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Sasaki–Einstein Metrics on $S^2 \times S^3$

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Abstract

We present a countably infinite number of new explicit co-homogeneity one Sasaki–Einstein metrics on $S^2 \times S^3$ of both quasi-regular and irregular type. These give rise to new solutions of type IIB supergravity which are expected to be dual to $\mathcal{N} = 1$ superconformal field theories in four dimensions with compact or non-compact R-symmetry and rational or irrational central charges, respectively.

1 Introduction

A Sasaki–Einstein five-manifold X_5 may be defined as an Einstein manifold whose metric cone is Ricci-flat and Kähler – that is, a Calabi–Yau threefold.

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Such manifolds provide interesting examples of the AdS/CFT correspondence [21]. In particular, $AdS_5 \times X_5$, with suitably chosen self-dual five-form field strength, is a supersymmetric solution of type IIB supergravity that is conjectured to be dual to an $\mathcal{N} = 1$ four-dimensional superconformal field theory arising from a stack of D3-branes sitting at the tip of the corresponding Calabi–Yau cone [19, 23, 11, 1]. It is striking that the only Sasaki–Einstein five-metrics that are explicitly known are the round metric on S^5 and the homogeneous metric $T^{1,1}$ on $S^2 \times S^3$, or certain quotients thereof. For S^5 the Calabi–Yau cone is simply $\mathbb{C}^3$ while for $T^{1,1}$ it is the conifold. Here we will present a countably infinite number of explicit cohomogeneity one Sasaki–Einstein metrics on $S^2 \times S^3$.

The new metrics were found rather indirectly. In [13] we analysed general supersymmetric solutions of $D = 11$ supergravity consisting of a metric and four-form field strength, whose geometries are the warped product of five-dimensional anti-de-Sitter space (AdS_5) with a six-dimensional manifold. A variety of different solutions were presented in explicit form. In one class of solutions, the six-manifold is topologically $S^2 \times S^2 \times T^2$. Dimensional reduction on one of the circle directions of the torus gives a supersymmetric solution of type IIA supergravity in $D = 10$. A further T-duality on the remaining circle of the original torus then leads to a supersymmetric solution of type IIB supergravity in $D = 10$ of the form $AdS_5 \times X_5$, with the only non-trivial fields being the metric and the self-dual five-form. It is known [1] that for such solutions to be supersymmetric, X_5 should be Sasaki–Einstein, and these are the metrics we will discuss here. The global analysis presented in this paper, requiring X_5 to be a smooth compact manifold, is then equivalent to requiring that the four-form field strength in the $D = 11$ supergravity solution is quantised, giving a good M-theory background.

Sasaki–Einstein geometries always possess a Killing vector of constant norm (for some general discussion see, for example, [7]). If the orbits are compact, then we have a $U(1)$ action. The quotient space is always locally Kähler–Einstein with positive curvature. If the $U(1)$ action is free then the quotient space is a Kähler–Einstein manifold with positive curvature. Such Sasaki–Einstein manifolds are called regular, and the five-dimensional compact variety are completely classified [12]. This follows from the fact that the smooth four-dimensional Kähler–Einstein metrics with positive curvature on the base have been classified by Tian and Yau [27, 28]. These include the special cases $\mathbb{C}P^2$ and $S^2 \times S^2$, with corresponding Sasaki–Einstein manifolds being the homogeneous manifolds S^5 (or $S^5/\mathbb{Z}_3$) and $T^{1,1}$ (or $T^{1,1}/\mathbb{Z}_2$), respectively. For the remaining metrics, the base is a del Pezzo surface obtained by blowing up $\mathbb{C}P^2$ at k generic points with $3 \leq k \leq 8$ and, although proven to exist, these metrics are not known explicitly.

More generally, since the $U(1)$ Killing vector has constant norm, the action it generates has finite isotropy subgroups. Only if the isotropy subgroup of every point is trivial is the action free. Thus, in general, the base – the space of leaves of the canonical $U(1)$ fibration – will have orbifold singularities. This class of metrics is called quasi-regular [5]. Recently [5, 8, 9, 6], a rich set of examples of quasi-regular Sasaki–Einstein metrics have been shown to exist on $\#l(S^2 \times S^3)$ with $l = 1, \ldots, 9$, but they have not yet been given in explicit form[1]. In particular, there are 14 inhomogeneous Sasaki–Einstein metrics on $S^2 \times S^3$.

Some of the new metrics presented here are foliated by $U(1)$ and are thus in the quasi-regular class. We do not yet know if they include any of the 14 discussed in [8]. We also find Sasaki–Einstein metrics where the Killing vector has non-compact orbits, which are hence *irregular*. These seem to be the first examples of such metrics.

It is worth emphasising that the isometries generated by the canonical Killing vector on the Sasaki–Einstein manifold are dual to the R-symmetry in the four-dimensional superconformal field theory. Thus the regular and quasi-regular examples are dual to field theories with compact $U(1)$ R-symmetry. In contrast, the irregular examples are dual to field theories with a non-compact R-symmetry. We will also calculate the volumes of the new Sasaki–Einstein metrics. In the dual conformal field theory these are inversely related to the central charge of the conformal field theory. We show that the metrics associated with compact $U(1)$ R-symmetry are associated with rational central charges while those with non-compact R-symmetry are associated with irrational central charges.

2 The metrics

Our starting point is the explicit local metric given by the line element [13]:

$$\begin{aligned} ds^2 = \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}[d\psi - \cos\theta d\phi]^2 \\ + w(y)\left[d\alpha + \frac{ac - 2y + y^2 c}{6(a - y^2)}[d\psi - \cos\theta d\phi]\right]^2 \end{aligned} \tag{2.1}$$

[1]Very recently an infinite class of explicit inhomogeneous Einstein metrics have been constructed in [17]. These include Einstein metrics on $S^2 \times S^3$, but they are not expected to be Sasaki–Einstein.

with

$$\begin{aligned} w(y) &= \frac{2(a-y^2)}{1-cy} \\ q(y) &= \frac{a-3y^2+2cy^3}{a-y^2} \ . \end{aligned} \tag{2.2}$$

A direct calculation shows that this metric is Einstein with $\mathrm{Ric} = 4g$, for all values of the constants a, c. Locally the space is also Sasaki. Note that the definition of Sasaki–Einstein can be given in several (more-or-less) equivalent ways. For example, one can define a Sasaki–Einstein geometry in terms of the existence of a certain contact structure, or in terms of the existence of a solution to the Killing spinor equation. The simplest way to demonstrate that we have a local Sasaki–Einstein metric is to write the metric in a canonical way, which we do in section 4. The corresponding local Killing spinors are then easily obtained (see the discussion in [14]).

The next step is to analyse when we can extend the local expression for the metric to a global metric on a complete manifold. In this section we will demonstrate that this can indeed be done, with the manifold being $S^2 \times S^3$. The point of section 4 will be to show that the Sasaki structure also extends globally, ensuring we have globally defined Killing spinors.

