Does noise improve well-posedness of fluid dynamic equations?

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Does noise improve well-posedness of fluid dynamic equations?

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Abstract

We discuss recent progresses in the study of well-posedness for PDEs by means of stochastic perturbation. We show a counterexample related to Burgers equation. We prove existence of L^1-weak solutions for a stochastic continuity equation involving a Hölder continuous vector field. For such equation there is no existence of L^1-weak solutions in the deterministic case.

1 Introduction

The question whether noise improves the well-posedness of certain models of fluid dynamics is natural, on one side by analogy with the effects of noise on deterministic ordinary differential equations, on the other side because noise may break the geometric idealization at the origin of certain phenomena (for instance, one can prepare initial configurations of finitely many vortices that collapse in finite time, but for generic initial conditions this does not happen). This question is relevant since most of the important equations of fluid dynamics are not known to be well-posed (see in particular one of the millennium prize problems [9]).

In spite of this, progresses on relevant models have been slow and only partial. We may identify two directions. The first one is based on the regularity of the transition probabilities, observed when a very non-degenerate *additive noise* is used; since this is not the main subject of this note, but it is related, we include a few words in the appendix. The second one is based on *multiplicative noise*. Its regularizing effect, at the level of distributions, is less clear, but a suitable multiplicative noise produces random rotations in phase space with the effect of redistribution of energy between modes. We do not know how to apply this to Euler or Navier-Stokes equations, but at least one can improve the theory of linear first order PDEs. The relevance of this result for fluid dynamics is still open, but at least it gives us some hope. However, no easy consequence on nonlinear equations can be expected, as counterexamples will show.

The material of Sections 2 and 5 is a review of recent results, while the content of Sections 3 and 4 is new.

2 Multiplicative noise in a transport equation

The transport equation in $\mathbb{R}^d \times [0,T]$

$$\partial_t u(x,t) + b(x,t) \cdot \nabla u(x,t) = 0, \quad (x,t) \in \mathbb{R}^d \times [0,T] \tag{1}$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d$$

driven by a vector field $b : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ which is Lipschitz continuous in x, uniformly in t, can be solved in several ways and in various functions spaces. Remarkable extensions have been obtained by Di Perna and P.L. Lions [7] and Ambrosio [1], who proved existence and uniqueness of weak solutions in $L^\infty\left(\mathbb{R}^d \times [0,T]\right)$, given $u_0 \in L^\infty\left(\mathbb{R}^d\right)$, weak*-continuous in time, when b is only $L^1\left(0,T; W^{1,1}_{loc}\left(\mathbb{R}^d\right)\right)$ or even $L^1\left(0,T; BV_{loc}\left(\mathbb{R}^d\right)\right)$, with $\operatorname{div} b \in L^1\left(0,T; L^\infty\left(\mathbb{R}^d\right)\right)$.

On the contrary, when b is only Hölder continuous in x, there are counterexamples: in $d=1$ the equation with

$$b(x) = 2\mathrm{sign}(x)\sqrt{|x| \wedge 1}, \quad u_0 = 1_{[0,\infty)} \tag{2}$$

has infinitely many weak solutions: solutions, for $(x,t) \in \mathbb{R} \times [0,1]$, are uniquely determined for $x > t^2$ and $x < -t^2$, where characteristics uniquely connect points (x,t) to points of the form $(x',0)$ with $x' \neq 0$. But in the region $|x| < t^2$ there is a high degree of indetermination. We shall see below three examples of solutions.

A multiplicative noise has here a strong regularizing effect. Consider the stochastic transport equation

$$du + b \cdot \nabla u \, dt + \sum_{i=1}^{d} e_i \cdot \nabla u \circ dW^i = 0, \quad u|_{t=0} = u_0 \tag{3}$$

where $(W^i)_{i=1,\ldots,d}$ are independent Brownian motions on a stochastic basis $(\Omega, (\mathcal{F}_t), \mathcal{F}, P)$, the stochastic integrals have to be understood in Stratonovich form, and $(e_i)_{i=1,\ldots,d}$ is the canonical basis of $\mathbb{R}^d$.

