Random-Field Solutions of Linear Parabolic Stochastic Partial Differential Equations with Polynomially Bounded Variable Coefficients

Publisher:

Springer

Published version:

DOI:10.1007/978-3-030-61346-4_2

(Article begins on next page)

Random-field Solutions of Linear Parabolic Stochastic Partial Differential Equations with Polynomially Bounded Variable Coefficients

Alessia Ascanelli, Sandro Coriasco, and André Süß

To Massimo and Michael, on occasion of their 60*th birthday*

Abstract We study random-field solutions of a class of stochastic partial differential equations, involving operators with polynomially bounded coefficients. We consider linear equations under suitable parabolicity hypotheses, and we provide conditions on the initial data and on the stochastic term, namely, on the associated spectral measure, so that these kind of solutions exist in suitably chosen functional classes. We also give a regularity result for the expected value of these solutions.

Key words: Parabolic stochastic partial differential equations; Random-field solutions; Variable coefficients; Fundamental solution
2010 Mathematics Subject Classification. Primary: 35L10, 60H15; Secondary: 35L40, 35S30.

1 Introduction

We consider linear stochastic partial differential equations (SPDEs in the sequel) of the general form

---

Alessia Ascanelli
Dipartimento di Matematica ed Informatica, Università di Ferrara, Via Machiavelli n. 30, 44121 Ferrara, Italy e-mail: alessia.ascanelli@unife.it

Sandro Coriasco
Dipartimento di Matematica "G. Peano", Università degli Studi di Torino, via Carlo Alberto n. 10, 10123 Torino, Italy e-mail: sandro.coriasco@unito.it

André Süß
C/O Dipartimento di Matematica ed Informatica, Università di Ferrara, Via Machiavelli n. 30, 44121 Ferrara, Italy e-mail: suess.andre@web.de

$$Lu(t,x) = [\partial_t + A(t)]u(t,x) = \gamma(t,x) + \sigma(t,x)\dot{\Xi}(t,x), \tag{1}$$

where:

- $A(t)$ is a continuous family of linear partial differential operators that contain partial derivatives in space ($x \in \mathbb{R}^d$, $d \geq 1$),
- γ, σ are real-valued functions, subject to certain regularity conditions,
- Ξ is an $\mathcal{S}'(\mathbb{R}^d)$-valued Gaussian process, white in time and coloured in space, with correlation measure Γ and spectral measure $\mathfrak{M}$ (see Section 2 for a precise definition),
- u is an unknown stochastic process called *solution* of the SPDE.

To give meaning to (1) we rewrite it in its corresponding integral form and look for *mild solutions of* (1), that is, stochastic processes $u(t,x)$ satisfying

$$\begin{aligned} u(t,x) = v_0(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y)dyds \\ &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)dyds, \end{aligned} \tag{2}$$

where:

- v_0 is a deterministic term, taking into account the initial condition;
- Λ is a suitable kernel, associated with the fundamental solution of the partial differential equation (PDE in the sequel) $Lu = [\partial_t + A(t)]u = 0$;
- the first integral in (2) is of deterministic type, while the second is a stochastic integral, and both are distributional integrals since $\Lambda(t,s,x,y)$ is, in general, a distribution with respect to the variables $(x,y) \in \mathbb{R}^{2d}$.

The kind of solution u we can construct for equation (1) depends on the approach we employ to make sense of the stochastic integral appearing in (2).

In the present paper we are looking for a *random-field solution* of (1), that is, we rely on the theory of stochastic integration with respect to a martingale measure developed in [8, 11, 21]. We are so going to define the stochastic integral in (2) through the martingale measure derived from the random noise $\dot{\Xi}$. Consequently, we are going to obtain a *random-field solution*, that is, a solution u defined as a map associating a random variable with each $(t,x) \in [0,T_0] \times \mathbb{R}^d$, where $T_0 > 0$ is the time horizon of the solution of the equation.

Recently, the existence of a random-field solution in the case of linear and semilinear hyperbolic SPDEs, involving operators with (t,x)-dependent coefficients, has been shown: first for linear operators with uniformly bounded coefficients [7], and subsequently for operators with polynomially bounded coefficients, both for linear equations [4] as well as for semilinear equations [5]. The main tools used for achieving this objective, namely, pseudodifferential and Fourier integral operators, come from microlocal analysis. To our knowledge, those are the first times that their full potential has been rigorously applied within the theory of random-field solutions to hyperbolic SPDEs.

Coming now to parabolic SPDEs, Dalang [11] studied random field solutions to parabolic equations with t-continous coefficients of the form

$$\begin{aligned}\partial_t u(t,x) - \left(\sum_{i,j=1}^{n} a_{i,j}(t)\partial^2_{x_i x_j} + \sum_{i=1}^{n} b_i(t)\partial_{x_i} + c(t)\right)u(t,x)\\ = \gamma(u(t,x)) + \sigma(u(t,x))\dot{\Xi}(t,x)\end{aligned} \tag{3}$$

under the coercivity assumption

$$\sum_{i,j=1}^{n} a_{i,j}(t)\xi_i\xi_j \geq \epsilon|\xi|^2, \quad (t,\xi) \in [0,T] \times \mathbb{R}^d,$$

for some constant $\epsilon > 0$. He obtained a random field solution of (3) under the condition

$$\int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{1+|\xi|^2} d\xi < \infty.$$

Furthermore, Sanz-Solé and Vuillermont [19] proved the existence and uniqueness of a variational random-field solution to a class of initial-boundary value problems for second order parabolic equations with variable coefficients of the form

$$\partial_t u(t,x) = \operatorname{div}\left(k(t,x)\nabla u(t,x)\right) + \gamma(u(t,x)) + \sigma(u(t,x))W(t,x),\ (t,x) \in [0,T]\times D,$$

with D a sufficiently regular bounded domain in $\mathbb{R}^d$, k a positive definite symmetric matrix, $W(t,x)$ a Wiener process.

In the present paper we deal with the existence of a random-field solution to linear parabolic SPDEs of the form (1) with (t,x)-dependent coefficients admitting, at most, a polynomial growth as $|x| \to \infty$. More precisely, here we treat *parabolic equations* of arbitrary order $m, \mu > 0$ of the form (1), whose coefficients are defined on the whole space $\mathbb{R}^d$, that is

$$L = \partial_t + A(t), \qquad A(t)u(t,x) = \sum_{|\alpha|\leq\mu} a_\alpha(t,x)\,(D^\alpha_x u)(t,x), \tag{4}$$

$D = -i\partial$, where $\mu \geq 1$, $a_\alpha \in C([0,T], C^\infty(\mathbb{R}^d))$ for $|\alpha| \leq \mu$, and, for all $\beta \in \mathbb{N}_0^d = (\mathbb{N}\cup\{0\})^d$, there exists a constant $C_{\alpha\beta} > 0$ such that

$$|\partial^\beta_x a_\alpha(t,x)| \leq C_{\alpha\beta}\langle x\rangle^{m-|\beta|},$$

for all $(t,x) \in [0,T]\times\mathbb{R}^d$, where $\langle x\rangle := (1+|x|^2)^{1/2}$. The parabolicity of L means that the parameter-dependent symbol $a(t,x,\xi)$ of the SG-operators family $A(t)$, defined here below, satisfies

$$a(t,x,\xi) := \sum_{|\alpha|\leq\mu} a_\alpha(t,x)\xi^\alpha \geq C\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}, \tag{5}$$

with $C > 0$, $m \geq m' > 0$, $\mu \geq \mu' > 0$, that is, a is SG-hypoelliptic. Postponing to the next Section 3 the precise characterization, we give here an example.

Example 1 An example of a SG-parabolic operator L is given by the generalized SG-heat operator, defined for every $m, \mu \in \mathbb{N} \setminus \{0\}$ by

$$L = \partial_t + \langle x \rangle^{2m} \langle D \rangle^{2\mu}, \qquad x \in \mathbb{R}^d.$$

In this case $m = m'$, $\mu = \mu'$, that is, a is SG-elliptic.

We study SPDEs of the form (1), (4), (5), and we derive conditions on the right-hand side terms γ and σ, and on the spectral measure $\mathfrak{M}$ (hence, on $\dot{\Xi}$), such that there exists a random-field (mild) solution to the corresponding Cauchy problem.

As customary for the classes of the associated deterministic PDEs, we are interested in the present paper in both the smoothness, as well as the decay/growth at spatial infinity of the solutions. Here we also obtain an analog of such *global regularity* properties, employing suitable *weighted Sobolev spaces*, namely, the so-called Sobolev-Kato spaces $H^{z,\zeta}(\mathbb{R}^d)$, $z, \zeta \in \mathbb{R}$ defined by

$$H^{z,\zeta}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) \colon \|u\|_{z,\zeta} = \|\langle \cdot \rangle^z \langle D \rangle^\zeta u\|_{L^2} < \infty\}. \tag{6}$$

The results proved in this paper expand the theory developed in [4, 7] to the cases of operators L which are parabolic and whose coefficients are not uniformly bounded, and expand the results of [11] to the case of space-depending coefficients with polynomial growth and of higher order equations (there, second order operators are considered). Our main result reads as follows (see Sections 3 and 4, and Theorem 6 below, for the precise definitions and statement).

Theorem Let us consider the Cauchy problem

$$\begin{cases} Lu(t,x) = \gamma(t,x) + \sigma(t,x)\dot{\Xi}(t,x), & (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{7}$$

for a SPDE associated with an SG-parabolic operator L of the form (4) with $m \geq m' > 0$, $\mu \geq \mu' > 0$. Let $u_0 \in H^{z,\zeta}(\mathbb{R}^d)$, with $z \geq 0$ and $\zeta > d/2$, and assume that $\gamma \in C([0,T]; H^{z,\zeta}(\mathbb{R}^d))$, $\sigma \in C([0,T], H^{0,\zeta}(\mathbb{R}^d))$, $s \mapsto \mathcal{F}\sigma(s) = \nu_s \in L^2([0,T], \mathcal{M}_b(\mathbb{R}^d))$, where $\mathcal{M}_b(\mathbb{R}^d)$ is the space of complex-valued measures with finite total variation. Assume that one of the following conditions on the spectral measure $\mathfrak{M}$, associated with the random noise $\dot{\Xi}$, holds:

(H0) either, for every $t \in [0,T]$,

$$\sup_{0 \leq s < t} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |e(t,s,x,\xi+\eta)|^2 \, \mathfrak{M}(d\xi) < \infty$$

and for every $0 \leq s < t$

$$\lim_{h\downarrow 0}\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\sup_{r\in(s,s+h)}|e(t,s,x,\xi+\eta)-e(t,r,x,\xi+\eta)|^2\,\mathfrak{M}(d\xi)=0,$$

where $e(t,s)$ is the (parameter-dependent) symbol of the fundamental solution of the operator L,

(H1) or

$$\int_{\mathbb{R}^d}\mathfrak{M}(d\xi)<\infty,$$

(H2) or $\mathfrak{M}$ is absolutely continuous, $|\nu_s|_{\mathrm{tv}}\in L^\infty(0,T)$, and

$$\int_{\mathbb{R}^d}\frac{\mathfrak{M}(d\xi)}{\langle\xi\rangle^{\mu'}}<\infty.$$

Then, there exists a random-field solution u of (7). Moreover, for any $\kappa\in[0,1)$,

$$\begin{aligned}\mathbb{E}[u]\in C([0,T],H^{z,\zeta}(\mathbb{R}^d))\cap C^1([0,T],H^{z-m,\zeta-\mu}(\mathbb{R}^d))\cap\\ \cap\, C^1((0,T],\mathcal{S}(\mathbb{R}^d))\cap L^1([0,T],H^{z+\kappa m',\zeta+\kappa\mu'}(\mathbb{R}^d)),\end{aligned}$$

and also $\partial_t\mathbb{E}[u]\in L^1([0,T],H^{z-m+\kappa m',\zeta-\mu+\kappa\mu'}(\mathbb{R}^d))$, $\kappa\in[0,1)$.

