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The role of the teacher in fostering students' evolution across different layers of generalization by means of argumentation

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Abstract

We address students' processes of generalization in early algebra and investigate how the teacher can support them in developing interpretations of non-canonical arithmetic representations, by means of argumentation. Data are constituted by grade 5 students written protocols and excerpts from video-recorded classroom discussions. The analysis is developed on qualitative base, referring to three main aspects: the layers of generalization that emerge in students' semiotic activities, the argumentation, with reference to the criteria of correctness, clearness, and completeness, and the roles played by the teacher to foster students' generalization and argumentation processes. Results point out three specific roles that revealed powerful for fostering students' evolution across different layers of generalization, by means of argumentation: *reflective guide*, *activator of reflective attitudes* and *activator of interpretative processes.*

Key-words: argumentation, early algebra, generalization, multimodality, teacher.

El papel del profesor en el fomento de la evolución de los estudiantes a través de diferentes capas de generalización mediante la argumentación

Resumen

Abordamos los procesos de generalización de los estudiantes en un contexto de álgebra temprana (*early algebra*) e investigamos cómo el profesor puede apoyarlos en el desarrollo de interpretaciones de representaciones aritméticas no canónicas, por medio de la argumentación. Los datos están constituidos por los protocolos escritos de los estudiantes de quinto grado y extractos de las discusiones en clase grabadas en video. El análisis se desarrolla sobre una base cualitativa, refiriéndose a tres aspectos principales: las capas de generalización que surgen en las actividades semióticas de los estudiantes, las argumentaciones, con referencia a los criterios de corrección, claridad e integridad, y los roles que desempeña el profesor para fomentar los procesos de generalización y argumentación de los estudiantes. Los resultados señalan tres roles específicos que revelan ser poderosos para fomentar la evolución de los estudiantes a través de diferentes capas de generalización, por medio de la argumentación: *guía reflexiva, activador de actitudes reflexivas y activador de procesos interpretativos.*

Palabras clave: argumentación, docente, early algebra, generalización, multimodalidad.

O papel do professor no fomento da evolução dos alunos em diferentes camadas de generalização por meio da argumentação

Resumo

Abordamos os processos de generalização dos estudantes num contexto de álgebra precoce *(early algebra)* e investigamos como o professor pode apoiá-los no desenvolvimento de interpretações de representações aritméticas não canônicas, por meio da argumentação. Os dados são constituídos por protocolos escritos pelos alunos da 5ª série e excertos de discussões em sala de aula gravadas em vídeo. A análise é desenvolvida em base qualitativa, referindo-se a três aspectos principais: as camadas de generalização que surgem em atividades semióticas dos alunos, os argumentos, com referência aos critérios de correção, clareza e integridade, e os papéis desempenhados pelo professor para promover a generalização dos alunos e processos de argumentação. Os resultados apontam três papéis específicos que revelaram poderoso para promover a evolução dos alunos através de diferentes camadas de generalização, por meio de argumentação: *guia reflexivo, ativador de atitudes reflexivas e ativador de processos interpretativos.*

Palavras chave: argumentação, early álgebra, generalização, multimodalidade, professor.

1 Introduction

Our study is set in the frame of early algebra, the strand of research (Kaput, Charraher & Blanton, 2008; Cai & Knut, 2011; Kieran et al, 2016) that, starting from the 90s, has investigated how the well-known difficulties associated to the formal aspects of algebra could be overcome through an algebrafication of arithmetic (Kaput and Blanton, 2001), which involves a focus on relational aspects of arithmetic (Linchevski, 1995) and on generational and global-meta-level activities (Kieran, 1996).

Since promoting the new habit of mind that characterizes early algebraic thinking (looking for regularities, conjecturing, expressing generalization, justifying referring to examples…) has to be achieved mainly through classroom interaction (Kieran et al, 2016), in the last 20 years, the focus of attention of research has shifted to the study of how students engage with their classmates and the teacher to elaborate their thinking during working group activities and classroom discussions (Blanton & Kaput, 2008; Cusi, Malara & Navarra, 2011). These studies have highlighted the key role played by the teacher in guiding students in sharing their ideas, understanding each other, challenging ideas and constructing argumentations to support their own conjectures.

The research presented in this paper is aimed at combining these elements, investigating the key role played by the teacher in supporting students' generalization processes.

To study these aspects, we will refer to a frame which is constituted by three main components that will be discussed in the next sections: (1) the development of algebraic thinking through generalization activities; (2) the role played by the teacher in guiding students during classroom discussions; (3) the role played by argumentation as a tool to support the sharing of ideas and the development of generalization.

2 Developing algebraic thinking through generalization activities

Generalizing constitute a core aspect of algebraic thinking (Linchevski, 1995; Kaput & Blanton, 2001; Kaput, 2008). Linchevski (1995) identifies three aspects of generalization and examines the activities through which these aspects could be integrated within early-algebra: (a) pattern generalizing, (b) use of paradigmatic examples, and (c) ratification and refutation of general rules by examples.

In his Theory of Knowledge Objectification, Radford introduces a typology of forms of algebraic thinking based on their level of generality and distinguishes between factual, contextual and standard algebraic thinking (Radford, 2010a).

In factual algebraic thinking formulas consists in pieces of embodied actions. For instance, in a sequence of pointing gestures coordinated with words, which are used by students in a first process of objectification. Algebraic thinking is factual if it operates at the level of particular numbers or facts. For instance, in a pattern generalization, students showing factual algebraic thinking are able to find the number of figure 10 or 100 of a given pattern, by referring to specific features of the pattern (e.g. the numbers of dots in a certain row, in a pattern formed by dots in rows). A certain generalization occurs, because students are able to go beyond the figures given in the pattern, but it can be considered as an elementary layer of generality, in which there is a constant reference to particular figures, like "Figure 100".