Definition 1 *Given $u_0 \in L^\infty(\mathbb{R}^d)$, an L^∞-solution of the Cauchy problem (3) is a function $u \in L^\infty(\mathbb{R}^d \times [0,T] \times \Omega)$ such that for all $f \in C_0^\infty(\mathbb{R}^d)$ the process $t \mapsto \int_{\mathbb{R}^d} u(x,t) f(x) dx$ is a continuous $\mathcal{F}_t$-adapted semimartingale and*

$$\begin{aligned}
&\int_{\mathbb{R}^d} u(x,t) f(x) dx - \int_{\mathbb{R}^d} u_0(x) f(x) dx \\
&= \int_0^t ds \int_{\mathbb{R}^d} u(x,s) \left[b(x,s) \cdot \nabla f(x) + \operatorname{div} b\,(x,s)\, f(x) \right] dx \\
&+ \sum_{i=1}^{d} \int_0^t \left(\int_{\mathbb{R}^d} u(x,s) D_i f(x) dx \right) \circ dW^i(s).
\end{aligned}$$

The following theorem is proved in [11] in a slightly more general form. It includes example (2).

Theorem 2 *If $b \in C\left([0,T]; C_b^\alpha\left(\mathbb{R}^d, \mathbb{R}^d\right)\right)$ for some $\alpha \in (0,1)$, with $\operatorname{div} b \in L^p\left([0,T] \times \mathbb{R}^d\right)$ for some $p \in (2,\infty)$ (or $\operatorname{div} b \in L^1_{loc}\left([0,T] \times \mathbb{R}^d\right)$ if $d=1$), the stochastic transport equation is well-posed in L^∞.*

Note that $C_b^\alpha\left(\mathbb{R}^d, \mathbb{R}^d\right)$ denotes the usual Banach space of all bounded $g : \mathbb{R}^d \to \mathbb{R}^d$, which are (globally) α-Hölder continuous. Moreover $\operatorname{div} b$ is understood in distributional sense.

The previous result is quite technical. We give the idea of the proof in the time-independent divergence-free case:

$$b \in C_b^\alpha\left(\mathbb{R}^d, \mathbb{R}^d\right), \quad \operatorname{div} b = 0.$$

The basic tool is the stochastic flow defined by the SDE

$$d\varphi_t(x) = b(\varphi_t(x))\,dt + dW_t, \quad \varphi_0(x) = x \in \mathbb{R}^d, \tag{4}$$

where $W = (W^i)_{i=1,\ldots,d}$. This equation is not included in the classical theory of stochastic flows (see [17]) but we may use the following trick. In the integral equation

$$\varphi_t(x) = x + \int_0^t b(\varphi_s(x))\,ds + W_t$$

we interpret $\int_0^t b(\varphi_s(x))\,ds$ as the corrector of an Itô formula, the usual (Tanaka like) method to capture the properties of local time of stochastic processes. This way, the x-dependence of $\int_0^t b(\varphi_s(x))\,ds$ is strongly improved.

Define

$$L^b u = \frac{1}{2}\triangle u + b \cdot \nabla u$$

and consider the vector valued elliptic equation in $\mathbb{R}^d$

$$\lambda g - L^b g = b$$

(to be interpreted componentwise). Under the assumption $b \in C_b^\alpha(\mathbb{R}^d, \mathbb{R}^d)$, by Schauder estimates, for any $\lambda > 0$, there exists a unique bounded classical solution g_λ which belongs to $C_b^{2+\alpha}(\mathbb{R}^d, \mathbb{R}^d)$.

Moreover, for λ large enough, we have $\sup_{x\in\mathbb{R}^d} |\nabla g_\lambda(x)| < 1$. We fix such λ and set $g = g_\lambda$.

By a classical Hadamard's theorem the mapping

$$G(x) = x + g(x)$$

is a C^2-diffeomorphism of $\mathbb{R}^d$. Moreover, one proves that G and G^{-1} have bounded first and second derivatives and that the second (Fréchet) derivative D^2G is globally α-Hölder continuous.

By Itô formula, the process

$$\psi_t(x) := G(\varphi_t(x))$$

satisfies

$$d\psi_t(x) = \left[DG\left(G^{-1}(\psi_t(x))\right)\right] dW_t + \lambda g\left(G^{-1}(\psi_t(x))\right) dt.$$

This equation lies in the framework of the classical flow theory (see [17]) and has a differentiable stochastic flow of diffeomorphisms. By a proper rewriting of the previous computations in reverse order one gets the existence of differentiable stochastic flow of diffeomorphisms $\varphi_t(x)$ associated to equation (4).