Remark 1 Notice that we find, for general parabolic SPDEs with coefficients in (t,x), possibly polynomially growing as $|x|\to\infty$, in the case of an absolutely continuous spectral measure and $|\nu_s|_{\mathrm{tv}}$ bounded, the same condition given in [11] on the spectral measure, with $\mu=\mu'=2$, see **(H2)**.

The main tools for proving the existence of random-field solutions to (1) will be pseudodifferential operators with symbols in the so-called SG classes. Such symbol classes have been introduced in the '70s by H.O. Cordes (see, e.g. [9]) and C. Parenti [17] (see also R. Melrose [16]). The strategy to prove the main theorem consists of the following steps:

1. construction of the fundamental solution of L in (4), and then (formally) of the solution u to (7);
2. proof of the fact that v_0 and the stochastic and deterministic integrals, appearing in the (formal) expression (2) of u, are well-defined.

For point (1) we need, on one hand, to perform compositions between pseudodifferential operators, using the theory developed, e.g., in [9], and, on the other hand, the construction of the fundamental solution of parabolic operators in the SG environment. The latter can be achieved in analogy to the theory developed in [14, Chapter 7, §4], but here, in addition, we obtain more precise information about the order of the pseudodifferential operator family $E(t,s)$ that defines the fundamental solution of L. For point (2) we rely on (a variant of) results proved in [7].

With the aim of giving a presentation as self-contained as possible, for the convenience of the reader, we provide various preliminaries from the existing literature. The paper is organized as follows.

In Section 2 we recall some notions about stochastic integration with respect to martingale measures and the corresponding concept of random-field solution to a SPDE. Since, in contrast to the classical references [11, 21], here we have to deal with integrands of the form $\Lambda(t,s,x,y)\sigma(s,y)$ with (t,x) fixed, we directly present here the conditions that Λ and σ have to satisfy to let the stochastic integral with respect to a martingale measure in (2) be well-defined.

In Section 3 we first give a brief summary of the main tools, coming from microlocal analysis, that we use for the construction of the fundamental solution operator and of its kernel $\Lambda(t,s,x,y)$ (these results come mainly from [9, 15]). Subsequently, we perform the construction of the fundamental solution of the SG-parabolic operator L. To our best knowledge, compared with the previously existing literature, such construction for this operator class, which is essential to us to prove our main theorem, has not appeared elsewhere.

In Section 4 we focus on the parabolic SPDE (1), (4), (5), and prove our main theorem, under appropriate assumptions (see Theorem 6). Finally, we mention that the results illustrated in Section 4 about the structure of the kernel $\Lambda(t,s,x,y)$ appearing in (2) are employed also in [6], where we look for function-valued solutions to the semilinear parabolic SPDEs

$$Lu(t,x) = \gamma(t,x,u(t,x)) + \sigma(t,x,u(t,x))\dot{\Xi}(t,x) \tag{8}$$

associated with (1).

2 Stochastic integration with respect to a martingale measure.

Let us consider a distribution-valued Gaussian process $\{\Xi(\phi);\ \phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with mean zero and covariance functional given by

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} \big(\phi(t) * \tilde{\psi}(t)\big)(x)\, \Gamma(dx)dt, \tag{9}$$

where $\tilde{\psi}(t,x) := \psi(t,-x)$, $*$ is the convolution operator and Γ is a nonnegative, nonnegative definite, tempered measure on $\mathbb{R}^d$ usually called *correlation measure*. Then [20, Chapter VII, Théorème XVIII] implies that there exists a nonnegative tempered measure $\mathfrak{M}$ on $\mathbb{R}^d$, usually called *spectral measure*, such that $\mathcal{F}\Gamma = \mathfrak{M}$, where $\mathcal{F}$ denotes the Fourier transform. By Parseval's identity, the right-hand side of (9) can be rewritten as

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} [\mathcal{F}\phi(t)](\xi) \cdot \overline{[\mathcal{F}\psi(t)](\xi)}\, \mathfrak{M}(d\xi)dt.$$

Definition 1 We call *(mild) random-field solution to* (1) an $L^2(\Omega)$-family of random variables $u(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}^d$, jointly measurable, satisfying the stochastic integral equation (2).

In this section we provide conditions to show that the stochastic integral in (2) is meaningful. This will be enough for our purposes, since the other two terms in (2) are deterministic, and will turn out to be well-defined by the theory of parabolic partial differential equations in our setting. For a complete set of conditions such that each term on the right-hand side of (2) is meaningful, when a general distribution Λ is involved, see [7].

We want to give a precise meaning to the stochastic integral in (2) by defining

$$\int_0^t\int_{\mathbb{R}^d}\Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)dsdy:=\int_0^t\int_{\mathbb{R}^d}\Lambda(t,s,x,y)\sigma(s,y)M(ds,dy),\tag{10}$$

where, on the right-hand side, we have a stochastic integral with respect to the martingale measure M related to Ξ. As explained in [12], by approximating indicator functions with C_0^∞-functions, the process Ξ can indeed be extended to a worthy martingale measure $M=(M_t(A);\ t\in\mathbb{R}_+, A\in\mathscr{B}_b(\mathbb{R}^d))$, where $\mathscr{B}_b(\mathbb{R}^d)$ denotes the bounded Borel subsets of $\mathbb{R}^d$. The stochastic integral with respect to the martingale measure M of stochastic processes f and g, indexed by $(t,x)\in[0,T]\times\mathbb{R}^d$ and satisfying suitable conditions, is constructed by steps (see [8, 11, 21]), starting from the class $\mathcal{E}$ of simple processes, and making use of the pre-inner product (defined for suitable f,g)

$$\begin{aligned}\langle f,g\rangle_0 &= \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d}\big(f(s)*\tilde{g}(s)\big)(x)\,\Gamma(dx)ds\right]\\ &= \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d}[\mathcal{F}f(s)](\xi)\cdot\overline{[\mathcal{F}g(s)](\xi)}\,\mathfrak{M}(d\xi)ds\right],\end{aligned}\tag{11}$$

with corresponding semi-norm $\|\cdot\|_0$. For a *simple process*

$$g(t,x;\omega)=\sum_{j=1}^m 1_{(a_j,b_j]}(t)1_{A_j}(x)X_j(\omega)\in\mathcal{E}$$

(with $m\in\mathbb{N}$, $0\le a_j<b_j\le T$, $A_j\in\mathscr{B}_b(\mathbb{R}^d)$, X_j bounded, and $\mathscr{F}_{A_j}$-measurable random variable for all $1\le j\le n$), the stochastic integral with respect to M is given by

$$(g\cdot M)_t:=\sum_{j=1}^m\big(M_{t\wedge b_j}(A_j)-M_{t\wedge a_j}(A_j)\big)X_j,$$

where $x\wedge y:=\min\{x,y\}$, and the fundamental isometry

$$\mathbb{E}\big[(g\cdot M)_t^2\big]=\|g\|_0^2\tag{12}$$

holds for all $g\in\mathcal{E}$. The Hilbert space $\mathcal{P}_0$ of integrable stochastic processes is defined as the completion of $\mathcal{E}$ with respect to $\langle\cdot,\cdot\rangle_0$. On $\mathcal{P}_0$, the stochastic integral with respect to M is constructed as an $L^2(\Omega)$-limit of simple processes via the isometry (12). Moreover, by Lemma 2.2 in [18] we know that $\mathcal{P}_0=L_p^2([0,T]\times\Omega,\mathcal{H})$, where here $L^2_p(\dots)$ stands for the predictable stochastic processes in $L^2(\dots)$ and $\mathcal{H}$ is the Hilbert space which is obtained by completing the Schwartz functions with respect to the inner product $\langle\cdot,\cdot\rangle_0$. Thus, $\mathcal{P}_0$ consists of predictable processes which may contain tempered distributions in the x-argument (whose Fourier transforms are functions, $\mathbb{P}$-almost surely).

Now, to give a meaning to the integral (10), we need to impose conditions on the distribution Λ and on the coefficient σ such that $\Lambda\sigma \in \mathcal{P}_0$. To this aim, we introduce the following space.

Definition 2 $\mathcal{S}'(\mathbb{R}^d)_\infty$ is the space of all the tempered distributions $T \in \mathcal{S}'(\mathbb{R}^d)$ such that, for every k, $\langle\cdot\rangle^k T$ is a bounded distribution on $\mathbb{R}^d$, i.e. it belongs to the dual space of $\{\varphi \in C^\infty(\mathbb{R}^d) | \forall\alpha \in \mathbb{N}^d\ \partial^\alpha\varphi \in L^1(\mathbb{R}^d)\}$.

It can be shown that $\mathcal{S}'(\mathbb{R}^d)_\infty = \mathcal{O}'_C(\mathbb{R}^d)$, where $\mathcal{O}'_C$ is the widest class of distributions such that the convolution with elements of $\mathcal{S}'$ is well-defined. A necessary and sufficient condition for $T \in \mathcal{S}'(\mathbb{R}^d)_\infty$, which is useful for us, is the following:

$$T \in \mathcal{O}'_C(\mathbb{R}^d) \iff \forall\chi \in C_0^\infty(\mathbb{R}^d)\, T * \chi \in \mathcal{S}(\mathbb{R}^d). \tag{13}$$

For more details, see [20] and the recent paper [3].

