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## Spacetime Positive Mass Theorems for Initial Data Sets with Non-Compact Boundary

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# SPACETIME POSITIVE MASS THEOREMS FOR INITIAL DATA SETS WITH NONCOMPACT BOUNDARY

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ABSTRACT. In this paper, we define an energy-momentum vector at the spatial infinity of either asymptotically flat or asymptotically hyperbolic initial data sets carrying a noncompact boundary. Under suitable dominant energy conditions imposed both on the interior and along the boundary, we prove the corresponding positive mass inequalities under the assumption that the underlying manifold is spin.

## CONTENTS

1. Introduction	1
2. Statement of results	3
3. The energy-momentum vectors	12
4. Spinors and the Dirac-Witten operator	17
5. The asymptotically flat case	23
6. The asymptotically hyperbolic case	33
7. Appendix: the Killing development of $M$	37
Funding	39
References	39

ABSTRACT. In this paper, we define an energy-momentum vector at the spatial infinity of either asymptotically flat or asymptotically hyperbolic initial data sets carrying a non-compact boundary. Under suitable dominant energy conditions (DECs) imposed both on the interior and along the boundary, we prove the corresponding positive mass inequalities under the assumption that the underlying manifold is spin. In the asymptotically flat case, we also prove a rigidity statement when the energy-momentum vector is light-like. Our treatment aims to underline both the common features and the differences between the asymptotically Euclidean and hyperbolic settings, especially regarding the boundary DECs.

## 1. INTRODUCTION

In General Relativity, positive mass theorems comprise the statement that, under suitable physically motivated energy conditions, the total mass of an isolated gravitational system, as measured at its spatial infinity, is non-negative and vanishes only in case the corresponding initial data set propagates in time to generate the Minkowski space. After the seminal contributions by Schoen-Yau [SY1, SY2,

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SY3] and Witten [Wi], who covered various important cases, the subject has blossomed in a fascinating area of research; see [Bar, PT, BC, CM, XD, Ei, EHLS, SY4, Lo, HL] for a sample of relevant contributions in the asymptotically flat setting. More recently, inspired by potential applications to the Yamabe problem on manifolds with boundary, a variant of the classical positive mass theorem for time-symmetric initial data sets carrying a non-compact boundary has been established in [ABdL], under the assumption that the double of the underlying manifold satisfies the standard (i.e. boundaryless) mass inequality. Hence, in view of the recent progress due to Schoen-Yau [SY4] and Lohkamp [Lo], the positive mass theorem in [ABdL] actually holds in full generality. We also note that an alternative approach to the main result in [ABdL], based on the theory of free boundary minimal hypersurfaces and hence only suited for low dimensions, is presented in [Ch].

Partly motivated by the so-called AdS/CFT correspondence in Quantum Gravity, there has been much interest in proving similar results in case the Minkowskian background is replaced by the anti-de Sitter spacetime. After preliminary contributions in [M-O, AD], the time-symmetric version has been established in [Wa, CH] under the spin assumption. We also refer to [Ma, CMT] for a treatment of the non-time-symmetric case, again in the spin context. Regarding the not necessarily spin case, we should mention the results in [ACG, CD, HJM]. Notice that in this asymptotically hyperbolic setting, the time-symmetric spin case in the presence of a non-compact boundary appears in [AdL]. As a consequence, a rigidity result in the conformally compact Einstein (c.c.E.) setting was proved there, thus extending a celebrated achievement in [AD], which has interesting applications to the classical AdS/CFT correspondence. By its turn, the rigidity result in [AdL, Theorem 1.3], which holds by merely assuming that the inner boundary is *mean convex*, may be of some interest in connection with recent developments involving the construction of a holograph dual of a conformal field theory defined on a manifold with boundary, the so-called AdS/BCFT correspondence [T, CMG, AS], where the problem of determining the structure of the moduli space of c.c.E. manifolds with a given conformal infinity and having a *minimal* inner boundary plays a key role.

The interesting applications of the purely Riemannian positive mass theorems in [ABdL, AdL] mentioned above motivate their extension to the spacetime, non-time-symmetric case. The purpose of this paper is precisely to carry out this formulation in general and to establish the corresponding mass inequalities under the assumption that the manifold underlying the given initial data set is spin; see Theorems 2.6 and 2.15 below. For this, we adapt the well-known Witten's spinorial method which, in each case, provides a formula for the energy-momentum vector in terms of a spinor suitably determined by means of boundary conditions imposed both at infinity and along the non-compact boundary. In particular, since the computations in [ABdL, AdL] are used in the present setting, our choice of boundary conditions along the non-compact boundary in each case matches those already considered there, namely, we employ MIT bag (respectively, chirality) conditions in the asymptotically flat (respectively, asymptotically hyperbolic) case; see also Remark 5.6 in this regard. We remark that extensive discussions of these boundary conditions for Dirac-type operators and their Dirac Laplacians may be found in [AE, Es, Ra, Gi, dL1, dL2]. We also emphasize that a key step in our approach is the selection of suitable dominant energy conditions along the

non-compact boundary which constitute natural extensions of the mean convexity assumption adopted in [ABdL, AdL]. In fact, the search for this kind of energy condition was one of the motivations we had to pursue the investigations reported here. In any case, it is quite fortunate that these energy conditions and the boundary conditions on spinors mentioned above fit together so as to allow for a somewhat unified approach to our main results.

Although we have been able to establish positive mass inequalities in full generality for initial data sets whose underlying manifolds are spin, a natural question that arises is whether this topological assumption may be removed. In the asymptotically flat case, one possible approach to this goal is to adapt, in the presence of the non-compact boundary, the classical technique based on MOTS (marginally outer trapped surfaces). Another promising strategy is to proceed in the spirit of the time-symmetric case treated in [ABdL] and improve the asymptotics in order to apply the standard positive mass inequality to the “double” of the given initial data set. We hope to address those questions elsewhere.

Now we briefly describe the content of this paper. Our main results are Theorems 2.6 and 2.15 which are proved in Sections 5 and 6, respectively. These are rather straightforward consequences of the Witten-type formulae in Theorems 5.4 and 6.9, whose proofs make use of the material on spinors and the Dirac-Witten operator presented in Section 4. We point out that, although the spinor argument only provides rigidity assuming the vanishing of the energy-momentum vector, using a doubling argument we are able to reduce the general case to that one by applying the results in [HL]. Sections 2 and 3 are of an introductory nature, as they contain the asymptotic definition of the energy-momentum vectors and a proof that these objects are indeed geometric invariants of the given initial data set. We also include in Section 2 a motivation for the adopted dominant energy conditions along the non-compact boundary which makes contact with the Hamiltonian formulation of General Relativity.

## 2. STATEMENT OF RESULTS

Let  $n \geq 3$  be an integer and consider  $(\overline{M}^{n+1}, \overline{g})$ , an oriented and time-oriented  $(n+1)$ -dimensional Lorentzian manifold carrying a non-compact, timelike boundary  $\overline{\Sigma}$ . We assume that  $\overline{M}$  carries a spacelike hypersurface  $M$  with non-compact boundary  $\Sigma = \overline{\Sigma} \cap M$ . Also, we suppose that  $M$  meets  $\overline{\Sigma}$  *orthogonally* along  $\Sigma$ ; see Remark 2.2 below. Let  $g = \overline{g}|_M$  be the induced metric and  $h$  be the second fundamental form of the embedding  $M \hookrightarrow \overline{M}$  with respect to the time-like, future directed unit normal vector field  $\mathbf{n}$  along  $M$ . As customary, we assume that  $\overline{g}$  is determined by extremizing the standard Gibbons-Hawking action

$$(1) \quad \overline{g} \mapsto \int_{\overline{M}} (R_{\overline{g}} - 2\Lambda + \mathcal{S}) d\overline{M} + 2 \int_{\overline{\Sigma}} (H_{\overline{g}} + \mathcal{S}) d\overline{\Sigma}.$$

Here,  $R_{\overline{g}}$  is the scalar curvature of  $\overline{g}$ ,  $\Lambda \leq 0$  is the cosmological constant,  $\Pi_{\overline{g}}$  is the second fundamental form of  $\overline{\Sigma}$  in the direction pointing towards  $\overline{M}$ , and  $H_{\overline{g}} = \text{tr}_{\overline{g}} \Pi_{\overline{g}}$  is its mean curvature. As usual, we have added to the purely gravitational action the stress-energy densities describing the non-gravitational contributions which are *independently* prescribed both in the interior of  $\overline{M}$  ( $\mathcal{S}$ ) and along the boundary  $\overline{\Sigma}$  ( $\mathcal{S}$ ). In the following, we often consider an orthonormal frame  $\{e_\alpha\}_{\alpha=0}^n$  along  $M$  which is adapted to the embedding  $M \hookrightarrow \overline{M}$  in the sense that

$e_0 = \mathbf{n}$ . We work with the index ranges

$$0 \leq \alpha, \beta, \dots \leq n, \quad 1 \leq i, j, \dots \leq n, \quad 1 \leq A, B, \dots \leq n-1, \quad 0 \leq a, b, \dots \leq n-1,$$

and the components of the second fundamental form  $h$  of  $M$  in the frame  $\{e_i\}$  are defined by

$$h_{ij} = \bar{g}(\bar{\nabla}_{e_i} e_0, e_j),$$

where  $\bar{\nabla}$  is the Levi-Civita connection of  $\bar{g}$ . Along  $\Sigma$ , we also assume that the frame is adapted in the sense that  $e_n = \varrho$ , where  $\varrho$  is the inward unit normal to  $\Sigma$ , so that  $\{e_A\} \subset T\Sigma$ .

In order to establish positive mass theorems, physical reasoning demands that the initial data set  $(M, g, h, \Sigma)$  should satisfy suitable dominant energy conditions (DECs). In the interior of  $M$ , this is achieved in the usual manner, namely, we consider the *interior constraint map*

$$\Psi_\Lambda(g, h) = 2(\rho_\Lambda(g, h), J(g, h)),$$

where

$$\rho_\Lambda(g, h) = \frac{1}{2}(R_g - 2\Lambda - |h|_g^2 + (\text{tr}_g h)^2), \quad J(g, h) = \text{div}_g h - d\text{tr}_g h$$

and  $R_g$  is the scalar curvature of  $g$ .

**Definition 2.1.** *We say that  $(M, g, h)$  satisfies the interior DEC if*

$$(2) \quad \rho_\Lambda \geq |J|$$

*everywhere along  $M$ .*

As we shall see below, prescribing DECs along  $\Sigma$  is a subtler matter. In the time-symmetric case, which by definition means that  $h = 0$ , the mass inequalities obtained in [ABdL, AdL] confirm that mean convexity of  $\Sigma$  (that is,  $H_g \geq 0$ , where  $H_g$  is the mean curvature of  $\Sigma \hookrightarrow M$  with respect to the inward pointing unit normal vector field  $\varrho$ ) qualifies as the right boundary DEC. In analogy with (2), this clearly suggests that, in the non-time-symmetric case considered here, the appropriate boundary DEC should be expressed by a pointwise lower bound for  $H_g$  in terms of the norm of a vector quantity constructed out of the geometry along  $\Sigma$ , which should vanish whenever  $h = 0$ . However, a possible source of confusion in devising this condition is that the momentum component of the energy-momentum vector, appearing in the positive mass theorems presented below, possesses a manifestly distinct nature depending on whether it comes from asymptotically translational isometries tangent to the boundary if  $\Lambda = 0$ , or asymptotically rotational isometries normal to the boundary if  $\Lambda < 0$ ; see Remark 2.12 below. Despite this difficulty, a reasonably unified approach may be achieved if, for the sake of motivation, we appeal to the so-called Hamiltonian formulation of General Relativity. Recall that, in this setting, the spacetime  $(\bar{M}, \bar{g})$  is constructed by infinitesimally deforming the initial data set  $(M, g, h, \Sigma)$  in a transversal, time-like direction with speed  $\partial_t = V\mathbf{n} + W^i e_i$ , where  $V$  is the lapse function and  $W$  is the shift vector. In terms of these quantities, and since  $M$  is supposed to meet  $\bar{\Sigma}$  orthogonally along  $\Sigma$ , the purely gravitational contribution  $\mathcal{H}_{\text{grav}}$  to the total Hamiltonian at each time slice is given by

$$(3) \quad \frac{1}{2} \mathcal{H}_{\text{grav}}(V, W) = \int_M (V\rho_\Lambda + W^i J_i) dM + \int_\Sigma (VH_g + W^i (\varrho \lrcorner \pi)_i) d\Sigma,$$

where  $\pi := h - (\text{tr}_g h)g$  is the conjugate momentum (also known as the Newton tensor of  $M \hookrightarrow \overline{M}$ ) and we assume for simplicity that  $M$  is compact in order to avoid the appearance of asymptotic terms in (3), which are not relevant for the present discussion. We refer to [HH] for a direct derivation of this formula starting from the action (1); the original argument, which relies on the Hamilton-Jacobi method applied to (1), appears in [BY]. We also mention that (3) may be derived in the framework of the formalism recently described in [HW], a sharpening of the celebrated Iyer-Wald covariant phase space method [IW] which properly takes into account the contributions coming from boundary terms.

Comparison of the interior and boundary integrands in (3) suggests the consideration of the *boundary constraint map*

$$\Phi(g, h) = 2(H_g, \varrho \lrcorner \pi).$$

The key observation now is that if we view  $(V, W)$  as the infinitesimal generator of a symmetry yielding an energy-momentum charge, then the boundary integrand in (3) suggests that the corresponding DEC should somehow select the component of  $\varrho \lrcorner \pi$  aligned with  $W$ . In this regard, we note that  $\varrho \lrcorner \pi$  admits a tangential-normal decomposition with respect to the embedding  $\Sigma \hookrightarrow M$ , namely,

$$\varrho \lrcorner \pi = ((\varrho \lrcorner \pi)^\top, (\varrho \lrcorner \pi)^\perp) = (\pi_{nA}, \pi_{nn}).$$

It turns out that the boundary DEC's employed here explore this natural decomposition. More precisely, as the lower bound for  $H_g$  mentioned above we take the norm  $|(\varrho \lrcorner \pi)^\top|$  of the tangential component if  $\Lambda = 0$  and the norm  $|(\varrho \lrcorner \pi)^\perp|$  of the normal component if  $\Lambda < 0$ ; see Definitions 2.5 and 2.13 below.

**Remark 2.2.** The orthogonality condition along  $\Sigma = M \cap \overline{\Sigma}$  is rather natural from the viewpoint of the Hamilton-Jacobi analysis put forward in [BY]. In fact, as argued there, it takes place for instance when we require that the corresponding Hamiltonian flow evolves the initial data set  $(M, g, h, \Sigma)$  in such a way that the canonical variables are not allowed to propagate across  $\Sigma$ . We also remark that this assumption is automatically satisfied in case the spacetime enjoys the time-symmetry  $t \mapsto -t$ , where  $t = 0$  stands for  $M$ .

For this first part of the discussion, which covers the asymptotically flat case, we assume that  $\Lambda = 0$  in (1). To describe the corresponding reference spacetime, let  $(\mathbb{L}^{n,1}, \overline{\delta})$  be the Minkowski space with coordinates  $X = (x_0, x)$ ,  $x = (x_1, \dots, x_n)$ , endowed with the standard flat metric

$$\langle X, X' \rangle_{\overline{\delta}} = -x_0 x'_0 + x_1 x'_1 + \dots + x_n x'_n.$$

We denote by  $\mathbb{L}_+^{n,1} = \{X \in \mathbb{L}^{n,1}; x_n \geq 0\}$  the *Minkowski half-space*, whose boundary  $\partial \mathbb{L}_+^{n,1}$  is a time-like hypersurface. Notice that  $\mathbb{L}_+^{n,1}$  carries the totally geodesic spacelike hypersurface  $\mathbb{R}_+^n = \{x \in \mathbb{L}_+^{n,1}; x_0 = 0\}$  which is endowed with the standard Euclidean metric  $\delta = \overline{\delta}|_{\mathbb{R}_+^n}$ . Notice that  $\mathbb{R}_+^n$  also carries a totally geodesic boundary  $\partial \mathbb{R}_+^n$ . One aim of this paper is to formulate and prove, under suitable dominant energy conditions and in the spin setting, a positive mass theorem for spacetimes whose spatial infinity is modelled on the embedding  $\mathbb{R}_+^n \hookrightarrow \mathbb{L}_+^{n,1}$ .

We now make precise the requirement that the spatial infinity of  $\overline{M}$ , as observed along the initial data set  $(M, g, h, \Sigma)$ , is modelled on the inclusion  $\mathbb{R}_+^n \hookrightarrow \mathbb{L}_+^{n,1}$ . For large  $r_0 > 0$  set  $\mathbb{R}_{+,r_0}^n = \{x \in \mathbb{R}_+^n; |x| > r_0\}$ , where  $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ .

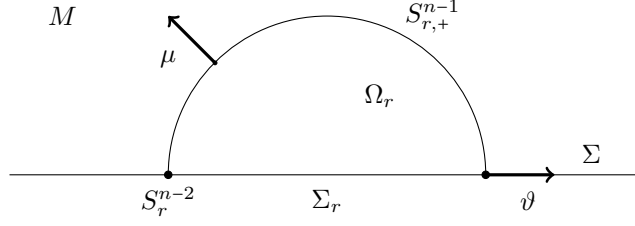


FIGURE 1. An initial data set with non-compact boundary.

**Definition 2.3.** We say that  $(M, g, h, \Sigma)$  is asymptotically flat (with a non-compact boundary  $\Sigma$ ) if there exist  $r_0 > 0$ , a region  $M_{\text{ext}} \subset M$ , with  $M \setminus M_{\text{ext}}$  compact, and a diffeomorphism

$$F : \mathbb{R}_{+,r_0}^n \rightarrow M_{\text{ext}}$$

satisfying the following:

(1) as  $|x| \rightarrow +\infty$ ,

$$(4) \quad |f|_\delta + |x| |\partial f|_\delta + |x|^2 |\partial^2 f|_\delta = O(|x|^{-\tau}),$$

and

$$|h|_\delta + |x| |\partial h|_\delta = O(|x|^{-\tau-1}),$$

where  $\tau > (n-2)/2$ ,  $f := g - \delta$ , and we have identified  $g$  and  $h$  with their pull-backs under  $F$  for simplicity of notation;

(2) there holds

$$(5) \quad \int_M |\Psi_0(g, h)| dM + \int_\Sigma |\Phi^\top(g, h)| d\Sigma < +\infty,$$

where

$$(6) \quad \Phi^\top(g, h) = 2 \begin{pmatrix} H_g \\ (\varrho \lrcorner \pi)^\top \end{pmatrix}.$$

Under these conditions, we may assign to  $(M, g, h, \Sigma)$  an energy-momentum-type asymptotic invariant as follows. Denote by  $S_{r,+}^{n-1}$  the upper hemisphere of radius  $r$  in the asymptotic region,  $\mu$  its outward unit normal vector field (computed with respect to  $\delta$ ),  $S_r^{n-2} = \partial S_{r,+}^{n-1}$  and  $\vartheta = \mu|_{S_r^{n-2}}$  its outward co-normal unit vector field (also computed with respect to  $\delta$ ); see Figure 1.

**Definition 2.4.** Under the conditions of Definition 2.3, the energy-momentum vector of the initial data set  $(M, g, h, \Sigma)$  is the  $n$ -vector  $(E, P)$  given by

$$(7) \quad E = \lim_{r \rightarrow +\infty} \left[ \int_{S_{r,+}^{n-1}} (\text{div}_\delta f - \text{dtr}_\delta f) (\mu) dS_{r,+}^{n-1} + \int_{S_r^{n-2}} f (\partial_{x_n}, \vartheta) dS_r^{n-2} \right],$$

and

$$(8) \quad P_A = \lim_{r \rightarrow +\infty} 2 \int_{S_{r,+}^{n-1}} \pi (\partial_{x_A}, \mu) dS_{r,+}^{n-1}, \quad A = 1, \dots, n-1.$$

If a chart at infinity  $F$  as above is fixed, the energy-momentum vector  $(E, P)$  may be viewed as a linear functional on the vector space  $\mathbb{R} \oplus \mathfrak{K}_\delta^+$ , where elements in the first factor are identified with time-like translations normal to  $\mathbb{R}_+^n \hookrightarrow \mathbb{L}_+^{n,1}$  and

$$(9) \quad \mathfrak{K}_\delta^+ = \left\{ W = \sum_{A=1}^{n-1} a_A \partial_{x_A}; a_A \in \mathbb{R} \right\}$$

corresponds to translational Killing vector fields on  $\mathbb{R}_+^n$  which are tangent to  $\partial\mathbb{R}_+^n$ . Under a change of chart, it will be proved that  $(E, P)$  is well defined (up to composition with an element of  $SO_{n-1,1}$ ); see Corollary 3.4 below. Thus, we may view  $(E, P)$  as an element of the Minkowski space  $\mathbb{L}^{n-1,1}$  at spatial infinity. Theorem 2.6 below determines the causal character of this vector under suitable dominant energy conditions, showing that it is future-directed and causal in case the manifold underlying the initial data set is spin.