Remark 3 *Formally the trick looks similar to the Zvonkin transformation, used by several authors to remove an irregular drift (see [24], [22], [16], [23]). Zvonkin approach is based on the transformation* $G_t : \mathbb{R}^d \to \mathbb{R}^d$, $t \in [0,T]$, *solution of the vector valued equation*

$$\frac{\partial G_t}{\partial t} + L^b G_t = 0 \text{ on } [0,T], \quad G_T(x) = x.$$

At time T*, the solution is an isomorphism by definition; one has to prove suitable regularity and invertibility of* G_t *for* $t \in [0,T]$*. Then* $\psi_t(x) := G_t(\varphi_t(x))$ *satisfies*

$$d\psi_t(x) = DG_t\left(G_t^{-1}(\psi_t(x))\right) dW_t.$$

The irregular drift has been removed. This approach, although successful, raises two delicate questions: i) one has to deal with unbounded initial conditions; ii) one has to prove some form of invertibility, which is not obvious. We do not have these difficulties in our approach. Notice also the fact that the motivation of our transformation is different, namely the use of a Tanaka-like method to exploit local time properties. Previous results in the one dimensional case, based on local time, to prove existence of stochastic flows for irregular drift are included in [14].

Having a smooth invertible flow $\varphi_t(x)$, one has to prove that

$$u(x,t) = u_0\left(\varphi_t^{-1}(x)\right)$$

is an L^∞-solution and that any L^∞-solution has this form. It is only here that the additional condition $\operatorname{div} b = 0$ simplifies the proof. To prove that $u(x,t) = u_0\left(\varphi_t^{-1}(x)\right)$ is a solution (this part is the *existence* of solutions for equation (3)), we have to prove that $\int_{\mathbb{R}^d} u(x,t) f(x) dx$ satisfies the equation in Definition 1, for every $f \in C_0^\infty(\mathbb{R}^d)$. From the assumption $\operatorname{div} b = 0$ one can deduce

$$\int_{\mathbb{R}^d} u_0\left(\varphi_t^{-1}(x)\right) f(x) dx = \int_{\mathbb{R}^d} u_0(y) f(\varphi_t(y)) dy.$$

Then all computations are performed by classical Itô formula on $f(\varphi_t(y))$. It is a simple exercise to check the equation.

To prove that any L^∞-solution $u(x,t)$ of equation (3) has the form $u(x,t) = u_0\left(\varphi_t^{-1}(x)\right)$ (*uniqueness* part), we mollify the equation for u by taking suitable test functions $f \in C_0^\infty(\mathbb{R}^d)$ in Definition 1. Call $u_\varepsilon(x,t)$ these regularizations of $u(x,t)$. We apply Itô-Wentzell formula (see [17]) to $u_\varepsilon(\varphi_t(x),t)$ and prove that $du_\varepsilon(\varphi_t(x),t)$ is equal to a term $R_\varepsilon(x,t)\,dt$ which vanishes as $\varepsilon \to 0$. By a rigorous control of this limit we prove that $u(\varphi_t(x),t) = u_0(x)$ and the proof is complete.

However we have to notice that this simple explanation is perhaps misleading. The uniqueness part is the most difficult one (also for deterministic transport equations, see [1], [7]), based on a commutator lemma which establish that the remainder $R_\varepsilon(x,t)$ converges to zero. In our case the Jacobian determinant $\det D\varphi_t(x)$ enters this commutator lemma. In the case $\operatorname{div} b = 0$, $\det D\varphi_t(x)$ is equal to one, so everything becomes very easy. Otherwise we need a control of $\det D\varphi_t(x)$. In [11] we prove that, P-a.s.,

$$\det D\varphi_t(x) \in L^1\left(0,T; W^{1,1}_{loc}\left(\mathbb{R}^d\right)\right).$$

This is a non-trivial fact, since it is related to second derivatives of the flow (and we only assume $b \in C_b^\alpha\left(\mathbb{R}^d,\mathbb{R}^d\right)$). The reason why we have this additional regularity relies on the formula

$$\det D\varphi_t(x) = e^{\int_0^t \operatorname{div} b(\varphi_s(x))ds}$$

where we may use the additional assumption of Theorem 2 and again the Tanaka-like trick.

We conclude the section by mentioning a consequence of the results in [11] which is of independent interest.