In [7], sufficient conditions for the existence of the integral on the right-hand side of (10) have been given, in the case that σ depends on the spatial argument y, assuming that the spatial Fourier transform of the function σ is a complex-valued measure with finite total variation. Namely, we assume that, for all $s \in [0,T]$,

$$|\mathcal{F}\sigma(\cdot,s)| = |\mathcal{F}\sigma(\cdot,s)|(\mathbb{R}^d) = \sup_\pi \sum_{A\in\pi} |\mathcal{F}\sigma(\cdot,s)|(A) < \infty,$$

where π is any partition on $\mathbb{R}^d$ into measurable sets A, and the supremum is taken over all such partitions. Let, in the sequel, $\nu_s := \mathcal{F}\sigma(\cdot,s)$, and let $|\nu_s|_{\mathrm{tv}}$ denote its total variation. We summarize such conditions in the following theorem (see [2, 4, 5, 7] for details).

Theorem 1 *Let Δ_T be the simplex given by $0 \le t \le T$ and $0 \le s \le t$. Let, for $(t,s,x) \in \Delta_T \times \mathbb{R}^d$, $\Lambda(t,s,x)$ be a deterministic function with values in $\mathcal{S}'(\mathbb{R}^d)_\infty$, and let σ be a function in $L^2([0,T], C_b(\mathbb{R}^d))$, where C_b stands for the space of continuous and bounded functions, such that:*

(A1) *the function $(t,s,x,\xi) \mapsto [\mathcal{F}\Lambda(t,s,x)](\xi)$ is measurable, the function $s \mapsto \mathcal{F}\sigma(s) = \nu_s$ belongs to $L^2([0,T], \mathcal{M}_b(\mathbb{R}^d))$, and, for every $t \in [0,T]$,*

$$\int_0^t \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |[\mathcal{F}\Lambda(t,s,x)](\xi+\eta)|^2\, \mathfrak{M}(d\xi) \right) |\nu_s|^2_{\mathrm{tv}}\, ds < \infty; \tag{14}$$

(A2) *Λ and σ are as in **(A1)** and, for every $t \in [0,T]$,*

$$\lim_{h\downarrow 0}\int_0^t \chi_{[0,t-h)}(s) \times\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\sup_{r\in(s,s+h)}|[\mathcal{F}(\Lambda(t,s,x)-\Lambda(t,r,x))](\xi+\eta)|^2\,\mathfrak{M}(d\xi)\right) \times|\nu_s|^2_{\mathrm{tv}}\,ds=0.$$

Then $\Lambda\sigma\in\mathcal{P}_0$. *In particular, the stochastic integral on the right-hand side of* (10) *is well-defined and*

$$\mathbb{E}\left[((\Lambda(t,\cdot,x,*)\sigma(\cdot,*))\cdot M)_t^2\right]\leq \leq\int_0^t\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}|[\mathcal{F}\Lambda(t,s,x)](\xi+\eta)|^2\,\mathfrak{M}(d\xi)\right)|\nu_s|^2_{\mathrm{tv}}\,ds.$$

Remark 2 In [7] conditions (**A1**) and (**A2**) are given in a slightly different way. Namely, an integral on $[0,T]$ appears there, in place of integrals on $[0,t]$ for every $t\in[0,T]$. Moreover, in (**A2**) a characteristic function naturally appears in the proof Theorem 2.3 in [7]. The present formulation is actually the minimal requirement needed to prove that theorem, see the corresponding proof.

Remark 3 If $\sigma=\sigma(s)$, then $\mathcal{F}\sigma(s)=(2\pi)^d\sigma(s)\delta_0$, where δ_0 is the Dirac delta distribution with total variation 1. In such case, the necessary condition becomes $\int_0^T\sigma(s)^2\int_{\mathbb{R}^d}|[\mathcal{F}\Lambda(t,s,x)](\xi)|^2\,\mathfrak{M}(d\xi)ds<\infty$, which is actually weaker than (14), in the sense that there is no supremum over η, and corresponds to the one given in [11, Example 9].

3 Microlocal analysis and fundamental solution to parabolic equations with polynomially bounded coefficients

3.1 Elements of the SG-calculus

We recall here the basic definitions and facts about the so-called SG-calculus of pseudodifferential operators, through standard material appeared, e.g., in [4, 5], and elsewhere (sometimes with slightly different notational choices). In the sequel, we will often write $A\lesssim B$ when $|A|\leq c\cdot|B|$, for a suitable constant $c>0$.

The class $S^{m,\mu}=S^{m,\mu}(\mathbb{R}^d)$ of SG symbols of order $(m,\mu)\in\mathbb{R}^2$ is given by all the functions $a(x,\xi)\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ with the property that, for any multiindices $\alpha,\beta\in\mathbb{N}_0^d$, there exist constants $C_{\alpha\beta}>0$ such that the conditions

$$|D_x^\alpha D_\xi^\beta a(x,\xi)|\leq C_{\alpha\beta}\langle x\rangle^{m-|\alpha|}\langle\xi\rangle^{\mu-|\beta|},\qquad (x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d, \tag{15}$$

hold. For $m,\mu\in\mathbb{R}$, $\ell\in\mathbb{N}_0$, $a\in S^{m,\mu}$, the quantities

$$\|a\|_{\ell}^{m,\mu} = \max_{|\alpha+\beta|\le\ell} \sup_{x,\xi\in\mathbb{R}^d} \langle x\rangle^{-m+|\alpha|}\langle\xi\rangle^{-\mu+|\beta|}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \tag{16}$$

are a family of seminorms, defining the Fréchet topology of $S^{m,\mu}$. The corresponding classes of pseudodifferential operators $\mathrm{Op}(S^{m,\mu}) = \mathrm{Op}(S^{m,\mu}\,(\mathbb{R}^d))$ are given, for $a \in S^{m,\mu}(\mathbb{R}^d)$, $u \in \mathcal{S}(\mathbb{R}^d)$, by

$$(\mathrm{Op}(a)u)(x) = (a(\cdot, D)u)(x) = (2\pi)^{-d}\int e^{\mathrm{i}x\xi} a(x,\xi)\hat{u}(\xi)d\xi, \tag{17}$$

where $\hat{u}$ stands for the Fourier transform of u, extended by duality to $\mathcal{S}'(\mathbb{R}^d)$. The operators in (17) form a graded algebra with respect to composition, i.e.,

$$\mathrm{Op}(S^{m_1,\mu_1}) \circ \mathrm{Op}(S^{m_2,\mu_2}) \subseteq \mathrm{Op}(S^{m_1+m_2,\mu_1+\mu_2}).$$

The symbol $c \in S^{m_1+m_2,\mu_1+\mu_2}$ of the composed operator $\mathrm{Op}(a)\circ\mathrm{Op}(b)$, $a \in S^{m_1,\mu_1}$, $b \in S^{m_2,\mu_2}$, admits the asymptotic expansion

$$c(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi), \tag{18}$$

which implies that the symbol c equals $a\cdot b$ modulo $S^{m_1+m_2-1,\mu_1+\mu_2-1}$.

The residual elements of the calculus are operators with symbols in

$$S^{-\infty,-\infty} = S^{-\infty,-\infty}(\mathbb{R}^d) = \bigcap_{(m,\mu)\in\mathbb{R}^2} S^{m,\mu}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^{2d}),$$

that is, those having kernel in $\mathcal{S}(\mathbb{R}^{2d})$, continuously mapping $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$. For any $a \in S^{m,\mu}$, $(m,\mu)\in\mathbb{R}^2$, $\mathrm{Op}(a)$ is a linear continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to itself, extending to a linear continuous operator from $\mathcal{S}'(\mathbb{R}^d)$ to itself, and from $H^{z,\zeta}(\mathbb{R}^d)$ to $H^{z-m,\zeta-\mu}(\mathbb{R}^d)$, where $H^{z,\zeta}(\mathbb{R}^d)$, $(z,\zeta)\in\mathbb{R}^2$, denotes the Sobolev-Kato (or *weighted Sobolev*) space defined in (6) with the naturally induced Hilbert norm. When $z \ge z'$ and $\zeta \ge \zeta'$, the continuous embedding $H^{z,\zeta} \hookrightarrow H^{z',\zeta'}$ holds true. It is compact when $z > z'$ and $\zeta > \zeta'$. Since $H^{z,\zeta} = \langle\cdot\rangle^z H^{0,\zeta} = \langle\cdot\rangle^z H^\zeta$, with H^ζ the usual Sobolev space of order $\zeta\in\mathbb{R}$, we find $\zeta > k + \dfrac{d}{2} \Rightarrow H^{z,\zeta} \hookrightarrow C^k$, $k \in \mathbb{N}_0$.

One also actually finds

$$\begin{aligned} &\bigcap_{z,\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d) = H^{\infty,\infty}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d),\\ &\bigcup_{z,\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d) = H^{-\infty,-\infty}(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d), \end{aligned} \tag{19}$$

as well as, for the space of *rapidly decreasing distributions*, see [3, 20],

$$\mathcal{S}'(\mathbb{R}^d)_\infty = \bigcap_{z\in\mathbb{R}} \bigcup_{\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d) = H^{+\infty,-\infty}(\mathbb{R}^d). \tag{20}$$

The continuity property of the elements of $\mathrm{Op}(S^{m,\mu})$ on the scale of spaces $H^{z,\zeta}(\mathbb{R}^d)$, $(m,\mu),(z,\zeta)\in\mathbb{R}^2$, is expressed more precisely in the next Theorem 2.

Theorem 2 *Let $a\in S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu)\in\mathbb{R}^2$. Then, for any $(z,\zeta)\in\mathbb{R}^2$, $\mathrm{Op}(a)\in\mathcal{L}(H^{z,\zeta}(\mathbb{R}^d),H^{z-m,\zeta-\mu}(\mathbb{R}^d))$, and there exists a constant $C>0$, depending only on d,m,μ,z,ζ, such that*

$$\|\mathrm{Op}(a)\|_{\mathscr{L}(H^{z,\zeta}(\mathbb{R}^d),H^{z-m,\zeta-\mu}(\mathbb{R}^d))} \le C\|a\|^{m,\mu}_{[\frac{d}{2}]+1}, \tag{21}$$

where $[t]$ denotes the integer part of $t\in\mathbb{R}$ and $\mathscr{L}(X,Y)$ stands for the space of linear and continuous maps from a space X to a space Y.

Cordes introduced the class $O(m,\mu)$ of the *operators of order* (m,μ) as follows, see, e.g., [9].