> Despite its apparently concrete nature, factual algebraic thinking is not a simple form of mathematical reflection. On the contrary, (…) it rests on highly evolved mechanisms of perception and a sophisticated rhythmic coordination of gestures, words, and symbols. The grasping of the regularity and the imagining of the figures in the course of the generalization results from, and remains anchored in, a profound sensuous mediated process—showing thereby the multi-modal nature of factual algebraic thinking." (ibid., p. 7).

When students go beyond particular figures and deal with a general figure, indeterminacy becomes part of explicit discourse and *contextual algebraic thinking* may appear. Typically, this happens when students are asked to write a message about a "general figure" and, in so doing, to render explicit things that may have remained implicit. Radford found that these written messages are characterized by key descriptive terms (in contrast with previous category, where action was prevalent), such as spatial deictics (e.g. top, bottom, right, left), such as for example: "You have to add one more circle than the number of the figure in the top row, and add one more circle than the top row to the one on the bottom"

When students express formulas using standard algebraic symbolism, they are within the so-called *standard algebraic thinking*. In contrast with previous types, in which students can make recourse to a variety of semiotic modes, here the unique mode of designation of the objects of discourse is the established and artificial algebraic standard symbolism. This drastic reduction of modes of signification is one of the most relevant causes of difficulties for students. Within standard algebraic thinking, Radford distinguishes two relevant cases: naïve induction and formula as a narrative. *Naïve induction* is a form of naïve arithmetic generalization which is related to guessing: it refers to those cases in which students by means of trial and errors guess a formula for a certain regularity (Radford, 2007). This practice cannot be considered algebraic because it is not based on an analytic approach to the indeterminate quantity, rather on a numerical match between a guessed formula and a few observed cases, a match that is hoped to hold for all numbers. *Formula as a narrative* indicate the case in which "the formula is not an abstract symbolic calculating artefact but rather a story that narrates, in a highly condensed manner, the students' mathematical experience" (Radford, 2010a, p. 10). In this case, the formula keeps a strong iconic relationship with the pattern: for instance, if the figures in the pattern are constituted by two rows, students use brackets to designate these two parts, even if these brackets are useless from an algebraic point of view. In this way, one of the strongest points of algebra, namely the detachment from the context in order to signify things in an abstract way, is not so evident.

Other research studies, which share similar theoretical premises on the idea of generalization, have stressed the importance of focusing on students' construction and interpretation of *arithmetic expressions* to foster algebraic generalization. The use of arithmetic expressions, in fact, can be qualified as algebraic when it is aimed not at calculation but at representing a generic example (Mason & Pimm, 1984; Kaput, Blanton & Moreno, 2008), that is when the arithmetic expression "ceases to be an example and gains the status of a generalized sentence even though no letters are used" (Linchevski, 1995, p.116).

Cusi, Malara and Navarra (2011) distinguish between the *canonical representation* of a number (for example, 12) and its *non-canonical representations* (2×6, 3×4, $11 + 1$, $13 - 1$, …), which bring with them further meanings that can be associated to the number (it is even, it is multiple of 3, it the subsequent of 11, it is the antecedent of 13…) and make sense in relation to the context in which the number is introduced: if 12 is, for example, the answer to a problem, a chosen non-canonical representation could be the expression that students construct to communicate the reasoning process that led to that result. According to these authors, fostering students' flexibility in recognizing and interpreting non-canonical representations can constitute the semantic basis for the understanding of symbolic algebraic expressions (such as ab, $c + 2d$, $\frac{e}{f}$, …) and the development of the meaning of equality sign '=' as indicator of equivalence. In fact, when students become flexible in using arithmetic expressions in this way, these expressions could represent generic examples (Mason & Pimm, 1984) for them, since students' attention could be "directed at a level of generality, despite the fact that they are ostensibly working with particular examples" (p.287).

This interpretation of arithmetic expressions as tools to support students in "seeing the general in the particular" (Mason & Pimm, 1984), suggested us to re-interpret Radford's ideas of naïve induction and formula as a narrative also to characterize students' use of non-canonical arithmetic representations of numbers as tools to express generality, when they face activities aimed at fostering generalization. In our interpretation, a non-canonical arithmetic representation of a number could be seen as the result of a naïve induction if students constructed it as the result of the observation of specific cases (and subsequent attempts to find out a formula that matches with the cases). In similar ways, non-canonical arithmetic representations of numbers can be seen as narratives if students are able to explicitly connect the representations with the reasoning process that they developed to find out the expression.

3 The role of the teacher in supporting the development of algebraic thinking

Making students able to shift from iconic formulas to non-canonical arithmetic representations, and to symbolic expressions, endowing formulas with new abstract meanings, is a fundamental didactic challenge for the teacher. The teacher can play a key-role in such a shift and may exploit a vast array of semiotic resources to this purpose, including the embodied ones. In (Radford, 2010a) we find the teacher making apparent for the students a new way of signifying formulas (their simplification) through a subtle coordination of gestures, words, drawings and coloured segments.

