Uniform approximation of uniformly continuous and bounded functions on Banach spaces

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UNIFORM APPROXIMATION OF UNIFORMLY CONTINUOUS AND BOUNDED FUNCTIONS ON BANACH SPACES

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ABSTRACT: We present some results concerning uniform approximation of uniformly continuous and bounded functions defined on infinite dimensional spaces, by means of smoother functions. To this purpose we use heat semigroup in abstract Wiener spaces. The subject has an interest in view of the treatment of PDE's with infinitely many variables.

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1 Introduction and basic notations

In this paper we consider approximations of uniformly continuous and bounded mappings by smoother mappings between some Banach spaces. Only uniform approximations are considered.

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be real Banach spaces, denote by $\mathcal{C}_b(E, F)$ the Banach space of all uniformly continuous and bounded maps between E and F, endowed with the sup norm:

$$\|f\|_0 \stackrel{\text{def}}{=} \sup_{x \in E} \|f(x)\|_F, \qquad f \in \mathcal{C}_b(E, F).$$

If $F = \mathbb{R}$, we set $\mathcal{C}_b(E, \mathbb{R}) = \mathcal{C}_b(E)$; this convention will be used for other functional spaces as well. This work develops into three parts.

The first (Section 2) gives density results for some subspaces of $\mathcal{C}_b(E, F)$, under suitable assumptions on E and F.

We recall briefly some known results about uniform approximation in Banach spaces (see also the introduction of [3]). First of all we point out that if dim $E < \infty$ then, using mollifiers and convolution with respect to the Lebesgue measure, it is easy to prove that $\mathcal{C}_b^\infty(E, F)$ (the space of all functions bounded together with all their derivatives of any order) is dense in $\mathcal{C}_b(E, F)$.

When E is infinite dimensional the situation is different. Even if $F = \mathbb{R}$, there exist many separable Banach spaces E, for which there is a function $f_0 \in \mathcal{C}_b(E)$ that is not uniformly approximable by Fréchet differentiable functions (for instance, take $E = \mathcal{C}([0, 1]$ endowed with the sup norm and $f_0(x) = \min(1, \|x\|_0)$, $x \in E$), for details see [2] and [4].

However Goodman was able to prove (see [8]) that, given any separable Banach space E, a function $f \in \mathcal{C}_b(E)$ can be approximated by bounded Lipschitz continuous functions which are differentiable in the Hadamard sense. We shall improve this result, showing that the approximating functions have uniformly continuous Hadamard derivatives, in a weak sense.

When E is a Hilbert space, possibly not separable, then the situation is better. Lasry and Lions have proved (see [14]) that $\mathcal{C}_b^{1,1}(E)$ (the space of all bounded Fréchet differentiable functions, having a Lipschitz continuous and bounded Fréchet derivative) is dense in $\mathcal{C}_b(E)$. However a result of Nemirowskii and Semenov (see [17]) implies that $\mathcal{C}_b^2(E)$ (the space of all functions in $\mathcal{C}_b^{1,1}(E)$ having a bounded, uniformly continuous second Fréchet derivative) is not dense even if E is separable. We shall improve the Lasry-Lions theorem, showing that $\mathcal{C}_s^2(E)$ (the space of all functions in $\mathcal{C}_b^{1,1}(E)$ having a weakly uniformly continuous second Hadamard derivative) is dense in $\mathcal{C}_b(E)$.

Approximation results for maps in $\mathcal{C}_b(E, F)$, where F is an infinite dimensional space, are available in literature. For any pair of Hilbert spaces H, K, a result of Valentine (see [20] and also [19]) implies that any function f in $\mathcal{C}_b(H, K)$ can be approximated by a sequence (f_n) of Lipschitz continuous and bounded functions. When H is separable, a theorem of Bogachev (see [3, §2]) implies that it is possible to choose each function f_n having a bounded Hadamard derivative in H. We shall show that each f_n can be choosen having also a weakly uniformly continuous Hadamard derivative.

In general, uniformly continuous functions from a separable Banach space E to a Hilbert space F cannot be approximated by Lipschitz continuous functions (according to [3, Remark 1]). However Hölder approximations of order 1/2 are always possible (see [16]). Other technical results in specific cases (concerning for instance maps between L^p spaces, see [19]) are available.

In the second part (Section 3), we establish a strict link between uniform approximation in $\mathcal{C}_b(E)$ by smooth functions and existence of smooth Urysohn functions on E.

We conclude the paper (Section 4), by proving an approximation result concerning real, bounded mappings on a separable Banach space E, which are uniformly continuous with respect to a locally convex topology weaker than the norm topology. This result implies that whenever E is reflexive, every real, bounded and $\sigma(E, E')$ ([1]) - uniformly continuous function can be approximated by functions in $\mathcal{C}_b^1(E)$ (the space of all functions in $\mathcal{C}_b(E)$ having a bounded, uniformly continuous Fréchet derivative).

Some applications of the density theorems proved here to PDE's with infinitely many variables for functions defined on an infinite dimensional space (see for instance [10] and [13]), will be given in forthcoming papers. .

Let us introduce some notations that will be used in the following.
Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ real Banach spaces. For any $f \in \mathcal{C}_b(E, F)$, we denote by $\omega_f : (0, \infty) \to [0, \infty)$, the continuity modulus of f, i.e.

$$\omega_f(r) \;=\; \sup_{x,y \in E,\, \|x-y\|_E \le r} \|f(x) - f(y)\|_F, \quad r \ge 0.$$

[1]We denote by $E' = \mathcal{L}(E, \mathbb{R})$, the dual topological space of E, and by $\sigma(E, E')$ the weakest topology on E that makes every $l \in E'$ continuous.