Definition 3 A linear continuous operator $A\colon\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$ belongs to the class $O(m,\mu)$, of the operators of order $(m,\mu)\in\mathbb{R}^2$ if, for any $(z,\zeta)\in\mathbb{R}^2$, it extends to a linear continuous operator $A_{z,\zeta}\colon H^{z,\zeta}(\mathbb{R}^d)\to H^{z-m,\zeta-\mu}(\mathbb{R}^d)$. We also define

$$O(\infty,\infty)=\bigcup_{(m,\mu)\in\mathbb{R}^2} O(m,\mu),\quad O(-\infty,-\infty)=\bigcap_{(m,\mu)\in\mathbb{R}^2} O(m,\mu).$$

Remark 4
1. Trivially, any $A\in O(m,\mu)$ admits a linear continuous extension $A_{\infty,\infty}\colon\mathcal{S}'(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$. In fact, in view of (19), it is enough to set $A_{\infty,\infty}|_{H^{z,\zeta}(\mathbb{R}^d)}=A_{z,\zeta}$.
2. Theorem 2 implies $\mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))\subset O(m,\mu)$, $(m,\mu)\in\mathbb{R}^2$.
3. $O(\infty,\infty)$ and $O(0,0)$ are algebras under operator multiplication, $O(-\infty,-\infty)$ is an ideal of both $O(\infty,\infty)$ and $O(0,0)$, and $O(m_1,\mu_1)\circ O(m_2,\mu_2)\subset O(m_1+m_2,\mu_1+\mu_2)$.

The following characterization of the class $O(-\infty,-\infty)$ is often useful, see [9].

Theorem 3 *The class $O(-\infty,-\infty)$ coincides with $\mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d))$ and with the class of smoothing operators, that is, the set of all the linear continuous operators $A\colon\mathcal{S}'(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$. All of them coincide with the class of linear continuous operators A admitting a Schwartz kernel K_A belonging to $\mathcal{S}(\mathbb{R}^{2d})$.*

An operator $A=\mathrm{Op}(a)$ and its symbol $a\in S^{m,\mu}$ are called *elliptic* (or $S^{m,\mu}$*-elliptic*) if there exists $R\ge 0$ such that

$$C\langle x\rangle^m\langle\xi\rangle^\mu\le|a(x,\xi)|,\qquad |x|+|\xi|\ge R,$$

for some constant $C>0$. If $R=0$, a^{-1} is everywhere well-defined and smooth, and $a^{-1}\in S^{-m,-\mu}$. If $R>0$, then a^{-1} can be extended to the whole of $\mathbb{R}^{2d}$ so that the extension $\tilde{a}_{-1}$ satisfies $\tilde{a}_{-1}\in S^{-m,-\mu}$. An elliptic SG operator $A\in\mathrm{Op}(S^{m,\mu})$ admits a parametrix $A_{-1}\in\mathrm{Op}(S^{-m,-\mu})$ such that

$$A_{-1}A = I + R_1, \quad AA_{-1} = I + R_2,$$

for suitable $R_1, R_2 \in \mathrm{Op}(S^{-\infty,-\infty})$, where I denotes the identity operator. In such a case, A turns out to be a Fredholm operator on the scale of functional spaces $H^{z,\zeta}(\mathbb{R}^d)$, $(z,\zeta) \in \mathbb{R}^2$.

Proposition 1 *Let $A = \mathrm{Op}(a)$ be a SG pseudodifferential operator, with symbol $a \in S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu) \in \mathbb{R}^2$, and let K_A denote its Schwartz kernel. Then, the Fourier transform with respect to the second argument of K_A, $\mathcal{F}_{\cdot\mapsto\eta}K_A(x,\cdot)$, is given by*

$$\mathcal{F}_{\cdot\mapsto\eta}K_A(x,\cdot) = \mathrm{e}^{-\mathrm{i}x\cdot\eta}a(x,-\eta). \tag{22}$$

The proof of Proposition 1 can be found, e.g., in [9]. The next Lemma 1 is a special case of the similar, more general result for the kernel of *SG* Fourier integral operators proved, for instance, in [5]. We give its direct proof here, for the convenience of the reader.

Lemma 1 *Let $A = \mathrm{Op}(a)$ be a SG pseudodifferential operator with symbol $a \in S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu) \in \mathbb{R}^2$, and let K_A denote its Schwartz kernel. Then, for every $x \in \mathbb{R}^d$, $K_A(x,\cdot) \in \mathcal{S}'(\mathbb{R}^d)_\infty$. More precisely, we find $K_A \in C^\infty(\mathbb{R}^d, \mathcal{S}'(\mathbb{R}^d)_\infty)$.*

Proof Given a fixed $x \in \mathbb{R}^d$, by [3, Theorem 3.3], to see that $K_A(x,\cdot) \in \mathcal{S}'(\mathbb{R}^d)_\infty$ it suffices to show that for every $\chi \in \mathcal{D}(\mathbb{R}^d)$, $K_A(x,\cdot) * \chi \in \mathcal{S}(\mathbb{R}^d)$. We already know [20, p. 244/245] that $K_A(x,\cdot) * \chi$ is a C^∞ function of slow growth. Computing now its Fourier transform (in the distributional sense), using Proposition 1 we see that

$$\mathcal{F}_{\cdot\mapsto\eta}(K_A(x,\cdot) * \chi)(\eta) = \mathcal{F}_{\cdot\mapsto\eta}K_A(x,\cdot) \cdot \widehat{\chi}(\eta) = \mathrm{e}^{-\mathrm{i}x\cdot\eta}a(x,-\eta)\widehat{\chi}(\eta) \in \mathcal{S}(\mathbb{R}^d_\eta).$$

It follows that, for its inverse Fourier transform, $K_A(x,\cdot) * \chi \in \mathcal{S}(\mathbb{R}^d)$, too. Finally, the fact that the map

$$x \mapsto K_A(x,y) = \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,\xi)đ\xi$$

belongs to $C^\infty(\mathbb{R}^d, \mathcal{S}'(\mathbb{R}^d)_\infty)$ is a consequence of the general properties of oscillatory integrals, taking into account that $x \cdot \xi$ and $a(x,\xi)$ are smooth functions with respect to x. This completes the proof. ◻

3.2 Construction of the fundamental solution of *SG*-parabolic operators

We work here with a class of operators with more general symbols than the (polynomial) ones appearing in (4). Namely, we consider operators of the form

$$L = \partial_t + A(t) = \partial_t + \mathrm{Op}(a(t)), \tag{23}$$

where, for $m, \mu > 0$, $A(t) = \mathrm{Op}(a(t))$ are SG pseudodifferential operators with parameter-dependent symbol $a \in C([0,T], S^{m,\mu}(\mathbb{R}^d))$. Notice that, of course, (4) is a special case of (23). The parabolicity condition on L is here expressed by means of the (SG-)hypoellipticity of $A(t)$, namely,

$$\begin{aligned} \exists C > 0 \quad & \mathrm{Re}\, a(t,x,\xi) \geq C\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}, \\ \forall \alpha,\beta \in \mathbb{N}^d\ \exists C_{\alpha\beta} > 0 \quad & \left|\frac{\partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)}{\mathrm{Re}\, a(t,x,\xi)}\right| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}. \end{aligned} \tag{24}$$

where $0 < m' \leq m$, $0 < \mu' \leq \mu$, $t \in [0,T]$, $x, \xi \in \mathbb{R}^d$. $A(t)$ is (SG-)elliptic if $m = m'$, $\mu = \mu'$, see above. Elements of the microlocal analysis of SG-parabolic operators can be found in [9, 15]. As customary, $A(t)$, $t \in [0,T]$, is considered as an unbounded operator in L^2 with dense domain $H^{m,\mu}$ (see [9, Ch. 3, Sec. 3-4]; see also [15] for the spectral theory of corresponding self-adjoint elliptic operators).

Definition 4 We say that $L = \partial_t + \mathrm{Op}(a(t))$, $a \in C([0,T], S^{m,\mu}(\mathbb{R}^d))$ is (SG-) parabolic, with respect to m, m', μ, μ', $0 < m' \leq m$, $0 < \mu' \leq \mu$, if a satisfies the (SG-)hypoellipticity condition (24).

We now prove our first main result, namely, the existence of the fundamental solution operator of a SG-parabolic operator L.

Theorem 4 *Let $L = \partial_t + \mathrm{Op}(a(t))$, $a \in C([0,T], S^{m,\mu}(\mathbb{R}^d))$ be (SG-)parabolic, with respect to m, m', μ, μ', $0 < m' \leq m$, $0 < \mu' \leq \mu$. Then, L admits a fundamental solution operator $E(t,s)$, $0 \leq s \leq t \leq T$, $0 \leq s < T$, that is, an operator family $E(t,s) = \mathrm{Op}(e(t,s))$ with $e(\cdot,s) \in C((s,T], S^{0,0}(\mathbb{R}^d)) \cap C^1((s,T], S^{m,\mu}(\mathbb{R}^d))$, with the following properties:*

1. *E satisfies the equation*

$$LE(t,s) = 0, \quad 0 \leq s < t \leq T; \tag{25}$$

2. *the symbol family $e(t,s)$ satisfies*

$$e(t,s,x,\xi) \longrightarrow 1 \text{ weakly in } S^{0,0}(\mathbb{R}^d) \text{ for } t \to s^+; \tag{26}$$

3. *writing $e(t,s)$ as*

$$e(t,s,x,\xi) = \exp\left(-\int_s^t a(\tau,x,\xi)\, d\tau\right) + r_0(t,s,x,\xi), \tag{27}$$

the symbol family $r_0(t,s)$ satisfies

$$r_0(t,s,x,\xi) \longrightarrow 0 \text{ weakly in } S^{-1,-1}(\mathbb{R}^d) \text{ for } t \to s^+, \tag{28}$$

$$\left\{\frac{r_0(t,s,x,\xi)}{t-s}\right\}_{0\leq s<t\leq T} \text{ is a bounded set in } S^{m-1,\mu-1}(\mathbb{R}^d). \tag{29}$$

Remark 5 1. It is enough that (24) is satisfied for $|x| + |\xi| \geq R > 0$. In fact, if this is the case, there exists $M > 0$ such that $a_M(t,x,\xi) = a(t,x,\xi) + M$ satisfies (24) everywhere. Let then $E_M(t,s)$ be the fundamental solution of $L_M = \partial_t + \mathrm{Op}(a_M(t))$. Then, $E(t,s) = e^{M(t-s)}E_M(t,s)$ is the fundamental solution of L and

$$e^{M(t-s)}e^{-\int_s^t [a(\tau)+M]\,d\tau} = e^{-\int_s^t a(\tau)\,d\tau},$$

so $E(t,s)$ has the properties stated in Theorem 4.

2. Similarly to the analogous result which holds true for parabolic operators defined by means of the Hörmander's symbols $S^m_{\rho,\delta}(\mathbb{R}^d)$, $0 \leq \delta < \rho \leq 1$, found in [14], Theorem 4 holds true, with simple modifications, for the generalized class of SG-symbols $S^{m,\mu}_{r,\rho}(\mathbb{R}^d)$, $r, \rho \geq 0$, $r + \rho > 0$, considered, e.g., in [10].