We find the global extensions in two steps. First we show that, for given range of a, one can choose the ranges of the coordinates (θ, ϕ, y, ψ) so that this "base space" B_4 (forgetting the α direction) is topologically the product space $S^2 \times S^2$. The second step is to show that, for a countably infinite number of values of a in this range, one can choose the period of α so that the five-dimensional space is then the total space of an S^1 fibration (with S^1 coordinate α) over B_4. Topologically this five-manifold turns out to be $S^2 \times S^3$. A detailed analysis of the possible values for a is the content of section 3.

As we see in section 5, when $c = 0$ the metric is the local form of the standard homogeneous metric on $T^{1,1}$ (the parameter a can always be rescaled by a coordinate transformation). Thus we will focus on the case $c \neq 0$. In this case we can, and will, rescale y to set $c = 1$. This leaves us with a local one-parameter family of metrics, parametrised by a.

The base B_4

First, we choose $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ so that the first two terms in (2.1), at fixed y, give the metric on a round two-sphere. Moreover, the two-dimensional (y, ψ)-space, defined by fixing θ and ϕ, is clearly fibred over this two-sphere. To analyse the fibre, we fix the range of y so that $1-y>0$, $a-y^2>0$, which implies that $w(y)>0$. We also demand that $q(y)\geq 0$ and that y lies between two zeroes of $q(y)$, i.e., $y_1\leq y\leq y_2$ with $q(y_i)=0$. Since the denominator of $q(y)$ is always positive, y_i are roots of the cubic

$$a-3y^2+2y^3=0\ . \tag{2.3}$$

All of these conditions can be met if we fix the range of a to be

$$0<a<1\ . \tag{2.4}$$

In particular, for this range, the cubic has three real roots, one negative and two positive, and $q(y)\geq 0$ if y_1 is taken to be the negative root and y_2 the smallest positive root. The case $a=1$ is special since the two positive roots coalesce into a single double root at $y=1$. In fact we will show in section 5 that when $a=1$ the metric (2.1) is locally that of S^5.

We now argue that by taking ψ to be periodic with period 2π, the (y,ψ)-fibre, at fixed θ and ϕ, is topologically a two-sphere. To see this, note the space is a circle fibred over an interval, $y_1\leq y\leq y_2$, with the circle shrinking to zero size at the endpoints, y_i. This will be diffeomorphic to a two-sphere provided that the space is free from conical singularities at the end-points. For $0<a<1$, near either root we have $q(y)\approx q'(y_i)(y-y_i)$. Fixing θ and ϕ in (2.1) (and ignoring the α direction for the time being), this gives the metric near the poles $y=y_i$ of the two-sphere

$$\frac{1}{w(y_i)q'(y_i)(y-y_i)}dy^2+\frac{q'(y_i)(y-y_i)}{9}d\psi^2\ . \tag{2.5}$$

Introducing the co-ordinate $R=[4(y-y_i)/w(y_i)q'(y_i)]^{1/2}$ this can be written as

$$dR^2+\frac{q'(y_i)^2w(y_i)R^2}{36}d\psi^2\ . \tag{2.6}$$

We now note the remarkable fact that since $q'(y_i)=-3/y_i$ and $w(y_i)=4y_i^2$, it follows that $q'(y_i)^2w(y_i)/36=1$ at any root of the cubic. Thus the potential conical singularities at the poles $y=y_1$ and $y=y_2$ can be avoided by choosing the period of ψ to be 2π.

Note that the properties of the function $q(y)$ allow the introduction of an angle $\zeta(y)$ defined by

$$\begin{aligned}\cos\zeta&=q(y)^{1/2}=\left(\frac{a-3y^2+2y^3}{a-y^2}\right)^{1/2}\\ \sin\zeta&=-\frac{2y}{w(y)^{1/2}}=-\sqrt{2}y\left(\frac{1-y}{a-y^2}\right)^{1/2}\end{aligned} \tag{2.7}$$

with ζ ranging from $\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$ between the two roots (*cf.* [13]). It is not simple to explicitly change coordinates from y to ζ, so we continue to work with y.

To argue that we have a bundle we must show that the (y, ψ)-sphere is properly fibred over the (θ, ϕ)-sphere. At a fixed value of y between the two roots, the (ψ, θ, ϕ)-space is a $U(1) = S^1$ bundle, parametrised by ψ, over the round two-sphere parametrised by θ and ϕ. Such bundles are, up to isomorphism, in one-to-one correspondence with $H^2(S^2; \mathbb{Z}) = \mathbb{Z}$, which is the Chern number, or equivalently the integral of the curvature two-form over the base space. In our case, this integral is

$$\frac{1}{2\pi} \int_{S^2} d(-\cos\theta d\phi) = 2 \ . \tag{2.8}$$

This identifies the three-space at fixed y as the Lens space $S^3/\mathbb{Z}_2 = \mathbb{R}P^3$. Equivalently, it is the total space of the bundle of unit tangent vectors of S^2. Since in the metric it is only the ψ-circle in the (y, ψ)-sphere that is twisted over the (θ, ϕ)-sphere, this establishes that the four-dimensional base manifold is indeed a two-sphere bundle over a two-sphere. Moreover, the $\mathbb{R}^2 = \mathbb{C}$ bundle over S^2 obtained by deleting the north pole from each two-sphere fibre is just the tangent bundle of S^2.

In general, oriented S^2 bundles over S^2 are classified up to isomorphism by an element of $\pi_1(SO(3)) = \mathbb{Z}_2$. One constructs any such bundle by taking trivial bundles, i.e., products, over the northern and southern hemispheres and gluing them together along the equator, with the appropriate group element. The gluing is given by a map from the equatorial S^1 into $SO(3)$. The topology of the bundle depends only on the homotopy type of the map and hence there are only two bundles, classified by an element in $\pi_1(SO(3)) = \mathbb{Z}_2$, one trivial and one non-trivial[2]. In fact in our case we have the trivial fibration and hence, topologically, the four-dimensional base space is simply $S^2 \times S^2$ as claimed earlier.

To see this, consider the gluing element of $\pi_1(SO(3))$ corresponding to the map

$$\phi \to \begin{pmatrix} \cos(N\phi) & -\sin(N\phi) & 0 \\ \sin(N\phi) & \cos(N\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

where ϕ is a coordinate on the equator of the base S^2, with $0 \leq \phi \leq 2\pi$. Here, the upper-left 2×2 block of the matrix describes the twisting of the

[2]The non-trivial bundle is obtained by adding a point to the fibres of the chiral spin bundle of S^2 and is not a spin manifold. It gives the same manifold as the Page instanton [24] on $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$.

equatorial plane in the fibre (the ψ coordinate in our metric) and the bottom-right entry refers to the "polar" direction. If one projects out the polar direction, leaving the 2×2 block in the upper-left corner, the above map then gives the element $N \in \pi_1(U(1)) = \mathbb{Z}$ corresponding to the Chern class of a $U(1) = SO(2)$ bundle over S^2 – this bundle is just a charge N Abelian monopole.