**Definition 2.5.** *We say that  $(M, g, h, \Sigma)$  satisfies the tangential boundary DEC if there holds*

$$(10) \quad H_g \geq |(\varrho \lrcorner \pi)^\top|$$

everywhere along  $\Sigma$ .

We may now state our main result in the asymptotically flat setting (for the definition of the weighted Hölder spaces  $C_{-\tau}^{k,\alpha}$ , we refer the reader to [HL]).

**Theorem 2.6.** *Let  $(M, g, h, \Sigma)$  be an asymptotically flat initial data set satisfying the DECs (2) and (10). Assume that  $M$  is spin. Then*

$$E \geq |P|.$$

Furthermore, if  $E = |P|$  and

$$(11) \quad \begin{aligned} g - \delta \in C_{-\tau}^{2,\alpha}(M), \quad h \in C_{-\tau-1}^{1,\alpha}(M) \quad \text{for some } \alpha \in (0, 1), \quad \alpha + \tau > n - 2, \\ \rho_0, J \in C_{-n-\varepsilon}^{0,\alpha}(M), \quad \text{for some } \varepsilon > 0, \end{aligned}$$

then  $E = |P| = 0$  and  $(M, g)$  can be isometrically embedded in  $\mathbb{L}_+^{n,1}$  with  $h$  as the induced second fundamental form. Moreover,  $\Sigma$  is totally geodesic (as a hypersurface in  $M$ ), lies on  $\partial\mathbb{L}_+^{n,1}$  and  $M$  is orthogonal to  $\partial\mathbb{L}_+^{n,1}$  along  $\Sigma$ . In particular,  $h_{nA}$  vanishes on  $\Sigma$ .

In the physically relevant case  $n = 3$ , the spin assumption poses no restriction whatsoever since any oriented 3-manifold is spin. Theorem 2.6 is the natural extension of Witten's celebrated result [Wi, GHHP, PT, He1, D, XD] to our setting, and its time-symmetric version appears in [ABdL, Section 5.2]. The mean convexity condition  $H_g \geq 0$ , which plays a prominent role in [ABdL], is deduced here as an immediate consequence of the boundary DEC (10), thus acquiring a justification on purely *physical* grounds; see also the next remark.

**Remark 2.7.** The DECs (2) and (10) admit a neat interpretation derived from the Lagrangian formulation. Indeed, after extremizing (1) we get the field equations

$$(12) \quad \begin{cases} \text{Ric}_{\bar{g}} - \frac{R_{\bar{g}}}{2} \bar{g} + \Lambda \bar{g} = T, & \text{in } \bar{M}, \\ \text{II}_{\bar{g}} - H_{\bar{g}} \bar{g}|_{\bar{\Sigma}} = S, & \text{on } \bar{\Sigma}. \end{cases}$$



Here,  $\text{Ric}_{\bar{g}}$  is the Ricci tensor of  $\bar{g}$  and  $T$  is the stress-energy tensor in  $\bar{M}$ , which is locally given by

$$T_{\alpha\beta} = \frac{1}{\sqrt{-\det \bar{g}}} \frac{\partial}{\partial \bar{g}^{\alpha\beta}} \left( \sqrt{-\det \bar{g}} \mathcal{L} \right).$$

Also, the boundary stress-energy tensor  $S$  on  $\bar{\Sigma}$  is similarly expressed in terms of  $\mathcal{L}$  and  $\bar{g}|_{\bar{\Sigma}}$ . It is well-known that restriction of the first system of equations in (12) to  $M$  yields the *interior constraint equations*

$$(13) \quad \begin{cases} \rho_\Lambda &= T_{00}, \\ J_i &= T_{0i}, \end{cases}$$

so that (2) is equivalent to saying that the vector  $T_{0\alpha}$  is causal and future directed. On the other hand, if  $\varrho = e_n$  and  $\bar{\varrho}$  are the inward unit normal vectors to  $\Sigma$  and  $\bar{\Sigma}$ , respectively, then the assumption that  $M$  meets  $\bar{\Sigma}$  orthogonally along  $\Sigma$  means that  $\bar{\varrho}|_\Sigma = \varrho$  and  $e_0$  is tangent to  $\bar{\Sigma}$ . We then have

$$\begin{aligned} S_{00} &= (\Pi_{\bar{g}})_{00} + H_{\bar{g}} = (\Pi_{\bar{g}})_{AA} \\ &= \langle \bar{\nabla}_{e_A} e_A, \bar{\varrho} \rangle = \langle \nabla_{e_A} e_A, \varrho \rangle = H_g \end{aligned}$$

and

$$\begin{aligned} S_{0A} &= (\Pi_{\bar{g}})_{0A} = \langle \bar{\nabla}_{e_A} e_0, \bar{\varrho} \rangle \\ &= \langle \bar{\nabla}_{e_A} e_0, \varrho \rangle = h(e_A, \varrho) \\ &= (\varrho \lrcorner h)_A = (\varrho \lrcorner \pi)_A, \end{aligned}$$

where in the last step we used that  $(\varrho \lrcorner g)_A = 0$ . Thus, we conclude that the restriction of the second system of equations in (12) to  $\Sigma$  gives the *boundary constraint equations*

$$(14) \quad \begin{cases} H_g &= S_{00}, \\ (\varrho \lrcorner \pi)_A &= S_{0A}. \end{cases}$$

As a consequence, (10) is equivalent to requiring that the vector  $S_{0a}$  is causal and future directed. We note however that the boundary DEC in the asymptotically hyperbolic case discussed below does not seem to admit a similar interpretation coming from the Lagrangian formalism; see Remark 2.14.

**Remark 2.8.** Similarly to what happens in (13), the boundary constraint equations in (14) seem to relate initial values of fields along  $\Sigma$  instead of determining how fields evolve in time. It remains to investigate the question of whether these boundary constraints play a role in the corresponding initial-boundary value Cauchy problem, so as to provide an alternative to the approaches in [RS, KW].

**Remark 2.9.** One could try a connection with the positive mass theorem with corners in [Sh, Section VI] as follows. Since the causal character of our energy-momentum vector of Definition 2.4 is invariant by the choice of asymptotically flat coordinates, we can choose one that satisfies  $g_{An} = 0$  on  $\Sigma$  for  $A = 1, \dots, n-1$ . In those coordinates, the integral on  $S_r^{n-2}$  vanishes in (2.7). By a doubling argument, we obtain an asymptotically flat manifold  $(\bar{M}, \bar{g})$  in the usual sense, i.e., without boundary and modeled on  $\mathbb{R}^n$  at infinity. Here, the metric  $\bar{g}$ , which is continuous but nonsmooth along  $\Sigma$ , is asymptotically flat in the sense of [LL, Sh]. However, this construction does not lead to a causal corner in the sense of [Sh, Definition 17]. This is because our boundary DEC is weaker than the one in [Sh]

as it does not include the  $n$ -th direction (across  $\Sigma$ ). The point is that the norm of our linear-momentum vector already gives a geometric invariant without including this direction. Although the orthogonal direction may lead to some other geometric invariant, we believe that the first  $n - 1$  components are the minimum required to obtain such geometric invariance.

Now we discuss the case of negative cosmological constant. As already mentioned, a positive mass inequality for time-symmetric asymptotically hyperbolic initial data sets endowed with a non-compact boundary has been proved in [AdL, Theorem 5.4]. Here, we pursue this line of research one step further and present a spacetime version of this result. In particular, we recover the mean convexity assumption along the boundary as an immediate consequence of the suitable boundary DEC.

To proceed, we assume that the initial data set  $(M, g, h, \Sigma)$  is induced by the embedding  $(M, g) \hookrightarrow (\overline{M}, \overline{g})$ , where  $\overline{g}$  extremizes (1) with  $\Lambda = \Lambda_n := -n(n - 1)/2$ . Recall that, using coordinates  $Y = (y_0, y)$ ,  $y = (y_1, \dots, y_n)$ , the *anti-de Sitter space* is the spacetime  $(\text{AdS}_+^{n,1}, \overline{b})$ , where

$$\overline{b} = -(1 + |y|^2)dy_0^2 + b, \quad b = (1 + |y|^2)^{-1}d|y|^2 + |y|^2h_0,$$

$|y| = \sqrt{y_1^2 + \dots + y_n^2}$  and, as usual,  $h_0$  is the standard metric on the unit sphere  $\mathbb{S}^{n-1}$ . Our reference spacetime now is the *AdS half-space*  $\text{AdS}_+^{n,1}$  defined by the requirement  $y_n \geq 0$ . Notice that this space carries a boundary  $\partial\text{AdS}_+^{n,1} = \{Y \in \text{AdS}_+^{n,1}; y_n = 0\}$  which is timelike and totally geodesic. Our aim is to formulate a positive mass inequality for spacetimes whose spatial infinity is modelled on the inclusion  $\mathbb{H}_+^n \hookrightarrow \text{AdS}_+^{n,1}$ , where  $\mathbb{H}_+^n = \{Y \in \text{AdS}_+^{n,1}; y_0 = 0\}$  is the totally geodesic spacelike slice which, as the notation suggests, can be identified with the hyperbolic half-space  $(\mathbb{H}_+^n, b)$  appearing in [AdL].

We now make precise the requirement that the spatial infinity of  $\overline{M}$ , as observed along the initial data set  $(M, g, h, \Sigma)$ , is modelled on the inclusion  $\mathbb{H}_+^n \hookrightarrow \text{AdS}_+^{n,1}$ . For all  $r_0 > 0$  large enough let us set  $\mathbb{H}_{+,r_0}^n = \{y \in \mathbb{H}_+^n; |y| > r_0\}$ .

**Definition 2.10.** *We say that the initial data set  $(M, g, h, \Sigma)$  is asymptotically hyperbolic (with a non-compact boundary  $\Sigma$ ) if there exist  $r_0 > 0$ , a region  $M_{\text{ext}} \subset M$ , with  $M \setminus M_{\text{ext}}$  compact, and a diffeomorphism*

$$F : \mathbb{H}_{+,r_0}^n \rightarrow M_{\text{ext}},$$

satisfying the following:

(1) as  $|y| \rightarrow +\infty$ ,

$$(15) \quad |f|_b + |\nabla_b f|_b + |\nabla_b^2 f|_b = O(|y|^{-\kappa}),$$

and

$$|h|_b + |\nabla_b h|_b = O(|y|^{-\kappa}),$$

where  $\kappa > n/2$ ,  $f := g - b$ , and we have identified  $g$  and  $h$  with their pull-backs under  $F$  for simplicity of notation;

(2) there holds

$$(16) \quad \int_M |y| |\Psi_{\Lambda_n}(g, h)| dM + \int_\Sigma |y| |\Phi^\perp(g, h)| d\Sigma < +\infty,$$

where

$$(17) \quad \Phi^\perp(g, h) = 2 \begin{pmatrix} H_g \\ (\varrho \lrcorner \pi)^\perp \end{pmatrix}$$

and  $|y|$  has been smoothly extended to  $M$ .

Under these conditions, we may assign to  $(M, g, h, \Sigma)$  an energy-momentum-type asymptotic invariant as follows. We essentially keep the previous notation and denote by  $S_{r,+}^{n-1}$  the upper hemisphere of radius  $r$  in the asymptotic region,  $\mu$  its outward unit vector field (computed with respect to  $b$ ),  $S_r^{n-2} = \partial S_{r,+}^{n-1}$  and  $\vartheta = \mu|_{S_r^{n-2}}$  its outward co-normal unit vector field (also computed with respect to  $b$ ). As in [AdL], we consider the space of static potentials

$$\mathcal{N}_b^+ = \left\{ V : \mathbb{H}_+^n \rightarrow \mathbb{R}; \nabla_b^2 V = Vb \text{ in } \mathbb{H}_+^n, \frac{\partial V}{\partial y_n} = 0 \text{ on } \partial\mathbb{H}_+^n \right\}.$$

Thus,  $\mathcal{N}_b^+$  is generated by  $\{V_{(a)}\}_{a=0}^{n-1}$ , where  $V_{(a)} = x_a|_{\mathbb{H}_+^n}$  and here we view  $\mathbb{H}_+^n$  embedded as the upper half hyperboloid in  $\mathbb{L}_+^{n,1}$  endowed with coordinates  $\{x_\alpha\}$ . Notice that  $V = O(|y|)$  as  $|y| \rightarrow +\infty$  for any  $V \in \mathcal{N}_b^+$ . Finally, we denote by  $\mathfrak{Kill}(\mathbb{H}^n)$  ( $\mathfrak{Kill}(\text{AdS}^{n,1})$ , respectively) the Lie algebra of Killing vector fields of  $\mathbb{H}^n$  ( $\text{AdS}^{n,1}$ , respectively) and set  $\mathcal{N}_b = \mathcal{N}_b^+ \oplus [x_n|_{\mathbb{H}^n}]$ . Note the isomorphism  $\mathfrak{Kill}(\text{AdS}^{n,1}) \cong \mathcal{N}_b \oplus \mathfrak{Kill}(\mathbb{H}^n)$ , where each  $V \in \mathcal{N}_b$  is identified with the Killing vector field in  $\text{AdS}_+^{n,1}$  whose restriction to the spacelike slice  $\mathbb{H}_+^n$  is  $V(1 + |y|^2)^{-1/2} \partial_{y_0}$ .

**Definition 2.11.** *The energy-momentum of the asymptotically hyperbolic initial data set  $(M, g, h, \Sigma)$  is the linear functional*

$$\mathbf{m}_{(g,h,F)} : \mathcal{N}_b^+ \oplus \mathfrak{K}_b^+ \rightarrow \mathbb{R}$$

given by

$$(18) \quad \begin{aligned} \mathbf{m}_{(g,h,F)}(V, W) &= \lim_{r \rightarrow +\infty} \left[ \int_{S_{r,+}^{n-1}} \tilde{\mathbb{U}}(V, f)(\mu) dS_{r,+}^{n-1} + \int_{S_r^{n-2}} V f(\varrho_b, \vartheta) dS_r^{n-2} \right] \\ &+ \lim_{r \rightarrow +\infty} 2 \int_{S_{r,+}^{n-1}} \pi(W, \mu) dS_{r,+}^{n-1}, \end{aligned}$$

where  $\varrho_b$  is the inward unit normal vector to  $\partial\mathbb{H}_+^n$ ,

$$(19) \quad \tilde{\mathbb{U}}(V, f) = V(\text{div}_b f - \text{dtr}_b f) - \nabla_b V \lrcorner f + \text{tr}_b f dV,$$

and  $\mathfrak{K}_b^+$  is the subspace of elements of  $\mathfrak{Kill}(\mathbb{H}^n)$  which are orthogonal to  $\partial\mathbb{H}_+^n$ .

**Remark 2.12.** As already pointed out, the energy-momentum invariant in Definition 2.4 may be viewed as a linear functional on the space of *translational* Killing vector fields  $\mathbb{R} \oplus \mathfrak{K}_\delta^+$ ; see the discussion surrounding (9). This should be contrasted to the Killing vector fields in the space  $\mathcal{N}_b^+ \oplus \mathfrak{K}_b^+$  appearing in Definition 2.11, which are *rotational* in nature. Besides, the elements of  $\mathfrak{K}_\delta^+$  are tangent to  $\partial\mathbb{R}_+^n$  whereas those of  $\mathfrak{K}_b^+$  are normal to  $\partial\mathbb{H}_+^n$ . Despite these notable distinctions between the associated asymptotic invariants, it is a remarkable feature of the spinorial approach that the corresponding mass inequalities can be established by quite similar methods.

**Definition 2.13.** *We say that  $(M, g, h, \Sigma)$  satisfies the normal boundary DEC if there holds*

$$(20) \quad H_g \geq |(\varrho \lrcorner \pi)^\perp|$$

everywhere along  $\Sigma$ .

**Remark 2.14.** Differently from what happens to the tangential boundary DEC in Definition 2.5, the requirement in (20), which involves the normal component of  $\varrho \lrcorner \pi$ , does not seem to admit an interpretation in terms of the Lagrangian formulation underlying the field equations (12), the reason being that only the variation of the tangential component of  $\bar{g}$  shows up in the boundary contribution to the variational formula for the action (1). This distinctive aspect of the Lagrangian approach explains why the second system of equations in (12) is explicated solely in terms of tensorial quantities acting on  $T\bar{\Sigma}$ , which leads to the argument in Remark 2.7 and eventually justifies the inclusion of the Hamiltonian motivation for the boundary DEC based on (3).

We now state our main result in the asymptotically hyperbolic case. This extends to our setting a previous result by Maerten [Ma].

**Theorem 2.15.** *Let  $(M, g, h, \Sigma)$  be an asymptotically hyperbolic initial data set as above and assume that the DEC (2) and (20) hold. Assume further that  $M$  is spin. Then there exists a quadratic map*

$$\mathbb{C}^d \xrightarrow{\mathcal{K}} \mathcal{N}_b^+ \oplus \mathfrak{R}_b^+,$$

where  $d = \lfloor \frac{n}{2} \rfloor$ , such that the composition

$$\mathbb{C}^d \xrightarrow{\mathcal{K}} \mathcal{N}_b^+ \oplus \mathfrak{R}_b^+ \xrightarrow{m_{(g,h,F)}} \mathbb{R}$$

is a hermitean quadratic form  $\tilde{\mathcal{K}}$  satisfying  $\tilde{\mathcal{K}} \geq 0$ . Also, if  $\tilde{\mathcal{K}} = 0$  then  $(M, g)$  is isometrically embedded in  $\text{AdS}_+^{n,1}$  with  $h$  as the induced second fundamental form and, besides,  $\Sigma$  is totally geodesic (as a hypersurface in  $M$ ), lies on  $\partial\text{AdS}_+^{n,1}$ ,  $\bar{\Sigma}$  is geodesic (as a hypersurface of  $\bar{M}$ ) in directions tangent to  $\Sigma$ , and  $h_{nA}$  vanishes on  $\Sigma$ .

**Remark 2.16.** The space  $\mathbb{C}^d$  should be viewed as the space of Killing spinors on  $\mathbb{H}^n$ ; see the paragraph above Proposition 6.8.

Differently from its counterpart in [Ma], the mass inequality  $\tilde{\mathcal{K}} \geq 0$  admits a nice geometric interpretation in any dimensions  $n \geq 3$  as follows. Either  $\mathcal{N}_b^+$  and  $\mathfrak{R}_b^+$  can be canonically identified with  $\mathbb{L}^{1,n-1}$  with its inner product

$$(21) \quad \langle\langle z, w \rangle\rangle = z_0 w_0 - z_1 w_1 - \dots - z_{n-1} w_{n-1}.$$

The identification  $\mathcal{N}_b^+ \cong \mathbb{L}^{1,n-1}$  is done as in [CH] by regarding  $\{V_{(a)}\}_{a=0}^{n-1}$  as an orthonormal basis and endowing  $\mathcal{N}_b^+$  with a time orientation by declaring  $V_{(0)}$  as future directed. Then the isometry group of the totally geodesic spacelike slice  $(\mathbb{H}_+^n, b, \partial\mathbb{H}_+^n)$ , which is formed by those isometries of  $\mathbb{H}^n$  preserving  $\partial\mathbb{H}_+^n$ , acts naturally on  $\mathcal{N}_b^+$  in such a way that the Lorentzian metric (21) is preserved (see [AdL]). On the other hand, as we shall see in Proposition 3.5, the identification  $\mathfrak{R}_b^+ \cong \mathbb{L}^{1,n-1}$  is obtained in a similar way. Thus, in the presence of a chart  $F$ , the mass functional  $m_{(g,h,F)}$  may be regarded as a pair of Lorentzian vectors  $(\mathcal{E}, \mathcal{P}) \in \mathbb{L}^{1,n-1} \oplus \mathbb{L}^{1,n-1}$  (see (26)). In terms of this geometric interpretation of the mass functional, Theorem 2.15 may be rephrased as the next result, whose proof also appears in Section 6.

**Theorem 2.17.** *Under the conditions of Theorem 2.15, the vectors  $\mathcal{E}$  and  $\mathcal{P}$ , viewed as elements of  $\mathbb{L}^{1,n-1}$ , are both causal and future directed. Moreover, if both of these vectors vanish then the rigidity statements in Theorem 2.15 hold true.*

**Remark 2.18.** If the initial data set in Theorem 2.15 is time-symmetric then the mass functional reduces to a map  $\mathfrak{m}_{(g,0,F)} : \mathcal{N}_b^+ \rightarrow \mathbb{R}$ . This is precisely the situation studied in [AdL]. Under the corresponding DEC, it follows from [AdL, Theorem 5.4] that  $\mathfrak{m}_{(g,0,F)}$ , viewed as an element of  $\mathcal{N}_b^+$ , is causal and future directed. In other words, there holds the mass inequality  $\langle\langle \mathfrak{m}_{(g,0,F)}, \mathfrak{m}_{(g,0,F)} \rangle\rangle \geq 0$ , which clearly is the time-symmetric version of the conclusion  $\tilde{\mathcal{K}} \geq 0$  in the broader setting of Theorem 2.15. Moreover, if  $\tilde{\mathcal{K}} = 0$  then  $\mathfrak{m}_{(g,0,F)}$  vanishes and the argument in [AdL, Theorem 5.4] implies that  $(M, g, \Sigma)$  is isometric to  $(\mathbb{H}_+^n, b, \partial\mathbb{H}_+^n)$ . We note however that in [AdL] this same isometry is achieved just by assuming that  $\mathfrak{m}_{(g,0,F)}$  is null (that is, lies on the null cone associated to (21)). Anyway, the rigidity statement in Theorem 2.15 implies that  $(M, g, 0, \Sigma)$  is isometrically embedded in  $\text{AdS}_+^{n,1}$ . We then conclude that in the time-symmetric case the assumption  $\tilde{\mathcal{K}} = 0$  actually implies that the embedding  $M \hookrightarrow \overline{M}$  reduces to the totally geodesic embedding  $\mathbb{H}_+^n \hookrightarrow \text{AdS}_+^{n,1}$ .