In other research studies considering the teacher's multimodal actions—which means including his/her gestures, gazes, and other embodied actions, together with utterances and written representations— the phenomenon of *semiotic game* between teacher and students has been identified (Arzarello & Paola 2007; Arzarello et al. 2009). In a semiotic game, the teacher tunes with the students' semiotic resources (e.g., words and gestures), and uses them to make the mathematical knowledge evolve towards scientifically shared meanings. Typically, the teacher repeats a gesture that one or more students have just made, and accompanies it with appropriate linguistic expressions and explanations, so to support the evolution of new meanings. Such semiotic games can develop if the students produce something meaningful with respect to the problem at hand, using some signs (words, gestures, drawings, etc.).

Other studies focused on the analysis of teachers' interventions during classroom discussions to support students' development of algebraic thinking. Cusi and Malara investigated the main characteristics of an aware and effective approach of a teacher who aims to foster students' development of key-competencies in using algebraic language as a thinking tool (Cusi&Malara, 2009, 2013, 2016). The research led to the elaboration of a theoretical construct – M-AEAB, acronym for "Model of Aware and Effective Attitudes and Behaviors" – which highlights the key-roles played by a teacher who consciously behave with the objective of "making thinking visible" (Collins, Brown & Newman, 1989), in order to guide her/his students in focusing not only on syntactical aspects but also on the effective strategies developed during classroom activities and on the meta-reflections on the actions which are performed.

These key-roles can be subdivided into two main groups. The first group refers to the roles that a teacher could play to pose her-himself not as an "expert", who proposes effective approaches, but rather as a learner, who faces problems with the main aim of making the hidden thinking visible and of sharing the objectives, the meaning of the strategies, and the interpretation of results. The role of *activator of interpretative processes* belongs to this group. This role is played when the teacher asks focused questions to clarify the meaning of expressions (symbolic or not) that students construct and to stimulate a continuous interpretation of the expressions and results that are obtained during the resolution of a problem.

The second group of roles refers to those phases of classroom activities during which the teacher guides her/his students to reflect on the approaches adopted during the activities and to become aware of the relationships between the activities in which they are involved and the knowledge they previously developed. Among them, we mention the roles of *reflective guide* and *activator of reflective attitudes.*

The teacher plays the role of *reflective guide* when she/he repeats parts of students' interventions and/or she/he poses questions with the aim of activating students' argumentative processes to make effective (or problematic) strategies/ways of reasoning explicit and to enable the students to identify, in this way, effective models from which they can draw their inspiration. For example, the teacher could ask to a student to explain to other students his way of reasoning in front of a specific problem, or she/he can ask to another student to explain this way of reasoning, or she/he can repeat/reformulate part of this way of reasoning.

The role of *activator of reflective attitudes* is strictly connected to that of *reflective guide* and is played when the teacher intervenes or makes questions to foster students' reflections on the meaning of the strategies that are discussed and a comparison between different strategies/ways of reasoning. In this way, students could become aware of their thinking processes and of the meaning of the activities in which they are involved.

The construct of M-AEAB is consistent with Mason's characterization of the approach of a teacher who is "mathematical with and in front of learners" (2008), with the aim of educating their awareness. Mason, in fact, suggests that the teacher should initially use particular repeated prompts, then use less and less direct prompts, or meta-questions aimed at helping students internalize these prompts so that they can autonomously refer to these stimuli in specific situations, even when the teacher does not explicitly remind students of them.

The M-AEAB construct proved to be an effective tool to analyze the teacher's roles during activities set within an early-algebraic frame (Cusi & Malara, 2013). In particular, we highlighted the importance of focusing on those roles which can better help students develop a deep awareness of the meaning of the processes within which they are involved.

4 The role of argumentation in developing generalization

In the previous section we have stressed on the importance of planning a teaching approach that supports students in "making their thinking visible". Fostering students' development of argumentative processes, through questions such as "justify your answers" or "explain your reasoning", represents an effective strategy to enable them to make their reasoning explicit. We refer to Stylianides et al.'s (2016) definition of mathematical argumentation as "the discourse or rhetorical means (not necessarily mathematical) used by an individual or a group to convince others that a statement is true or false" (p. 316). We interpret the world "statement" also with the meaning of "answer" (given by a student). Therefore, we focus also on the discourses developed by students to justify their answers, sharing the reasoning processes that led to them.

If reasoning is made explicit, students can share their answers, compare them, reflect on the effectiveness of specific strategies, becoming aware of their ways of reasoning and of the meanings subtended to the approaches proposed by their classmates. Following this line of thought, we considered argumentation as a tool for formative assessment, and investigated the role played by the teacher in activating formative assessment strategies during classroom discussions (Cusi, Morselli & Sabena, 2017, 2018).

During our teaching experiments, classroom discussions were designed starting from the collection and analysis (by the teacher and by a researcher who was in the class) of students' written answers to a given task, in which they were asked also to justify their answers. Then, students' answers were selected and grouped (according, for example, to the kind of argumentations that characterized them) to be displayed (on an IWB). The discussion was, then, focused on these selections of students' answers (for further details about this teaching methodology, see Cusi, Morselli & Sabena, 2017).

Important foci of the analysis collectively developed during these discussions were the characteristics of students' arguments: are they correct? (that is, they do not contain any mathematical mistake); are they clear? (that is, they are understandable by the interlocutors, peers or teacher); are they complete? (that is, they express the main steps that lead from the data to its conclusion).

We are convinced that focusing on argumentation can represent an effective strategy also when the classroom discussion is aimed at promoting the development of different levels of generalization, for two main reasons. On the one side, when students are prompted to justify a formula they constructed, they make their ways of interpreting the formula explicit, therefore they share their objectifications of the formula. On the other side, when the teacher fosters a collective analysis of students' argumentations on a specific formula, students can develop a more advanced interpretation of the formula itself and, therefore, to move forward in the levels of generalization.