Let $\mathcal{L}(E,F)$ be the Banach space of all linear, continuous operators from E to F endowed with the norm

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{\|u\|\le 1} \|Tu\|_F \quad T \in \mathcal{L}(E,F).$$

We shall often use another locally convex topology in $\mathcal{L}(E,F)$: the *strong topology* ([2]). We denote by $\mathcal{L}_s(E,F)$, the space $\mathcal{L}(E,F)$, endowed with the strong topology. Let G be a Banach space, it easy to verify, by definition, that a map

$$T \ : G \ \to \quad \mathcal{L}_s(E,F)$$

is uniformly continuous if and only if $T(\cdot)(u) : G \ \to F$ is uniformly continuous, for any $u \in E$.
With the above notation, we introduce the set $\mathcal{C}_s(G, \mathcal{L}_s(E,F))$ of all uniformly continuous function T from G to $\mathcal{L}_s(E,F)$ such that:

$$\|T\|_0 \stackrel{\text{def}}{=} \sup_{u\in G} \|Tu\|_{\mathcal{L}(E,F)} < \infty. \tag{1.1}$$

In view of the uniform boundedness principle, $T \in \mathcal{C}_s(G, \mathcal{L}_s(E,F))$ if and only if:

$$T(\cdot)(u) \ \in \mathcal{C}_b(G,F), \qquad u \in E.$$

We will need the following straightforward lemma. We give here the simple proof for the reader's convenience.

Lemma 1.1 *Let E, G, F be Banach spaces, then for a map $T \ : G \ \to \mathcal{L}_s(E,F)$ the following conditions are equivalent:*

(i) T belongs to $\mathcal{C}_s(G, \mathcal{L}_s(E,F))$;
(ii) for any compact set K in E, the map $\sup_{u\in K} T(\cdot)(u)$ belongs to $\mathcal{C}_b(G,F)$;
(iii) for any compact set K in E, the map $T(\cdot)(\cdot)$ belongs to $\mathcal{C}_b(G\times K, F)$.

Each above condition implies that the map:

$$T(\cdot)(\cdot) \quad : \ G \times E \ \to \ F \quad \text{is continuous.} \tag{1.2}$$

Proof We only prove that (i) $\Rightarrow$ (iii).
Boundedness of $T(\cdot)(\cdot)$ is clear so we verify uniform continuity. Fix a compact set K in E, then for any $\epsilon > 0$, there exists a finite set $L = \{v_1, \dots v_n\}$ in K such that for $v \in K$ we can find $v_k \in L$ with $\|v - v_k\|_E \le \epsilon$. Take $\delta > 0$ such that $\omega_{T(\cdot)(v_i)} \le \epsilon, \ \ i = 1 \dots n$.
Thus for any $x, y \in G$, with $\|x - y\|_G \le \delta, \quad u, v \in K$ with $\|u - v\|_E \le \epsilon$, we can choose
$v_k \in L$ such that $\|u - v_k\|_E \le \epsilon$ and we get

$$\|T(x)(u) - T(y)(v)\|_F \ \le \ \|T(x)[u - v_k]\|_F \ + \ \|[T(x) - T(y)](v_k)\|_F$$

[2]We define the strong topology by nets. A net $(T_i : i \in I)$ in $\mathcal{L}(E,F)$ converges to $T \in \mathcal{L}(E,F)$, with respect to the strong topology, if for any $v \in E$, $\lim_{i\in I} T_i(v) = T(v)$ in F. (see for instance [22, §IV.7]).

$$+ \|T(y)[v_k - v]\|_F \ \leq \ 2\epsilon \|T\|_0 \ + \ \epsilon.$$

Finally condition (1.2) follows from the following inequality:
$\|T(x)(u) - T(z)(v)\|_F \leq \| \ [T(x) - T(z)](u)\|_F \ + \|T(z)[u-v]\|_F, \ \ u, v \in E, \ x, z \in G.$
∎

Recall that a map $f \ : E \ \to \ F$ is said to be Gâteaux differentiable at the point x of E if there exists $Df(x) \in \mathcal{L}(E,F)$ such that:

$$\lim_{s \to 0^+} \frac{f(x+sv) - f(x)}{s} \ = \ Df(x)(v), \quad v \in E. \tag{1.3}$$

Moreover if for any compact set $K \subset E$ (resp. bounded set $B \subset E$) the limit in (1.3) is uniform in $v \in K$ (resp. in $v \in B$), then $Df(x)$ is said to be the Hadamard (resp. Fréchet) derivative of f at x. We remark that the Hadamard derivative is rarely considered elsewhere despite its many advantages. For a detailed exposition of the subject we refer to [7].

We introduce the following functions spaces:

$$\mathcal{C}_b^{0,1}(E,F) \stackrel{\text{def}}{=} \{ f \in \mathcal{C}_b(E,F), \ \text{ such that } \mathrm{Lip}(f) \stackrel{\text{def}}{=} \sup_{x,y \in E, x \neq y} \frac{\|f(x) - f(y)\|_F}{\|x-y\|_E} < \infty \},$$

$\mathcal{C}_b^1(E,F) \stackrel{\text{def}}{=} \{ \ f \in \mathcal{C}_b(E,F)$, Fréchet differentiable in E, having the Fréchet derivative $\ Df \in \mathcal{C}_b(E, \mathcal{L}(E,F)) \ \}$,

$\mathcal{C}_b^{1,1}(E) \stackrel{\text{def}}{=} \{ f \in \mathcal{C}_b^1(E)$, having the Fréchet derivative $Df \in \mathcal{C}_b^{0,1}(E,E') \ \}$,

$\mathcal{C}_b^2(E) \stackrel{\text{def}}{=} \{ f \in \mathcal{C}_b^1(E)$, twise Fréchet differentiable in E, having the second Fréchet derivative $\ D^2 f \in \mathcal{C}_b(E, \mathcal{L}(E,E')) \}$,

$\mathcal{C}_s^1(E,F) \stackrel{\text{def}}{=} \{ \ f \in \mathcal{C}_b(E,F)$, Hadamard differentiable in E, having the Hadamard derivative $\ Df \in \mathcal{C}_s(E, \mathcal{L}_s(E,F)) \ \}$,

$\mathcal{C}_s^2(E) \stackrel{\text{def}}{=} \{ f \in \mathcal{C}_b^{1,1}(E)$, having the second Hadamard derivative $D^2 f(x)$, at any point $x \in E$ and $D^2 f \in \ \mathcal{C}_s(E, \mathcal{L}_s(E,E') \ \}$.