### 3. THE ENERGY-MOMENTUM VECTORS

In this section we indicate how the asymptotic invariants considered in the previous section are well defined in the appropriate sense.

Let  $(M, g, h, \Sigma)$  be an initial data set either asymptotically flat, as in Definition 2.3, or asymptotically hyperbolic, as in Definition 2.10. In the former case the model will be  $(\mathbb{R}_+^n, \delta, 0, \partial\mathbb{R}_+^n)$ , and in the latter it will be  $(\mathbb{H}_+^n, b, 0, \partial\mathbb{H}_+^n)$ . We will denote these models by  $(\mathbb{E}_+^n, g_0, 0, \partial\mathbb{E}_+^n)$ , so that  $\mathbb{E}_+^n$  stands for either  $\mathbb{R}_+^n$  or  $\mathbb{H}_+^n$ , and  $g_0$  by either  $\delta$  or  $b$ . In particular, Definitions 2.3 and 2.10 ensure that  $f = g - g_0$  has appropriate decay order. Finally, we denote by  $\varrho_{g_0}$  the inward unit normal vector to  $\partial\mathbb{E}_+^n$ .

We define  $\mathcal{N}_{g_0}^+$  as a subspace of the vector space of solutions  $V \in C^\infty(\mathbb{E}_+^n)$  to

$$(22) \quad \begin{cases} \nabla_{g_0}^2 V - (\Delta_{g_0} V)g_0 - V\text{Ric}_{g_0} = 0 & \text{in } \mathbb{E}_+^n, \\ \frac{\partial V}{\partial \varrho_{g_0}} \gamma_0 + V\Pi_{g_0} = 0 & \text{on } \partial\mathbb{E}_+^n, \end{cases}$$

chosen as follows. Set  $\mathcal{N}_b^+$  to be the full space itself and set  $\mathcal{N}_\delta^+$  to be the space of constant functions. So,  $\mathcal{N}_b^+ \cong \mathbb{L}^{1,n-1}$  and  $\mathcal{N}_\delta^+ \cong \mathbb{R}$ .

**Remark 3.1.** Although the second fundamental form  $\Pi_{g_0}$  of  $\partial\mathbb{E}_+^n$  vanishes for either  $g_0 = \delta$  or  $g_0 = b$ , we will keep this term in this section in order to preserve the generality in our calculations.

We define  $\mathfrak{K}_{g_0}^+$  as a subspace of the space of  $g_0$ -Killing vector fields as follows. Choose  $\mathfrak{K}_b^+$  as the subset of all such vector fields of  $\mathbb{H}^n$  which are orthogonal to  $\partial\mathbb{H}_+^n$ . The Killing fields  $\{L_{an}\}_{a=0}^{n-1}$  displayed in the proof of Proposition 3.5 below constitute a basis for  $\mathfrak{K}_b^+$ ; see also Remark 3.6. Choose  $\mathfrak{K}_\delta^+$  as the translations of  $\mathbb{R}^n$  which are tangent to  $\partial\mathbb{R}_+^n$ , so that  $\mathfrak{K}_\delta^+$  is generated by  $\{\partial_{x_A}\}_{A=1}^{n-1}$ .

Let  $F$  be a chart at infinity for  $(M, g, h, \Sigma)$ . As before, we identify  $g$  and  $h$  with their pull-backs by  $F$ , and set  $f = g - g_0$ .

**Definition 3.2.** For  $(V, W) \in \mathcal{N}_{g_0}^+ \oplus \mathfrak{K}_{g_0}^+$  we define the charge density

$$(23) \quad \begin{aligned} \mathbb{U}_{(f,h)}(V, W) = & V \left( \text{div}_{g_0} f - d\text{tr}_{g_0} f \right) - \nabla_{g_0} V \lrcorner f + (\text{tr}_{g_0} f) dV \\ & + 2(W \lrcorner h - (\text{tr}_{g_0} h) W_\flat), \end{aligned}$$

where  $W_b = g_0(\cdot, W)$ , and the energy-momentum functional

$$(24) \quad \mathbf{m}_{(g,h,F)}(V, W) = \lim_{r \rightarrow \infty} \left\{ \int_{S_{r,+}^{n-1}} \mathbb{U}_{(f,h)}(V, W)(\mu) + \int_{S_r^{n-2}} Vf(\varrho_{g_0}, \vartheta) \right\}.$$

**Agreement.** In this section we are omitting the volume elements in the integrals, which are all taken with respect to  $g_0$ .

**Proposition 3.3.** *The limit in (24) exists. In particular, it defines a linear functional*

$$\mathbf{m}_{(g,h,F)} : \mathcal{N}_{g_0}^+ \oplus \mathfrak{K}_{g_0}^+ \rightarrow \mathbb{R}.$$

If  $\tilde{F}$  is another asymptotic coordinate system for  $(M, g, h, \Sigma)$  then

$$(25) \quad \mathbf{m}_{(g,h,\tilde{F})}(V, W) = \mathbf{m}_{(g,h,F)}(V \circ A, A^*W)$$

for some isometry  $A : \mathbb{E}_+^n \rightarrow \mathbb{E}_+^n$  of  $g_0$ .

Before proceeding to the proof of Proposition 3.3, we state two immediate corollaries, one for the asymptotically flat case and the other for the asymptotically hyperbolic one.

In the the asymptotically flat case, observe that the energy-momentum vector  $(E, P)$  of Definition 2.4 is given by

$$E = \mathbf{m}_{(g,h,F)}(1, 0), \quad P_A = \mathbf{m}_{(g,h,F)}(0, \partial_{x_A}), \quad A = 1, \dots, n-1.$$

**Corollary 3.4.** *If  $(M, g, h, \Sigma)$  is an asymptotically flat initial data, then  $(E, P)$  is well defined (up to composition with an element of  $\text{SO}_{n-1,1}$ ). In particular, the causal character of  $(E, P) \in \mathbb{L}^{n-1,1}$  and the quantity*

$$\langle (E, P), (E, P) \rangle = -E^2 + P_1^2 + \dots + P_{n-1}^2$$

do not depend on the chart  $F$  at infinity chosen to compute  $(E, P)$ .

In the asymptotically hyperbolic case, observe that the functional  $\mathbf{m}_{(g,h,F)}$  coincides with the one of Definition 2.11. In order to understand the space  $\mathfrak{K}_b^+$ , we state the following result:

**Proposition 3.5.** *If  $\mathfrak{Kill}(\mathbb{H}^n)$  is the space of Killing vector fields on  $\mathbb{H}^n$ , then there are isomorphisms*

$$\mathfrak{Kill}(\mathbb{H}^n) \cong \mathfrak{K}_b^+ \oplus \mathfrak{Kill}(\mathbb{H}^{n-1}), \quad \mathfrak{K}_b^+ \cong \mathbb{L}^{1,n-1}.$$

Moreover, the space  $\text{Isom}(\mathbb{H}_+^n)$  of isometries of  $\mathbb{H}_+^n$  acts on  $\mathfrak{Kill}(\mathbb{H}^n)$  preserving the decomposition  $\mathfrak{K}_b^+ \oplus \mathfrak{Kill}(\mathbb{H}^{n-1})$ . In particular,  $\text{Isom}(\mathbb{H}_+^n)$  acts on  $\mathfrak{K}_b^+$  by isometries of  $\mathbb{L}^{1,n-1}$ .

*Proof.* Observe that  $\mathfrak{Kill}(\mathbb{H}^n)$  is generated by

$$L_{0j} = x_0 \partial_{x_j} + x_j \partial_{x_0}, \quad L_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i}, \quad i < j,$$

where  $x_0, \dots, x_n$  are the coordinates of  $\mathbb{L}^{n,1}$ , and  $\mathbb{H}^n \hookrightarrow \mathbb{L}^{n,1}$  is represented in the hyperboloid model. Restricting to  $x_n = 0$  we obtain

$$L_{0B}|_{x_n=0} = x_0 \partial_{x_B} + x_B \partial_{x_0}, \quad L_{AB}|_{x_n=0} = x_A \partial_{x_B} - x_B \partial_{x_A}, \quad A < B,$$

and

$$L_{0n}|_{x_n=0} = x_0 \partial_{x_n}, \quad L_{An}|_{x_n=0} = x_A \partial_{x_n}, \quad A, B = 1, \dots, n-1.$$

This shows that  $\{L_{an}|_{\mathbb{H}_+^n}\}_{a=0}^{n-1}$  is a basis for  $\mathfrak{K}_b^+$ . Moreover, since  $V_{(a)} = x_a|_{\mathbb{H}_+^n}$ , we obtain the isomorphisms  $\mathfrak{K}_b^+ \cong \mathcal{N}_b^+ \cong \mathbb{L}^{1,n-1}$ .  $\square$

**Remark 3.6.** We may also provide an explicit basis for  $\mathfrak{K}_b^+$  in terms of the Poincaré half-ball model

$$\mathbb{H}_+^n = \{z = (z_1, \dots, z_n) \in \mathbb{R}^n; |z| < 1, z_n \geq 0\}$$

with boundary  $\partial\mathbb{H}_+^n = \{z \in \mathbb{H}_+^n; z_n = 0\}$ . In this representation,

$$b = \frac{4}{(1 - |z|^2)^2} \delta,$$

and the anti-de Sitter space  $\text{AdS}_+^{n,1} = \mathbb{R} \times \mathbb{H}_+^n$  is endowed with the metric

$$\bar{b} = - \left( \frac{1 + |z|^2}{1 - |z|^2} \right)^2 dz_0^2 + b, \quad z_0 \in \mathbb{R}.$$

It follows that  $\mathfrak{K}_b^+$  is generated by

$$L_{0n} = \frac{1 + |z|^2}{2} \partial_{z_n} - z_n z_j \partial_{z_j}, \quad L_{0n}|_{\partial\mathbb{H}_+^n} = \underbrace{\frac{1 + |z|^2}{1 - |z|^2} e_n}_{=V_{(0)}}$$

and

$$L_{An} = z_A \partial_{z_n} - z_n \partial_{z_A}, \quad L_{An}|_{\partial\mathbb{H}_+^n} = \underbrace{\frac{2z_A}{1 - |z|^2} e_n}_{=V_{(A)}}$$

We define the energy-momentum vector  $(\mathcal{E}, \mathcal{P}) \in \mathcal{N}_b^+ \oplus \mathfrak{K}_b^+ \cong \mathbb{L}^{1,n-1} \oplus \mathbb{L}^{1,n-1}$  by

$$(26) \quad \mathcal{E}_a = \mathbf{m}_{(g,h,F)}(V_{(a)}, 0), \quad \mathcal{P}_a = \mathbf{m}_{(g,h,F)}(0, W_{(a)}),$$

where  $V_{(a)} = x_a|_{\mathbb{H}_+^n}$  and  $W_{(a)} = L_{an}|_{\mathbb{H}_+^n}$ ,  $a = 0, \dots, n-1$ , are the generators of  $\mathcal{N}_b^+$  and  $\mathfrak{K}_b^+$ , respectively.

**Corollary 3.7.** *If  $(M, g, h, \Sigma)$  is an asymptotically hyperbolic initial data, then  $\mathcal{E}$  and  $\mathcal{P}$  are well defined up to composition with an element of  $\text{SO}_{1,n-1}$ . In particular, the causal characters of  $\mathcal{E}$  and  $\mathcal{P}$  and the quantities*

$$\langle \langle \mathcal{E}, \mathcal{E} \rangle \rangle = \mathcal{E}_0^2 - \mathcal{E}_1^2 - \dots - \mathcal{E}_{n-1}^2, \quad \langle \langle \mathcal{P}, \mathcal{P} \rangle \rangle = \mathcal{P}_0^2 - \mathcal{P}_1^2 - \dots - \mathcal{P}_{n-1}^2$$

do not depend on the chart  $F$  at infinity chosen to compute  $(\mathcal{E}, \mathcal{P})$ .

The rest of this section is devoted to the proof of Proposition 3.3. Define the constraint maps  $\Psi_\Lambda$  and  $\Phi$  as in Section 2, namely

$$\Psi_\Lambda : \mathcal{M} \times \mathcal{S}_2(M) \rightarrow C^\infty(M) \times \Gamma(T^*M), \quad \Phi : \mathcal{M} \times \mathcal{S}_2(M) \rightarrow C^\infty(\Sigma) \times \Gamma(T^*M|_\Sigma)$$

$$\Psi_\Lambda(g, h) = \left( \begin{array}{c} R_g - 2\Lambda - |h|_g^2 + (\text{tr}_g h)^2 \\ 2(\text{div}_g h - d\text{tr}_g h) \end{array} \right), \quad \Phi(g, h) = \left( \begin{array}{c} 2H_g \\ 2\rho \lrcorner (h - (\text{tr}_g h)g) \end{array} \right),$$

where  $\mathcal{S}_2(M)$  is the space of symmetric bilinear forms on  $M$ ,  $\mathcal{M} \subset \mathcal{S}_2(M)$  is the cone of (positive definite) metrics and  $\Lambda = 0$  or  $\Lambda = \Lambda_n$  according to the case.

We follow the perturbative approach in [Mi]. For notational simplicity we omit from now on the reference to the cosmological constant in  $\Psi_\Lambda$ . Thus, in the asymptotic region we expand the constraint maps  $(\Psi, \Phi)$  around  $(\mathbb{E}_+^n, g_0, 0, \partial\mathbb{E}_+^n)$  to deduce that

$$(27) \quad \begin{aligned} \Psi(g, h) &= D\Psi_{(g_0,0)}(f, h) + \mathfrak{R}_{(g_0,0)}(f, h), \\ \Phi(g, h) &= D\Phi_{(g_0,0)}(f, h) + \tilde{\mathfrak{R}}_{(g_0,0)}(f, h), \end{aligned}$$

where  $\mathfrak{R}_{(g_0,0)}$  and  $\tilde{\mathfrak{R}}_{(g_0,0)}$  are remainder terms that are at least quadratic in  $(f, h)$  and

$$D\Psi_{(g_0,0)}(f, h) = \left. \frac{d}{dt} \right|_{t=0} \Psi(g_0 + tf, th), \quad D\Phi_{(g_0,0)}(f, h) = \left. \frac{d}{dt} \right|_{t=0} \Phi(g_0 + tf, th).$$

Using formulas for metric variation in [Mi, CEM], we get

$$(28) \quad \begin{aligned} D\Psi_{(g_0,0)}(f, h) &= \begin{pmatrix} \operatorname{div}_{g_0}(\operatorname{div}_{g_0} f - \operatorname{dtr}_{g_0} f) - \langle \operatorname{Ric}_{g_0}, f \rangle_{g_0} \\ 2(\operatorname{div}_{g_0} h - \operatorname{dtr}_{g_0} h) \end{pmatrix} \\ D\Phi_{(g_0,0)}(f, h) &= \begin{pmatrix} (\operatorname{div}_{g_0} f - \operatorname{dtr}_{g_0} f)(\varrho_{g_0}) + \operatorname{div}_{\gamma_0}((\varrho_{g_0} \lrcorner f)^\top) - \langle \Pi_{g_0}, f \rangle_{\gamma_0} \\ 2\varrho_{g_0} \lrcorner (h - (\operatorname{tr}_{g_0} h)g_0) \end{pmatrix}. \end{aligned}$$

As in [Mi], we obtain

$$(29) \quad \langle D\Psi_{(g_0,0)}(f, h), (V, W) \rangle = \operatorname{div}_{g_0}(\mathbb{U}_{(f,h)}(V, W)) + \langle (f, h), \mathcal{F}(V, W) \rangle_{g_0},$$

where  $\mathbb{U}_{(f,h)}(V, W)$  is defined by (23) and

$$(30) \quad \mathcal{F}(V, W) = \begin{pmatrix} \nabla_{g_0}^2 V - (\Delta_{g_0} V)g_0 - V\operatorname{Ric}_{g_0} \\ -\mathcal{L}_W g_0 + 2(\operatorname{div}_{g_0} W)g_0 \end{pmatrix},$$

is the formal adjoint of  $D\Psi_{(g_0,0)}$ .

For large  $r$  we set  $S_{r,+}^{n-1} = \{x \in \mathbb{E}_+^n; |x| = r\}$ , where  $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ , and for  $r' > r$  we define

$$A_{r,r'} = \{x \in \mathbb{E}_+^n; r \leq |x| \leq r'\}, \quad \Sigma_{r,r'} = \{x \in \partial\mathbb{E}_+^n; r \leq |x| \leq r'\}$$

so that

$$\partial A_{r,r'} = S_{r,+}^{n-1} \cup \Sigma_{r,r'} \cup S_{r',+}^{n-1}.$$

We represent by  $\mu$  the outward unit normal vector field to  $S_{r,+}^{n-1}$  or  $S_{r',+}^{n-1}$ , computed with respect to the reference metric  $g_0$ . Also, we consider  $S_{r,+}^{n-2} = \partial S_{r,+}^{n-1} \subset \partial\mathbb{E}_{+,r}^n$ ; see Figure 1. Using (27), (29) and the divergence theorem, we have that

$$\mathcal{A}_{r,r'}(g, h) := \int_{A_{r,r'}} \Psi(g, h)(V, W) + \int_{\Sigma_{r,r'}} \Phi(g, h)(V, W)$$

is given by

$$\begin{aligned} \mathcal{A}_{r,r'}(g, h) &= \int_{S_{r',+}^{n-1}} \mathbb{U}_{(f,h)}(V, W)(\mu) - \int_{S_{r,+}^{n-1}} \mathbb{U}_{(f,h)}(V, W)(\mu) \\ &\quad + \int_{\Sigma_{r,r'}} \left[ \langle D\Phi_{(g_0,0)}(f, h), (V, W) \rangle - \mathbb{U}_{(f,h)}(V, W)(\varrho_{g_0}) \right] \\ &\quad + \int_{A_{r,r'}} \langle (f, h), \mathcal{F}(V, W) \rangle + \int_{A_{r,r'}} \mathfrak{R}_{(g_0,0)}(f, h) \\ &\quad + \int_{\Sigma_{r,r'}} \tilde{\mathfrak{R}}_{(g_0,0)}(f, h). \end{aligned}$$

From (28) and (23), the third integrand in the right-hand side above can be written as

$$f(\varrho_{g_0}, \nabla_{g_0} V - \nabla_{\gamma_0} V) - (\operatorname{tr}_{g_0} f)dV(\varrho_{g_0}) - \langle V\Pi_{g_0}, f \rangle_{\gamma_0} + \operatorname{div}_{\gamma_0}(V(\varrho_{g_0} \lrcorner f)^\top),$$



which clearly equals

$$\langle -dV(\varrho_{g_0})\gamma_0 - V\Pi_{g_0}, f \rangle_{\gamma_0} + \operatorname{div}_{\gamma_0}(V(\varrho_{g_0} \lrcorner f)^\top).$$

Plugging this into the expression for  $\mathcal{A}_{r,r'}$  above and using the divergence theorem, we eventually get

$$\begin{aligned} \mathcal{A}_{r,r'}(g, h) &= \int_{S_{r',+}^{n-1}} \mathbb{U}_{(f,h)}(V, W)(\mu) - \int_{S_{r,+}^{n-1}} \mathbb{U}_{(f,h)}(V, W)(\mu) \\ &\quad + \int_{S_{r'}^{n-2}} Vf(\varrho_{g_0}, \vartheta) - \int_{S_r^{n-2}} Vf(\varrho_{g_0}, \vartheta) \\ &\quad + \int_{A_{r,r'}} \mathfrak{R}_{(g_0,0)}(f, h) + \int_{\Sigma_{r,r'}} \tilde{\mathfrak{R}}_{(g_0,0)}(f, h) \\ &\quad + \int_{A_{r,r'}} \langle (f, h), \mathcal{F}(V, W) \rangle_{g_0} + \int_{\Sigma_{r,r'}} \langle -dV(\varrho_{g_0})\gamma_0 - V\Pi_{g_0}, f \rangle_{\gamma_0}. \end{aligned}$$

The last line vanishes due to (22) and the fact that  $W$  is Killing. Making use of the decay assumptions coming from Definitions 2.3 and 2.10, we are led to

$$(31) \quad \begin{aligned} o_{r,r'}(1) &= \int_{S_{r',+}^{n-1}} \mathbb{U}_{(f,h)}(V, W)(\mu) - \int_{S_{r,+}^{n-1}} \mathbb{U}_{(f,h)}(V, W)(\mu) \\ &\quad + \int_{S_{r'}^{n-2}} Vf(\varrho_{g_0}, \vartheta) - \int_{S_r^{n-2}} Vf(\varrho_{g_0}, \vartheta), \end{aligned}$$

where  $o_{r,r'}(1) \rightarrow 0$  as  $r, r' \rightarrow +\infty$ . The first statement of Proposition 3.3 follows at once.

