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CONTRACTIVITY FOR SMOLUCHOWSKI'S COAGULATION EQUATION WITH SOLVABLE KERNELS

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Abstract. We show that the Smoluchowski coagulation equation with the solvable kernels $K(x,y)$ equal to 2, $x+y$ or xy is contractive in suitable Laplace norms. In particular, this proves exponential convergence to a self-similar profile in these norms. These results are parallel to similar properties of Maxwell models for Boltzmann-type equations, and extend already existing results on exponential convergence to self-similarity for Smoluchowski's coagulation equation.

1. Introduction

Smoluchowski's coagulation equation describes the growth of clusters in systems of merging particles in a broad range of applications (see Banasiak et al. (2019) for general references on the matter). Precisely, the equation is given by

$$\begin{aligned}\partial_t n(t,x) &= \frac{1}{2}\int_0^x K(x-y,y)n(t,x-y)n(t,y)\,\mathrm{d}y - \int_0^\infty K(x,y)n(t,x)n(t,y)\,\mathrm{d}y \\ &=: \mathcal{C}(n(t,\cdot),n(t,\cdot))(x), \qquad x>0\end{aligned} \tag{1.1}$$

where $n(t,x)$ is the density of clusters of size $x>0$ at time $t \geqslant 0$ and the integral kernel $K(x,y)\geqslant 0$ describes the rate at which clusters of sizes x and y merge. In applications, the latter function is usually homogeneous of a certain degree γ, i.e. $K(ax,ay)=a^\gamma K(x,y)$ for all $a,x,y>0$. In this paper we are concerned with the so-called *solvable kernels*:

$$\begin{aligned}
K(x,y) &= 2 && \text{(constant kernel)},\\
K(x,y) &= x+y && \text{(additive kernel)},\\
K(x,y) &= xy && \text{(multiplicative kernel)}.
\end{aligned}$$

For these kernels an explicit solution to (1.1) may be found by using the Laplace transform, see Menon and Pego (2004); Banasiak et al. (2019).

In this note we show that for these kernels, Equation (1.1) satisfies new contractivity properties in suitable weak distances which we define below. When one considers the usual change of scale to self-similar variables, we show that

$$\|g_1(\tau,\cdot)-g_2(\tau,\cdot)\| \leqslant e^{-\lambda t}\|g_1(0,\cdot)-g_2(0,\cdot)\|,$$

where $g_1=g_1(\tau,x)$, $g_2=g_2(\tau,x)$ are obtained from two finite-mass solutions to (1.1) through the change of variables, and $\|\cdot\|$ is a suitable weighted norm of the Laplace transform of g. Precise statements are given at the end of this introduction. In particular, this provides explicit exponential rates of convergence towards self-similarity with respect to this norm, always for solutions with finite mass. This exponential convergence was already known in other norms since Cañizo et al. (2010); Srinivasan (2011), and in fact our arguments have a similar flavour to those in Srinivasan (2011).

What is remarkable is that the calculation involving these Laplace-based distances is much simpler, especially for the additive kernel, and yields contractivity of the whole flow, not just of the distance to self-similarity. These distances are inspired by analogous norms based on the Fourier transform which have been used in Carrillo and Toscani (2007) to study several models related to the Boltzmann equation with constant collision kernels (the *Maxwell cases*), and which to our knowledge have not been exploited in proving the convergence to self-similarity for coagulation equations.

Let us give some background on Smoluchowski's equation in order describe our results more precisely. An important property of (1.1) is the (formal) conservation of the total mass

$$M_1[n(t)] = M_1[n(0)] \qquad \forall t \geqslant 0$$

where, for any nonnegative function $f\,:\mathbb{R}_+ \to \mathbb{R}$, we set

$$M_\ell[f] = \int_0^\infty x^\ell f(x)\,\mathrm{d}x, \qquad \ell \geqslant 0.$$

Indeed, multiplying (1.1) by x and integrating over $(0,\infty)$ we obtain that $\frac{\mathrm{d}}{\mathrm{d}t}M_1[n(t)] = 0$ by formally interchanging the order of integration. However, for homogeneity degree $\gamma > 1$ this procedure cannot be made rigorous and solutions in fact lose mass after some finite time, a phenomenon which is known as *gelation*, see for example Escobedo et al. (2002); Breschi and Fontelos (2014); van Dongen and Ernst (1986), and Banasiak et al. (2019, Chapter 9) for a thorough discussion of this topic. In fact, one defines the *gelation time* T_* as

$$T_* = \inf\{t \geqslant 0\ |M_1(t) < M_1(0)\}.$$

If $\gamma \leqslant 1$ one sets $T_* = \infty$.