Now we need the notion of abstract Wiener space. For details see [9], [10] and [13].

Let $(E, \|\cdot\|_E)$ be a separable Banach space and $(H, \|\cdot\|_H)$ be a separable Hilbert space, such that $H \hookrightarrow E$ (i.e. H continuously and densely embedded in E). We identify H with H', so that we have the following inclusions:

$$E' \ \hookrightarrow \ H' \ \simeq \ H \ \hookrightarrow \ E. \tag{1.4}$$

Let p_1 be a gaussian measure on $\mathcal{B}(E)$ ([3]) such that $\mathrm{supp}(p_1) = \ E$ and such that each $l \in E'$ be normally distributed with mean 0 and covariance $\|l\|_H^2$ with respect to p_1.

[3] $\mathcal{B}(E)$ denotes the σ-algebra of all Borel sets in E.

The triple (E, H, p_1), is called an *abstract Wiener space*. We use the following result (see [13, §I.4.4]): for any separable Banach space E, there exists a separable Hilbert space H and a gaussian measure p_1 such that (E, H, p_1) is an abstract Wiener space.
In (E, H, p_1) is defined the family of gaussian measures $(p_t)_{t>0}$ on $\mathcal{B}(E)$,

$$p_t(B) \ = \ p_1(\frac{B}{\sqrt{t}}), \qquad B \in \mathcal{B}(E).$$

It is easy to verify that for any $t > 0$, $\text{supp}(p_t) = E$ and each $l \in E'$ is normally distributed with mean 0 and covariance $t\|l\|_H^2$ with respect to p_t.

If E' is equipped with the norm inherited from H, for any p_t, the linear map: $R^t : E' \to L^2(E, p_t)$ is an isometry (up to the constant t). Therefore this map extends uniquely to an isometry, denoted again by R^t,

$$h \ \mapsto \ R_h^t, \quad h \in H, \tag{1.5}$$

from H to $L^2(E, p_t)$. In the sequel we write R instead of R^t to semplify the notation.
We define for any $x \in E$, the gaussian measure $p_t(x, \cdot)$, $p_t(x, B) \ = \ p_t(B - x)$, $B \in \mathcal{B}(E)$. The Cameron - Martin theorem states that: for any $t > 0$, the gaussian measures $p_t = \ p_t(0, \cdot)$ and $p_t(z, \cdot)$, $z \in E$ are either equivalent or singular. They are equivalent if and only if $z \in H$; moreover if $h \in H$, the Radon - Nikodym derivative of $p_t(h, \cdot)$, with respect to p_t is given by the formula:

$$\frac{dp_t(h, \cdot)}{dp_t}(x) \ = \ \exp\Big[- \frac{1}{2t}\|h\|_H^2 \ + \ \frac{1}{t} \ R_h(x) \Big], \quad x \in E, \ \ p_t - \text{a.e.}. \tag{1.6}$$

When E is a Hilbert space, with scalar product $< \cdot, \cdot >_E$, p_t admits a covariance operator tQ on E,

$$< tQ\, u, v > = \int_E < u, y >< v, y > \ p_t(dy), \quad t > 0,$$

where Q is a positive self-adjoint trace class operator in E. We take $H \ = \ Q^{1/2}E$ endowed with scalar product:

$$< x, y >_H \ \stackrel{\text{def}}{=} \ < Q^{-1/2}x, Q^{-1/2}y >_E, \quad x, y \in H.$$

2 Density theorems

We start with a Lemma. Statement (a) is essentially known (see [3, Lemma 1] and [7, §4.2]). However we give the proof for the sake of completeness.

Lemma 2.1 *Let* $(E, \ \|\cdot\|_E)$, $(F, \ \|\cdot\|_F)$ *be Banach spaces,* $D \subset E$ *a dense linear subspace and let* $f \in \mathcal{C}_b^{0,1}(E, F)$. *Suppose that:*

(i) for any $x \in E$, $v \in D$, *there exists:*

$$\lim_{s \to 0^+} \ \frac{f(x + sv) - f(x)}{s} \ = \ A(x, v) \in F;$$

(ii) for any fixed $x \in E$, $A(x, \ \cdot)$ *is linear from* D *in* F.
Then it holds:

(a) f is Hadamard differentiable in E and $\|Df(x)\|_{\mathcal{L}(E,F)} \leq \mathrm{Lip}(f)$.
If moreover

(iii) the limit in (i) is uniform in $x \in E$,

then we have

(b) $f \in \mathcal{C}^1_s(E,F)$

Proof Assume that (i) and (ii) hold, then fix $x, v \in E$ and take $(v_n)_{n\geq 1} \subset D$ such that $v_n \to v$. Consider the mappings:

$$\psi_n,\ \psi\ :\ (0,1] \to F, \qquad n \geq 1,$$

$$\psi_n(s) \stackrel{\text{def}}{=} \frac{f(x+sv_n) - f(x)}{s}, \quad \psi(s) \stackrel{\text{def}}{=} \frac{f(x+sv) - f(x)}{s}, \quad s \in (0,1].$$

It turns out that $\psi_n \to \psi$ uniformly in $s \in (0,1]$. Indeed

$$\sup_{s\in(0,1]} \|\psi_n(s) - \psi(s)\|_F \leq \mathrm{Lip}(f)\, \|v_n - v\|_E \quad \to 0,$$

as $n \to \infty$. By hypothesis (i), there exists $\lim_{s\to 0^+} \psi_n(s) = A(x, v_n)$ in F, so we can deduce that there exists

$$B(x,v) \stackrel{\text{def}}{=} \lim_{s\to 0^+} \psi(s) = \lim_{n\to\infty} A(x, v_n), \quad x, v \in E.$$

