A unitarity compatible approach to one-loop amplitudes with massive fermions

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ABSTRACT: We explain how one-loop amplitudes with massive fermions can be computed using only on-shell information. We first use the spinor-helicity formalism in six dimensions to perform generalised unitarity cuts in d dimensions. We then show that divergent wave-function cuts can be avoided, and the remaining ambiguities in the renormalised amplitudes can be fixed, by matching to universal infrared poles in $4 - 2\epsilon$ dimensions and ultraviolet poles in $6 - 2\epsilon$ dimensions. In the latter case we construct an effective Lagrangian in six dimensions and reduce the additional constraint to an on-shell tree-level computation.

Contents

1 Introduction

Precise predictions for the production of strongly interacting massive particles are in high demand for current experimental analyses at the LHC. The current precision level of predictions is in relatively good shape, with top quark pair production now known differentially at NNLO in QCD [1, 2] and a full range of off-shell decays known at NLO in QCD with an additional jet [3]. Modern one-loop techniques are also able to explore high multiplicity final states where the current state-of-the-art is top quark pair production in association
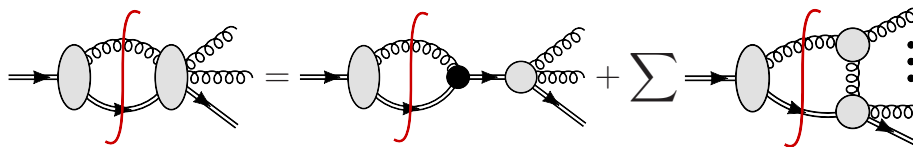
Figure 1. Decomposing the tree amplitude appearing on the left hand side of a wavefunction cut reveals a divergent graph.

with three jets [4]. The GoSam collaboration has also been able to produce NLO predictions for the challenging $t\bar{t}H + j$ final state [5]. A more complete overview of the current status can be found in reference [6].

On the other hand, these processes are often overlooked by more formal studies of amplitudes in gauge theory which can uncover hidden simplicity and structure. While it is well known that on-shell techniques like unitarity [7], spinor integration [8, 9] and BCFW recursion apply equally well to massive amplitudes, explicit computations are relatively few [10–12]. Nevertheless some computations using these approaches have produced compact analytic results useful for phenomenological applications [11, 13]. While elements of these computations use unitarity cuts and on-shell trees, Feynman diagrams techniques were also employed to compute the UV counterterms necessary for mass and wavefunction renormalisation. To the best of our knowledge the only computations not to do this are those with a massive internal loop where a UV matching prescription was used [7, 10].

The obstacle is that the traditional approach to renormalisation requires the amputation of wavefunction graphs, and the addition of counterterm diagrams. This procedure breaks gauge invariance during intermediate steps and therefore causes problems for methods based on (generalised) unitarity [14–16], which construct amplitudes from on-shell tree-level building blocks. Naive attempts to amputate wavefunction graphs in generalised unitarity are precluded by the presence of an on-shell propagator, leading to a factor 1/0: this is depicted explicitly in figure 1, where the on-shell tree amplitude appearing on the right hand side of a two-particle cut is expanded to reveal a divergent propagator inside. Consequently, the favoured method is still to follow an approach based on Feynman diagrams; then the amputation of wavefunction graphs is straightforward.

Two solutions to this problem have been proposed. Ellis, Giele, Kunszt and Melnikov showed that modifying the tree-level input entering the double cuts of the wavefunction graphs allowed a simple implementation of the on-shell renormalisation scheme [17]. All cuts can then be performed but gauge invariance is only restored at the end of the computation. Since the removal of the unwanted graphs is extremely easy to implement within a Berends-Giele construction of the tree-level amplitudes in the cuts this method is quite efficient numerically. A second solution, proposed by Britto and Mirabella [18], is to regulate the divergent tree by introducing a momentum shift. This procedure allows us to preserve gauge invariance but introduces an additional variable into the calculation which will cancel when combined with the mass-renormalisation counterterms. In either case a set of extra two- and single-particle cuts is necessary together with the counterterms to fully determine the amplitude in comparison to the massless case.

Despite both of these solutions there is still an open question: is it possible to compute amplitudes with masses using only on-shell gauge invariant building blocks and without introducing additional regulators? Both of the approaches mentioned above follow the on-shell renormalisation scheme where divergences can be absorbed into additional terms in the Lagrangian. In this paper we will seek an alternative way to absorb the divergences by appealing to an effective six dimensional version of QCD.

This procedure relies on first computing a full set of finite d-dimensional unitarity cuts. We show how this can be done efficiently in the six-dimensional spinor-helicity formalism [19] by embedding the additional mass into the higher dimensions and performing cuts in six dimensions. In particular we show how these results can be dimensionally reduced to d-dimensional amplitudes keeping the spin dimension of the gluon d_s arbitrary.[1] This generalises the previous approaches used for massless cuts in six-dimensions [20, 21].

Our paper is organised as follows. We begin by reviewing the structure of one-loop amplitudes in dimensional regularisation and the integrand level representations. In section 3 we review the spinor-helicity formalisms in four and six dimensions, and show how Dirac spinors for massive fermions can be represented as massless Weyl spinors in six dimensions. We then discuss a simple example of a pair of massive fermions coupling to an off-shell vertex at one-loop. This example allows us to show how computations in six dimensions can be performed, and how they can be dimensionally reduced to results with an arbitrary spin dimension d_s. In section 4 we summarise the generalised unitarity method in six dimensions and explain some of the key features needed to apply it to the case of $gg \rightarrow t\bar{t}$ scattering. Section 5 describes the procedure of fixing the remaining ambiguities using the universal epsilon pole structure in $d = 4 - 2\epsilon$ dimensions and the corresponding epsilon pole structure of the effective theory in $6 - 2\epsilon$ dimensions. Following our conclusions, the appendices give further details of the spinor-helicity method, the tree-level amplitudes in six dimensions and multiple cut solutions. We also include an appendix with the Feynman rules for the dimension six operators used in section 5.

In addition we provide a MATHEMATICA workbook as supplementary material to this article. This workbook contains a basic implementation of all the ingredients used in the paper and runs through numerical examples for two leading colour primitive one-loop amplitudes contributing to $gg \rightarrow t\bar{t}$ scattering. Analytic formulae for these two primitive amplitudes are also provided in a computer readable form including the full dependence on the loop momentum dimension d and spin dimension d_s.

2   Review of one-loop amplitudes and integrands

In this section we review the basics of one-loop integrand parametrisations in d dimensions. While many good reviews of the subject are available, e.g. [22–24], this section also introduces the notation we will use in the rest of the paper. The amplitudes we will consider in this paper are QCD amplitudes with one massive fermion flavour. In this case there are

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[1]The distinction between the spin dimension, d_s, and spacetime dimension, d, is motivated by different regularisation schemes. We find it to be very convenient to maintain the distinction throughout our calculations. To be clear, we define the spin dimension such that the gluon has $d_s - 2$ physical polarization states.

only two possible basis integrals which go beyond those appearing in the massless case,

$$A_n^{(1)} = B_n^{(1)} + c_{2;m^2} I_{2,m^2} + c_1 I_1. \tag{2.1}$$

The amplitude labelled $B_n^{(1)}$ is the part that can be constructed from finite d-dimensional unitarity cuts. The additional basis integrals depend only on the fermion mass and in dimensional regularisation are,

$$I_{2,m^2} = \mu_R^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2((k-p)^2 - m^2)} \overset{d=4-2\epsilon}{=} ic_\Gamma \left(\frac{1}{\epsilon} + \log\left(\frac{\mu_R^2}{m^2}\right) + 2 \right) + \mathcal{O}(\epsilon), \tag{2.2}$$

$$I_1 = \mu_R^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} \overset{d=4-2\epsilon}{=} ic_\Gamma m^2 \left(\frac{1}{\epsilon} + \log\left(\frac{\mu_R^2}{m^2}\right) + 1 \right) + \mathcal{O}(\epsilon), \tag{2.3}$$

where $c_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)}$.

The amplitudes $B_n^{(1)}$ can be written in the usual integrand basis of irreducible scalar products including extra dimensional terms following the OPP [25]/EGKM [26, 27] constructions,

$$B_n^{(1)} = \mu_R^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \left\{ \sum_{1 \le i_1 < i_2 < i_3 < i_4 < i_5 \le n} \frac{\Delta_{\{i_1,i_2,i_3,i_4,i_5\}}}{D_{i_1} D_{i_2} D_{i_3} D_{i_4} D_{i_5}} + \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} \frac{\Delta_{\{i_1,i_2,i_3,i_4\}}}{D_{i_1} D_{i_2} D_{i_3} D_{i_4}} \right.$$

$$\left. + \sum_{1 \le i_1 < i_2 < i_3 \le n} \frac{\Delta_{\{i_1,i_2,i_3\}}}{D_{i_1} D_{i_2} D_{i_3}} + \sum_{\substack{1 \le i_1 < i_2 \le n \\ i_2 - i_1 \bmod n > 1}} \frac{\Delta_{\{i_1,i_2\}}}{D_{i_1} D_{i_2}} \right\}. \tag{2.4}$$

For renormalisable gauge theories we can give a complete parametrisation of the numerators,

$$\Delta_{\{i_1,i_2,i_3,i_4,i_5\}} = c^{(0)}_{i_1,i_2,i_3,i_4,i_5} \mu^2, \tag{2.5a}$$

$$\begin{aligned} \Delta_{\{i_1,i_2,i_3,i_4\}} = {}& c^{(0)}_{i_1,i_2,i_3,i_4} + c^{(1)}_{i_1,i_2,i_3,i_4} (k \cdot w_{1;i_1,i_2,i_3}) + c^{(2)}_{i_1,i_2,i_3,i_4} \mu^2 \\ & + c^{(3)}_{i_1,i_2,i_3,i_4} \mu^2 (k \cdot w_{1;i_1,i_2,i_3}) + c^{(4)}_{i_1,i_2,i_3,i_4} \mu^4, \end{aligned} \tag{2.5b}$$

$$\begin{aligned} \Delta_{\{i_1,i_2,i_3\}} = {}& c^{(0)}_{i_1,i_2,i_3} + c^{(1)}_{i_1,i_2,i_3} (k \cdot w_{1;i_1,i_2}) + c^{(2)}_{i_1,i_2,i_3} (k \cdot w_{2;i_1,i_2}) \\ & + c^{(3)}_{i_1,i_2,i_3} (k \cdot w_{1;i_1,i_2})(k \cdot w_{2;i_1,i_2}) + c^{(4)}_{i_1,i_2,i_3} \left((k \cdot w_{1;i_1,i_2})^2 - (k \cdot w_{2;i_1,i_2})^2 \right) \\ & + c^{(5)}_{i_1,i_2,i_3} (k \cdot w_{1;i_1,i_2})^2 (k \cdot w_{2;i_1,i_2}) + c^{(6)}_{i_1,i_2,i_3} (k \cdot w_{1;i_1,i_2})(k \cdot w_{2;i_1,i_2})^2 \\ & + c^{(7)}_{i_1,i_2,i_3} \mu^2 (k \cdot w_{1;i_1,i_2}) + c^{(8)}_{i_1,i_2,i_3} \mu^2 (k \cdot w_{2;i_1,i_2}) + c^{(9)}_{i_1,i_2,i_3} \mu^2, \end{aligned} \tag{2.5c}$$

$$\begin{aligned} \Delta_{\{i_1,i_2\}} = {}& c^{(0)}_{i_1,i_2} + c^{(1)}_{i_1,i_2} (k \cdot w_{1;i_1}) + c^{(2)}_{i_1,i_2} (k \cdot w_{2;i_1}) + c^{(3)}_{i_1,i_2} (k \cdot w_{3;i_1}) \\ & + c^{(4)}_{i_1,i_2} (k \cdot w_{1;i_1})(k \cdot w_{2;i_1}) + c^{(5)}_{i_1,i_2} (k \cdot w_{1;i_1})(k \cdot w_{3;i_1}) \\ & + c^{(6)}_{i_1,i_2} (k \cdot w_{2;i_1})(k \cdot w_{3;i_1}) + c^{(7)}_{i_1,i_2} \left((k \cdot w_{1;i_1})^2 - (k \cdot w_{3;i_1})^2 \right) \\ & + c^{(8)}_{i_1,i_2} \left((k \cdot w_{2;i_1})^2 - (k \cdot w_{3;i_1})^2 \right) + c^{(9)}_{i_1,i_2} \mu^2. \end{aligned} \tag{2.5d}$$

The irreducible numerators $k \cdot w_{x;i_1,\dots,i_s}$ can be constructed using the spurious directions of van Neerven and Vermaseren [28] and vanish after integration. The spurious directions

