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DENSITY AND SPECTRUM OF MINIMAL SUBMANIFOLDS IN SPACE FORMS

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Abstract. Let M^m be a minimal, properly immersed submanifold in a space form N^n_k of curvature $-k \le 0$. In this paper, we are interested in the relation between the density function $\Theta(r)$ of M^m and the spectrum of the Laplace-Beltrami operator. In particular, we prove that if $\Theta(r)$ has subexponential growth (when $k < 0$) or sub-polynomial growth ($k = 0$) along a sequence, then the spectrum of M^m is the same as that of the space form N^m_k. Notably, the result applies to Anderson's (smooth) solutions of Plateau's problem at infinity on the hyperbolic space, independently of their boundary regularity. We also give a simple condition on the second fundamental form that ensures M to have finite density. In particular, we show that minimal submanifolds of the hyperbolic space that have finite total curvature have also finite density.

Contents

1. Introduction

Let M^m be a minimal, properly immersed submanifold in a complete ambient space N^n. Observe that, in this case, M is complete. In the present paper, we are interested in the case when N is close, in a sense made precise below, to a space form $\mathbb{N}^n_k$ of curvature $-k \le 0$. In particular, our focus is the study of the spectrum of the Laplace

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Beltrami operator $-\Delta$ on M and its relationship with the density at infinity of M, that is, the limit as $r \to +\infty$ of the (monotone) quantity

$$\Theta(r) \doteq \frac{\mathrm{vol}(M \cap B_r)}{V_k(r)}, \tag{1.1}$$

where B_r indicates a geodesic ball of radius r in N^n and $V_k(r)$ is the volume of a geodesic ball of radius r in $\mathbb{N}_k^m$. Hereafter, we will say that M has finite density if

$$\Theta(+\infty) \doteq \lim_{r\to+\infty} \Theta(r) < +\infty.$$

To properly put our results into perspective, we briefly recall few facts about the spectrum of the Laplacian on a geodesically complete manifold. It is known by works of P. Chernoff [15] and R.S. Strichartz [49] that $-\Delta$ on a complete manifold is essentially self-adjoint on the domain $C_c^\infty(M)$, and thus it admits a unique self-adjoint extension, which we still call $-\Delta$. Since $-\Delta$ is positive and self-adjoint, its spectrum is the set of $\lambda \ge 0$ such that $\Delta + \lambda I$ does not have bounded inverse. Sometimes we say spectrum of M rather than spectrum of $-\Delta$ and we denote it by $\sigma(M)$. The well-known Weyl's characterization for the spectrum of a self-adjoint operator in a Hilbert space implies the following

Lemma 1. [19, Lemma 4.1.2] *A number $\lambda \in \mathbb{R}$ lies in $\sigma(M)$ if and only if there exists a sequence of nonzero functions $u_j \in \mathrm{Dom}(-\Delta)$ such that*

$$\|\Delta u_j + \lambda u_j\|_2 = o\big(\|u_j\|_2\big) \qquad \text{as } j \to +\infty. \tag{1.2}$$

In the literature, characterizations of the whole $\sigma(M)$ are known only in few special cases. Among them, the Euclidean space, for which $\sigma(\mathbb{R}^m) = [0, \infty)$, and the hyperbolic space $\mathbb{H}_k^m$, for which

$$\sigma(\mathbb{H}_k^m) = \left[\frac{(m-1)^2 k}{4}, +\infty\right). \tag{1.3}$$

The approach to guarantee that $\sigma(M) = [c, +\infty)$, for some $c \ge 0$, usually splits into two parts. The first one is to show that $\inf \sigma(M) \ge c$ via, for instance, the Laplacian comparison theorem from below ([42, 5]), and the second one is to produce a sequence like in lemma 1 for each $\lambda > c$. This step is accomplished by considering radial functions of compact support, and, at least in the first results on the topic like the one in [21], uses the comparison theorems on both sides for $\Delta\rho$, ρ being the distance from a fixed origin $o \in M$. Therefore, the method needs both a pinching on the sectional curvature and the smoothness of ρ, that is, that o is a pole of M (see [21, 25, 37] and Corollary 2.17 in [8]), which is a severe topological restriction. Since then, various efforts were made to weaken both the curvature and the topological assumptions. We briefly overview some of the main achievements.

In [35], Kumura observed that to perform the second step (and just for it) it is enough that there exists a relatively compact, mean convex, smooth open set Ω with the property that the normal exponential map realizes a global diffeomorphism $\partial\Omega \times \mathbb{R}_0^+ \to M\backslash\Omega$. Conditions of this kind seem, however, unavoidable for his techniques to work. On the other hand, in [36] the author drastically weakened the curvature requirements needed to establish Step 2, by replacing the two-sided pinching on the sectional curvature with a combination of a lower bound on a suitably weighted volume and an L^p-bound on the Ricci curvature.

Regarding the need for a pole, major recent improvements have been made in a series of papers ([50, 55, 41, 11]): their guiding idea was to replace the L^2-norm in relation (1.2) with the L^1-norm, which via a trick in [55, 41] enables to use smoothed distance functions to construct sequences as in Lemma 1. Building on deep function-theoretic results due to Sturm [50] and Charalambous-Lu [11], in [55, 41] the authors proved that $\sigma(M) = [0, \infty)$ when

$$\liminf_{\rho(x)\to+\infty} \mathrm{Ricc}_x = 0 \tag{1.4}$$

in the sense of quadratic forms, without any topological assumption. This remarkable result improves on [37] and [25] (see also Corollary 2.17 in [8]), where M was assumed to have a pole. Further refinements of (1.4) have been given in [11]. However, when (1.4) does not hold, the situation is more delicate and is still the subject of an active area of research. In this respect, we also quote the general function-theoretic criteria developed by H. Donnelly [22], and K.D. Elworthy and F-Y. Wang [24] to ensure that a half-line belongs to the spectrum of M.

The main concern in this paper is to achieve, in the above-mentioned setting of minimal submanifolds, a characterization of the whole $\sigma(M)$ free from curvature or topological conditions on M. It is known by [18] and [5] that for a minimal immersion $\varphi : M^m \to \mathbb{N}^n_k$ the fundamental tone of M, $\inf \sigma(M)$, is at least that of $\mathbb{N}^m_k$, i.e.,

$$\inf \sigma(M) \ge \frac{(m-1)^2 k}{4}. \tag{1.5}$$

Moreover, as a corollary of [35] and [4, 6], if the second fundamental form II satisfies the decay estimate

$$\begin{aligned} &\lim_{\rho(x)\to+\infty} \rho(x)|\mathrm{II}(x)| = 0 \qquad && \text{if } k = 0 \\ &\lim_{\rho(x)\to+\infty} |\mathrm{II}(x)| = 0 && \text{if } k > 0 \end{aligned} \tag{1.6}$$

($\rho(x)$ being the intrinsic distance with respect to some fixed origin $o \in M$), then M has the same spectrum that a totally geodesic submanifold $\mathbb{N}^m_k \subset \mathbb{N}^n_k$, that is,

$$\sigma(M) = \left[\frac{(m-1)^2 k}{4}, +\infty\right). \tag{1.7}$$

According to [1, 20], (1.6) is ensured when M has finite total curvature, that is, when

$$\int_M |\mathrm{II}|^m < +\infty. \tag{1.8}$$

Remark 1. A characterization of the essential spectrum, similar to (1.7), also holds for submanifolds of the hyperbolic space $\mathbb{H}^n_k$ with constant (normalized) mean curvature $H < \sqrt{k}$. There, condition (1.8) is replaced by the finiteness of the L^m-norm of the traceless second fundamental form. For deepening, see [10].

Inspecting the proofs of the above results it seemed to us that, for (1.7) to hold, condition (1.8) and more generally (1.6) could be substantially weakened. Here, we identify a suitable growth condition on the density function $\Theta(r)$ *along a sequence* as a natural candidate to replace them, see (1.10). As a very special case, (1.7) holds when M has finite density. It might be surprising that just a volume growth condition along a sequence could control the whole spectrum of M; for this to happen, the minimality condition enters in a crucial and subtle way.

Regarding the relation between (1.8) and the finiteness of $\Theta(+\infty)$, we remark that their interplay has been investigated in depth for minimal submanifolds of $\mathbb{R}^n$, but the case of $\mathbb{H}^n_k$ seems to be partly unexplored. In the next section, we will briefly discuss the state of the art, to the best of our knowledge. As a corollary of Theorem 2 below, we will show the following

Corollary 1. *Let M^m be a minimal properly immersed submanifold in $\mathbb{H}^n_k$. If M has finite total curvature, then $\Theta(+\infty) < +\infty$.*

As far as we know, this result was previously known just in dimension $m = 2$ via a Chern-Osserman type inequality, see the next section for further details.

We now come to our results, beginning with defining the ambient spaces which we are interested in: these are manifolds with a pole, whose radial sectional curvature is suitably pinched to that of the model $\mathbb{N}^n_k$.

Definition 1. *Let N^n possess a pole $\bar{o}$ and denote with $\bar{\rho}$ the distance function from $\bar{o}$. Assume that the radial sectional curvature $\bar{K}_{\mathrm{rad}}$ of N, that is, the sectional curvature restricted to planes π containing $\bar{\nabla}\bar{\rho}$, satisfies*

$$-G\big(\bar{\rho}(x)\big) \le \bar{K}_{\mathrm{rad}}(\pi_x) \le -k \le 0 \qquad \forall\, x \in N\backslash\{\bar{o}\}, \tag{1.9}$$

for some $G \in C^0(\mathbb{R}^+_0)$. We say that

(*i*) N has a pointwise (respectively, integral) pinching to $\mathbb{R}^n$ if $k = 0$ and
$$sG(s) \to 0 \ \text{ as } \ s \to +\infty \qquad \big(\text{respectively, } \ sG(s) \in L^1(+\infty)\big);$$

(*ii*) N has a pointwise (respectively, integral) pinching to $\mathbb{H}^n_k$ if $k > 0$ and
$$G(s) - k \to 0 \ \text{ as } \ s \to +\infty \qquad \big(\text{respectively, } \ G(s) - k \in L^1(+\infty)\big).$$

Hereafter, given an ambient manifold N with a pole $\bar{o}$, the density function $\Theta(r)$ will always be computed by taking extrinsic balls centered at $\bar{o}$.

Our main achievements are the following two theorems. The first one characterizes $\sigma(M)$ when the density of M grows subexponentially (respectively, sub-polynomially) along a sequence. Condition (1.10) below is very much in the spirit of a classical volume growth hypothesis due to R. Brooks [9] and Y. Higuchi [31] to bound from above the infimum of the essential spectrum of $-\Delta$. However, we stress that Theorem 1 below seems to be the first result in the literature characterizing the whole spectrum of M under just a mild volume growth assumption.

Theorem 1. *Let $\varphi : M^m \to N^n$ be a minimal properly immersed submanifold, and suppose that N has a pointwise or an integral pinching to a space form. If either*

$$\begin{array}{lll} N \textit{ is pinched to } \mathbb{H}^n_k, \textit{ and} & \displaystyle\liminf_{s\to+\infty} \frac{\log\Theta(s)}{s} = 0, & \textit{or} \\[2ex] N \textit{ is pinched to } \mathbb{R}^n, \textit{ and} & \displaystyle\liminf_{s\to+\infty} \frac{\log\Theta(s)}{\log s} = 0. & \end{array} \tag{1.10}$$

then

$$\sigma(M) = \left[\frac{(m-1)^2 k}{4}, +\infty\right). \tag{1.11}$$

The above theorem is well suited for minimal submanifolds constructed via Geometric Measure Theory since, typically, their existence is guaranteed by controlling the density function $\Theta(r)$. As an important example, Theorem 1 applies to all solutions of Plateau's problem at infinity $M^m \to \mathbb{H}^n_k$ constructed in [2], provided that they are smooth. Indeed, because of their construction, $\Theta(+\infty) < +\infty$ (see [2], part [A] at p. 485) and they are proper (it can also be deduced as a consequence of $\Theta(+\infty) < +\infty$, see Remark 5). By standard regularity theory, smoothness of M^m is automatic if $m \le 6$.