The next Theorem 5 is an immediate consequence of Theorem 4, by a Duhamel's argument and the properties of the fundamental solution E.

Theorem 5 *Let $u_0 \in H^{z,\zeta}(\mathbb{R}^d)$, $f \in C([0,T], H^{z,\zeta}(\mathbb{R}^d))$, $z, \zeta \geq 0$, and $L = \partial_t + A(t)$ satisfy the same assumptions as in Theorem 4. Then, the Cauchy problem*

$$\begin{cases} Lu(t,x) = f(t,x), & (t,x) \in (s,T] \times \mathbb{R}^d, \\ u(s,x) = u_0(x), & x \in \mathbb{R}^d, s \in [0,T), \end{cases} \tag{30}$$

admits a solution given by

$$u(t,x) = E(t,s)u_0(x) + \int_s^t E(t,\tau)\, f(\tau,x)\,d\tau, \quad s \leq t \leq T, \tag{31}$$

with $E(t,s)$ the fundamental solution operator obtained in Theorem 4. Moreover, such solution satisfies

$$u \in C([s,T], H^{z,\zeta}(\mathbb{R}^d)) \cap C^1([s,T], H^{z-m,\zeta-\mu}(\mathbb{R}^d)).$$

Remark 6 Recall that the initial condition in (30) is understood as

$$\lim_{t\to s^+} u(t) = u_0 \quad \text{in } L^2(\mathbb{R}^d).$$

We prove Theorem 4 by extending to the SG setting the argument given in [14] for the analogous result in the $S^m_{\rho,\delta}$ setting. Similarly to the mentioned proof scheme, we rely on the next three technical lemmas, which are, essentially, consequences of the SG-calculus. In particular, the proof of Lemma 4 requires the properties of the multiproducts of SG pseudodifferential operators (see [1]). For the sake of brevity, we only sketch the corresponding arguments.

Lemma 2 *Assume that $a \in C([0,T], S^{m,\mu}(\mathbb{R}^d))$ satisfies* (24)*, $0 < m' \leq m$, $0 < \mu' \leq \mu$, $t \in [0,T]$, $x, \xi \in \mathbb{R}^d$. Set*

$$e_0(t,s,x,\xi) = \exp\left(-\int_s^t a(\tau,x,\xi)\,d\tau\right),$$

and define inductively $\{e_j(t,s)\}_{j=1}^{\infty}$, $\{q_j(t,s)\}_{j=1}^{\infty}$, $0 \le s \le t \le T$ *by*

$$q_j(t,s,x,\xi) = \sum_{k=0}^{j-1} \sum_{|\alpha|+k=j} \frac{1}{\alpha!}\partial_\xi^\alpha a(t,x,\xi)\cdot D_x^\alpha e_k(t,s,x,\xi), \quad j \ge 1, \tag{32}$$

and

$$\begin{cases} [\partial_t + a(t,x,\xi)]e_j(t,s,x,\xi) = -q_j(t,s,x,\xi), \\ \qquad e_j(s,s,x,\xi) = 0, \qquad j \ge 1. \end{cases} \tag{33}$$

Then, for any $\alpha, \beta \in \mathbb{N}^d$, *there exist* $C_{\alpha\beta}, C'_{\alpha\beta} > 0$ *such that*

$$|\partial_x^\alpha \partial_\xi^\beta e_j(t,s,x,\xi)| \le \begin{cases} C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}, & j \ge 0 \\ C'_{\alpha\beta}(t-s)\langle x\rangle^{m-|\alpha|-j}\langle\xi\rangle^{\mu-|\beta|-j}, & j+|\alpha+\beta| \ge 1. \end{cases} \tag{34}$$

The proof of Lemma 2 follows from an accurate usage of the trivial estimate $s^\kappa e^{-s} \le C_\kappa < \infty$ for every $s \ge 0$, with constants $C_\kappa > 0$, $\kappa \in [0,+\infty)$, and from condition (24). By explicitly writing

$$q_1(t,s,x,\xi) = -\sum_{j=1}^{n} \partial_{\xi_j} a(t,x,\xi)\, e_0(t,s,x,\xi) \int_s^t \partial_{x_j} a(\tau,x,\xi)\,d\tau,$$

observing that

$$|e_0(t,s,x,\xi)| = e^{-\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\,d\tau} \le e^{-C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} \le 1, \tag{35}$$

$$\begin{aligned}\left|e_0(t,s,x,\xi)\int_s^t \partial_{x_j} a(\tau,x,\xi)\,d\tau\right| &\lesssim e^{-\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\,d\tau} \int_s^t \langle x\rangle^{-1}\mathrm{Re}\, a(\tau,x,\xi)\,d\tau \\ &\le C_1\langle x\rangle^{-1} \lesssim \langle x\rangle^{-1},\end{aligned}$$

and similarly estimating derivatives, one can prove $q_1(t,s,x,\xi) \in SG^{m-1,\mu-1}$ (and, inductively, $q_j(t,s,x,\xi) \in SG^{m-j,\mu-j}$). Now, solving (33), it follows

$$e_j(t,s,x,\xi) = -e_0(t,s,x,\xi)\int_s^t \frac{q_j(\tau,s,x,\xi)}{e_0(\tau,s,x,\xi)}\,d\tau, \quad j \ge 1. \tag{36}$$

On one hand, we can estimate

$$|e_j(t,s,x,\xi)| \le \int_s^t |q_j(\tau,s,x,\xi)|\,d\tau \le C(t-s)\langle x\rangle^{m-j}\langle\xi\rangle^{\mu-j}, \quad j \ge 1.$$

On the other hand, by explicitly writing q_1 and using (24) twice, we get

$$|e_1(t,s,x,\xi)| \lesssim \langle x\rangle^{-1}\langle\xi\rangle^{-1} e^{-\int_s^t \operatorname{Re} a(\tau,x,\xi)\,d\tau}\left(\int_s^t \operatorname{Re} a(\tau,x,\xi)\,d\tau\right)^2$$
$$\leq C_2\langle x\rangle^{-1}\langle\xi\rangle^{-1} \lesssim \langle x\rangle^{-1}\langle\xi\rangle^{-1}.$$

Similar arguments work for the derivatives of e_j, $j \geq 2$, so that we can actually conclude $e_j(t,s,x,\xi) \in SG^{-j,-j}$, $j \geq 1$.

Lemma 3 *Let, for* $N \geq 1$*,*

$$E_N(t,s) = \sum_{j=0}^{N-1} \operatorname{Op}(e_j(t,s)),$$

and

$$R_N(t,s) = \operatorname{Op}(r_N(t,s)) = LE_N(t,s), \tag{37}$$

with L *from Theorem 4 and* $\{e_j(t,s)\}_{j=1}^{\infty}$ *from Lemma 2. Then,*

$$r_N(\cdot,s) \in C((s,T], S^{m-N,\mu-N}(\mathbb{R}^d)), \quad 0 \leq s < T, \tag{38}$$

$$\left\{\frac{r_N(t,s)}{t-s}\right\}_{0\leq s<t\leq T} \text{ is bounded in } S^{2m-N,2\mu-N}(\mathbb{R}^d). \tag{39}$$

The proof of Lemma 3 is straightforward, in view of Lemma 2. Indeed, by the SG-calculus, employing the asymptotic expansion of the symbol of $\operatorname{op}(a(t))E_j(t,s)$,

$$LE_N(t,s) = \sum_{j=0}^{N-1} \operatorname{op}(\partial_t e_j(t,s) + a(t)e_j(t,s))$$
$$+ \sum_{j=0}^{N-1}\sum_{|\alpha|=1}^{N-j} \frac{i^{|\alpha|}}{\alpha!}\operatorname{op}(D_\xi^\alpha a(t) D_x^\alpha e_j(t,s)) + \sum_{j=0}^{N-1} R_{N,j}(t,s),$$

with $r_{N,j}(t,s,x,\xi) \in SG^{m-N-1,\mu-N-1}$, since $e_j \in SG^{-j,-j}$ for every $j \geq 0$, and $r_{N,j}(t,s,x,\xi) \in SG^{2m-N-1,2\mu-N-1}$, for every $j \geq 1$, by the second inequality in (34). By the choice of q_j in (32) and by equation (33), we see that $LE_N(t,s) = \sum_{j=0}^{N-1} R_{N,j}(t,s) = R_N(t,s)$, and formulae (38) and (39) hold.

Lemma 4 *Let* $R_N(t,s) = \operatorname{Op}(r_N(t,s))$ *be defined by* (37)*, with*

$$N \geq 1 \text{ such that } \max\{m,\mu\} - N \leq 0. \tag{40}$$

Define inductively the sequence of operator families $\{W_\nu(t,s)\}_{\nu=1}^{\infty} = \{\operatorname{Op}(w_\nu(t,s))\}_{\nu=1}^{\infty}$, $0 \leq s \leq t \leq T$*, by*

$$W_1(t,s) = -R_N(t,s) = -\operatorname{Op}(r_N(t,s)), \tag{41}$$

$$W_\nu(t,s) = \int_s^t W_1(t,\tau)\circ W_{\nu-1}(\tau,s)\,d\tau, \quad \nu \geq 2. \tag{42}$$

Then, for $l \ge 1$, $0 \le s \le t \le T$,

$$\sum_{\nu=1}^{l} W_\nu(t,s) = -R_N(t,s) - \int_s^t R_N(t,\tau) \sum_{\nu=1}^{l-1} W_\nu(\tau,s)\, d\tau, \tag{43}$$

and, for any $\alpha, \beta \in \mathbb{N}^d$, there exist constants $A_{\alpha\beta}, A'_{\alpha\beta} > 0$ such that, for $0 \le s \le t \le T$, $x, \xi \in \mathbb{R}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta w_\nu(t,s,x,\xi)| \le (A_{\alpha\beta})^\nu \frac{(t-s)^{\nu-1}}{(\nu-1)!} \langle x\rangle^{m-N-|\alpha|} \langle \xi\rangle^{\mu-N-|\beta|}, \tag{44}$$

$$|\partial_x^\alpha \partial_\xi^\beta w_\nu(t,s,x,\xi)| \le (A'_{\alpha\beta})^\nu \frac{(t-s)^{\nu}}{(\nu-1)!} \langle x\rangle^{2m-N-|\alpha|} \langle \xi\rangle^{2\mu-N-|\beta|}. \tag{45}$$

Formula (43) follows readily by definitions (41) and (42). To get (44) and (45) we need to write

$$W_\nu(t,s) = \int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-2}} W_1(t,t_1) \cdots W_1(t_{\nu-1},s) dt_{\nu-1} \cdots dt_1.$$

By the choice of N, we can look at $W_1(t,t_1)$ as an operator of order either $(m-N, \mu-N)$ or $(2m-N, 2\mu-N)$ according to (37) or (38), respectively, and we can look at $W_1(t_1,t_2), \dots, W_1(t_{\nu-1},s)$ as operators of order $(0,0)$. By integrating on the symplex $s \le t_{\nu-1} \le \cdots \le t_1 \le t$, formulae (44) and (45) follow.