Now, the above $SO(3)$ matrix is well-known to correspond (with a choice of sign) to the $SU(2)$ matrix

$$\begin{pmatrix} e^{iN\phi/2} & 0 \\ 0 & e^{-iN\phi/2} \end{pmatrix} \tag{2.10}$$

where we recall that $SU(2)$ is the simply-connected double-cover of $SO(3)$. It follows that, for N even, the resulting curve in $SU(2) = S^3$ is a closed cycle, and therefore corresponds to the trivial element of $\pi_1(SO(3)) = H_1(SO(3);\mathbb{Z}) = \mathbb{Z}_2$. Thus all even N give topologically product spaces $S^2 \times S^2$. For N odd, the curve in $SU(2)$ is not closed – it starts at one pole of S^3 and finishes at the antipodal pole. Of course, when projected to $SO(3)$ this curve now becomes closed and thus represents a non-trivial cycle. In fact this is the generator of $\pi_1(SO(3))$. Thus all the odd N give the non-trivial S^2 bundle over S^2. In particular, recall that for us the $U(1)$ bundle corresponding to ψ was $\mathbb{R}P^3$ with $N = 2$, and thus we have topologically a product space

$$B_4 = S^2 \times S^2 \ . \tag{2.11}$$

In what follows it will be useful for us to have an explicit basis for the homology group $H_2(B_4;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ of two-dimensional cycles on $B_4 = S^2 \times S^2$. The natural choice is simply the two S^2 cycles C_1, C_2 themselves, but since our metric on B_4 is not a product metric, it is not immediately clear where these two two-spheres are. In fact, we can take C_1 to be the fibre S^2 at some fixed value of θ and ϕ on the round S^2. Returning to the metric (2.1) we note that there are two other natural copies of S^2 located at the south and north poles of the fibre, i.e., at the roots $y = y_1$ and y_2, or equivalently $\zeta = \frac{1}{2}\pi$ and $\zeta = -\frac{1}{2}\pi$. Call these S_1 and S_2. Then the other cycle[3] is just $C_2 = S_2 + C_1 = S_1 - C_1$. Thus we have

$$2C_1 = S_1 - S_2 \qquad 2C_2 = S_1 + S_2 \ . \tag{2.12}$$

For completeness let us also give explicitly the dual elements in the coho-

[3] One way to check this is to work out the intersection numbers of the cycles. We have $C_1^2 = C_2^2 = 0$, $C_1 \cdot C_2 = 1$, $S_1^2 = 2$, $S_2^2 = -2$, $S_1 \cdot S_2 = 0$, $S_1 \cdot C_1 = S_2 \cdot C_1 = 1$.

mology $H^2(S^2 \times S^2; \mathbb{Z})$. We have

$$\begin{aligned} \omega_1 &= \frac{1}{4\pi} \cos\zeta d\zeta \wedge (d\psi - \cos\theta d\phi) + \frac{1}{4\pi} \sin\zeta \sin\theta d\theta \wedge d\phi \\ \omega_2 &= \frac{1}{4\pi} \sin\theta d\theta \wedge d\phi \end{aligned} \tag{2.13}$$

with $\int_{C_i} \omega_j = \delta_{ij}$.

The circle fibration

Now we turn to the fibre direction, α, of the full five-dimensional space. It is convenient to write the five-dimensional metric (2.1) in the form

$$ds^2 = ds^2(B_4) + w(y)(d\alpha + A)^2 \tag{2.14}$$

where $ds^2(B_4)$ is the non-trivial metric on $S^2 \times S^2$ just described, and the local one-form A is given by

$$A = \frac{a - 2y + y^2}{6(a - y^2)}[d\psi - \cos\theta d\phi] \ . \tag{2.15}$$

Note that the norm-squared of the Killing vector $\partial/\partial\alpha$ is $w(y)$, which is nowhere-vanishing.

In order to get a compact manifold, we would like the α coordinate to describe an S^1 bundle over B_4. We take

$$0 \leq \alpha \leq 2\pi\ell \tag{2.16}$$

where the period $2\pi\ell$ of α is *a priori* arbitrary. Thus, rescaling by ℓ^{-1}, we have that $\ell^{-1}A$ should be a connection on a $U(1)$ bundle over $B_4 = S^2 \times S^2$. However, this puts constraints on A. In general, such $U(1)$ bundles are completely specified topologically by the gluing on the equators of the two S^2 cycles, C_1 and C_2. These are measured by the corresponding Chern numbers in $H^2(S^2; \mathbb{Z}) = \mathbb{Z}$ which we label p and q. The corresponding five-dimensional spaces will be denoted by $Y^{p,q}$. In general, we will find that for any p and q, such that $0 < q/p < 1$, we can always choose ℓ and the parameter $0 < a < 1$ such that $\ell^{-1}A$ is a *bona fide* $U(1)$ connection. For p and q relatively prime, which we can always achieve by taking an appropriate ℓ, i.e., the maximal period, it turns out that $Y^{p,q}$ are all topologically $S^2 \times S^3$. We now fill in the details of this argument.

The essential point is to show that $\ell^{-1}A$ is a connection on a $U(1)$ bundle. As mentioned above, a $U(1)$ bundle over $B_4 = S^2 \times S^2$ is characterised by the two Chern numbers p and q. These are given by the integrals of the $U(1)$-curvature two-form $\ell^{-1}dA/2\pi$ over the two-cycles C_1 and C_2 which form the basis of $H_2(S^2 \times S^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Let us define the two periods P_1 and P_2 as

$$P_i = \frac{1}{2\pi} \int_{C_i} dA \tag{2.17}$$

which, in general, will be functions of a. The corresponding integrals of $\ell^{-1}dA/2\pi$ must give the Chern numbers p and q. That is, we require $P_1 = \ell p$ and $P_2 = \ell q$. Since we are free to choose ℓ, the only constraint is that we must find a such that

$$P_1/P_2 = p/q \ . \tag{2.18}$$

In particular, we will choose ℓ so that p and q are coprime. Then with

$$\ell = P_1/p = P_2/q \tag{2.19}$$

we get a five-dimensional manifold which is an S^1 bundle over $B_4 = S^2 \times S^2$ with winding numbers p and q, denoted by $Y^{p,q}$. We will find that (2.18) can be satisfied for a countably infinite number of values of a.