Corollary 2. *Let $\Sigma \subset \partial_\infty \mathbb{H}^n_k$ be a closed, integral $(m-1)$ current in the boundary at infinity of $\mathbb{H}^n_k$ such that, for some neighbourhood $U \subset \mathbb{H}^n_k$ of $\mathrm{supp}(\Sigma)$, Σ does not bound in U, and let $M^m \hookrightarrow \mathbb{H}^n_k$ be the solution of Plateau's problem at infinity constructed in* [2] *for Σ. If M is smooth, then* (1.11) *holds.*

An interesting fact of Corollary 2 is that M is *not* required to be regular up to $\partial_\infty \mathbb{H}^n_k$, in particular it might have infinite total curvature. In this respect, we observe that if M be C^2 up to $\partial_\infty \mathbb{H}^n_k$, then M would have finite total curvature (Lemma 5 in Appendix 1). By deep regularity results, this is the case if, for instance, $M^m \to \mathbb{H}^{m+1}_k$ is a smooth hypersurface that solves Plateau's problem for Σ, and Σ is a $C^{2,\alpha}$ (for $\alpha > 0$), embedded compact hypersurface of $\partial_\infty \mathbb{H}^n_k$. See Appendix 1 for details.

The spectrum of solutions of Plateau's problems has also been considered in [3] for minimal surfaces in $\mathbb{R}^3$. In this respect, it is interesting to compare Corollary 2 with (3) of Corollary 2.6 therein.

Remark 2. The solution M of Plateau's problem in [2] is constructed as a weak limit of a sequence M_j of minimizing currents for suitable boundaries Σ_j converging to Σ. and property $\Theta(+\infty) < +\infty$ is a consequence of a uniform upper bound for the mass of a sequence M_j (part [A], p. 485 in [2]). Such a bound is achieved because of the way the boundaries Σ_j are constructed, in particular, since they are all sections of the same cone. One might wonder whether $\Theta(+\infty) < +\infty$, or at least the subexponential growth in (1.10), is satisfied by all solutions of Plateau's problem. In this respect, we just make this simple observation: in the hypersurface case $n = m+1$, if $M \cap B^{m+1}_r$ is volume-minimizing then clearly

$$\Theta(r) = \frac{\mathrm{vol}(M \cap B^{m+1}_r)}{V_k(r)} \le \frac{\mathrm{vol}(\partial B^{m+1}_r \subset \mathbb{H}^{m+1}_k)}{V_k(r)} = c_k \frac{\sinh^m(\sqrt{k}r)}{V_k(r)},$$

but this last expression diverges exponentially fast as $r \to +\infty$ (differently from its Euclidean analogous, which is finite). This might suggest that a general solution of Plateau's problem does not automatically satisfies $\Theta(+\infty) < +\infty$, and maybe not even (1.10).

In our second result we focus on the particular case when $\Theta(+\infty) < +\infty$, and we give a sufficient condition for its validity in terms of the decay of the second fundamental form. Towards this aim, we shall restrict to ambient spaces with an integral pinching.

Theorem 2. *Let $\varphi : M^m \to N^n$ be a minimal immersion, and suppose that N has an integral pinching to a space form. Denote with $\rho(x)$ the intrinsic distance from some reference origin $o \in M$. Assume that there exist $c > 0$ and $\alpha > 1$ such that the second fundamental form satisfies, for $\rho(x) >> 1$,*

$$\begin{aligned} |\mathrm{II}(x)|^2 &\le \frac{c}{\rho(x)\log^\alpha \rho(x)} \qquad \text{if } N \text{ is pinched to } \mathbb{H}^n_k; \\ |\mathrm{II}(x)|^2 &\le \frac{c}{\rho(x)^2\log^\alpha \rho(x)} \qquad \text{if } N \text{ is pinched to } \mathbb{R}^n. \end{aligned} \tag{1.12}$$

Then, φ is proper, M is diffeomorphic to the interior of a compact manifold with boundary, and $\Theta(+\infty) < +\infty$.

The assertions that φ be proper and M have finite topology is well-known under assumptions even weaker than (1.12) and not necessarily requiring the minimality, see for instance [4, 6]. Former results are due to [1] ($N = \mathbb{R}^m$) and [20, 10] ($N = \mathbb{H}_k$). Here, our original contribution is to show that M has finite density. Because of a result in [20, 46], if $\varphi : M \to \mathbb{H}^n_k$ has finite total curvature then $|II(x)| = o(\rho(x)^{-1})$ as $\rho(x) \to +\infty$. Hence, (1.12) is met and Corollary 1 follows at once.

We briefly describe the strategy of the proof of Theorem 1. In view of (1.5), it is enough to show that each $\lambda > (m-1)^2 k/4$ lies in $\sigma(M)$. To this end, we follow an approach inspired by a general result due to K.D. Elworthy and F-Y. Wang [24]. However, Elworthy-Wang's theorem is not sufficient to conclude, and we need to considerably refine the criterion in order to fit in the present setting. To construct the sequence as in Lemma 1, a key step is to couple the volume growth requirement (1.10) with a sharpened form of the monotonicity formula for minimal submanifolds, which improves on the classical ones in [48, 2]. Indeed, in Proposition 3 we describe three monotone quantities other than $\Theta(s)$, that might be useful beyond the purpose of the present paper. For example, in the very recent [27] the authors discovered and used some of the relations in Proposition 3 to show interesting comparison results for the capacity and the first eigenvalue of minimal submanifolds.

1.1. Finite density and finite total curvature in $\mathbb{R}^n$ and $\mathbb{H}^n$.

The first attempt to extend the classical theory of finite total curvature surfaces in $\mathbb{R}^n$ (see [44, 32, 16, 17]) to the higher-dimensional case is due to M.T. Anderson. In [1], the author drew from (1.8) a number of topological and geometric consequences, and here we focus on those useful to highlight the relationship between total curvature and density. First, he showed that (1.8) implies the decay

$$\lim_{\rho(x)\to+\infty} \rho(x)|\mathrm{II}(x)| = 0, \tag{1.13}$$

where $\rho(x)$ is the intrinsic distance from a fixed origin, and as a consequence M is proper, the extrinsic distance function r has no critical points outside some compact set and $|\nabla r| \to 1$ as r diverges, so by Morse theory M is diffeomorphic to the interior of a compact manifold with boundary. Moreover, he proved that M has finite density via a higher-dimensional extension of the Chern-Osserman identity [16, 17], namely the following relation linking the Euler characteristic $\chi(M)$ and the Pfaffian form Ω ([1], Theorem 4.1):

$$\chi(M) = \int_M \Omega + \lim_{r\to+\infty} \frac{\mathrm{vol}(M \cap \partial B_r)}{V_0'(r)}. \tag{1.14}$$

Observe that, since $|\nabla r| \to 1$, by coarea's formula the limit in the right hand-side coincides with $\Theta(+\infty)$. We underline that property $\Theta(+\infty) < +\infty$ plays a fundamental role to apply the machinery of manifold convergence to get information on the limit structure of the ends of M ([1, 47, 54]). For instance, $\Theta(+\infty)$ is related to the number $\mathcal{E}(M)$ of ends of M: if we denote with $V_1, \ldots, V_{\mathcal{E}(M)}$ the (finitely many) ends of M, (1.8) implies for $m \geq 3$ the identities

$$\Theta(+\infty) = \sum_{i=1}^{\mathcal{E}(M)} \lim_{r\to+\infty} \frac{\mathrm{vol}(V_i \cap \partial B_r)}{V_0'(r)} \equiv \mathcal{E}(M), \tag{1.15}$$

and thus M is totally geodesic provided that it has only one end and finite total curvature ([1], Thm 5.1 and its proof). Further information on the mutual relationship between the finiteness of the total curvature and $\Theta(+\infty) < +\infty$ can be deduced under the additional requirement that M is stable or it has finite stability index. For example, by work of J. Tysk [54], if M^m has finite index and $m \le 6$, then

$$\Theta(+\infty) < +\infty \qquad \text{if and only if} \qquad \int_M |\mathrm{II}|^m < +\infty. \tag{1.16}$$

Remark 3. Indeed, the main result in [54] states that, when $\Theta(+\infty) < +\infty$ and $m \le 6$, M has finite index if and only if it has finite total curvature. However, since the finite total curvature condition alone implies both that M has finite index and $\Theta(+\infty) < +\infty$ (in any dimension*), the characterization in (1.16) is equivalent to Tysk's theorem. We underline that it is still a deep open problem whether or not, for $m \ge 3$, stability or finite index alone implies the finiteness of the density at infinity.

Since then, efforts were made to investigate analogous properties for minimal submanifolds of finite total curvature immersed in $\mathbb{H}^n_k$. There, some aspects show strong analogy with the $\mathbb{R}^n$ case, while others are strikingly different. For instance, minimal immersions $\varphi : M^m \to \mathbb{H}^n_k$ with finite total curvature enjoy the same decay property (1.13) with respect to the intrinsic distance $\rho(x)$ ([20], see also [46]), which is enough to deduce that they are properly immersed and diffeomorphic to the interior of a compact manifold with boundary. Moreover, Anderson [2] proved the monotonicity of $\Theta(r)$ in (1.1). In order to show (among other things) that complete, finite total curvature surfaces $M^2 \hookrightarrow \mathbb{H}^n$ have finite density, in [13, 14] the authors obtained the following Chern-Osserman type (in)equality:

$$\chi(M) \ge -\frac{1}{4\pi}\int_M |\mathrm{II}|^2 + \Theta(+\infty), \tag{1.17}$$

see also [28]. However, in the higher dimensional case we found no analogous of (1.14), (1.17) in the literature, and adapting the proof of (1.14) to the hyperbolic ambient space seems to be subtler than what we expected. In fact, an *equality* like (1.14) is not even possible to obtain, since there exist minimal submanifolds of $\mathbb{H}^n_k$ with finite density but whose density at infinity depends on the chosen reference origin [26]. We point out that, on the contrary, inequality (1.17) holds for each choice of the reference origin in $\mathbb{H}^n$. This motivated the different route that we follow to prove Theorem 2 and Corollary 1. Among the results in [1] that could not admit a corresponding one in $\mathbb{H}^n_k$, in view of the solvability of Plateau's problem at infinity on $\mathbb{H}^n_k$ we stress that a relation like (1.15) cannot hold for each minimal submanifold of $\mathbb{H}^n_k$ with finite total curvature. Indeed, there exist a wealth of properly immersed minimal submanifolds in $\mathbb{H}^n_k$ with finite total curvature and one end: for example, referring to the upper half-space model, the graphical solution of Plateau's problem for $\Sigma^{m-1} \subset \partial_\infty \mathbb{H}^n_k$ being the boundary of a convex set (constructed at the end of [2]) has finite total curvature, as follows from Lemma 5 and the regularity results recalled in Appendix 1. It shall be observed, however, that when II decays sufficiently fast at infinity with respect to the extrinsic distance function $r(x)$:

$$\lim_{r(x)\to+\infty} e^{2\sqrt{k}r(x)}|\mathrm{II}(x)| = 0, \tag{1.18}$$

then the inequality $\Theta(+\infty) \le \mathcal{E}(M)$ still holds for minimal hypersurfaces in $\mathbb{H}^n_k$ as shown in [29], and in particular M is totally geodesic provided that it has only one

*As said, finite total curvature implies $\Theta(+\infty) < +\infty$ by (1.14), while the finiteness of the index can be seen as an application of the generalized Cwikel-Lieb-Rozembljum inequality (see [38]) to the stability operator $L = -\Delta - |\mathrm{II}|^2$, recalling that a minimal submanifold $M^m \to \mathbb{R}^n$ satisfies a Sobolev inequality. We refer to [45] for deepening.

end, as first observed in [33, 34]. We remark that there exists an infinite family of complete minimal cylinders $\varphi_\lambda : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{H}^3$ whose second fundamental form II_λ decays exactly of order $\exp\{-2r(x)\}$, see [43].

2. Preliminaries

Let $\varphi : (M^m, \langle\, , \rangle) \to (N^n, (\, ,))$ be an isometric immersion of a complete m-dimensional Riemannian manifold M into an ambient manifold N of dimension n and possessing a pole $\bar{o}$. We denote with $\nabla, \mathrm{Hess}\,, \Delta$ the connection, the Riemannian Hessian and the Laplace-Beltrami operator on M, while quantities related to N will be marked with a bar. For instance, $\bar{\nabla}, \overline{\mathrm{dist}}, \overline{\mathrm{Hess}}$ will identify the connection, the distance function and the Hessian in N. Let $\bar{\rho}(x) = \overline{\mathrm{dist}}(x, \bar{o})$ be the distance function from $\bar{o}$. Geodesic balls in N of radius R and center y will be denoted with $B^N_R(y)$. Moreover, set

$$r : M \to \mathbb{R}, \qquad r(x) = \bar{\rho}\big(\varphi(x)\big), \tag{2.1}$$

for the extrinsic distance from $\bar{o}$. We will indicate with Γ_s the extrinsic geodesic spheres restricted to M: $\Gamma_s \doteq \{x \in M;\ r(x) = s\}$. Fix a base point $o \in M$. In what follows, we shall also consider the intrinsic distance function $\rho(x) = \mathrm{dist}(x, o)$ from a reference origin $o \in M$.