Proof (of Theorem 4) Lemma 4 implies that

$$W(t,s) = \sum_{\nu=1}^{\infty} W_\nu(t,s)$$

converges in the topology of $\mathrm{Op}(S^{m-N,\mu-N})$, since, by (44), $\sum_\nu w_\nu(t,s)$ converges in the topology of $S^{m-N,\mu-N}$, for any fixed N satisfying (40) and $0 \le s \le t \le T$. With $E_N(t,s)$ from Lemma 3, define, for $0 \le s < t \le T$, $N \ge 1$, $\max\{m,\mu\} - N \le 0$,

$$E(t,s) = E_N(t,s) + \int_s^t E_N(t,\tau) \circ W(\tau,s)\, d\tau. \tag{46}$$

Then, by (37),

$$\begin{aligned} LE(t,s) &= LE_N(t,s) + W(t,s) + \int_s^t [LE_N(t,\tau)] \circ W(\tau,s)\, d\tau \\ &= R_N(t,s) + W(t,s) + \int_s^t R_N(t,\tau) \circ W(\tau,s)\, d\tau. \end{aligned} \tag{47}$$

By letting $l \longrightarrow +\infty$ in (43), we find, for any N satisfying (40),

$$W(t,s) = -R_N(t,s) - \int_s^t R_N(t,\tau) \circ W(\tau,s)\, d\tau,$$

so that, by (47), it follows $LE(t,s) = 0$, $0 \le s < t \le T$, as claimed. All the properties of the symbol $e(t,s)$ of the operator family $E(t,s)$ are then consequences of (46) and Lemmas 2, 3, and 4. □

Remark 7 Clearly, by construction, $e(t,s)$ (and $E(t,s)$) are continuous also with respect to s, $0 \le s \le t \le T$ (see Lemmas 2, 3, and 4, and the proof of Theorem 4).

In the next Lemma 5, we obtain further estimates for the symbol family $e(t,s)$, showing that, actually, for $0 \le s < t \le T$, it gives rise to (a C^1 family of) operators in $\mathcal{O}(-\infty,-\infty)$. This, of course, cannot be extended by continuity up to $t = s$, but some L^1 regularity with respect to t, that we employ in Section 4, can still be achieved.

Lemma 5 *For every $j \in \mathbb{N}$, $\alpha,\beta \in \mathbb{N}^d$, we have, for suitable constants $C'_{j\alpha\beta} > 0$,*

$$|\partial_x^\alpha \partial_\xi^\beta e_j(t,s,x,\xi)| \le C'_{j\alpha\beta}\sqrt{|e_0(t,s,x,\xi)|}\,\langle x\rangle^{-j-|\alpha|}\langle\xi\rangle^{-j-|\beta|}, \tag{48}$$

with $0 \le s \le t \le T$, $(x,\xi) \in \mathbb{R}^d$. Moreover, for every $j \in \mathbb{N}$, $0 \le s < T$, $e_j(\cdot,s) \in C^1((s,T],\mathcal{S}(\mathbb{R}^{2d}))$ and $e(\cdot,s) \in L^1([s,T],S^{-\kappa m',-\kappa\mu'}(\mathbb{R}^d))$, $\partial_t e(\cdot,s) \in L^1([s,T],S^{m-\kappa m',\mu-\kappa\mu'}(\mathbb{R}^d))$, for every $\kappa \in [0,1)$.

Proof From (35), for every $m',\mu' > 0$ we see that, for every $\kappa \in [0,1)$,

$$\begin{aligned}
&|e_0(t,s,x,\xi)| \le \\
&\quad \le e^{-C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}}(C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'})^{\kappa}(C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'})^{-\kappa} \\
&\quad \lesssim \frac{C_\kappa}{(t-s)^\kappa}\langle x\rangle^{-\kappa m'}\langle\xi\rangle^{-\kappa\mu'} \lesssim (t-s)^{-\kappa}\langle x\rangle^{-\kappa m'}\langle\xi\rangle^{-\kappa\mu'},
\end{aligned} \tag{49}$$

where C_κ is the upper bound of $s^\kappa e^{-s}$, $s \ge 0$, which gives $\langle x\rangle^{\kappa m'}\langle\xi\rangle^{\kappa\mu'}e_0(\cdot,s,x,\xi) \in L^1([s,T])$, and similarly for the derivatives with respect to x and ξ. By induction, (48) follows. Let us perform part of the induction step for $j = 1$, leaving the remaining details to the reader. We have:

$$\begin{aligned}
|e_1(t,s,x,\xi)| &\le \sum_{j=1}^d |e_0(t,s,x,\xi)| \left|\int_s^t \frac{\partial_{\xi_j}a(\tau,x,\xi)\cdot D_{x_j}e_0(\tau,s,x,\xi)}{e_0(\tau,s,x,\xi)}\,d\tau\right| \\
&\le \sum_{j=1}^d |e_0(t,s,x,\xi)| \int_s^t |\partial_{\xi_j}a(\tau,x,\xi)|\cdot\left|\int_s^\tau D_{x_j}\mathrm{Re}\,a(r,x,\xi)\,dr\right| d\tau \\
&\lesssim \langle\xi\rangle^{-1}\langle x\rangle^{-1}|e_0(t,s,x,\xi)|\int_s^t |\mathrm{Re}\,a(\tau,x,\xi)|\cdot\left(\int_s^\tau \mathrm{Re}\,a(r,x,\xi)\,dr\right) d\tau \\
&\le \langle\xi\rangle^{-1}\langle x\rangle^{-1}|e_0(t,s,x,\xi)|^{\frac12}\left[|e_0(t,s,x,\xi)|\left(\int_s^t \mathrm{Re}\,a(\tau,x,\xi)\,d\tau\right)^4\right]^{\frac12} \\
&\le \sqrt{C_4}\,\langle\xi\rangle^{-1}\langle x\rangle^{-1}|e_0(t,s,x,\xi)|^{\frac12} \lesssim \langle\xi\rangle^{-1}\langle x\rangle^{-1}|e_0(t,s,x,\xi)|^{\frac12},
\end{aligned}$$

with C_4 the upper bound of the function s^4e^{-s}, $s \ge 0$. This implies

$$|e_1(t,s,x,\xi)| \lesssim \sqrt{e_0(t,s,x,\xi)} \leq e^{-\frac{C}{2}(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} \lesssim (t-s)^{-\kappa}\langle x\rangle^{-\kappa m'}\langle\xi\rangle^{-\kappa\mu'},$$

for every $\kappa \in [0,1)$, and similar estimates hold for the derivatives of e_1, and for e_j, $j \geq 2$. From the definition of E_N in Lemma 3, we have $E_N(t,s) \in \mathrm{Op}(S^{-\kappa m',-\kappa\mu'})$. Again, reading $W(t,s)$ as an operator of order $(0,0)$, from equation (46) we now see that $E(\cdot,s) \in L^1([s,T],\mathrm{Op}(S^{-\kappa m',-\kappa\mu'}))$, that is, $e(\cdot,s) \in L^1([s,T];S^{-\kappa m',-\kappa\mu'})$. That $\partial_t e(\cdot,s) \in L^1([s,T],S^{m-\kappa m',\mu-\kappa\mu'})$ follows then by the result for $e(\cdot,s)$, recalling $\partial_t E(t,s) = -\mathrm{Op}(a(t))E(t,s)$, $0 \leq s < t \leq T$, by Theorem 4, and $a \in C([0,T],S^{m,\mu})$, by hypothesis.

Arguing similarly, using (24), (48), and (49), it follows, that, for all $j, M \in \mathbb{N}$, $\alpha,\beta \in \mathbb{N}^d$, there exists $C''_{jM\alpha\beta} > 0$ such that, for any $x,\xi \in \mathbb{R}^d$, $0 \leq s < t \leq T$,

$$|(\langle x\rangle\langle\xi\rangle)^M \partial_x^\alpha \partial_\xi^\beta e_j(t,s,x,\xi)| \leq C''_{jM\alpha\beta}(t-s)^{-\frac{M}{\min\{\{m',\mu'\}}},$$

and analogous estimates for $\partial_t e_j(t,s,x,\xi)$, which imply $e_j(\cdot,s) \in C^1((s,T],\mathcal{S}(\mathbb{R}^d))$, as claimed. □

Corollary 1 *Under the same hypothesis of Theorem 5, the solution of the Cauchy problem* (30) *described there satisfies, for any* $\kappa \in [0,1)$,

$$u \in C([s,T],H^{z,\zeta}(\mathbb{R}^d)) \cap C^1([s,T],H^{z-m,\zeta-\mu}(\mathbb{R}^d)) \cap$$
$$\cap\, C^1((s,T],\mathcal{S}(\mathbb{R}^d)) \cap L^1([s,T],H^{z+\kappa m',\zeta+\kappa\mu'}(\mathbb{R}^d)).$$

It also satisfies $L^1([s,T],H^{z-m+\kappa m',\zeta-\mu+\kappa\mu'}(\mathbb{R}^d))$, $\kappa \in [0,1)$.

Proof The claim is an immediate consequence of Lemma 5 and Duhamel's formula (31) from Theorem 5, using (19) and Theorem 2. □

4 Existence of a random-field solution

In the next Theorem 6 we prove our second main result, the existence of a random-field solution of the SPDE (1), under the assumptions of (SG-)parabolicity for the operator L, see Definition 4. We consider, in the $L^2(\mathbb{R}^d)$ environment, the corresponding Cauchy problem

$$\begin{cases} Lu(t,x) = f(t,x) = \gamma(t,x) + \sigma(t,x)\dot{\Xi}(t,x), & (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{50}$$

with the aim of finding conditions on L, on the stochastic noise $\dot{\Xi}$, and on σ, γ, u_0, such that (50) admits a random-field solution. The conditions on the stochastic noise will be given on the spectral measure $\mathfrak{M}$ corresponding to the correlation measure Γ related to the noise $\dot{\Xi}$.