**Remark 3.8.** It is important to stress that, for (31) to hold, no boundary condition is imposed on the Killing field  $W$ . Indeed, as we shall see later, the requirement that  $W \in \mathfrak{K}_{g_0}^+$  appearing in Definition 3.2 arises as a consequence of the spinor approach, and in particular it is not a necessary condition for the limit defining the energy-momentum vector to exist.

For the proof of (25), we first consider the asymptotically hyperbolic case which is more involved than the asymptotically flat one. The next two results are Lemmas 3.3 and 3.2 in [AdL].

**Lemma 3.9.** *If  $\phi : \mathbb{H}_+^n \rightarrow \mathbb{H}_+^n$  is a diffeomorphism such that*

$$\phi^*b = b + O(|y|^{-\kappa}), \text{ as } |y| \rightarrow \infty,$$

*for some  $\kappa > 0$ , then there exists an isometry  $A$  of  $(\mathbb{H}_+^n, b)$  which preserves  $\partial\mathbb{H}_+^n$  and satisfies*

$$\phi = A + O(|y|^{-\kappa}),$$

*with similar estimate holding for the first order derivatives.*

**Lemma 3.10.** *If  $(V, W) \in \mathcal{N}_b^+ \oplus \mathfrak{K}_b^+$  and  $\zeta$  is a vector field on  $\mathbb{H}_+^n$ , tangent to  $\partial\mathbb{H}_+^n$ , then*

$$\mathbb{U}_{(L_\zeta b, 0)}(V, W) = \operatorname{div}_b \mathbb{V},$$

*with  $\mathbb{V}_{ik} = V(\zeta_{i;k} - \zeta_{k;i}) + 2(\zeta_k V_i - \zeta_i V_k)$ .*

We now follow the lines of [AdL, Theorem 3.4]. Suppose  $F_1$  and  $F_2$  are asymptotic coordinates for  $(M, g, h, \Sigma)$  as in Definition 2.10 and set  $\phi = F_1^{-1} \circ F_2$ . It follows from Lemma 3.9 that  $\phi = A + O(|y|^{-\kappa})$ , for some isometry  $A$  of  $(\mathbb{H}_+^n, b)$ . By composing with  $A^{-1}$ , one can assume that  $A$  is the identity map of  $\mathbb{H}_+^n$ . In particular,  $\phi = \exp \circ \zeta$  for some vector field  $\zeta$  tangent to  $\partial\mathbb{H}_+^n$ . Set

$$f_1 = F_1^* g - b, \quad h_1 = F_1^* h, \quad f_2 = F_2^* g - b, \quad h_2 = F_2^* h.$$

Then

$$f_2 - f_1 = \phi^* F_1^* g - F_1^* g = \phi^*(b + f_1) - (b + f_1) = L_\zeta b + O(|y|^{-2\kappa}).$$

Similarly,  $h_2 - h_1 = O(|y|^{-2\kappa-1})$ . This implies

$$\mathbb{U}_{(f_2, h_2)}(V, W) - \mathbb{U}_{(f_1, h_1)}(V, W) = \mathbb{U}_{(L_\zeta b, 0)}(V, W) + O(|y|^{-2\kappa+1}).$$

By Lemma 3.10 and Stokes' theorem,

$$\begin{aligned} (32) \quad \lim_{r \rightarrow \infty} \int_{S_{r,+}^{n-1}} \mathbb{U}_{(f_2, h_2)}(V, W)(\mu) - \lim_{r \rightarrow \infty} \int_{S_{r,+}^{n-1}} \mathbb{U}_{(f_1, h_1)}(V, W)(\mu) \\ = \lim_{r \rightarrow \infty} \int_{S_{r,+}^{n-1}} \mathbb{U}_{(L_\zeta b, 0)}(V, W)(\mu) \\ = \lim_{r \rightarrow \infty} \int_{\Sigma_r} \mathbb{U}_{(L_\zeta b, 0)}(V, W)(\varrho_b) \end{aligned}$$

where  $\Sigma_r = \{y \in \Sigma; |y| \leq r\}$ . Observe that  $\varrho_b \lrcorner \mathbb{V}$  is tangent to the boundary and set  $\beta = b|_{\partial\mathbb{H}_+^n}$ . Direct computations yield

$$\mathbb{U}_{(L_\zeta b, 0)}(V, W)(\varrho_b) = \operatorname{div}_\beta(\varrho_b \lrcorner \mathbb{V}),$$

so another integration by parts shows that the right-hand side of (32) is

$$\lim_{r \rightarrow \infty} \int_{S_r^{n-2}} \mathbb{V}(\varrho_b, \vartheta) = \lim_{r \rightarrow \infty} \int_{S_r^{n-2}} V(-\zeta_{\alpha;n}) \vartheta_\alpha = \lim_{r \rightarrow \infty} \int_{S_r^{n-2}} V(f_1 - f_2)(\varrho_b, \vartheta).$$

This proves (25) for the asymptotically hyperbolic case. The asymptotically flat one is simpler: Lemma 3.10 is not necessary because  $\mathcal{N}_\delta^+ \cong \mathbb{R}$ , and Lemma 3.9 has a similar version in [ABdL, Proposition 3.9].

#### 4. SPINORS AND THE DIRAC-WITTEN OPERATOR

Here we describe the so-called Dirac-Witten operator, which will play a central role in the proofs of Theorems 2.6 and 2.15. Although most of the material presented in this section is already available in the existing literature (see for instance [CHZ, D, XD] and the references therein) we insist on providing a somewhat detailed account as this will help us to carefully keep track of the boundary terms, which are key for this paper.

We start with a few preliminary algebraic results. As before, let  $(\mathbb{L}^{n,1}, \bar{\delta})$  be the Minkowski space endowed with the standard orthonormal frame  $\{\partial_{x_\alpha}\}_{\alpha=0}^n$ . Let  $\operatorname{Cl}_{n,1}$  be the Clifford algebra of the pair  $(\mathbb{L}^{n,1}, \bar{\delta})$  and  $\mathbb{C}\operatorname{Cl}_{n,1} = \operatorname{Cl}_{n,1} \otimes \mathbb{C}$  its complexification. Thus,  $\mathbb{C}\operatorname{Cl}_{n,1}$  is the unital complex algebra generated over  $\mathbb{L}^{n,1}$  under the Clifford relations:

$$(33) \quad X \cdot X' \cdot \cdot + X' \cdot X \cdot \cdot = -2\langle X, X' \rangle_{\bar{\delta}}, \quad X, X' \in \mathbb{L}^{n,1},$$

where the dot represents Clifford multiplication. Since  $\text{Cl}_{n,1}$  can be explicitly described in terms of matrix algebras, its representation theory is quite easy to understand. In fact, if  $n+1$  is even then  $\text{Cl}_{n,1}$  carries a unique irreducible representation whereas if  $n+1$  is odd then it carries precisely two inequivalent irreducible representations.

Let  $\text{SO}_{n,1}^0$  be the identity component of the subgroup of isometries of  $(\mathbb{L}^{n,1}, \bar{\delta})$  fixing the origin. Passing to its simply connected double cover we obtain a Lie group homomorphism

$$\bar{\chi} : \text{Spin}_{n,1} \rightarrow \text{SO}_{n,1}^0$$

The choice of the time-like unit vector  $\partial_{x_0}$  gives the identification

$$\mathbb{R}^n = \{X \in \mathbb{L}^{n,1}; \langle X, \partial_{x_0} \rangle = 0\}$$

so we obtain a Lie group homomorphism

$$\chi := \bar{\chi}|_{\text{Spin}_n} : \text{Spin}_n \rightarrow \text{SO}_n.$$

Hence,  $\text{Spin}_n$  is the universal double cover of  $\text{SO}_n$ , the rotation group in dimension  $n$ . Summarizing, we have the diagram

$$\begin{array}{ccc} \text{Spin}_n & \xrightarrow{\bar{\gamma}} & \text{Spin}_{n,1} \\ \downarrow \chi & & \downarrow \bar{\chi} \\ \text{SO}_n & \xrightarrow{\gamma} & \text{SO}_{n,1}^0 \end{array}$$

where the horizontal arrows are inclusions and the vertical arrows are two-fold covering maps.

We now recall that  $\text{Spin}_{n,1}$  can be realized as a multiplicative subgroup of  $\text{Cl}_{n,1} \subset \mathbb{C}\ell_{n,1}$ . Thus, by restricting any of the irreducible representations of  $\text{Cl}_{n,1}$  described above we obtain the so-called spin representation  $\bar{\sigma} : \text{Spin}_{n,1} \times S \rightarrow S$ . It turns out that the vector space  $S$  comes with a natural positive definite, hermitian inner product  $\langle \cdot, \cdot \rangle$  satisfying

$$\langle X \cdot \psi, \phi \rangle = \langle \psi, \theta(X) \cdot \phi \rangle,$$

where

$$\theta(a_0 \partial_{x_0} + a_i \partial_{x_i}) = a_0 \partial_{x_0} - a_i \partial_{x_i}.$$

In particular,  $\langle \cdot, \cdot \rangle$  is  $\text{Spin}_n$  but not  $\text{Spin}_{n,1}$ -invariant; see [D]. A way to partly remedy this is to consider another hermitean inner product on  $S$  given by

$$(\psi, \phi) = \langle \partial_{x_0} \cdot \psi, \phi \rangle,$$

which is clearly  $\text{Spin}_{n,1}$ -invariant. But notice that  $(\cdot, \cdot)$  is *not* positive definite. We remark that  $\partial_{x_0} \cdot$  is hermitean with respect to  $\langle \cdot, \cdot \rangle$  whereas  $\partial_{x_i} \cdot$  is skew-hermitean with respect to  $\langle \cdot, \cdot \rangle$ . On the other hand, any  $\partial_{x_\alpha} \cdot$  is hermitean with respect to  $(\cdot, \cdot)$ .

We now work towards globalizing the algebraic picture above. Consider a space-like embedding

$$i : (M^n, g) \hookrightarrow (\bar{M}^{n+1}, \bar{g})$$

endowed with a time-like unit normal vector  $e_0$ . Here,  $(\bar{M}, \bar{g})$  is a Lorentzian manifold. Let  $P_{\text{SO}}(T\bar{M})$  (respectively,  $P_{\text{SO}}(TM)$ ) be the principal  $\text{SO}_{n,1}^0$ - (respectively,  $\text{SO}_n$ -) frame bundle of  $T\bar{N}$  (respectively,  $TM$ ). Also, set

$$\widetilde{P_{\text{SO}}(T\bar{M})} := i^* P_{\text{SO}}(T\bar{M})$$

to be the restricted principal  $\mathrm{SO}_{n,1}^0$ -frame bundle. In order to lift  $\widetilde{P_{\mathrm{SO}}(TN)}$  to a principal  $\mathrm{Spin}_{n,1}$ -bundle we note that the choice of  $e_0$  provides the identification

$$\widetilde{P_{\mathrm{SO}}(TM)} = P_{\mathrm{SO}}(TM) \times_{\gamma} \mathrm{SO}_{n,1}^0.$$

Now, as  $M$  is supposed to be spin, there exists a twofold lift

$$P_{\mathrm{Spin}}(TM) \longrightarrow P_{\mathrm{SO}}(TM),$$

so  $P_{\mathrm{Spin}}(TM)$  is the principal spin bundle of  $TM$ . We then set

$$P_{\mathrm{Spin}}(\widetilde{TM}) := P_{\mathrm{Spin}}(TM) \times_{\overline{\gamma}} \mathrm{Spin}_{n,1},$$

which happens to be the desired lift of  $\widetilde{P_{\mathrm{SO}}(TM)}$ . The corresponding *restricted* spin bundle is defined by means of the standard associated bundle construction, namely,

$$\mathbb{S}_M := P_{\mathrm{Spin}}(\widetilde{TM}) \times_{\overline{\sigma}} S.$$

This comes endowed with the hermitean metric  $(\cdot, \cdot)$  and a compatible connection  $\overline{\nabla}$  (which is induced by the extrinsic Levi-Civita connection  $\overline{\nabla}$  of  $(\overline{M}, \overline{g})$ ). Finally, we also have

$$\mathbb{S}_M = P_{\mathrm{Spin}}(TM) \times_{\sigma} S,$$

where  $\sigma = \overline{\sigma}|_{\mathrm{Spin}_n}$ . Hence,  $\mathbb{S}_M$  is also endowed with the metric  $(\cdot, \cdot)$  and a compatible connection  $\nabla$  (which is induced by the intrinsic Levi-Civita connection  $\nabla$  of  $(M, g)$ ) satisfying

$$(34) \quad \nabla_{e_i} = e_i + \frac{1}{4} \Gamma_{ij}^k e_j \cdot e_k.$$

In particular,  $\overline{\nabla}$  is compatible with  $(\cdot, \cdot)$  but *not* with  $\langle \cdot, \cdot \rangle$ . We finally remark that in terms of an adapted frame  $\{e_\alpha\}$  there holds

$$(35) \quad \overline{\nabla}_{e_i} = \nabla_{e_i} - \frac{1}{2} h_{ij} e_0 \cdot e_j,$$

where  $h_{ij} = g(\overline{\nabla}_i e_0, e_j)$  are the components of the second fundamental form. This is the so-called *spinorial Gauss formula*.

We are now ready to introduce the main character of our story.

**Definition 4.1.** *The Dirac-Witten operator  $\overline{\mathcal{D}}$  is defined by the composition*

$$\Gamma(\mathbb{S}_M) \xrightarrow{\overline{\nabla}} \Gamma(TM \otimes \mathbb{S}_M) \xrightarrow{\cdot} \Gamma(\mathbb{S}_M).$$

Locally,

$$\overline{\mathcal{D}} = e_i \cdot \overline{\nabla}_{e_i}.$$

The key point here is that  $\overline{\mathcal{D}}$  has the same symbol as the intrinsic Dirac operator  $\mathcal{D} = e_i \cdot \nabla_{e_i}$  but in its definition  $\overline{\nabla}$  is used instead of  $\nabla$ .

Usually we view  $\overline{\mathcal{D}}$  as acting on spinors satisfying a suitable boundary condition along  $\Sigma$ .

In what follows we discuss the one to be used for the asymptotically flat case. The one for the asymptotically hyperbolic will be discussed in Section 6.

**Definition 4.2.** Let  $\omega = i\rho \cdot$  be the (pointwise) hermitean involution acting on  $\Gamma(\mathbb{S}_M|_\Sigma)$ . We say that a spinor  $\psi \in \Gamma(\mathbb{S}_M)$  satisfies the MIT bag boundary condition if any of the identities

$$(36) \quad \omega\psi = \pm\psi$$

holds along  $\Sigma$ .

**Remark 4.3.** Note that a spinor  $\psi$  satisfying a MIT bag boundary condition also enjoys  $(\rho \cdot \psi, \psi) = 0$  on  $\Sigma$ . Indeed, the fact that  $\rho \cdot$  is Hermitian for  $(\cdot, \cdot)$  implies

$$(\rho \cdot \psi, \psi) = (\rho \cdot (\pm i\rho \cdot \psi), \pm i\rho \cdot \psi) = -(\psi, \rho \cdot \psi) = -(\rho \cdot \psi, \psi).$$

**Proposition 4.4.** Let  $\overline{\mathcal{D}}_\pm$  be the Dirac-Witten operator acting on spinors satisfying the MIT bag boundary condition (36). Then  $\overline{\mathcal{D}}_+$  and  $\overline{\mathcal{D}}_-$  are adjoints to each other with respect to  $\langle \cdot, \cdot \rangle$ .

*Proof.* We use a local frame such that  $\nabla_{e_i} e_j|_p = 0$  for a given  $p \in M$ . It is easy to check that at this point we also have  $\overline{\nabla}_{e_i} e_j = h_{ij} e_0$  and  $\overline{\nabla}_{e_i} e_0 = h_{ij} e_j$ . Now, given spinors  $\phi, \xi \in \Gamma(\mathbb{S}_M)$ , consider the  $(n-1)$ -form

$$\widehat{\theta} = \langle e_i \cdot \phi, \xi \rangle e_i \lrcorner dM.$$

Thus,

$$d\widehat{\theta} = e_i \langle e_i \cdot \phi, \xi \rangle dM = e_i (e_0 \cdot e_i \cdot \phi, \xi) dM = e_i (e_i \cdot \phi, e_0 \cdot \xi) dM.$$

Using that  $\overline{\nabla}$  is compatible with  $(\cdot, \cdot)$ , we have

$$\begin{aligned} d\widehat{\theta} &= ((\overline{\nabla}_{e_i} e_i \cdot \phi, e_0 \cdot \xi) + (e_i \cdot \overline{\nabla}_{e_i} \phi, e_0 \cdot \xi) \\ &\quad + (e_i \cdot \phi, \overline{\nabla}_{e_i} e_0 \cdot \xi) + (e_i \cdot \phi, e_0 \cdot \overline{\nabla}_{e_i} \xi)) dM \\ &= (h_{ii} (e_0 \cdot \phi, e_0 \cdot \xi) + (\overline{\mathcal{D}}\phi, e_0 \cdot \xi) \\ &\quad + h_{ij} (e_i \cdot \phi, e_j \cdot \xi) - (e_0 \cdot \phi, e_i \cdot \overline{\nabla}_{e_i} \xi)) dM \\ &= (h_{ii} (\phi, \xi) + (e_0 \cdot \overline{\mathcal{D}}\phi, \xi) \\ &\quad + h_{ij} (e_j \cdot e_i \cdot \phi, \xi) - (e_0 \cdot \phi, \overline{\mathcal{D}}\xi)) dM. \end{aligned}$$

Now, the first and third terms cancel out due to the Clifford relations (33) so we end up with

$$(37) \quad d\widehat{\theta} = (\langle \overline{\mathcal{D}}\phi, \xi \rangle - \langle \phi, \overline{\mathcal{D}}\xi \rangle) dM.$$

Hence, assuming that  $\phi$  and  $\psi$  are compactly supported we get

$$\begin{aligned} \int_M \langle \overline{\mathcal{D}}\phi, \xi \rangle dM - \int_M \langle \phi, \overline{\mathcal{D}}\xi \rangle dM &= \int_\Sigma \langle e_i \cdot \phi, \xi \rangle e_i \lrcorner dM \\ &= \int_\Sigma \langle \rho \cdot \phi, \xi \rangle d\Sigma. \end{aligned}$$

where we used an adapted frame such that  $e_n = \rho$ . If  $\omega\phi = \phi$  and  $\omega\xi = -\xi$  we have

$$\begin{aligned} \langle \rho \cdot \phi, \xi \rangle &= \langle \rho \cdot (i\rho \cdot \phi), -i\rho \cdot \xi \rangle \\ &= \langle \phi, \rho \cdot \xi \rangle \\ &= -\langle \rho \cdot \phi, \xi \rangle, \end{aligned}$$

that is,  $\langle \rho \cdot \phi, \xi \rangle = 0$ . □

**Proposition 4.5.** *Given a spinor  $\psi \in \Gamma(\mathbb{S}_M)$ , define the  $(n-1)$ -forms*

$$\theta = \langle e_i \cdot \overline{\mathcal{D}}\psi, \psi \rangle e_i \lrcorner dM, \quad \eta = \langle \overline{\nabla}_{e_i}\psi, \psi \rangle e_i \lrcorner dM,$$

then

$$(38) \quad d\theta = \left( \langle \overline{\mathcal{D}}^2\psi, \psi \rangle - |\overline{\mathcal{D}}\psi|^2 \right) dM,$$

and

$$(39) \quad d\eta = \left( -\langle \overline{\nabla}^*\overline{\nabla}\psi, \psi \rangle + |\overline{\nabla}\psi|^2 \right) dM,$$

where  $\overline{\nabla}^*\overline{\nabla} = \overline{\nabla}_{e_i}^*\overline{\nabla}_{e_i}$  is the Bochner Laplacian acting on spinors. Here,

$$\overline{\nabla}_{e_i}^* = -\overline{\nabla}_{e_i} + h_{ij}e_j \cdot e_0.$$

is the formal adjoint of  $\overline{\nabla}_{e_i}$  with respect to  $\langle \cdot, \cdot \rangle$ .