$w_{x;i_1,\ldots,i_p}$ are orthogonal to the p dimensional physical space spanned by the momenta entering vertices $i_1,\ldots,i_s$ where $x = 1,\ldots,s$ with $s + p = 4$. $\mu^2 = -\tilde{k}\cdot\tilde{k}$ is the extra dimensional scalar product where $k = \bar{k} + \tilde{k}$. These give rise to dimension shifted integrals which in turn lead to rational terms in $d = 4 - 2\epsilon$ dimensions. The coefficients c are rational functions of the external kinematics and can be extracted from generalised unitarity cuts. After elimination of vanishing integrals over the spurious directions, the d-dimensional representation of the amplitude is,

$$
\begin{aligned}
B_n^{(1)}(d, d_s) =& \sum_{1 \le i_1 < i_2 < i_3 < i_4 < i_5 \le n} c_{i_1,i_2,i_3,i_4,i_5}^{(0)} I_{i_1,i_2,i_3,i_4,i_5}^d[\mu^2] \\
&+ \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} c_{i_1,i_2,i_3,i_4}^{(0)} I_{i_1,i_2,i_3,i_4}^d[1] + c_{i_1,i_2,i_3,i_4}^{(2)} I_{i_1,i_2,i_3,i_4}^d[\mu^2] + c_{i_1,i_2,i_3,i_4}^{(4)} I_{i_1,i_2,i_3,i_4}^d[\mu^4] \\
&+ \sum_{1 \le i_1 < i_2 < i_3 \le n} c_{i_1,i_2,i_3}^{(0)} I_{i_1,i_2,i_3}^d[1] + c_{i_1,i_2,i_3}^{(9)} I_{i_1,i_2,i_3}^d[\mu^2] \\
&+ \sum_{\substack{1 \le i_1 < i_2 \le n \\ i_2 - \bar{i}_1 \bmod n > 1}} c_{i_1,i_2}^{(0)} I_{i_1,i_2}^d[1] + c_{i_1,i_2}^{(9)} I_{i_1,i_2}^d[\mu^2],
\end{aligned}
\tag{2.6}
$$

where

$$
I_{i_1,i_2,\ldots,i_n}^d[N] = \mu_R^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{N}{D_{i_1} D_{i_2} \cdots D_{i_n}}.
\tag{2.7}
$$

Explicitly the dimension shifting relations are,[2]

$$
I_{i_1,i_2,\ldots,i_n}^d[\mu^2] = \frac{d-4}{2}(4\pi) I_{i_1,i_2,\ldots,i_n}^{d+2}[1],
\tag{2.8}
$$

$$
I_{i_1,i_2,\ldots,i_n}^d[\mu^4] = \frac{(d-4)(d-2)}{4}(4\pi)^2 I_{i_1,i_2,\ldots,i_n}^{d+4}[1].
\tag{2.9}
$$

3 Massive fermion spinors

In this section we describe how we can use massless six dimensional momenta to obtain amplitudes in four dimensions with massive particles. Before getting started it is helpful to recall how massive fermion wavefunctions can be incorporated within the four-dimensional spinor-helicity formalism commonly used for massless amplitudes. We follow the notation used previously in reference [12] while the formalism itself was established long before that, see for example [29–33].

Starting from a massive 4-momentum p with $p^2 = m^2$, we can define a massless projection with respect to a light-like reference vector η,

$$
p^\flat = p - \frac{m^2}{2p \cdot \eta}\eta,
\tag{3.1}
$$

---

[2]A simple derivation of this fact is shown in appendix A.2 of reference [7].

such that $(p^\flat)^2 = 0$. A complete set of solutions of the Dirac equation for the massive momentum p can then be constructed from the Weyl spinors of $p^\flat$ and η:

$$\bar{u}_+(p,m) = \frac{\langle\eta|(\not{p}+m)}{\langle\eta p^\flat\rangle}, \qquad\qquad \bar{u}_-(p,m) = \frac{[\eta|(\not{p}+m)}{[\eta p^\flat]},$$
$$v_+(p,m) = \frac{(\not{p}-m)|\eta\rangle}{\langle p^\flat\eta\rangle}, \qquad\qquad v_-(p,m) = \frac{(\not{p}-m)|\eta]}{[p^\flat\eta]}. \tag{3.2}$$

These Dirac spinors maintain several of the simplifications which are familiar in the massive case. The tree-level helicity amplitudes for $gg \to t\bar{t}$ scattering, for example, take the relatively simple forms

$$-i\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle A^{(0)}(1_t^+, 2^+, 3^+, 4_{\bar{t}}^+) = -\frac{m_t^3 s_{23}\langle\eta_1\eta_4\rangle}{2p_1 \cdot p_2\langle 23\rangle^2}, \tag{3.3}$$

$$-i\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle A^{(0)}(1_t^+, 2^+, 3^-, 4_{\bar{t}}^+) = \frac{m_t\langle\eta_1 3\rangle\langle\eta_4 3\rangle\langle 3|1|2]}{2p_1 \cdot p_2\langle 23\rangle} + \frac{m_t\langle\eta_1\eta_4\rangle\langle 3|1|2]^2}{2p_1 \cdot p_2 s_{23}}. \tag{3.4}$$

We will now show that these results can be rewritten in terms of amplitudes of massless fermions in six dimensions.

3.1 Massive fermions from massless six dimensional spinors

In this section we will use the six dimensional spinor-helicity formalism introduced by Cheung and O'Connell [19] to find representations of four-dimensional massive fermion wave-functions and amplitudes. Further discussion of the six dimensional spinor-helicity formalism can be found in references [20, 21, 34–38]. We begin our discussion by looking at a free massive fermion field in four dimensions,

$$\mathcal{L}^{4d} = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x). \tag{3.5}$$

In the Weyl representation the Dirac γ matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & \tilde{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \tag{3.6}$$

where $\sigma^\mu = (1, \sigma^i)$, $\tilde{\sigma}^\mu = (1, -\sigma^i)$ and the usual Pauli matrices, σ^i, $i = 1, 2, 3$, are written in full in appendix A. For the spinors associated with external fermions we seek solutions to the massive Dirac equation

$$(\gamma \cdot \bar{p} - m)u_s(\bar{p}) = 0 \text{ and } \bar{u}_s(\bar{p})(\gamma \cdot \bar{p} - m) = 0, \tag{3.7}$$

where the bar on the momentum $\bar{p}$ denotes that the vector is in four dimensions.

Alternatively we can consider a massless fermion field in six dimensions, with Lagrangian

$$\mathcal{L}^{6d} = \overline{\Psi}(x)(i\Gamma^M\partial_M)\Psi(x). \tag{3.8}$$

Note that for six dimensions we use capital Greek letters and M runs from 0 to 5. In six spacetime dimensions the Dirac matrices are 8×8 objects, which we choose to be

$$\Gamma^M = \begin{pmatrix} 0 & \tilde{\Sigma}^M \\ \Sigma^M & 0 \end{pmatrix}, \tag{3.9}$$

where the Σ matrices are defined by taking outer products of Pauli matrices and are listed explicitly in appendix A. This representation of the Γ matrices is simply related to the four dimensional γ-matrices. The relation for the first four Σ matrices is

$$-\tilde{\Sigma}^{5,AX} \Sigma^\mu_{XB} = (\gamma^\mu)^A{}_B = \tilde{\Sigma}^{\mu,AX} \Sigma^5_{XB}, \tag{3.10}$$

where we have adopted the convention that Σ^M carry lower spinor indices while $\tilde{\Sigma}^M$ carry upper indices. For the remaining two Σ matrices we have

$$-\tilde{\Sigma}^{5,AX} \Sigma^4_{XB} = (-\gamma^0 \gamma^1 \gamma^2 \gamma^3)^A{}_B = i(\gamma^5)^A{}_B, \tag{3.11}$$

$$-\tilde{\Sigma}^{5,AX} \Sigma^5_{XB} = \mathbf{1}^A{}_B. \tag{3.12}$$

In the present case, there is no six dimensional mass term. Moreover, in our Weyl basis for the Γ matrices (3.9) we can decompose $\Psi = (\Psi_1, \Psi_2)$ and see that the two fields decouple:

$$\mathcal{L}^{6d} = \overline{\Psi}_1(x)(i\Sigma^M \partial_M)\Psi_1(x) + \overline{\Psi}_2(x)(i\tilde{\Sigma}^M \partial_M)\Psi_2(x). \tag{3.13}$$

Hence the two Ψ_i are essentially copies of each other. The spinors associated with these fields each have four components, and the Dirac equation in momentum space associated with $\Psi_1(x)$ reads

$$(\Sigma \cdot p)_{AB} \lambda^B_a(p) = 0, \tag{3.14}$$

where the SU(4) indices A, B run from 1 to 4, and the lower case index, a, is a helicity index taking two values. Thus, we may regard the six dimensional spinors as 4×2 matrices. For the spinors associated with $\Psi_2(x)$ the corresponding equation is

$$(\tilde{\Sigma} \cdot p)^{AB} \tilde{\lambda}_{B\dot{a}}(p) = 0. \tag{3.15}$$

We embed the massive four dimensional momentum $\bar{p}$ into a six dimensional massless momentum by declaring that

$$p = (\bar{p}, 0, m), \text{ so } p^2 = \bar{p}^2 - m^2 = 0. \tag{3.16}$$

Having made this choice, it is consistent to relate the chiral spinor λ to the anti-chiral spinor $\tilde{\lambda}$ by defining

$$\lambda^A = i\tilde{\Sigma}^{4,AB} \tilde{\lambda}_B. \tag{3.17}$$

It is straightforward to verify that this is a solution to the Dirac equation by inserting eq. (3.17) into eq. (3.14) and using the Clifford algebra:

$$
\begin{aligned}
0 &= (\Sigma \cdot p)_{AB} \lambda_a^B(p) \\
&= i(\Sigma \cdot p)_{AB} \tilde{\Sigma}^{4,BC} \tilde{\lambda}_C \\
&= -i\Sigma_{AB}^4 (\tilde{\Sigma} \cdot p)^{BC} \tilde{\lambda}_C \\
\Rightarrow 0 &= (\tilde{\Sigma} \cdot p)^{BC} \tilde{\lambda}_C.
\end{aligned}
\tag{3.18}
$$

Having embedded our massive four-dimensional momentum into six dimensions, it is instructive to understand in detail how massless six-dimensional spinors relate to the usual massive four-dimensional Dirac spinors. We begin by writing the massless six-dimensional Dirac equation (3.14) in detail as

$$
(\Sigma \cdot p)_{AB} \lambda_a^B(p) = \left(\Sigma^\mu p_\mu - \Sigma^5 p^{(5)} \right)_{AB} \lambda_a^B(p) = 0.
\tag{3.19}
$$

Multiplying from the left by $-\tilde{\Sigma}^{5,XA}$ we obtain

$$
(\gamma \cdot \bar{p} - p_1^{(5)})^X_{\ B} \lambda^B(p) = 0.
\tag{3.20}
$$

Notice how the sign on the sixth component of momentum determines whether $\lambda(p)$ should be associated with the four-dimensional spinor for a fermion $u(p)$ or an anti-fermion $v(p)$:

$$
\lambda(p) = \begin{cases} u(\bar{p}), & p^{(5)} = m \\ v(\bar{p}), & p^{(5)} = -m \end{cases}.
\tag{3.21}
$$

A similar calculation shows how to identify massless six-dimensional spinors with the conjugate four-dimensional Dirac spinors:

$$
\begin{aligned}
0 &= \lambda^A(p)(\Sigma^\mu p_\mu - \Sigma^5 p^{(5)})_{AB} \\
&= \lambda^A(p)(-\Sigma^5 \tilde{\Sigma}^5)^X_A (\Sigma^\mu p_\mu - \Sigma^5 p^{(5)})_{XB} \\
&= \lambda^A(p)\Sigma_{AX}^5 (\gamma \cdot \bar{p} - p^{(5)})^X_{\ B}.
\end{aligned}
\tag{3.22}
$$

Again the sixth momentum component determines whether $\lambda(p)\Sigma^5$ should be identified with $\bar{u}(p)$ or $\bar{v}(p)$:

$$
\lambda(p)\Sigma^5 = \begin{cases} \bar{u}(\bar{p}), & p^{(5)} = m \\ \bar{v}(\bar{p}), & p^{(5)} = -m \end{cases}.
\tag{3.23}
$$

In the following, we find it useful to write an explicit representation for $\lambda^A(p)$ that allows us to make direct connection with the specific four-dimensional Dirac spinors given in (3.2). We use a massless (in the four dimensional sense) reference vector η, as introduced in (3.1), with Weyl spinors $\kappa_\alpha(\eta), \tilde{\kappa}^{\dot{\alpha}}(\eta)$ and define the six dimensional spinors:

$$
\lambda^{Aa}(\eta, \bar{p}^\flat) = \begin{pmatrix} 0 & \frac{\tilde{\kappa}^{\dot{\alpha}}(\eta)}{[p^\flat \eta]} \\ \frac{\kappa_\alpha(\eta)}{\langle p^\flat \eta \rangle} & 0 \end{pmatrix}, \quad \tilde{\lambda}_{A\dot{a}}(\eta, \bar{p}^\flat) = \begin{pmatrix} 0 & \frac{\tilde{\kappa}_{\dot{\alpha}}(\eta)}{[p^\flat \eta]} \\ \frac{\kappa^\alpha(\eta)}{\langle p^\flat \eta \rangle} & 0 \end{pmatrix}.
\tag{3.24}
$$

Using $(\Sigma \cdot p)_{AB}(\tilde{\Sigma} \cdot p)^{BC} = 0$ we see that the Dirac equation (3.14) is solved by setting

$$\lambda^A(p) = (\tilde{\Sigma} \cdot p)^{AB}\tilde{\lambda}_B(\eta, \bar{p}^{\flat}). \tag{3.25}$$

The anti-chiral case is completely analogous:

$$\tilde{\lambda}_A(p) = (\Sigma \cdot p)_{AB}\lambda^B(\eta, \bar{p}^{\flat}). \tag{3.26}$$

The discussion following (3.19) showed how these six dimensional spinors solve the massive Dirac equation in four dimensions with the appropriate choice of sign for $p^{(5)}$.