2.1. **Target spaces.** Hereafter, we consider an ambient space N possessing a pole $\bar{o}$ and, setting $\bar{\rho}(x) \doteq \mathrm{dist}(x, \bar{o})$, we assume that (1.9) is met for some $k \geq 0$ and some $G \in C^0(\mathbb{R}^+_0)$. Let $\mathrm{sn}_k(t)$ be the solution of

$$\begin{cases} \mathrm{sn}_k'' - k\,\mathrm{sn}_k = 0 \quad \text{on } \mathbb{R}^+, \\ \mathrm{sn}_k(0) = 0, \quad \mathrm{sn}_k'(0) = 1, \end{cases} \tag{2.2}$$

that is

$$\mathrm{sn}_k(t) = \begin{cases} t & \text{if } k = 0, \\ \sinh(\sqrt{k}t)/\sqrt{k} & \text{if } k > 0. \end{cases} \tag{2.3}$$

Observe that $\mathbb{R}^n$ and $\mathbb{H}^n_k$ can be written as the differentiable manifold $\mathbb{R}^n$ equipped with the metric given, in polar geodesic coordinates $(\rho, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$ centered at some origin, by

$$\mathrm{d}s_k^2 = \mathrm{d}\rho^2 + \mathrm{sn}_k^2(\rho)\,\mathrm{d}\theta^2,$$

$\mathrm{d}\theta^2$ being the metric on the unit sphere $\mathbb{S}^{n-1}$.

We also consider the model M^n_g associated with the lower bound $-G$ for $\bar{K}_{\mathrm{rad}}$, that is, we let $g \in C^2(\mathbb{R}^+_0)$ be the solution of

$$\begin{cases} g'' - Gg = 0 \quad \text{on } \mathbb{R}^+, \\ g(0) = 0, \quad g'(0) = 1, \end{cases} \tag{2.4}$$

and we define M^n_g as being $(\mathbb{R}^n, \mathrm{d}s_g^2)$ with the C^2-metric $\mathrm{d}s_g^2 = \mathrm{d}\rho^2 + g^2(\rho)\mathrm{d}\theta^2$ in polar coordinates. Condition (1.9) and the Hessian comparison theorem (Theorem 2.3 in [45], or Theorem 1.15 in [8]) imply

$$\frac{\mathrm{sn}_k'(\bar{\rho})}{\mathrm{sn}_k(\bar{\rho})}\Big((\, ,) - \mathrm{d}\bar{\rho} \otimes \mathrm{d}\bar{\rho}\Big) \leq \overline{\mathrm{Hess}}\,(\bar{\rho}) \leq \frac{g'(\bar{\rho})}{g(\bar{\rho})}\Big((\, ,) - \mathrm{d}\bar{\rho} \otimes \mathrm{d}\bar{\rho}\Big). \tag{2.5}$$

The next proposition investigates the ODE properties that follow from the assumptions of pointwise or integral pinching.

Proposition 1. *Let N^n satisfy* (1.9), *and let* sn_k, g *be solutions of* (2.3), (2.4). *Define*

$$\zeta(s) \doteq \frac{g'(s)}{g(s)} - \frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}. \tag{2.6}$$

Then, $\zeta(0^+) = 0$, $\zeta \geq 0$ *on* $\mathbb{R}^+$. *Moreover,*

(*i*) *If N has a pointwise pinching to $\mathbb{H}^n_k$ or $\mathbb{R}^n$, then $\zeta(s) \to 0$ as $s \to +\infty$.*
(*ii*) *If N has an integral pinching to $\mathbb{H}^n_k$ or $\mathbb{R}^n$, then $g/\mathrm{sn}_k \to C$ as $s \to +\infty$ for some $C \in \mathbb{R}^+$, and*

$$\zeta(s) \in L^1(\mathbb{R}^+), \qquad \zeta(s)\frac{\mathrm{sn}_k(s)}{\mathrm{sn}_k'(s)} \to 0 \quad \text{as} \ \ s \to +\infty. \tag{2.7}$$

Proof. The non-negativity of ζ, which in particular implies that g/sn_k is non-decreasing, follows from $G \geq k$ via Sturm comparison, and $\zeta(0^+) = 0$ depends on the asymptotic relations $\mathrm{sn}_k'/\mathrm{sn}_k = s^{-1} + o(1)$ and $g'/g = s^{-1} + o(1)$ as $s \to 0^+$, which directly follow from the ODEs satisfied by sn_k and g. To show (i), differentiating ζ we get

$$\zeta'(s) = R(s) - \zeta(s)B(s), \tag{2.8}$$

where $R(s) \doteq G(s) - k$ and $B(s) \doteq \dfrac{g'(s)}{g(s)} + \dfrac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}$. Thus, integrating on $[1, s]$, we can rewrite ζ as follows:

$$\zeta(s) = \zeta(1)e^{-\int_1^s B} + e^{-\int_1^s B}\int_1^s R(\sigma)e^{\int_1^\sigma B}\mathrm{d}\sigma \tag{2.9}$$

Using that $B \notin L^1([1, +\infty))$, and applying de l'Hopital's theorem, we infer

$$\lim_{s\to+\infty}\zeta(s) = \lim_{s\to+\infty}\frac{R(s)}{B(s)} \leq \lim_{s\to+\infty}\frac{\mathrm{sn}_k(s)[G(s)-k]}{\mathrm{sn}_k'(s)}.$$

In our pointwise pinching assumptions on $G(s)$, for both $k = 0$ and $k > 0$ the last limit is zero, hence $\zeta(s) \to 0$ as s diverges. To show (ii), suppose that N has an integral pinching to $\mathbb{H}^n_k$ or to $\mathbb{R}^n$. We first observe that the boundedness of g/sn_k on $\mathbb{R}^+$ equivalent to the property $\zeta \in L^1(+\infty)$, as it follows from

$$\log\frac{g(s)}{\mathrm{sn}_k(s)} = \int_0^s \frac{\mathrm{d}}{\mathrm{d}\sigma}\log\left(\frac{g(\sigma)}{\mathrm{sn}_k(\sigma)}\right)\mathrm{d}s = \int_0^s \zeta \tag{2.10}$$

(we used that $(g/\mathrm{sn}_k)(0^+) = 1$). The boundedness of g/sn_k is the content of Corollary 4 and Remark 16 in [7], but we prefer here to present a direct proof. Integrating (2.9) on $[1, s]$ and using Fubini's theorem, the monotonicity of g/sn_k and the expression of B we obtain

$$\begin{aligned}
\int_1^s \zeta &= \zeta(1)\int_1^s \frac{g(1)\mathrm{sn}_k(1)}{g(\sigma)\mathrm{sn}_k(\sigma)}\mathrm{d}\sigma + \int_1^s e^{-\int_1^\sigma B}\int_1^\sigma R(\tau)e^{\int_1^\tau B}\mathrm{d}\tau\,\mathrm{d}\sigma \\
&\leq \zeta(1)\mathrm{sn}_k(1)^2\int_1^s \frac{\mathrm{d}\sigma}{\mathrm{sn}_k^2(\sigma)} + \int_1^s\left[\int_\tau^s e^{-\int_1^\sigma B}R(\tau)e^{\int_1^\tau B}\mathrm{d}\sigma\right]\mathrm{d}\tau \\
&\leq C + \int_1^s R(\tau)g(\tau)\mathrm{sn}_k(\tau)\left[\int_\tau^s \frac{\mathrm{d}\sigma}{g(\sigma)\mathrm{sn}_k(\sigma)}\right]\mathrm{d}\tau \\
&\leq C + \int_1^s R(\tau)g(\tau)\mathrm{sn}_k(\tau)\left[\int_\tau^{+\infty} \frac{\mathrm{d}\sigma}{g(\sigma)\mathrm{sn}_k(\sigma)}\right]\mathrm{d}\tau
\end{aligned} \tag{2.11}$$

for some $C > 0$, where we have used that $\mathrm{sn}_k^{-2}, g^{-1}\mathrm{sn}_k^{-1} \in L^1(+\infty)$. Next, since $g\,\mathrm{sn}_k/\mathrm{sn}_k^2$ is non-decreasing, Proposition 3.12 in [8] ensures the validity of the following inequality:

$$g(\tau)\mathrm{sn}_k(\tau)\left[\int_\tau^{+\infty}\frac{\mathrm{d}\sigma}{g(\sigma)\mathrm{sn}_k(\sigma)}\right] \le \mathrm{sn}_k^2(\tau)\left[\int_\tau^{+\infty}\frac{\mathrm{d}\sigma}{\mathrm{sn}_k^2(\sigma)}\right].$$

It is easy to show that the last expression is bounded if $k > 0$, and diverges at the order of τ if $k = 0$. In other words, it can be bounded by $C_1\mathrm{sn}_k/\mathrm{sn}_k'$ on $[1, +\infty)$, for some large $C_1 > 0$. Therefore, by (2.11)

$$\int_1^s \zeta \le C + C_1\int_1^s R(\tau)\frac{\mathrm{sn}_k(\tau)}{\mathrm{sn}_k'(\tau)}\mathrm{d}\tau = C + C_1\int_1^s \big[G(\tau) - k\big]\frac{\mathrm{sn}_k(\tau)}{\mathrm{sn}_k'(\tau)}\mathrm{d}\tau.$$

In our integral pinching assumptions, both for $k = 0$ and for $k > 0$ it holds $(G - k)\mathrm{sn}_k/\mathrm{sn}_k' \in L^1(+\infty)$, and thus $\zeta \in L^1(+\infty)$. Next, we use (2.8) and the non-negativity of ζ, B to obtain

$$\begin{aligned}\left(\frac{\zeta(s)\mathrm{sn}_k(s)}{\mathrm{sn}_k'(s)}\right)' &= \big[G(s) - k - \zeta(s)B(s)\big]\frac{\mathrm{sn}_k(s)}{\mathrm{sn}_k'(s)} + \zeta(s)\left[1 - k\left(\frac{\mathrm{sn}_k(s)}{\mathrm{sn}_k'(s)}\right)^2\right] \\ &\le \frac{\big[G(s) - k\big]\mathrm{sn}_k(s)}{\mathrm{sn}_k'(s)} + \zeta(s) \in L^1(+\infty),\end{aligned}$$

hence $\zeta\mathrm{sn}_k/\mathrm{sn}_k' \in L^\infty(\mathbb{R}^+)$ by integrating. This implies that the function B in (2.8) satisfies $B \le C\mathrm{sn}_k'/\mathrm{sn}_k$ for some constant $C > 0$. Therefore, from (2.8) we get $\zeta' \ge -\zeta B \ge -C\zeta\mathrm{sn}_k'/\mathrm{sn}_k$. Integrating on $[s, t]$ and using the monotonicity of $\mathrm{sn}_k'/\mathrm{sn}_k$ we obtain

$$-C\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\int_s^t \zeta \le \zeta(t) - \zeta(s).$$

Since $\zeta \in L^1(\mathbb{R}^+)$, we can choose a divergent sequence $\{t_j\}$ such that $\zeta(t_j) \to 0$ as $j \to +\infty$. Setting $t = t_j$ into the above inequality and taking limits we deduce

$$\zeta(s) \le C\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\int_s^{+\infty}\zeta,$$

thus letting $s \to +\infty$ we get the second relation in (2.7). □

2.2. **A transversality lemma.** This subsection is devoted to an estimate of the measure of the critical set

$$S_{t,s} = \Big\{x \in M \;:\; t \le r(x) \le s,\ |\nabla r(x)| = 0\Big\},$$

with the purpose of justifying some coarea's formulas for integrals over extrinsic annuli. We begin with the next

