiable because $t > 0$ in all the pairwise choices. Tanaka et al. (2010) estimate the following Dlogit model:

$$P^{\text{Dlog}}((x,0) \geq (y,t)) = \frac{1}{1 + \exp\lambda\left(-x + y\beta(1-(1-\theta)\rho t)^{\frac{1}{1-\theta}}\right)}$$

corresponding to a Dlogit with $w(x) = x$. The discount function $D(0) = 1$, $D(t) \triangleq \beta(1-(1-\theta)\rho t)^{\frac{1}{1-\theta}}$ for $t > 0$, where ρ is the discount rate, β the present bias factor and θ the hyperbolic coefficient, encompasses geometric, hyperbolic and quasi-hyperbolic discounting. For $\theta \to 1$, it corresponds to the quasi-hyperbolic discounting model, $D(t) = \beta e^{-\rho t}$. If $\theta \to 1$ and $\beta = 1$, it becomes the standard geometric discounting, $D(t) = e^{-\rho t}$. For $\theta = 2$ and $\beta = 1$, it gives the hyperbolic discounting model, $D(t) = 1/(1+\rho t)$. We estimate our DLuce model:

$$P^{\text{DLuce}}((x,0) \geq (y,t)) = \frac{x}{x + y\beta(1-(1-\theta)\rho t)^{\frac{1}{1-\theta}}}.$$

We fitted the DLuce and the Dlogit by using non-linear least-squares. Table 4 reports the estimates provided by the two models.[16] In the exponential discounting model $D(t) = e^{-\rho t}$, the DLuce rule produces a higher discount rate ρ, hence more impatient individuals, than the Dlogit. The same is true for the hyperbolic discounting, $D(t) = \frac{1}{1+\rho t}$. Moving to the quasi-hyperbolic discounting model, the DLuce rule estimates a smaller present-bias coefficient $\beta = 0.53$ with respect to the discounted logit where $\beta = 0.64$, as well as a higher discount rate (0.013 vs 0.008). Therefore, for to the DLuce, individuals are more impatient and more present-biased with respect to the Dlogit. In the last specification, the shape parameter θ is considerably smaller in the DLuce, hence

[16]In Table 4, we have omitted the estimates of the parameter λ (see Remark 1) for the Dlogit, which can be found in Tanaka et al. (2010, Table 6). For completeness, they are $\lambda = 6.26 \cdot 10^{-6}$ for exponential discount, $\lambda = 7.60 \cdot 10^{-6}$ for the hyperbolic discount, $\lambda = 8.58 \cdot 10^{-6}$ for quasi-hyperbolic discount and $\lambda = 8.70 \cdot 10^{-6}$ for the general discount function D (all coefficients are significantly different from zero).

	Exponential		Hyperbolic		Quasi-hyperbolic		General	
	DLuce	*Dlogit*	*DLuce*	*Dlogit*	*DLuce*	*Dlogit*	*DLuce*	*Dlogit*
ρ	0.025***	0.021***	0.076***	0.046***	0.013***	0.008***	0.056***	0.078
	(0.001)	(0.001)	(0.008)	(0.004)	(0.001)	(0.001)	(0.018)	(0.074)
β					0.529***	0.644***	0.673***	0.820***
					(0.023)	(0.019)	(0.043)	(0.070)
θ							2.715***	5.070***
							(0.268)	(0.659)
$\hat{R}^2$	0.522	0.515	0.534	0.519	0.535	0.522	0.536	0.523

Table 1: Comparison of the discounted Luce and discounted logit estimates of time preference parameters. Data from Tanaka et al. (2010). Robust standard errors in parenthesis. $\hat{R}^2$ is the adjusted R^2. *** means significant at the 1 percent level.

the discount function is closer to hyperbolic discounting in the DLuce than in the Dlogit. Overall, the time preference parameters estimated by the DLuce model are *uniformly more impatient* than those estimated by the Dlogit. The difference is substantial and should be taken into account in applications, for example, if such estimates are used to inform public policies. Andreoni et al. (2016), for example, used time preference estimates to customize vaccination incentives in Pakistan. Different point estimates imply different calibration of incentives, and the choice of one model over another ultimately affects the success of a policy based on the estimates.

5. Consumption streams

The dated reward setting is rather restrictive and, intertemporal choices, beyond experimental settings involving simple choices (such as the MPL), often concern consumption streams. For example, Warner and Pleeter (2001) use choices between consumption streams, lump sum payments versus annuities, to estimate discount rates. Alternatively, choices over consumption streams are observed in the Convex Time Budget design of Andreoni and Sprenger (2012). In this setting, subjects are given an amount of "tokens" and decide how to allocate them between a sooner and a later date yielding different “token exchange rates." For example, a subject given 100 tokens can assign N tokens to a sooner time yielding a_t per token and $100-N$ to a later time yielding a_{t+k} per token. The experimenter exogenously fixes the “exchange rates" a_t and a_{t+k}. This translates into a choice from a menu containing, for example, the following options: a choice between a consumption stream paying \$19 now and \$0 in 5 weeks, one paying \$15.2 now and \$4 in 5 weeks and one paying \$11.4 now and \$8 in 5 weeks.

The first consideration is that, differently from the dated outcomes case, the DLuce is no longer monotone in the sense of Apesteguia and Ballester (2018).[17] However, the intuition concerning the Weak Stationarity property is still valid. Indeed, the Dlogit is incompatible with the (suitably adapted version of the) weak stationarity condition, while the DLuce is the only model that satisfies Weak Stationarity plus some additional technical axioms.