Since the helicity indices for external particles are not contracted in the six dimensional spinor-helicity formalism, we obtain all helicity configurations in one amplitude. In six dimensions gluons have four polarisation states denoted by two helicity indices. The polarisation vector for a gluon with momentum p is given by

$$\varepsilon^M_{a\dot{a}} = \frac{1}{\sqrt{2}}\langle p_a|\Sigma^M|q_b\rangle(\langle q_b|p^{\dot{a}}])^{-1}, \tag{3.27}$$

where q is a reference vector satisfying $p \cdot q \neq 0$. Because we use Weyl spinors the fermions still have two states. The amplitude for a quark pair and two gluons is given by [21]

$$A(1_{q,a}, 2_{b\dot{b}}, 3_{c\dot{c}}, 4_{\bar{q},d}) = \frac{-i}{2s_{12}s_{23}}\langle 1_a 2_b 3_c 4_d\rangle[1_{\dot{x}}2_{\dot{b}}3_{\dot{c}}1^{\dot{x}}]. \tag{3.28}$$

The results in four dimensions are obtained by picking out helicity configurations $(a, b\dot{b}, c\dot{c}, d) = (2, 22, 22, 2),\ (2, 11, 22, 2)$ for the all plus (3.3) and single minus (3.4) configurations respectively. Other relevant six dimensional trees are given in appendix B.

3.2 Interactions and state-sum reduction

We introduce interactions as always by replacing the derivative with the covariant derivative. In six dimensions

$$\partial_M \to D_M = \partial_M - igA^i_M(x)t^i, \tag{3.29}$$

where $A^i_M(x)$ are the gauge fields and t^i are the generators of the gauge group. We dimensionally reduce the six-dimensional gauge field to four dimensions by treating its last two entries as scalar fields,

$$A_M(x) = A^i_M(x)t^i = (A_\mu(x), \phi_1(x), \phi_2(x)), \tag{3.30}$$

leading to the following interaction terms for Ψ_1 (dropping dependence on position for simplicity):

$$\begin{aligned}
\mathcal{L}^{6d}_{\text{int},\Psi_1} &= -ig\overline{\Psi}_1\Sigma^M A_M\Psi_1 \\
&= -ig\overline{\Psi}_1\left(\Sigma^\mu A_\mu - \Sigma^4\phi_1 - \Sigma^5\phi_2\right)\Psi_1 \\
&= -ig\overline{\Psi}_1\Sigma^\mu A_\mu\Psi_1 + g\overline{\Psi}_1\phi_1\Psi_2 - ig\overline{\Psi}_1\phi_2\gamma_5\Psi_2,
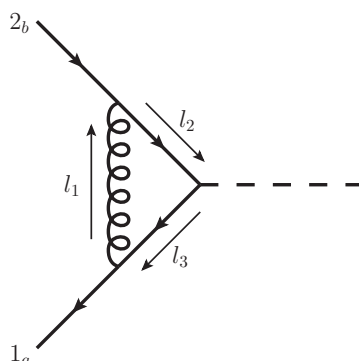\end{aligned} \tag{3.31}$$

Figure 2. Feynman diagram for one-loop contribution to the coupling between a massive fermion pair and an off-shell scalar. All external momenta are outgoing.

where, in the last line, we have used the relation between chiral and anti-chiral spinors (3.17), which for the fields reads $\Psi_1 = i\tilde{\Sigma}^4 \Psi_2$. The last two terms give rise to the three-point amplitudes given in (B.8) and (B.9). While the first term resembles the four dimensional interaction term the two last terms are additional contributions arising from the extra momentum components. For internal lines these contributions correspond to additional gluon polarisation states that should be subtracted to obtain the four-dimensional result. This procedure is known as state-sum reduction.

The contraction of Lorentz indices over internal propagators leads to explicit dependence on the spin dimension d_s. Working explicitly in six dimensions this dependence will be lost but can be recovered through state-sum reduction. The general procedure is described in [21, 27]. Gluons in six dimensions have $6 - 2 = 4$ polarisation states, so for each extra dimension introduced we get one more state. Each of these states correspond to the contribution from replacing gluons in the loop by a scalar. By subtracting these scalars the number of polarisation states can be reduced to $d_s - 2$. In our set-up, the scalar associated with the mass direction should be subtracted separately and we arrive at the state-sum reduction prescription

$$\boldsymbol{c} = \boldsymbol{c}^{6d} - (5 - d_s)\boldsymbol{c}_{\phi_1} - \boldsymbol{c}_{\phi_2}. \tag{3.32}$$

3.3 A one-loop example calculation

Let us now illuminate this higher dimensional formalism with a worked example: the one-loop amplitude for a massive fermion pair coupling to an off-shell scalar, $A^{(1)}$. This calculation involves only one Feynman diagram (figure 2), which, using the colour-ordered four dimensional Feynman rules, is given by

$$A^{(1),4d} = \int \frac{d^d \ell_1}{(2\pi)^d} \, \bar{u}_1 \gamma^\mu \frac{(\gamma \cdot \ell_3 + m)}{\ell_3^2 - m^2} \frac{(\gamma \cdot \ell_2 + m)}{\ell_2^2 - m^2} \gamma^\nu v_2 \frac{\eta_{\mu\nu}}{\ell_1^2} \equiv \int \frac{d^d \ell_1}{(2\pi)^d} \frac{N^{4d}}{D_1 D_2 D_3}, \tag{3.33}$$

where $\ell_2 = \ell_1 - p_2$, $\ell_3 = \ell_1 + p_1$, $D_i = \ell_i^2 - m_i^2$, and N^{4d} is the numerator. We will write the result in terms of the scalar integrals using the notation of [39]

$$\boldsymbol{I} = \left\{ I_3(m^2, s, m^2; 0, m^2, m^2), F_2(s, m^2), I_2(m^2; 0, m^2) \right\}, \tag{3.34}$$

where $F_2(s, m^2) = I_2(s; m^2, m^2) - I_2(m^2; 0, m^2)$. The result is $A^{(1),4d} = \boldsymbol{c}^{(d_s)} \cdot \boldsymbol{I} A^{(0),4d}$ where the integral coefficients are given by

$$\boldsymbol{c}^{(d_s)} = \left\{ -2(s - 2m^2), (d_s - 4) - \frac{8m^2}{s\beta^2}, d_s \right\}. \tag{3.35}$$

We have set $\beta^2 = 1 - \frac{4m^2}{s}$ and d_s, as usual, is the polarisation state dimension. Using the relation between γ^μ and the Σ- and $\tilde{\Sigma}$-matrices (3.10) we may simplify the numerator by insertion of $\boldsymbol{1}^A_{\ B} = -\tilde{\Sigma}^{5,AX} \Sigma^5_{XB}$ in (3.33):

$$\begin{aligned} N^{4d} &= \bar{u}_1 \gamma^\mu (\gamma \cdot \bar{\ell}_3 + m)(\gamma \cdot \bar{\ell}_2 + m) \gamma_\mu v_2 \\ &= \bar{u}_1 \boldsymbol{1} \gamma^\mu (\gamma \cdot \bar{\ell}_3 + m) \boldsymbol{1} (\gamma \cdot \bar{\ell}_2 + m) \boldsymbol{1} \gamma_\mu v_2 \\ &= -\bar{u}_1 \tilde{\Sigma}^5 \Sigma^\mu (\tilde{\Sigma}^\nu \bar{\ell}_{3\nu} - \tilde{\Sigma}^5 m) \Sigma^5 (\tilde{\Sigma}^\rho \bar{\ell}_{2\rho} - \tilde{\Sigma}^5 m) \Sigma_\mu v_2 \\ &= \lambda_1 \Sigma^\mu (\tilde{\Sigma} \cdot \ell_3) \Sigma^5 (\tilde{\Sigma} \cdot \ell_2) \Sigma_\mu \lambda_2. \end{aligned} \tag{3.36}$$

Note the leftover Σ^5 which is associated with the scalar interaction. Hence the tree level amplitude in six dimensions is given by

$$A^{(0),6d} = \lambda_1 \Sigma^5 \lambda_2. \tag{3.37}$$

As we discussed in section 3.2, the contraction of the six-dimensional Lorentz indices of internal gluon lines includes contributions from the extra dimensions. The procedure of reducing the sum over internal states allows us to obtain the explicit dependence on spacetime dimensionality. In the case at hand, the numerator in the six dimensional calculation is:

$$N^{6d} = \lambda_1 \Sigma^M (\tilde{\Sigma} \cdot \ell_3) \Sigma^5 (\tilde{\Sigma} \cdot \ell_2) \Sigma_M \lambda_2. \tag{3.38}$$

Comparing with N^{4d} in equation (3.36), the extra contributions in six dimensions are evidently

$$\begin{aligned} N^{6d}_{\phi_1} &= -\lambda_1 \Sigma^4 (\tilde{\Sigma} \cdot \ell_3) \Sigma^5 (\tilde{\Sigma} \cdot \ell_2) \Sigma^4 \lambda_2, \\ N^{6d}_{\phi_2} &= -\lambda_1 \Sigma^5 (\tilde{\Sigma} \cdot \ell_3) \Sigma^5 (\tilde{\Sigma} \cdot \ell_2) \Sigma^5 \lambda_2. \end{aligned} \tag{3.39}$$

It follows from (3.31) that contributions from the scalars can equivalently by obtained with

$$\begin{aligned} N^{6d}_{\phi_1} &= -\lambda_1 (\Sigma \cdot \ell_3) \tilde{\Sigma}^5 (\Sigma \cdot \ell_2) \lambda_2, \\ N^{6d}_{\phi_2} &= \lambda_1 \gamma_5 (\Sigma \cdot \ell_3) \tilde{\Sigma}^5 (\Sigma \cdot \ell_2) \tilde{\gamma}_5 \lambda_2, \end{aligned} \tag{3.40}$$

where $(\tilde{\gamma}_5)^A_{\ B} = -i\tilde{\Sigma}^{4,AX} \Sigma^5_{XB}$. Using the integral basis in (3.34) the result is

$$\begin{aligned} A^{(1),6d} &= \boldsymbol{c}^{6d} \cdot \boldsymbol{I} A^{(0),6d} \\ &= \left\{ -2s, -\frac{16m^2}{s\beta^2}, 4 \right\} \cdot \boldsymbol{I} A^{(0),6d}, \end{aligned} \tag{3.41}$$

$$\begin{aligned} A^{(1)}_{\phi_1} &= \boldsymbol{c}_{\phi_1} \cdot \boldsymbol{I} A^{(0),6d} \\ &= \{0, 1, 1\} \cdot \boldsymbol{I} A^{(0),6d}, \end{aligned} \tag{3.42}$$

$$\begin{aligned} A^{(1)}_{\phi_2} &= \boldsymbol{c}_{\phi_2} \cdot \boldsymbol{I} A^{(0),6d} \\ &= \left\{ -4m^2, -1 - \frac{8m^2}{s\beta^2}, -1 \right\} \cdot \boldsymbol{I} A^{(0),6d}. \end{aligned} \tag{3.43}$$

$$A^{[L]}(1_t, 2, 3, 4_{\bar{t}}) = \quad\quad\quad\quad + \ldots, \quad\quad A^{[R]}(1_t, 2, 3, 4_{\bar{t}}) = \quad\quad\quad\quad + \ldots$$

Figure 3. Configurations for left- and right-moving primitive amplitudes contributing to $gg \to t\bar{t}$ scattering.