A well-known conjecture, known as the *scaling hypothesis*, states that the behaviour of solutions n to (1.1) is self-similar as $t \to T_*$ (perhaps under additional conditions on the initial datum). That is: there exists a *self-similar profile* $\widehat{n}$, a scaling function $s(t) \to \infty$ as $t \to T_*$ and a constant $\alpha > 0$ such that

$$\big(s(t)\big)^\alpha n(t, s(t)x) \longrightarrow \widehat{n}(x) \qquad \text{if } t \to T_*. \tag{1.2}$$

However, this claim is still unproven for most kernels K. The only cases where (1.2) is well understood are the solvable kernels $K(x,y) = 2$, $K(x,y) = x + y$ and $K(x,y) = xy$. In fact, for these rate kernels, the scaling hypothesis was verified in Menon and Pego (2004), i.e. there exists one unique fast decaying self-similar profile (up to normalisation) which attracts all solutions with initial condition satisfying $\int_0^\infty x^{\gamma+1} n(0,x)\,\mathrm{d}x < \infty$ with respect to weak convergence. A more precise statement can be found in Menon and Pego (2004), where in addition the existence of *fat-tailed profiles* was established and the corresponding domains of attraction were characterised. These proofs heavily rely on Laplace transform methods which allow to compute solution formulas for (1.1) rather explicitly.

The results on the fast-decaying profiles were further improved in Menon and Pego (2005) by showing that (1.2) also holds uniformly with respect to $x \in \mathbb{R}_+$ (i.e. (1.2) is obtained in $L^\infty(\mathbb{R}^+)$). For the constant kernel, the scaling hypothesis was also verified by a different approach which relies on spectral gap estimates for the linearised coagulation operator (Cañizo et al. (2010)), yielding explicit rates of convergence to self-similarity. Moreover, Srinivasan (2011) later provided rates of convergence for the primitive of n, for all three solvable kernels, using explicit calculations inspired in arguments related to the central limit theorem in probability.

The only known full verification of the scaling hypothesis for a class of non-solvable kernels was recently given in Cañizo and Throm (2019) where bounded perturbations $K(x,y) = 2 + \epsilon W(x,y)$ with $\|W\|_{L^\infty} \leqslant 1$ and small ϵ have been considered. The proof again relies on spectral gap estimates and provides explicit rates of convergence towards the self-similar profile.

Assume the coagulation kernel has homogeneity degree γ. It is known (Menon and Pego, 2004; Escobedo and Mischler, 2006) that if a finite-mass solution satisfies (1.2) with a finite-mass profile $\widehat{n}$, it must happen that $\alpha = 2$. Introducing a change of unknown

$$g(\tau, z) := s(t)^2 n\Big(t, s(t)z\Big), \qquad t = t(\tau) > 0$$

and using that, for a coagulation kernel homogeneous of degree γ,

$$s(t)^2\mathcal{C}(n(t), n(t))(x)\Big|_{t=t(\tau), x=s(t(\tau))z} = s(t(\tau))^{\gamma-1}\left[\mathcal{C}(g(\tau), g(\tau)\right](z)$$

$$\text{and} \qquad z\partial_z g(\tau, z) = \Big(s^2(t)\, x\partial_x n(t,x)\Big)\Big|_{t=t(\tau), x=s(t(\tau))z}$$

we obtain that, if $n(t,x)$ satisfies (1.1),

$$\begin{aligned}
&\partial_\tau g(\tau, z) \\
&\quad = \frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau}\Big\{2\dot{s}(t)\, s(t)\, n(t,x) + s^2(t)\mathcal{C}(n(t), n(t)) + s(t)\dot{s}(t) x\partial_x n(t,x)\Big\}\Big|_{t=t(\tau), x=s(t(\tau))z} \\
&\qquad = \frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau}\left\{2\frac{\dot{s}(t(\tau))}{s(t(\tau))} g(\tau, z) + s^{\gamma-1}(t(\tau))\mathcal{C}(g(\tau), g(\tau))(z) + \frac{\dot{s}(t(\tau))}{s(t(\tau))} z\partial_z g(\tau, z)\right\}.
\end{aligned}$$

where $\dot{s}$ denotes the derivative with respect to the original variable t. Therefore, choosing $s(t)$ and $t(\tau)$ such that

$$\frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau}\frac{\dot{s}(t(\tau))}{s(t(\tau))} = 1, \qquad \frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau} s^{\gamma-1}(t(\tau)) = \frac{1}{k}, \tag{1.3}$$