Now, for any $x \in E$, $B(x,\cdot)$ is linear from E into F (by (ii)) and it is continuous since we have, for any $x,\ v \in E,\ s \in (0,1]$,

$$\|\psi(s)\|_F \leq \mathrm{Lip}(f)\, \|v\|_E, \quad \text{and so} \qquad \|B(x,v)\|_F \leq \mathrm{Lip}(f)\, \|v\|_E.$$

Thus the Gâteaux differentiability of f in E is proved. Denote by Df the Gâteaux derivative of f. Now we check that f is also Hadamard differentiable in E. Fix $x \in E$, a compact set $K \subset E$ and consider the mappings,

$\eta_s :\ K \to F, \quad s \in (0,1]$,

$$\eta_s(v) \stackrel{\text{def}}{=} \frac{f(x+sv) - f(x)}{s}, \qquad v \in K.$$

We show that for any sequence $(s_n) \subset (0,1]$ such that $s_n \to 0$, there exists

$$\lim_{n\to\infty} \sup_{v\in K} \|\eta_{s_n}(v) - Df(x)(v)\|_F = 0 \quad \text{uniformly in } v \in K. \tag{2.1}$$

Take any subsequence (s^1_n) of (s_n). Since f is Lipschitz continuous, $(\eta_{s^1_n})$ is an equicontinuous sequence of mappings in $\mathcal{C}_b(K,F)$. Moreover for any $v \in K$, the sequence $\{\eta_{s^1_n}(v)\}$ is relatively compact in F, since there exists $\lim_{s\to 0^+} [\,\eta_s(v) - Df(x)(v)\,] = 0$ in F.

Therefore applying the Arzela - Ascoli theorem (see for instance [1, §A8.5]) we can deduce that there is a subsequence (s^2_n) of s^1_n such that

$$\lim_{n\to\infty} \sup_{v\in K} \|\eta_{s^2_n}(v) - Df(x)(v)\|_F = 0.$$

In this way we have proved formula (2.1). The Hadamard differentiability at $x \in E$ is proved.

Assume now that also (iii) holds, then fix $v \in E$ and take $(v_n) \subset D$, such that $v_n \to v$ in E. Define the maps:
$\phi_n,\ \phi:\ (0,1] \to \mathcal{C}_b(E,F), \quad n \geq 1$ such that:

$$\phi_n(s) \stackrel{\text{def}}{=} \frac{f(\cdot + sv_n) - f(\cdot)}{s}, \quad \phi(s) \stackrel{\text{def}}{=} \frac{f(\cdot + sv) - f(\cdot)}{s}, \quad s \in (0,1]. \tag{2.2}$$

Arguing as for (ψ_n), we get that $\phi_n \to \phi$ uniformly in $s \in (0,1]$. Indeed

$$\lim_{n\to\infty} \sup_{s\in(0,1]} \|\phi_n(s) - \phi(s)\|_{\mathcal{C}_b(E,F)} = \lim_{n\to\infty} \sup_{s\in(0,1]} \sup_{x\in E} \|\frac{f(x+ sv_n) - f(x+ sv)}{s}\|_F \leq$$

$$\leq \mathrm{Lip}(f) \lim_{n\to\infty} \|v_n - v\|_E = 0.$$

By hypothesis (iii), fixing $n \geq 1$, we have that $\lim_{s\to 0^+} \phi_n(s) = A(\cdot, v_n)$, uniformly in $x \in E$. Consequently

$$A(\cdot, v_n) \in \mathcal{C}_b(E,F), \quad n \geq 1.$$

Hence the following limit exists in $\mathcal{C}_b(E,F)$,

$$\lim_{s\to 0^+} \phi(s) = Df(\cdot)(v) = \lim_{n\to\infty} A(\cdot, v_n). \tag{2.3}$$

We have just proved that for any $v \in E$, $Df(\cdot)(v) \in \mathcal{C}_b(E,F)$, so the conclusion follows. ■

We present our first density result. Proof uses as tool the heat semigroup on abstract Wiener space as in Goodman's theorem.

Theorem 2.2 *Let E be a separable Banach space, then $\mathcal{C}^1_s(E)$ is dense in $\mathcal{C}_b(E)$.*

Proof We shall use the fact that $\mathcal{C}^{0,1}_b(E)$ is dense in $\mathcal{C}_b(E)$ (see for instance [10, §3.2.1].
Moreover we consider E as an abstract Wiener space (E, H, p_1). Gross has proved (see [10]) that if we set:

$$O_t f(x) = \int_E f(x+y)\, p_t(dy), \quad f \in \mathcal{C}_b(E),\ x \in E\ t > 0, \tag{2.4}$$

then O_t is a strongly continuous linear semigroup on $\mathcal{C}_b(E)$. We call it *the heat semigroup* in $\mathcal{C}_b(E)$.
We show that

$$O_t\,(\,\mathcal{C}^{0,1}_b(E)\,) \subset \mathcal{C}^1_s(E), \quad t > 0. \tag{2.5}$$

By this fact the assertion follows. Indeed for any $g \in \mathcal{C}_b(E)$, for any $\epsilon > 0$ there exists $h \in \mathcal{C}^{0,1}_b(E)$ such that $\|g - h\|_0 \leq \epsilon$. Therefore the inequality

$$\|g - O_t h\|_0 \leq \|g - h\|_0 + \|h - O_t h\|_0, \tag{2.6}$$

allows us to conclude, using that O_t is strongly continuous.

Fix $f \in \mathcal{C}_b^{0,1}(E)$ and $t > 0$. We shall apply Lemma 2.1, using the density of H in E. By the Cameron-Martin formula (1.6), see also [10, §9] for details, we know that for any $h \in H,\ x \in E,\ \ t > 0$, there exists

$$\lim_{s \to 0^+} \frac{O_t f(x + sh) - O_t f(x)}{s}$$

$$= \lim_{s \to 0^+} \frac{1}{s} \int_E f(x+y) \Big[\exp\Big(- \frac{s^2}{2t} \|h\|_H^2 + \frac{s}{t} R_h(y) \Big) - 1 \Big] p_t(dy) \tag{2.7}$$

$$= \frac{1}{t} \int_E f(x+y)\, R_h(y)\, p_t(dy),$$

where R_h was defined in the introduction (take into account that R_h is a gaussian random variable with respect to p_t and so in particular $\exp(|R_h|)$ is p_t - integrable). Formula (2.7) also holds when f is only a Borel bounded function.