Remark 4 *If $u_0 \in C^1\left(\mathbb{R}^d\right)$ then the unique weak solution u verifies $u(t,\cdot) \in C^1\left(\mathbb{R}^d\right)$, $t \in [0,T]$, P-a.s.; moreover u is continuous on $[0,T]\times\mathbb{R}^d$, P-a.s.. Thus blow-up in $C^1\left(\mathbb{R}^d\right)$ cannot occur (contrary to examples in the deterministic case).*

3 Stochastic continuity equation

Given a vector field $b(x,t)$ on $\mathbb{R}^d \times [0,T]$, the continuity equation is the following linear PDE

$$\partial_t p + \operatorname{div}(bp) = 0, \quad p|_{t=0} = p_0.$$

It can be interpreted in a distributional way and it has a meaning even for measure valued functions $t \mapsto p_t$. It is dual to a backward transport equation. If b is so regular that the equation

$$\frac{d\varphi_t(x)}{dt} = b(\varphi_t(x)), \quad \varphi_0(x) = x \tag{5}$$

defines a flow $\varphi_t(x)$, then, given a probability measure p_0, the image law $p_t := \varphi_t p_0$ solves the continuity equation. An advanced account is given in [1] in the case when b is only $L^1\left(0,T;BV_{loc}\left(\mathbb{R}^d\right)\right)$ with $\operatorname{div} b \in L^1\left(0,T;L^\infty\left(\mathbb{R}^d\right)\right)$.

If b is only Hölder continuous, we may have pathological behaviors. Consider the example

$$b(x) = -2\operatorname{sign}(x)\sqrt{|x| \wedge 1}, \quad p_0 = 1_{[-1/2,1/2]}. \tag{6}$$

The deterministic ODE is simply the time reversal of example (2). Solutions $\varphi_t(x)$ of (5) coalesce as t increase: for instance, $\varphi_1(1) = \varphi_1(-1) = 0$. The only generalized solution p_t of the continuity equation is not a function but a probability measure with a concentrated mass at $x = 0$, equal to the portion of mass of p_0 given to those points that at time t have coalesced at zero. This is an *example of non existence* of L^1-solutions of the deterministic continuity equation. It is dual to the non-uniqueness in L^∞ for the dual backward transport equation (example (2)).

The situation is different under random perturbations. Consider the stochastic continuity equation

$$dp + \operatorname{div}(bp)\,dt + \sum_{i=1}^{d} \operatorname{div}(e_i p) \circ dW^i = 0, \quad p|_{t=0} = p_0. \tag{7}$$

Denote by $L^1_+(\mathbb{R}^d)$ the space of all probability densities on $\mathbb{R}^d$. The following definition requires boundedness of b (assumed below in the theorem).

Definition 5 *Given $p_0 \in L^1_+(\mathbb{R}^d)$, a weak L^1_+-solution of the Cauchy problem (7) is a non-negative measurable function $p : \mathbb{R}^d \times [0,T] \times \Omega \to \mathbb{R}$, such that $p(.,t,\omega) \in L^1_+(\mathbb{R}^d)$, for all $t \in [0,T]$ and P-a.s. $\omega \in \Omega$, $t \mapsto \int_{\mathbb{R}^d} p(x,t)g(x,t)dx$ is integrable for P-a.s. $\omega \in \Omega$ for every bounded continuous function $g : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, for all $f \in C_0^\infty(\mathbb{R}^d)$ the process*

$t \mapsto \int_{\mathbb{R}^d} p(x,t)f(x)dx$ is a continuous adapted semimartingale and

$$\int_{\mathbb{R}^d} p(x,t)f(x)dx - \int_{\mathbb{R}^d} p_0(x)f(x)dx$$
$$= \int_0^t ds \int_{\mathbb{R}^d} p(x,s)b(x,s)\cdot\nabla f(x)dx$$
$$+ \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} p(x,s)D_i f(x)dx\right)\circ dW^i(s).$$

For the purpose of this note we limit ourselves to an existence result, which includes example (6) and thus makes a difference with respect to the deterministic case. Uniqueness, as for the transport equation, requires considerable more work and will be treated elsewhere.