(for any $k > 0$ to be chosen later) we obtain that $g(\tau, z)$ satisfies the self-similar Smoluchowski equation

$$\partial_\tau g(\tau, z) = 2g(\tau, z) + z\partial_z g(\tau, z) + \frac{1}{k}\mathcal{C}(g(\tau), g(\tau))(z), \qquad z > 0, \qquad \tau > 0 \tag{1.4}$$

and the stationary solutions to this equation are the self-similar profiles $\widehat{n} = \widehat{n}(z)$ appearing in (1.2). We refer to Menon and Pego (2005) and Banasiak et al. (2019) for details on this subject. After solving, (1.3) yields

$$t(\tau) = \frac{\tau}{k} \qquad \text{and} \qquad s(t) = e^{kt} \qquad \text{if } \gamma = 1$$

whereas

$$t(\tau) = \frac{1}{k(1-\gamma)}\left(e^{(1-\gamma)\tau} - 1\right), \qquad\qquad s(t) = (1 + k(1-\gamma)t)^{\frac{1}{1-\gamma}} \qquad \text{if } \gamma \neq 1.$$

Notice that $s(t(\tau)) = e^{\tau}$. The gelation time T_* is then equal to $+\infty$ if $\gamma \leqslant 1$, and is finite for $\gamma > 1$. The self-similar profile $\widehat{n} = \widehat{n}(z)$ must then satisfy the equation

$$2\widehat{n} + z\partial_z \widehat{n} + \frac{1}{k}\mathcal{C}(\widehat{n}, \widehat{n}) = 0. \tag{1.5}$$

In the three cases which concern us in this paper, and always considering finite-mass solutions, this becomes the following:

(1) For the constant case $K = 2$ (so homogeneity $\gamma = 0$), the value of k is irrelevant (since the convergence (1.2) holds for all $k > 0$). We choose then $k = 1$ and obtain

$$t(\tau) = e^{\tau} - 1, \qquad g(\tau, z) := e^{2\tau} n(e^{\tau} - 1, e^{\tau} z), \tag{1.6}$$

which satisfies then

$$\partial_\tau g = 2g + z\partial_z g + \mathcal{C}_{\mathsf{const}}(g, g) \tag{1.7}$$

where $\mathcal{C}_{\mathsf{const}}$ is the coagulation operator in (1.1) for $K = 2$. For this model, mass is conserved for solutions to (1.9), i.e. $M_1[g(\tau)] = M_1[g(0)]$ for all $\tau \geqslant 0$. Moreover, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} M_0[g(\tau)] = M_0[g(\tau)]\ (1 - M_0[g(\tau)])$$

and thus, rescaling such that $M_0[g(0)] = 1$, also the moment of order zero is conserved, i.e. in summary we get

$$\int_0^\infty g(\tau, z) \begin{bmatrix} 1 \\ z \end{bmatrix} \mathrm{d}z = \int_0^\infty g(0, z) \begin{bmatrix} 1 \\ z \end{bmatrix} \mathrm{d}z \qquad \forall \tau > 0.$$

(2) For the linear case $K(x, y) = x + y$ (for which $\gamma = 1$) assuming the solution n has mass 1 requires that $k = 2$ in order to have a solution. Hence

$$t(\tau) = \tfrac{1}{2}\tau, \qquad g(\tau, z) := e^{2\tau} n\left(\tfrac{1}{2}\tau, e^{\tau} z\right), \tag{1.8}$$

which satisfies then

$$\partial_\tau g = 2g + z\partial_z g + \tfrac{1}{2}\mathcal{C}_{\mathsf{add}}(g, g) \tag{1.9}$$

where $\mathcal{C}_{\mathsf{add}}$ is the coagulation operator in (1.1) associated to $K(x, y) = x + y$. In that case, one sees that the first moment is conserved $M_1[g(\tau)] = M_1[g(0)]$ for any $\tau > 0$ whereas

$$\frac{\mathrm{d}}{\mathrm{d}\tau} M_2[g(\tau)] = M_2[g(\tau)]\ (M_1[g(\tau)] - 1)\,.$$

This means that, if $M_1[g(0)] = 1$ then both the first and second moments are conserved for solutions to (1.9), i.e.

$$M_1[g(0)] = 1 \Longrightarrow M_1[g(\tau)] = 1, \quad M_2[g(\tau)] = M_2[g(0)], \qquad \forall \tau > 0. \tag{1.10}$$

(3) For the multiplicative case $K(x,y) = xy$ (corresponding to $\gamma = 2$), assuming the solution n has mass 1 and initial second moment equal to 1, then we must choose $k = 1$ in order to have the correct gelation time. Hence

$$t(\tau) = 1 - e^{-\tau}, \qquad g(\tau, z) := e^{2\tau} n(1 - e^{-\tau}, e^{\tau} z), \tag{1.11}$$

which satisfies then

$$\partial_\tau g = 2g + z\partial_z g + \mathcal{C}_{\mathsf{mult}}(g, g) \tag{1.12}$$

where $\mathcal{C}_{\mathsf{mult}}$ is the coagulation operator in (1.1) associated to $K(x,y) = xy$.