The coefficients above are the ingredients needed to perform the state-sum reduction and reproduce (3.35).

4 Generalised unitarity cuts in six dimensions

To illustrate our method we consider two gauge invariant primitive amplitudes relevant for the $gg \to t\bar{t}$ one-loop scattering amplitude. Helicity amplitudes for this process have been previously presented in reference [12]. Using the usual colour decomposition [40] we define the ordered partial amplitudes $A_{4;1}^{(1)}$ and $A_{4;3}^{(1)}$ by,

$$\mathcal{A}^{(1)}(1_t, 2, 3, 4_{\bar{t}}) = \sum_{P(2,3)} (T^{a_2} T^{a_3})_{i_1}^{\bar{i}_4} A_{4;1}^{(1)}(1_t, 2, 3, 4_{\bar{t}}) + \text{tr}(T^{a_2} T^{a_3})\, \delta_{i_1}^{\bar{i}_4} A_{4;3}^{(1)}(1_t, 4_{\bar{t}}; 2, 3), \quad (4.1)$$

where $P(2,3)$ is the permutations over the order of gluons. These partial amplitudes can be further decomposed into gauge invariant primitive amplitudes,

$$A_{4;1}^{(1)}(1_t, 2, 3, 4_{\bar{t}}) = N_c A^{[L]}(1_t, 2, 3, 4_{\bar{t}}) - \frac{1}{N_c} A^{[R]}(1_t, 2, 3, 4_{\bar{t}})$$
$$- N_f A^{[f]}(1_t, 2, 3, 4_{\bar{t}}) - N_H A^{[H]}(1_t, 2, 3, 4_{\bar{t}}), \quad\quad\quad (4.2)$$

$$A_{4;3}^{(1)}(1_t 4_{\bar{t}}; 2, 3) = \sum_{P(2,3)} \left(A^{[L]}(1_t, 2, 3, 4_{\bar{t}}) + A^{[L]}(1_t, 2, 4_{\bar{t}}, 3) + A^{[R]}(1_t, 2, 3, 4_{\bar{t}}) \right), \quad (4.3)$$

where N_c is the number of colours, while N_f and N_H are the number of light and heavy fermion flavours, respectively. The left-moving $A^{[L]}$ and right-moving $A^{[R]}$ primitive amplitudes are labelled according to the direction of the fermion current as it enters the loop, following the convention of reference [40]. Representative diagrams for these amplitudes are shown in figure 3. We will not consider the fermion loop contributions $A^{[f]}$ and $A^{[H]}$ in this article as they do not present any further technical difficulties.

Each primitive amplitude can be decomposed at the integrand level into the basis of integrals described in section 2. To capture the full d-dimensional dependence, we first compute generalised cuts in six dimensions using the spinor-helicity formalism described in the previous section. We then compute the two additional scalar loop contributions and perform the state sum reduction onto a general dimension d according to eq. (3.32). The complete set of generalised cuts needed for the amputated primitives $B^{[L]}$ and $B^{[R]}$, cf. $B_n^{(1)}$ in equation (2.1), are shown in figure 4 and 5, in which the divergent two-particle and one-particle cuts are removed.
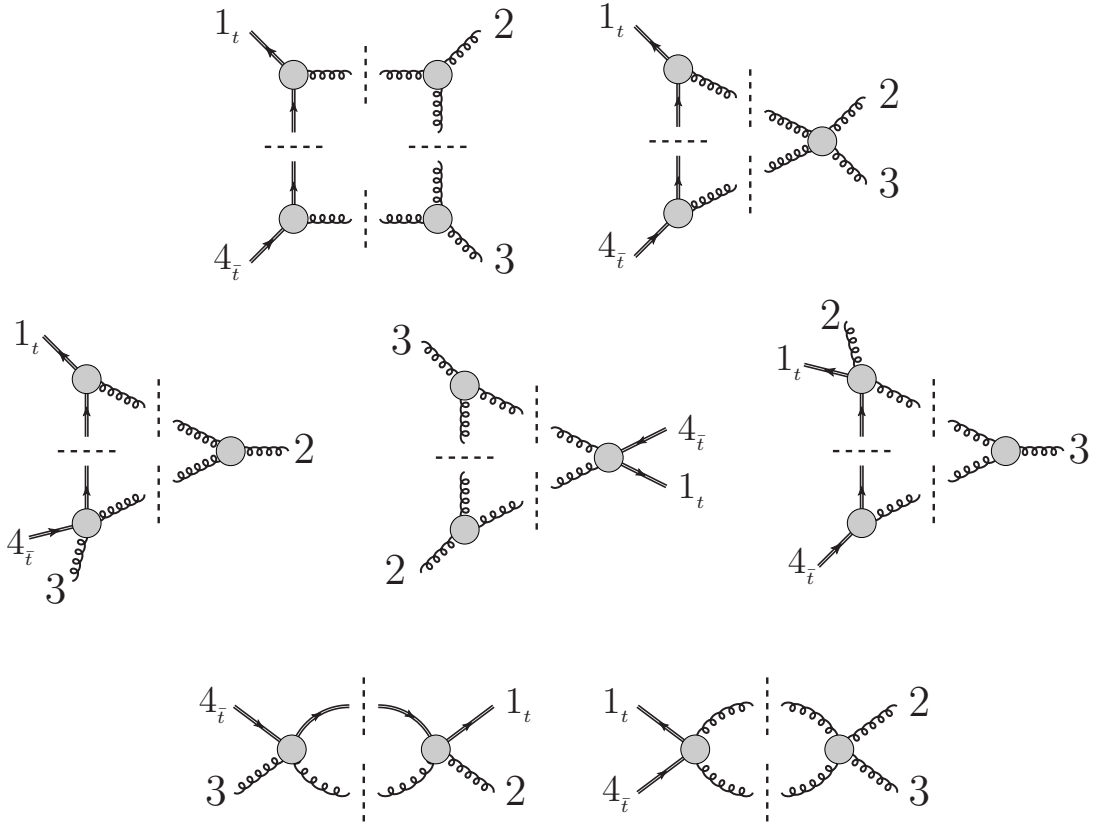
Figure 4. The complete set of cuts for $B^{[L]}(1_t, 2, 3, 4_{\bar{t}})$. Double lines represent massive fermions.

Each six-dimensional cut is associated with a set of loop momenta ℓ_i which enter the tree-level amplitudes. These momenta are determined by solving the system of on-shell equations $\{\ell_i^2 = 0, i \in S\}$. The complete set of loop momenta for our ordered amplitudes are labelled as,

$$\ell_i^\mu \equiv \ell_0^\mu - P_i^\mu, \qquad P_i^\mu = \sum_{n=1}^{i} p_n^\mu,$$
$$\ell_0^\mu \equiv k^\mu, \tag{4.4}$$

where p_n^μ are the external momenta and k is the loop integration momentum.

The internal particles are embedded into six dimensions by allowing the mass to flow in the sixth component, following our convention in eq. (3.16), and the $(d-4)$ part of the loop momentum to flow in the fifth component,

$$\text{gluon loop momentum:} \quad \ell = \{\bar{\ell}, \mu, 0\},$$
$$\text{fermion loop momentum:} \quad \ell = \{\bar{\ell}, \mu, m\}. \tag{4.5}$$

The gluon and fermion loop propagators can then be expanded into a four-dimensional part and an effective mass term μ^2,

$$\text{gluon propagator:} \quad \ell^2 = \bar{\ell}^2 - \mu^2, \tag{4.6}$$
$$\text{fermion propagator:} \quad \ell^2 = \bar{\ell}^2 - \mu^2 - m^2. \tag{4.7}$$
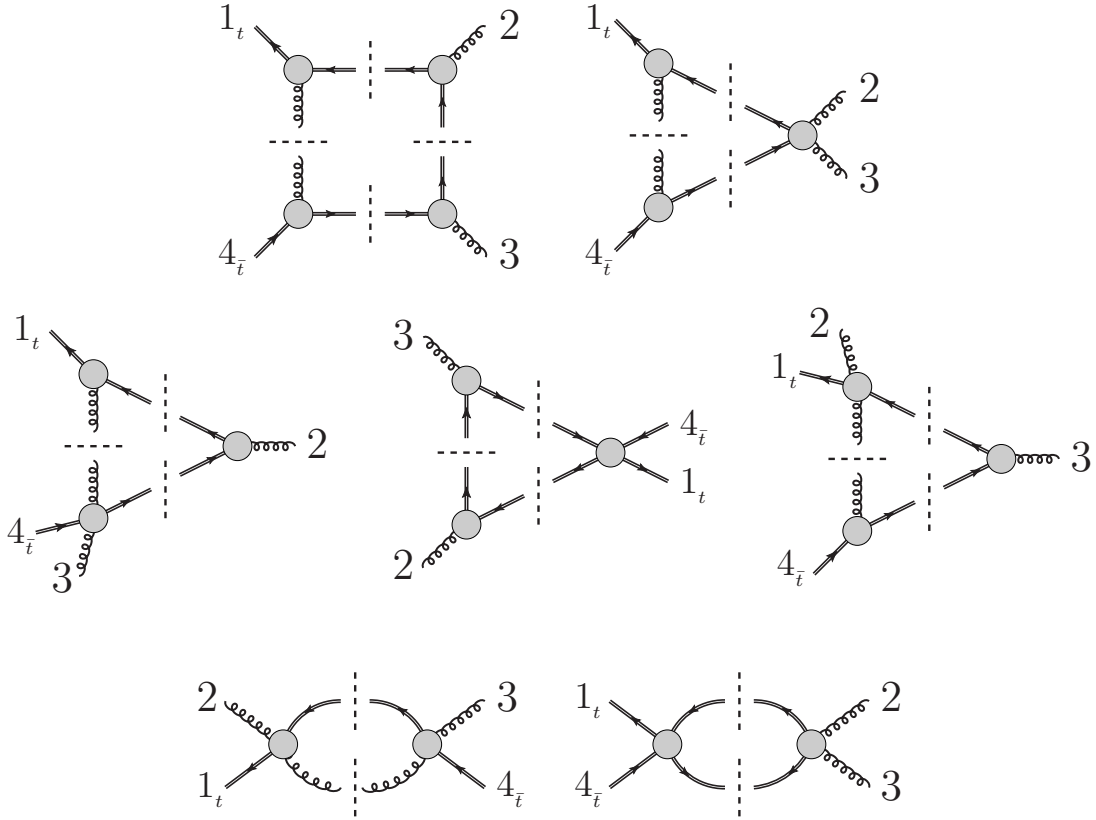
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Figure 5. The complete set of cuts for $B^{[R]}(1_t, 2, 3, 4_{\bar{t}})$. Double lines represent massive fermions.

This choice is particularly convenient when requiring momentum conservation and orthogonality of the -2ϵ component with respect to the external massive fermions momenta expressed in the six dimensional representation, as shown in figure 6.

As an explicit example we will describe the computation of the quadruple cuts. The on-shell equations for these cuts in the left- and right-moving configurations are,

$$S_{4;0123}^L = \begin{cases} \ell_0^2 = \ell_1^2 = \ell_2^2 = \ell_3^2 = 0 \\ \ell_0^{(5)} = m \end{cases}, \qquad S_{4;0123}^R = \begin{cases} \ell_0^2 = \ell_1^2 = \ell_2^2 = \ell_3^2 = 0 \\ \ell_0^{(5)} = 0 \end{cases}. \tag{4.8}$$

The constraint on the sixth component of the loop momentum ℓ_0 distinguishes between the two different configurations.