Let us first check that dA is properly globally defined. Recall that, in general, a connection one-form is not a globally well-defined one-form – if it is globally well-defined, the curvature is exact and the bundle is topologically trivial. Rather, a connection one-form is defined only locally in patches, with gauge transformations between the patches. However, the curvature is a globally well-defined smooth two-form. Let us now check this is true for our metric. At fixed value of y between the two roots, $y_1 < y < y_2$, we see that A is proportional to the "global angular form" on the $U(1)$ bundle with fibre ψ and base parametrised by (θ, ϕ). This is actually globally well-defined since a gauge transformation on ψ is cancelled by the corresponding gauge transformation of the connection $-\cos\theta d\phi$. Thus in fact dA is exact on a slice of the four-manifold B_4 at fixed y. This is also clearly true on the whole of the total space of the "cylinder bundle" obtained by deleting the north and south poles of the fibres of the S^2 bundle. One must now check that dA is smooth as one approaches the poles of the fibres. There are two terms, and the only term of concern smoothly approaches a form proportional to $dy \wedge d\psi$ near the poles, at fixed value of θ and ϕ. We must now recall that the proper radial coordinate is (*e.g.* at the south pole) $R \propto (y - y_i)^{1/2}$. Thus $dy \propto RdR$, and this piece of dA is a smooth function times the canonical volume form $RdR \wedge d\psi$ on the open subset of $\mathbb{R}^2$ in the fibre near to the poles. Thus dA is a globally-defined smooth two-form on $B_4 = S^2 \times S^2$, and thus represents an element of $H^2_{\text{de Rham}}(B_4)$.

To calculate the periods P_i it is easiest to first calculate the integrals of dA over the cycles S_i at the north and south poles of the (y, ψ) fibre. We find

$$\frac{1}{2\pi}\int_{S_i} dA = \frac{1}{2\pi}\int_{S_i} \frac{a-2y_i+y_i^2}{6(a-y_i^2)}\sin\theta d\theta d\phi = \frac{y_i-1}{3y_i} \tag{2.20}$$

and hence, given the relations (2.12), we have

$$\begin{aligned} P_1 &= \frac{y_1-y_2}{6y_1y_2} \\ P_2 &= \frac{2y_1y_2-y_1-y_2}{6y_1y_2} = -\frac{(y_1-y_2)^2}{9y_1y_2} \end{aligned} \tag{2.21}$$

and so

$$\frac{P_1}{P_2} = \frac{3}{2(y_2-y_1)} \ . \tag{2.22}$$

Thus our requirement (2.18) is that

$$y_2 - y_1 \text{ is rational.} \tag{2.23}$$

In the next section we will show that there are infinitely many values of a for which this is true, in the range $0 < a < 1$ and with $0 < q/p < 1$. Furthermore, as shown in Appendix A, for any p and q coprime, the space $Y^{p,q}$ is topologically $S^2 \times S^3$. This then completes our argument about the regularity of the metrics on $S^2 \times S^3$.

Our new metrics admit a number of Killing vectors which generate isometries. Clearly there is a $U(1)$ generated by $\partial/\partial\alpha$. In addition there is an $SU(2)\times U(1)$ action. To see this, we write the metric in terms of left-invariant one-forms σ_i, $i = 1, 2, 3$ on $SU(2)$

$$\begin{aligned} ds^2 = \frac{1-y}{6}(\sigma_1^2+\sigma_2^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}\sigma_3^2 \\ + w(y)\left[d\alpha + \frac{a-2y+y^2}{6(a-y^2)}\sigma_3\right]^2 \ . \end{aligned} \tag{2.24}$$

This displays the fact that the metrics admit an $SU(2)$ left-action and a $U(1)$ right-action. Together with the $U(1)$ isometry generated by $\partial/\partial\alpha$, this gives an isometry group $SU(2) \times_{\mathbb{Z}_2} U(1)^2$, where we have noted that the element $(-1_2, -1, -1)$ acts trivially.

We end by noting that the volume of these spaces is given by

$$\mathrm{vol} = \frac{4\pi^3}{9}\ell\, (y_1-y_2)(y_1+y_2-2) \ . \tag{2.25}$$

3 All solutions for the parameter a

We have shown that it is necessary and sufficient that $P_1/P_2 = p/q$ is rational in order to get metrics on a complete manifold. Clearly it is sufficient that the roots y_1 and y_2 of the cubic (2.3) (and hence all three of the roots) are rational. This leads to a number theoretic analysis which is presented below, and we find an infinite number of values of a for which the roots are rational. In these cases the volume of the manifolds are rationally related to the volume of the round five-sphere. We will argue later that these values of a give rise to quasi-regular Sasaki–Einstein manifolds. However, it is also possible to achieve rational P_1/P_2 for an infinite number of values of a when the roots are *irrational* as the following general analysis reveals. We will see later that these cases give rise to irregular Sasaki–Einstein metrics.

General case

Assume $P_1/P_2 = p/q$ is rational. As discussed above, this implies that

$$y_2 - y_1 = \frac{3q}{2p} \equiv \lambda \ . \tag{3.1}$$

We first observe that the three roots of the cubic satisfy:

$$\begin{aligned} y_1 + y_2 + y_3 &= \tfrac{3}{2} \\ y_1 y_2 + y_1 y_3 + y_2 y_3 &= 0 \\ 2 y_1 y_2 y_3 &= -a \ . \end{aligned} \tag{3.2}$$

Next note that $y_1 + y_2 = 0$ if and only if $a = 0$, which is excluded from our considerations. Then we deduce that

$$y_1 + y_2 = \frac{2}{3}(y_1^2 + y_1 y_2 + y_2^2) \tag{3.3}$$

with the third root given by $y_3 = \frac{3}{2} - y_1 - y_2$. One may now solve for y_1 in terms of λ. Since y_1 is required to be the smallest root of the cubic, one takes the smaller of the two roots to obtain

$$y_1 = \frac{1}{2}\left(1 - \lambda - \sqrt{1 - \lambda^2/3}\right) \ . \tag{3.4}$$

Since $y_2 > y_1$ this means that $0 < \lambda \leq \sqrt{3}$, where the upper bound ensures that y_1 is real. Notice that $y_1 < 0$, as it should be.

We now require that y_1 be a root of the cubic. To ensure this, we simply define

$$a = a(\lambda) = 3y_1^2(\lambda) - 2y_1^3(\lambda) \ . \tag{3.5}$$

One requires that $0 < a < 1$ which is in fact automatic for the range of λ already chosen. However, some values of a are covered twice since the function $a(\lambda)$ has a maximum of 1 at the value $\lambda = \frac{3}{2}$. Specifically, this range is easily computed to be $\frac{1}{2} \leq a \leq 1$. The range $0 < a < \frac{1}{2}$ is covered only once. Finally, we must ensure that $y_2 = y_1 + \lambda$ is the smallest positive root. Comparing with $y_3 = \frac{3}{2} - y_1 - y_2$, we see that this implies that $\lambda < \frac{3}{2}$.