Before characterizing the DLuce in the consumption streams case, we need to formalize the setting. We enlarge the set X of rewards by an element z that will play a special role in the following analysis, and we denote by W the set of all rewards $W = X \cup z$. For some $T > 0$, let $W^{T+1} = W \times W \times \cdots W$ be a $T+1$ product of W. An element of W^{T+1} represents a consumption stream $\mathbf{x} = (x_0, x_1, \ldots, x_T)$. A choice set is an element of $\mathscr{A} = 2^{W^{T+1}} \setminus \{\emptyset\}$. In the consumption stream setting, the DLuce has the following form:

$$P^{\text{DLuce}}(\mathbf{x}|A) = \frac{\sum_{t=0}^{T} D(t)\, v(x_t)}{\sum_{\mathbf{y}\in A} \sum_{t=0}^{T} D(t)\, v(y_t)}$$

where D is a discount function. The probability of selecting a consumption stream $\mathbf{x}$ from A is given by its relative weight in the choice set A. The weight of $\mathbf{x}$ is its discounted value. The first two axioms are standard:

(Positivity). For all $A \in \mathscr{A}$ with $\mathbf{x} \in A$ and $\mathbf{x} \neq \mathbf{z}$, $P(\mathbf{x}|A) > 0$.

Choice probabilities are always strictly positive except for the probability of se-

[17]Take, for example $v(x) = x$, $D(t) = \delta^t$ and two consumption streams $\mathbf{x} = (2,0,0,1)$ and $\mathbf{y} = (1,2,0,0)$. Then $P^{\text{DLuce}}(\mathbf{x}|A)$, with $A = \mathbf{x} \cup \mathbf{y}$ is equal to 0.53 for $\delta = 0.4$, to 0.4913 for $\delta = 0.8$ and to 0.498 for $\delta = 0.99$.

lecting the stream $\mathbf{z} = (z, z, z, \ldots, z)$. The standard Luce choice axiom or Independence of Irrelevant Alternatives (IIA) with Positivity becomes:

(IIA). For all $A, B \in \mathscr{A}$ and $\mathbf{x}, \mathbf{y} \in A \cap B$ with $\mathbf{x}, \mathbf{y} \neq \mathbf{z}$,

$$\frac{P(\mathbf{x}|A)}{P(\mathbf{y}|A)} = \frac{P(\mathbf{x}|B)}{P(\mathbf{y}|B)}.$$

Before introducing the adapted version of the Weak Stationarity axiom, we introduce additional notation. For an arbitrary $x \in W$, we denote by $\mathbf{x}(t)$ the consumption stream $\mathbf{x}(t) = (z, z, \ldots, x, z, \ldots, z)$ that pays x at time t and z otherwise. For any $\mathbf{x} \in W^{T+1}$, we denote by $(\mathbf{x_{-t}}, z)$ a consumption stream equal to $\mathbf{x}$ for all $s \neq t$ and equal to z in t, i.e. $(\mathbf{x_{-t}}, z) = (x_0, x_1, \ldots, x_{t-1}, z, x_{t+1}, \ldots, x_T)$. Lastly, for each $A \in \mathscr{A}$, we define $A_{+1} = \{(z, \mathbf{x}) : \mathbf{x} \in A\}$, where the consumption stream $(z, \mathbf{x})$ is a push-forward of $\mathbf{x}$, $(z, \mathbf{x}) = (z, x_0, \ldots, x_{T-1})$. Extending the argument from dated rewards to consumption streams, we introduce the weak stationarity axiom:

(Weak Stationarity). For all $x, y \in X$, $0 \leq t \leq T-1$ and all $A \in \mathscr{A}$:

$$\frac{P(\mathbf{x}(t)|A)}{P(\mathbf{y}(t)|A)} = \frac{P((z, \mathbf{x}(t))|A_{+1})}{P((z, \mathbf{y}(t))|A_{+1})}.$$

Suppose that $v(z) = 0$, then the discounted Luce rule satisfies the Weak Stationarity axiom, indeed:

$$\begin{aligned}\frac{P^{\text{DLuce}}(\mathbf{x}(t)|A)}{P^{\text{DLuce}}(\mathbf{y}(t)|A)} &= \frac{\sum_{s=0}^{t-1} D(s)\,v(z) + D(t)\,v(x) + \sum_{s=t+1}^{T} D(s)\,v(z)}{\sum_{s=0}^{t-1} D(s)\,v(z) + D(t)\,v(y) + \sum_{s=t+1}^{T} D(s)\,v(z)} = \frac{D(t)\,v(x)}{D(t)\,v(y)} \\ &= \frac{D(t+1)\,v(x)}{D(t+1)\,v(y)} = \frac{P^{\text{DLuce}}((z, \mathbf{x}(t))|A_{+1})}{P^{\text{DLuce}}((z, \mathbf{y}(t))|A_{+1})}.\end{aligned}$$

Notice that we did not assume $v(z) = 0$, however the following axiom, akin to additive separability, implies $v(z) = 0$ (see Lemma 1).

(Separability). For all $\mathbf{x}, \mathbf{y} \in W^{T+1}$, with $\mathbf{y} \neq \mathbf{z}$, and all $0 \leq t \leq T$:

$$\frac{P(\mathbf{x}|A)}{P(\mathbf{y}|A)} = \frac{P(\mathbf{x_t}(t)|B) + P(\mathbf{x_{-t}}, z|B)}{P(\mathbf{y}|B)}.$$

Separability implies that the relative probability of choosing $\mathbf{x}$ from a menu A containing $\mathbf{y}$ can be decomposed into the probability of choosing its "components" $\mathbf{x_t}(t) = (z, z, \ldots, x_t, z, \ldots, z)$. To gain intuition, consider two periods $T = 1$, $\mathbf{x} = (x_0, x_1)$, $\mathbf{y} = (y_0, y_1)$, $A = \{(x_0, x_1), (y_0, y_1)\}$ and $B = \{(x_0, z), (z, x_1), (y_0, y_1)\}$. Separability implies

$$\frac{P((x_0, x_1)|A)}{P((y_0, y_1)|A)} = \frac{P((x_0, z)|B) + P((z, x_1)|B)}{P((y_0, y_1)|B)}.$$

It is straightforward to observe that the Dlogit violates Separability as well as Weak Stationarity. Lastly, we define a stochastic notion of impatience similar to that presented in the delayed rewards setting:

(Stochastic Impatience). For all $x, y \neq z \in X$, $t \geq 0$ and $A \in \mathscr{A}$, with $\mathbf{x}(t), \mathbf{x}(t+1) \notin A$, $P(\mathbf{x}(t)|A \cup \mathbf{x}(t)) \geq P(\mathbf{x}(t+1)|A \cup \mathbf{x}(t+1))$.