*Proof.* By setting  $\phi = \overline{\mathcal{D}}\psi$  and  $\xi = \psi$  in (37), (38) follows. To prove (39) we note that

$$d\eta = e_i \langle \overline{\nabla}_{e_i}\psi, \psi \rangle dM = e_i (e_0 \cdot \overline{\nabla}_{e_i}\psi, \psi) dM = e_i (\overline{\nabla}_{e_i}\psi, e_0 \cdot \psi) dM,$$

so that

$$\begin{aligned} d\eta &= ((\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\psi, e_0 \cdot \psi) + (\overline{\nabla}_{e_i}\psi, \overline{\nabla}_{e_i}e_0 \cdot \psi) + (\overline{\nabla}_{e_i}\psi, e_0 \cdot \overline{\nabla}_{e_i}\psi)) dM \\ &= ((e_0 \cdot \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\psi, \psi) + h_{ij}(\overline{\nabla}_{e_i}\psi, e_j \cdot \psi) + (e_0 \cdot \overline{\nabla}_{e_i}\psi, \overline{\nabla}_{e_i}\psi)) dM. \end{aligned}$$

The term in the middle equals

$$h_{ij}(e_j \cdot \overline{\nabla}_{e_i}\psi, \psi) = h_{ij}\langle e_0 \cdot e_j \cdot \overline{\nabla}_{e_i}\psi, \psi \rangle = -h_{ij}\langle e_j \cdot e_0 \cdot \overline{\nabla}_{e_i}\psi, \psi \rangle,$$

so we end up with

$$d\eta = \left( \underbrace{\langle (-\overline{\nabla}_{e_i} + h_{ij}e_j \cdot e_0) \overline{\nabla}_{e_i}\psi, \psi \rangle}_{\overline{\nabla}_{e_i}^*} + |\overline{\nabla}\psi|^2 \right) d\text{vol},$$

which completes the proof of (39).  $\square$

Another key ingredient is the following Weitzenböck-Lichnerowicz formula.

**Proposition 4.6.** *One has*

$$(40) \quad \overline{\mathcal{D}}^2 = \overline{\nabla}^*\overline{\nabla} + \mathcal{R},$$

where the symmetric endomorphism  $\mathcal{R}$  is given by

$$\mathcal{R} = \frac{1}{4}(R_{\overline{g}} + 2\text{Ric}_{\overline{g}0\alpha}e_\alpha \cdot e_0).$$

*Proof.* See [PT].  $\square$

**Remark 4.7.** We recall that the DEC (2.1) with  $\Lambda = 0$  implies that  $\mathcal{R} \geq 0$ . Indeed, a simple computation shows that

$$\mathcal{R} = \frac{1}{2}(\rho_0 + J_i e_i \cdot e_0).$$

Hence, if  $\phi$  is an eigenvector of  $\mathcal{R}$  then it is an eigenvector of  $\mathcal{J} := J_i e_i \cdot e_0$  as well, say with eigenvalue equal to  $\lambda$ . Thus,

$$\begin{aligned} \lambda^2 |\phi|^2 &= \langle J_i e_i \cdot e_0 \cdot \phi, J_j e_j \cdot e_0 \cdot \phi \rangle \\ &= J_i J_j \langle e_i \cdot \phi, e_j \cdot \phi \rangle \\ &= -J_i J_j \langle e_j \cdot e_i \cdot \phi, \phi \rangle \\ &= \left( \sum_i J_i^2 \right) |\phi|^2, \end{aligned}$$

so that  $\lambda = \pm |J|$ . The claim follows.

By putting together the results above, and using a standard approximation procedure, we obtain a fundamental integration by parts formula which will play a key role in the proof of Theorem 2.6. Hereafter,  $H^k(\mathbb{S}_M)$  denotes the Sobolev space of spinors whose derivatives up to order  $k$  are in  $L^2$ . We refer to [ABdL, GN] for its definition and basic properties.

**Proposition 4.8.** *If  $\psi \in H_{\text{loc}}^2(\mathbb{S}_M)$  and  $\Omega \subset M$  is compact then*

$$(41) \quad \int_{\Omega} (|\bar{\nabla}\psi|^2 + \langle \mathcal{R}\psi, \psi \rangle - |\bar{\mathcal{D}}\psi|^2) dM = \int_{\partial\Omega} \langle (\bar{\nabla}_{e_i} + e_i \cdot \bar{\mathcal{D}})\psi, \psi \rangle e_i \lrcorner dM.$$

The next proposition describes how the operator in the right-hand side of (41) decomposes into its intrinsic and extrinsic components. This is a key step toward simplifying our approach to Theorems 2.6 and 2.15 as it allows us to make use of the ‘‘intrinsic’’ computations in [ABdL, AdL].

**Proposition 4.9.** *One has*

$$(42) \quad \bar{\nabla}_{e_i} + e_i \cdot \bar{\mathcal{D}} = \nabla_{e_i} + e_i \cdot \mathcal{D} - \frac{1}{2} \pi_{ij} e_0 \cdot e_j,$$

where  $\mathcal{D} = e_j \cdot \nabla_{e_j}$  is the intrinsic Dirac operator.

*Proof.* From (35) we have

$$\begin{aligned} \bar{\mathcal{D}} &= e_k \cdot \bar{\nabla}_{e_k} = e_k \cdot \left( \nabla_{e_k} - \frac{1}{2} h_{kl} e_0 \cdot e_l \right) \\ &= \mathcal{D} - \frac{1}{2} h_{kl} e_k \cdot e_0 \cdot e_l \end{aligned}$$

so that

$$\begin{aligned} \bar{\nabla}_{e_i} + e_i \cdot \bar{\mathcal{D}} &= \nabla_{e_i} - \frac{1}{2} h_{ij} e_0 \cdot e_j + e_i \cdot \left( \mathcal{D} - \frac{1}{2} h_{kl} e_k \cdot e_0 \cdot e_l \right) \\ &= \nabla_{e_i} - \frac{1}{2} h_{ij} e_0 \cdot e_j + e_i \cdot \mathcal{D} - \frac{1}{2} h_{kl} e_i \cdot e_k \cdot e_0 \cdot e_l \\ &= \nabla_{e_i} + e_i \cdot \mathcal{D} - \frac{1}{2} h_{ij} e_0 \cdot e_j + \frac{1}{2} h_{kl} e_i \cdot e_k \cdot e_l \cdot e_0 \\ &= \nabla_{e_i} + e_i \cdot \mathcal{D} - \frac{1}{2} h_{ij} e_0 \cdot e_j + \frac{1}{2} h_{kk} e_0 \cdot e_i \\ &\quad + \frac{1}{2} \sum_{k \neq l} h_{kl} e_i \cdot \underbrace{(e_k \cdot e_l + e_l \cdot e_k)}_{=0} e_0, \end{aligned}$$

and the result follows.  $\square$

## 5. THE ASYMPTOTICALLY FLAT CASE

In this section, we prove Theorem 2.6. Assume that the embedding  $(M, g) \hookrightarrow (\overline{M}, \overline{g})$  is asymptotically flat in the sense of Definition 2.3. In particular, in the asymptotic region we have  $g_{ij} = \delta_{ij} + a_{ij}$  with

$$|a_{ij}| + |x||\partial_{x_k} a_{ij}| + |x|^2|\partial_{x_k} \partial_{x_l} a_{ij}| = O(|x|^{-\tau}),$$

where  $\tau > (n-2)/2$ . Thus, we may orthonormalize the standard frame  $\{\partial_{x_i}\}$  by means of

$$(43) \quad e_i = \partial_{x_i} - \frac{1}{2} a_{ij} \partial_{x_j} + O(|x|^{-\tau}) = \partial_{x_i} + O(|x|^{-\tau}),$$

and we can further assume that  $e_n = \varrho$  is the inward pointing normal vector to  $\Sigma \hookrightarrow M$ . We denote with a hat the extension, to the spinor bundle, of the linear isometry

$$(44) \quad T\mathbb{R}_{+,r_0}^n \rightarrow TM_{\text{ext}}, \quad X^i \partial_{x_i} \mapsto X^i e_i.$$

Note that a spinor  $\phi$  on  $\mathbb{S}_{\mathbb{R}_{+,r_0}^n}$  satisfies the MIT bag boundary condition (36) with  $\omega = i\partial_{x_n} \cdot$ , if and only if its image  $\hat{\phi}$  on  $\mathbb{S}_{M_{\text{ext}}}$  satisfies (36) with  $\omega = ie_n \cdot$ .

We begin by specializing the identity in Proposition 4.8 to the case  $\Omega = \Omega_r$ , the compact region in an initial data set  $(M, g, h, \Sigma)$  determined by the coordinate hemisphere  $S_{r,+}^{n-1}$ ; see Figure 1. Notice that  $\partial\Omega_r = S_{r,+}^{n-1} \cup \Sigma_r$ , where  $\Sigma_r$  is the portion of  $\Sigma$  contained in  $\Omega_r$ .

**Proposition 5.1.** *Assume that  $\psi \in H_{\text{loc}}^2(\mathbb{S}_M)$  satisfies the boundary condition (36) along  $\Sigma$ . Then*

$$(45) \quad \int_{\Omega_r} (|\overline{\nabla}\psi|^2 + \langle \mathcal{R}\psi, \psi \rangle - |\overline{\mathcal{D}}\psi|^2) dM = \int_{S_{r,+}^{n-1}} \langle (\overline{\nabla}_{e_i} + e_i \cdot \overline{\mathcal{D}})\psi, \psi \rangle e_i \lrcorner dM - \frac{1}{2} \int_{\Sigma_r} \langle (H_g + \mathcal{U})\psi, \psi \rangle d\Sigma,$$

where  $\mathcal{U} = \pi_{A_n} e_0 \cdot e_A \cdot$  and  $e_n = \varrho$ .

*Proof.* We must work out the contribution of the right-hand side of (41) over  $\Sigma_r$ . By (42),

$$\int_{\Sigma_r} \langle (\overline{\nabla}_{e_i} + e_i \cdot \overline{\mathcal{D}})\psi, \psi \rangle e_i \lrcorner dM = \int_{\Sigma_r} \langle (\nabla_{e_i} + e_i \cdot \mathcal{D})\psi, \psi \rangle e_i \lrcorner dM - \frac{1}{2} \int_{\Sigma_r} \langle \mathcal{U}\psi, \psi \rangle d\Sigma - \frac{1}{2} \int_{\Sigma_r} \pi_{nn} \langle e_0 \cdot e_n \cdot \psi, \psi \rangle d\Sigma.$$

Because of Remark 4.3, the MIT bag boundary condition guarantees that

$$\langle e_0 \cdot e_n \cdot \psi, \psi \rangle = (\varrho \cdot \psi, \psi) = 0,$$

and the last integral vanishes. On the other hand, it is known that

$$\int_{\Sigma_r} \langle (\nabla_{e_i} + e_i \cdot \mathcal{D})\psi, \psi \rangle e_i \lrcorner dM = \int_{\Sigma_r} \left\langle \left( D^\top - \frac{H_g}{2} \right) \psi, \psi \right\rangle d\Sigma,$$

where  $D^\top$  is a certain Dirac-type operator associated to the embedding  $\Sigma \hookrightarrow M$ ; see [ABdL, p.697] for a detailed discussion of this operator. However,  $D^\top$  intertwines the projections defining the boundary conditions and this easily implies that  $\langle D^\top \psi, \psi \rangle = 0$ .  $\square$



**Remark 5.2.** Let  $\psi \in \Gamma(\mathbb{S}_\Sigma)$  be an eigenvector of the linear map  $\mathcal{U} = \pi_{A_n} e_0 \cdot e_A$ , say with eigenvalue  $\lambda$ . We then have

$$\begin{aligned} \lambda^2 |\psi|^2 &= \langle \pi_{A_n} e_0 \cdot e_A \cdot \psi, \pi_{B_n} e_0 \cdot e_B \cdot \psi \rangle \\ &= \pi_{A_n} \pi_{B_n} \langle e_A \cdot \psi, e_B \cdot \psi \rangle \\ &= -\pi_{A_n} \pi_{B_n} \langle e_B \cdot e_A \cdot \psi, \psi \rangle \\ &= \left( \sum_A \pi_{A_n}^2 \right) |\psi|^2. \end{aligned}$$

Thus, the eigenvalues of  $\mathcal{U}$  are  $\pm \sqrt{\sum_A \pi_{A_n}^2} = \pm |(\varrho \lrcorner \pi)^\top|$ . In particular, if the DEC (10) holds then  $H_g + \mathcal{U} \geq 0$ .

The next step involves a judicious choice of a spinor  $\psi$  to be used in (45) above. Denote with  $\Gamma_c(\mathbb{S}_M)$  the space of smooth spinors with compact support in  $M$ , and with  $\mathcal{H}$  its completion with respect to the norm

$$\|\psi\|_{\mathcal{H}} := \|\nabla \psi\|_2 + \left\| \frac{\psi}{r} \right\|_2,$$

where  $r(x) > 0$  for  $x \in M$  and  $r(x) = |x|$  in a fixed asymptotic chart, and  $\|\cdot\|_2$  is the  $L^2$ -norm on the entire  $M$ .

**Proposition 5.3.** *Assume that the DEC (2) and (10) hold. If  $\eta \in L^2(\mathbb{S}_M)$ , there exists a unique  $\varphi \in \mathcal{H}$  solving any of the boundary value problems*

$$(46) \quad \begin{cases} \bar{D}\varphi = \eta & \text{on } M \\ \omega\varphi = \pm\varphi & \text{on } \Sigma. \end{cases}$$

Moreover,  $\varphi \in H_{\text{loc}}^2(\mathbb{S}_M)$  whenever  $\eta \in H_{\text{loc}}^1(\mathbb{S}_M)$ .

*Proof.* We adapt arguments in [GN, PT, BaC]. First, observe the validity of the Euclidean Hardy (weighted Poincaré) inequality

$$(47) \quad \int_{\mathbb{R}_+^n} \frac{(n-2)^2}{4|x|^2} |\psi|^2 \leq \int_{\mathbb{R}_+^n} |\nabla \psi|^2 \quad \forall \psi \in \Gamma_c(\mathbb{S}_{\mathbb{R}_+^n}).$$

Indeed, it is sufficient to extend  $\varphi$  by reflection across  $\partial\mathbb{R}_+^n$  and to apply the Hardy inequality for spinors in  $\mathbb{S}_{\mathbb{R}^n}$ . We sketch a brief argument to prove the latter, for the sake of completeness: consider a constant spinor  $\psi_0$  of norm 1 on  $\mathbb{R}_+^n$  and set  $\tau := w\psi_0$ , with  $w(x) = |x|^{\frac{2-n}{2}}$  being the square root of the Green kernel of  $-\Delta_\delta$  on  $\mathbb{R}^n$  with pole at the origin. A computation shows that

$$D\tau = \nabla w \cdot \psi_0, \quad D^2\tau = \frac{(n-2)^2}{4|x|^2} \tau \quad \text{on } \mathbb{R}^n,$$

Given  $\psi \in \Gamma_c(\mathbb{S}_{\mathbb{R}^n})$  supported outside of the origin, integrating by parts against  $(|\psi|^2/w)\psi_0$  with respect to the Euclidean measure we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{(n-2)^2}{4|x|^2} |\psi|^2 &\leq \int_{\mathbb{R}^n} \frac{|\psi|^2}{w} \langle \psi_0, \mathcal{D}^2 \tau \rangle \\ &= \int_{\mathbb{R}^n} \frac{|\psi|^2}{w^2} \langle \psi_0, \nabla w \cdot \mathcal{D} \tau \rangle - 2 \int_{\mathbb{R}^n} \frac{\langle \psi, \nabla_{\partial_{x_i}} \psi \rangle}{w} \langle \psi_0, \partial_{x_i} \cdot \mathcal{D} \tau \rangle \\ &\leq - \int_{\mathbb{R}^n} \frac{|\psi|^2}{w^2} |\nabla w|^2 + 2 \int_{\mathbb{R}^n} \frac{|\psi| |\nabla \psi|}{w} |\nabla w| \\ &\leq \int_{\mathbb{R}^n} |\nabla \psi|^2, \end{aligned}$$

where in the second and third lines we used, respectively, Cauchy-Schwarz and Young inequalities. Pick an asymptotic chart  $\mathbb{R}_{+,r_0}^n \rightarrow M_{\text{ext}}$ , and write for convenience  $M_R = \{x \in M : |x| > R\}$ . Let  $\delta$  denote a metric on  $M$  that is Euclidean on  $M_{\text{ext}}$ , and let  $r$  be a positive function on  $M$  that equals  $|x|$  on  $M_{\text{ext}}$ . Define also a family of cut-offs  $\{\eta_R\} \subset C_c^\infty(M)$  satisfying

$$(48) \quad 0 \leq \eta_R \leq 1, \quad \eta_R = 1 \text{ on } M \setminus M_R, \quad \text{spt } \eta_R \subset M \setminus M_{2R}, \quad |\nabla \eta_R| \leq \frac{\kappa}{R}$$

for some constant  $\kappa > 0$ . With the aid of the isometry (44), we deduce from (47) the existence of a constant  $C > 0$  only depending on  $\|g - \delta\|_\infty$  such that

$$(49) \quad \left\| \frac{\varphi}{r} \right\|_2 \leq C \|\nabla \varphi\|_2 \quad \forall \varphi \in \Gamma_c(\mathbb{S}_{M_R}),$$

where now the norms and connections are taken with respect to  $g$ . A gluing argument with the aid of the cutoff  $\eta_R$  in (48) and of the Poincaré inequality on  $M \setminus M_R$  guarantees that the weighted Poincaré inequality

$$(50) \quad \left\| \frac{\varphi}{r} \right\|_2 \leq C \|\nabla \varphi\|_2 \quad \forall \varphi \in \Gamma_c(\mathbb{S}_M)$$

holds for some constant  $C > 0$ . From the spinorial Gauss Equation and using  $|h| \leq C_1 r^{-\tau-1}$ ,

$$(51) \quad \begin{aligned} \|\bar{\nabla} \psi\|_2 &\leq \|\nabla \psi\|_2 + \|h|\psi\|_2 \leq \|\nabla \psi\|_2 + C \left\| \frac{\psi}{r} \right\|_2 \leq C_1 \|\psi\|_{\mathcal{H}}^2 \quad \forall \psi \in \Gamma_c(\mathbb{S}_M), \\ \|\bar{\nabla} \psi\|_2 &\geq \|\nabla \psi\|_2 - \|h|\psi\|_2 \geq \|\nabla \psi\|_2 - \frac{C_1}{R^\tau} \left\| \frac{\psi}{r} \right\|_2 \quad \forall \psi \in \Gamma_c(\mathbb{S}_{M_R}). \end{aligned}$$

In particular, for  $R$  large enough, because of (50) the term containing  $R^{-\tau}$  in the second line can be absorbed into the norm  $\|\cdot\|_{\mathcal{H}}$ , yielding to

$$(52) \quad C_2 \|\psi\|_{\mathcal{H}}^2 \leq \|\bar{\nabla} \psi\|_2 \quad \forall \psi \in \Gamma_c(\mathbb{S}_{M_R}).$$

The gluing argument in [PT, Lemma 5.5] (cf. also [BaC, Theorem 9.5]) guarantees that

$$C_3 \|\psi\|_{\mathcal{H}}^2 \leq \|\bar{\nabla} \psi\|_2 \leq C_1 \|\psi\|_{\mathcal{H}}^2 \quad \forall \psi \in \Gamma_c(\mathbb{S}_M)$$

for some constant  $C_3 > 0$ . Define  $\mathcal{H}_\pm$  as the  $\|\cdot\|_{\mathcal{H}}$ -closure of the subspaces

$$\Gamma_c(\mathbb{S}_M)_\pm := \{\psi \in \Gamma_c(\mathbb{S}_M) : \omega \psi = \pm \psi \text{ on } \Sigma\},$$

respectively.