Throughout this work, we will use the sub- and superscripts const, add and mult to denote quantities related to the constant, additive and multiplicative kernel respectively.

We introduce the following spaces

$$\mathbb{Y}_{\mathsf{const}} = \left\{ g \in L^1(\mathbb{R}^+) \; ; \int_0^\infty g(x)\,\mathrm{d}x = \int_0^\infty x g(x)\,\mathrm{d}x = 0, \quad \int_0^\infty x^2 |g(x)|\,\mathrm{d}x < \infty \right\}$$

$$\mathbb{Y}_{\mathsf{add}} = \left\{ g \in L^1_{\mathrm{loc}}(\mathbb{R}^+) \; ; \int_0^\infty x g(x)\,\mathrm{d}x = \int_0^\infty x^2 g(x)\,\mathrm{d}x = 0, \quad \int_0^\infty x^3 |g(x)|\,\mathrm{d}x < \infty \right\}$$

$$\mathbb{Y}_{\mathsf{mult}} = \left\{ g \in L^1_{\mathrm{loc}}(\mathbb{R}^+) \; ; \int_0^\infty x^2 g(x)\,\mathrm{d}x = \int_0^\infty x^3 g(x)\,\mathrm{d}x = 0, \quad \int_0^\infty x^4 |g(x)|\,\mathrm{d}x < \infty \right\}.$$

Finally, given $\kappa \in \mathbb{R}$, we define the following

$$\begin{gathered} [\![G]\!]_\kappa = \sup_{\eta > 0} |\eta|^{-\kappa} |G(\eta)| \\ \|g\|_{\mathsf{const},\kappa} = [\![\mathcal{L}[g]]\!]_\kappa, \qquad \|g\|_{\mathsf{add},\kappa} = [\![\mathcal{B}[g]]\!]_\kappa \qquad \text{and} \qquad \|g\|_{\mathsf{mult},\kappa} = [\![\mathcal{B}[xg]]\!]_\kappa. \end{gathered} \tag{1.13}$$

where $\mathcal{L}$ and $\mathcal{B}$ denote the Laplace and desingularised Laplace (Bernstein) transform, i.e.

$$\mathcal{L}[g](\eta) = \int_0^\infty e^{-\eta x} g(x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{B}[g](\eta) = \int_0^\infty (1 - e^{-\eta x}) g(x)\,\mathrm{d}x \qquad \eta > 0.$$

We state here a first obvious result where the uniqueness property $\|u\| = 0$ comes from the fact that both $\mathcal{L}$ and $\mathcal{B}$ are one-to-one:

Proposition 1.1. *The following holds*

(1) For $\kappa \in [0,2]$, $\|\cdot\|_{\mathsf{const},\kappa}$ is a norm on $\mathbb{Y}_{\mathsf{const}}$.
(2) For $\kappa \in [0,3]$, $\|\cdot\|_{\mathsf{add},\kappa}$ is a norm on $\mathbb{Y}_{\mathsf{add}}$ and $\|\cdot\|_{\mathsf{mult},\kappa}$ is a norm on $\mathbb{Y}_{\mathsf{mult}}$.

Remark 1.2. *The constraint on the range of κ for which the above quantities are norms is needed to ensure their finiteness (see Carrillo and Toscani (2007, Proposition 2.6) for similar considerations on Fourier-like metrics). Notice that, in the above listed cases, the spaces $(\mathbb{Y}_{\mathsf{const}}, \|\cdot\|_{\mathsf{const},\kappa})$, $(\mathbb{Y}_{\mathsf{add}}, \|\cdot\|_{\mathsf{add},\kappa})$ and $(\mathbb{Y}_{\mathsf{mult}}, \|\cdot\|_{\mathsf{const},\kappa})$ are not necessarily Banach spaces. However, we will not need such a property in our analysis.*

Similar norms have also been used in Niethammer et al. (2016); Throm (2017) to prove the uniqueness of self-similar profiles for perturbations of the constant kernel.

Our main results assert the *exponential contractivity* of the above norm for the difference of two solutions to (1.4) in the spirit of Carrillo and Toscani (2007, Sections 5.2 & 5.3). Our method of proof differs here from that of Carrillo and Toscani (2007), who investigate contractivity properties of the operator itself. We rather exploit a simple Duhamel representation for the difference of two solutions g_1, g_2 to (1.4).