The next theorem characterizes the DLuce model in the domain of consumption streams:

Theorem 3. *The choice probabilities P satisfy Positivity, the IIA axiom, Weak Stationarity, Separability and Stochastic Impatience if and only if there exists a random scale $v : X \to \mathbb{R}_{++}$ with $v(z) = 0$ and a (unique) discount function $D : \{0, 1, \ldots, T\} \to (0, 1]$ such that:*

$$P(\mathbf{x}|A) = \frac{\sum_{t=0}^{T} D(t)\, v(x_t)}{\sum_{\mathbf{y} \in A} \sum_{t=0}^{T} D(t)\, v(y_t)}.$$

As for dated rewards, applying the Dlogit to weakly stationary choice data will result in a biased estimation of the preference parameters. Indeed, even if $w(z) = 0$, the equality $\frac{P^{\text{Dlog}}(\mathbf{x}(t)|A)}{P^{\text{Dlog}}(\mathbf{y}(t)|A)} = \frac{P^{\text{Dlog}}((z, \mathbf{x}(t))|A_{+1})}{P^{\text{Dlog}}((z, \mathbf{y}(t))|A_{+1})}$ is equivalent to $e^{D(t)(w(x) - w(y))} = e^{D(t+1)(w(x) - w(y))}$, that holds only if $w(x) = w(y)$ or $D(t) = D(t+1)$. Again, the Dlogit is incompatible with weakly stationary choice probabilities.

5.1. Geometric and quasi-hyperbolic discounting

Similarly to the case of dated outcomes, axioms that strengthen the Weak Stationarity condition identify geometric and quasi-hyperbolic discounting in the DLuce model. Intuitively, we deem a stochastic choice rule to be stationary if the relative probability of choosing a consumption stream $\mathbf{x}$ over $\mathbf{y}$ in a set A remains unvaried when all the elements in A are "shifted" by one period. For $\mathbf{x} \in W^{T+1}$, we denote by $(\mathbf{x}, z)$, the consumption stream $(x_0, x_1, \ldots, x_{T-1}, z)$ and by $(z, \mathbf{x}, z)$ the stream $((z, \mathbf{x}), z) = (z, x_0, x_1, \ldots, x_{T-2}, z)$. We say that a stochastic choice rule satisfies Fishburn Stationarity if:

(Fishburn Stationarity). For all $\mathbf{x}, \mathbf{x}' \in W^{T+1}$ with $\mathbf{x}' \neq \mathbf{z}$, all $A \in \mathscr{A}$ with $(\mathbf{x}, z), (\mathbf{x}', z) \in A$:

$$\frac{P((\mathbf{x}, z)|A)}{P((\mathbf{x}', z)|A)} = \frac{P((z, \mathbf{x})|A_{+1})}{P((z, \mathbf{x}')|A_{+1})}.$$

This definition is the stochastic counterpart of Fishburn's definition of stationarity for deterministic choice over finite-horizon consumption streams (see Fishburn, 1970, Def. 7.3). Suppose that $v(z) = 0$, then the DLuce rule satisfies Fishburn Stationarity when the discount function is geometric, indeed:

$$\frac{P^{\text{DLuce}}((z, \mathbf{x})|A_{+1})}{P^{\text{DLuce}}((z, \mathbf{x}')|A_{+1})} = \frac{v(z) + \delta v(x_0) + \delta^2 v(x_1) + \sum_{t=3}^{T} \delta^t v(x_{t-1})}{v(z) + \delta v(x'_0) + \delta^2 v(x'_1) + \sum_{t=3}^{T} \delta^t v(x'_{t-1})} = \frac{P^{\text{DLuce}}((\mathbf{x}, z)|A)}{P^{\text{DLuce}}((\mathbf{x}', z)|A)}.$$

The converse result is also true: replacing Weak Stationarity with Fishburn Stationarity characterizes geometric discounting.

Proposition 6. *The choice probabilities P satisfy Positivity, the IIA axiom, Fishburn Stationarity, Separability and Stochastic Impatience if and only if there exists a ratio scale $v : X \to \mathbb{R}_+$, with $v(z) = 0$ and a (unique) $\delta \in (0, 1]$ such that:*

$$P(\mathbf{x}|A) = \frac{\sum_{t=0}^{T} \delta^t v(x_t)}{\sum_{\mathbf{y} \in A} \sum_{t=0}^{T} \delta^t v(y_t)}.$$

In the consumption stream setting, too, we are interested in determining the observable restrictions of quasi-hyperbolic discounting. Indeed, although $v(z) = 0$, quasi-hyperbolic discounting violates Fishburn Stationarity.[18] The following relaxation of Fishburn Stationarity is parallel to that introduced in the delayed rewards setting and includes all the intuitive features of a present-biased stochastic choice rule. These features are comparable with the axioms