From the very definition of  $\overline{\mathcal{D}}$ ,  $\|\overline{\mathcal{D}}\psi\|_2^2 \leq n\|\nabla\psi\|_2^2 \leq C\|\psi\|_{\mathcal{H}}^2$ , thus  $\overline{\mathcal{D}}$  extends to a continuous operator  $\overline{\mathcal{D}}: \mathcal{H} \rightarrow L^2(\mathbb{S}_M)$ . The identity

$$(53) \quad \int_M (|\nabla\psi|^2 + \langle \mathcal{R}\psi, \psi \rangle - |\overline{\mathcal{D}}\psi|^2) dM = -\frac{1}{2} \int_{\Sigma} \langle (H_g + \mathcal{U})\psi, \psi \rangle d\Sigma$$

$$\forall \psi \in \Gamma_c(\mathbb{S}_M), \omega\psi = \pm\psi \text{ on } \Sigma$$

in Proposition 5.1 together with the DEC's imply that  $\|\overline{\mathcal{D}}\psi\|_2 \geq \|\nabla\psi\|_2$  for every  $\psi \in \Gamma_c(\mathbb{S}_M)_{\pm}$ , hence

$$(54) \quad \|\overline{\mathcal{D}}\psi\|_2 \geq C'\|\psi\|_{\mathcal{H}} \quad \forall \psi \in \mathcal{H}_{\pm}.$$

To analyse more closely the boundary trace of  $\psi \in \mathcal{H}$ , define the conformally deformed metric  $\hat{g} := r^{-\frac{4}{n}}g = e^{2u}g$ , where we set for convenience  $u := -\frac{2}{n}\log r$ . Denote quantities referring to  $\hat{g}$  with a hat and let  $\hat{\mathbb{S}}_M$  be the space of spinors for  $\hat{g}$ . To each  $X \in TM$ , set  $\hat{X} = e^{-u}X$ . It is known (cf. [Hi, Section 4]) that there exists an isomorphism  $\hat{\cdot}: \mathbb{S}_M \rightarrow \hat{\mathbb{S}}_M$ , which is an isometry on fibers and satisfies

$$\widehat{X \cdot \psi} = \hat{X} \cdot \hat{\psi}, \quad \widehat{\nabla_X \psi} = \widehat{\nabla_X \psi} - \frac{1}{2}X \cdot \widehat{\nabla u} \cdot \psi - \frac{1}{2}X(u)\hat{\psi}.$$

Let  $\{e_i\}$  be an orthonormal frame for  $g$ , and  $\hat{e}_i = e^{-u}e_i$  the corresponding orthonormal frame for  $\hat{g}$ . We compute

$$\begin{aligned} |\widehat{\nabla \psi}|_{\hat{g}}^2 &= \sum_i |\widehat{\nabla_{\hat{e}_i} \psi}|_{\hat{g}}^2 \leq 8e^{-2u} \sum_i \left( |\widehat{\nabla_{e_i} \psi}|_{\hat{g}}^2 + \frac{1}{4}|e_i \cdot \widehat{\nabla u} \cdot \psi|_{\hat{g}}^2 + \frac{1}{4}|e_i(u)|^2 |\hat{\psi}|_{\hat{g}}^2 \right) \\ &\leq 8e^{-2u} (|\nabla\psi|^2 + |\nabla u|^2 |\psi|^2). \end{aligned}$$

Therefore,

$$(55) \quad \begin{aligned} \int_M |\hat{\psi}|_{\hat{g}}^2 dM_{\hat{g}} &= \int_M |\psi|^2 r^{-2} dM \leq \|\psi\|_{\mathcal{H}}^2, \\ \int_M |\widehat{\nabla \psi}|_{\hat{g}}^2 dM_{\hat{g}} &\leq C \int_M e^{-2u} (|\nabla\psi|^2 + |\nabla u|^2 |\psi|^2) e^{nu} dM \leq C\|\psi\|_{\mathcal{H}}^2, \end{aligned}$$

where we used that  $u$  is bounded from above and  $|\nabla u| \leq C|\nabla r|/r \leq C'/r$ . Hence,  $\hat{\cdot}$  induces an inclusion

$$\hat{\cdot}: \mathcal{H} \longrightarrow H^1(\mathbb{S}_M, \hat{g})$$

into a subspace  $\hat{\mathcal{H}}$  of the Sobolev space  $H^1$  of spinors for  $(M, \hat{g})$ . The manifold  $(M, \hat{g})$  has bounded geometry in the sense of [GN, Definition 2.2], so the trace theorem in [GN, Theorem 3.7] implies that the restriction

$$R: \psi \in \mathcal{H} \longmapsto \hat{\psi}|_{\Sigma} \in H^{\frac{1}{2}}(\mathbb{S}_M|_{\Sigma}, \hat{g})$$

is a bounded operator. Furthermore, we claim that the functional

$$(\psi, \xi) \in \Gamma_c(\mathbb{S}_M) \times \Gamma_c(\mathbb{S}_M) \longmapsto \int_{\Sigma} \langle (H_g + \mathcal{U})\psi, \xi \rangle d\Sigma$$

extends continuously on  $\mathcal{H} \times \mathcal{H}$ , so in particular (53) extends by density to  $\psi \in \mathcal{H}_{\pm}$ . Using  $|h| \leq Cr^{-1-\tau}$ , we compute

$$\begin{aligned} \left| \int_{\Sigma} \langle (H + \mathcal{U})\psi, \xi \rangle d\Sigma \right| &\leq C \int_{\Sigma} r^{-1-\tau} |\langle \psi, \xi \rangle| d\Sigma = C \int_{\Sigma} r^{-1-\tau} |\langle \hat{\psi}, \hat{\xi} \rangle_{\hat{g}}| r^{\frac{2n-1}{n}} d\Sigma_{\hat{g}} \\ &\leq C \int_{\Sigma} r^{-\frac{(n-2)^2}{2n}} |\langle \hat{\psi}, \hat{\xi} \rangle_{\hat{g}}| d\Sigma_{\hat{g}}, \end{aligned}$$

where, in the last inequality, we used that  $\tau > (n-2)/2$ . By [GN, Lemma 3.8], the  $L^2$  product

$$(\hat{\psi}, \hat{\xi}) \mapsto \int_{\Sigma} \langle \hat{\psi}, \hat{\xi} \rangle_{\hat{g}} d\Sigma_{\hat{g}}$$

on  $\Gamma_c(\mathbb{S}_M, \hat{g})$  extends continuously to  $H^1(\mathbb{S}_M, \hat{g}) \times H^{-1}(\mathbb{S}_M, \hat{g})$ , hence to  $H^1(\mathbb{S}_M, \hat{g}) \times H^1(\mathbb{S}_M, \hat{g})$  in view of [GN, Remark 3.6]. Concluding, from (55) we get

$$(56) \quad \left| \int_{\Sigma} \langle (H + \mathcal{U})\psi, \xi \rangle d\Sigma \right| \leq C \|\psi\|_{\mathcal{H}} \|\xi\|_{\mathcal{H}},$$

as claimed.

Denote by  $\hat{\varrho}$  the unit normal vector to  $\Sigma \hookrightarrow (M, \hat{g})$  corresponding to  $\varrho$ , and define the MIT boundary operators

$$\begin{aligned} \mathcal{B}_{\pm} &: H^{\frac{1}{2}}(\mathbb{S}_M|_{\Sigma}, \hat{g}) \rightarrow H^{\frac{1}{2}}(\mathbb{S}_M|_{\Sigma}, \hat{g}), & \eta &\mapsto \hat{\varrho}\eta \mp \eta, \\ \mathcal{K}_{\pm} &:= \mathcal{B}_{\pm} \circ R, & \mathcal{K}_{\pm} &: \mathcal{H} \rightarrow H^{\frac{1}{2}}(\mathbb{S}_M|_{\Sigma}, \hat{g}), \end{aligned}$$

with  $\hat{\omega}\eta := i\hat{\varrho}\hat{\cdot}\eta$ . Note that  $\mathcal{B}_{\pm}$  and  $\mathcal{K}_{\pm}$  are continuous. The commutativity  $\hat{\varrho}\hat{\cdot}\hat{\psi} = \widehat{\varrho \cdot \psi}$  guarantees that  $\mathcal{H}_{\pm} = \ker \mathcal{K}_{\pm}$ , hence because of (54) the operator

$$(\overline{\mathcal{D}}, \mathcal{K}_{\pm}) : \mathcal{H} \longrightarrow L^2(\mathbb{S}_M) \times H^{\frac{1}{2}}(\mathbb{S}_M|_{\Sigma}, \hat{g})$$

is injective. We then proceed as in [BaC]. By (54), the quadratic form

$$(\psi, \xi) \in \mathcal{H}_{\pm} \times \mathcal{H}_{\pm} \longmapsto (\overline{\mathcal{D}}\psi, \overline{\mathcal{D}}\xi)_2 \in \mathbb{C}$$

is coercive, so for given  $\eta \in L^2(\mathbb{S}_M)$  and associated continuous functional  $\psi \in \mathcal{H}_{\pm} \mapsto (\overline{\mathcal{D}}\psi, \eta)_2$ , there exists  $\varphi \in \mathcal{H}_{\pm}$  such that

$$(\overline{\mathcal{D}}\psi, \overline{\mathcal{D}}\varphi - \eta)_2 = 0 \quad \forall \psi \in \mathcal{H}_{\pm},$$

namely,  $\Psi := \overline{\mathcal{D}}\varphi - \eta \in L^2(\mathbb{S}_M)$  weakly solves the adjoint Dirac equation  $(\overline{\mathcal{D}}\psi, \Psi)_2 = 0$  for every  $\psi \in \mathcal{H}_{\pm}$ . The identity in the proof of Proposition 4.4, that can be rewritten as

$$\begin{aligned} (\overline{\mathcal{D}}\psi, \xi)_2 - (\psi, \overline{\mathcal{D}}\xi)_2 &= \int_{\Sigma} \langle \varrho \cdot \psi, \xi \rangle d\Sigma \\ &= \frac{1}{2} \int_{\Sigma} [\langle \varrho \cdot \psi, \mathcal{K}_- \xi \rangle - \langle \varrho \cdot \mathcal{K}_+ \psi, \xi \rangle] d\Sigma \quad \forall \psi, \xi \in \Gamma_c(\mathbb{S}_M), \end{aligned}$$

shows that the formal adjoint of  $(\overline{\mathcal{D}}, \mathcal{K}_{\pm})$  is  $(\overline{\mathcal{D}}, \mathcal{K}_{\mp})$ . In particular,

$$\overline{\mathcal{D}}\Psi = 0 \quad \text{in } L^2_{\text{loc}}(\mathbb{S}_M) \quad \text{and} \quad \mathcal{K}_{\mp}\Psi = 0 \quad \text{in } H^{\frac{1}{2}}_{\text{loc}}(\mathbb{S}_M|_{\Sigma}).$$

Let  $\{\eta_k\}$  be the cut-offs in (48) with indices  $R = k \in \mathbb{N}$ , and define  $\Psi_k = \eta_k \Psi \in H^1_c(\mathbb{S}_M) \cap \mathcal{H}_{\mp}$ . By (54),

$$\begin{aligned} C' \|\Psi_k - \Psi_j\|_{\mathcal{H}}^2 &\leq \|\overline{\mathcal{D}}(\Psi_k - \Psi_j)\|_2^2 = C \int_M [|\nabla \eta_k|^2 |\Psi|^2 + |\nabla \eta_j|^2 |\Psi|^2] dM \\ &\rightarrow 0 \quad \text{as } j, k \rightarrow \infty, \end{aligned}$$

thus  $\{\Psi_k\}$  is a Cauchy hence convergent sequence in  $\mathcal{H}$ . Concluding,  $\Psi \in \mathcal{H}_{\mp}$ , hence  $\Psi \in \ker(\overline{\mathcal{D}}, \mathcal{K}_{\mp}) = 0$ , so that  $\varphi$  solves (46). In view of the higher regularity estimates in [BaC] (cf. Theorems 3.8, 6.6 and Remark 6.7 therein),  $\varphi \in H^2_{\text{loc}}(\mathbb{S}_M)$  whenever  $\eta \in H^1_{\text{loc}}(\mathbb{S}_M)$ .  $\square$

We proceed by choosing a non-trivial *parallel* spinor  $\phi \in \Gamma(\mathbb{S}_{\mathbb{R}^n_{+,r_0}})$  satisfying  $i\partial_{x_n} \cdot \phi = \pm\phi$ , and transplant it to a spinor  $\hat{\phi} \in \mathbb{S}_{M_{\text{ext}}}$  satisfying (36). We extend  $\hat{\phi}$  to the rest of  $\Sigma$  so that the boundary condition holds everywhere, and finally extend  $\hat{\phi}$  to the rest of  $M$  in an arbitrary manner. It follows from (34) and (35) that

$$\bar{\nabla}_{e_i} \hat{\phi} = \partial_{x_i} \hat{\phi} + \frac{1}{4} \Gamma_{ij}^k \partial_{x_j} \cdot \partial_{x_k} \cdot \hat{\phi} - \frac{1}{2} h_{ij} e_0 \cdot e_j \cdot \hat{\phi}.$$

Since  $\partial_{x_i} \hat{\phi} = 0$ , by (43) we then have  $\bar{\nabla}_{e_i} \hat{\phi} = O(|x|^{-\tau-1})$ , that is,  $\bar{\nabla} \hat{\phi} \in L^2(\mathbb{S}_M)$ . By Proposition 5.3 we can find a spinor  $\varphi \in \mathcal{H} \cap H_{\text{loc}}^2(M)$  such that  $\bar{\mathcal{D}}\varphi = -\bar{\mathcal{D}}\hat{\phi}$  and satisfying (36) along  $\Sigma$ . We define

$$(57) \quad \psi = \hat{\phi} + \varphi \in H_{\text{loc}}^2(\mathbb{S}_M).$$

Thus,  $\psi$  is harmonic ( $\bar{\mathcal{D}}\psi = 0$ ), satisfies (36) along  $\Sigma$ , and asymptotes  $\hat{\phi}$  at infinity in the sense  $\psi - \hat{\phi} \in \mathcal{H}$ .

The next result gives a nice extension of Witten's celebrated formula for the energy-momentum vector of an asymptotically flat initial data set in the presence of a non-compact boundary. More precisely, it is the spacetime version of [ABdL, Theorem 5.2].

**Theorem 5.4.** *If the asymptotically flat initial data set  $(M, g, h, \Sigma)$  satisfies the DECs (2) and (10) and  $\psi$  is the harmonic spinor in (57) then*

$$(58) \quad \begin{aligned} \frac{1}{4} (E|\phi|^2 - \langle \phi, P_A \partial_{x_0} \cdot \partial_{x_A} \cdot \phi \rangle) &= \int_M (|\bar{\nabla}\psi|^2 + \langle \mathcal{R}\psi, \psi \rangle) d\text{vol} \\ &+ \frac{1}{2} \int_{\Sigma} \langle (H_g + \mathcal{U})\psi, \psi \rangle d\Sigma. \end{aligned}$$

*Proof.* From (42) and (45) we get

$$\begin{aligned} \int_{\Omega_r} (|\bar{\nabla}\psi|^2 + \langle \mathcal{R}\psi, \psi \rangle) dM &= \int_{S_{r,+}^{n-1}} \langle (\nabla_{e_i} + e_i \cdot \mathcal{D})\psi, \psi \rangle e_i \lrcorner dM \\ &- \frac{1}{2} \int_{S_{r,+}^{n-1}} \pi_{ij} \langle e_0 \cdot e_i \cdot \psi, \psi \rangle e_j \lrcorner dM \\ &- \frac{1}{2} \int_{\Sigma_r} \langle (H_g + \mathcal{U})\psi, \psi \rangle d\Sigma. \end{aligned}$$

First, the computation in [ABdL, Section 5.2] shows that

$$\lim_{r \rightarrow +\infty} \int_{S_{r,+}^{n-1}} \langle (\nabla_{e_i} + e_i \cdot \mathcal{D})\psi, \psi \rangle e_i \lrcorner dM = \frac{1}{4} E|\phi|^2.$$

Also,

$$(59) \quad \begin{aligned} \lim_{r \rightarrow +\infty} \int_{S_{r,+}^{n-1}} \pi_{ij} \langle e_0 \cdot e_i \cdot \psi, \psi \rangle e_j \lrcorner dM &= \lim_{r \rightarrow +\infty} \int_{S_{r,+}^{n-1}} \pi_{ij} \langle e_0 \cdot e_i \cdot \hat{\phi}, \hat{\phi} \rangle e_j \lrcorner dM \\ &= \lim_{r \rightarrow +\infty} \int_{S_{r,+}^{n-1}} \pi(\partial_{x_i}, \mu) \langle \partial_{x_0} \cdot \partial_{x_i} \cdot \phi, \phi \rangle dS_{r,+}^{n-1} \\ &= \lim_{r \rightarrow +\infty} \int_{S_{r,+}^{n-1}} \pi(\partial_{x_A}, \mu) \langle \partial_{x_0} \cdot \partial_{x_A} \cdot \phi, \phi \rangle dS_{r,+}^{n-1} \\ &= \frac{1}{2} \langle P_A \partial_{x_0} \cdot \partial_{x_A} \cdot \phi, \phi \rangle, \end{aligned}$$

where we used (8) together with the fact that, by Remark 4.3 and since  $\phi \in \Gamma(\mathbb{S}_{\mathbb{R}_+^n, r_0}^n)$  is constant,  $\langle \partial_{x_0} \cdot \partial_{x_n} \cdot \phi, \phi \rangle = 0$  on the entire  $\mathbb{R}_+^n$ .  $\square$

**Remark 5.5.** The argument leading to the first identity in (59) needs a little justification, since a-priori  $|\psi - \hat{\phi}| \in \mathcal{H}$ , in particular we only know that

$$\lim_{j \rightarrow \infty} r_j \int_{S_{r_j, +}^{n-1}} \frac{|\psi - \hat{\phi}|^2}{r_j^2} dS_{r_j, +}^{n-1} = 0$$

for some diverging sequence  $\{r_j\}$ . Using

$$|\langle e_0 \cdot e_i \cdot \psi, \psi \rangle - \langle e_0 \cdot e_i \cdot \hat{\phi}, \hat{\phi} \rangle| \leq [|\psi| + |\hat{\phi}|] |\psi - \hat{\phi}| \leq [2|\hat{\phi}| + |\psi - \hat{\phi}|] |\psi - \hat{\phi}|,$$

we readily deduce that

$$\begin{aligned} & \left| \int_{S_{r_j, +}^{n-1}} \pi_{ij} [\langle e_0 \cdot e_i \cdot \psi, \psi \rangle - \langle e_0 \cdot e_i \cdot \hat{\phi}, \hat{\phi} \rangle] e_j \lrcorner dM \right| \\ & \leq Cr_j^{-\tau-1} \int_{S_{r_j, +}^{n-1}} [2\|\hat{\phi}\|_\infty |\psi - \hat{\phi}| + |\psi - \hat{\phi}|^2] dS_{r_j, +}^{n-1} \\ & \leq Cr_j^{-\tau-1} \left\{ 2\|\hat{\phi}\|_\infty \sqrt{|S_{r_j, +}^{n-1}|} r_j \left( \int_{S_{r_j, +}^{n-1}} \frac{|\psi - \hat{\phi}|^2}{r_j} dS_{r_j, +}^{n-1} \right)^{\frac{1}{2}} + r_j \int_{S_{r_j, +}^{n-1}} \frac{|\psi - \hat{\phi}|^2}{r_j} dS_{r_j, +}^{n-1} \right\} \rightarrow 0 \end{aligned}$$

as  $j \rightarrow \infty$ , where we used the full strength of the inequality  $\tau > (n-2)/2$ . This is enough to conclude that the claimed equality holds if the limit is evaluated along the sequence  $\{r_j\}$ . In particular, each of the limits in (59) should be read as a sequential limit along  $\{r_j\}$ , which nevertheless suffices to guarantee (58).

In order to complete the proof of Theorem 2.6, we make one last assumption on the parallel spinor  $\phi$  used in the construction of  $\psi$  in (57). As in Remark 5.2, one checks that the operator  $\mathcal{T} = P_A \partial_{x_0} \cdot \partial_{x_A} \cdot$  has  $\pm|P|$  as eigenvalues. Also, it satisfies  $\mathcal{T}(i\partial_{x_n}) = (i\partial_{x_n})\mathcal{T}$ . In particular,  $\mathcal{T}$  and  $i\partial_{x_n} \cdot$  are simultaneously diagonalizable. Thus we may choose  $\phi$  constant (parallel) in  $\mathbb{R}_{+, r_0}^n$  satisfying both  $\mathcal{T}\phi = |P|\phi$  and one of the MIT bag boundary conditions  $i\partial_{x_n} \cdot \phi = \pm\phi$ . Using this  $\phi$  in (58) we get

$$(60) \quad \frac{1}{4} (E - |P|) |\phi|^2 = \int_M (|\bar{\nabla}\psi|^2 + \langle \mathcal{R}\psi, \psi \rangle) dM + \frac{1}{2} \int_\Sigma \langle (H_g + \mathcal{U})\psi, \psi \rangle d\Sigma.$$

Since the right-hand side is nonnegative by Remarks 4.7 and 5.2, we obtain the mass inequality

$$(61) \quad E \geq |P|.$$

Suppose next that  $E = |P|$ , so there exists  $\psi \in H_{\text{loc}}^2(\mathbb{S}_M)$  satisfying

$$(62) \quad \begin{aligned} \bar{\nabla}\psi &= 0, & \langle \mathcal{R}\psi, \psi \rangle &\equiv 0 & \text{on } M, \\ i e_n \cdot \psi &= \pm\psi, & \langle (H_g + \mathcal{U})\psi, \psi \rangle &= 0, & (e_n \cdot \psi, \psi) = 0 & \text{on } \Sigma, \end{aligned}$$

and that (11) is in force. By bootstrapping,  $\bar{\nabla}\psi = 0$  readily implies that  $\psi \in C_{\text{loc}}^{2, \alpha}(\mathbb{S}_M)$ . Our first goal is to prove that  $E = |P| = 0$ . To do so, we avail of the following identity in [dLGM]:

$$(63) \quad E = d_n \cdot \lim_{r \rightarrow \infty} \left[ \int_{\mathbb{S}_{r, +}^{n-1}} G(r\partial_r, \mu) d\mathbb{S}_{r, +}^{n-1} - \int_{\mathbb{S}_r^{n-2}} N(r\partial_r, \vartheta) d\mathbb{S}_r^{n-2} \right],$$

where  $G$  is the Einstein tensor of  $M$ ,  $N$  is the first Newton tensor of  $\Sigma \hookrightarrow M$  in the direction  $\varrho = e_n$ , and  $d_n = \frac{2}{2-n}$ . We consider the lapse-shift pair associated to  $\psi$  :

$$V = \langle \psi, \psi \rangle, \quad W = \sum_{j=1}^n (e_j \cdot \psi, \psi) e_j = W^j e_j.$$

In a frame  $\{e_i\}$  that is  $\nabla$ -normal at a given point,

(64)

$$(i) \quad V_i = e_i(e_0 \cdot \psi, \psi) = (\bar{\nabla}_{e_i} e_0 \cdot \psi, \psi) = h_{ik}(e_k \cdot \psi, \psi) = h_{ik} W^k$$

$$(ii) \quad \nabla_{e_i} W = e_i(e_j \cdot \psi, \psi) e_j = (\bar{\nabla}_{e_i} e_j \cdot \psi, \psi) e_j = h_{ij}(e_0 \cdot \psi, \psi) e_j = h_{ij} V e_j,$$

so

$$(65) \quad \mathcal{L}_W g - 2Vh = 0, \quad d(V^2 - |W|^2) = 0 \quad \text{on } M.$$

Furthermore,  $W^n = (e_n \cdot \psi, \psi) = 0$ .