We construct explicit solutions for the six-dimensional spinors of ℓ_i by introducing arbitrary two-component reference spinors x_a and $\tilde{x}_{\dot{a}}$. These solutions, which have a similar form to those presented in refs. [41, 42], take a simple form,

$$\ell_0^M = \frac{\langle x.4|\Sigma^M\,1\,2\,3|4.\tilde{x}]}{\langle x.4|2\,3|4.\tilde{x}]}, \qquad \ell_1^M = \frac{\langle x.4|1\,\tilde{\Sigma}^M\,2\,3|4.\tilde{x}]}{\langle x.4|2\,3|4.\tilde{x}]},$$

$$\ell_2^M = \frac{\langle x.4|1\,2\,\Sigma^M\,3|4.\tilde{x}]}{\langle x.4|2\,3|4.\tilde{x}]}, \qquad \ell_3^M = \frac{\langle x.4|1\,2\,3\,\tilde{\Sigma}^M|4.\tilde{x}]}{\langle x.4|2\,3|4.\tilde{x}]}, \tag{4.9}$$
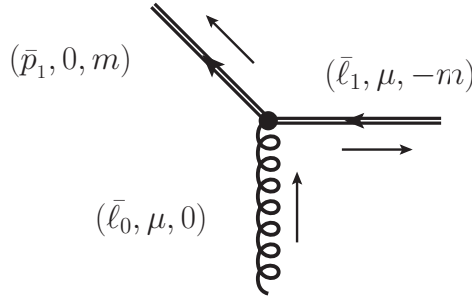
Figure 6. To perform the unitarity cuts of the six dimensional propagators involving internal fermions, we allow the $(d-4)$ part, μ, of the loop momentum to flow in the fifth component and the mass term to flow in the sixth component, in order to easily impose momentum conservation.

where $\langle x.4| = x^a\langle 4_a|$, $|4.\tilde{x}] = |4^{\dot{a}}]\tilde{x}_{\dot{a}}$ and the spinor product strings have the following expression (for n even)

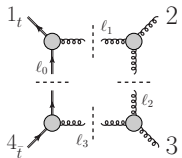$$\langle 1_a|2\,3\ldots(n-1)|n^{\dot{b}}] = \lambda_a^A(p_1)(\Sigma \cdot p_2)_{AB}(\tilde{\Sigma}\cdot p_3)^{BC}\ldots(\tilde{\Sigma}\cdot p_{n-1})^{CA}\tilde{\lambda}_A^{\dot{b}}(p_n). \quad (4.10)$$

The expressions for the two reference spinors can generically be chosen to be

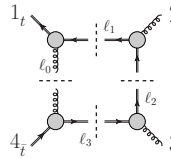$$x_a = (1,\tau_1), \qquad\qquad \tilde{x}_{\dot{a}} = (1,y), \qquad (4.11)$$

where y is fixed, for left and right, by the mass constraint for $\ell_0^{(5)}$ specified in (4.8). Because we have a system of 5 equations for 6 dimensional momenta, the parameter τ_1 is left unconstrained.

On the quadruple cut the amplitudes factorise into products of four tree-level amplitudes,

$$C_{4;0123}^L = A(-\ell_{0a},1_t^\alpha,\ell_1^{b\dot{b}})A(-\ell_{1b\dot{b}},2^{\beta\dot{\beta}},\ell_2^{c\dot{c}})A(-\ell_{2c\dot{c}},3^{\gamma\dot{\gamma}},\ell_3^{d\dot{d}})A(-\ell_{3d\dot{d}},4_{\bar{t}}^\delta,\ell_0^a),$$
$$(4.12)$$

and

$$C_{4;0123}^R = A(-\ell_{0a\dot{a}},1_t^\alpha,\ell_1^b)A(-\ell_{1b},2^{\beta\dot{\beta}},\ell_2^c)A(-\ell_{2c},3^{\gamma\dot{\gamma}},\ell_3^d)A(-\ell_{3d},4_{\bar{t}}^\delta,\ell_0^{a\dot{a}}),$$
$$(4.13)$$

where in both cases the repeated SU(2) spinor indices are summed over the six dimensional polarisation states.

The integrand reduction method then proceeds to extract the five independent coefficients in the integrand parametrisation from eq. (2.6) by evaluating both the product of trees and the irreducible scalar products μ^2 and $k \cdot w_{1;123}$ using the on-shell solution in

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eq. (4.9) and comparing the resulting rational functions in τ_1. We encounter an interesting subtlety when following this procedure since the six-dimensional cut contains additional terms which are linear in the extra-dimensional component of the loop momentum μ. These terms are spurious and integrate to zero, but require additional coefficients to be added at the integrand level if this direct approach is taken. A slightly simpler approach is to cancel the linear part of the cut by averaging over the two different flows of the momentum in the fifth component,

$$\frac{1}{2}\left(C_{4;0123}\big|_{S_{4;0123}^+} + C_{4;0123}\big|_{S_{4;0123}^-}\right) = \Delta_{\{0,1,2,3\}}\big|_{S_{4;0123}}, \tag{4.14}$$

where

$$S^+ = \left\{\ell_i^2 = 0, \ell_i = \{\ldots, \mu, \ldots\}\right\}, \qquad S^- = \left\{\ell_i^2 = 0, \ell_i = \{\ldots, -\mu, \ldots\}\right\}. \tag{4.15}$$

The triangle and bubble coefficients follow using the OPP method to systematically remove all singularities from the cut amplitude using the previously computed irreducible numerators. The mass dependence of the propagators is now dictated by six dimensional momentum conservation applied to the loop momenta, so all propagators are simply ℓ_i^2. To remove the terms linear in μ, we average over the two directions for the extra-dimensional component, as described above. Thus,

$$\frac{1}{2}\sum_{\sigma=\pm} C_{4;0123}\big|_{S_{4;0123}^\sigma} = \Delta_{\{0,1,2,3\}}\big|_{S_{4;0123}}, \tag{4.16a}$$

$$\frac{1}{2}\sum_{\sigma=\pm} C_{3;012}\bigg|_{S_{3;012}^\sigma} - \frac{\Delta_{\{0,1,2,3\}}}{\ell_3^2}\bigg|_{S_{3;012}} = \Delta_{\{0,1,2\}}\big|_{S_{3;012}}, \tag{4.16b}$$

$$\frac{1}{2}\sum_{\sigma=\pm} C_{2;02}\bigg|_{S_{3;02}^\sigma} - \left(\frac{\Delta_{\{0,1,2\}}}{\ell_1^2} + \frac{\Delta_{\{0,2,3\}}}{\ell_3^2} + \frac{\Delta_{\{0,1,2,3\}}}{\ell_1^2\ell_3^2}\right)\bigg|_{S_{2;02}} = \Delta_{\{0,2\}}\big|_{S_{2;02}}, \tag{4.16c}$$

where the parametrisations for each irreducible numerator are those of equation (2.5). The remaining triple and double cuts follow by permuting the equations (4.16). Further details on the on-shell cut solutions are given in appendix C and a full set of numerical results for the six dimensional cuts are listed in the MATHEMATICA notebook accompanying this article.

The final step to dimensionally reduce the coefficients from 6 to a general dimension d is to remove the extra degrees of freedom contained in the six dimensional loop momentum according to eq. (3.32). The computation of these extra cuts is done using the same procedure as above, but the internal gluon lines in figures 4 and 5 are replaced with scalar lines. For example, the quadruple cuts are given by the following expressions

$$C_{4;0123}^{L,\phi_{(1,2)}} = A(-\ell_{0\dot{a}}, 1_t^\alpha, \ell_1)A(-\ell_1, 2^{\beta\dot{\beta}}, \ell_2)A(-\ell_2, 3^{\gamma\dot{\gamma}}, \ell_3)A(-\ell_3, 4_{\bar{t}}^\delta, \ell_0^{\dot{a}}), \tag{4.17}$$

$$C_{4;0123}^{R,\phi_{(1,2)}} = A(-\ell_0, 1_t^\alpha, \ell_1^{\dot{b}})A(-\ell_{1\dot{b}}, 2^{\beta\dot{\beta}}, \ell_2^{\dot{c}})A(-\ell_{2\dot{c}}, 3^{\gamma\dot{\gamma}}, \ell_3^{\dot{d}})A(-\ell_{3\dot{d}}, 4_{\bar{t}}^\delta, \ell_0). \tag{4.18}$$

A complete set of fermion and scalar integrand coefficients are presented in the attached notebook.

5 Determining the remaining integral coefficients

At this point, let us pause to take stock of what has been achieved, and what remains to be done. To do so, we return to equation (2.1), the standard expression for a one-loop amplitude, expanded in a basis of scalar integrals:

$$A_n^{(1)} = B_n^{(1)} + c_{2;m^2} I_{2,m^2} + c_1 I_1. \tag{5.1}$$

By definition, $B_n^{(1)}$ is the part of the amplitude which can be computed using finite d-dimensional unitarity cuts; its expansion in terms of an integral basis was explicitly given in equation (2.4). We have therefore computed $B_n^{(1)}$ explicitly in section 4. A complete construction of the amplitude requires us to find the integral coefficients $c_{2;m^2}$ and c_1. This is the task of the present section.

5.1 Fixing c_{2,m^2} by matching the poles in $4 - 2\epsilon$ dimensions

Our first source of additional information is the universal pole structure of four dimensional amplitudes. The poles of general one-loop QCD amplitudes in four dimensions were inferred from the corresponding real-radiation contributions to the NLO cross-section in full generality by Catani, Dittmaier and Trocsanyi [43],

$$A^{(1),4-2\epsilon} = c_\Gamma I^{(1)}(\epsilon) A^{(0)} + \text{finite}. \tag{5.2}$$

The integrals I_{2,m^2} and I_1 appearing in equation (5.1) are divergent, and therefore the coefficients $c_{2;m^2}$ and c_1 contribute to the pole structure of our amplitude. This will allow us to constrain them.

For the simplified case of $t\bar{t} + n(g)$ with n_f light quark flavours and one heavy flavour of mass m, the function $I^{(1)}(\epsilon)$ appearing in the univeral pole formula is, explicitly,

$$I^{(1)}(\epsilon) = \frac{n_g \beta_0 (n_f + 1)}{2\epsilon} + \sum_{i,j} \left(\frac{\mu_R^2}{s_{ij}}\right)^\epsilon \mathcal{V}_{ij} - n_g \Gamma_g - 2\Gamma_t + \text{finite}. \tag{5.3}$$

Following Catani et al. [43], this formula corresponds to partially renormalised amplitudes. The first term contains UV poles related to charge renormalisation, the second term corresponds to soft-collinear poles and takes the familiar dipole form in colour space. The last terms contain poles given by the anomalous dimensions,

$$\Gamma_g = \frac{\beta_0(n_f)}{2\epsilon} + \frac{2T_R}{3} \log\left(\frac{\mu_R^2}{m_t^2}\right), \tag{5.4}$$

$$\Gamma_t = C_F \left(\frac{1}{\epsilon} - \frac{1}{2} \log\left(\frac{\mu_R^2}{m_t^2}\right) - 2\right). \tag{5.5}$$

The QCD β function appears as a function of the active fermion flavours $\beta_0(n_f) = (11C_A - 4T_R n_f)/3$. For the purposes of this paper we will not require the finite parts of $I^{(1)}$ which depend on the dimensional regularisation scheme, e.g. CDR or FDH/DR. The exact form of the function $\mathcal{V}$ is a little more complicated and not of direct relevance here. Clearly there

is an enormous amount of information contained in this result and further details can be found by consulting the original reference [43].

The simple observation relevant for our approach is that this universal information can be compared to the integral basis in equation (5.1), enabling a partial determination of the unknown coefficients of wavefunction bubble and tadpole integrals. These integrals give rise to single poles in ϵ and single logarithms in the mass m. This comparison is however insufficient to constrain both c_{2,m^2} and c_1.

It is convenient to modify the integral basis slightly, introducing finite bubble and tadpole functions defined by

$$F_{2;i_1,i_2} = I_{2,i_1,i_2} - I_{2,m^2}, \tag{5.6}$$

$$F_1 = I_1 - m^2 I_{2,m^2}. \tag{5.7}$$

The result of this modification is that only the finite bubble integrals and the wavefunction integral contribute to the $\log(\mu_R^2/m_t^2)$ dependence of the universal pole structure (5.3). Upon matching the amplitude with the universal pole structure, we find that the amplitude takes the explicit expression

$$A^{(1)} = A^{6D,(1)}\bigg|_{I_2 \to F_2} + \frac{d_s - 2}{4} A^{(0)} I_{2,m^2} + c_1 F_1, \tag{5.8}$$

where the only missing information now lies in the tadpole coefficient c_1.

5.2 Counterterms for QCD in six dimensions

Because of our exploitation of the universal four-dimensional pole structure, the one-loop amplitude, in the form given in equation (5.8), has the property that its infrared and ultraviolet poles have been correctly determined. In addition, all logs in the mass m_t are correctly reproduced. Indeed, the unknown coefficient c_1 now multiplies an integral F_1 which we may explicitly compute:

$$F_1 \overset{d=4-2\epsilon}{=} -ic_\Gamma m^2 + \mathcal{O}(\epsilon) = -\frac{im^2}{(4\pi)^2} + \mathcal{O}(\epsilon). \tag{5.9}$$

Since c_1 is also a rational function, the part of the amplitude which remains to be determined is simply a rational function of the external momenta and masses.