To summarise, any rational value of $\lambda = 3q/2p$ is allowed within the range $0 < \lambda < \frac{3}{2}$, and this is achieved by choosing a given by (3.4), (3.5). Moreover, the range of a is $0 < a < 1$ and is covered in a monotonic increasing fashion. The period of the coordinate α is given by $2\pi\ell$ where

$$\ell = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}} \ . \tag{3.6}$$

Note that we can also recast our formula for the volume in terms of p and q to give

$$\mathrm{vol} = \frac{q^2[2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]}\pi^3 \tag{3.7}$$

which is generically an irrational fraction of the volume π^3 of a unit round S^5. This implies that we have irrational central charges in the dual superconformal field theory. In fact the result is stronger: the central charge can be written only in terms of square-roots of rational numbers. Note that by setting $q = 1$ and letting p become large, we see that the volume can be arbitrarily small. The largest volume given by our metrics on $Y^{p,q}$ occurs for $p = 2, q = 1$ with vol $\approx 0.29\pi^3$, and corresponds to an irrational case.

Case of rational roots

We now show that for a countably infinite number of values of a the roots y_1 and y_2 are rational. First note from (3.4) that y_1 is rational if and only if $1 - \lambda^2/3$ is the square of a rational. Then y_2 (and also y_3) is rational since λ is necessarily rational. Substituting $\lambda = 3q/2p$, the problem reduces to finding all solutions to the quadratic diophantine

$$4p^2 - 3q^2 = n^2 \tag{3.8}$$

where $p, q \in \mathbb{N}$, $n \in \mathbb{Z}$, $(p, q) = 1$, $q < p$.

To find the general solution, we proceed as follows. First define $r = 2p$, so that (3.8) may be written

$$(r - n)(r + n) = 3q^2 \ . \tag{3.9}$$

We first argue that a prime factor $t > 3$ of $r + n$ must appear with an even power. Suppose it did not. We have t divides q^2 and so t divides q. Then, since the power of t in q^2 is clearly even, t must divide $r - n$. This is now a contradiction since t now divides both p and q which are, by assumption, coprime. Using a similar argument for $t = 3$ we conclude that

$$\begin{aligned} r + n &= 3A^2 2^{k_1} \\ r - n &= B^2 2^{k_2} \end{aligned} \tag{3.10}$$

where $A, B, k_i \in \mathbb{N}$ and we have used the freedom of switching the sign of n in fixing the factor of 3. Moreover, A, B satisfy

$$(A, B) = 1, \quad B \neq 0 \mod 3, \quad A, B \neq 0 \mod 2\ . \tag{3.11}$$

We deal with the factors of 2 in two steps. First, if n is odd then $k_1 = k_2 = 0$ and we have the following solutions

$$p = \frac{1}{4}(3A^2 + B^2), \quad q = AB, \tag{3.12}$$

with $n = \frac{1}{2}(3A^2 - B^2)$. To ensure that $p > q$ we need to also impose

$$A > B, \quad \text{or} \quad B > 3A\ . \tag{3.13}$$

Alternatively, for solutions with n even, one can show that q is always even and p is then always odd. A little effort reveals two families of solutions. The first is:

$$p = 3A^2 + B^2 2^{2k}, \quad q = AB2^{k+2} \tag{3.14}$$

with $n = 2(3A^2 - B^2 2^{2k})$, $k \geq 1$ and A, B satisfying (3.11) and also

$$A > B2^k, \quad \text{or} \quad B2^k > 3A\ . \tag{3.15}$$

The second is:

$$p = 3A^2 2^{2k} + B^2, \quad q = AB2^{k+2} \tag{3.16}$$

with $n = 2(3A^2 2^{2k} - B^2)$, $k \geq 1$ and A, B satisfying (3.11) and also

$$A2^k > B, \quad \text{or} \quad B > 3A2^k\ . \tag{3.17}$$

Finally, returning to the volume formula (3.7), we note that when the roots are rational, since $4p^2 - 3q^2 = n^2$, clearly the volume is a rational fraction of that of the unit round S^5. This corresponds to rational central charges in the dual superconformal field theory.

4 The metrics in canonical form

At this stage we have shown that we have a countably infinite number of new Einstein metrics on $S^2 \times S^3$. We now establish that the geometries admit a Sasaki structure which extends globally and that the metrics admit globally defined Killing spinors. We do this by first showing that the metric can be written in a canonical form implying it has a local Sasaki structure. We then show that this Sasaki–Einstein structure, defined in terms of contact structures, is globally well-defined. Finally, given we have a simply-connected five-dimensional manifold with a spin structure, using theorem 3 of [12], we see that this implies we have global Killing spinors.

Employing the change of coordinates $\alpha = -\frac{1}{6}\beta - \frac{1}{6}c\psi'$, $\psi = \psi'$ the local metric (2.1) becomes

$$\begin{aligned} ds^2 = \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{dy^2}{w(y)q(y)} + \frac{1}{36}w(y)q(y)(d\beta + c\cos\theta d\phi)^2 \\ + \frac{1}{9}[d\psi' - \cos\theta d\phi + y(d\beta + c\cos\theta d\phi)]^2 \end{aligned} \tag{4.1}$$

where we have temporarily reinstated the constant c. This has the standard form

$$ds^2 = ds_4^2 + \left(\tfrac{1}{3}d\psi' + \sigma\right)^2 \tag{4.2}$$

where ds_4^2 is a local Kähler–Einstein metric and the form σ satisfies $d\sigma = 2J_4$. We have the local Kähler form

$$J_4 = \frac{1}{6}(1-cy)\sin\theta\, d\theta \wedge d\phi + \frac{1}{6}dy \wedge (d\beta + c\cos\theta d\phi)\ . \tag{4.3}$$

In the quasi-regular case, J_4 extends globally to the Kähler form on the four-dimensional base orbifold.

One can check explicitly that the four-dimensional metric is Kähler–Einstein and has Ricci-form equal to six times the Kähler form. This form of the metric (4.2) is the standard one for a locally Sasaki–Einstein metric, with $\partial/\partial\psi'$ the constant norm Killing vector. As noted earlier, the normalisation is canonical with the Ricci tensor being four times the metric.

Note that the $SU(2)\times U(1)^2$ isometry group of our metrics (2.24) implies that we can introduce a set of $SU(2)$ left-invariant one-forms, $\tilde{\sigma}_i$, and write ds_4^2 in (4.2) locally as a bi-axially squashed $SU(2)$-invariant Kähler–Einstein metric. The most general metric of this type was found in [15], where it was shown to depend on two parameters (one being the overall scale). Global properties of such metrics were then discussed, to some extent, in [25]. To recast our metric in the form given in [15], we introduce $\rho^2 = 2(1-y)/3$ and $\tilde{\sigma}_i$ to get, with $c = 1$,

$$\begin{aligned} ds_4^2 &= \frac{1}{\Delta}d\rho^2 + \frac{\rho^2}{4}\left(\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + \Delta\tilde{\sigma}_3^2\right) \\ \Delta &= 1 + \frac{4(a-1)}{27}\frac{1}{\rho^4} - \rho^2 \ . \end{aligned} \tag{4.4}$$