We prove that for any $h \in H$, the limit in (2.7) is uniform in $x \in E$ as well.

$$\lim_{s \to 0^+} \sup_{x \in E} \Big| \frac{O_t f(x+sh) - O_t f(x)}{s} - \frac{1}{t} \int_E f(x+y) R_h(y) p_t(dy) \Big|$$

$$\le \|f\|_0 \lim_{s \to 0^+} \int_E \Big| \frac{1}{s} \Big[\exp\Big(- \frac{s^2}{2t} \|h\|_H^2 + \frac{s}{t} R_h(y) \Big) - 1 \Big] - \frac{1}{t} R_h(y) \Big| \, p_t(dy) = 0. \tag{2.8}$$

We have applied the dominated convergence theorem.

Now since $f \in \mathcal{C}_b^{0,1}(E)$ we have, clearly, that $O_t f \in \mathcal{C}_b^{0,1}(E)$, for $t > 0$ as well. By Lemma 2.1 and by (2.8), we get that there exists the Hadamard derivative $DO_t f \in \mathcal{C}_s(E, E'),\ \ t > 0$. Thus $O_t f \in \mathcal{C}_s^1(E)$ and the proof is complete. ■

Let $f \in \mathcal{C}_b^{0,1}(E)$, by the above proof and applying formula (2.3) we also obtain an explicit formula for the Hadamard derivative $DO_t f,\ \ t > 0$.
For any $u \in E$, take any sequence $(h_n) \subset H$, such that $h_n \to u$, then we find

$$DO_t f(x)(u) = \lim_{n \to \infty} \frac{1}{t} \int_E f(x+y)\, R_{h_n}(y)\, p_t(dy), \quad x \in E,\ t > 0. \tag{2.9}$$

and the limit is uniform in $x \in E$.

Remark 2.3 V. Goodman actually proved that for any separable Banach space E, $Q^1(E)$ is dense in $\mathcal{C}_b(E)$ (see [8]).

To introduce the space $Q^1(E)$, he used the notion of "quasi-differentiability" that we briefly recall. A function $f : E \to \mathbb{R}$ is said to be quasi-differentiable at $x \in E$, if there exists $\eta_x \in E'$ such that for each function g from a neighbourhood of 0 in $\mathbb{R}$ to E, which takes value x at 0 and has derivative at 0, the function $f \circ g$ has a derivative at 0 equal to $\eta_x(g'(0))$. The functional η_x is then called the quasi-derivative of f at x.

$Q^1(E)$ is the space of all functions f in $\mathcal{C}_b(E)$, that are quasi-differentiable in each point of E with quasi-derivative Df bounded and such that

$$Df(\cdot)(\cdot) \ : E \times E \ \to \mathbb{R} \text{ is continuous.}$$

Now it is know (see [7](§4.2.8)) that quasi-differentiability is equivalent to the Hadamard differentiability. Further invoking Lemma 1.1, assertion (1.2), we can state that
$\mathcal{C}^1_s(E) \subset Q^1(E)$; moreover the inclusion is strict even when $E = \mathbb{R}$ as easily can be checked.

Remark 2.4 We point out that heat semigroup on $\mathcal{C}_b(E)$ does not help us to uniformly aproximate any function $f \in \mathcal{C}_b(E)$ by mappings in $\mathcal{C}^{0,1}_b(E)$.

Indeed we prove that for any $t > 0$ we have:

$$O_t(\, \mathcal{C}_b(E)\,) \not\subset \mathcal{C}^{0,1}_b(E).$$

Denote by $\mathcal{A}$ the infinitesimal generator of O_t in $\mathcal{C}_b(E)$. Gross has proved ([10, Theorem 3]) that for any $g \in \mathcal{C}^{0,1}_b(E)$ then $O_t g \in \mathrm{Dom}(\mathcal{A})$, for all $t > 0$.

Assume, by contradiction, that $O_{t_0}(\mathcal{C}_b(E)) \subset \mathcal{C}^{0,1}_b(E)$ for such a $t_0 > 0$. Then we deduce that

$$O_{t_0+\epsilon} f \;=\; O_\epsilon O_{t_0} f \,\in \mathrm{Dom}(\mathcal{A}), \quad f \in \mathcal{C}_b(E).$$

But this is not true, since O_t is not *eventually differentiable* ([4]), see [21], [6], [11].

To prove other density theorems, note that the heat semigroup can be extended in a natural way to $\mathcal{C}_b(E, F)$ where $E = (E, H, p_1)$ is an abstract Wiener space and F is any Banach space. We denote by $\hat{O}_t$ this semigroup.

$$\hat{O}_t f(x) \stackrel{\text{def}}{=} \int_E f(x + y) p_t(dy), \quad f \in \mathcal{C}_b(E, F), \;\; t > 0, \;\; x \in E, \tag{2.10}$$

where the integral is in Bochner's sense, using the fact that the E is separable and so the range of f is also separable in F for any $f \in \mathcal{C}_b(E, F)$, (for instance see [5, §1.1]).

It is possible to prove that $\hat{O}_t$ is a strongly continuous semigroup on $\mathcal{C}_b(E, F)$ in the same way as for O_t in $\mathcal{C}_b(E)$ (see [10]).

Theorem 2.5 *Let E , K be Hilbert spaces and assume that E is separable. Then $\mathcal{C}^1_s(E, K)$ is dense in $\mathcal{C}_b(E, K)$.*

Proof We shall use the fact that $\mathcal{C}^{0,1}_b(E, K)$ is dense in $\mathcal{C}_b(E, K)$ (see [20] and [19]).