[18] In general, $\frac{P^{\text{DLuce}}((\mathbf{x}, z)|A)}{P^{\text{DLuce}}(\mathbf{x}', z)|A)} = \frac{v(x_0) + \beta \sum_{t=1}^{T-1} \delta^t v(x_t) + \beta \delta^T v(z)}{v(x'_0) + \beta \sum_{t=1}^{T-1} \delta^t v(x'_t) + \beta \delta^T v(z)} \neq \frac{v(z) + \beta \sum_{t=1}^{T} \delta^t v(x_{t-1})}{v(z) + \beta \sum_{t=1}^{T} \delta^t v(x'_{t-1})} = \frac{P^{\text{DLuce}}((z, \mathbf{x})|A_{+1})}{P^{\text{DLuce}}((z, \mathbf{x}')|A_{+1})}$.

characterizing quasi-hyperbolic discounting in deterministic choice (e.g. Montiel Olea and Strzalecki, 2014). The axiom establishes two properties: Quasi-stationarity and Present Bias.

(Quasi-hyperbolic Stationarity).

1. (Quasi-stationarity): for all $A \in \mathscr{A}$, with $(z,\mathbf{x}), (z,\mathbf{x}') \in A$, if $P((z,\mathbf{x}',z)|A) > 0$ and $P((z,z,\mathbf{x}')|A_{+1}) > 0$:
$$\frac{P((z,\mathbf{x},z)|A)}{P((z,\mathbf{x}',z)|A)} = \frac{P((z,z,\mathbf{x})|A_{+1})}{P((z,z,\mathbf{x}')|A_{+1})}.$$

2. (Present Bias): for all $A \in \mathscr{A}$, with $(\mathbf{x},z,z), (z,\mathbf{x}',z) \in A$, if $P((z,\mathbf{x},z)|A_{+1}) > 0$ and $P((z,z,\mathbf{x}')|A_{+1}) > 0$:
$$\frac{P((\mathbf{x},z,z)|A)}{P((z,\mathbf{x}',z)|A)} \geq \frac{P((z,\mathbf{x},z)|A_{+1})}{P((z,z,\mathbf{x}')|A_{+1})}.$$

We have the following result:

Proposition 7. *The choice probabilities P satisfy Positivity, the IIA axiom, Weak Stationarity, Quasi-hyperbolic Stationarity, Separability and Stochastic Impatience if and only if there exists a ratio scale $v : X \to \mathbb{R}_{++}$ with $v(z) = 0$ and (unique) $\beta, \delta \in (0,1]$ such that:*

$$P(\mathbf{x}|A) = \frac{v(x_0) + \beta \sum_{t=1}^{T} \delta^t v(x_t)}{\sum_{\mathbf{y}\in A} \left[v(y_0) + \beta \sum_{t=1}^{T} \delta^t v(y_t) \right]}.$$

6. Conclusion

The choice of a random utility model to estimate time preferences is a crucial step of the analysis. Often, such a choice is not properly justified and is driven by heuristics or popularity. In our analysis, we argue theoretically and support empirically the conclusion that the discounted logit may cause problems in estimating time preferences. Under common interpretations of stochastic choice, the discounted Luce is more appropriate as a structural model.

Appendix A. Proofs

Proof of Theorem 1. Suppose that the equality $\frac{P^{\text{Dlog}}((x,t+r)|A_r)}{P^{\text{Dlog}}((y,t+r)|A_r)} = \frac{P^{\text{Dlog}}((x,t)|A)}{P^{\text{Dlog}}((x,t)|A)}$ holds. By the IIA axiom, the previous equality implies $\frac{P^{\text{Dlog}}((x,t+r)|(y,t+r))}{P^{\text{Dlog}}((y,t+r)|(x,t+r))} = \frac{P^{\text{Dlog}}((x,t)|(y,t))}{P^{\text{Dlog}}((y,t)|(x,t))}$ or equivalently $e^{D(t+r)w(x)-D(t+r)w(y)} = e^{D(t)w(x)-D(t)w(y)}$ which can be true only if $D(t+r) = D(t)$ or $w(y) = w(x)$. In the former case, since D is weakly decreasing, $D(s)$ is constant for all $s \in [t, t+r]$. The opposite direction is straightforward. □

Proof of Proposition 1. Necessity is straightforward. For sufficiency, by the IIA axiom and Positivity there exists a random scale $u : Z \to \mathbb{R}_{++}$ such that $P((x,t)|A) = \frac{u(x,t)}{\sum_{(y,s)\in A} u(y,s)}$. For an arbitrary $t \geq 0$ and $x \in X$, let us define $D_x(t) \equiv \frac{P((x,t)|A)}{P((x,0)|A)} = \frac{u(x,t)}{u(x,0)}$. Let us define $v(x) \equiv u(x,0)$ for all $x \in X$, then $u(x,t) = D_x(t)v(x)$. Take $A = \{(x,0),(y,0)\}$, then by the Weak Stationarity axiom:

$$\frac{u(x,0)}{u(y,0)} = \frac{u(x,r)}{u(y,r)}$$

or $\frac{v(x)}{v(y)} = \frac{D_x(r)v(x)}{D_y(r)v(y)}$ that implies $D_x(\cdot) = D_y(\cdot)$ for all $x, y \in X$. Hence we denote by $D(\cdot)$ such a common function. By Stochastic Impatience, $D(t)$ is decreasing. Substituting $D(t)v(x) = u(x,t)$ gives the result. Uniqueness of $D(\cdot)$ follows from the fact that u is a ratio scale. □