We would now like to confirm that the Sasaki structure extends globally. To do so, we will work with the definition of a Sasaki–Einstein manifold given in terms of a contact structure. Given the metric extends globally, this is equivalent to the condition that the Killing vector $\partial/\partial\psi'$ and hence the dual one-form $\frac{1}{3}d\psi' + \sigma$ with $d\sigma = 2J_4$ in (4.2) are globally defined. To show this we simply consider these objects in the original coordinates (2.1). We first observe that the Killing vector field of the Sasaki structure is given by

$$\frac{\partial}{\partial\psi'} = \frac{\partial}{\partial\psi} - \frac{1}{6}\frac{\partial}{\partial\alpha} \tag{4.5}$$

which is globally well defined, since both of the vectors on the right hand side are globally well defined. Since this vector has constant norm, the dual one-form

$$[d\psi' - \cos\theta d\phi + y(d\beta + c\cos\theta d\phi)] \tag{4.6}$$

must also be globally well defined. It is interesting to see this in the original coordinates. This one-form can be written as the sum of two one-forms:

$$\begin{aligned} -6y&\left[d\alpha + \frac{ac - 2y + y^2c}{6(a-y^2)}(d\psi - \cos\theta d\phi)\right] \\ &\qquad + \frac{a - 3y^2 + 2cy^3}{a - y^2}\left(d\psi - \cos\theta d\phi\right) \ . \end{aligned} \tag{4.7}$$

From the analysis of section 2, we conclude that the first one-form in this expression is globally well-defined since it is, up to a smooth function, just the dual of the globally defined vector $\partial/\partial\alpha$. This form is also the so-called global angular form on the total space of the $U(1)$ bundle $U(1) \hookrightarrow Y^{p,q} \rightarrow S^2 \times S^2$. Now, the one-form $(d\psi - \cos\theta d\phi)$ is a global angular form on the $U(1)$ bundle $U(1) \hookrightarrow \mathbb{R}P^3 \rightarrow S^2$ at fixed y between the two roots/poles. Although this one-form is not well defined at the poles $y = y_i$, the second one-form in (4.7) *is* globally defined, since the pre-factor vanishes smoothly at the poles. Finally, the exterior derivative of (4.6) is also clearly well defined and is equal to $6J_4$. Thus we conclude that the Sasaki structure is globally well defined and that we have a countably infinite number of Sasaki–Einstein manifolds.

Recall that in the original co-ordinates we argued that ψ was periodic with period 2π and that α was periodic with period $2\pi\ell = 2\pi P_1/p = 2\pi P_2/q$. The form of (4.5) shows that, for general values of a discussed in section 3, the orbits of the vector $\partial/\partial\psi'$ are not closed. In this case the Sasaki–Einstein metric is irregular. There is no Kähler–Einstein "base manifold" or even "base orbifold" for this case. Since the orbits of $\partial/\partial\psi'$ are dense in the torus defined by the Killing vectors $\partial/\partial\psi$ and $\partial/\partial\alpha$, these Sasaki–Einstein metrics are, more precisely, irregular of rank two.

However, in the special case that

$$\frac{6P_1}{p} = \frac{6P_2}{q} = \frac{s}{r} \tag{4.8}$$

for suitably chosen integers r, s, we can choose ψ' to be periodic with period $2\pi s$. This case occurs only when the roots of the cubic are rational. The Sasaki–Einstein metrics are then quasi-regular. To see that we are not in the regular class, observe that if the base were a manifold it would have to be in Tian and Yau's list. Notice, however, that our metrics admit an $SU(2)$ action. As discussed in [10], these two facts are mutually exclusive, except in the cases $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ with canonical metrics. Thus we must be in the quasi-regular class. The base space is then an orbifold – it would be interesting to better understand its geometry.

5 Special cases

We discuss here the two special cases that were mentioned earlier.

Case 1: $T^{1,1}$

First consider $c = 0$, and then rescale to set $a = 3$. Starting with (2.1) and introducing the coordinates $\cos\omega = y$, $\nu = 6\alpha$ we obtain

$$\begin{aligned} ds^2 &= \frac{1}{6}(d\theta^2 + \sin^2\theta d\phi^2 + d\omega^2 + \sin^2\omega d\nu^2) \\ &\quad + \frac{1}{9}[d\psi - \cos\theta d\phi - \cos\omega d\nu]^2 \ . \end{aligned} \tag{5.1}$$

If the period of ν is 2π we see that the four-dimensional base metric orthogonal to ∂_ψ is now the canonical metric on $S^2 \times S^2$, and if we choose the period of ψ to be 4π we recover the metric on $T^{1,1}$. If the period is taken to be 2π, as in the rest of the family with $c \neq 0$, we get $T^{1,1}/\mathbb{Z}_2$. Both of these metrics are well known to be Sasaki–Einstein.

Case 2: S^5

Next, returning to $c = 1$ and setting $a = 1$, we introduce the new coordinates $1 - y = \frac{3}{2}\sin^2\sigma$, $\psi = \psi'' - \beta$ and $\alpha = -\frac{1}{6}\psi''$ in (2.1) with $0 \leq \sigma \leq \frac{1}{2}\pi$ to get

$$
\begin{aligned}
ds^2 = d\sigma^2 &+ \frac{1}{4}\sin^2\sigma(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{4}\cos^2\sigma\sin^2\sigma(d\beta + \cos\theta d\phi)^2 \\
&+ \frac{1}{9}\left[d\psi'' - \frac{3}{2}\sin^2\sigma(d\beta + \cos\theta d\phi)\right]^2 .
\end{aligned}
\tag{5.2}
$$

If the period of β is taken to be 4π the base metric is the Fubini–Study metric on $\mathbb{C}P^2$. If the period of ψ'' is taken to be 6π we get the round metric on S^5. If the period of ψ'' is taken to be 2π we obtain the Lens space $S^5/\mathbb{Z}_3$. Both of these metrics are also well known to be Sasaki–Einstein.

We can view this case as a limit $a = 1$ of our family of solutions. In this case ψ has period 2π implying β has period 2π. Since $y_1 = -\frac{1}{2}$, $y_2 = 1$ we take $P_1 = P_2 = \frac{1}{2}$ and $p = q = 1$ and hence the period α to be π. This implies that the period of ψ'' is 6π and thus the five-dimensional space is the orbifold $S^5/\mathbb{Z}_2$. In other words, the mildly singular $D = 11$ spaces that we constructed in section 5.1 of [13] with $a = c = 1$ are in fact related, after dimensional reduction and T-duality, to $S^5/\mathbb{Z}_2$.

6 Discussion

We have presented an infinite number of new Sasaki–Einstein metrics of cohomogeneity one on $S^2 \times S^3$, both in the quasi-regular and irregular classes. As far as we know these are the first examples of irregular metrics. As type IIB backgrounds both classes should provide supergravity duals of a family of $\mathcal{N} = 1$ superconformal field theories. Let us make some general comments about the field theories. First we recall that the geometries generically admit an $SU(2) \times U(1)^2$ isometry. We also get additional baryonic $U(1)_B$ flavour symmetry factors from reducing the RR four-form gauge potential on independent three-cycles in X_5. Since here $X_5 = S^2 \times S^3$ this gives a single $U(1)_B$ factor. Thus the continuous global symmetry group of the dual field theory for all our examples is, modulo discrete identifications,

$$
SU(2) \times U(1)^2 \times U(1)_B . \tag{6.1}
$$

From the geometry we can also identify the R-symmetry. It is generated by the Sasaki Killing vector[4] $\partial/\partial\psi'$ which is a linear combination (4.5) of $\ell\partial/\partial\alpha$ and $\partial/\partial\psi$ which generate the $U(1)^2$ isometry group.