Lemma 2. *Let $\varphi : M^m \to N^n$ be an isometric immersion, and let $r(x) = \overline{\mathrm{dist}}(\varphi(x), \bar{o})$ be the extrinsic distance function from $\bar{o} \in N$. Denote with $\Gamma_\sigma \doteq \{x \in M;\ r(x) = \sigma\}$. Then, for each $f \in L^1(\{t \le r \le s\})$,*

$$\int_{\{t\le r\le s\}} f\,\mathrm{d}x = \int_{S_{t,s}} f\,\mathrm{d}x + \int_t^s\left[\int_{\Gamma_\sigma}\frac{f}{|\nabla r|}\right]\mathrm{d}\sigma. \tag{2.12}$$

In particular, if

$$\mathrm{vol}(S_{t,s}) = 0, \tag{2.13}$$

then

$$\int_{\{t\le r\le s\}} f\,\mathrm{d}x = \int_t^s \left[\int_{\Gamma_\sigma} \frac{f}{|\nabla r|}\right]\mathrm{d}\sigma. \tag{2.14}$$

Proof. We prove (2.12) for $f \ge 0$, and the general case follows by considering the positive and negative part of f. By the coarea's formula, we know that for each $g \in L^1(\{t \le r \le s\})$,

$$\int_{\{t\le r\le s\}} g|\nabla r|\,\mathrm{d}x = \int_t^s \left[\int_{\Gamma_\sigma} g\right]\mathrm{d}\sigma. \tag{2.15}$$

Fix j and consider $A_j = \{|\nabla r| > 1/j\}$ and the function

$$g = f1_{A_j}/|\nabla r| \in L^1(\{t \le r \le s\}).$$

Applying (2.15), letting $j \to +\infty$ and using the monotone convergence theorem we deduce

$$\int_{\{t\le r\le s\}\setminus S_{t,s}} f\,\mathrm{d}x = \int_t^s \left[\int_{\Gamma_\sigma\setminus S_{t,s}} \frac{f}{|\nabla r|}\right]\mathrm{d}\sigma = \int_t^s \left[\int_{\Gamma_\sigma} \frac{f}{|\nabla r|}\right]\mathrm{d}\sigma, \tag{2.16}$$

where the last equality follows since $\Gamma_\sigma \cap S_{t,s} = \emptyset$ for a.e. $\sigma \in [t,s]$, in view of Sard's theorem. Formula (2.12) follows at once.

□

Let now N possess a pole $\bar{o}$ and satisfy (1.9), and consider a minimal immersion $\varphi : M \to N$. Since, by the Hessian comparison theorem, geodesic spheres in N centered at $\bar{o}$ are positively curved, it is reasonable to expect that the "transversality" condition (2.13) holds. This is the content of the next

Proposition 2. *Let $\varphi : M^m \to N^n$ be a minimal immersion, where N possesses a pole $\bar{o}$ and satisfies* (1.9)*. Then,*

$$\mathrm{vol}(S_{0,+\infty}) = 0. \tag{2.17}$$

Proof. Suppose by contradiction that $\mathrm{vol}(S_{0,+\infty}) > 0$. By Stampacchia and Rademacher's theorems,

$$\nabla|\nabla r|(x) = 0 \qquad \text{for a.e. } x \in S_{0,+\infty}. \tag{2.18}$$

Pick one such x and a local Darboux frame $\{e_i\}, \{e_\alpha\}$, $1 \le i \le m$, $m+1 \le \alpha \le n$ around x, that is, $\{e_i\}$ is a local orthonormal frame for TM and $\{e_\alpha\}$ is a local orthonormal frame for the normal bundle $TM^\perp$. Since $\nabla r(x) = 0$, then $\bar{\nabla}\bar{\rho}(x) \in T_xM^\perp$. Up to rotating $\{e_\alpha\}$, we can suppose that $\bar{\nabla}\bar{\rho}(x) = e_n(x)$. Fix i and consider a unit speed geodesics $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = x$, $\dot{\gamma}(0) = e_i$. Identify γ with its image $\varphi \circ \gamma$ in N. By Taylor's formula and (2.18),

$$|\nabla r|(\gamma(t)) = o(t) \qquad \text{as } t \to 0^+.$$

Using that $|\nabla r| = \sqrt{1 - \sum_\alpha (\bar{\nabla}\bar{\rho}, e_\alpha)^2}$, we deduce

$$1 - \sum_\alpha (\bar{\nabla}\bar{\rho}, e_\alpha)^2_{\gamma(t)} = o(t^2). \tag{2.19}$$

Since $\bar{\nabla}\bar{\rho}(x) = e_n(x)$, we deduce from (2.20) that also

$$u(t) \doteq 1 - (\bar{\nabla}\bar{\rho}, e_n)^2_{\gamma(t)} = o(t^2), \tag{2.20}$$

thus $\dot u(0) = \ddot u(0) = 0$. Computing,

$$\begin{aligned}
\dot u(t) &= 2(\bar\nabla\bar\rho, e_n)\left[(\bar\nabla_{\dot\gamma}\bar\nabla\bar\rho, e_n) + (\bar\nabla\bar\rho, \bar\nabla_{\dot\gamma}e_n)\right] \\
\ddot u(t) &= 2\left[(\bar\nabla_{\dot\gamma}\bar\nabla\bar\rho, e_n) + (\bar\nabla\bar\rho, \bar\nabla_{\dot\gamma}e_n)\right]^2 \\
&\quad +2(\bar\nabla\bar\rho, e_n)\left[(\bar\nabla_{\dot\gamma}\bar\nabla_{\dot\gamma}\bar\nabla\bar\rho, e_n) + 2(\bar\nabla_{\dot\gamma}\bar\nabla\bar\rho, \bar\nabla_{\dot\gamma}e_n) + (\bar\nabla\bar\rho, \bar\nabla_{\dot\gamma}\bar\nabla_{\dot\gamma}e_n)\right].
\end{aligned}$$

Evaluating at $t = 0$ we deduce

$$0 = \ddot u(0)/2 = (\bar\nabla_{e_i}\bar\nabla_{e_i}\bar\nabla\bar\rho, \bar\nabla\bar\rho) + 2(\bar\nabla_{e_i}\bar\nabla\bar\rho, \bar\nabla_{e_i}e_n) + (e_n, \bar\nabla_{e_i}\bar\nabla_{e_i}e_n).$$

Differentiating twice $1 = |e_n|^2 = |\bar\nabla\bar\rho|^2$ along e_i we deduce the identities $(e_n, \bar\nabla_{e_i}\bar\nabla_{e_i}e_n) = -|\bar\nabla_{e_i}e_n|^2$ and $(\bar\nabla_{e_i}\bar\nabla_{e_i}\bar\nabla\bar\rho, \bar\nabla\bar\rho) = -|\bar\nabla_{e_i}\bar\nabla\bar\rho|^2$, hence

$$0 = \ddot u(0)/2 = -|\bar\nabla_{e_i}\bar\nabla\bar\rho|^2 + 2(\bar\nabla_{e_i}\bar\nabla\bar\rho, \bar\nabla_{e_i}e_n) - |\bar\nabla_{e_i}e_n|^2 = -\big|\bar\nabla_{e_i}\bar\nabla\bar\rho - \bar\nabla_{e_i}e_n\big|^2,$$

which implies $\bar\nabla_{e_i}\bar\nabla\bar\rho = \bar\nabla_{e_i}e_n$. Therefore, at x,

$$(\mathrm{II}(e_i,e_i), e_n) = -(\bar\nabla_{e_i}e_n, e_i) = -(\bar\nabla_{e_i}\bar\nabla\bar\rho, e_i) = \overline{\mathrm{Hess}}(\bar\rho)(e_i,e_i).$$

Tracing with respect to i, using that M is minimal and (2.5) we conclude that

$$0 \ge \frac{\mathrm{sn}_k'(r(x))}{\mathrm{sn}_k(r(x))}(m - |\nabla r(x)|^2) = m\frac{\mathrm{sn}_k'(r(x))}{\mathrm{sn}_k(r(x))} > 0,$$

a contradiction. □

3. Monotonicity formulae and conditions equivalent to $\Theta(+\infty) < +\infty$

Our first step is to improve the classical monotonicity formula for $\Theta(r)$, that can be found in [48] (for $N = \mathbb{R}^m$) and [2] (for $N = \mathbb{H}^n_k$). For $k \ge 0$, let v_k, V_k denote the volume function, respectively, of geodesic spheres and balls in the space form of sectional curvature $-k$, i.e.,

$$v_k(s) = \omega_{m-1}\mathrm{sn}_k(s)^{m-1}, \qquad V_k(s) = \int_0^s v_k(\sigma)\mathrm{d}\sigma, \tag{3.1}$$

where ω_{m-1} is the volume of the unit sphere $\mathbb{S}^{m-1}$. Although we shall not use all the four monotone quantities in (3.3) below, nevertheless they have independent interest, and for this reason we state the result in its full strength. We define the *flux* $J(s)$ of ∇r over the extrinsic sphere Γ_s:

$$J(s) \doteq \frac{1}{v_k(s)}\int_{\Gamma_s}|\nabla r|. \tag{3.2}$$

Proposition 3 (The monotonicity formulae)**.** *Suppose that N has a pole $\bar o$ and satisfies* (1.9)*, and let $\varphi : M^m \to N^n$ be a proper minimal immersion. Then, the functions*

$$\Theta(s), \qquad \frac{1}{V_k(s)}\int_{\{0\le r\le s\}}|\nabla r|^2 \tag{3.3}$$

are absolutely continuous and monotone non-decreasing. Moreover, $J(s)$ coincides, on an open set of full measure, with the absolutely continuous function

$$\bar J(s) \doteq \frac{1}{v_k(s)}\int_{\{r\le s\}}\Delta r$$

and $\bar J(s)$, $V_k(s)\big[\bar J(s) - \Theta(s)\big]$ are non-decreasing. In particular, $J(s) \ge \Theta(s)$ a.e. on $\mathbb{R}^+$.

Remark 4. To the best of our knowledge, the monotonicity of $J(s)$ (aside from its differentiability properties) has first been shown, in the Euclidean setting, in a paper by V. Tkachev [51].

Proof. We first observe that, in view of Lemma 2 and Proposition 2 applied with $f = \Delta r$,

$$v_k(s)\bar{J}(s) \doteq \int_{\{r\le s\}} \Delta r \equiv \int_0^s \left[\int_{\Gamma_\sigma} \frac{\Delta r}{|\nabla r|}\right] \mathrm{d}\sigma \tag{3.4}$$

is absolutely continuous, and by the divergence theorem it coincides with $v_k(s)J(s)$ for regular values of s. Consider

$$f(s) = \int_0^s \frac{V_k(\sigma)}{v_k(\sigma)}\mathrm{d}\sigma = \int_0^s \frac{1}{v_k(\sigma)}\left[\int_0^\sigma v_k(\tau)\mathrm{d}\tau\right]\mathrm{d}\sigma \tag{3.5}$$

which is a C^2 solution of

$$f'' + (m-1)\frac{\mathrm{sn}_k'}{\mathrm{sn}_k}f' = 1 \quad \text{on } \mathbb{R}^+, \quad f(0)=0, \quad f'(0)=0,$$

and define $\psi(x) = f(r(x)) \in C^2(M)$. Let $\{e_i\}$ be a local orthonormal frame on M. Since φ is minimal, by the chain rule and the lower bound in the Hessian comparison theorem 2.5

$$\Delta r = \sum_{j=1}^m \overline{\mathrm{Hess}}\,(\bar\rho)\big(\mathrm{d}\varphi(e_j), \mathrm{d}\varphi(e_j)\big) \ge \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\big(m - |\nabla r|^2\big). \tag{3.6}$$

We then compute

$$\begin{aligned} \Delta\psi &= f''|\nabla r|^2 + f'\Delta r \ge f''|\nabla r|^2 + f'\frac{\mathrm{sn}_k'}{\mathrm{sn}_k}(m - |\nabla r|^2) \\ &= 1 + \big(1 - |\nabla r|^2\big)\left(f'(r)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} - f''(r)\right). \end{aligned} \tag{3.7}$$

It is not hard to show that the function

$$z(s) \doteq f'(s)\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} - f''(s) = \frac{m}{m-1}\frac{V_k(s)v_k'(s)}{v_k^2(s)} - 1.$$

is non-negative and non-decreasing on $\mathbb{R}^+$. Indeed, from

$$z(0) = 0, \qquad z'(s) = \frac{m}{v_k(s)}\left[kV_k(s) - \frac{1}{m-1}v_k'(s)z(s)\right] \tag{3.8}$$

we deduce that $z' > 0$ when $z < 0$, which proves that $z \ge 0$ on $\mathbb{R}^+$. Fix $0 < t < s$ regular values for r. Integrating (3.7) on the smooth compact set $\{t \le r \le s\}$ and using the divergence theorem we deduce

$$\frac{V_k(s)}{v_k(s)}\int_{\Gamma_s}|\nabla r| - \frac{V_k(t)}{v_k(t)}\int_{\Gamma_t}|\nabla r| \ge \mathrm{vol}\big(\{t \le r \le s\}\big). \tag{3.9}$$