Theorem 6 *If $b \in C([0,T];C_b^\alpha(\mathbb{R}^d,\mathbb{R}^d))$ for some $\alpha\in(0,1)$, then the stochastic continuity equation has at least one weak L^1_+-solution.*

Proof. We know from the previous section that the SDE (4) defines a differentiable invertible stochastic flow $\varphi_t(x)$. Let μ_0 be the probability law on $\mathbb{R}^d$ with density p_0 with respect to Lebesgue measure. Let μ_t be the image probability law of μ_0 under φ_t. Notice that μ_t is a random probability measure. See [3] for general facts about random measures and [10] for some detail related to their use for random continuity equations. Since φ_t is differentiable and invertible, μ_t is absolutely continuous with respect to Lebesgue measure and its density $p(x,t)$ is given by the formula

$$p(x,t) = |\det D\varphi_t^{-1}(x)|\cdot p_0\left(\varphi_t^{-1}(x)\right) \tag{8}$$

and satisfies the identity

$$\int_{\mathbb{R}^d} p(x,t)g(x)dx = \int_{\mathbb{R}^d} p_0(x)\,g(\varphi_t(x))dx \tag{9}$$

for all bounded continuous function $g:\mathbb{R}^d\to\mathbb{R}$. Let us prove that $p(x,t)$ is a weak L^1_+-solution. The measurability in all arguments can be deduced from (8), which could be used to prove additional regularity results that we omit in this note. The property $p(.,t,\omega)\in L^1_+(\mathbb{R}^d)$, for all $t\in[0,T]$ and P-a.s. $\omega\in\Omega$ is true by definition of p. Identity (9) implies one of the conditions of Definition 5 and, restricted to $g=f\in C_0^\infty(\mathbb{R}^d)$, implies the continuous semimartingale property. Finally, again from (9) and Itô formula in Stratonovich form (see [17]) we have

$$\int_{\mathbb{R}^d} p(x,t) f(x) dx = \int_{\mathbb{R}^d} p_0(x) f(\varphi_t(x)) dx = \int_{\mathbb{R}^d} p_0(x) f(x) dx$$

$$+ \int_{\mathbb{R}^d} p_0(x)\, dx \left[\int_0^t \nabla f(\varphi_s(x)) \cdot b(\varphi_s(x), s) ds + \sum_{k=1}^d \int_0^t D_k f(\varphi_s(x)) \circ dW^k(s) \right]$$

and now, in the last expression, the first term is equal to

$$\int_0^t ds \int_{\mathbb{R}^d} p_0(x) \nabla f(\varphi_s(x)) \cdot b(\varphi_s(x), s) dx$$
$$= \int_0^t ds \int_{\mathbb{R}^d} p(x,s) \nabla f(x) \cdot b(x,s) dx;$$

and the second one is equal to

$$\sum_{k=1}^d \int_0^t \left[\int_{\mathbb{R}^d} p_0(x) D_k f(\varphi_t(x)) dx \right] \circ dW^k(s)$$
$$= \sum_{k=1}^d \int_0^t \left[\int_{\mathbb{R}^d} p(x,s) D_k f(x) dx \right] \circ dW^k(s).$$

We have used Fubini theorem both in the classical and stochastic version, taking advantage of the boundedness of all terms except the L^1 function $p_0(x)$. Thus the equation in Definition 5 is satisfied. The proof is complete. ■

The stochastic continuity equation is related to the SPDE studied by [19]. However, in our case we insist on an irregular drift, while the main examples of generalized flows of [19] are related to irregular diffusion terms.

4 A counterexample for Burgers equation

One of the main open problems is the generalization to nonlinear equations, where b depends on u. This is not just a technical generalization. Let us show the difficulties by means of one of the simplest examples, the one-dimensional stochastic Burgers equation

$$\partial_t u + u \partial_x u dt + \partial_x u \circ dW_t = 0, \quad u|_{t=0} = u_0. \tag{10}$$

Another example is given in [11].

Consider the stochastic flow formally associated to (10)

$$d\varphi_t(x) = u(\varphi_t(x), t)\,dt + dW_t.$$

Assume that u is sufficiently regular and $\varphi_t(x)$ is well-defined. Then, by Itô-Wentzell formula (see [17])

$$\begin{aligned} du(\varphi_t(x), t) &= \partial_t u + \partial_x u \circ d\varphi_t \\ &= -u\partial_x u dt - \partial_x u \circ dW_t + u\partial_x u dt + \partial_x u \circ dW_t = 0 \end{aligned}$$

hence

$$u(\varphi_t(x), t) = u_0(x).$$

But this implies that the equation for the flow is

$$d\varphi_t(x) = u_0(x)\,dt + dW_t.$$

The drift is a direct function of the initial position.