Moreover we consider E as an abstract Wiener space (E, H, p_1). Denoting by $\hat{O}_t$ the heat semigroup in $\mathcal{C}_b(E, K)$ (defined by (2.10)), we argue as in proof of Theorem 2.2.

Any map $f \in \mathcal{C}_b(E, K)$ can be pointwise approximated by a sequence of simple functions (f_n) such that $\|f_n(x) - f(x)\|_K \downarrow 0$, for any $x \in E$ (see [5, Lemma 1.1]). Thus, using the Cameron-Martin formula, it holds:

$$\hat{O}_t f(x+h) \;=\; \int_E f(x+y)\; \exp\Big(-\frac{1}{2t}\|h\|^2_H + \frac{1}{t} R_h(y)\Big) p_t(dy), \quad x \in E, \quad h \in H, \;\; t > 0.$$

[4] Let P_t be a strongly continuous linear semigroup on a Banach space X. P_t is said to be *eventually differentiable* if there exists $\hat{t} \geq 0$ such that for any $x \in X$, the map $t \mapsto P_t x$, from $(\hat{t}, \infty)$ to X is differentiable.

Therefore formula (2.8) also holds for $\hat{O}_t$ and $f \in \mathcal{C}_b(E, K)$. Now applying Lemma 2.1 we can prove that

$$\hat{O}_t(\,\mathcal{C}_b^{0,1}(E, K)\,) \subset \mathcal{C}_s^1(E, K), \qquad t > 0. \tag{2.11}$$

Arguing as for formula (2.6) we can conclude. ∎

It is known that for any Hilbert space E, $\mathcal{C}_b^{1,1}(E)$ is dense in $\mathcal{C}_b(E)$ (see [14]). We use this result to prove the following theorem.

Theorem 2.6 *Let E, be a separable Hilbert space, with inner product $< \cdot, \cdot >$, then $\mathcal{C}_s^2(E)$ is dense in $\mathcal{C}_b(E)$.*

Proof We consider E as an abstract Wiener space (E, H, p_1). Let us indentify E with E'. Consider the following two heat semigroups:

O_t on $\mathcal{C}_b(E)$ and $\hat{O}_t$ on $\mathcal{C}_b(E, E)$,

both defined by integrals with respect to p_t (see (2.10)).

Arguing as in the proof of Theorem 2.2, formula (2.6) to prove our assertion it is enough to show that for any $f \in \mathcal{C}_b^{1,1}(E)$ and $t > 0$, we have $O_t f \in \mathcal{C}_s^2(E)$. To this end fix $f \in \mathcal{C}_b^{1,1}(E)$ and fix $t > 0$.

First we deduce the Gâteaux differentiability of $O_t f$ in E. Denote by Df the Fréchet derivative of f. Let $x \in E$, for any $v \in E$, $s \in (0, 1]$ we have

$$\lim_{s \to 0^+} \frac{O_t f(x + sv) - O_t f(x)}{s} = \int_E < Df(x + y), v > \; p_t(dy),$$

since f is a Lipschitz continuous map and so we can pass to the limit under the integral, by the dominated convergence theorem. In this way we obtain that there exists the Gâteaux derivative: $D_G O_t f(x)$ at any $x \in E$ and further that it holds for every $v \in E$

$$< D_G O_t f(x), v > = O_t(< Df(\cdot), v >)\,(x) = < \hat{O}_t(Df)(x), v > \text{ and so}$$

$$D_G O_t f(x) = \hat{O}_t Df(x), \quad x \in E, \quad . \tag{2.12}$$

Now $Df \in \mathcal{C}_b(E, E)$ implies that $\hat{O}_t Df \in \mathcal{C}_b(E, E)$, with $\omega_{\hat{O}_t Df} \leq \omega_{Df}$. Invoking a well known result about differentiability we can deduce that $O_t f$ is also Fréchet differentiable in E and (2.12) holds with the Gâteaux derivative replaced by the Fréchet derivative.

Remark that, by the assumption, $Df \in \mathcal{C}_b^{0,1}(E, E)$. Applying formula (2.11) in the proof of Theorem 2.5 (with $K = E$) we have that $\hat{O}_t Df \in \mathcal{C}_s^1(E, E)$. This is equivalent, by (2.12), to state that there exists the second Hadamard derivative of $O_t f$ and that it is in $\mathcal{C}_s(E, \mathcal{L}_s(E))$. Thus the proof is complete. ∎

Remark 2.7 We want to show how the heat semigroup can be used to improve other approximation results.

In [12] it is mentioned, without proof, the following Tsar'kov result:

let E, K be Hilbert spaces, then any uniformly continuous map $g : E \to K$ is uniformly approximable by Fréchet differentiable maps $f : H \to K$, having a bounded derivative.

We define the space $\mathcal{C}^1_{s,F}(E,K) \ = \ \{f \in \mathcal{C}^1_s(E,K)$ which are Fréchet differentiable in $E\}$. Invoking the Tsar'kov theorem we can prove:

let E, K be Hilbert space and assume that E be separable, then $\mathcal{C}^1_{s,F}(E,K)$ is dense in $\mathcal{C}_b(E,K)$.

Let $\hat{O}_t$ the heat semigroup in $\mathcal{C}_b(E,K)$ as in Theorem 2.5. Fix $g \in \mathcal{C}_b(E,K)$. For any $\epsilon > 0$, by the above Tsar'kov theorem, we can choose a function $f \in \mathcal{C}^{0,1}_b(E,K)$ that is Fréchet differentiable in E (with a bounded derivative) and such that $\|g - f\|_0 \le \epsilon$.

Using the inequality: $\|g - \hat{O}_t f\|_0 \le \|g - f\|_0 + \|f - \hat{O}_t f\|_0, \quad t > 0$, to get the assertion it is enough to verify that $\hat{O}_t f \in \mathcal{C}^1_{s,F}(E,K)$ for any $t > 0$.