5 Research aims and methodology

In this paper we focus on the roles played by the teacher in fostering, during classroom discussions, students' processes of generalization. In particular, we investigate how the teacher can guide the discussion on the argumentations produced by students in order to support them in making their interpretation of non-canonical arithmetic representations or symbolic expressions explicit, and in developing and sharing new interpretations, which may constitute new layers of generality for students.

In order to develop this investigation, we carried out a teaching experiment aimed at fostering the development of algebraic thinking through argumentation, in two 5th grade primary school classes.

During the teaching experiment, the first author was always in the class as both an observer and a participant, to support the teacher in the management of the classroom discussions. All lessons were video-recorded. Other collected data were students' written answers (examples in table 3) and field notes from the observers.

Excerpts from video-recordings were selected by the two authors, who identified episodes that are rich in terms of sharing and analysis of students' argumentation, reflections developed by students, teachers' ability of conducting the discussion in a way that support students' generalization processes. The selected class episodes were transcribed and analysed independently by the two authors, who then compared their analysis and discussed possible problematic codes.

The teaching experiment focused on a sequence of tasks, the "Rivabella parking lot"-Activity, which is our adaptation of an activity within the ArAl Unit "Sequences as functions" (Malara, Navarra & Sini, 2012).

In this paper we will present the analysis of excerpts from a classroom discussion developed in one of the two classes involved in the teaching experiment. The discussion was focused on students' answers to the first task (see section 6) of the activity "The Rivabella parking lot".

The analysis will be developed referring to three main aspects:

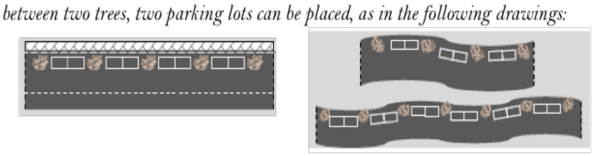
a) The layers of generalization (Radford, 2010a) that emerge in students' semiotic activities;
b) The argumentations that they develop, with reference to the criteria of correctness, clearness, and completeness (Cusi, Morselli & Sabena, 2017 and 2018);
c) The roles played by the teacher to foster students' generalization and argumentation processes, according to the M-AEAB construct (Cusi & Malara 2009, 2013, 2016), and the enacted semiotic means, within a multimodal semiotic perspective (Arzarello et al., 2009).

6 The Rivabella parking lots – Task 1

The "Rivabella parking lots" activity requires: (a) to investigate the relationship between two variables for the analysis of specific cases, (b) to represent this relationship through a symbolic expression; (c) to investigate the inverse relationship; (d) to interpret and use other representations of the same relationships (tabular, graphical, sagittal …).

The Rivabella's town council approves to build some parking lots, according to the following regulation:

Parking lots should be built in this way: in a specific lane, next to the sidewalk, some trees have to be planted. The distance between each pair of trees should be such that, between two trees, two parking lots can be placed, as in the following drawings:



Along a boulevard there are 37 trees, planted at the right distances.
How many parking lots can be built in the boulevard? Justify your answer.

Figure 1: The "Rivabella parking lot", Task 1

Task 1 (see Figure 1) requires students to study a specific sequence in which trees and parking lots alternate according to this kind of general structure: TPPTPPTPPT…TPPT (every sequence starts and ends with a tree).

The main aim of this task is making students to investigate a linear relationship between two variables (the number of trees and the corresponding number of parking lots). This relationship could be modeled through different symbolic expressions, corresponding to different ways of looking at the drawings. If we represent the number of trees with t and the number of parking lots with p, possible symbolic expressions that model the relationship are: $p = 2t - 2$, $p = 2(t - 1)$, $p = t + (t - 2)$.

Roughly speaking, students can face this task in two main ways: (a) looking at the drawings and generalizing the ways in which the number of parking lots can be "counted", or (b) collecting ordered pairs of corresponding values (for example, within a table), and trying to highlight the law through which the number of parking lots can be determined starting from the number of trees.

The first two expressions above can be determined through strategy (a):

$p = 2t - 2$ can be associated to this argumentation: "Since every tree has two parking lots at its

Table 1: Collection of ordered pairs of corresponding values

t	2	3	4	5	6	7	8	…
p	2	4	6	8	10	12	14	

right, except the last one, if we multiply the number of trees by 2, we obtain two more trees, that should be subtracted from the double of the number of trees";

$p = 2(t-1)$ can be justified through this counting strategy: "Since every tree has two parking lots at its right, except the last one, we subtract 1 to the number of trees and double the result".

The third expression, $p = t + (t-2)$, can be determined mainly through strategy (b): if students collect some ordered pairs of corresponding values in a table (Table 1), in fact, they can observe, for example, that the difference between p and t increases in a way that the difference between two corresponding terms is the consecutive of the difference between the previous corresponding terms ($2-2=0$, $4-3=1$, $6-4=2$, $8-5=3$, $10-6=4$, $12-7=5$, $14-8=6$, …).