Having made heavy use of higher dimensional methods so far in our computation, it is natural to regard the four-dimensional result we wish to determine as a specialisation of an amplitude that exists in higher dimensions. Indeed, a quantum field theory which is an analogue of QCD exists in six dimensions. Moreover, in six dimensions the integral F_1 is no longer simply a finite rational function. It has an epsilon-pole given by

$$F_1 \overset{d=6-2\epsilon}{=} -\frac{im^4}{(4\pi)^3}\frac{1}{6\epsilon} + \mathcal{O}(\epsilon). \tag{5.10}$$

We may therefore find c_1 by comparison with the universal epsilon-pole structure of the amplitude in six dimensions.

Thus, we are motivated to consider QCD in six dimensions. Above four dimensions QCD ceases to be renormalisable, so to determine the universal epsilon-pole structure in six dimensions we must include higher (mass-)dimension operators[3] and treat the theory as an effective theory. By power counting, these operators have one or two powers of momentum more than in the usual QCD Lagrangian, so that they have mass-dimension five or six. The point of view we adopt is that the role of the additional operators is simply to provide counterterms, subtracting the infinities from any one-loop amplitude in the theory. Once all the counterterms have been determined, the epsilon-pole structure of any one-loop amplitude is known.

We therefore begin by constructing a basis of the dimension five and six operators which are required for renormalising QCD amplitudes in six dimensions. These operators contain either two quark fields and three derivatives, such as $\mathcal{O}_1 \equiv i\bar{\psi}\slashed{D}\slashed{D}\slashed{D}\psi$, or are purely bosonic operators such as $\mathrm{tr}\, F^{\mu\nu}F_{\nu\rho}F^\rho{}_\mu$.[4] A full list of potential operators appears in table 1.

Since we are only concerned with poles of on-shell amplitudes, rather than of off-shell correlation functions, we need only study operators which lead to independent contributions to the S matrix. It is a well known fact that operators which are related by the classical equations of motion of the theory lead to the same contribution to the S matrix, to all orders of perturbation theory [44–48]. Thus we may simplify the list of operators in table 1 using the equations of motion

$$i\slashed{D}\psi = m\psi, \tag{5.11}$$

$$D^\mu F^a_{\mu\nu} = -g\bar{\psi}\gamma^\nu T^a\psi. \tag{5.12}$$

It is straightforward to see that many operators in table 1 are related to other operators in our Lagrangian. For example,

$$\mathcal{O}_1 \equiv i\bar{\psi}\slashed{D}\slashed{D}\slashed{D}\psi = -im^2\bar{\psi}\slashed{D}\psi, \tag{5.13}$$

so that $\mathcal{O}_1$ does not lead to a new, independent counterterm. It may therefore be omitted.

Our task now is to construct a basis of operators which are independent under the use of the equations of motion, integration by parts etc. To construct such a basis, we consider several categories of operators. Firstly, we will focus on operators containing two quark fields. We classify these operators further according to the powers of derivatives, or of derivatives and field strength insertions as shown in detail in table 1. We will begin by examining operators containing the largest number of derivatives or field strengths, as the use of the equations of motion may reduce these operators to simpler operators containing fewer derivatives (or field strengths.)

Each of the derivatives contained in operators of type $[\bar{\psi}D^3\psi]$ has one Lorentz index which we must contract using either metric tensors or gamma matrices. By making use of the equations of motion, we may ignore the options of contracting the left-most or right-most D index against a gamma matrix — such a contraction would reduce to an operator

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[3]It is linguistically unfortunate that we are now dealing with operators of mass-dimension five and six (using the usual four-dimensional classification of operator dimension) in a theory defined in six spacetime dimensions. We hope that context will make the meaning of the word "dimension" clear.

[4]Recall that a field strength F counts as two derivatives since $[D_\mu, D_\nu] = -igF_{\mu\nu}$.

Quark fields	Operator	Operator class name
Two quarks	$i\bar{\psi}\slashed{D}\slashed{D}\slashed{D}\psi$	$[\bar{\psi}D^3\psi]$
	$i\bar{\psi}\slashed{D}D^2\psi$	
	$i\bar{\psi}D^\mu\slashed{D}D_\mu\psi$	
	$\bar{\psi}\gamma^\mu\gamma^\nu F_{\mu\nu}\slashed{D}\psi$	$[\bar{\psi}DF\psi]$
	$\bar{\psi}D^\mu F_{\mu\nu}\gamma^\nu\psi$	
	$\bar{\psi}F_{\mu\nu}\gamma^\mu D^\nu\psi$	
	$\bar{\psi}\slashed{D}\slashed{D}\psi$	$[\bar{\psi}D^2\psi]$
	$\bar{\psi}D^2\psi$	
	$i\bar{\psi}\gamma^\mu\gamma^\nu F_{\mu\nu}\psi$	$[\bar{\psi}F\psi]$
Zero quarks	$i\,\mathrm{tr}\,F^{\mu\nu}F_{\nu\rho}F^\rho{}_\mu$	
	$\mathrm{tr}\,F^{\mu\nu}D^2 F_{\mu\nu}$	
	$\mathrm{tr}(D^\mu F_{\mu\nu})(D^\rho F_\rho{}^\nu)$	

Table 1. Table of potential higher dimension operators in the 6 dimensional QCD effective Lagrangian. We have ignored four quark operators, which are not relevant for $t\bar{t}$ + gluons scattering at this order, and operators related to those in our table by integration-by-parts or Hermitian conjugation. We have also imposed the parity symmetry of QCD.

with fewer derivatives which we will analyze below. We are left with the unique possibility $\bar{\psi}D^\mu\slashed{D}D_\mu\psi$. However, this operator is equivalent to a linear combination of operators of class $[\bar{\psi}DF\psi]$ and $[\bar{\psi}D^2\psi]$ upon use of the equations of motion, since

$$\bar{\psi}D^\mu\slashed{D}D_\mu\psi = \bar{\psi}\left(-imD^\mu D_\mu - igD^\mu\gamma^\nu F_{\mu\nu}\right)\psi. \tag{5.14}$$

Therefore, the class $[\bar{\psi}D^3\psi]$ can be completely reduced to simpler operators.

Next, consider the class $[\bar{\psi}DF\psi]$. In this case we again have three possible Lorentz indices which must be contracted against gamma matrices or metric tensors. We may ignore the possibility of contracting the Lorentz index of the covariant derivative against a gamma matrix because of the equations of motion. We are left with two potential operator structures: $\bar{\psi}D^\mu F_{\mu\nu}\gamma^\nu\psi$ and $\bar{\psi}F_{\mu\nu}D^\mu\gamma^\nu\psi$. But

$$\bar{\psi}D^\mu F_{\mu\nu}\gamma^\nu\psi = \bar{\psi}(-g\bar{\psi}\gamma_\nu\psi)\gamma^\nu\psi + \bar{\psi}F_{\mu\nu}D^\mu\gamma^\nu\psi, \tag{5.15}$$

using the Yang-Mills equation. Since we are only interested in processes with two quarks, we will systematically ignore four quark operators. Therefore, we may replace the operator $\bar{\psi}D^\mu F_{\mu\nu}\gamma^\nu\psi$ with $\bar{\psi}F_{\mu\nu}D^\mu\gamma^\nu\psi$. This is the only member of the class $[\bar{\psi}DF\psi]$ which is of interest to us.

We now turn to operator structures containing two quark fields but only one extra power of derivatives or gauge fields. Thus the available operator structures are $[\bar{\psi}DD\psi]$ and $[\bar{\psi}F\psi]$. Up to equations of motion, there is only one operator of the first type: $\bar{\psi}D^\mu D_\mu\psi$. However, this is a reducible operator:

$$\bar{\psi}D^\mu D_\mu\psi = \bar{\psi}\slashed{D}\slashed{D}\psi - \frac{ig}{2}\bar{\psi}F_{\mu\nu}\gamma^\nu\gamma^\mu\psi. \tag{5.16}$$

Thus, up to equations of motion, we may reduce the $[\bar{\psi}DD\psi]$ class to the $[\bar{\psi}F\psi]$ class. Because of the antisymmetry of the field strength tensor, there is only one operator in the $[\bar{\psi}F\psi]$ class, namely $\bar{\psi}F_{\mu\nu}\gamma^\nu\gamma^\mu\psi$.

Finally, we must consider operators containing no quark fields. There are three gauge invariant possibilities: $\operatorname{tr} F^{\mu\nu}F_{\nu\rho}F^\rho{}_\mu$, $\operatorname{tr} F_{\mu\nu}D^2F^{\mu\nu}$, and $\operatorname{tr}(D^\mu F_{\mu\nu})(D^\rho F_\rho{}^\nu)$. The last of these three operators is equivalent to a four quark operator using the Yang-Mills equation, and is therefore of no interest to us. Meanwhile, the second of the three is equivalent to the other two:

$$\operatorname{tr} F_{\mu\nu}D^2F^{\mu\nu} = -2\operatorname{tr}(D^\mu F_{\mu\nu})D_\alpha F^{\alpha\nu} - 2ig\operatorname{tr} F_{\nu\mu}F^\mu{}_\alpha F^{\alpha\nu}. \tag{5.17}$$
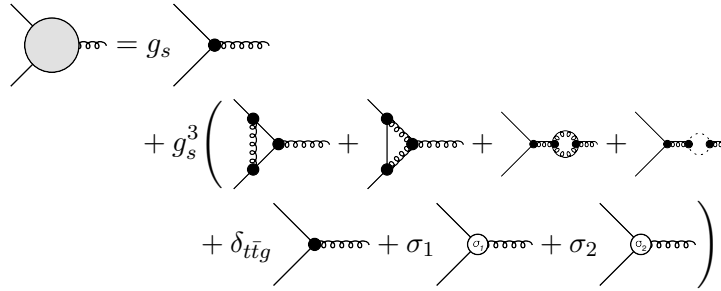
As a result, we may also ignore this operator, leaving only $\operatorname{tr} F^{\mu\nu}F_{\nu\rho}F^\rho{}_\mu$.

In summary, there are only three higher dimension operators that contribute to the on-shell amplitudes. We may therefore take the full QCD Lagrangian in six dimensions, at one loop order, to be

$$\mathcal{L}^6_{\text{QCD}} = \bar{\psi}(i\slashed{D} - m)\psi - \frac{1}{2}\operatorname{tr} F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\sigma_1\, g_s^3 m_t\, \bar{\psi}\gamma^\mu\gamma^\nu F_{\mu\nu}\psi$$
$$+ i\sigma_2\, g_s^3\, \bar{\psi}F_{\mu\nu}\gamma^\mu D^\nu\psi + \frac{i}{6}\gamma\, g_s^3\operatorname{tr}\left(F^{\mu\nu}[F_{\mu\lambda}, F_\nu{}^\lambda]\right). \tag{5.18}$$

A selection of the resulting Feynman rules are listed in appendix D.

We adopt the point of view that σ_1, σ_2 and γ are couterterms which remove the divergences in loop amplitudes. In addition there are the usual counterterms from the dimension four vertices $t\bar{t}g$ and ggg. We can compute the constants $\delta_{t\bar{t}g}, \delta_{ggg}, \sigma_1, \sigma_2$ and γ from simple one-loop vertex graphs. For example, expanding the $t\bar{t}g$ vertex to $\mathcal{O}(g_s^3)$ leads to,
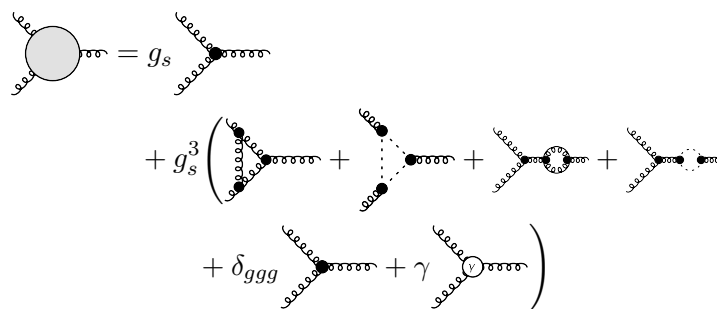

$$+ \mathcal{O}(g_s^5). \tag{5.19}$$

Renormalising this correlation function off-shell would require the inclusion of all possible counterterms (before use of the equations of motion.) For us, it is simpler to compute the on-shell three point vertex, in which case all infinities can be absorbed in our effective Lagrangian, equation (5.18). This presents a minor problem since the three point vertex is not well defined for real momenta. The computation may be performed using complex external kinematics or alternatively performed with the gluon taken off-shell and the constants extracted by taking the on-shell limit $p^2 \to 0$ at the end of the computation. We find this amplitude is UV finite in $6 - 2\epsilon$ dimensions for the values:

$$\delta_{t\bar{t}g} = \frac{m_t^2}{24(4\pi)^3\epsilon}C_F(3d_s + 2), \tag{5.20}$$

$$\sigma_1 = -\frac{1}{12(4\pi)^3\epsilon}\left(C_A(d_s - 5) - \frac{C_F}{2}(3d_s - 14)\right), \tag{5.21}$$

where $C_F = \frac{N_c^2-1}{2N_c}$ and $C_A = N_c$. A similar computation for the three gluon vertex,

$$+ \mathcal{O}(g_s^5), \tag{5.22}$$

results in

$$\delta_{ggg} = 0, \tag{5.23}$$

$$\gamma = \frac{1}{12(4\pi)^3\epsilon} C_A \frac{(d_s - 2)}{5}. \tag{5.24}$$

5.3 Fixing c_1 by matching poles in $6 - 2\epsilon$ dimensions

We finally apply this knowledge of the universal epsilon poles in six dimensions to determine the remaining unknown coefficient, c_1 in equation (5.8). The six-dimensional leading colour partial amplitude $A_{4;1}^{(1),6-2\epsilon}(1_t, 2, 3, 4_{\bar{t}})$ can be decomposed into gauge invariant primitives

$$A_{4;1}^{(1),6-2\epsilon}(1_t, 2, 3, 4_{\bar{t}}) = N_c A^{[L],6-2\epsilon}(1_t, 2, 3, 4_{\bar{t}}) - \frac{1}{N_c} A^{[R],6-2\epsilon}(1_t, 2, 3, 4_{\bar{t}}), \tag{5.25}$$

precisely as in four dimensions (we ignore fermion loops as they present no technical difficulties.) Because the epsilon-poles are universal, we know that the poles of this amplitude are

$$A_{4;1}^{(1),6-2\epsilon}(1_t, 2, 3, 4_{\bar{t}}) = g_s^4 \left(2\delta_{t\bar{t}g} A^{(0)}(1_t, 2, 3, 4_{\bar{t}}) + \sigma_1 A^{[\sigma_1]}(1_t, 2, 3, 4_{\bar{t}}) \right.$$

$$\left. + \sigma_2 A^{[\sigma_2]}(1_t, 2, 3, 4_{\bar{t}}) + \gamma A^{[\gamma]}(1_t, 2, 3, 4_{\bar{t}}) \right) + \mathcal{O}(\epsilon^0), \tag{5.26}$$

where the tree-type amplitudes $A^{[\sigma_1]}(1_t, 2, 3, 4_{\bar{t}})$, $A^{[\sigma_2]}(1_t, 2, 3, 4_{\bar{t}})$ and $A^{[\gamma]}(1_t, 2, 3, 4_{\bar{t}})$ are associated with the three higher dimension operators in the effective 6d QCD Lagrangian, equation (5.18). They are explicitly defined by the diagrams shown in figure 7. In a similar fashion to the vertex computation we find that $A^{[\sigma_2]}(1_t, 2, 3, 4_{\bar{t}}) = 0$. By collecting in powers of N_c, and inserting the known expressions for $\delta_{t\bar{t}g}$, σ_1 and γ given in equations (5.20), (5.21) and (5.24) we find,

$$A^{[L],6-2\epsilon}(1_t, 2, 3, 4_{\bar{t}}) = \frac{g_s^4}{48(4\pi)^3\epsilon} \left(2(3d_s + 2)m_t^2 A^{(0)}(1_t, 2, 3, 4_{\bar{t}}) + \frac{4(d_s - 2)}{5} A^{[\gamma]}(1_t, 2, 3, 4_{\bar{t}}) \right.$$

$$\left. - (d_s - 6) A^{[\sigma_1]}(1_t, 2, 3, 4_{\bar{t}}) \right) + \mathcal{O}(\epsilon^0) \tag{5.27}$$

$$A^{[\sigma_1]}(1_t, 2, 3, 4_{\bar{t}}) =$$

$$A^{[\gamma]}(1_t, 2, 3, 4_{\bar{t}}) =$$

Figure 7. The Feynman diagrams contributing to the tree-level amplitudes appearing the pole structure of the one-loop $t\bar{t}gg$ amplitudes in $6 - 2\epsilon$ dimensions. Solid vertices correspond to the usual QCD interactions while the open vertices are those resulting from the corresponding dimension six operators in $\mathcal{L}_{\mathrm{QCD}}^6$ of eq. (5.18).

for the left-moving ordering and

$$A^{[R], 6-2\epsilon}(1_t, 2, 3, 4_{\bar{t}}) = \frac{g_s^4}{48(4\pi)^3\epsilon}\left(2(3d_s + 2)m_t^2 A^{(0)}(1_t, 2, 3, 4_{\bar{t}})\right.$$

$$\left. + (3d_s - 14)A^{[\sigma_1]}(1_t, 2, 3, 4_{\bar{t}})\right) + \mathcal{O}(\epsilon^0) \qquad (5.28)$$

for the right-moving case.

The tree amplitudes $A^{[\sigma_1]}(1_t, 2, 3, 4_{\bar{t}})$ and $A^{[\gamma]}(1_t, 2, 3, 4_{\bar{t}})$ are easily determined by calculating the diagrams in figure 7. Written in terms of four dimensional spinor products, the independent helicity amplitudes are

$$-i\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle A^{[\sigma_1]}(1_t^+, 2^+, 3^+, 4_{\bar{t}}^+) =$$
$$\frac{-2m_t(2m_t^2 - 4p_1\cdot p_2 - s_{23})s_{23}\langle\eta_1 2\rangle\langle\eta_4 3\rangle}{\langle 23\rangle^3} + \frac{2m_t(m_t^2 - 2p_1\cdot p_2)s_{23}\langle\eta_1\eta_4\rangle}{m_t\langle 23\rangle^2}$$
$$- \frac{m_t(m_t^2 - 2p_1\cdot p_2)s_{23}\langle\eta_1 3\rangle\langle\eta_4 3\rangle\langle 2|1|3]}{p_1\cdot p_2\langle 23\rangle^3} + \frac{m_t(m_t^2 - 2p_1\cdot p_2)s_{23}\langle\eta_1 2\rangle\langle\eta_4 2\rangle\langle 3|1|2]}{p_1\cdot p_2\langle 23\rangle^3}, \qquad (5.29)$$

$$-i\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle A^{[\sigma_1]}(1_t^+, 2^+, 3^-, 4_{\bar{t}}^+) = \frac{(-4m_t(p_1\cdot p_2)^2 + m_t^2 s_{23} - 2p_1\cdot p_2 s_{23})\langle\eta_1 3\rangle\langle\eta_4 3\rangle\langle 3|1|2]}{p_1\cdot p_2 s_{23}\langle 23\rangle}$$
$$- \frac{2m_t\langle\eta_1\eta_4\rangle\langle 3|1|2]^2}{m_t s_{23}} + \frac{m_t(4p_1\cdot p_2 + s_{23})\langle\eta_1 2\rangle\langle\eta_4 3\rangle\langle 3|1|2]^2}{p_1\cdot p_2 s_{23}\langle 23\rangle} - \frac{m_t\langle\eta_1 2\rangle\langle\eta_4 2\rangle\langle 3|1|2]^3}{p_1\cdot p_2 s_{23}\langle 23\rangle}, \qquad (5.30)$$

$$-i\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle A^{[\gamma]}(1_t^+, 2^+, 3^+, 4_{\bar{t}}^+) = \frac{m_t s_{23}^2\langle\eta_1 2\rangle\langle\eta_4 3\rangle}{2\langle 23\rangle^3} + \frac{m_t p_1\cdot p_2 s_{23}\langle\eta_1\eta_4\rangle}{\langle 23\rangle^2}, \qquad (5.31)$$

$$-i\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle A^{[\gamma]}(1_t^+, 2^+, 3^-, 4_{\bar{t}}^+) = 0, \qquad (5.32)$$

$$-i\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle A^{[\sigma_2]}(1_t^+, 2^+, 3^+, 4_{\bar{t}}^+) = 0, \qquad (5.33)$$

$$-i\langle\eta_1 1^\flat\rangle\langle\eta_4 4^\flat\rangle A^{[\sigma_2]}(1_t^+, 2^+, 3^-, 4_{\bar{t}}^+) = 0. \qquad (5.34)$$

We note that the amplitudes of the σ_2 operator vanish in the cases we have considered. We still include it in our analysis since the operator remains in the Lagrangian after using the equations of motion despite not playing a role for the amplitudes in this paper.

The final step necessary to determine the tadpole coefficient is to evaluate the poles of the basis integrals of the one-loop amplitude in $6 - 2\epsilon$ dimensions. We find

$$I_1^{6-2\epsilon}[1](m^2) = \frac{-im^4}{2(4\pi)^3\epsilon} + \mathcal{O}(\epsilon^0) \tag{5.35}$$

$$I_2^{6-2\epsilon}[1](P^2, m_1^2, m_2^2) = i\frac{P^2 - 3(m_1^2 + m_2^2)}{6(4\pi)^3\epsilon} + \mathcal{O}(\epsilon^0) \tag{5.36}$$

$$I_2^{6-2\epsilon}[\mu^2](P^2, m_1^2, m_2^2) = i\frac{P^4 - 5P^2(m_1^2 + m_2^2) + 10\left((m_1^2 + m_2^2)^2 - m_1^2 m_2^2\right)}{60(4\pi)^3\epsilon} + \mathcal{O}(\epsilon^0), \tag{5.37}$$

$$I_3^{6-2\epsilon}[1] = \frac{-i}{2(4\pi)^3\epsilon} + \mathcal{O}(\epsilon^0) \tag{5.38}$$

$$I_3^{6-2\epsilon}[\mu^2](P_1^2, P_2^2, P_3^2, m_1^2, m_2^2, m_3^2) = -i\frac{P_1^2 + P_2^2 + P_3^2 - 4(m_1^2 + m_2^2 + m_3^2)}{24(4\pi)^3\epsilon} + \mathcal{O}(\epsilon^0), \tag{5.39}$$

$$I_4^{6-2\epsilon}[1] = \mathcal{O}(\epsilon^0), \tag{5.40}$$

$$I_4^{6-2\epsilon}[\mu^2] = \frac{i}{6(4\pi)^3\epsilon} + \mathcal{O}(\epsilon^0). \tag{5.41}$$

We do not list the formulae for box integrals in 10 dimensions (μ^4) since they do not appear in amplitudes with a fermion pair and any number of gluons. The formulae are easy to derive using the dimensional recurrence relation implemented in LiteRed [49] in any case.

The only unknowns in equations (5.27) and (5.28) are then the left- and right-moving tadpole coefficients c_1, allowing a direct determination of these rational functions. The results are somewhat lengthy formulae which are explicitly derived in the Mathematica workbook. We have checked that this procedure matches the expected result by comparing with the previous computation of reference [12].

6 Conclusions

In this paper we have explored a new technique for the computation of one-loop amplitudes with massive fermions. Our methods are designed to be compatible with on-shell generalised unitarity.

The six-dimensional spinor-helicity scheme proved to be an efficient way to describe the tree-level input into the d-dimensional generalised unitarity method. Divergent wave-function cuts were avoided, and the remaining ambiguities in the amplitudes were fixed by matching to the universal physical pole structure. The $4 - 2\epsilon$ pole structure of Catani et al. [43] is sufficient to constrain all remaining logarithms in the fermion mass while additional information is needed to fix the remaining finite corrections connected to tadpole integrals. We obtained this second constraint by allowing the loop momenta in our integrals to be defined in a higher dimension spacetime, and imposing the universality of ultraviolet divergences in this higher dimensional quantum field theory. Since six is the next even dimension above four it was natural to study QCD as an effective theory in $6 - 2\epsilon$ dimensions. We used the on-shell equations of motion to find a minimal set of additional dimension six operators in this theory, and computed the required counterterms essentially following the

textbooks. We applied our method to a variety of simple cases and validated it on helicity amplitudes for top quark pair production.

The methods we used in this paper are flexible, and it is clear that they apply more generally than to $gg \to t\bar{t}$ scattering. It would be interesting to work out the extension to more general cases with multiple fermions and multiple masses, as well as to higher loops. In the presence of more fermions, four quark operators would need to be included in the effective Lagrangian, while at higher loops one would need to consider operators of mass-dimension greater than six.

Since this method can compute amplitudes with fewer cuts than other known approaches it has the potential to optimise existing numerical and analytical approaches. However, since the main computational bottleneck in most phenomenological collider studies at NLO lies in the integration over the unresolved phase-space, the technique is probably best suited to find compact analytic expressions where the improvement in stability and speed over existing numerical approaches is particularly beneficial.