Let us take the Lipschitz continuous initial condition

$$u_0(x) = \begin{cases} 1 & \text{for} \quad x \le 0 \\ 1-x & \text{for} \quad 0 \le x \le 1 \\ 0 & \text{for} \quad x \ge 1 \end{cases}.$$

In the deterministic case

$$\partial_t u + u\partial_x u dt = 0, \quad u|_{t=0} = u_0$$

one easily shows that a unique Lipschitz continuous solution exists until time $t = 1$, but then characteristics meet and a discontinuity emerges (see for instance [18]).

In the stochastic case, the flow is given by

$$\begin{aligned} &\varphi_t(x) = x + t + W_t \quad \text{for } x \le 0 \\ &\varphi_t(x) = x + W_t \quad \text{for } x \ge 1 \end{aligned}$$

hence $\varphi_1(0) = \varphi_1(1)$, exactly as in the deterministic case. The effect of the noise is just a background space translation. This kind of noise cannot improve the well-posedness and regularity theory.

5 Zero noise limit, superposition solutions

In some one-dimensional example of linear transport equation, it is possible to analyze also the zero-noise limit of equation (3). Consider again example (2) and consider the equation

$$d_t u^\varepsilon + b \cdot \partial_x u^\varepsilon dt = \varepsilon \partial_x u^\varepsilon \circ dW_t, \quad u^\varepsilon|_{t=0} = u_0.$$

One can prove (see [2]) that the law μ^ε of the solution u^ε in the space $L^1_{loc}([0,T] \times \mathbb{R})$ weakly converges to the probability measure $\mu = \frac{1}{2}\delta_{u_1} + \frac{1}{2}\delta_{u_2}$ where $u_1(x,t) = 1_{\{x \geq t^2\}}$, $u_2(x,t) = 1_{\{x > -t^2\}}$. The functions u_1 and u_2 are two weak solutions of equation (1). The result is related and based on a similar result for one-dimensional SDE,

Following a terminology used in the finite dimensional case, see [1] (see also [10]), one may call *superposition solutions* of the deterministic Cauchy problem (1) all probability measures μ on $L^\infty([0,T] \times \mathbb{R})$ such that $\mu(C(u_0)) = 1$, where $C(u_0)$ is the set of all weak solutions of (1). One could also call *true* superposition solution a superposition solution which is not a delta Dirac. The previous result states that the limit in law of solutions to the stochastic transport equation, in the example above, is a true superposition solution. If we accept the general viewpoint that the 'physical' objects (solutions, invariant measures, see [8]), in case of non uniqueness, are those obtained in the zero-noise limit, then we see that true superposition solutions are the right objects for certain PDEs.

An intriguing fact (see [2]) is that zero-viscosity solutions may be different: in the example above, the solution of the parabolic PDE

$$d_t u^\varepsilon + b \cdot \partial_x u^\varepsilon dt = \varepsilon \partial_x^2 u^\varepsilon, \quad u^\varepsilon|_{t=0} = u_0$$

converges to the average of u_1 and u_2. It is thus important to identify other criteria to decide which solutions of equation (1) are more relevant.

6 Appendix: additive noise in 3D Navier-Stokes equations

The break-through on this problem is due to Da Prato and Debussche [4], followed by contributions also of the same and other authors, [5], [6], [20],

[12], [13], [21], [15]. The equations considered are the 3D Navier-Stokes equations

$$\frac{\partial u}{\partial t}+u\cdot\nabla u+\nabla p-\nu\triangle u=\xi$$
$$\operatorname{div} u=0$$

(for simplicity on the torus $[0,L]^3$ with periodic boundary conditions) where ξ is a white noise formally of the form

$$\xi\left(x,t\right)=\sum_{i=1}^{\infty}\sigma_i h_i\left(x\right)\frac{dW^i\left(t\right)}{dt}$$

where $\left(h_i\right)_{i\in\mathbb{N}}$ is a complete orthonormal system in H of eigenfunctions of the linear part of the previous equations, H being an L^2 space with suitable additional conditions, $\left(W^i\right)_{i\in\mathbb{N}}$ is a family of independent Brownian motions and the intensities $\left(\sigma_i\right)_{i\in\mathbb{N}}$ decay with suitable power law. Existence of weak (in both the probabilistic and analytic sense) solutions is a classical result, but their uniqueness is still open, as in the deterministic case. However there is a remarkable new result with respect to the deterministic case. We do not specify the space $\mathcal{W}$ of next theorem because it depends on the paper (among those quoted above) and it is a rather technical issue; we just say that it is a suitable subspace of an L^2 space.