Students can work with *non-canonical arithmetic representations* (Cusi, Malara & Navarra, 2011) of the numbers of parking lots in order to progressively see "the general in the particular" (Mason & Pimm, 1984), highlighting how the number of parking lots

Table 2: Use of non-canonical arithmetic representations to highlight a relationship between p and t

t	2	3	4	5	6	7	8	…	t
p	2=2+0= =2+(2-2)	4=3+1= =3+(3-2)	6=4+2= =4+(4-2)	8=5+3= =5+(5-2)	10=6+4= =6+(6-2)	12=7+5= =7+(7-2)	14=8+6= =8+(8-2)		p=t+(t-2)

table 3: Some students' written answers to Rivabella task 1and their analysis

Students' written answers (produced in the group-work phase)	**Double-level analysis: generalization and argumentation**
A) *It is possible to insert 72 parking lots because we made the drawing* (they add a drawing with 37 trees and the parking lots between them), *we counted the spaces between the trees and we obtained 36. Then we multiplied 36 × 2 = 72, which are the parking lots.*	The task has been solved by means of a drawing with 37 trees: students simply counted the spaces between the trees, finding that they are 36. Then, they multiplied 36 by 2 to find out the number of parking lots. No process of generalization has been developed, so we cannot speak of algebraic thinking. Argumentation of the numerical result 72 is based on an empirical stance, namely the observation of the drawing; argumentation of why the result is related to 36x2 is missing.
B) *It Is possible to insert 72 parking lots because, if you multiply 37, which are the parking lots, and you multiply it by 2, you find the double, that is 74. But you have to subtract 2 to the number of parking lots because the spaces are delimited by 2 trees.* C) *Initially we made 37x2, but we understood it was a mistake. Then we reasoned on the examples and we observed that, if you multiply the number of trees by 2 and then subtract 2, which are the first and the last trees that delimit the beginning and the end of the car park, the result is the number of parking lots: 37x2=74 74-2=72. To verify our answer, we make another example* (they add the drawing in the case of 4 trees): 4x2=8. 8-2=6.	We comment answers B and C together because they both implicitly refer to the expression 37×2-2. Moreover, both answers highlight a *naïf inductive approach* to the identification of the number of parking lots. In answer C, this approach is more evident because the students explicitly refer to their need of looking at the examples to correct their initial mistake and they propose another example to further verify their answer. It is interesting to notice that the request of justifying their answers has boosted students' interpretation of the meaning of "-2" in relation to a possible general strategy of counting the parking lots. However, both groups' argumentations are incomplete. The authors of answer B grasped that the two "objects" that are subtracted are 2 parking lots, but the final part of their argumentation is unclear and incomplete (in the sentence "the spaces are delimited by 2 trees" it is not clear which are these trees, and it is not explained why two parking lots have to be subtracted). The authors of answer C force their interpretation of "-2" proposing an incorrect argumentation, namely that what is subtracted to the result of 37x2 is the number of the trees that delimit the sequence (the one at the beginning and the one at the end).

can be determined starting from the corresponding number of trees (Table 2).

Also the expressions $p = 2t - 2$ and $p = 2(t - 1)$ could be determined looking at table 1, if students observe that, if we consider a value of the variable t in the table and we double it, we obtain the value of p which belongs to the next column (on the right); moreover, each value of t can be obtained subtracting 1 to the following value of t in the table, while each value of p can be determined subtracting 2 from the following value of p in the table.

Students have to coordinate the different representations that they use to face the problem: tables of data, drawings, verbal expressions, arithmetic expressions, symbolic expressions.

7 Analysis

According to the adopted didactic methodology, first students faced "Rivabella parking lots"-Task 1 within small group work (2-3 students), then the teacher selected some answers and projected them at the IWB, as a basis for the discussion.

All the groups identified the correct answer to the question, that is "we can insert 72 parking lots". Four groups out of nine referred to the expression $(37 - 1) \times 2$, and some of them also proposed good argumentations, with respect to clear-complete-correct criteria. Other four groups referred to the expression $37 \times 2 - 2$. Two of these groups proposed incomplete or partly incorrect argumentations. Finally, one group proposed 36 x 2, without giving a proper argumentation. We focus on these last three cases, for which the teacher's guide becomes crucial. They are presented in Table 2, and they are analyzed according to a double-level: the first level concerns characteristics of the argumentation, the second level regards layers of generalization.

The discussion is guided by the teacher-researcher (in the following, T-R) and is aimed at i) developing in all students a contextual level of argumentation, included those students who did not previously provide any generalization, as in case A discussed above, and ii) providing correct, clear and complete arguments for the claims made in these answers.

Initially the T-R, who takes the role of the teacher, asks students to identify possible common elements of the three displayed written answers. The students agree that all these answers refer to the same strategy, which can be summarized with the expression $37 \times 2 - 2$. The first excerpt we are going to analyze starts with a question that T-R poses to the students that did *not* propose this kind of strategy. For these students, the task consists in developing a *narrative* that can explain the given formula with respect to the given situation. Possibly, they are asked to focus on a way to interpret the drawings, that is different with respect to their previous one.

In the transcripts, besides words we report the gestures performed by the students and by the T-R; underlined words are co-timed with reported gestures.

1 T-R: Those who did not reason in this way, have understood why they did 37x2-2?

2 S: In my opinion it is right to do times 2 minus 2 because if…there are all the trees that…between two trees there are two parking lots, but between the first and the last…between the first and the last one there are not two parking lots

3 T-R: Would you like to come at the whiteboard, so you explain with the drawing, which may be easier?

S agrees to come at the whiteboard to explain her idea.