By the definition of $J(s)$ and $\Theta(s)$, and since $J(s) \equiv \bar{J}(s)$ for regular values, the above inequality rewrites as follows:

$$V_k(s)\bar{J}(s) - V_k(t)\bar{J}(t) \ge V_k(s)\Theta(s) - V_k(t)\Theta(t),$$

or in other words,

$$V_k(s)\big[\bar{J}(s) - \Theta(s)\big] \ge V_k(t)\big[\bar{J}(t) - \Theta(t)\big].$$

Since all the quantities involved are continuous, the above relation extends to all $t, s \in \mathbb{R}^+$, which proves the monotonicity of $V_k[\bar{J} - \Theta]$. Letting $t \to 0$ we then deduce that $\bar{J}(s) \geq \Theta(s)$ on $\mathbb{R}^+$. Next, by using $f \equiv 1$ and $f \equiv |\nabla r|^2$ in Lemma 2 and exploiting again Proposition 2 we get

$$\mathrm{vol}\big(\{t \leq r \leq s\}\big) = \int_t^s \left[\int_{\Gamma_\sigma} \frac{1}{|\nabla r|}\right] \mathrm{d}\sigma, \qquad \int_{\{0 \leq r \leq s\}} |\nabla r|^2 = \int_0^s \left[\int_{\Gamma_\sigma} |\nabla r|\right] \mathrm{d}\sigma, \tag{3.10}$$

showing that the two quantities in (3.3) are absolutely continuous. Plugging into (3.9), letting $t \to 0$ and using that $z \geq 0$ we deduce

$$\frac{V_k(s)}{v_k(s)} \int_{\Gamma_s} |\nabla r| \geq \int_0^s \left[\int_{\Gamma_\sigma} \frac{1}{|\nabla r|}\right] \mathrm{d}\sigma, \tag{3.11}$$

for regular s, which together with the trivial inequality $|\nabla r|^{-1} \geq |\nabla r|$ and with (3.10) gives

$$\begin{aligned} V_k(s) \int_{\Gamma_s} |\nabla r| &\geq v_k(s) \int_0^s \left[\int_{\Gamma_\sigma} |\nabla r|\right] \mathrm{d}\sigma, \\ V_k(s) \left[\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{vol}\big(\{r \leq s\}\big)\right] &\geq v_k(s) \mathrm{vol}\big(\{r \leq s\}\big). \end{aligned} \tag{3.12}$$

Integrating the second inequality we obtain the monotonicity of $\Theta(s)$, while integrating the first one and using (3.10) we obtain the monotonicity of the second quantity in (3.3). To show the monotonicity of $\bar{J}(s)$, by (3.6) and using the full information coming from (2.5) we obtain

$$\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} \big(m - |\nabla r|^2\big) \leq \Delta r \leq \frac{g'(r)}{g(r)} \big(m - |\nabla r|^2\big). \tag{3.13}$$

In view of the identity (3.4), we consider regular $s > 0$, we divide (3.13) by $|\nabla r|$ and integrate on Γ_s to get

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} \int_{\Gamma_s} \frac{m - |\nabla r|^2}{|\nabla r|} \leq \big(v_k(s)\bar{J}(s)\big)' \leq \frac{g'(s)}{g(s)} \int_{\Gamma_s} \frac{m - |\nabla r|^2}{|\nabla r|}. \tag{3.14}$$

Writing $m - |\nabla r|^2 = m(1 - |\nabla r|^2) + (m-1)|\nabla r|^2$, setting for convenience

$$v_g(s) = \omega_{m-1} g(s)^{m-1}, \qquad T(s) \doteq \frac{\int_{\Gamma_s} |\nabla r|^{-1}}{\int_{\Gamma_s} |\nabla r|} - 1, \tag{3.15}$$

rearranging we deduce the two inequalities

$$\begin{aligned} \big(v_k(s)\bar{J}(s)\big)' &\geq v_k'(s)\bar{J}(s) + m\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} T(s) v_k(s) \bar{J}(s) \\ \big(v_k(s)\bar{J}(s)\big)' &\leq \frac{v_g'(s)}{v_g(s)} v_k(s)\bar{J}(s) + m\frac{g'(s)}{g(s)} T(s) v_k(s) \bar{J}(s). \end{aligned} \tag{3.16}$$

Expanding the derivative on the left-hand side, we deduce

$$\begin{aligned} \bar{J}'(s) &\geq m\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} T(s) \bar{J}(s), \\ \left(\frac{v_k(s)}{v_g(s)} \bar{J}(s)\right)' &\leq m\frac{g'(s)}{g(s)} T(s) \left(\frac{v_k(s)}{v_g(s)} \bar{J}(s)\right). \end{aligned} \tag{3.17}$$

The first inequality together with the non-negativity of T implies the desired $\bar{J}' \geq 0$, concluding the proof. The second inequality in (3.17), on the other hand, will be useful in awhile. ☐

Remark 5. The properness of φ is essential in the above proof to justify integrations by parts. However, if φ is non-proper, at least when N is Cartan-Hadamard with sectional curvature $\bar{K} \leq -k$ the function Θ is still monotone in an extended sense. In fact, as it has been observed in [54] for $N = \mathbb{R}^{m+1}$, $\Theta(s) = +\infty$ for each s such that $\{r < s\}$ contains a limit point of φ. Briefly, if $\bar{x} \in N$ is a limit point with $\bar{\rho}(\bar{x}) < s$, choose $\varepsilon > 0$ such that $2\varepsilon < s - \bar{\rho}(\bar{x})$, and a diverging sequence $\{x_j\} \subset M$ such that $\varphi(x_j) \to \bar{x}$. We can assume that the balls $B_\varepsilon(x_j) \subset M$ are pairwise disjoint. Since $\mathrm{dist}(\varphi(x), \varphi(x_j)) \leq \mathrm{dist}(x, x_j)$, we deduce that $\varphi(B_\varepsilon(x_j)) \subset \{r < s\}$ for j large enough, and thus

$$\mathrm{vol}\big(\{r \leq s\}\big) \geq \sum_j \mathrm{vol}(B_\varepsilon(x_j)).$$

However, using that $\bar{K} \leq -k$ and since N is Cartan-Hadamard, we can apply the intrinsic monotonicity formula (see Proposition 7 in Appendix 2 below) with chosen origin $\varphi(x_j)$ to deduce that $\mathrm{vol}(B_\varepsilon(x_j)) \geq V_k(\varepsilon)$ for each j, whence $\mathrm{vol}(\{r \leq s\}) = +\infty$.

We next investigate conditions equivalent to the finiteness of the density.

Proposition 4. *Suppose that N has a pole and satisfies* (1.9). *Let $\varphi : M^m \to N^n$ be a proper minimal immersion. Then, the following properties are equivalent:*

(1) $\Theta(+\infty) < +\infty$;
(2) $\bar{J}(+\infty) < +\infty$.

Moreover, both (1) *and* (2) *imply that*

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\left[\frac{\int_{\Gamma_s} |\nabla r|^{-1}}{\int_{\Gamma_s} |\nabla r|} - 1\right] \in L^1(\mathbb{R}^+). \tag{3}$$

If further N has an integral pinching to $\mathbb{R}^n$ or $\mathbb{H}^n_k$, then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

Proof. We refer to the proof of the previous proposition for notation and formulas.
(2) $\Rightarrow$ (1) is obvious since, by the previous proposition, $\bar{J}(s) \geq \Theta(s)$.
(1) $\Rightarrow$ (2). Note that the limit in (2) exists since $\bar{J}$ is monotone. Suppose by contradiction that $\bar{J}(+\infty) = +\infty$, let $c > 0$ and fix s_c large enough that $\bar{J}(s) \geq c$ for $s \geq s_c$. From (3.10) and (3.2), and since $\bar{J} \equiv J$ a.e.,

$$\begin{aligned}\Theta(s) &= \frac{1}{V_k(s)}\int_0^s \left[\int_{\Gamma_\sigma} \frac{1}{|\nabla r|}\right] \mathrm{d}\sigma \geq \frac{1}{V_k(s)}\int_0^s v_k(\sigma)J(\sigma)\mathrm{d}\sigma \\ &\geq \frac{1}{V_k(s)}\int_{s_c}^s v_k(\sigma)J(\sigma)\mathrm{d}\sigma \geq c\frac{V_k(s)-V_k(s_c)}{V_k(s)}.\end{aligned}$$

Letting $s \to +\infty$ we get $\Theta(+\infty) \geq c$, hence $\Theta(+\infty) = +\infty$ by the arbitrariness of c, contradicting (1).
(2) $\Rightarrow$ (3). Integrating (3.17) on $[1, s]$ we obtain

$$c_1 \exp\left\{m\int_1^s \frac{\mathrm{sn}_k'(\sigma)}{\mathrm{sn}_k(\sigma)}T(\sigma)\mathrm{d}\sigma\right\} \leq \bar{J}(s) \leq c_2\frac{v_g(s)}{v_k(s)}\exp\left\{m\int_1^s \left[\frac{g'(\sigma)}{g(\sigma)}\right]T(\sigma)\mathrm{d}\sigma\right\}, \tag{3.18}$$

for some constants $c_1, c_2 > 0$, where $v_g(s), T(s)$ is as in (3.15). The validity of (2) and the first inequality show that $\mathrm{sn}_k' T/\mathrm{sn}_k \in L^1(+\infty)$, that is, (3) is satisfied.
(3) $\Rightarrow$ (2). In our pinching assumptions on N, (ii) in Proposition 1 gives

$$\frac{g'}{g} = \frac{\mathrm{sn}_k'}{\mathrm{sn}_k} + \zeta, \quad \text{with} \quad \zeta \le C\frac{\mathrm{sn}_k'}{\mathrm{sn}_k} \ \text{ on } \mathbb{R}^+, \quad \text{and} \quad g \le C\mathrm{sn}_k \ \text{ on } \mathbb{R}^+,$$

for some $C > 0$. Plugging into (3.18) and recalling the definition of v_g we obtain

$$\bar{J}(s) \le c_3 \exp\left\{ c_4 \int_1^s \left[\frac{\mathrm{sn}_k'(\sigma)}{\mathrm{sn}_k(\sigma)}\right] T(\sigma)\mathrm{d}\sigma \right\},$$

for some $c_3, c_4 > 0$, and (3) $\Rightarrow$ (2) follows by letting $s \to +\infty$. □

Remark 6. It is worth to observe that a version of Propositions 3 and 4 that covers most of the material presented above has also been independently proved in the very recent [27], see Theorems 2.1 and 6.1 therein. We mention that their results are stated for more general ambient spaces subjected to specific function-theoretic requirements, and that, in Proposition 4, it holds in fact $\bar{J}(+\infty) \equiv \Theta(+\infty)$. For an interesting characterization, when $N = \mathbb{R}^n$, of the limit $\bar{J}(+\infty)$ in terms of an invariant called the projective volume of M we refer to [51].

4. Proof of Theorem 1

Let M^m be a minimal properly immersed submanifold in N^n, and suppose that N has a pointwise or integral pinching towards a space form. Because of the upper bound in (1.9), by [18] and [5] the bottom of $\sigma(M)$ satisfies

$$\inf \sigma(M) \ge \frac{(m-1)^2 k}{4}. \tag{4.1}$$

Briefly, the lower bound in (3.13) implies

$$\Delta r \ge (m-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)} \ge (m-1)\sqrt{k} \qquad \text{on } M.$$

Integrating on a relatively compact, smooth open set Ω and using the divergence theorem and $|\nabla r| \le 1$, we deduce $\mathcal{H}^{m-1}(\partial\Omega) \ge (m-1)\sqrt{k}\mathrm{vol}(\Omega)$. The desired (4.1) then follows from Cheeger's inequality:

$$\inf \sigma(M) \ge \frac{1}{4}\left(\inf_{\Omega \Subset M} \frac{\mathcal{H}^{m-1}(\partial\Omega)}{\mathrm{vol}(\Omega)}\right)^2 \ge \frac{(m-1)^2 k}{4}.$$

To complete the proof of the theorem, since $\sigma(M)$ is closed it is sufficient to show that each $\lambda > (m-1)^2 k/4$ lies in $\sigma(M)$.