Proof of Proposition 3. The fact that the representation implies the axioms is straightforward. We prove that the axioms are sufficient. By Positivity and the IIA axioms there exists $u : Z \to \mathbb{R}_{++}$ such that $P((x,t)|A) = \frac{u(x,t)}{\sum_{(y,s)\in A} u(y,s)}$. Let us consider $A = \{(x,1),(x,0)\}$. Then, for all $t \geq 1$, the Stationarity axiom implies

$$\frac{P((x,1)|A)}{P((x,0)|A)} = \frac{P((x,t)|A_{t-1})}{P((x,t-1)|A_{t-1})}$$

or

$$\frac{u(x,1)}{u(x,0)} = \frac{u(x,t)}{u(x,t-1)} \tag{A.1}$$

Now, let us write

$$u(x,t) = u(x,0)\frac{u(x,1)}{u(x,0)}\frac{u(x,2)}{u(x,1)}\frac{u(x,3)}{u(x,2)}\cdots\frac{u(x,t)}{u(x,t-1)}$$

By Eq. (A.1) and by defining $\delta_x \equiv \frac{u(x,1)}{u(x,0)}$, we have $u(x,t) = u(x,0)\delta_x^t$. By Stochastic Impatience for $t = 0$ $\delta_x = \frac{u(x,1)}{u(x,0)} \leq 1$. Plugging $u(x,t) = \delta_x^t u(x,0)$ into the Stationarity axiom gives $\frac{\delta_x^t}{\delta_y^s} = \frac{\delta_x^{t+r}}{\delta_y^{s+r}}$ or $1 = \frac{\delta_x^r}{\delta_y^r}$ that is true only when $\delta_x = \delta_y$. Hence, defining $v(x) \equiv u(x,0)$ gives the result. The uniqueness of δ follows from the fact that u is a ratio scale. □

Proof of Proposition 4. Necessity is straightforward. For sufficiency, by Positivity, the IIA axiom and Weak Stationarity, choice probabilities have a DLuce representation, for some random scale $v : X \to \mathbb{R}_{++}$ and some discount function D. Let us define $\delta \equiv \frac{D(2)}{D(1)}$, Quasi-stationarity implies

$$\frac{D(2)\,v(x)}{D(1)\,v(y)} = \frac{D(2+r)\,v(x)}{D(1+r)\,v(y)}$$

for all $x, y \in X$ and $r \geq 0$. Equivalently, $\delta = \frac{D(r+2)}{D(r+1)}$ for all $r \geq 0$. For $r = 1$, $\delta = \frac{D(2)}{D(1)} = \frac{D(3)}{D(2)}$, or $D(3) = D(2)\delta = D(1)\delta^2$. Repeating this argument gives $D(t) = D(1)\delta^{t-1}$, for all $t \geq 1$. Present Bias implies

$$\frac{1}{D(1)} \geq \frac{1}{\delta}$$

then there exists $\beta \in (0,1]$ with $D(1) = \delta\beta$. Substituting in the previous equality, $D(t) = \beta\delta^t$ for $t \geq 1$ concludes the proof. The uniqueness follows from the uniqueness properties of Proposition 1.

□

Proof of Proposition 5. Necessity is straightforward. For sufficiency, by Positivity and the IIA axioms, $P((x,t)|A) = \frac{u(x,t)}{\sum_{(y,s)\in A} u(y,s)}$ for some $u : X \to \mathbb{R}_{++}$. Since there is a pair $x^*, y^* \in X$ that is 0-distinguishable, let us define

$$D_{x^*,y^*}(t) \equiv \frac{l_{x^*,y^*}(t)}{l_{x^*,y^*}(0)}$$

which is a well-defined real number. By Log-odds Stationarity, $D_{x^*,y^*}(t) = \frac{l_{x^*,y^*}(t)}{l_{x^*,y^*}(0)} = \frac{l_{x',y^*}(t)}{l_{x',y^*}(0)} = D_{x',y^*}(t)$, hence the function D_{x^*,y^*} is independent of x^*. Now define $w(x) \equiv l_{x,y^*}(0)$ if x and y^* are 0-distinguishable and $w(x) \equiv 0$ otherwise (since there is a 0-distinguishable pair, w is not identically zero.) Define $\lambda(t) \equiv u(y^*,t)$. It follows that $\lambda(t)e^{D(t)w(x)} = u(y^*,t)e^{\frac{l_{x,y^*}(t)}{l_{x,y^*}(0)}l_{x,y^*}(0)} = u(y^*,t)e^{l_{x,y^*}(t)} = u(y^*,t)\frac{P((x,t)|A)}{P((y^*,t)|A)} = u(x,t)$, hence the conclusion. □

Before proving Theorem 3, we consider a consequence of Separability. Let $\mathbf{z} = (z, z, \ldots, z)$, then:

Lemma 1. *For the* $z \in W$ *satisfying Separability,* $P(\mathbf{z}|A) = 0$ *for all* $A \in \mathscr{A}$ *with* $\mathbf{x}, \mathbf{z} \in A$ *and* $\mathbf{x} \neq \mathbf{z}$*.*

Proof of Lemma 1. Take $A = B$, by Separability, $P(\mathbf{z}|A) = P(\mathbf{z_t}(t)|A) + P(\mathbf{z_t}, z|A)$ for some A containing $\mathbf{z}$, but $\mathbf{z_t}(t) = \{\mathbf{z_t}, z\} = \mathbf{z}$. Hence, $P(\mathbf{z}|A) = 2P(\mathbf{z}|A)$ and this can be true only if $P(\mathbf{z}|A) = 0$. □