4 T-R: Let's take one of these drawings and reason on it

5 S: Because…in between these two, for instance (pointings, Fig. 2a and b), there is a…a parking lot (gestures as in Fig. 2c), but between the first (pointing, Fig. 2a) and the last one (pointing, Fig. 2b), here and

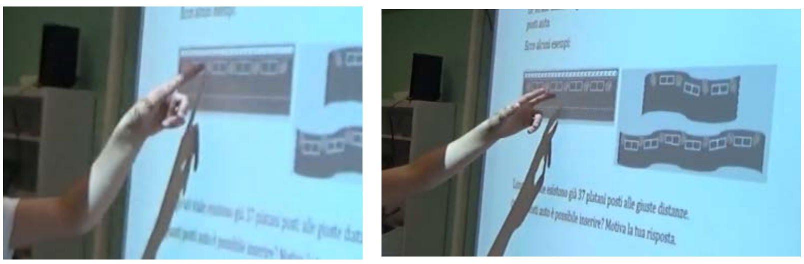


Figure 2 a-b: Pointing to the first and to the last tree in the drawing

Figure 2c: Pointing touching the space between two trees.

here (repeating the same pointing as in Fig. 3a and b), there is no parking lot, so minus 2 (almost whispering)

6 T-R: So, S was saying: "Between the first one and the last one (gesture as in Fig. 4) there is no parking lots, so I have to take away 2! (gesture as in Fig. 5)"

The T-R (line 1) poses a question which represents a typical intervention of a teacher as a *reflective guide*: her aim, in fact, is to make the students to interpret the way of reasoning of their classmates, in order to reason on possible effective strategic approaches to the problem.

In her answer, S (line 2) does not refer to the particular case of 37 trees, but to a general "rule": "times 2, minus 2"): she is able to seize the general structure of the formula. However, her argumentation to justify the strategy, in which she explicitly refers to the role played by the first and last trees, is incomplete.

Therefore, the T-R asks her to come at the IWB (line 3) and suggests her to refer to the drawings (line 3-4) in order to better make her thinking visible and share it with the teacher and the classmates. Again, the T-R is playing the role of *reflective guide*, but also of *activator of interpretative processes*, because her aim is to make S interpret the expression $37 \times 2 - 2$ with reference to the given figures.

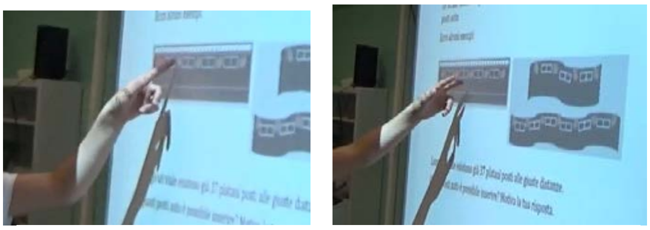


Figure 3 a-b: Quick pointings to two consecutive trees "inside" the drawing

Now S is placed in the condition of exploiting also the drawn figures and while repeating her argument she performs two quick pointing gestures to two consecutive trees "inside" the drawing (Fig 2 a-b): she is considering any pair of consecutive trees of the line, except for the first and the last one. For any such pair of trees, she identifies a space, which she refers to with the words "parking lot" (line 5). From the gesture accompanying her speech (Fig. 2c: her pointing is materially touching the drawing exactly in the line separating the 2 consecutive parking lots), we can see that she is actually meaning the space containing 2 parking lots: through her speech and her gestures, and referring to the given figures, she is establishing the relation between 2 consecutive trees and 1 space between them. Then, she identifies the first and the last tree as somehow different—notice that they are also referred to differently through gestures (Fig 3 a-b), not only through specific words— and so they must not be counted: therefore, they must be subtracted. Even if S is materially referring to a specific drawing (in particular by means of her gestures), she is actually considering a more general case, as it happens in contextual generalizations.

The T-R again activates herself as a *reflective guide* when she repeats, with emphasis, part of S's argument (line 6 and Fig. 4), but she plays also the role of an *activator of reflective attitudes*, because, even if she is not posing questions directly to the other

Figure 4: Pointing with two open hands to the first and to the last tree.

Figure 5: Rotating "fuzzy" gesture

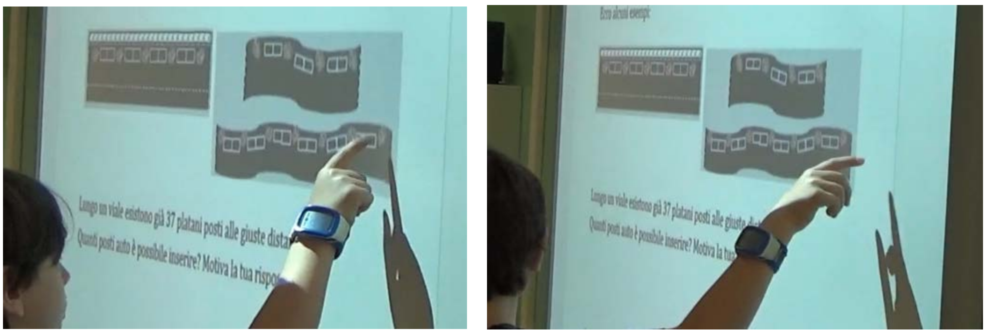


Figure 6a: M is pointing to the last tree.

Figure 6b: M is pointing to the space to the right of the last tree.

students, her aim is to make them focus their attention on S's argument and reflect about its correctness and completeness. By repeating the same words and by giving the same deictic reference with gestures (Fig. 4), the teacher is repeating with emphasis only a part of S's argument, which needs clarification, so to focus the students' attention on it. Her last gesture (Fig 5) is not referred to the figure: it possibly signals that something needs to be clarified in the corresponding part of the argument.