This geometrical picture matches the discussion of R-charges in $\mathcal{N}=1$ superconformal field theories by Intriligator and Wecht [18]. Consider such a field theory with the above global symmetries. In [18] it is argued that the R-symmetry is not expected to mix with the non-Abelian and baryonic factors $SU(2)$ and $U(1)_B$. Thus it should be some combination of the $U(1)^2$ factors precisely as we see in the supergravity solutions. There are then two distinct possibilities. For the quasi-regular metrics the Sasaki–Einstein Killing vector generates a compact $U(1)_R$ symmetry in the field theory and the R-charges of the fields must be rational. For the irregular metrics, the R-symmetry is non-compact and we are allowed irrational R-charges. Note that from (3.6) and (4.5) we see that we have a relation between the Killing vectors involving only quadratic algebraic numbers (by which we mean they are *square-roots* of rational numbers). This implies that the R-charges are also quadratic algebraic numbers, in agreement with the analysis in [18].

In general, the AdS/CFT correspondence implies that the ratio of the central charges associated to our spaces $Y^{p,q}$ to the central charge of $\mathcal{N}=4$ super-Yang–Mills theory with gauge group $SU(N)$, is equal to the ratio of the volumes of S^5 and $Y^{p,q}$ [16]. This is simply given by the right-hand side of the formula (3.7) without the π^3. If we denote the central charge of the superconformal field theory associated to X_5 as $a(X_5)$ we find

$$a(S^5) < a(T^{1,1}) < a(S^5/\mathbb{Z}_2) < a(T^{1,1}/\mathbb{Z}_2) < a(Y^{2,1}) < a(Y^{p,q}) \qquad (6.2)$$

for $(p,q) \neq (2,1)$. Since the volumes of our new spaces can be arbitrarily small, the corresponding central charge can be arbitrarily large. Assuming an a-theorem[5], we see that none of the conformal field theories associated with $Y^{p,q}$ can arise in the IR via RG flow of a perturbation of the known field theories[6] associated with S^5, $T^{1,1}$, $S^5/\mathbb{Z}_2$ or $T^{1,1}/\mathbb{Z}_2$. This is to be contrasted with the field theory associated with $T^{1,1}$ which has been argued to arise via a perturbation of the field theory associated with the orbifold $S^5/\mathbb{Z}_2$ [19].

[4]This follows from the fact that the supercharges of the CFT are identified with the Killing spinors on X_5, and these indeed are charged with respect to the canonical Sasaki direction [14].

[5]For recent progress in this direction, see [20].

[6]It is interesting to observe that for any Einstein manifold $X_5 \neq S^5$ with the same Ricci curvature as S^5, then $\mathrm{vol}(X_5) < \mathrm{vol}(S^5)$ (see section G of chapter 12 of [3]). Hence the field theories dual to a type IIB solution of the direct product form $AdS_5 \times X_5$ should not arise as perturbations of $\mathcal{N}=4$ SYM theory.

In accordance with the R-charge discussion above it is simple to see, from eq. (3.7), that the central charges of the quasi-regular metrics that we have constructed are rational, while the central charges of the irregular metrics are irrational. In fact, the latter are quadratic algebraic – hence we find perfect agreement with the field theory prediction of [18]. We emphasize that only the irregular Sasaki–Einstein manifolds are possible candidate duals of a superconformal field theory with irrational R-charges and hence central charges. In particular this requires that the volumes of regular and quasi-regular Sasaki–Einstein metrics, which give rational R-charges and hence central charges, must always be rational fractions of that of S^5, with the same curvature, which is always the case. Indeed, if X_5 is a regular Sasaki–Einstein manifold, with Kähler–Einstein base B_4 and with the $U(1)$ fibration being that corresponding to the canonical bundle of B_4, we have (see also [2])

$$\mathrm{vol}(X_5) = \frac{\gamma(B_4)}{27}\mathrm{vol}(S^5) \tag{6.3}$$

where

$$\gamma(B_4) = \int_{B_4} c_1^2 \tag{6.4}$$

is a Chern number of B_4, and $c_1 \in H^2(B_4;\mathbb{Z})$ is the first Chern class. In particular, note that $\gamma(B_4)$ is always an integer. As an example, take $B_4 = \mathbb{C}P^1 \times \mathbb{C}P^1$. Then $\gamma(B_4) = 8$, and (6.3) gives the ratio $\frac{8}{27}$, which is indeed correct for $T^{1,1}/\mathbb{Z}_2$ [16]. The quasi-regular case is more subtle, but a similar result still holds since now the Ricci form (divided by 2π) has rational, rather than integral, periods: $c_1 \in H^2(B_4;\mathbb{Q})$. Also note that similar formulae hold in higher dimensions.

As mentioned in the introduction, these type IIB solutions $AdS_5 \times Y^{p,q}$ are T-dual to type IIA solutions which can in turn be lifted to solutions of $D = 11$ supergravity. Indeed, it was the reverse route that originally led us to $Y^{p,q}$ [13]. For the type IIA and $D = 11$ solutions to be good string or M-theory backgrounds it is necessary that they have quantised fluxes. However, under T-duality this is simply a consequence of the regularity of the spaces $Y^{p,q}$ (see for example [4]).

For the solutions to provide good *supergravity* duals to superconformal field theories, X_5 should not have any small cycles. As usual this requires that the volume of $Y^{p,q}$ is chosen to be large, by an overall scaling the solution, and corresponds to large 't Hooft coupling g_sN where g_s is the string coupling and N is the number of D3-branes. However we must also consider the size of the α circle. From (3.6) it is clear that this can potentially be small. However, we can always choose the overall scaling of the solution so that there is a good type IIB supergravity description. For example, consider the large λ limit $p \gg q$. We see that the overall volume of the manifold scales as $1/p$. The period ℓ of α scales as $1/q$ while $w(y)$ scales as q^2/p^2 and, hence, the length of the α-circle also scales as $1/p$. The dependence of the overall volume on p is due solely to the size of the α-circle; the other dimensions are all of order one. Hence choosing $(g_s N)^{1/4} \gg p$ all cycles are large and we have a good IIB supergravity solution. However, if we choose $p \gg (g_s N)^{1/4} \gg 1$ then the α-circle remains small and a good supergravity description of the field theory will be provided by the T-dual type IIA solution (or the $D = 11$ solution when the IIA string coupling is large).