One important point of the method is that it strongly exploits the fact that, for all the three solvable kernels considered here, *two* different moments are conserved by the flow of solution. This allows to *fix* two such initial moments and work in the various spaces $\mathbb{Y}_{\mathsf{mult}}$, $\mathbb{Y}_{\mathsf{const}}$ and $\mathbb{Y}_{\mathsf{add}}$. More precisely, we prove here the following three statements:

Theorem 1.3 (***constant kernel***)**.** *Let $n_1(t,x)$ and $n_2(t,x)$ be solutions to (1.1) with constant kernel $K = 2$ such that $n_\ell(0,\cdot) \in L^1(\mathbb{R}^+)$ and*

$$\int_0^\infty n_\ell(0,x)\,\mathrm{d}x = \int_0^\infty x n_\ell(x)\,\mathrm{d}x = 1, \qquad \int_0^\infty x^2 n_\ell(x)\,\mathrm{d}x < \infty \qquad \text{for } \ell = 1,2.$$

For $\ell = 1,2$ let furthermore g_ℓ be the rescaling of n_ℓ as given by (1.6). Then, for each $\kappa \in (1,2]$ we have

$$\|g_1(\tau,\cdot) - g_2(\tau,\cdot)\|_{\mathsf{const},\kappa} \leqslant \exp\left(-(\kappa-1)\tau\right)\|g_1(0,\cdot) - g_2(0,\cdot)\|_{\mathsf{const},\kappa}, \qquad \forall \tau \geqslant 0.$$

In particular, this shows exponential convergence towards the unique self-similar profile $G_{\mathsf{const}}(x) = e^{-x}$ with respect to $\|\cdot\|_{\mathsf{const},\kappa}$.

Theorem 1.4 (***additive kernel***)**.** *Let $n_1(t,x)$ and $n_2(t,x)$ be solutions to (1.1) with additive kernel $K(x,y) = x + y$ such that $n_\ell(0,\cdot) \in L^1(\mathbb{R}^+)$ and*

$$\int_0^\infty x n_\ell(0,x)\,\mathrm{d}x = \int_0^\infty x^2 n_\ell(0,x)\,\mathrm{d}x = 1, \qquad \int_0^\infty x^3 n_\ell(0,x)\,\mathrm{d}x < \infty \qquad \text{for } \ell = 1,2.$$

Let g_ℓ be the corresponding rescaling as specified in (1.8). Then, for each $\kappa \in (2,3)$ we have

$$\|g_1(\tau,\cdot) - g_2(\tau,\cdot)\|_{\mathsf{add},\kappa} \leqslant \exp\left(-\tfrac{1}{2}(\kappa-2)\tau\right)\|g_1(0,\cdot) - g_2(0,\cdot)\|_{\mathsf{add},\kappa}, \qquad \forall \tau \geqslant 0.$$

This shows in particular exponential convergence towards the unique self-similar profile

$$G_{\mathsf{add}}(x) = \frac{1}{\sqrt{2\pi}} x^{-3/2} e^{-x/2}$$

with respect to $\|\cdot\|_{\mathsf{add},\kappa}$.

For the multiplicative kernel, one can resort to a well-known change of variables linking solutions to (1.11) to the solutions to (1.8) to deduce from Theorem 1.4 the following

Theorem 1.5 (***multiplicative kernel***)**.** *Let $n_1(t,x)$ and $n_2(t,x)$ be solutions to (1.1) with multiplicative kernel $K(x,y) = xy$ such that $n_\ell(0,\cdot) \in L^1(\mathbb{R}^+)$ and*

$$\int_0^\infty x^2 n_\ell(0,x)\,\mathrm{d}x = \int_0^\infty x^3 n_\ell(0,x)\,\mathrm{d}x = 1 \qquad \int_0^\infty x^4 n_\ell(0,x)\,\mathrm{d}x < \infty \quad \text{for } \ell = 1,2.$$

Let g_ℓ be the rescaling of n_ℓ as given in (1.11). Then, for each $\kappa \in (2,3)$ we have

$$\|g_1(\tau,\cdot)-g_2(\tau,\cdot)\|_{\mathsf{mult},\kappa} \leqslant \exp\left(-\tfrac{1}{2}(\kappa-2)\tau\right)\|g_1(0,\cdot)-g_2(0,\cdot)\|_{\mathsf{mult},\kappa} \qquad \forall \tau \geqslant 0.$$

In particular, this proves the convergence towards the unique self-similar profile

$$G_{\mathsf{mult}}(x) = \frac{1}{\sqrt{2\pi}} x^{-5/2} e^{-x/2}$$

with respect to $\|\cdot\|_{\mathsf{mult},\kappa}$.