After the intervention of another student, who tries to propose his interpretation and interacts with T-R to explain his reasoning, but then asks for more time to reflect, M asks to intervene and comes to the IWB.

15 M: Well, let's take the same example. Between here and here (quickly pointing to the first and to the second tree) there is a parking lot. We…

16 T-R (interrupt): That is, two parking lots, you mean. A space with two parking lots

17 M: Yes. These are seven (counting the trees)

18 T-R: Seven trees

19 M: Seven trees. But here (pointing as in Fig. 6a) there is not another parking lot (pointing as in Fig. 6b)

20 T-R: You say that to the right of the last tree (pointing to the last tree) there are not parking lots

21 M: Yes

22 T-R: And so?

23 M: We must stop and take 2 away: there is not one at the beginning and not one at the end

24 T-R: Tell me if I understood well. M says: "To the right of every tree there are two parking lots (gestures as in Fig. 7). But if I have 7 trees and I make times 2 (gesture as in Fig. 8), I am counting also those slots that are here (gestures as in Fig. 9), that must be taken away. Did M reason right?

Most of the students declare that M is right, so T-R asks to some of them to propose again a complete argumentation to explain the arithmetic non-canonical expression $37 \times 2 - 2$ (in particular, the role played by "-2" in this expression). After some interventions by his classmates, also M asks to better explain why 2 must be subtracted from 37×2:

29 M: I should always do times 2, but also the last one is times 2, so I am counting also a …parking lot (pointing to the right of the figure, as in Fig. 10) that does not exist

M begins by saying that he will refer to the same example as S (line 25), but indeed he choose another one (to the right, as we can see in Fig. 6): this is another indication that, pushed by the explicit focus on *explaining* the interpretation of the given formula with respect to the figures and giving clear, correct and complete arguments, the students are exploiting the given figures to reason on a general case, and they are shifting from a *contextual* layer of generalization to *provide a narrative to the formula.*

The student then repeats the first part of S's argument, showing that between two consecutive trees there is a space with two parking lots (lines 15-18). The most interesting part is the subsequent one, from line 19: M points to the last tree (Fig. 6a) and highlights that after this tree there no parking lots (Fig. 6b). This explains the "-2" part of the formula,

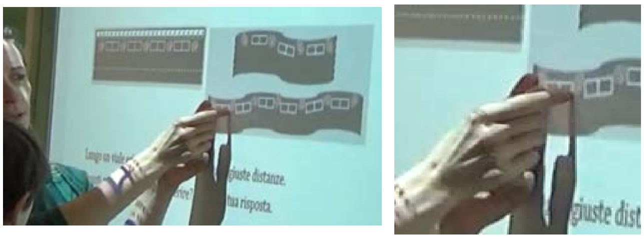


Figure 7: Left hand kept pointed and still on the first tree. Right hand: dynamic pointing to the two parking lots after the first tree.

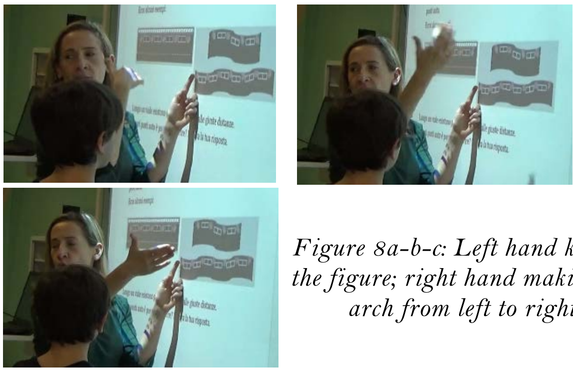


Figure 8a-b-c: Left hand kept to the figure; right hand making an arch from left to right

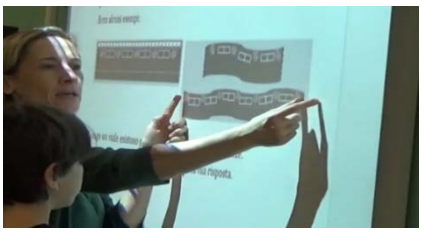


Figure 9: Pointing gesture to the space after the last tree

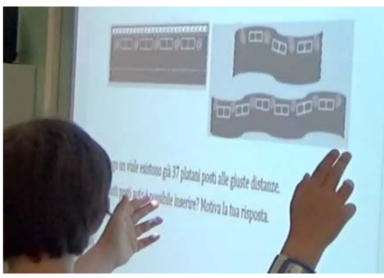


Figure 10: M's is gesturing at the right of the figure, but detached from the IWB

or, as M says, that "we must stop and take 2 away" (line 19). But then M continues and proposes also S's argument for subtracting 2, referring to the first and to the last tree.

With the aim of making students focus on M's argument and reflect on it, in lines 16 and 18 T-R poses herself as an *activator of reflective attitudes*, reformulating parts of M's argument to render it into clear terms. This is evident when, by means of a semiotic game, T-R (line 20) interprets M's gestures, adding an information ("to the right"). Her aim is also to support him in completing his argumentation.

M's reasoning is based on considering two parking lots at the right of every tree, except the last one (line 23).