**Claim 1:** We have  $V^2 \geq |W|^2$ .

*Proof:* The claim is obvious at points where  $\psi = 0$ . On the other hand, if  $|\psi|^2 > 0$ , then  $\{e_j \cdot \psi\}$  is an orthogonal set of spinors for  $\langle \cdot, \cdot \rangle$ , with  $|e_j \cdot \psi|^2 = V$  for every  $j$  (in fact,  $\Re(e_i \cdot \eta, e_j \cdot \eta) = \delta_{ij} |\eta|^2$ ,  $\forall \eta \in \Gamma(\mathbb{S}_M)$ ). Thus

$$(66) \quad |W|^2 = \sum_j (e_j \cdot \psi, \psi)^2 = \sum_j (e_j \cdot \psi, e_0 \cdot \psi)^2 \leq V |e_0 \cdot \psi|^2 = V^2.$$

**Claim 2:** If  $E = |P|$ , then  $V > 0$  on  $M$  and

$$\Sigma \hookrightarrow M \text{ is totally geodesic,} \quad \pi_{A_n} = 0 \quad \text{on } \Sigma, \quad \forall A \in \{1, \dots, n-1\}.$$

Consequently,  $e_n(V) = 0$  on  $\Sigma$ .

*Proof:* We use (64) and (66) to compute

$$(67) \quad \begin{aligned} \Delta V &= (h_{ik} W^k)_i = |h|^2 V + h_{ik,i} W^k \\ &\leq |h|^2 V + |\nabla h| |W| \leq (|h|^2 + |\nabla h|) V \quad \text{on } M. \end{aligned}$$

By the strong maximum principle, we deduce that either  $V \equiv 0$  or  $V > 0$  on  $\text{Int } M$ . The first case cannot occur, because otherwise  $\psi \equiv 0$  and thus  $\psi - \hat{\phi} \notin \mathcal{H}$ . Next, we differentiate the equality  $i_{e_n} \cdot \psi = \pm \psi$  and use  $\bar{\nabla} \psi = 0$  to get

$$(68) \quad \begin{aligned} 0 &= \bar{\nabla}_{e_A} (e_n \cdot \psi) = (\bar{\nabla}_{e_A} e_n) \cdot \psi \\ &= -b_{AB} e_B \cdot \psi - \bar{g}(\bar{\nabla}_{e_A} e_n, e_0) e_0 \cdot \psi = (-b_{AB} e_B + \pi_{An} e_0) \cdot \psi. \end{aligned}$$

where  $b_{AB}$  is the second fundamental form of  $\Sigma \hookrightarrow M$  in the inward direction  $e_n$  (so  $H_g = \text{tr}_g b$ ). Making the product  $\langle \cdot, \cdot \rangle$  with, respectively,  $e_B \cdot \psi$  and  $e_0 \cdot \psi$ , we obtain the system

$$(69) \quad \begin{cases} (i) & -b_{AB} V + \pi_{An} W^B = 0 & \forall A, B \\ (ii) & -b_{AB} W^B + \pi_{An} V = 0 & \forall A. \end{cases}$$

Suppose that  $V(x) = 0$  for some  $x \in \Sigma$ . From (i), we then deduce  $\pi_{An} W^A = 0$  at  $x$ , and in view of the identity  $W^n = 0$  on  $\Sigma$  we infer

$$e_n(V) = h_{nj} W^j = \pi_{nA} W^A = 0 \quad \text{at } x,$$

contradicting the Hopf maximum principle. This shows that  $V > 0$  on  $\Sigma$ , hence on the entire  $M$ . Differentiating the pointwise inequality  $\langle (H_g + \mathcal{U})(\psi + t\eta), \psi + t\eta \rangle \geq 0$

at  $t = 0$ , because of (62) we get  $\Re\langle (H_g + \mathcal{U})\psi, \eta \rangle \equiv 0$  on  $\Sigma$  for each  $\eta \in C_{\text{loc}}^2(\mathbb{S}_M)$ . Using as a test spinor, respectively,  $\eta = \psi$  and  $\eta = e_0 \cdot e_B \cdot \psi$  we get

$$(70) \quad \begin{cases} (iii) & 0 = \Re\langle (H_g + \mathcal{U})\psi, \psi \rangle = H_g V + \pi_{An} W^A \\ (iv) & 0 = \Re\langle (H_g + \mathcal{U})\psi, e_0 \cdot e_B \cdot \psi \rangle = H_g W^B + \pi_{Bn} V. \end{cases}$$

Tracing (i) in  $A, B$  and comparing with (iii) we deduce  $H_g V = 0$ , hence  $H_g = 0$  on  $\Sigma$ . Plugging into (iv) gives  $\pi_{Bn} = 0$ , in particular  $e_n(V) = \pi_{Bn} W^B = 0$ . By (i), we conclude  $b_{AB} = 0$  on  $\Sigma$ .

**Claim 3:** If  $E = |P|$ , then

$$(71) \quad \rho_0 V = J(W), \quad \rho_0 W = V J^\sharp, \quad \rho_0 = |J| \quad \text{on } M.$$

Furthermore,  $V^2 \equiv |W|^2$  unless  $\rho_0 = |J| \equiv 0$  on  $M$ .

*Proof:* We differentiate the pointwise inequality  $\langle \mathcal{R}(\psi + t\eta), \psi + t\eta \rangle \geq 0$  at  $t = 0$  and use the second identity in (62) to deduce  $\Re\langle \mathcal{R}\psi, \eta \rangle \equiv 0$  on  $M$  for each  $\eta \in C_{\text{loc}}^2(\mathbb{S}_M)$ . Using as a test spinor, respectively,  $\eta = \psi$  and  $\eta = e_0 \cdot e_k \cdot \psi$  we get

$$\begin{aligned} 0 &= 2\Re\langle \mathcal{R}\psi, \psi \rangle = \rho_0 V + J_i \Re\langle e_i \cdot e_0 \cdot \psi, \psi \rangle = \rho_0 V - J_i W^i \\ 0 &= 2\Re\langle \mathcal{R}\psi, e_0 \cdot e_k \cdot \psi \rangle = \rho_0 \langle \psi, e_k \cdot \psi \rangle + J_i \Re\langle e_i \cdot e_0 \cdot \psi, e_0 \cdot e_k \cdot \psi \rangle \\ &= \rho_0 W_k - J_i \Re\langle e_i \cdot \psi, e_k \cdot \psi \rangle = \rho_0 W_k - J_k V, \end{aligned}$$

proving the first two identities in (71). Multiplying the second one in (71) by  $J^\sharp$  we deduce that

$$\rho_0^2 V = \rho_0 J(W) = V |J|^2,$$

and  $\rho_0 = |J|$  follows since  $V > 0$  on  $M$ . Similarly, multiplying that identity by  $W$  we get  $\rho_0 |W|^2 = V J(W) = \rho_0 V^2$ . If  $\rho_0 \neq 0$ , then  $V^2 \equiv |W|^2$  on the entire  $M$ , in view of the constancy of  $V^2 - |W|^2$ .

Let  $(M', g)$  consists of two copies of  $M$  glued along the totally geodesic boundary  $\Sigma$ . Then,  $M'$  is asymptotically flat and the metric  $g$  extended by reflection across  $\Sigma$  is  $C_{\text{loc}}^{2,1}$ .

**Claim 4:** If  $E = |P|$ , then  $g - \delta \in C_{-\tau}^{2,\alpha}(M')$  and  $h \in C_{-\tau-1}^{1,\alpha}(M')$ , where  $h_{ij}$  is extended by reflection along  $\widehat{\Sigma}$ .

*Proof.* We consider the manifold

$$\widehat{M} = M \times \mathbb{R}$$

endowed with the following Riemannian metric  $\widehat{g}$ . Let  $\{x_i\}$  be local smooth coordinates around a point  $p \in \widehat{M}$ . We define

$$(72) \quad \widehat{g} = V^2 du^2 + g_{ij}(dx^i + W^i du) \otimes (dx^j + W^j du),$$

where  $W = W^i \partial_{x_j}$ . Hereafter, the functions  $g_{ij}$  and  $W^i$  are meant to be functions on  $\widehat{M}$  by pre-composing with the projection  $\widehat{M} \rightarrow M$  onto the first factor. The manifold  $(\widehat{M}, \widehat{g})$  is a Riemannian analogue of the Killing development of  $(M, g)$ , described in [CM] and recalled in the Appendix. Direct calculations show that

$$\partial_u g(\partial_u, \partial_u) = \partial_u g(\partial_u, \partial_{x_A}) = \partial_u g(\partial_{x_B}, \partial_{x_A}) = 0, \quad \text{on } \widehat{\Sigma} := \Sigma \times \mathbb{R},$$

which implies that  $\widehat{\Sigma}$  is totally geodesic with respect to  $\widehat{g}$ .



We consider again the double  $(\widehat{M}', \widehat{g})$  of  $(\widehat{M}, \widehat{g})$  along  $\widehat{\Sigma}$ . Here,  $\widehat{g}$  extends by reflection across  $\widehat{\Sigma}$  with regularity  $\widehat{g} \in C_{loc}^{2,1}(\widehat{M}')$  as  $\widehat{\Sigma}$  is totally geodesic. As in its Lorentzian counterpart, the second fundamental form of each slice  $\{u = \text{const}\} \subset \widehat{M}'$  is given by  $h_{ij}$ , so we conclude that  $h_{ij} \in C_{-\tau-1}^{1,\alpha}(M')$ , thus proving the claim.

We now conclude that  $E = |P| = 0$ . First, using Claims 2 and 3, and since  $\rho_0$  and  $J$  are clearly continuous across  $\Sigma$ , we get  $\rho_0, J \in C^{0,\alpha}$  across  $\Sigma$ . Consequently, by (11),

$$\rho_0, J \in C_{-n-\varepsilon}^{0,\alpha}(M').$$

The data  $(g, h, \rho_0, J)$  on  $M'$  therefore satisfies the requirements to apply the rigidity result in [HL, Theorem 2]. Denote with  $(E', P')$  the energy-momentum vector of  $M'$ . From (63), the fact that  $\Sigma$  is totally geodesic, and the identity corresponding to (63) in the boundaryless case (due to [AH, Hu, MT, He2]), we get

$$(73) \quad E = d_n \cdot \lim_{r \rightarrow \infty} \int_{\mathbb{S}_{r,+}^{n-1}} G(r\partial_r, \mu) d\mathbb{S}_{r,+}^{n-1} = \frac{1}{2} E'.$$

On the other hand, by symmetry, the component  $P'_n$  of the momentum vector of  $M'$  satisfies

$$P'_n = 2 \lim_{r \rightarrow \infty} \int_{\mathbb{S}_r^{n-1}} \pi(\partial_{x_n}, \mu) d\mathbb{S}_r^{n-1} = 0,$$

and evidently  $P'_A = 2P_A$  for  $1 \leq A \leq n-1$ . Hence,  $E' = |P'|$ , and [HL, Theorem 2] guarantees the vanishing of  $E'$  and  $|P'|$ .

Having shown that  $E = |P| = 0$ , choose a basis of parallel spinors  $\{\phi_m\}$  for  $\mathbb{S}_{\mathbb{R}_+^n}$ , with each  $\phi_m$  satisfying (36). From (58), the corresponding harmonic spinors  $\psi_m \in \Gamma(\mathbb{S}_M)$  satisfy

$$\int_M (|\bar{\nabla}\psi_m|^2 + \langle \mathcal{R}\psi_m, \psi_m \rangle) dM + \frac{1}{2} \int_{\Sigma} \langle (H_g + \mathcal{U})\psi_m, \psi_m \rangle d\Sigma = 0 \quad \forall m,$$

and are therefore parallel, in particular pointwise linearly independent everywhere. Combining this with

$$\begin{aligned} 0 &= \bar{R}_{e_i, e_j} \psi_m := (\bar{\nabla}_{e_i} \bar{\nabla}_{e_j} - \bar{\nabla}_{e_j} \bar{\nabla}_{e_i} - \bar{\nabla}_{[e_i, e_j]}) \psi_m \\ &= \left\{ -\frac{1}{4} [R_{ijks} + h_{ik} h_{js} - h_{is} h_{jk}] e_k \cdot e_s + \frac{1}{2} [h_{kj, i} - h_{ki, j}] e_k \cdot e_0 \right\} \psi_m, \end{aligned}$$

we deduce

$$\bar{R}_{ijks} := R_{ijks} + h_{ik} h_{js} - h_{is} h_{jk} \equiv 0, \quad \bar{R}_{k0ij} := h_{kj, i} - h_{ki, j} \equiv 0.$$

From the above two identities it readily follows that the Killing development  $\bar{M}$  of  $M$  (as defined in [BC, CM], see also the Appendix) is flat, and that  $\partial\bar{M} \hookrightarrow \bar{M}$  is totally geodesic. Moreover,  $M$  meets  $\partial\bar{M}$  orthogonally along  $\Sigma$ . Since  $M$  is asymptotically flat, the only possibility is that  $\bar{M} = \mathbb{L}_+^{n,1}$ . This completes the proof of Theorem 2.6.

**Remark 5.6.** Nowhere in the argument leading to Theorem 2.6 we assume that the noncompact boundary is connected, so we can allow for finitely many *compact* components as part of  $\Sigma$  without compromising the final result, including the rigidity statement. We may additionally envisage a situation in which an extra finite family of *inner* compact boundary components, say  $\{\Gamma_k\}_{k=1}^l$ , should be viewed as (past or future) trapped hypersurfaces. Along these boundary components we

must instead impose chirality boundary conditions (as in Definition 6.1 below) so that the extra term appearing in the right-hand side of (45) becomes

$$-\frac{1}{2} \sum_k \int_{\Gamma_k} (H_g \pm \pi_{nn}) |\psi|^2 d\Gamma_k;$$

see Remark 6.3 below for a justification of the cancellation leading to this boundary contribution. Thus, in order to make sure that this term has the right sign needed to carry out the reasoning we must impose the usual trapped condition  $H_g \pm \pi_{nn} \geq 0$  along the inner boundaries. Since it is well-known that the existence of such trapped inner boundaries foretells, under suitable global conditions, the existence of a black hole region in the Cauchy development of the given initial data set, this argument extends to our setting well-known black hole positive mass theorems previously established in case  $\Sigma$  is empty [GHHP, He1, XD]. Notice however that the rigidity statement does not seem to persist here, and we are led to infer that the presence of trapped surfaces forces the corresponding energy-moment vector to be necessarily time-like. This clearly suggests that a Penrose-type inequality might hold in this setting, a proposition that, at least in the time-symmetric case with  $n = 3$ , has been recently confirmed [Ko].

## 6. THE ASYMPTOTICALLY HYPERBOLIC CASE

In this section we prove Theorems 2.15 and 2.17. We begin by proving Theorem 2.15 which is inspired by [Ma]; see also [CMT]. Let  $(M, g, h, \Sigma)$  be an initial data set with  $(M, g) \hookrightarrow (\bar{M}, \bar{g})$  as in the statement of Theorem 2.15. As in Section 4, over the spin slice  $M$  we have both an extrinsic and an intrinsic description of the restricted spin bundle  $\mathbb{S}_M$ . Thus,  $\mathbb{S}_M$  comes endowed with the inner products  $(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle$  and the connections  $\bar{\nabla}$  and  $\nabla$ , which allow us to define the Dirac-Witten and the intrinsic Dirac operators  $\bar{\mathcal{D}}$  and  $\mathcal{D}$ , respectively. We then define the *Killing connections* on  $\mathbb{S}_M$  by

$$\bar{\nabla}_X^\pm = \bar{\nabla}_X \pm \frac{i}{2} X \cdot$$

and the corresponding *Killing-Dirac-Witten operators* by

$$\bar{\mathcal{D}}^\pm = e_i \cdot \bar{\nabla}_{e_i}^\pm.$$

It is clear that

$$(74) \quad \bar{\mathcal{D}}^\pm = \bar{\mathcal{D}} \mp \frac{ni}{2},$$

which after (37) gives

$$(75) \quad d\hat{\theta} = (\langle \bar{\mathcal{D}}^\pm \phi, \xi \rangle - \langle \phi, \bar{\mathcal{D}}^\mp \xi \rangle) dM.$$

We now introduce the relevant boundary condition on spinors. Consider the chirality operator  $Q = e_0 \cdot : \Gamma(\mathbb{S}_M) \rightarrow \Gamma(\mathbb{S}_M)$ . This is a (pointwise) selfadjoint involution, which is parallel (with respect to  $\bar{\nabla}$ ) and anti-commutes with Clifford multiplication by tangent vectors to  $M$ . We then define the (pointwise) hermitean involution

$$\mathcal{Q} = Q \varrho \cdot = e_0 \cdot \varrho \cdot,$$

acting on spinors restricted to  $\Sigma$ .

**Definition 6.1.** We say that  $\psi \in \Gamma(\mathbb{S}_M)$  satisfies the chirality boundary condition if along  $\Sigma$  it satisfies any of the identities

$$(76) \quad \mathcal{Q}\psi = \pm\psi.$$

**Proposition 6.2.** The operators  $\overline{\mathcal{D}}^+$  and  $\overline{\mathcal{D}}^-$  are formally adjoints to each other under any of the boundary conditions (76).

*Proof.* If  $\phi$  and  $\xi$  are compactly supported we have

$$\int_M \langle \overline{\mathcal{D}}^+ \phi, \xi \rangle dM - \int_M \langle \phi, \overline{\mathcal{D}}^+ \xi \rangle dM = \int_\Sigma \langle \varrho \cdot \phi, \xi \rangle d\Sigma.$$

However, if either  $\mathcal{Q}\phi - \phi = 0 = \mathcal{Q}\xi - \xi$  or  $\mathcal{Q}\phi + \phi = 0 = \mathcal{Q}\xi + \xi$  then it is easy to check that  $\langle \varrho \cdot \phi, \xi \rangle = 0$  on  $\Sigma$ .  $\square$

**Remark 6.3.** Note that if a spinor  $\psi \in \Gamma(\mathbb{S}_M)$  satisfies any of the chirality boundary conditions (76) then  $\langle e_0 \cdot e_A \cdot \psi, \psi \rangle = 0$  along  $\Sigma$ . Indeed,

$$\begin{aligned} \langle e_0 \cdot e_A \cdot \psi, \psi \rangle &= \langle e_0 \cdot e_A \cdot e_0 \cdot e_n \cdot \psi, e_0 \cdot e_n \cdot \psi \rangle \\ &= \langle e_A \cdot e_0 \cdot e_n \cdot \psi, e_n \cdot \psi \rangle \\ &= \langle e_n \cdot e_A \cdot e_0 \cdot \psi, e_n \cdot \psi \rangle \\ &= \langle e_A \cdot e_0 \cdot \psi, \psi \rangle \\ &= -\langle e_0 \cdot e_A \cdot \psi, \psi \rangle. \end{aligned}$$

**Proposition 6.4.** Given a spinor  $\psi \in \Gamma(\mathbb{S}_M)$ , define the  $(n-1)$ -forms

$$\theta^+ = \langle e_i \cdot \overline{\mathcal{D}}^+ \psi, \psi \rangle e_i \lrcorner dM \quad \text{and} \quad \eta^+ = \langle \overline{\nabla}_{e_i}^+ \psi, \psi \rangle e_i \lrcorner dM.$$

Then it holds

$$(77) \quad d\theta^+ = \left( \langle (\overline{\mathcal{D}}^+)^2 \psi, \psi \rangle - |\overline{\mathcal{D}}^+ \psi|^2 \right) dM,$$

and

$$(78) \quad d\eta^+ = \left( -\langle (\overline{\nabla}^+)^* \overline{\nabla}^+ \psi, \psi \rangle + |\overline{\nabla}^+ \psi|^2 \right) dM.$$

*Proof.* Straightforward computations similar to those of Proposition 4.5.  $\square$

We now combine this with the corresponding Weitzenböck formula, namely,

$$(79) \quad (\overline{\mathcal{D}}^+)^2 = (\overline{\nabla}^+)^* \overline{\nabla}^+ + \mathcal{W},$$

where the symmetric endomorphism  $\mathcal{W}$  is given by

$$\mathcal{W} = \frac{1}{4} (R_{\overline{g}} + n(n-1) + 2\text{Ric}_{\overline{g}_{0\alpha}} e_\alpha \cdot e_0).$$

**Remark 6.5.** As in Remark 4.7, we see that the DEC (2) with  $\Lambda = -n(n-1)/2$  implies that  $\mathcal{W} \geq 0$ .