Set for convenience $\beta \doteq \sqrt{\lambda - (m-1)^2 k/4}$ and, for $0 \le t < s$, let $A_{t,s}$ denote the extrinsic annulus

$$A_{t,s} \doteq \big\{ x \in M \ : \ r(x) \in [t,s] \big\}.$$

Define the weighted measure $\mathrm{d}\mu_k \doteq v_k(r)^{-1}\mathrm{d}x$ on $\{r \ge 1\}$. Hereafter, we will always restrict to this set. Consider

$$\psi(s) \doteq \frac{e^{i\beta s}}{\sqrt{v_k(s)}}, \qquad \text{which solves} \qquad \psi'' + \psi'\frac{v_k'}{v_k} + \lambda\psi = a(s)\psi, \tag{4.2}$$

where

$$a(s) \doteq \frac{(m-1)^2 k}{4} + \frac{1}{4}\left(\frac{v_k'(s)}{v_k(s)}\right)^2 - \frac{1}{2}\frac{v_k''(s)}{v_k(s)} \to 0 \tag{4.3}$$

as $s \to +\infty$. For technical reasons, fix $R > 1$ large such that $\Theta(R) > 0$. Fix t, s, S such that

$$R + 1 < t < s < S - 1,$$

and let $\eta \in C^\infty_c(\mathbb{R})$ be a cut-off function satisfying

$$\begin{array}{lll} 0 \le \eta \le 1, & \eta \equiv 0 \ \text{ outside of } \ (t-1, S), & \eta \equiv 1 \ \text{ on } \ (t,s), \\ |\eta'| + |\eta''| \le C_0 \quad \text{on } \ [t-1, s], & & |\eta'| + |\eta''| \le \frac{C_0}{S-s} \quad \text{on } \ [s, S] \end{array}$$

for some absolute constant C_0 (the last relation is possible since $S - s \ge 1$). The value S will be chosen later in dependence of s. Set $u_{t,s} \doteq \eta(r)\psi(r) \in C^\infty_c(M)$. Then, by (4.2),

$$\begin{aligned} \Delta u_{t,s} + \lambda u_{t,s} &= (\eta''\psi + 2\eta'\psi' + \eta\psi'')|\nabla r|^2 + (\eta'\psi + \eta\psi')\Delta r + \lambda\eta\psi \\ &= \left(\eta''\psi + 2\eta'\psi' - \frac{v_k'}{v_k}\eta\psi' - \lambda\eta\psi + a\eta\psi\right)(|\nabla r|^2 - 1) + a\eta\psi \\ &\quad + (\eta'\psi + \eta\psi')\left(\Delta r - \frac{v_k'}{v_k}\right) + \left(\eta''\psi + 2\eta'\psi' + \eta'\psi\frac{v_k'}{v_k}\right). \end{aligned}$$

Using that there exists an absolute constant c for which $|\psi| + |\psi'| \le c/\sqrt{v_k}$, the following inequality holds:

$$\begin{aligned} \|\Delta u_{t,s} + \lambda u_{t,s}\|_2^2 &\le C\left(\int_{A_{t-1,S}} \left[(1 - |\nabla r|^2)^2 + \left(\Delta r - \frac{v_k'}{v_k}\right)^2 + a(r)^2\right] \mathrm{d}\mu_k \right. \\ &\quad \left. + \frac{\mu_k(A_{s,S})}{(S-s)^2} + \mu_k(A_{t-1,t})\right), \end{aligned}$$

for some suitable C depending on c, C_0. Since $\|u_{t,s}\|_2^2 \ge \mu_k(A_{t,s})$ and $(1 - |\nabla r|^2)^2 \le 1 - |\nabla r|^2$, we obtain

(4.4)

$$\begin{aligned} \frac{\|\Delta u_{t,s} + \lambda u_{t,s}\|_2^2}{\|u_{t,s}\|_2^2} &\le C\left(\frac{1}{\mu_k(A_{t,s})}\int_{A_{t-1,S}} \left[1 - |\nabla r|^2 + \left(\Delta r - \frac{v_k'}{v_k}\right)^2 + a(r)^2\right] \mathrm{d}\mu_k \right. \\ &\quad \left. + \frac{1}{(S-s)^2}\frac{\mu_k(A_{s,S})}{\mu_k(A_{t,s})} + \frac{\mu_k(A_{t-1,t})}{\mu_k(A_{t,s})}\right) \end{aligned}$$

Next, using (2.5),

$$\Delta r = \sum_{j=1}^m \overline{\mathrm{Hess}}(\bar\rho)(e_i, e_i) = \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(m - |\nabla r|^2) + T(x) = \frac{v_k'(r)}{v_k(r)} + \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(1 - |\nabla r|^2) + T(x),$$

where, by Proposition 1,

$$\begin{aligned} 0 \le T(x) &\doteq \sum_{j=1}^m \overline{\mathrm{Hess}}(\bar\rho)(e_i, e_i) - \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(m - |\nabla r|^2) \\ &\le \left(\frac{g'(r)}{g(r)} - \frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}\right)(m - |\nabla r|^2) = \zeta(r)(m - |\nabla r|^2) \le m\zeta(r). \end{aligned} \tag{4.5}$$

We thus obtain, on the set $\{r \ge 1\}$,

$$
\begin{aligned}
\left(\Delta r - \frac{v_k'}{v_k}\right)^2 + 1 - |\nabla r|^2 + a(r)^2 &\le \left[\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(1-|\nabla r|^2) + m\zeta(r)\right]^2 \\
&\quad + 1 - |\nabla r|^2 + a(r)^2 \\
&\le C\Big(\zeta(r)^2 + 1 - |\nabla r|^2 + a(r)^2\Big)
\end{aligned} \tag{4.6}
$$

for some absolute constant C. Note that, in both our pointwise or integral pinching assumptions on N, by Proposition 1 it holds $\zeta(s) \to 0$ as $s \to +\infty$. Set

$$
F(t) \doteq \sup_{\sigma \in [t-1,+\infty)} [a(\sigma)^2 + \zeta(\sigma)^2],
$$

and note that $F(t) \to 0$ monotonically as $t \to +\infty$. Integrating (4.6) we get the existence of $C > 0$ independent of s, t such that

$$
\begin{aligned}
&\int_{A_{t-1,S}} \left[\left(\Delta r - \frac{v_k'}{v_k}\right)^2 + 1 - |\nabla r|^2 + a(r)^2\right] \mathrm{d}\mu_k \\
&\qquad \le C\left(F(t)\int_{A_{t-1,S}} \frac{1}{v_k(r)} + \int_{A_{t-1,S}} \frac{1-|\nabla r|^2}{v_k(r)}\right).
\end{aligned} \tag{4.7}
$$

Using the coarea's formula and the transversality lemma, for each $0 \le a < b$

$$
\mu_k(A_{a,b}) = \int_{A_{a,b}} \frac{1}{v_k(r)} = \int_a^b J\big[1+T\big], \qquad \int_{A_{a,b}} \frac{1-|\nabla r|^2}{v_k(r)} = \int_a^b JT, \tag{4.8}
$$

where J and T are defined, respectively, in (3.2) and (3.15). Summarizing, in view of (4.7) and (4.8) we deduce from (4.4) the following inequalities:

$$
\begin{aligned}
\frac{\|\Delta u_{t,s} + \lambda u_{t,s}\|_2^2}{\|u_{t,s}\|_2^2} &\le C\left(\frac{1}{\int_t^s J\big[1+T\big]}\left[F(t)\int_{t-1}^S J\big[1+T\big] + \int_{t-1}^S JT\right]\right. \\
&\qquad \left. + \frac{\int_s^S J\big[1+T\big]}{(S-s)^2\int_t^s J\big[1+T\big]} + \frac{\int_{t-1}^t J\big[1+T\big]}{\int_t^s J\big[1+T\big]}\right) \doteq \mathcal{Q}(t,s).
\end{aligned} \tag{4.9}
$$

If we can guarantee that

$$
\liminf_{t\to+\infty}\liminf_{s\to+\infty} \frac{\|\Delta u_{t,s} + \lambda u_{t,s}\|_2^2}{\|u_{t,s}\|_2^2} = 0, \tag{4.10}
$$

then we are able to construct a sequence of approximating eigenfunctions for λ as follows: fix $\varepsilon > 0$. By (4.10) there exists a divergent sequence $\{t_i\}$ such that, for $i \ge i_\varepsilon$,

$$
\liminf_{s\to+\infty} \frac{\|\Delta u_{t_i,s} + \lambda u_{t_i,s}\|_2^2}{\|u_{t_i,s}\|_2^2} < \varepsilon/2.
$$

For $i = i_\varepsilon$, pick then a sequence $\{s_j\}$ realizing the liminf. For $j \ge j_\varepsilon(i_\varepsilon, \varepsilon)$

$$
\|\Delta u_{t_i,s_j} + \lambda u_{t_i,s_j}\|_2^2 < \varepsilon\|u_{t_i,s_j}\|_2^2, \tag{4.11}
$$

Writing $u_\varepsilon \doteq u_{t_{i_\varepsilon},s_{j_\varepsilon}}$, by (4.11) from the set $\{u_\varepsilon\}$ we can extract a sequence of approximating eigenfunctions for λ, concluding the proof that $\lambda \in \sigma(M)$. To show (4.10), by (4.9) it is enough to prove that

$$
\liminf_{t\to+\infty}\liminf_{s\to+\infty} \mathcal{Q}(t,s) = 0. \tag{4.12}
$$

Suppose, by contradiction, that (4.12) were not true. Then, there exists a constant $\delta > 0$ such that, for each $t \geq t_\delta$, $\liminf_{s\to+\infty} \mathcal{Q}(t,s) \geq 2\delta$, and thus for $t \geq t_\delta$ and $s \geq s_\delta(t)$

$$F(t)\int_{t-1}^{S} J\big[1+T\big] + \int_{t-1}^{S} JT + \int_{s}^{S} \frac{J\big[1+T\big]}{(S-s)^2} + \int_{t-1}^{t} J\big[1+T\big] \geq \delta\int_{t}^{s} J\big[1+T\big], \tag{4.13}$$

and rearranging

$$(F(t)+1)\int_{t-1}^{S} J\big[1+T\big] - \int_{t-1}^{S} J + \int_{s}^{S} \frac{J\big[1+T\big]}{(S-s)^2} + \int_{t-1}^{t} J\big[1+T\big] \geq \delta\int_{t}^{s} J\big[1+T\big]. \tag{4.14}$$

We rewrite the above integrals in order to make $\Theta(s)$ appear. Integrating by parts and using again the coarea's formula and the transversality lemma,

$$\begin{aligned} \int_a^b J\big[1+T\big] &= \int_{A_{a,b}} \frac{1}{v_k(r)} = \int_a^b \frac{1}{v_k(\sigma)}\left[\int_{\Gamma_\sigma} \frac{1}{|\nabla r|}\right] \mathrm{d}\sigma = \int_a^b \frac{\big(V_k(\sigma)\Theta(\sigma)\big)'}{v_k(\sigma)}\mathrm{d}\sigma \\ &= \frac{V_k(b)}{v_k(b)}\Theta(b) - \frac{V_k(a)}{v_k(a)}\Theta(a) + \int_a^b \frac{V_k v_k'}{v_k^2}\Theta. \end{aligned} \tag{4.15}$$

To deal with the term containing the integral of J alone in (4.14), we use the inequality $J(s) \geq \Theta(s)$ coming from the monotonicity formulae in Proposition 3. This passage is crucial for us to conclude. Inserting (4.15) and $J \geq \Theta$ into (4.14) we get

$$\begin{aligned} &(F(t)+1)\frac{V_k(S)}{v_k(S)}\Theta(S) - (F(t)+1)\frac{V_k(t-1)}{v_k(t-1)}\Theta(t-1) + (F(t)+1)\int_{t-1}^{S} \frac{V_k v_k'}{v_k^2}\Theta \\ &-\int_{t-1}^{S}\Theta + \frac{1}{(S-s)^2}\left[\frac{V_k(S)}{v_k(S)}\Theta(S) - \frac{V_k(s)}{v_k(s)}\Theta(s) + \int_s^S \frac{V_k v_k'}{v_k^2}\Theta\right] + \frac{V_k(t)}{v_k(t)}\Theta(t) \\ &-\frac{V_k(t-1)}{v_k(t-1)}\Theta(t-1) + \int_{t-1}^{t} \frac{V_k v_k'}{v_k^2}\Theta \geq \delta\frac{V_k(s)}{v_k(s)}\Theta(s) - \delta\frac{V_k(t)}{v_k(t)}\Theta(t) + \delta\int_t^s \frac{V_k v_k'}{v_k^2}\Theta. \end{aligned} \tag{4.16}$$