Fix $t > 0$, by formula 2.11 we know that $\hat{O}_t f \in \mathcal{C}^1_s(E,K)$. Thus we only prove that $\hat{O}_t f$ is Fréchet differentiable in E and it holds:

$$D\hat{O}_t f(x)(v) \ = \ \hat{O}_t[Df(\cdot)(v)](x), \quad x, \ v \in E, \tag{2.13}$$

where $D\hat{O}_t f$ and Df are Fréchet derivatives. To establish (2.13) we can not argue as for proving (2.12), since Df is not supposed to be continuous. However fix $x \in E$ and let C be the unit closed ball of E. The assertion (2.13) is equivalent to prove:

$$\lim_{s\to 0^+} \sup_{v\in C} \Big| \frac{\hat{O}_t f(x+sv) - \hat{O}_t f(x)}{s} - \int_E Df(x+y)(v) p_t(dy) \Big| = \ 0.$$

We define $\Theta \, : \, (0,1] \times C \times E \ \to \ \mathbb{R}$, for any $s \in (0,1], \ v \in C, \ y \in E$,

$$\Theta(s,v,y) \ = \ \| \frac{f(x+y+sv) - f(x+y)}{s} \ - \ Df(x+y)(v) \|_K.$$

With this notation to get (2.13), it is sufficient to verify that

$$\lim_{s\to 0^+} \sup_{v\in C} \int_E \ |\Theta(s,v,y)| \ p_t(dy) = \ 0.$$

To this purpose, take a countable dense set D in C. Then $\Theta(s,\cdot,y)$ is uniformly continuous for any $s \in (0,1], \ y \in E$ so that we have

$$\sup_{v\in D} |\Theta(s,v,y)| \ = \ \sup_{v\in C} |\Theta(s,v,y)|, \quad s \in (0,1], \ y \in E.$$

Now for any $s \in (0,1], \ v \in C, \ \Theta(s,v,\cdot)$ is a Borel function and so $\sup_{v\in D} |\Theta(s,v,y)|$ is still Borel, since D is countable.

Moreover we have $|\Theta(s,v,y)| \le \ 2\|Df\|_0, \quad s \in (0,1], \ v \in C, \ y \in E$. Now by the inequality:

$$\sup_{v\in C} \int_E \ |\Theta(s,v,y)| p_t(dy) \ \le \ \int_E \ \sup_{v\in D} |\Theta(s,v,y)| \ p_t(dy),$$

as $s \to 0^+$ we get 0, using the Lebesgue's theorem. Thus (2.13) follows.

Remark 2.8 By the previous density theorems we can also approximate mappings defined only on a subset of a separable Banach space E.

Let S be any subset E. We can define the Banach space $\mathcal{C}_b(S)$, endowed with the sup-norm, in an obvious way. We recall the following McShane result (see [15]).

Let (M,ρ) a metric space with metric ρ and A a subset of M, then any map

$f: A \to \mathbb{R}$ uniformly continuous and bounded can be extended to a map $\hat{f}: M \to \mathbb{R}$ (i.e. $\hat{f}(x) = f(x), \ \ x \in A$) uniformly continuous and bounded, having the same bounds and the same continuity modulus of f.

As a direct consequence of McShane's theorem and of Theorems 2.2 and 2.6, we find:

(i) the restrictions to S of functions which are in $\mathcal{C}^1_s(E)$ are dense in $\mathcal{C}_b(S)$;

(ii) if E is a separable Hilbert space, then the restrictions to S of functions that are in $\mathcal{C}^2_s(E)$ are dense in $\mathcal{C}_b(S)$.

Let Ω be an open set of a separable Hilbert space E. We point out that $\mathcal{C}^2_b(\Omega)$ ([5]) is not dense in $\mathcal{C}_b(\Omega)$.

This is a consequence of a A.S. Nemirovskii and S. M. Semenov's result (see [17]). They have constructed a map $f_0 \in \mathcal{C}_b(B)$, where B denotes the unit open ball of E, such that f_0 is not uniformly approximable by maps in $\mathcal{C}^2_b(B)$.

Take an open ball $\hat{B} \in \Omega$ and easily construct a map $\hat{f} \in \mathcal{C}_b(\hat{B})$, using the map f_0, that is not approximable by maps in $\mathcal{C}^2_b(\hat{B})$. Now using the above McShane result we can extend $\hat{f}$ to a map $g \in \mathcal{C}_b(\Omega)$ that is not approximable by maps in $\mathcal{C}^2_b(\Omega)$.

3 Uniform approximation and Urysohn maps

We introduce a connection between uniform approximation of real, uniformly continuous and bounded functions and existence of Urysohn maps. Let (M, d) a metric space with metric d and denote $\mathcal{C}_b(M)$ the Banach space of all real, bounded, uniformly continuous mappings endowed with the sup norm.

Two non empty closed sets A, B are said to be *separated* if $\inf_{x\in A, y \in B} \ d(x,y) > 0$. A function $f \in \mathcal{C}_b(M)$ is said to be an *Urysohn function* for the pair (A, B) of separated closed sets if

$$f : M \to [0,1], \quad f(x) = 1 \text{ for any } x \in A, \quad f(y) = 0 \ \text{ for any } y \in B.$$

Remark that for any separated closed sets A, B in M, the function

$$f_{A,B} \ = \ \frac{d(x,B)}{d(x,A) + d(x,B)}, \quad x \in M,$$

is a Lipschitz continuous Urysohn function for (A, B).

It is known (see for instance [18]) that uniform approximation of functions which belong to $\mathcal{C}_b(E)$, where E is a Banach space, by smooth functions implies existence of smooth Urysohn functions. We present the following two propositions for the sake of completeness.

Proposition 3.1 *Let E be a separable Hilbert space, for any separated closed sets A, B, there exists a Urysohn map g for (A, B) that is in $\mathcal{C}^2_s(H)$.*

[5] $\mathcal{C}^2_b(\Omega)$ stand for the space of all maps f in $\mathcal{C}_b(\Omega)$, having a first bounded Fréchet derivative and a second Fréchet derivative, $D^2 f : \Omega \to \mathcal{L}(E)$ that is bounded and uniformly continuous.