Theorem 6 *Let us consider the Cauchy problem* (50) *for a SPDE associated with a SG-parabolic operator L of the form* (23)*. Assume also, for the initial conditions, that* $u_0 \in H^{z,\zeta}(\mathbb{R}^d)$*, with* $z \geq 0$ *and* $\zeta > d/2$*. Furthermore, assume that* $\gamma \in C([0,T]; H^{z,\zeta}(\mathbb{R}^d))$*,* $\sigma \in C([0,T], H^{0,\zeta}(\mathbb{R}^d))$*,* $s \mapsto \mathcal{F}\sigma(s) = \nu_s \in L^2([0,T], \mathcal{M}_b(\mathbb{R}^d))$*. Assume that one of the following conditions on the spectral measure* $\mathfrak{M}$*, associated with the random noise* $\dot{\Xi}$*, hold true:*

(H0) *either, for every* $t \in [0,T]$*,*

$$\sup_{0\leq s<t} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |e(t,s,x,\xi+\eta)|^2 \,\mathfrak{M}(d\xi) < \infty \tag{51}$$

and

$$\lim_{h\downarrow 0} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta)-e(t,r,x,\xi+\eta)|^2 \,\mathfrak{M}(d\xi) = 0, \quad 0 \leq s < t, \tag{52}$$

where $e(t,s)$ *is the (parameter-dependent) symbol of the fundamental solution of the operator L,*

(H1) *or*

$$\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) < \infty, \tag{53}$$

(H2) *or* $\mathfrak{M}$ *is absolutely continuous,* $|\nu_s|_{\mathrm{tv}} \in L^\infty(0,T)$*, and*

$$\int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{\langle\xi\rangle^{\mu'}} < \infty. \tag{54}$$

Then, there exists a random-field solution u of (7)*. Moreover, for any* $\kappa \in [0,1)$*,*

$$\begin{aligned}\mathbb{E}[u] \in C([0,T], H^{z,\zeta}(\mathbb{R}^d)) \cap C^1([0,T], H^{z-m,\zeta-\mu}(\mathbb{R}^d)) \cap \\ \cap\, C^1((0,T], \mathcal{S}(\mathbb{R}^d)) \cap L^1([0,T], H^{z+\kappa m',\zeta+\kappa\mu'}(\mathbb{R}^d)).\end{aligned}$$

It also satisfies $\partial_t \mathbb{E}[u] \in L^1([0,T], H^{z-m+\kappa m',\zeta-\mu+\kappa\mu'}(\mathbb{R}^d))$*,* $\kappa \in [0,1)$*.*

Remark 8 The class of the stochastic noises which are admissible, if we want to obtain a random-field solution of the Cauchy problem for a SPDE through our method, is described by (51) and (52) for all *SG*-parabolic operators *L*, by (53) or (54) under some additional assumptions. Conditions (51), (53), and (54) can be understood as *compatibility conditions* between the noise and the equation.

Proof (of Theorem 6) Let us insert $f(t,x) = \gamma(t,x) + \sigma(t,x)\dot{\Xi}(t,x)$ in (31), so that, formally,

$$
\begin{aligned}
u(t,x) = v_0(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y)\,dyds \\
&+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)\,dyds \\
= v_0(t,x) &+ v_1(t,x) + v_2(t,x),
\end{aligned}
\tag{55}
$$

where we indicated by $\Lambda(t,s)$ the Schwartz kernel of $E(t,s)$ and $v_0 = E(t,s)u_0$.

In view of the special structure of Λ (kernel of a smooth family of certain SG-pseudodifferential operators, as described in the previous section), the fact that the deterministic integral in (55) and v_0 are well-defined directly follows by the general theory of SG equations, under the assumptions on γ given in the statement of Theorem 6. By Theorem 5, recalling also Theorem 2 and (26), we find, for any $\kappa \in [0,1)$,

$$
\begin{aligned}
v_0 \in C([0,T], H^{z,\zeta}) &\cap C^1([0,T], H^{z-m,\zeta-\mu}) \cap \\
&\cap C^1((0,T], \mathcal{S}) \cap L^1([0,T], H^{z+\kappa m',\zeta+\kappa\mu'}) \subset C([0,T], L^2),
\end{aligned}
$$

which is a continuous function in $(t,x) \in [0,T] \times \mathbb{R}^d$. This implies that $v_0(t,x)$ is finite for every $(t,x) \in [0,T] \times \mathbb{R}^d$. Since $\gamma \in C([0,T], H^{z,\zeta})$, by the properties of $E(t,s)$ we find that v_1 is of the same regularity class of v_0, namely, it is a well-defined, continuous function in $(t,x) \in [0,T] \times \mathbb{R}^d$. For this term, since we also have $E(t,\cdot) \in L^1([0,T], O(-\kappa m', -\kappa\mu'))$, we additionally find $v_1 \in C([0,T], H^{z+\kappa m',\zeta+\kappa\mu'})$. We can rewrite v_2 in (55) as

$$
v_2(t,x) = \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)M(ds,dy),
$$

where M is the martingale measure associated with the stochastic noise Ξ, as defined in Section 2. Then, we prove that conditions **(A1)**, **(A2)**, from Section 2 hold true, to achieve that such stochastic integral is well-defined. To this aim, we first observe that, by Proposition 1 and Theorem 4,

$$
|\mathcal{F}_{y\mapsto\eta}\Lambda(t,s,x,\cdot)(\eta)|^2 = \left|\mathrm{e}^{-\mathrm{i}x\cdot\eta} e(t,s,x,-\eta)\right|^2 = |e(t,s,x,-\eta)|^2 \le C_{t,s}, \tag{56}
$$

where $C_{t,s}$ can be chosen to be continuous in s and t, in view of the properties of $e(t,s)$, see Lemmas 2-5.

1. Using (56), we get that condition **(A1)**, with $\Lambda(t,s)$ being the Schwartz kernel of $E(t,s)$, is satisfied if for every $t \in [0,T]$

$$
J = \int_0^t \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |e(t,s,x,\eta+\xi)|^2 \,\mathfrak{M}(d\xi) \right) |\nu_s|^2_{\mathrm{tv}}\, ds < \infty.
$$

If we assume the hypothesis **(H0)**, we find, by the assumptions on σ, for every $t \in [0,T]$,

$$J \leq \left(\sup_{0\leq s<t} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |e(t,s,x,\eta+\xi)|^2 \,\mathfrak{M}(d\xi) \right) \int_0^t \, |\nu_s|^2_{\mathrm{tv}}\, ds < \infty,$$

and **(A1)** holds true.
If we assume the hypothesis **(H1)**, we find, again by the assumptions on σ, taking into account that $e(t,s) \in S^{0,0}$, $0 \leq s \leq t \leq T$,

$$J \lesssim \int_0^t \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) |\nu_s|^2_{\mathrm{tv}}\, ds = \left(\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) \int_0^t \, |\nu_s|^2_{\mathrm{tv}}\, ds < \infty,$$

showing that **(A1)** holds true as well in this second case.
Finally, if we assume the hypothesis **(H2)**, using the absolute continuity of $\mathfrak{M}$, the uniform boundedness of $|\nu_s|_{\mathrm{tv}}$, and Lemma 5, first we observe that (46) implies $e(t,s) = e_N(t,s) \mod C([s,T], S^{-\infty,-\infty})$, and compute, for any $M \geq \max\{m', \mu'\} > 0$, and a suitable $C_{t,s}$, continuous with respect to s, t, $0 \leq s \leq t \leq T$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} |e(t,s,x,\xi)|^2 \,\mathfrak{M}(d\xi)\, |\nu_s|^2_{\mathrm{tv}}\, ds \\
&\qquad \lesssim \int_0^t \int_{\mathbb{R}^d} \left[|e_N(t,s,x,\xi)|^2 \mod C_{t,s} \cdot S^{-\infty,-\infty} \right] \mathfrak{M}(d\xi)\, ds \\
&\qquad \lesssim \int_{\mathbb{R}^d} \int_0^t \left[e_0(t,s,x,\xi) + \frac{C_{t,s}}{(\langle x\rangle\langle\xi\rangle)^M} \right] ds\, \mathfrak{M}(d\xi) \\
&\qquad \lesssim \int_{\mathbb{R}^d} \left[\frac{1 - e^{-Ct\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}}}{\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} + \frac{1}{(\langle x\rangle\langle\xi\rangle)^M} \right] \mathfrak{M}(d\xi) \\
&\qquad \lesssim \int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{\langle\xi\rangle^{\mu'}} < \infty, \\
\Rightarrow J &= \int_0^t \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |e(t,s,x,\xi+\eta)|^2 \,\mathfrak{M}(d\xi) \right) |\nu_s|^2_{\mathrm{tv}}\, ds \\
&= \int_0^t \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |e(t,s,x,\xi)|^2 \,\mathfrak{M}(d\xi) \right) |\nu_s|^2_{\mathrm{tv}}\, ds \\
&\lesssim \left(\int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{\langle\xi\rangle^{\mu'}} \right) \int_0^t \, |\nu_s|^2_{\mathrm{tv}}\, ds < \infty,
\end{aligned}$$

proving that **(A1)** holds true also in this last case.

2. Using (56), we get that condition **(A2)**, with $\Lambda(t,s)$ being the Schwartz kernel of $E(t,s)$, is satisfied if

$$
\begin{aligned}
K = \lim_{h\downarrow 0} \int_0^t & \chi_{[0,t-h)}(s) \\
& \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \, \mathfrak{M}(d\xi) \right) \\
& \times |\nu_s|_{\mathrm{tv}}^2 \, ds = 0.
\end{aligned}
$$

If we assume the hypothesis **(H0)**, we find, by regularity of e with respect to (t,s), (52), the assumptions on σ, and, recalling (51), Lebesgue's Dominated Convergence Theorem, for every $t \in [0,T]$,

$$
\begin{aligned}
K &= \lim_{h\downarrow 0} \int_0^t \chi_{[0,t-h)}(s) \\
&\qquad \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \, \mathfrak{M}(d\xi) \right) \\
&\qquad \times |\nu_s|_{\mathrm{tv}}^2 \, ds \\
&= \int_0^t \lim_{h\downarrow 0} \chi_{[0,t-h)}(s) \\
&\qquad \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \, \mathfrak{M}(d\xi) \right) \\
&\qquad \times |\nu_s|_{\mathrm{tv}}^2 \, ds = 0,
\end{aligned}
$$

and **(A2)** holds true.
If we assume the hypothesis **(H1)**, it suffices to show that

$$
\sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \le C_{t,s,h}^2, \tag{57}
$$

with $C_{t,s,h}$ a continuous function with respect to s, t, h, such that $C_{t,s,h} \to 0$ as $h \downarrow 0$ and $C_{t,s,h} \le C_T$ for every $h \in [0, t-s]$, $0 \le s < t \le T$. Indeed, since $e(t,s)$ is regular with respect to s and t, if (57) holds true we find, for $0 \le t \le T$,

$$
\begin{aligned}
\int_0^t & \chi_{[0,t-h)}(s) \\
& \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \, \mathfrak{M}(d\xi) \right) \\
& \times |\nu_s|_{\mathrm{tv}}^2 \, ds \\
& \le \int_0^t C_{t,s,h}^2 \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) |\nu_s|_{\mathrm{tv}}^2 \, ds = \left(\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) \int_0^t |\nu_s|_{\mathrm{tv}}^2 \, C_{t,s,h}^2 \, ds,
\end{aligned}
$$