To reach the desired contradiction, the idea is to prove that (1.10) cannot hold by showing that

$$\int_{t-1}^{S} \Theta \tag{4.17}$$

must grow sufficiently fast as $S \to +\infty$. To do so, we need to simplify (4.16) in order to find a suitable differential inequality for (4.17).
We first observe that, both for $k > 0$ and for $k = 0$, there exists an absolute constant $\hat{c}$ such that $\hat{c}^{-1} \leq V_k v_k'/v_k^2 \leq \hat{c}$ on $[1,+\infty)$. Furthermore, by the monotonicity of Θ,

$$\int_s^S \frac{V_k v_k'}{v_k^2}\Theta \leq \hat{c}(S-s)\Theta(S). \tag{4.18}$$

Next, we deal with the two terms in the left-hand side of (4.16) that involve (4.17):

$$\begin{aligned} (F(t)+1)\int_{t-1}^{S} \frac{V_k v_k'}{v_k^2}\Theta - \int_{t-1}^{S}\Theta &= F(t)\int_{t-1}^{S} \frac{V_k v_k'}{v_k^2}\Theta + \int_{t-1}^{S} \frac{V_k v_k' - v_k^2}{v_k^2}\Theta \\ &\leq \hat{c}F(t)\int_{t-1}^{S}\Theta + \int_{t-1}^{S} \frac{V_k v_k' - v_k^2}{v_k^2}\Theta. \end{aligned}$$

The key point is the following relation:

$$\frac{V_k(s)v_k'(s) - v_k(s)^2}{v_k(s)^2} \begin{cases} = -1/m & \text{if } k = 0; \\ \to 0 \text{ as } s \to +\infty, & \text{if } k > 0. \end{cases} \tag{4.19}$$

Define

$$\omega(t) \doteq \sup_{[t-1,+\infty)} \frac{V_k v_k' - v_k^2}{v_k^2}, \qquad \chi(t) \doteq \hat{c}F(t) + \omega(t).$$

Again by the monotonicity of Θ,

$$\begin{aligned} (F(t)+1)\int_{t-1}^S \frac{V_k v_k'}{v_k^2}\Theta - \int_{t-1}^S \Theta \quad &\leq \quad \big[\hat{c}F(t) + \omega(t)\big]\int_{t-1}^S \Theta = \chi(t)\int_{t-1}^S \Theta \\ &\leq \quad \chi(t)\Theta(t) + \chi(t)\int_t^S \Theta. \end{aligned} \tag{4.20}$$

For simplicity, hereafter we collect all the terms independent of s in a function that we call $h(t)$, which may vary from line to line. Inserting (4.18) and (4.20) into (4.16) we infer

$$\begin{aligned} &\left[\left(F(t) + 1 + \frac{1}{(S-s)^2}\right)\frac{V_k(S)}{v_k(S)} + \frac{\hat{c}}{S-s}\right]\Theta(S) + \chi(t)\int_t^S \Theta \\ &\geq h(t) + \left(\delta + \frac{1}{(S-s)^2}\right)\frac{V_k(s)}{v_k(s)}\Theta(s) + \delta\hat{c}^{-1}\int_t^s \Theta. \end{aligned} \tag{4.21}$$

Summing $\delta\hat{c}^{-1}(S-s)\Theta(S)$ to the two sides of the above inequality, using the monotonicity of Θ and getting rid of the term containing $\Theta(s)$ we obtain

$$\begin{aligned} &\left[\left(F(t) + 1 + \frac{1}{(S-s)^2}\right)\frac{V_k(S)}{v_k(S)} + \frac{\hat{c}}{S-s} + \delta\hat{c}^{-1}(S-s)\right]\Theta(S) + \chi(t)\int_t^S \Theta \\ &\geq h(t) + \delta\hat{c}^{-1}\int_t^S \Theta. \end{aligned} \tag{4.22}$$

Using (4.19), the definition of $\chi(t)$ and the properties of $\omega(t), F(t)$, we can choose t_δ sufficiently large to guarantee that

$$\delta\hat{c}^{-1} - \chi(t) \geq c_k \doteq \begin{cases} \frac{1}{m} + \frac{\delta\hat{c}^{-1}}{2} & \text{if } k = 0, \\ \frac{\delta\hat{c}^{-1}}{2} & \text{if } k > 0, \end{cases} \tag{4.23}$$

hence

$$\left[\left(F(t) + 1 + \frac{1}{(S-s)^2}\right)\frac{V_k(S)}{v_k(S)} + \frac{\hat{c}}{S-s} + \delta\hat{c}^{-1}(S-s)\right]\Theta(S) \geq h(t) + c_k\int_t^S \Theta. \tag{4.24}$$

We now specify $S(s)$ depending on whether $k > 0$ or $k = 0$.

The case $k > 0$.
We choose $S \doteq s + 1$. In view of the fact that V_k/v_k is bounded above on $\mathbb{R}^+$, (4.24) becomes

$$\bar{c}\Theta(s+1) \geq h(t) + c_k\int_t^{s+1} \Theta \geq \frac{c_k}{2}\int_t^{s+1} \Theta, \tag{4.25}$$

for some $\bar{c}$ independent of t, s. Note that the last inequality is satisfied provided $s \geq s_\delta(t)$ is chosen to be sufficiently large, since the monotonicity of Θ implies that $\Theta \notin L^1(\mathbb{R}^+)$. Integrating and using again the monotonicity of Θ, we get

$$(s+1-t)\Theta(s+1) \geq \int_t^{s+1} \Theta \geq \left[\int_t^{s_0+1} \Theta\right] \exp\left\{\frac{c_k}{2\bar{c}}(s-s_0)\right\},$$

hence $\Theta(s)$ grows exponentially. Ultimately, this contradicts our assumption (1.10).

The case $k = 0$.
We choose $S \doteq s + \sqrt{s}$. Since $V_k(S)/v_k(S) = S/m$, from (4.24) we infer

$$\left[\left(F(t) + 1 + \frac{1}{s}\right)\frac{S}{m} + \frac{\hat{c}}{\sqrt{s}} + \delta\hat{c}^{-1}\sqrt{s}\right]\Theta(S) \geq h(t) + c_k \int_t^S \Theta. \tag{4.26}$$

Using the expression of c_k and the fact that $F(t) \to 0$, up to choosing t_δ and then $s_\delta(t)$ large enough we can ensure the validity of the following inequality:

$$\left[\left(F(t) + 1 + \frac{1}{s}\right)\frac{S}{m} + \frac{\hat{c}}{\sqrt{s}} + \delta\hat{c}^{-1}\sqrt{s}\right] < \left[\frac{1}{m} + \frac{\delta\hat{c}^{-1}}{4}\right]S = \left[c_k - \frac{\delta\hat{c}^{-1}}{4}\right]S$$

for $t \geq t_\delta$ and $s \geq s_\delta(t)$. Plugging into (4.24), and using that $\Theta \notin L^1(\mathbb{R}^+)$,

$$S\Theta(S) \geq h(t) + \frac{c_k}{c_k - \delta\hat{c}^{-1}/4}\int_t^S \Theta \geq (1+\varepsilon)\int_t^S \Theta,$$

for a suitable $\varepsilon > 0$ independent of t, S, and provided that $S \geq s_\delta(t)$ is large enough. Integrating and using again the monotonicity of Θ,

$$S\Theta(S) \geq (S-t)\Theta(S) \geq \int_t^S \Theta \geq \left[\int_t^{S_0} \Theta\right]\left(\frac{S}{S_0}\right)^{1+\varepsilon},$$

hence $\Theta(S)$ grows polynomially at least with power ε, contradicting (1.10).
Concluding, both for $k > 0$ and for $k = 0$ assuming (4.13) leads to a contradiction with our assumption (1.10), hence (4.10) holds, as required.

5. Proof of Theorem 2

We first show that φ is proper and that M is diffeomorphic to the interior of a compact manifold with boundary. Both the properties are consequence of the following lemma due to [6], which improves on [1, 20, 10, 4].

Lemma 3. *Let $\varphi : M^m \to N^n$ be an immersed submanifold into an ambient manifold N with a pole and suppose that N satisfies* (1.9) *for some $k \geq 0$. Denote by $B_s = \{x \in M;\ \rho(x) \leq s\}$ the intrinsic ball on M. Assume that*

$$\begin{array}{lll} (i) & \limsup_{s\to+\infty} s\|\mathrm{II}\|_{L^\infty(\partial B_s)} < 1 & \text{if } k = 0 \text{ in (1.9), or} \\ (ii) & \limsup_{s\to+\infty} \|\mathrm{II}\|_{L^\infty(\partial B_s)} < \sqrt{k} & \text{if } k > 0 \text{ in (1.9).} \end{array} \tag{5.1}$$

Then, φ is proper and there exists $R > 0$ such that $|\nabla r| > 0$ on $\{r \geq R\}$, where r is the extrinsic distance function. Consequently, the flow

$$\Phi : \mathbb{R}^+ \times \{r = R\} \to \{r \geq R\}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\Phi_s(x) = \frac{\nabla r}{|\nabla r|^2}\big(\Phi_s(x)\big) \tag{5.2}$$

is well defined, and M is diffeomorphic to the interior of a compact manifold with boundary.

The properness of φ enables us to apply Proposition 4. Therefore, to show that $\Theta(+\infty) < +\infty$ it is enough to check that

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)} \frac{\int_{\Gamma_s} \left[|\nabla r|^{-1} - |\nabla r|\right]}{\int_{\Gamma_s} |\nabla r|} \in L^1(+\infty). \tag{5.3}$$

To achieve (5.3), we need to bound from above the rate of approaching of $|\nabla r|$ to 1 along the flow Φ in Lemma 3. We begin with the following

Lemma 4. *Suppose that N has a pole and radial sectional curvature satisfying* (1.9)*, and that $\varphi : M^m \to N^n$ is a proper minimal immersion such that $|\nabla r| > 0$ outside of some compact set $\{r \le R\}$. Let Φ denote the flow of $\nabla r/|\nabla r|^2$ as in* (5.2) *and let $\gamma : [R, +\infty) \to M$ be a flow line starting from some $x_0 \in \{r = R\}$. Then, along γ,*

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{sn}_k(r)\sqrt{1-|\nabla r|^2}\big) \le \mathrm{sn}_k(r)|\mathrm{II}(\gamma(s))| \tag{5.4}$$

Proof. Observe that $r(\gamma(s)) = s - R$. By the chain rule and the Hessian comparison theorem 2.5,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}|\nabla r|^2 &= 2\mathrm{Hess}\, r(\nabla r, \dot\gamma) = \frac{2}{|\nabla r|^2}\mathrm{Hess}\, r(\nabla r, \nabla r) \\
&= \frac{2}{|\nabla r|^2}\overline{\mathrm{Hess}}\,(\bar\rho)\big(\mathrm{d}\varphi(\nabla r), \mathrm{d}\varphi(\nabla r)\big) + \frac{2}{|\nabla r|^2}\big(\bar\nabla\bar\rho, \mathrm{II}(\nabla r, \nabla r)\big) \\
&\ge 2\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(1-|\nabla r|^2) + 2|\bar\nabla^\perp\bar\rho||\mathrm{II}|,
\end{aligned}$$

where $\bar\nabla^\perp\bar\rho$ is the component of $\bar\rho$ perpendicular to $\mathrm{d}\varphi(TM)$ and $|\bar\nabla^\perp\rho| = \sqrt{1-|\nabla r|^2}$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}s}|\nabla r|^2 \ge 2\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}(1-|\nabla r|^2) + 2|\mathrm{II}|\sqrt{1-|\nabla r|^2}.$$

Multiplying by $\mathrm{sn}_k^2(r)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{sn}_k^2(r)(1-|\nabla r|^2)\big) \le 2\mathrm{sn}_k^2(r)|\mathrm{II}|\sqrt{1-|\nabla r|^2},$$

which implies (5.4). □

The above lemma relates the behaviour of $|\nabla r|$ to that of the second fundamental form. The next result makes this relation explicit in the two cases considered in Theorem 2.