By putting together the results above we obtain a fundamental integration by parts formula. This is the analogue of Proposition 4.8.

**Proposition 6.6.** If  $\psi \in \Gamma(\mathbb{S}_M)$  and  $\Omega \subset M$  is compact then

$$(80) \quad \int_\Omega (|\overline{\nabla}^+ \psi|^2 + \langle \mathcal{W}\psi, \psi \rangle - |\overline{\mathcal{D}}^+ \psi|^2) dM = \int_{\partial\Omega} \langle (\overline{\nabla}_{e_i}^+ + e_i \cdot \overline{\mathcal{D}}^+) \psi, \psi \rangle e_i \lrcorner dM.$$

As in Proposition 5.1, we now specialize (80) to the case in which  $\Omega = \Omega_r$ , the compact region in a initial data set  $(M, g, h, \Sigma)$  determined by the coordinate hemisphere  $S_{r,+}^{n-1}$  in the asymptotic region; see Figure 1 for a similar configuration.

**Proposition 6.7.** *With the notation above assume that  $\psi \in \Gamma(\mathbb{S}_M)$  satisfies the boundary condition (76) along  $\Sigma$ . Then*

$$(81) \quad \int_{\Omega_r} (|\bar{\nabla}^+ \psi|^2 + \langle \mathcal{W}\psi, \psi \rangle - |\bar{\mathcal{D}}^+ \psi|^2) dM = \int_{S_{r,+}^{n-1}} \langle (\bar{\nabla}_{e_i}^+ + e_i \cdot \bar{\mathcal{D}}^+) \psi, \psi \rangle e_i \lrcorner dM - \frac{1}{2} \int_{\Sigma_r} (H_g \pm \pi_{nn}) |\psi|^2 d\Sigma,$$

where the  $\pm$  sign agrees with the one in (76).

*Proof.* We first observe that, using (74) and similarly to (42),

$$(82) \quad \bar{\nabla}_{e_i}^+ + e_i \cdot \bar{\mathcal{D}}^+ = \nabla_{e_i} + e_i \cdot \mathcal{D} - \frac{1}{2} \pi_{ij} e_0 \cdot e_j \cdot + \frac{n-1}{2} \mathbf{i} e_i,$$

so that

$$(83) \quad \begin{aligned} \int_{\Sigma_r} \langle (\bar{\nabla}_{e_i}^+ + e_i \cdot \bar{\mathcal{D}}^+) \psi, \psi \rangle e_i \lrcorner dM &= \int_{\Sigma_r} \langle (\nabla_{e_i} + e_i \cdot \mathcal{D}) \psi, \psi \rangle e_i \lrcorner dM \\ &\quad - \frac{1}{2} \int_{\Sigma_r} \pi_{ij} \langle e_0 \cdot e_j \cdot \psi, \psi \rangle e_i \lrcorner dM \\ &\quad + \frac{n-1}{2} \mathbf{i} \int_{\Sigma_r} \langle \varrho \cdot \psi, \psi \rangle d\Sigma \\ &= \int_{\Sigma_r} \left\langle \left( D^\top - \frac{H_g}{2} \right) \psi, \psi \right\rangle d\Sigma \\ &\quad - \frac{1}{2} \int_{\Sigma_r} \pi(e_n, e_j) \langle e_0 \cdot e_j \cdot \psi, \psi \rangle d\Sigma \\ &\quad + \frac{n-1}{2} \mathbf{i} \int_{\Sigma_r} \langle \varrho \cdot \psi, \psi \rangle d\Sigma. \end{aligned}$$

However, as in the proof of Propositions 5.1 and 6.2, and by Remark 6.3, the boundary condition (see [AdL] for further discussions on this) implies that  $\langle D^\top \psi, \psi \rangle = 0$ ,  $\langle \varrho \cdot \psi, \psi \rangle = 0$  and  $\langle e_0 \cdot e_A \cdot \psi, \psi \rangle = 0$ . So, the equation (83) becomes

$$\int_{\Sigma_r} \langle (\bar{\nabla}_{e_i}^+ + e_i \cdot \bar{\mathcal{D}}^+) \psi, \psi \rangle e_i \lrcorner dM = -\frac{1}{2} \int_{\Sigma_r} (H_g |\psi|^2 + \pi_{nn} \langle \mathcal{Q}\psi, \psi \rangle) d\Sigma,$$

from which the result follows.  $\square$

We now proceed with the proof of Theorem 2.15. We start by picking a *Killing* spinor  $\phi$  in the restricted reference spin bundle  $\mathbb{S}_{\mathbb{H}_+^n}$ , which by definition means that  $\bar{\nabla}^+ \phi = 0$  for the metric  $b$ . We assume that, along  $\partial\mathbb{H}_+^n$ ,  $\phi$  satisfies one of the chirality boundary conditions (76). Thus,

$$(84) \quad \bar{e}_0 \cdot \bar{e}_n \cdot \phi = \pm \phi,$$

where, here and in the next proposition,  $\{\bar{e}_\alpha\}$  is an adapted orthonormal frame with respect to  $\bar{b}$ . As the space of Killing spinors in  $\mathbb{H}^n$  is identified with  $\mathbb{C}^d$ ,  $d = \lfloor \frac{n}{2} \rfloor$ , the quadratic form  $\mathcal{K} : \mathbb{C}^d \rightarrow \mathcal{N}_b^+ \oplus \mathfrak{K}_b^+$  is given by the following proposition:

**Proposition 6.8.** *Each Killing spinor  $\phi$  as above gives rise to an element*

$$\mathcal{K}(\phi) := (V_\phi, W_\phi) \in \mathcal{N}_b^+ \oplus \mathfrak{K}_b^+ \cong \mathbb{L}^{1,n-1} \oplus \mathbb{L}^{1,n-1}$$

by means of the prescriptions

$$(85) \quad V_\phi = \langle \phi, \phi \rangle, \quad W_\phi = \langle \bar{e}_0 \cdot \bar{e}_i \cdot \phi, \phi \rangle \bar{e}_i.$$

Moreover, any  $V \in \mathcal{N}_b^+$  or  $W \in \mathfrak{K}_b^+$  on the corresponding future light cone may be obtained in this way.

*Proof.* Define a 1-form on  $\text{AdS}_+^{n,1}$  by

$$\alpha_\phi(Z) = \langle \bar{e}_0 \cdot Z \cdot \phi, \phi \rangle = (Z \cdot \phi, \phi)$$

A simple computation shows that

$$(\bar{\nabla}_Z \alpha_\phi)(Z') = \frac{\mathbf{i}}{2} ((Z \cdot Z' \cdot -Z' \cdot Z) \phi, \phi) = -(\bar{\nabla}_{Z'} \alpha_\phi)(Z),$$

so the dual vector field

$$\widetilde{W}_\phi = \langle \bar{e}_0 \cdot \bar{e}_\alpha \cdot \phi, \phi \rangle \bar{e}_\alpha$$

is Killing (with respect to  $\bar{b}$ ). Since  $\langle \bar{e}_0 \cdot \bar{e}_0 \cdot \phi, \phi \rangle = V_\phi$ , we have  $\widetilde{W}_\phi = V_\phi \bar{e}_0 + W_\phi$ , which we identify to  $\mathcal{K}(\phi) = (V_\phi, W_\phi)$ . It is easy to check that

$$dV_\phi(X) = \mathbf{i} \langle X \cdot \phi, \phi \rangle, \quad X \in \Gamma(T\mathbb{H}_+^n),$$

so that, along  $\partial\mathbb{H}_+^n$ ,  $\partial V_\phi / \partial y_n = \mathbf{i} \langle \partial_{y_n} \cdot \phi, \phi \rangle = 0$ , where in the last step we used the chirality boundary condition. This shows that  $V_\phi \in \mathcal{N}_b^+$ . Also, from Remark 6.3 we get that, along  $\partial\mathbb{H}_+^n$ ,  $W_\phi = \pm V_\phi \bar{e}_n$ , which means that  $W_\phi \in \mathfrak{K}_b^+$ . Finally, the last assertion of the proposition for  $V \in \mathcal{N}_b^+$  is well-known (see [AdL, Proposition 5.1]) and the corresponding statement for  $W \in \mathfrak{K}_b^+$  follows from the isomorphism  $\mathfrak{K}_b^+ \cong \mathcal{N}_b^+$  already established in Proposition 3.5.  $\square$

Under the conditions of Theorem 2.15, the standard analytical argument allows us to obtain a spinor  $\psi \in \Gamma(\mathbb{S}_M)$  which is Killing harmonic ( $\bar{\mathcal{D}}^+ \psi = 0$ ), asymptotes  $\phi$  at infinity, and satisfies one of the chirality boundary conditions (76) (with sign agreeing with the one for  $\phi$ ). Note that, on asymptotically hyperbolic manifolds, for given  $\eta \in L^2(\mathbb{S}_M)$  a solution of

$$\begin{cases} \bar{\mathcal{D}}^+ \varphi = \eta & \text{on } M \\ \mathcal{Q}\varphi = \pm \varphi & \text{on } \Sigma \end{cases}$$

can be found in  $H^1(\mathbb{S}_M)$  and not in a larger space like the completion  $\mathcal{H}$  appearing in the proof of Proposition 5.3. This makes the proof much simpler, in the sense that it follows directly from the existence theory in [GN], and relies on the fact that the weight in the Hardy inequality on  $\mathbb{H}^n$  has a positive lower bound, cf. [BaC, Theorem 9.10].

Replacing this  $\psi$  in (81) is the first step in proving the following Witten-type formula, which extends results in [CH, AdL, Ma]:

**Theorem 6.9.** *Under the conditions above, there holds*

$$(86) \quad \begin{aligned} \frac{1}{4} \widetilde{\mathcal{K}}(\phi) &= \int_M (|\bar{\nabla}^+ \psi|^2 + \langle \mathcal{W}\psi, \psi \rangle) dM \\ &+ \frac{1}{2} \int_\Sigma (H_g \pm \pi_{nn}) |\psi|^2 d\Sigma, \end{aligned}$$

where  $\tilde{\mathcal{K}} = \mathfrak{m}_{(g,h,F)} \circ \mathcal{K}$ .

*Proof.* From (81) we have

$$\begin{aligned} \lim_{r \rightarrow +\infty} \int_{S_{r,+}^{n-1}} \langle (\bar{\nabla}_{e_i}^+ + e_i \cdot \bar{\mathcal{D}}^+) \psi, \psi \rangle e_i \lrcorner dM &= \int_M (|\bar{\nabla}^+ \psi|^2 + \langle \mathcal{W} \psi, \psi \rangle) dM \\ &+ \frac{1}{2} \int_{\Sigma} (H_g \pm \pi_{nn}) |\psi|^2 d\Sigma. \end{aligned}$$

Hence, in order to prove (86) we must check that

$$(87) \quad \lim_{r \rightarrow +\infty} \int_{S_{r,+}^{n-1}} \langle (\bar{\nabla}_{e_i}^+ + e_i \cdot \bar{\mathcal{D}}^+) \psi, \psi \rangle e_i \lrcorner dM = \frac{1}{4} \mathfrak{m}_{(g,h,F)}(V_\phi, W_\phi).$$

We note that (82) may be rewritten as

$$(88) \quad \bar{\nabla}_{e_i}^+ + e_i \cdot \bar{\mathcal{D}}^+ = \nabla_{e_i}^+ + e_i \cdot \mathcal{D}^+ - \frac{1}{2} \pi_{ij} e_0 \cdot e_j,$$

where

$$\nabla_X^+ = \nabla_X + \frac{\mathbf{i}}{2} X, \quad \mathcal{D}^+ = \mathcal{D} - \frac{n\mathbf{i}}{2},$$

are the intrinsic Killing connection and the intrinsic Killing Dirac operator, respectively. Since the computation in [AdL, Section 5] gives

$$\lim_{r \rightarrow +\infty} \int_{S_{r,+}^{n-1}} \langle (\nabla_{e_i}^+ + e_i \cdot \mathcal{D}^+) \psi, \psi \rangle e_i \lrcorner dM = \frac{1}{4} \mathfrak{m}_{(g,h,F)}(V_\phi, 0),$$

and it is clear from (18) and (85) that

$$\begin{aligned} \lim_{r \rightarrow +\infty} \frac{1}{2} \int_{S_{r,+}^{n-1}} \pi_{ij} \langle e_0 \cdot e_j \cdot \psi, \psi \rangle e_i \lrcorner dM &= - \lim_{s \rightarrow +\infty} \frac{1}{2} \int_{S_{s,+}^{n-1}} \pi(\mu, e_j) \langle e_0 \cdot e_j \cdot \phi, \phi \rangle d\Sigma \\ &= -\frac{1}{4} \mathfrak{m}_{(g,h,F)}(0, W_\phi), \end{aligned}$$

we readily obtain (87).  $\square$

The inequality  $\tilde{\mathcal{K}} \geq 0$  in Theorem 2.15 is an immediate consequence of (86) and the assumed DEC. Indeed, given any Killing spinor  $\phi \in \Gamma(\mathbb{S}_{\mathbb{H}^n})$  satisfying (84), by Remark 6.5 and using that  $(\varrho \lrcorner \pi)^\perp = \pi_{nn}$ , the r.h.s. of (86) is non-negative. Using both signs on (86), this covers the full space  $\mathbb{C}^d$  of Killing spinors on  $\mathbb{H}^n$  and proves that  $\tilde{\mathcal{K}} \geq 0$ . Observe that the argument also works when  $\bar{\mathcal{D}}^+$  is replaced by  $\bar{\mathcal{D}}^-$ .

As for the rigidity statement, the assumption  $\tilde{\mathcal{K}} = 0$  implies that  $\mathbb{S}_M$  is trivialized by the Killing spinors  $\{\psi_m\}$  associated to the basis  $\{\phi_m\}$ . From this point on, the argument is pretty much like that in [Ma], so it is omitted. As for the remaining properties of  $\Sigma$ , they are readily checked by combining the arguments in the proofs of [AdL, Theorem 5.4] and Theorem 2.6 above. This proves Theorem 2.15.

Lastly, we prove Theorem 2.17. That the inequality  $\tilde{\mathcal{K}} \geq 0$  implies the mentioned causal character of  $(\mathcal{E}, \mathcal{P})$  follows from the last statement in Proposition 6.8. Also,  $\mathcal{E} = \mathcal{P} = 0$  clearly implies that  $\tilde{\mathcal{K}} = 0$ .

7. APPENDIX: THE KILLING DEVELOPMENT OF  $M$ 

Computationally, the Killing development is quite similar to its Riemannian counterpart, described in Claim 4 of Section 5. Having fixed a local orthonormal frame  $\{e_i\}$  on  $M$ , with dual coframe  $\{\theta^j\}$  and connection forms  $\omega_j^i$ , we consider  $\widetilde{M} = M \times \mathbb{R}$  endowed with the tensor

$$\widetilde{g} = -V^2 du^2 + \sum_i (\theta^i - W^i du) \otimes (\theta^i - W^i du),$$

where  $W = W^i e_i$  and we implicitly assume  $W^i, V, \theta^i, \omega_j^i$  to be pulled-back to  $\widetilde{M}$  via the projection  $\mathfrak{p} : \widetilde{M} \rightarrow M$  onto the first factor. Clearly,  $\widetilde{g}$  is a Lorentzian metric on  $\{V > 0\} \times \mathbb{R}$ , in particular on the entire  $\widetilde{M}$  in the assumptions of Theorem 2.6 for  $E = |P|$  (that we showed to imply  $V > 0$  on  $M$ ). One readily checks the following properties:

- The frame

$$\widetilde{e}_0 = \frac{1}{V} (\partial_u + W^i e_i), \quad \widetilde{e}_i = e_i$$

is orthonormal for  $\widetilde{g}$  ( $e_j$  being the field on  $\widetilde{M}$  tangent to fibers  $\{u = \text{const}\}$  and projecting to  $e_j$ ), dual to the coframe

$$\widetilde{\theta}^0 = V du, \quad \widetilde{\theta}^i = \theta^i - W^i du.$$

In particular,  $\widetilde{e}_0$  a timelike, unit normal field to each slice  $\{u = \text{const}\}$ .

- The forms

$$\begin{cases} \widetilde{\omega}_0^j = \widetilde{\omega}_j^0 = \frac{V_j}{V} \widetilde{\theta}^0 + \frac{(\mathcal{L}_W g)_{jk}}{2V} \widetilde{\theta}^k \\ \widetilde{\omega}_j^i = \omega_j^i + \frac{W_j^i - W_i^j}{2V} \widetilde{\theta}^0 \end{cases}$$

satisfy the structure equations

$$\begin{cases} d\widetilde{\theta}^\alpha = -\widetilde{\omega}_\beta^\alpha \wedge \widetilde{\theta}^\beta \\ \widetilde{\omega}_j^i = -\widetilde{\omega}_i^j, \quad \widetilde{\omega}_0^j = \widetilde{\omega}_j^0, \end{cases}$$

and are therefore the connection forms of the Levi-Civita connection  $\widetilde{\nabla}$  of  $\widetilde{g}$ . When (65) holds,

$$(89) \quad \begin{cases} \widetilde{\omega}_0^j = \widetilde{\omega}_j^0 = \frac{V_j}{V} \widetilde{\theta}^0 + h_{jk} \widetilde{\theta}^k = h_{jk} \theta^k \\ \widetilde{\omega}_j^i = \omega_j^i. \end{cases}$$

- The second fundamental form of each slice in the  $\widetilde{e}_0$  direction is  $h_{ij} \theta^i \otimes \theta^j$ , and the field  $\partial_u$  is  $\widetilde{\nabla}$ -parallel.
- The components  $\widetilde{R}_{\alpha\beta\gamma\delta}$  of the curvature tensor  $\widetilde{R}$  of  $\widetilde{g}$  satisfy

$$(90) \quad \begin{aligned} \widetilde{R}_{ijkl} &= R_{ijkl} + h_{ik} h_{jt} - h_{it} h_{jk}, \\ \widetilde{R}_{ijk0} &= [R_{ijkl} + h_{ik} h_{jt} - h_{it} h_{jk}] \frac{W^t}{V} = h_{kj,i} - h_{ki,j} \\ \widetilde{R}_{i0k0} &= (h_{it,k} - h_{ik,t}) \frac{W^t}{V}. \end{aligned}$$

Here, we use the agreement

$$\begin{aligned} \widetilde{R}(\widetilde{e}_\alpha, \widetilde{e}_\beta) \widetilde{e}_\gamma &= \widetilde{\nabla}_{\widetilde{e}_\alpha} \widetilde{\nabla}_{\widetilde{e}_\beta} \widetilde{e}_\gamma - \widetilde{\nabla}_{\widetilde{e}_\beta} \widetilde{\nabla}_{\widetilde{e}_\alpha} \widetilde{e}_\gamma - \widetilde{\nabla}_{[\widetilde{e}_\alpha, \widetilde{e}_\beta]} \widetilde{e}_\gamma = R_{\gamma\alpha\beta}^\delta \widetilde{e}_\delta \\ \widetilde{R}_{\delta\gamma\alpha\beta} &= \widetilde{g}(\widetilde{R}(\widetilde{e}_\alpha, \widetilde{e}_\beta) \widetilde{e}_\gamma, \widetilde{e}_\delta). \end{aligned}$$

In the setting of Theorem 2.6, for  $E = |P|$ , the immersion  $\widetilde{\Sigma} := \Sigma \times \mathbb{R} \hookrightarrow \widetilde{M}$  is totally geodesic: indeed, by Claim 2 and (89),

$$\langle \widetilde{\nabla}_{\widetilde{e}_A} \widetilde{e}_B, \widetilde{e}_n \rangle = b_{AB} = 0, \quad \langle \widetilde{\nabla}_{\widetilde{e}_A} \widetilde{e}_0, \widetilde{e}_n \rangle = h_{An} = 0,$$

and

$$\langle \widetilde{\nabla}_{\widetilde{e}_0} \widetilde{e}_0, \widetilde{e}_n \rangle = \langle \frac{V_j}{V} \widetilde{e}_j, \widetilde{e}_n \rangle = \frac{V_n}{V} = 0.$$

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