Perhaps a more interesting direction would be to look into the implications of the higher dimensional pole structure on the spurious singularities appearing in integral reductions. As a result of matching to a tree-level computation with an effective Lagrangian, we find non-trivial relations between the d-dimensional integral coefficients in which all spurious poles cancel. These cancellations had to occur, since the effective theory contains only local operators. This information could be useful in finding compact and stable representations of one-loop amplitudes.

Acknowledgments

A Conventions and spinor construction

We use the mostly minus metrics

$$\eta^{\mu\nu} = \text{diag}\{1, -1, -1, -1\},$$
$$\eta^{MN} = \text{diag}\{1, -1, -1, -1, -1, -1\}, \tag{A.1}$$

where lower-case Greek letters are used for four dimensional Lorentz indices and upper-case for six dimensions.

We use the following Pauli matrices:

$$\sigma^1_{\alpha\dot\alpha} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \ \sigma^2_{\alpha\dot\alpha} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \sigma^3_{\alpha\dot\alpha} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.2}$$

and $(\tilde\sigma^\mu)^{\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}$ with $\epsilon_{12} = 1$. Using the Pauli matrices we define the Σ-matrices

$$\begin{aligned} \Sigma^0 &= i\sigma^1 \times \sigma^2 & \tilde\Sigma^0 &= -i\sigma^1 \times \sigma^2 \\ \Sigma^1 &= i\sigma^2 \times \sigma^3 & \tilde\Sigma^1 &= i\sigma^2 \times \sigma^3 \\ \Sigma^2 &= \sigma^2 \times \sigma^0 & \tilde\Sigma^2 &= -\sigma^2 \times \sigma^0 \\ \Sigma^3 &= -i\sigma^2 \times \sigma^1 & \tilde\Sigma^3 &= -i\sigma^2 \times \sigma^1 \\ \Sigma^4 &= -\sigma^3 \times \sigma^2 & \tilde\Sigma^4 &= \sigma^3 \times \sigma^2 \\ \Sigma^5 &= -i\sigma^0 \times \sigma^2 & \tilde\Sigma^5 &= -i\sigma^0 \times \sigma^2, \end{aligned} \tag{A.3}$$

which obey the Clifford algebra

$$\Sigma^M\tilde\Sigma^N + \Sigma^N\tilde\Sigma^M = 2\eta^{MN}. \tag{A.4}$$

B Tree-level amplitudes in six dimensions

In this section we list the tree-level amplitudes in six dimensions used in our calculation.

B.1 Three-point amplitudes

- $A^{(0)}(1_q, 2_{\bar q}, 3_g)$

$$A^{(0)}(1_q^a, 2_{\bar q}^b, 3_g^{c\dot c}) = \frac{i}{s_{r3}}\langle 1^a 2^b 3^c r^x\rangle\langle r_x|3^{\dot c}] \tag{B.1}$$

where r is a massless reference vector satisfying $s_{r3} \neq 0$.

- $A^{(0)}(1_g, 2_g, 3_g)$

$$A^{(0)}(1_{a\dot a}, 2_{b\dot b}, 3_{c\dot c}) = i\Gamma_{abc}\tilde\Gamma_{\dot a\dot b\dot c} \quad \Gamma_{abc} = u_{1a}u_{2b}w_{3c} + u_{1a}w_{2b}u_{3c} + w_{1a}u_{2b}u_{3c} \tag{B.2}$$
$$\tilde\Gamma_{\dot a\dot b\dot c} = \tilde u_{1\dot a}\tilde u_{2\dot a}\tilde w_{3\dot c} + \tilde u_{1\dot a}\tilde w_{2\dot a}\tilde u_{3\dot c} + \tilde w_{1\dot a}\tilde u_{2\dot a}\tilde u_{3\dot c}, \tag{B.3}$$

where the tensors Γ and $\tilde\Gamma$ are written in terms of the SU(2) spinors $u, \tilde u$ satisfying the following properties, defined on a cyclic order $\{ijk\}$,

$$u_{ia}\tilde u_{jb} = \langle i_a|j_b], \qquad\qquad u_{ja}\tilde u_{ib} = -\langle j_a|i_b], \tag{B.4}$$

and $w, \tilde w$ are the inverse of the $u, \tilde u$

$$\epsilon_{ab} = u_a w_b - u_b w_a, \qquad\qquad \epsilon_{\dot a\dot b} = \tilde u_{\dot a}\tilde w_{\dot b} - \tilde u_{\dot b}\tilde w_{\dot a}, \tag{B.5}$$

for which we impose momentum conservation

$$0 = \tilde w_{1\dot a}\tilde\lambda^{\dot a}_{1A} + \tilde w_{2\dot a}\tilde\lambda^{\dot a}_{2A} + \tilde w_{3\dot a}\tilde\lambda^{\dot a}_{3A}. \tag{B.6}$$

- $A^{(0)}(1_{\phi_{1,2}}, 2_{\phi_{1,2}}, 3_g)$

$$A^{(0)}(1_{\phi_{1,2}}, 2_{\phi_{1,2}}, 3_g^{a\dot{a}}) = \frac{-i}{2s_{r3}} \langle 3^a | (1-2) | r | 3^{\dot{a}}] \tag{B.7}$$

where r is a massless reference vector satisfying $s_{r3} \neq 0$.

- $A^{(0)}(1_{\phi_1}, 2_{\bar{q}}, 3_q)$

$$A^{(0)}(1_{\phi_1}, 2_{\bar{q}}^a, 3_q^{\dot{b}}) = \frac{i}{\sqrt{2}} \langle 1^a | 2^{\dot{b}}]. \tag{B.8}$$

- $A^{(0)}(1_{\phi_2}, 2_{\bar{q}}, 3_q)$

$$A^{(0)}(1_{\phi_1}, 2_{\bar{q}}^a, 3_q^{\dot{b}}) = \frac{i}{\sqrt{2}} \langle 1^a | \gamma^5 | 2^{\dot{b}}]. \tag{B.9}$$

B.2 Four-point amplitudes

- $A^{(0)}(1_g, 2_g, 3_g, 4_g)$

$$A^{(0)}(1_{a\dot{a}}, 2_{b\dot{b}}, 3_{c\dot{c}}, 4_{d\dot{d}}) = \frac{-i}{s_{12}s_{23}} \langle 1_a 2_b 3_c 4_d \rangle [1_{\dot{a}} 2_{\dot{b}} 3_{\dot{c}} 4_{\dot{d}}] \tag{B.10}$$

- $A(1_q, 2_g, 3_g, 4_{\bar{q}})$

$$A^{(0)}(1_{q,a}, 2_{b\dot{b}}, 3_{c\dot{c}}, 4_{\bar{q},d}) = \frac{i}{2s_{12}s_{23}} \langle 1_a 2_b 3_c 4_d \rangle [1_{\dot{x}} 2_{\dot{b}} 3_{\dot{c}} 1^{\dot{x}}]. \tag{B.11}$$

- $A^{(0)}(1_g, 2_g, 3_{\phi_{1,2}}, 4_{\phi_{1,2}})$

$$A^{(0)}(1_{a\dot{a}}, 2_{b\dot{b}}, 3, 4) = \frac{i}{4s_{12}s_{23}} \langle 1_a 2_b 3_x 3^x \rangle [1_{\dot{a}} 2_{\dot{b}} 3_{\dot{x}} 3^{\dot{x}}] \tag{B.12}$$

C   Cut solutions in six dimensions

In this section we give details on the solutions for the triple- and double-cuts in six dimensions. We will describe the parametrisation used to get the solutions without writing down any explicit expression for them. The implementation is given in the attached MATHEMATICA notebook. Notice that all the cut solutions are rational functions of the kinematics and the free parameters and contain no square roots.

C.1   Triple cut

We write the loop momentum ℓ_i^μ in the following basis,

$$\beta = \left\{ v^\mu, w^\mu, \langle v^1 | \Sigma^\mu | w_1 \rangle, \langle v^1 | \Sigma^\mu | w_2 \rangle, \langle v^2 | \Sigma^\mu | w_1 \rangle, \langle v^2 | \Sigma^\mu | w_2 \rangle \right\}, \tag{C.1}$$

where v and w are six dimensional massless momenta and use the parametrisation

$$\ell_i = \beta \cdot \{y_1, y_2, y_3, y_4, \tau_1, \tau_2\}. \tag{C.2}$$

We impose the cut conditions

$$S_{ijk} = \begin{cases} \ell_i^2 = \ell_j^2 = \ell_k^2 = 0 \\ \ell_i^{(5)} = \begin{cases} 0 & \text{if } i \text{ gluon} \\ \pm m & \text{if } i \text{ fermion} \end{cases} \end{cases}, \tag{C.3}$$

where $\{ijk\}$ is the set of the three cut propagators and the sign of the mass component depends on the kinematic configuration. This system of equations for ℓ_i only constrains 4 parameters so solving for the y_i's, τ_1, τ_2 are left as free parameters.

C.2 Double cut

For the double cut solutions we use the basis in (C.1) and use the following parametrisation

$$\ell_i = \beta \cdot \{y_1, \tau_1, y_2, \tau_2, y_3, \tau_3\}. \tag{C.4}$$

The y_i's are fixed by the double cut constraints

$$S_{ij} = \begin{cases} \ell_i^2 = \ell_j^2 = 0 \\ \ell_i^{(5)} = \begin{cases} 0 & \text{if } i \text{ gluon} \\ \pm m & \text{if } i \text{ fermion} \end{cases} \end{cases}, \tag{C.5}$$

where $\{ij\}$ is the set of the two cut propagators and the sign of the mass component depends on the kinematic configuration. The parameters τ_1, τ_2, τ_3 are unconstrained.

D Feynman rules for the effective Lagrangian

In this appendix we present selected Feynman rules for the six dimensional effective theory of interest to us, defined by the Lagrangian

$$\mathcal{L}_{\text{QCD}}^6 = \bar{\psi}(i\slashed{D} - m)\psi - \frac{1}{2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \sigma_1 \, g_s^3 m_t \, \bar{\psi} \gamma^\mu \gamma^\nu F_{\mu\nu} \psi$$
$$+ i\sigma_2 \, g_s^3 \, \bar{\psi} F_{\mu\nu} \gamma^\mu D^\nu \psi + \frac{i}{6} \gamma \, g_s^3 \operatorname{tr} \left(F^{\mu\nu} [F_{\mu\lambda}, F_\nu{}^\lambda] \right). \tag{D.1}$$

We further define

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c, \tag{D.2}$$

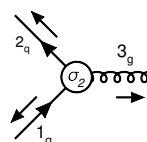$$\sigma^{\mu\nu} = \frac{i}{2} \left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \right). \tag{D.3}$$

These rules were derived with the help of FeynCalc [50, 51] and FeynRules [52, 53]. The vertices are colour ordered and all momenta are considered to be out-going. We include the
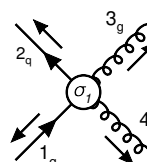
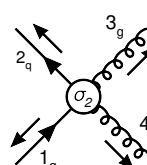coupling constants here for clarity though in the main text they are stripped off.

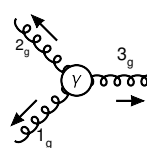$$= -g_s^3 \sigma_1 \, m_t \sigma^{\mu_3 \nu} p_{3\nu} \tag{D.4}$$

$$= -i g_s^3 \sigma_2 \left(p_2^{\mu_3} \slashed{p}_3 - p_2 \cdot p_3 \gamma^{\mu_3} \right) \tag{D.5}$$

$$= g_s^4 \sigma_1 \, m_t \sigma^{\mu_3 \mu_4} \tag{D.6}$$

$$= -i g_s^4 \sigma_2 \left(g^{\mu_3 \mu_4} \slashed{p}_3 - \gamma^{\mu_4} p_1^{\mu_3} + \gamma^{\mu_3} (p_1^{\mu_4} - p_3^{\mu_4}) \right) \tag{D.7}$$

$$= -\frac{i}{2} g_s^3 \gamma \left(g^{\mu_1 \mu_2} \left(p_1 \cdot p_3 \, p_2^{\mu_3} - p_2 \cdot p_3 \, p_1^{\mu_3} \right) \right.$$
$$+ g^{\mu_2 \mu_3} \left(p_2 \cdot p_1 \, p_3^{\mu_1} - p_3 \cdot p_1 \, p_2^{\mu_1} \right)$$
$$+ g^{\mu_3 \mu_1} \left(p_3 \cdot p_2 \, p_1^{\mu_2} - p_1 \cdot p_2 \, p_3^{\mu_2} \right)$$
$$\left. - p_3^{\mu_1} p_1^{\mu_2} p_2^{\mu_3} + p_2^{\mu_1} p_3^{\mu_2} p_1^{\mu_3} \right). \tag{D.8}$$

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