It is interesting to note that under T-duality the canonical Sasaki Killing direction associated with the R-symmetry does not get mapped to the canonical Killing direction which generates the R-symmetry of the $D = 11$ solution. As shown in [13], the latter is generated by $\partial/\partial\psi$ and this always generates a compact symmetry for the solutions we are discussing. The discrepancy is proportional to the period ℓ of α which indicates that there are string theory corrections to the R-symmetry when the α-circle is small.

Beyond these general properties one would of course like to identify the dual conformal field theories in more detail. They arise from N D3-branes at the tip of a singular Calabi–Yau geometry given by the metric cone over X_5 with the field content determined by the form of the singularity. It is suggestive to note that we can view our $Y^{p,q}$ manifolds as a Lens space $L(p, 1)$ fibred over a base S^2. The Lens space is parametrised by the coordinates (y, ψ, α) in the metric (2.1) and the S^2 base by (θ, ϕ). Topologically $L(p, 1)$ is the base of the four-dimensional cone describing an A_{p-1} singularity $\mathbb{C}^2/\mathbb{Z}_p$. Thus perhaps one way to view the singular Calabi–Yau geometry is as an A_{p-1} singularity fibred over a collapsing S^2. The Chern number p describes the order of the singularity A_{p-1} while q encodes the fibration over S^2.

If this is correct, one would then expect to find a related smooth Calabi–Yau manifold by first resolving the base S^2 (in analogy with the conifold) and then blowing up the singular A_{p-1} fibres. This resolution, and possibly others, should have important consequences for the dual field theories and the generalisations obtained by adding fractional branes. It would be of interest to construct these smooth Calabi-Yau manifolds, which are expected to be co-homogeneity two.

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A The topology of the manifolds

In this appendix we show that the total space $Y^{p,q}$ of the $U(1)$ bundle over $S^2 \times S^2$ with relatively prime winding numbers p and q over the two two-cycles is topologically $S^2 \times S^3$. This is a simple consequence of Smale's Theorem [26] and the Gysin sequence for the $U(1)$ fibration. We include the argument here for completeness.

Let E be the complex line bundle over $S^2 \times S^2$ with winding numbers p and q, where $(p, q) = 1$. The boundary ∂E of E is then our space $Y^{p,q}$. We denote the projections $\Pi : E \to B$, $\pi : \partial E \to B$, where $B = S^2 \times S^2$.

We first show that $Y^{p,q}$ is simply-connected. Consider the following part of the long exact cohomology sequence for the pair $(E, \partial E)$:

$$\cdots \to H^4(E, \partial E; \mathbb{Z}) \xrightarrow{f} H^4(E; \mathbb{Z}) \xrightarrow{i^*} H^4(\partial E; \mathbb{Z}) \to H^5(E, \partial E; \mathbb{Z}) \to \cdots \tag{A.1}$$

where $i : \partial E \to E$ denotes embedding and the map f forgets that a class is a relative class (has compact support). The Thom isomorphism theorem states that $H^*(B; \mathbb{Z}) \overset{\cup\Phi}{\cong} H^{*+2}(E, \partial E; \mathbb{Z})$ for a complex line bundle E over base B, where the isomorphism is the cup product with the Thom class $\Phi \in H^2(E, \partial E; \mathbb{Z}) \cong \mathbb{Z}$. Thus the last term of the above sequence is $H^3(B; \mathbb{Z}) = 0$, since $B = S^2 \times S^2$. Exactness of the sequence now implies

$$H^4(\partial E; \mathbb{Z}) \cong H^4(B; \mathbb{Z}) / \left[c_1 \cup H^2(B; \mathbb{Z})\right] \tag{A.2}$$

where we have used the fact that $H^2(B; \mathbb{Z}) \overset{\cup\Phi}{\cong} H^4(E, \partial E; \mathbb{Z})$ and also that $f(\Phi) = \Pi^*(c_1) \in H^2(E; \mathbb{Z})$. Now B is a deformation retract of E, so $H^4(E; \mathbb{Z}) \cong H^4(B; \mathbb{Z}) \cong \mathbb{Z}$, and $H^2(B; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ where the generators are dual to the two S^2 cycles. Thus $c_1 \cup H^2(B; \mathbb{Z}) \subset \mathbb{Z}$. This subgroup is generated by all elements of the form $pb + qa$ where $a, b \in \mathbb{Z}$, since c_1 is p times the first generator of $H^2(B;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and q times the second. The subgroup is therefore $(p,q)\mathbb{Z}$, where (p,q) denotes highest common factor. Since the latter is 1, we find that $H_1(\partial E;\mathbb{Z}) \cong H^4(\partial E;\mathbb{Z}) = 0$ is trivial. It follows that $\pi_1(\partial E)$ is also trivial, so ∂E is simply-connected.

One can similarly show that $H_2(\partial E;\mathbb{Z}) \cong H^3(\partial E;\mathbb{Z}) \cong \mathbb{Z}$. For this we need the following part of the long exact sequence:

$$0 \rightarrow H^3(\partial E;\mathbb{Z}) \stackrel{\delta^*}{\rightarrow} H^4(E, \partial E;\mathbb{Z}) \stackrel{f}{\rightarrow} H^4(E;\mathbb{Z}) \rightarrow 0 \tag{A.3}$$

where we have now used the fact that ∂E is simply-connected. The homomorphism δ^* is the so-called connecting homomorphism of the long exact sequence. The last two terms are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}$, respectively. First, note that $H^3(\partial E;\mathbb{Z})$ can contain no torsion. For, suppose that $nx = 0$ for some $x \in H^3(\partial E;\mathbb{Z})$ and $n \in \mathbb{Z}$. Then $n\delta^*(x) = 0$ as δ^* is a homomorphism. But this implies that $\delta^*(x) = 0$ as it is an element of $\mathbb{Z} \oplus \mathbb{Z}$. But now exactness of the sequence implies that x must be trivial. Finally, the rank of $H^3(\partial E;\mathbb{Z})$ is $\dim \mathrm{Im}\delta^* = \dim \ker f = 1$.

Note also that $w_2(\partial E) = \pi^* w_2(B) = 0$, where w_2 denotes the second Stiefel-Whitney class. Thus ∂E is a spin manifold.

We now use Smale's theorem [26]. This states that any simply-connected compact 5-manifold which is spin and has no torsion in the second homology group is diffeomorphic to $S^5 \# l(S^2 \times S^3)$, for some non-negative integer l. Here # denotes connected sum, which means one excises small balls from each manifold, and then glues the remaining sphere boundaries together, with appropriate orientation. Clearly, taking the connected sum of any manifold with S^5 gives back the same manifold – notice that $S^5 \# l(S^2 \times S^3)$, with $l = 0$, is simply S^5. It is straightforward to show[7] that the second homology group of $S^5 \# l(S^2 \times S^3)$ is $\mathbb{Z}^l$. Thus, from the analysis above, we see that the topology of $\partial E = Y^{p,q}$ is $S^2 \times S^3$, for all $p, q \in \mathbb{Z}$ with $(p,q) = 1$.

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