Proposition 5. *In the assumptions of the above proposition, suppose further that either*

$$\begin{array}{llll} & (i) & \|\mathrm{II}\|_{L^\infty(\partial B_s)} \le \dfrac{C}{s\log^{\alpha/2} s} & \text{if } k = 0 \text{ in (1.9), or} \\ (5.5) & & & \\ & (ii) & \|\mathrm{II}\|_{L^\infty(\partial B_s)} \le \dfrac{C}{\sqrt{s}\log^{\alpha/2} s} & \text{if } k > 0 \text{ in (1.9).} \end{array}$$

for $s \ge 1$ and some constants $C > 0$ and $\alpha > 0$. Here, ∂B_s is the boundary of the intrinsic ball $B_s(o)$. Then, $|\nabla r|(\gamma(s)) \to 1$ as s diverges, and if $s > 2R$ and R is *sufficiently large,*

$$
\begin{aligned}
&\textit{in the case } (i), \qquad && 1-|\nabla r(\gamma(s))|^2 \le \frac{\hat{C}}{\log^\alpha s} \\
&\textit{in the case } (ii), \qquad && 1-|\nabla r(\gamma(s))|^2 \le \frac{\hat{C}}{s\log^\alpha s}
\end{aligned}
\tag{5.6}
$$

for some constant $\hat{C}$ depending on R.

Proof. We begin by observing that, in (5.5), ∂B_s can be replaced by Γ_s. Indeed, since $r(x) \le r(o) + \rho(x)$, we can choose R large enough depending on $r(o), \alpha$ in such a way that, for instance in (i),

$$
|\mathrm{II}(x)| \le \frac{C}{\rho(x)\log^{\alpha/2}\rho(x)} \le \frac{C_1}{r(x)\log^{\alpha/2} r(x)}
$$

for some absolute C_1 and for each $r \ge R$. Thus, from (i) and (ii) we infer the bounds

$$
\|\mathrm{II}\|_{L^\infty(\Gamma_s)} \le \frac{C_1}{s\log^{\alpha/2} s} \quad \text{for } (i), \qquad \|\mathrm{II}\|_{L^\infty(\Gamma_s)} \le \frac{C_1}{\sqrt{s}\log^{\alpha/2} s} \quad \text{for } (ii).
\tag{5.7}
$$

Because of (5.7), up to enlarging R further there exists a uniform constant $C_2 > 0$ such that, on $[R, +\infty)$,

$$
\mathrm{sn}_k(s)|\mathrm{II}(\gamma(s))| \le \begin{cases} \dfrac{C_1}{\log^{\alpha/2} s} \le C_2 \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{s}{\log^{\alpha/2} s}\right) & \text{if } k = 0; \\[2ex] \dfrac{C_1 \mathrm{sn}_k(s)}{\sqrt{s}\log^{\alpha/2} s} \le C_2 \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{\mathrm{sn}_k(s)}{\sqrt{s}\log^{\alpha/2} s}\right) & \text{if } k > 0. \end{cases}
\tag{5.8}
$$

Integrating on $[R, s]$ and using (5.4) we get

$$
\sqrt{1-|\nabla r(\gamma(s))|^2} \le \begin{cases} \dfrac{C_3(R)}{s} + \dfrac{C_4}{\log^{\alpha/2} s} \le \dfrac{C_5}{\log^{\alpha/2} s} & \text{if } k = 0, \\[2ex] \dfrac{C_3(R)}{\mathrm{sn}_k(s)} + \dfrac{C_4}{\sqrt{s}\log^{\alpha/2} s} \le \dfrac{C_5}{\sqrt{s}\log^{\alpha/2} s} & \text{if } k > 0, \end{cases}
$$

for some absolute constants $C_4, C_5 > 0$ and if $s > 2R$ and R is large enough. The desired (5.6) follows by taking squares. □

We are now ready to conclude the proof of Theorem 2 by showing that M has finite density or, equivalently, that (5.3) holds.

Let $\eta(s)$ be either

$$
\frac{1}{\log^\alpha s} \quad \text{when } k = 0, \text{ or } \quad \frac{1}{s\log^\alpha s} \quad \text{when } k > 0,
\tag{5.9}
$$

where $\alpha > 1$ and C is a large constant. In our assumptions, we can apply Lemma 4 and Proposition 5 to deduce, according to (5.6), that, for large enough R,

$$
1-|\nabla r(\gamma(s))|^2 \le C\eta(s) \qquad \text{on } (R, +\infty),
$$

where $\gamma(s)$ is a flow curve of Φ in (5.2) and $C = C(R)$ is large enough. In particular, $|\nabla r(\gamma(s))| \to 1$ as $s \to +\infty$. We therefore deduce the existence of a constant $C_2(R) > 0$

such that

$$\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\frac{\int_{\Gamma_s}\left[|\nabla r|^{-1}-|\nabla r|\right]}{\int_{\Gamma_s}|\nabla r|} \le C\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\eta(s)\frac{\int_{\Gamma_s}|\nabla r|^{-1}}{\int_{\Gamma_s}|\nabla r|} \le C_2\frac{\mathrm{sn}_k'(s)}{\mathrm{sn}_k(s)}\eta(s).$$

In both our cases $k = 0$ and $k > 0$, since $\alpha > 1$ it is immediate to check that $\mathrm{sn}_k'\eta/\mathrm{sn}_k \in L^1(+\infty)$, proving (5.3).

6. Appendix 1: finite total curvature solutions of Plateau's problem

In this appendix, we show that (smooth) solutions of Plateau's problem at infinity $M^m \to \mathbb{H}^n$ have finite total curvature whenever M is a hypersurface and the boundary datum $\Sigma \subset \partial_\infty\mathbb{H}^n$ is sufficiently regular. Consider the Poincaré model of $\mathbb{H}^n$, and let $M \to \mathbb{H}^n$ be a proper minimal submanifold. We say that M is $C^{k,\alpha}$ up to $\partial_\infty\mathbb{H}^n$ if its closure $\overline{M}$ in the topology of the closed unit ball $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial_\infty\mathbb{H}^n$ is a $C^{k,\alpha}$-manifold with boundary. We begin with a lemma, whose proof have been suggested to the second author by L. Mazet.

Lemma 5. *Let $\varphi : M^m \to \mathbb{H}^n$ be a proper minimal submanifold. If M is of class C^2 up to $\partial_\infty\mathbb{H}^n$, then M has finite total curvature.*

Proof. The Euclidean metric $\overline{\langle\, ,\,\rangle}$ is related to the Poincaré metric $\langle\, ,\,\rangle$ by the formula

$$\overline{\langle\, ,\,\rangle} = \lambda^2\langle\, ,\,\rangle, \qquad \text{with} \quad \lambda = \frac{1-|x|^2}{2}.$$

Given a proper, minimal submanifold $\varphi : (M^m, g) \to (\mathbb{H}^n, \langle\, ,\,\rangle)$, we associate the isometric immersion $\bar\varphi : (M, (\lambda^2\circ\varphi)g) \to (\mathbb{H}^n, \overline{\langle\, ,\,\rangle})$, $\bar\varphi(x) \doteq \varphi(x)$. Fix a local Darboux frame $\{e_i, e_\alpha\}$ on (M, g) for φ, with $\{e_i\}$ tangent to M and $\{e_\alpha\}$ in the normal bundle, and let $\bar e_i = e_i/\lambda$, $\bar e_\alpha = e_\alpha/\lambda$ be the corresponding Darboux frame on $(M, \lambda^2 g)$ for $\bar\varphi$. Let $\mathrm{d}V$ and $\mathrm{d}\bar V = \lambda^m\mathrm{d}V$ be the volume forms of (M, g) and $(M, \lambda^2 g)$, and denote with h^α_{ij} and $\bar h^\alpha_{ij}$ the coefficients of the second fundamental forms of φ and $\bar\varphi$, respectively. A standard computation shows that

$$\bar h^\alpha_{ij} = \frac{1}{\lambda}h^\alpha_{ij} - \frac{\lambda_\alpha}{\lambda}\delta_{ij},$$

where $\lambda_\alpha = e_\alpha(\lambda)$. Evaluating the norms of II and $\bar{\mathrm{II}}$, since h^α_{ij} is trace-free by minimality we obtain

$$|\bar{\mathrm{II}}|^2 = \lambda^{-2}|\mathrm{II}|^2 + m|\nabla^\perp \log\lambda|^2 \ge \lambda^{-2}|\mathrm{II}|^2,$$

and thus $|\bar{\mathrm{II}}|^m\mathrm{d}\bar V \ge |\mathrm{II}|^m\mathrm{d}V$. Integrating on M it holds

$$\int_M |\mathrm{II}|^m\mathrm{d}V \le \int_M |\bar{\mathrm{II}}|^m\mathrm{d}\bar V.$$

However, the last integral is finite since M is C^2 up to $\partial_\infty\mathbb{H}^n$, and thus φ has finite total curvature. □

In view of Lemma 5, we briefly survey on some boundary regularity results for solutions of Plateau's problem. To the best of our knowledge, we just found regularity results *for hypersurfaces*. Let $M^m \to \mathbb{H}^{m+1}$ be a solution of Plateau's problem for a compact, $(m-1)$-dimensional submanifold $\Sigma^{m-1} \subset \partial_\infty\mathbb{H}^{m+1}$. Then, a classical result of Hardt and Lin [30] states that if $\Sigma^{m-1} \hookrightarrow \partial_\infty\mathbb{H}^n$ is properly embedded and $C^{1,\alpha}$, with $0 \le \alpha \le 1$, near Σ each solution $M^m \to \mathbb{H}^n$ of Plateau's problem is a finite collection of $C^{1,\alpha}$-manifolds with boundary, which are disjoint except at the boundary.

Therefore, near Σ, M can locally be described as a graph, and the higher regularity theory in [39, 40, 53, 52], applies to give the following: if Σ is $C^{j,\alpha}$, then M is $C^{j,\alpha}$ up to $\partial_\infty \mathbb{H}^n$ whenever

- $1 \le j \le m-1$ and $0 \le \alpha \le 1$, or
- $j = m$ and $0 < \alpha < 1$, or
- $j \ge m+1$ and $0 < \alpha < 1$, under a further condition on Σ if m is odd.

(see the statement and references in [40]). In particular, because of Lemma 5, if Σ is $C^{2,\alpha}$ for some $0 < \alpha < 1$ then M has finite total curvature (provided that it is smooth).

7. Appendix 2: the intrinsic monotonicity formula

We conclude by recalling an intrinsic version of the monotonicity formula. To state it, we premit the following observation due to H. Donnelly and N. Garofalo, Proposition 3.6 in [23].

Proposition 6. *For $k \ge 0$, the function*

$$\frac{V_k(s)}{v_k(s)} \qquad \text{is non-decreasing on } \mathbb{R}^+. \tag{7.1}$$

Proof. The ratio v_k'/v_k is monotone decreasing by the very definition of v_k. Then, since $v_k' > 0$, the desired monotonicity follows from a lemma at p. 42 of [12]. □

Proposition 7 (The intrinsic monotonicity formula)**.** *Suppose that N has a pole $\bar{o}$ and satisfies* (1.9)*, and let $\varphi : M^m \to N^n$ be a complete, minimal immersion. Suppose that $\bar{o} \in \varphi(M)$, and choose $o \in M$ be such that $\varphi(o) = \bar{o}$. Then, denoting with ρ the intrinsic distance function from o and with $B_s = \{\rho \le s\}$,*

$$\frac{\mathrm{vol}(B_s)}{V_k(s)} \tag{7.2}$$

is monotone non-decreasing on $\mathbb{R}^+$.

Proof. We refer to Proposition 3 for definitions and computations. We know that the function $\psi = f \circ r$, with f as in (3.5), solves $\Delta\psi \ge 1$ on M. Integrating on B_s and using the definition of ψ we obtain

$$\mathrm{vol}(B_s) \le \int_{B_s} \Delta\psi = \int_{\partial B_s} \langle \nabla\psi, \nabla\rho \rangle \le \int_{\partial B_s} \frac{V_k(r)}{v_k(r)}.$$

Next, since $\bar{o} = \varphi(o)$, it holds $r(x) \le \rho(x)$ on M. Using then Proposition 6, we deduce

$$\mathrm{vol}(B_s) \le \frac{V_k(s)}{v_k(s)} \mathrm{vol}(\partial B_s).$$

Integrating we obtain the monotonicity of the desired (7.2). □

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