Proof of Theorem 3. By the IIA and Positivity axioms, there exists a random scale $u : W^{T+1} \to \mathbb{R}_+$ such that $u(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{z}$ and $P(\mathbf{x}|A) = \frac{u(x_0, x_1, \ldots, x_T)}{\sum_{\mathbf{y} \in A} u(y_0, y_1, \ldots, y_T)}$. By Lemma 1, $u(z, z, \ldots, z) = 0$. For an arbitrary $x \in X$, let define $v(x) \equiv u(x, z, z, \ldots) = u(\mathbf{x}(0))$ and define $D_x(t) \equiv \frac{u(\mathbf{x}(t))}{u(\mathbf{x}(0))}$. By Weak Stationarity, for all $x, y \in X$:

$$\frac{P(\mathbf{x}(0)|A)}{P(\mathbf{y}(0)|A)} = \frac{P((z, \mathbf{x}(0))|A_{+1})}{P((z, \mathbf{y}(0))|A_{+1})}.$$

The latter ratio is equal to

$$\frac{P((z, \mathbf{x}(0))|A_{+1})}{P((z, \mathbf{y}(0))|A_{+1})} = \frac{P(\mathbf{x}(1)|A_{+1})}{P(\mathbf{y}(1)|A_{+1})} = \frac{u(\mathbf{x}(1))}{u\mathbf{y}(1)}$$

Repeated applications of Weak Stationarity imply that, for all $x, y \in X$,

$$\frac{u(\mathbf{x}(t))}{u(\mathbf{x}(0))} = \frac{u(\mathbf{y}(t))}{u(\mathbf{y}(0))}$$

and this implies $D_x(t) = D_y(t)$ for all $x, y \in X$. Hence, we can write $D(t)$ for $D_x(t)$. Lastly, let us define $D_z(t) \equiv D(t)$ for $z \in W$ and all $t \leq T$. By Stochastic Impatience $D(t) \leq 1$ for all $t \leq T$. By definition $u(\mathbf{x}(t)) = \frac{u(\mathbf{x}(0))}{u(\mathbf{x}(0))} u(\mathbf{x}(t)) = v(x)D(t)$. To conclude, we need to prove that $u(x_0, x_1, x_2, \ldots) = \sum_{t=0}^{T} D(t) v(x_t)$. To see this, consider $P((x_0, x_1, \ldots, x_T)|A)$ for some A and apply Separability twice:

$$\begin{aligned}\frac{u(x_0, x_1, \ldots, x_T)}{u(y_0, y_1, \ldots, y_T)} &= \frac{v(x_0) + u(z, x_1, x_2, \ldots, x_T)}{u(y_0, y_1, \ldots, y_T)} = \\ &= \frac{v(x_0)}{u(y_0, y_1, \ldots, y_T)} + \frac{v(x_1)D(1)}{u(y_0, y_1, \ldots, y_T)} + \frac{u(z, z, x_2, x_3, \ldots, x_T)}{u(y_0, y_1, \ldots, y_T)}\end{aligned}$$

repeating the argument until T gives

$$u(x_0, x_1, \ldots, x_T) = \sum_{t=0}^{T} u(\mathbf{x_t}(t)) = \sum_{t=0}^{T} D(t) v(x_t)$$

The uniqueness of D follows from the fact that u is a ratio scale.

□

Proof of Proposition 6. By the IIA and Positivity axioms, there exists a random scale $u : W^{T+1} \to \mathbb{R}_+$ such that $u(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{z}$ and $P(\mathbf{x}|A) = \frac{u(x_0, x_1, \ldots, x_T)}{\sum_{\mathbf{y} \in A} u(y_0, y_1, \ldots, y_T)}$. For an arbitrary $x \in X$, let us define $v(x) \equiv u(x, z, z, \ldots) = u(\mathbf{x}(0))$. By Lemma 1, $u(z, z, z, \ldots, z) = 0$. For an $x \in X$, let us define $\delta_x \equiv \frac{u(\mathbf{x}(1))}{u(\mathbf{x}(0))}$. By Fishburn Stationarity and $x, y \in X$,

$$\frac{P(\mathbf{x}(1)|A_{+1})}{P(\mathbf{y}(1)|A_{+1})} = \frac{P(\mathbf{x}(0)|A)}{P(\mathbf{y}(0)|A)}$$

or equivalently,

$$\frac{P(\mathbf{x}(1)|A_{+1})}{P(\mathbf{x}(0)|A)} = \frac{P(\mathbf{y}(1)|A_{+1})}{P(\mathbf{y}(0)|A)} \quad \Longrightarrow \quad \frac{u(\mathbf{x}(1))}{u(\mathbf{x}(0))} = \frac{u(\mathbf{y}(1))}{u(\mathbf{y}(0))}$$

and this implies $\delta_x = \delta_y = \delta$ for all $x, y \in X$ and define $\delta_z \equiv \delta$ for $z \in W$. By Stochastic Impatience $\delta \leq 1$. Fishburn Stationarity implies

$$\frac{P(\mathbf{x}(1)|A)}{P(\mathbf{x}(0)|A)} = \frac{P(\mathbf{x}(2)|A_{+1})}{P(\mathbf{x}(1)|A_{+1})}$$

another application of Fishburn Stationarity implies

$$\frac{P(\mathbf{x}(2)|A_{+1})}{P(\mathbf{x}(1)|A_{+1})} = \frac{P(\mathbf{x}(3)|(A_{+1})_{+1})}{P(\mathbf{x}(2)|(A_{+1})_{+1})}$$

equivalently,

$$\frac{u(\mathbf{x}(1))}{u(\mathbf{x}(0))} = \frac{u(\mathbf{x}(2))}{u(\mathbf{x}(1))} = \frac{u(\mathbf{x}(3))}{u(\mathbf{x}(2))}$$