The difference in the range of parameters for which each of the above results holds is due to two different restrictions. The upper bound on the allowed κ is due to the choice of the conserved moments, and ensures the finiteness of the respective Laplace-based norm (recall for instance that, $\|g_1(\tau)-g_2(\tau)\|_{\mathsf{const},\kappa} < \infty$ for $\kappa \in (0,2]$ whereas $\|g_1(\tau)-g_2(\tau)\|_{\mathsf{add},\kappa} < \infty$ for $\kappa \in (0,3]$). More interestingly, the lower bound on the range of κ comes from the different behaviour of the semigroup associated with the shifted drift operator $g \mapsto z\partial_z g + 2g$ in the various spaces $\mathbb{Y}_{\mathsf{add}}, \mathbb{Y}_{\mathsf{const}}, \mathbb{Y}_{\mathsf{mult}}$.

1.1. **Organization of the paper.** After this Introduction, Section 2 is devoted to the proof of Theorem 1.3, Section 3 is devoted to the proof of Theorem 1.4 and Section 4 to that of Theorem 1.5.

2. Proof for the constant kernel

Proof of Theorem 1.3. Let n be a solution to (1.1) with constant kernel $K = 2$ and let g be the rescaled solution according to (1.6). It is easy to check that the corresponding Laplace transform

$$N(t,\lambda) = \int_0^\infty \exp(-\lambda\, t) n(t,x)\,\mathrm{d}x, \qquad \lambda \in \mathbb{R}$$

satisfies the equation

$$\partial_t N(t,\lambda) = N^2(t,\lambda) - 2N(t,0)N(t,\lambda), \qquad \lambda \geqslant 0.$$

Taking the limit $\lambda \to 0$ this yields the relation

$$\partial_t N(t,0) = -N^2(t,0), \qquad t \geqslant 0$$

where $N(t,0) = \int_0^\infty n(t,x)\,\mathrm{d}x$ is the moment of order zero. By assumption we have

$$\int_0^\infty n(0,x)\,\mathrm{d}x = \int_0^\infty x n(0,x)\,\mathrm{d}x = 1$$

which yields

$$N(t,0) = \frac{1}{t+1} \qquad \forall t \geqslant 0. \tag{2.1}$$

One directly verifies that

$$\int_0^\infty g(\tau,z)\,\mathrm{d}z = e^\tau \int_0^\infty n(e^\tau - 1, x)\,\mathrm{d}x \quad \text{and} \quad \int_0^\infty z g(\tau,z)\,\mathrm{d}z = \int_0^\infty x n(e^\tau - 1, x)\,\mathrm{d}x \equiv 1.$$

Together with (1.6) and (2.1) this gives

$$\int_0^\infty g(\tau,z)\,\mathrm{d}z = \int_0^\infty z g(\tau,z)\,\mathrm{d}z = 1, \qquad \forall \tau \geqslant 0 \tag{2.2}$$

Denoting

$$U(\tau,\eta) = \int_0^\infty g(\tau,z)\exp(-\eta z)\,\mathrm{d}z.$$

we have the relation

$$U(\tau,\eta) = e^{\tau} N(e^{\tau}-1, \eta e^{-\tau})$$

and U satisfies the equation

$$\partial_\tau U(\tau,\eta) + \eta\,\partial_\eta U(\tau,\eta) + U(\tau,\eta) = U^2(\tau,\eta) \tag{2.3}$$

with initial datum $U_0(\eta) = U(0,\eta)$ while we also exploit that $U(\tau,0) = \int_0^\infty g(\tau,z)\,\mathrm{d}z = 1$ according to (2.2). For two solutions n_1 and n_2 with rescalings g_1 and g_2 and corresponding Laplace transforms U_1 and U_2 we introduce $u(\tau,\eta) = U_1(\tau,\eta) - U_2(\tau,\eta)$ which solves the equation

$$\partial_\tau u(\tau,\eta) + \eta\,\partial_\eta u(\tau,\eta) + u(\tau,\eta) = u(\tau,\eta)\left(U_1(\tau,\eta) + U_2(\tau,\eta)\right) \tag{2.4}$$

We define the semigroup

$$\mathbf{T}_\tau v(\eta) = e^{-\tau} v(\eta e^{-\tau})$$

so that

$$u(\tau,\eta) = \mathbf{T}_\tau u_0(\eta) + \int_0^\tau \mathbf{T}_{\tau-s}\left[u(s,\cdot)\big(U_1(s,\cdot) + U_2(\tau,\cdot)\big)\right](\eta)\,\mathrm{d}s. \tag{2.5}$$

One easily checks that

$$[\![\mathbf{T}_\tau v]\!]_\kappa = \exp(-(1+\kappa)\tau)[\![v]\!]_\kappa$$

and thus, for the solution to (2.5)

$$[\![u(\tau)]\!]_\kappa \leqslant e^{-(1+\kappa)\tau}[\![u_0]\!]_\kappa + \int_0^\tau e^{-(1+\kappa)(\tau-s)}[\![u(s,\cdot)\big(U_1(s,\cdot) + U_2(\tau,\cdot)\big)]\!]_\kappa\,\mathrm{d}s.$$