By means of an interplay between words, gestures and reference to the drawing, the T-R enacts the role of *reflective guide*, making explicit for all students a correct, clear and complete argument for the given formula. Let us analyse in details how this is accomplished. First, she repeats only a (correct) part in M's argument (line 24), introducing the word "every". She uses gestures as well, in synergy with her words: in Figure 8 we can see how she is purposely coordinating the two hands in order to stress the relation between 1 tree (pointed at with the left hand) and 2 parking lots (indicated with the right hand fingers), which provides a narrative for the "37×2" part of the formula, as it is emphasized also in gestures shown in Figure 8. Here, the T-R's right hand is flipping in the air rightwards: this can be interpreted with reference to the mathematical operation "times 2" through possible metaphoric reference to the number line; at the same time, it keeps an iconic link with the way the diagram is being looked at (for any tree, considering the space at its right). Her left hand, kept pointed and still, provides a reference to the modeled situation. With the same emphasis, the T-R is repeating the same right-hand gesture after the last tree (Fig. 9), while her words are slowing down. This repetition of gestures, or 'catchment' as it is called in gesture studies (McNeill et al, 2001), is used by the T-R to highlight a structure which is preserved in the discourse.

Moreover, with her final question (Did M reason right?), she aims at *activating reflective attitudes* in students. As a result, also M feels the need of correcting is previous argumentation (line 29), stressing on the fact that, if 37 is multiplied by 2, two more "imaginary" parking lots are added at the right of the last tree (M speaks about a parking lot that "does not exist"). In this way the level of completeness of M's argumentation has become higher, since he makes the role played by "-2" explicit, with clear reference to the part of the classroom discussion in which the argumentation has been shared.

8 Conclusions

Endowing students with thinking and material tools to address generality in analytical ways is far from being an easy and straightforward endeavor, as well-known to school teachers and from many research studies in mathematics education. Within the Theory of Objectification, Radford underlines the importance of the students' activity in different layers of generality (Radford 2010a) and points to the teacher as a creator of "the conditions of possibility for the students to transform the object of knowledge into an object of consciousness" (Radford 2010b, p. 5). In our research design, this is accomplished through an explicit focus on argumentation and considering the teacher as a model of aware and effective attitudes and behaviors (Cusi & Malara, 2013, 2016). By means of a multimodal semiotic lens (Arzarello et al., 2009) we investigated how the teacher can guide the discussion on the argumentations produced by students in order to support them in making their interpretation of non-canonical arithmetic representations expressions explicit, and in developing and sharing new interpretations, which may constitute new layers of generality for students.

From the corpus of our data, out of which we presented some written protocols and excerpt above, it emerges that asking students to provide their answers with clear, correct and complete argumentations constitutes a fruitful means for them i) to objectify the meanings they associate with each numerical expression, and ii) to make these meaning evolve from the contextual to the standard level of generalization.

On the other hand, such a highly demand on the argumentative level—which requires to take a long time for facing tasks and for the discussions—becomes a powerful tool for the teacher: it is in the tension towards justification (prompted by the explicit focus on argumentation) that emerges a possibility for the teacher to develop the students' contextual discourse towards providing an *explicative narrative for a formula* (see M's case). Furthermore, beside getting information on the students' reasoning that lead to a certain formula, the request of argumentation opens to the teacher the possibility of working within students' Zone of Proximal Development (Vygotsky, 1978) and to guide them towards clear, correct and, in our case, more complete argumentations.

As shown in the excerpts above, we remark three specific roles that revealed powerful for fostering students' evolution across different layers of generalization, by means of argumentation: *reflective guide*, *activator of reflective attitudes*, and *activator of interpretative processes*. This is accomplished by means of an interplay of words, gestures, and the reference to the given drawings. For instance, after S's intervention, we see that the teacher (who in our case was a teacher-researcher) repeats, with emphasis with words and gestures, a certain part of the student's argument (line 6 and Fig. 4), so to focus the students' attention on it and to prompt further clarification. Again, short afterwards (line 24) she repeats only a (correct) part in M's argument, introducing in the discourse the word "every". As we can see in Figure 6, she uses gestures as well, in synergy with her words, purposely coordinating the two hands in order to stress the relation between 1 tree (pointed at with the left hand) and 2 parking lots (indicated with the right hand fingers), which provides a narrative for the "37×2" part of the formula. In the cases of semiotic game as reported in the literature (Arzarello & Paola, 2007; Arzarello et al., 2009), the teacher typically repeats some student's gesture and accompanies it with proper words or mathematical symbols; here, differently from these cases, when assuming the role of *reflective guide* the teacher is *reproducing only partially* students' gestures, and she is *introducing new gestures* in order to push students to reinterpret the drawings in a certain way, so to endow them with a new layer of objectification of the generality of the numerical expression. In this way, through a coordination between words and gestures, the teacher plays another fundamental role to foster students' generalization: the role of *activator of interpretative processes.* The interplay between the roles of reflective guide, activator of reflective attitudes and activator of interpretative processes is synergic, because, fostering students' interpretative processes enable them to share and compare their ways of thinking, while stimulating students' reflections support them in developing different interpretations of the representations on which the discussion is focused.

Furthermore, we remark how in order to convey the idea of something repeating along the entire sequence of trees, the teacher makes use of a catchment, i.e. gestural repetition (Figg. 8 and 9). Catchments have been identified and interpreted by Arzarello and Sabena (2014) as playing a key-role in providing structural cohesion in students' argumentation. In the present study, we could identify a key-role for catchment also in the teacher's hands.

As a final remark, we recognize that in our study the role of the teacher has been assumed by a teacher-researcher, well-acquainted with results from literature on the main theoretical stances underpinning this work—namely layers of generalization, the role of argumentation, the importance of multimodal resources in the teacher-students interaction, and the teacher's models for developing aware and effective attitudes and behaviors. We are convinced about the importance of making teachers aware of the intentionality of their didactic practices, and of the means through which it unfolds in classroom context. We hope to have contributed with some insight into this direction.

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