Theorem 7 *There exists Markov selections in H and all of them are strongly Feller in $\mathcal{W}$.*

The continuous dependence on initial conditions in $\mathcal{W}$, of the elements of a Markov selection, is a property without anything similar in the deterministic case, where lack of (proof of) uniqueness goes parallel to lack of (proof of) any continuous dependence. Using these tools one can also construct new criteria of uniqueness.

References

[1] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, *Invent. Math.* **158** (2004), no. 2, 227–260.

[2] S. Attanasio, F. Flandoli, Zero-noise solutions of linear transport equations without uniqueness: an example, (2008) preprint.

[3] H. Crauel, *Random Probability Measures on Polish Spaces*, Stochastic Monographs, Vol. 11, Taylor & Francis, London 2002.

[4] G. Da Prato, A. Debussche, Ergodicity for the 3D stochastic Navier-Stokes equations, *J. Math. Pures Appl.* (9) **82** (2003), no. 8, 877–947.

[5] G. Da Prato, A. Debussche, On the martingale problem associated to the 2D and 3D stochastic Navier-Stokes equations, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei* (9) *Mat. Appl.* **19** (2008), no. 3, 247–264.

[6] A. Debussche, C. Odasso, Markov solutions for the 3D stochastic Navier-Stokes equations with state dependent noise, *J. Evol. Equ.* **6** (2006), no. 2, 305–324.

[7] R. J. DiPerna, P. L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* **98** (1989), no. 3, 511–547.

[8] J.-P. Eckmann, D. Ruelle, Ergodic theory of chaos and strange attractors, *Rev. Modern Phys.* **57** (1985), no. 3, part 1, 617–656.

[9] C. L. Fefferman, Existence and smoothness of the Navier-Stokes equations, the millennium prize problems, Clay Math. Inst., Cambridge 2006, 57-67.

[10] F. Flandoli, Remarks on uniqueness and strong solutions to deterministic and stochastic differential equations, to appear on Metrika.

[11] F. Flandoli, M. Gubinelli, E. Priola, Well posedness of the transport equation by stochastic perturbations, (2008) preprint.

[12] F. Flandoli, M. Romito, Regularity of transition semigroups associated to a 3D stochastic Navier-Stokes equation, in: Stochastic differential equations: theory and applications, 263–280, Interdiscip. Math. Sci., 2, World Sci. Publ., Hackensack, NJ, 2007.

[13] F. Flandoli, M. Romito, Markov selections for the 3D stochastic Navier-Stokes equations, *Probab. Theory Related Fields* **140** (2008), no. 3-4, 407–458.

[14] F. Flandoli, F. Russo, Generalized calculus and SDEs with non regular drift, *Stoch. Stoch. Rep.* **72** (2002), no. 1-2, 11–54.

[15] B. Goldys, M. Röckner, X. Zhang, Martingale solutions and Markov selections for stochastic partial differential equations, (2008) to appear.

[16] N. V. Krylov, M. Röckner, Strong solutions of stochastic equations with singular time dependent drift, *Probab. Theory Related Fields* **131** (2005), no. 2, 154–196.

[17] H. Kunita, Stochastic differential equations and stochastic flows of diffeomorphisms, Ecole d'été de probabilités de Saint-Flour, XII—1982, 143–303, Lecture Notes in Math., 1097, Springer, Berlin, 1984.

[18] P. D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973.

[19] Y. Le Jan, O. Raymond, Integration of Brownian vector fields, *Ann. Probab.* **30** (2002), n. 2, 826-873.

[20] C. Odasso, Exponential mixing for the 3D stochastic Navier-Stokes equations, *Comm. Math. Phys.* **270** (2007), no. 1, 109–139.

[21] R. Romito, Analysis of equilibrium states of Markov solutions to the 3D Navier-Stokes equations driven by additive noise, *J. Stat. Phys.* **131** (2008), no. 3, 415-444.

[22] Yu. A., Veretennikov, On strong solution and explicit formulas for solutions of stochastic integral equations, *Math. USSR Sb.* **39** (1981) 387-403.

[23] X. Zhang, Homeomorphic flows for multi-dimensional SDEs with non-Lipschitz coefficients, Stochastic Processes and their Applications **115** (2005) 435448.

[24] Zvonkin A.K., A transformation of the phase space of a diffusion process that removes the drift, *Mat. Sb.* **93** (1) (1974), 129-150.