The first equality implies $u(\mathbf{x}(2)) = u(\mathbf{x}(1)) \cdot \frac{u(\mathbf{x}(1))}{u(\mathbf{x}(0))}$ and the second, $u(\mathbf{x}(3)) = u(\mathbf{x}(2)) \cdot \frac{u(\mathbf{x}(1))}{u(\mathbf{x}(0))}$, and together $u(\mathbf{x}(3)) = u(\mathbf{x}(1)) \cdot \left(\frac{u(\mathbf{x}(1))}{u(\mathbf{x}(0))}\right)^2$. Repeating the argument up to t gives

$$u(\mathbf{x}(t)) = u(\mathbf{x}(1)) \cdot \left(\frac{u(\mathbf{x}(1))}{u(\mathbf{x}(0))}\right)^{t-1}$$

multiplying and dividing by $u(\mathbf{x}(0))$, gives

$$u(\mathbf{x}(t)) = u(\mathbf{x}(0)) \cdot \left(\frac{u(\mathbf{x}(1))}{u(\mathbf{x}(0))}\right)^{t} \tag{A.2}$$

in our notation this becomes $u(\mathbf{x}(t)) = \delta^t v(x)$. The proof that $u(x_0, x_1, x_2, \ldots) = v(x_0) + \sum_{t=1}^{T} \delta^t v(x_t)$ follows the same steps of the proof of Theorem 3.

□

Proof of Proposition 7. By the IIA and Positivity axioms, there exists a random scale $u : W^{T+1} \to \mathbb{R}$ such that $u(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{z}$ and $P(\mathbf{x}|A) = \frac{u(x_0, x_1, \ldots, x_T)}{\sum_{\mathbf{y} \in A} u(y_0, y_1, \ldots, y_T)}$. By Lemma 1, $u(z, z, z, \ldots, z) = 0$. For a given $x \in X$, let us define $v(x) \equiv u(x, z, z, \ldots)$ and $\delta_x = \frac{u(\mathbf{x}(2))}{u(\mathbf{x}(1))}$ By Quasi-stationarity,

$$\frac{P(\mathbf{x}(2)|A_{+1})}{P(\mathbf{y}(2)|A_{+1})} = \frac{P(\mathbf{x}(1)|A)}{P(\mathbf{y}(1)|A)}$$

or equivalently,

$$\frac{P(\mathbf{x}(2)|A_{+1})}{P(\mathbf{x}(1)|A)} = \frac{P(\mathbf{y}(2)|A_{+1})}{P(\mathbf{y}(1)|A)} \implies \frac{u(\mathbf{x}(2))}{u(\mathbf{x}(1))} = \frac{u(\mathbf{y}(2))}{u(\mathbf{y}(1))}$$

and this implies $\delta_x = \delta_y = \delta$ for all $x, y \in X$ and define $\delta_z \equiv \delta$. By Stochastic Impatience $\delta \leq 1$. Quasi-stationarity again implies

$$\frac{P(\mathbf{x}(2)|A)}{P(\mathbf{x}(1)|A)} = \frac{P(\mathbf{x}(3)|A_{+1})}{P(\mathbf{x}(2)|A_{+1}))}$$

and another application of Quasi-stationarity implies

$$\frac{P(\mathbf{x}(3)|A_{+1})}{P(\mathbf{x}(2)|A_{+1}))} = \frac{P(\mathbf{x}(4)|(A_{+1})_{+1})}{P(\mathbf{x}(3)|(A_{+1})_{+1})}$$

which is equivalent to $\frac{u(\mathbf{x}(2))}{u(\mathbf{x}(1))} = \frac{u(\mathbf{x}(3))}{u(\mathbf{x}(2))} = \frac{u(\mathbf{x}(4))}{u(\mathbf{x}(3))}$. The first equality implies $u(\mathbf{x}(3)) = u(\mathbf{x}(2)) \cdot \frac{u(\mathbf{x}(2))}{u(\mathbf{x}(1))}$ and the second, $u(\mathbf{x}(4)) = u(\mathbf{x}(3)) \cdot \frac{u(\mathbf{x}(2))}{u(\mathbf{x}(1))}$, and together $u(\mathbf{x}(4)) = u(\mathbf{x}(2)) \cdot \left(\frac{u(\mathbf{x}(2))}{u(\mathbf{x}(1))}\right)^2$, repeating the same argument for an arbitrary $t > 1$, gives

$$u(\mathbf{x}(t)) = u(\mathbf{x}(2)) \cdot \left(\frac{u(\mathbf{x}(2))}{u(\mathbf{x}(1))}\right)^{t-2}$$

multiplying and dividing by $u(\mathbf{x}(1))$, gives

$$u(\mathbf{x}(t)) = u(\mathbf{x}(1)) \cdot \left(\frac{u(\mathbf{x}(2))}{u(\mathbf{x}(1))}\right)^{t-1} \tag{A.3}$$

for all $t > 0$. By Present Bias, with $\mathbf{x}' = \mathbf{y}(0)$ and $\mathbf{x} = \mathbf{x}(0)$:

$$\frac{u(\mathbf{x}(0))}{u(\mathbf{x}(1))} \geq \frac{u(\mathbf{y}(1))}{u(\mathbf{y}(2))}$$

hence $\frac{u(\mathbf{x}(1))}{u(\mathbf{x}(0))} \leq \frac{u(\mathbf{y}(2))}{u(\mathbf{y}(1))} = \delta$. Then, there exists $\beta_x \in [0,1]$ such that $u(\mathbf{x}(0)) = \beta_x \delta u(\mathbf{x}(1))$. Multiplying and dividing Eq. (A.3) by $u(\mathbf{x}(0))$ and using the equality $u(\mathbf{x}(0)) = \beta_x \delta u(\mathbf{x}(1))$, gives

$$u(\mathbf{x}(t)) = u(\mathbf{x}(0)) \cdot \beta_x \left(\frac{u(\mathbf{x}(2))}{u(\mathbf{x}(1))} \right)^t$$

for all $t > 0$. In our notation $u(\mathbf{x}(t)) = \beta_x \delta^t v(x)$ for $t > 0$ and $u(\mathbf{x}(0)) = v(x)$. By Weak Stationarity,