Since $U_\ell(\tau,\eta) \leqslant U(\tau,0) = 1$ one has

$$|U_1(s,\eta) + U_2(s,\eta)| \leqslant 2 \qquad \forall \eta > 0$$

so that

$$[\![u(s,\cdot)\big(U(s,\cdot) + U_2(s,\cdot)\big)]\!]_\kappa \leqslant 2[\![u(s,\cdot)]\!]_\kappa$$

and therefore

$$[\![u(\tau)]\!]_\kappa \leqslant e^{-(1+\kappa)\tau}[\![u_0]\!]_\kappa + 2\int_0^\tau e^{-(1+\kappa)(\tau-s)}[\![u(s)]\!]_\kappa\,\mathrm{d}s.$$

From Gronwall's lemma applied to $w(t) := [\![u(\tau)]\!]_k e^{(1+\kappa)\tau}$,

$$[\![u(\tau)]\!]_\kappa \leqslant e^{(1-\kappa)t}[\![u(0)]\!]_\kappa$$

so that exponential convergence holds for $\kappa > 1$. Notice that (2.2) yields the relation $u(t,0) = -\partial_\eta u(t,0) = 0$ which ensures that

$$[\![u(t)]\!]_\kappa < \infty \qquad \text{for } \kappa \in (0,2]$$

which gives contractivity for all $\kappa \in (1,2]$ and thus finishes the proof. $\square$

3. Proof for the additive kernel

Proof of Theorem 1.4. Let n be a solution to (1.1) with additive kernel $K(x,y) = x + y$ and g the corresponding rescaled solution according to (1.8) such that $\int_0^\infty xn(t,x)\,\mathrm{d}x = 1$ for all $t \geqslant 0$ (note that mass is conserved). Let us denote by N and U the corresponding desingularised Laplace (Bernstein) transforms, i.e.

$$N(t,\lambda) = \int_0^\infty (1-e^{-x\lambda})n(t,x)\,\mathrm{d}x \quad \text{and} \quad U(\tau,\eta) = \int_0^\infty (1-e^{-\eta z})g(\tau,z)\,\mathrm{d}z, \qquad \eta \in \mathbb{R}.$$

One easily checks that N satisfies the equation

$$\partial_t N(t,\lambda) = -N(t,\lambda) + N(t,\lambda)(\partial_\lambda N)(t,\lambda),$$

since the total mass is normalised to one. In Laplace variables, the rescaling (1.8) translates into $U(\tau,\eta) = e^\tau N(\frac{1}{2}\tau, e^{-\tau}\eta)$ such that U solves

$$\partial_t U(t,\eta) = \tfrac{1}{2}\left[(U-2\eta)\partial_\eta U + U\right] \qquad \eta \geqslant 0.$$

Recall that, due to the choice of the initial condition, the equation (1.8) preserves first and second moments (see (1.10)). Let now n_1 and n_2 be two solutions with corresponding rescalings g_1 and g_2 normalised according to Theorem 1.4 which yields

$$\int_0^\infty zg_1(\tau,z)\,\mathrm{d}z = \int_0^\infty zg_2(\tau,z)\,\mathrm{d}z = 1 \qquad \text{for all } \tau \geqslant 0,$$
$$\int_0^\infty z^2 g_1(\tau,z)\,\mathrm{d}z = \int_0^\infty z^2 g_2(\tau,x)\,\mathrm{d}z = 1 \qquad \text{for all } t \geqslant 0.$$

Let U_1 and U_2 be the associated Bernstein transforms which consequently satisfy

$$U_1(\tau,0) = U_2(\tau,0) = 0 \qquad \text{for all } \tau \geqslant 0,$$
$$\partial_\eta U_1(\tau,0) = \partial_\eta U_2(\tau,0) = 1 \qquad \text{for all } \tau \geqslant 0,$$
$$\partial_\eta^2 U_1(\tau,0) = \partial_\eta^2 U_2(\tau,0) = -1 \qquad \text{for all } t \geqslant 0.$$

Let $u(\tau,\eta) = U_1(\tau,\eta) - U_2(\tau,\eta)$ be the difference of U_1 and U_2 which solves

$$2\partial_\tau u = (U_1 - 2\eta)\partial_\eta u + u\partial_\eta U_2 + u. \tag{3.1}$$

In order to rewrite (3.1) we view it as an equation for u, with coefficients which depend on U_1, U_2. We define the characteristic curves $\tau \mapsto X(\tau;\tau_0,\eta_0)$ as the solution to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}X := -\tfrac{1}{2}(U_1(\tau,X) - 2X), \qquad X(\tau_0) = \eta_0.$$