which implies

$$
\begin{aligned}
0 &\le K \\
&= \lim_{h\downarrow 0} \int_0^t \chi_{[0,t-h)}(s) \\
&\qquad \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2\, \mathfrak{M}(d\xi) \right) \\
&\qquad \times |\nu_s|_{\mathrm{tv}}^2\, ds \\
&\le \left(\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) \lim_{h\downarrow 0} \int_0^t |\nu_s|_{\mathrm{tv}}^2\, C_{t,s,h}^2\, ds \\
&= \left(\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) \int_0^t |\nu_s|_{\mathrm{tv}}^2 \left(\lim_{h\downarrow 0} C_{t,s,h}^2 \right) ds = 0,
\end{aligned}
$$

via Lebesgue's Dominated Convergence Theorem, showing that **(A2)** holds true as well in this second case. The proof of (57) is actually a simpler version of the analogous inequality proved in [4, 5], so we omit it here.
If we assume hypothesis **(H2)**, it suffices to show that

$$
\sup_{r\in(s,s+h)} |e_0(t,s,x,\xi) - e_0(t,r,x,\xi)|^2 \le C_{s,h}\, e^{-C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}}, \tag{58}
$$

where $C_{s,h}$ is a positive function, continuous with respect to h, s, $h \in [0, t-s]$, $0 \le s < t \le T$, and such that $C_{s,h} \to 0$ as $h \to 0$, while C is the costant which appears in (24). Indeed, if (58) hods true, writing as above $e(t,s) = e_N(t,s)$ mod $C([s,T], S^{-\infty,-\infty})$, choosing $M \ge \max\{m', \mu'\} > 0$, with $A_{t,s}$ a suitable continuous function of s, t, $0 \le s \le t \le T$, we find, for $0 \le s < t \le T$,

$$
\begin{aligned}
&\int_0^t \chi_{[0,t-h)}(s) \\
&\qquad \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2\, \mathfrak{M}(d\xi) \right) \\
&\qquad \times |\nu_s|_{\mathrm{tv}}^2\, ds \\
&\le \int_0^t \chi_{[0,t-h)}(s) \left(\int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi) - e(t,r,x,\xi)|^2\, \mathfrak{M}(d\xi) \right) |\nu_s|_{\mathrm{tv}}^2\, ds \\
&\lesssim \int_0^t \chi_{[0,t-h)}(s) \\
&\qquad \times \left[\int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} \left(|e_0(t,s,x,\xi) - e_0(t,r,x,\xi)|^2 + \frac{|A_{t,s} - A_{t,r}|^2}{(\langle x\rangle\langle\xi\rangle)^{2M}} \right) \mathfrak{M}(d\xi) \right] \\
&\qquad \times |\nu_s|_{\mathrm{tv}}^2\, ds
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \int_0^t \left[\int_{\mathbb{R}^d} \left(C_{s,h} e^{-C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} + \frac{B_{t,s,h}}{(\langle x\rangle\langle\xi\rangle)^{2M}}\right) \mathfrak{M}(d\xi)\right] ds \\
&\lesssim \widetilde{C}_{t,h} \int_{\mathbb{R}^d} \left[\int_0^t \left(e^{-C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} + \frac{1}{(\langle x\rangle\langle\xi\rangle)^{2M}}\right) ds\right] \mathfrak{M}(d\xi) \\
&\lesssim \widetilde{C}_{t,h} \int_{\mathbb{R}^d} \left(\frac{1 - e^{-Ct\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}}}{\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} + \frac{1}{(\langle x\rangle\langle\xi\rangle)^{2M}}\right) \mathfrak{M}(d\xi) \\
&\lesssim \widetilde{C}_{t,h} \int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{\langle\xi\rangle^{\mu'}},
\end{aligned}
$$

where $\widetilde{C}_{t,h} = \max_{0\le s\le t}(C_{s,h} + B_{t,s,h})$, $\widetilde{C}_{t,h} \to 0$ for $h \downarrow 0$. This implies, by (54),

$$
\begin{aligned}
0 &\le K \\
&= \lim_{h\downarrow 0} \int_0^t \chi_{[0,t-h)}(s) \\
&\qquad \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \,\mathfrak{M}(d\xi)\right) \\
&\qquad \times |\nu_s|^2_{\mathrm{tv}}\, ds \\
&\lesssim \left(\int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{\langle\xi\rangle^{\mu'}}\right) \lim_{h\downarrow 0} \widetilde{C}_{t,h} = 0,
\end{aligned}
$$

proving that **(A2)** holds true also in this last case. Let us then show that (58) holds true. We have:

$$
\begin{aligned}
|e_0(t,s,x,\xi) - e_0(t,r,x,\xi)| &= \left| e^{-\int_s^t a(\tau,x,\xi)\,d\tau} - e^{-\int_r^t a(\tau,x,\xi)\,d\tau} \right| \\
&= e^{-\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\,d\tau} \left| 1 - e^{\int_s^r a(\tau,x,\xi)\,d\tau} \right| \\
&\le e^{-\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\,d\tau} \int_s^r \mathrm{Re}\, a(\tau,x,\xi)\,d\tau \\
&\le e^{-\frac{1}{2}\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\,d\tau} \left(e^{-\frac{1}{2}\int_s^r \mathrm{Re}\, a(\tau,x,\xi)\,d\tau} \int_s^r \mathrm{Re}\, a(\tau,x,\xi)\,d\tau \right) \\
&\le e^{-\frac{C}{2}(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} C_{s,r}
\end{aligned}
$$

with a function $C_{s,r}$, continuous in s, r and such that $C_{s,r} \le \sqrt{C_2}$, C_2 the supremum of $s^2 e^{-s}$, $s \ge 0$. This implies

$$
\sup_{r\in(s,s+h)} |e_0(t,s,x,\xi) - e_0(t,r,x,\xi)|^2 \le C_{s,h}\, e^{-C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}}
$$

with $C_{s,h} = \sup_{r\in(s,s+h)} C^2_{s,r}$, which clearly has all the requested properties.

Summing up, v_2 in (55) is well-defined, as a stochastic integral with respect to the martingale measure canonically associated with $\mathfrak{M}$, under either one of the hypotheses **(H0)**, **(H1)**, or **(H2)**. Since $\mathbb{E}[v_2] = 0$, the regularity of $\mathbb{E}[u]$ is the same as the one of the solution of the associated deterministic Cauchy problem, described in Theorem 5. The proof is complete. □

Acknowledgements The first and second author have been partially supported by their own National Research Fund FFABR 2017.

References

1. A. Abdeljawad. Global microlocal analysis on $\mathbb{R}^d$ with applications to hyperbolic partial differential equations and modulation spaces. Ph.D. Thesis, Università degli Studi di Torino (2019), http://hdl.handle.net/2318/1718409
2. A. Abdeljawad, A. Ascanelli, S. Coriasco. Deterministic and Stochastic Cauchy problems for a class of weakly hyperbolic operators on $\mathbb{R}^n$. *Monatshefte für Mathematik* 192 (2020), 1–38.
3. A. Ascanelli, S. Coriasco, A Süß. On temperate distributions decaying at infinity Volume: *Generalized functions and Fourier analysis*, Series: "Operator Theory: Advances and Applications" (Editors: M.Oberguggenberger, J.Toft, J.Vindas, P.Wahlberg), Vol. 260 (2017), Birkhäuser/Springer, pp. 1–18.
4. A. Ascanelli, S. Coriasco, A. Süß. Random-field solutions of weakly hyperbolic stochastic partial differential equations with polynomially bounded coefficients. *J. Pseudo-Differ. Oper. Appl.* 11, 1 (2020) , 387–424.
5. A. Ascanelli, S. Coriasco, A. Süß. Solution theory to semilinear hyperbolic stochastic partial differential equations with polynomially bounded coefficients. *Nonlinear Analysis, Theory, Methods and Applications* 189 (2019), 111–574.
6. A. Ascanelli, S. Coriasco, A Süß. Solution theory to semilinear parabolic stochastic partial differential equations with polynomially bounded coefficients. In preparation (2020).
7. A. Ascanelli, A. Süß. Random-field solutions to linear hyperbolic stochastic partial differential equations with variable coefficients. *Stochastic Processes and their Applications* 128 (2018), 2605–2641.
8. D. Conus, R. C. Dalang. The non-linear stochastic wave equation in high dimensions. *Electronic Journal of Probability* 13 (2008), 629–670.
9. H. O. Cordes. The Technique of Pseudodifferential Operators. Cambridge Univ. Press, 1995.
10. S. Coriasco, J. Toft. Asymptotic expansions for Hörmander symbol classes in the calculus of pseudo- differential operators. *J. Pseudo-Differ. Oper. Appl.* 5, 1 (2014), 27–41.
11. R. C. Dalang. Extending Martingale Measure Stochastic Integral with Applications to Spatially Homogeneous SPDEs. *Electronic Journal of Probability* 4 (1999), 1–29.
12. R. C. Dalang, N. Frangos. The stochastic wave equation in two spatial dimensions. *Annals of Probability* 26, 1 (1998) :187–212.
13. G. DaPrato, J. Zabczyk. Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and its Applications 45. Cambridge University Press, 2008.
14. H. Kumano-go. Pseudo-Differential Operators. MIT Press, 1981.
15. L. Maniccia, P. Panarese. Eigenvalue asymptotics for a class of md-elliptic ψdos on manifolds with cylindrical exits. *Annali di Matematica* 181 (2002), 283–308.
16. R. Melrose. Geometric scattering theory. Stanford Lectures. Cambridge University Press, Cambridge, 1995.

17. C. Parenti. Operatori pseudodifferenziali in $\mathbb{R}^n$ e applicazioni. *Ann. Mat. Pura Appl.* 93 (1972), 359–389.
18. M. Sanz-Solé, A. Süß. The stochastic wave equation in high dimensions: Malliavin differentiability and absolute continuity. *Electronic Journal of Probability* 18 (2013), Paper no 64.
19. M. Sanz-Solé, P. A. Vuillermot. Equivalence and Hölder-Sobolev regularity of solutions for a class of non-autonomous stochastic partial differential equations. *Inst. H. Poincaré Probab. Statist.* 39, 4 (2003), 703–742.
20. L. Schwartz. *Théorie des Distributions*. Hermann, 2nd edition, 2010.
21. J. B. Walsh. *École d'été de Probabilités de Saint Flour XIV, 1984*, volume 1180 of *Lecture Notes in Math*, chapter An Introduction to Stochastic Partial Differential Equations. Springer, 1986.