$$\frac{P(\mathbf{x}(1)|A_{+1})}{P(\mathbf{x}(0)|A)} = \frac{P(\mathbf{y}(1)|A_{+1})}{P(\mathbf{y}(0)|A)}$$

and,

$$\frac{u(\mathbf{x}(1))}{u(\mathbf{x}(0))} = \frac{u(\mathbf{y}(1))}{u(\mathbf{y}(0))} \iff \frac{v(x)\beta_x\delta}{v(x)} = \frac{v(y)\beta_y\delta}{v(y)}$$

that implies $\beta_x = \beta_y$ for all $x, y \in X$. Lastly, let us define $\beta_z \equiv \beta$. The fact that $u(x_0, x_1, x_2, \ldots) = v(x_0) + \beta \sum_{t=1}^{T} \delta^t v(x_t)$ follows from the same argument in the proof of Theorem 3. The uniqueness of δ and β are guaranteed by the fact that u is a ratio scale.

$\square$

Appendix B. Dealing with zero probabilities

In the DLuce, we assumed that all options are selected with non-zero probability. However, in experiments using the Multiple Price List method, however, some options are strictly dominated. For example, the subjects had to choose between (x, t) and $(x, 0)$. Despite the potential randomness of choices, a strictly dominated option is rarely chosen, therefore zero-probabilities cannot be excluded. It is well known that the Luce model does not satisfactorily handle zero probabilities. If $(x, 0)$ is selected in A with zero probability, the Luce model predicts that it will be selected with zero probability in any choice set. Various generalizations of the Luce model have recently been proposed to satisfactory handle zero probability.[19] In this section, we use the approach of Cerreia-Vioglio et al. (2017). We write $P(B|A) = \sum_{(x,t)\in B} P((x,t)|A)$, for any $B \subseteq A$. The next axiom is the well-known Luce Choice axiom of Luce (1959).

[19] Cerreia-Vioglio et al. (2017); Dogan and Yildiz (2019); Echenique and Saito (2019); Horan (2018)

(Luce Choice Axiom). For all $B \subseteq A$ in $\mathscr{A}$ and $(x,t) \in B$:

$$P((x,t)|A) = P((x,t)|B)P(B|A)$$

We denote by $r((x,t)|(y,s)) \triangleq \frac{P((x,t)|(y,s))}{P((y,s)|(x,t))}$ the odds ratio between (x,t) and (y,s). The next axiom is the desired restriction characterizing the generalized discounted Luce rule:

(Strong Stationarity). For all $(x,0),(y,s),(x,1) \in Z$ with $\frac{r((x,1)|(y,s))}{r((x,0)|(y,s))} \in (0,\infty)$:

$$\frac{r((x,1)|(y,s))}{r((x,0)|(y,s))} = \frac{r((x,t+1)|(y,s))}{r((x,t)|(y,s))}$$

for all $t \geq 0$.

The axiom has two consequences: it imposes a restriction similar to Stationarity and it forces certain positive probabilities to remain positive: if $P((x,0)|(y,s)) > 0$, then $P((x,t)|(y,s)) > 0$ for all $t \geq 0$.

Theorem 4. *Choice probabilities P satisfy the Luce Choice Axiom and Strong Stationarity if and only if there exist $v : X \to \mathbb{R}_{++}$, a correspondence $\Gamma : \mathscr{A} \to \mathscr{A}$ and $\delta > 0$ such that:*

$$P((x,t)|A) = \begin{cases} \dfrac{\delta^t v(x)}{\sum\limits_{(y,s)\in\Gamma(A)} \delta^s v(y)} & \textit{if } (x,t) \in \Gamma(A) \\ 0 & \textit{otherwise} \end{cases} \tag{B.1}$$

Proof of Theorem 4. By Cerreia-Vioglio et al. (2017), if choice probabilities satisfy the Luce choice axiom, then there exist a ratio scale $u : Z \to \mathbb{R}_{++}$ and a correspondence $\Gamma : \mathscr{A} \rightrightarrows \mathscr{A}$ such that $P((x,t)|A) = \frac{u(x,t)}{\sum_{(y,s)\in\Gamma(A)} u(y,s)}$ if $(x,t) \in \Gamma(A)$ and $P((x,t)|A) = 0$ otherwise. By the properties of Γ (see Cerreia-Vioglio et al. (2017)), the binary relation $\succcurlyeq$ defined as $(x,t) \succcurlyeq (y,s)$ if and only if $(x,t) \in \Gamma(\{(x,t),(y,s)\}$ is a weak order. Moreover, $(x,t) \sim (y,s)$ if and only if $P((x,t)|(y,s)) > 0$ and $P((y,s)|(x,t)) > 0$ if and only if $r((x,t)|(y,s)) \in (0,\infty)$. Again following Cerreia-Vioglio et al. (2017), we consider the partition of Z induced by $\succcurlyeq$ and for each cell Z_i of the partition we take $(x_i,t_i) \in Z_i$. Any $(x,t) \in Z$ belongs to one cell and we define $u(x,t) = \frac{P((x,t)|(x_i,t_i))}{P((x_i,t_i)|(x,t))} = r((x,t)|(x_i,t_i))$ for the selected $(x_i,t_i) \sim (x,t)$. By Strong Stationarity,

$$\frac{u(x,t)}{u(x,t-1)} = \frac{r((x,t)|(x_i,t_i))}{r((x,t-1)|(x_i,t_i))} = \frac{r((x,1)|(x_i,t_i))}{r((x,0)|(x_i,t_i))} = \frac{u(x,1)}{u(x,0)}$$

Therefore, the same argument in the proof of Theorem 3 can be repeated. □

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