The solution u to (3.1) can then be written as

$$u(\tau,\eta) = u_0(X(0;\tau,\eta))\exp\left(\frac{1}{2}\int_0^\tau (1+\partial_\eta U_2(s,X(s;\tau,\eta)))\,\mathrm{d}s.\right)$$

Using that

$$\partial_\eta U_2(\tau,\eta) = \int_0^\infty zg_2(\tau,z)e^{-\eta z}\,\mathrm{d}z \leqslant \int_0^\infty zg_2(\tau,z)\,\mathrm{d}z = 1$$

we have

$$u(\tau,\eta) \leqslant u_0(X(0;\tau,\eta))e^{\tau}.$$

Hence,

$$\begin{aligned}\llbracket u(\tau,\cdot)\rrbracket_\kappa &\leqslant e^{\tau}\sup_{\eta>0}\frac{|u_0(X(0;\tau,\eta))|}{|\eta|^\kappa}\\ &= e^{\tau}\sup_{\eta>0}\frac{|u_0(X(0;\tau,\eta))|}{|X(0;\tau,\eta)|^\kappa}\frac{|X(0;\tau,\eta)|^\kappa}{|\eta|^\kappa} \leqslant \exp((1-\frac{1}{2}\kappa)\tau)\llbracket u_0\rrbracket_\kappa,\end{aligned}$$

since

$$|X(0;\tau,\eta)| \leqslant \eta e^{-\tau/2}.$$

This last inequality can be seen as follows: using that

$$U_1(\tau,\eta) = \int_0^\infty (1-\exp(-\eta\, z))g_1(\tau,z)\,\mathrm{d}z \leqslant \int_0^\infty \eta z g_1(\tau,z)\,\mathrm{d}z = \eta$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}s}X(s;\tau,\eta) = -\tfrac{1}{2}(U_1(\tau,X(s;\tau,\eta)) - 2X(s;\tau,\eta)) \geqslant \tfrac{1}{2}X(s;\tau,\eta),$$

and hence

$$X(\tau;\tau,\eta) \geqslant X(0;\tau,\eta)e^{\frac{1}{2}\tau},$$

that is,

$$X(0;\tau,\eta) \leqslant \eta e^{-\frac{1}{2}\tau}.$$

We finally obtain

$$\llbracket u(\tau,\cdot)\rrbracket_\kappa \leqslant \exp((1-\frac{1}{2}\kappa)\tau)\llbracket u_0\rrbracket_\kappa,$$

which gives a contractivity for $2<\kappa<3$. □

4. Proof for the multiplicative kernel

Proof of Theorem 1.5. It is well known (see e.g. Menon and Pego (2004)) that the choice $\int_0^\infty x^2 n(0,x)\,\mathrm{d}x = 1$ fixes the gelation time to $T_* = 1$ and additionally that solutions n_{add} and n_{mult} to Smoluchowski's coagulation equation for the additive and multiplicative kernel respectively are related by the change of variables

$$n_{\mathsf{mult}}(t,x) = \frac{1}{(1-t)x}n_{\mathsf{add}}\left(\log\left(\frac{1}{1-t}\right),x\right).$$

If we switch to self-similar variables the corresponding solutions g_{add} and g_{mult} respectively, satisfy the following relation:

$$g_{\mathsf{mult}}(\tau,z) = \frac{1}{z}g_{\mathsf{add}}(\tau,z).$$

Note that this change also transforms the time domain for g_{mult} to $(0,\infty)$. We thus obtain that $z\, g_{\mathsf{mult}}(\tau,z)$ satisfies (1.9) and consequently the second and third moment are preserved (since the second moment has been chosen to be one).

We denote now by $U(\tau,\eta) = \int_0^\infty (1-e^{-\eta z})zg_{\mathsf{mult}}(z)\,\mathrm{d}z$ and for two solutions $g_{1,\mathsf{mult}}$ and $g_{2,\mathsf{mult}}$ we denote the difference $u(\tau,\eta) = U_1(\tau,\eta) - U_2(\tau,\eta)$. Thus, arguing exactly as for the additive kernel (where now the first moment is replaced by the second one) we again obtain

$$[\![u(\tau,\cdot)]\!]_\kappa \leqslant \exp((1-\tfrac{1}{2}\kappa)\tau)[\![u_0]\!]_\kappa,$$

i.e. a contractivity for $2<\kappa<3$. We also note, that the rate of convergence to the self-similar profile at gelation time in the original time variable $t=1-e^{-\tau}$ is given by $(1-t)^{\kappa-2}$ as $t\to 1=T_*$. □

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