Proof Take a Lipschitz continuous Urysohn map f for (A,B), then by Theorem 2.5, there exists a map $h \in \mathcal{C}^2_s(E)$ such that: $h(x) > 3/4$ when $x \in A$ and $h(y) < 1/4$ when $y \in B$. Consider a function $j \in \mathcal{C}^\infty_b(\mathbb{R})$ such that:
$j : \mathbb{R} \to [0,1]$, $j(s) = 1$ for $|s| \geq 3/4$, $j(s) = 0$, $|s| \leq 1/4$.
Finally define $g = j \circ h$, $g \in \mathcal{C}^2_s(E)$ is the map looked for. ∎

In the same way we can get the next proposition.

Proposition 3.2 *Let E be a separable Banach space, for any separated closed sets A, B, there exists a Urysohn map g for (A,B) that is in $\mathcal{C}^1_s(E)$.*

Thus on one hand it is clear the link between uniform approximation and existence Urysohn functions. On the other hand there is a connection as well. The following theorem shows that existence of regular Urysohn functions implies existence of regular uniform approximations.

Theorem 3.3 *Let (M,d) be a metric space, with metric d, and $\mathcal{S}(M)$ a linear subspace of $\mathcal{C}_b(M)$. If for any pair of separated closed sets A, B there exists a Urysohn function $f \in \mathcal{S}(M)$ for (A,B), then $\mathcal{S}(M)$ is dense in $\mathcal{C}_b(M)$.*

Proof We use an inductive argument as in [10] (Lemma 3.2.1).
Since for any $g \in \mathcal{C}_b(M)$,

$$g = \max(g,0) - \max(-g,0) \quad \text{and} \quad g = (g - \inf_{x\in M} g(x)) + \inf_{x \in M} g(x),$$

it suffices to consider only non negative functions f in $\mathcal{C}_b(M)$ with $\inf_{x\in M} f(x) = 0$. Fix $\epsilon > 0$ and let

$$\Lambda_n = \{f \in \mathcal{C}_b(M), \ / \ f(x) \leq n\epsilon, \ x \in M\}.$$

We shall show by induction on n that for any function in Λ_n there exists a map $h \in \mathcal{S}(M)$ such that $\|f - h\|_0 \leq 2\epsilon$.

The assertion is true if $n = 2$, taking $h = 0$, so suppose that the assumption is satisfied for all $n \leq k$ and prove it for $n = k+1$, where $k \geq 2$. Let f be in Λ_{k+1} but not in Λ_k and set

$$A = \{x \in M, \ / \ f(x) \geq k\epsilon\}, \quad B = \{x \in M, \ / \ f(x) \leq (k-1)\epsilon\}.$$

A, B are two non empty closed sets. They are also separated for the uniform continuity of f. Hence take a map $l \in \mathcal{S}(M)$ that is Urysohn map for (A,B) and consider $\epsilon\, l$. We have that $\epsilon l(x) = \epsilon$ for any $x \in A$ and $\epsilon l(x) = 0$ for any $x \in B$ so that we get:

$$0 \leq f(x) - \epsilon l(x) \leq k\epsilon, \quad x \in M.$$

Thus $f - \epsilon l \in \Lambda_k$ and by induction hypothesis there is a map $g \in \mathcal{S}(M)$ such that:

$$\|f - \epsilon l - g\|_0 \leq 2\epsilon.$$

Taking $h = \epsilon l + g$, we can conclude. ∎

Now using Theorem 3.3 and Remark 2.8 we deduce the following result.

Corollary 3.4 *Let E be a separable Hilbert space, then there exist two separated closed sets A, B in E such that they do not admit any Urysohn function in $\mathcal{C}^2_b(E)$.*

4 Uniform approximation of σ-uniformly continuous maps

We recall some notions on locally convex topologies. Let $(E, \|\cdot\|_E)$ be a Banach space and let σ be a locally convex Haussdorff topology on E. Γ_σ denotes the family of all seminorms on E which are continuous with respect to σ.

We consider three different spaces of real functions on E.

$\mathcal{C}_\sigma(E) \stackrel{\text{def}}{=} \{f : (E, \sigma) \to \mathbb{R}$, uniformly continuous ([6]) and bounded $\}$.
$\mathcal{C}_\sigma(E)$ turns out to be a Banach space endowed with the sup norm.

$\mathcal{C}^{0,1}_\sigma(E) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_\sigma(E)$, for which there exists $q_f \in \Gamma_\sigma$, a constant $\mathrm{L}(f) > 0$,
such that $|f(x) - f(z)| \le \mathrm{L}(f)\, q_f(x - z), \quad x, z \in E \}$,

$\mathcal{C}^1_\sigma(E) \stackrel{\text{def}}{=} \{f \in \mathcal{C}_\sigma(E)$, having the Fréchet derivative Df in E
such that $Df : (E, \sigma(E, E')) \to E'$ is uniformly continuous and bounded $\}$.

Clearly if σ is weaker than the norm topology of E we have:
$\mathcal{C}_\sigma(E) \subset \mathcal{C}_b(E)$, $\mathcal{C}^{0,1}_\sigma(E) \subset \mathcal{C}^{0,1}_b(E)$, $\mathcal{C}^1_\sigma(E) \subset \mathcal{C}^1_b(E)$ and the inclusions are strict. The following lemma shows that $\mathcal{C}^{0,1}_\sigma(E)$ is dense in $\mathcal{C}_\sigma(E)$. It is a straightforward variation of [10, Lemma 3.2.1], we state it without proof.

Lemma 4.1 *Let (V, σ) be a real locally convex Haussdorff space, then $\mathcal{C}^{0,1}_\sigma(V)$ is dense in $\mathcal{C}_\sigma(V)$.*

Now we are ready to prove the following result.

Theorem 4.2 *Let E be a separable Banach space, with unit closed ball C, and σ be a locally convex Haussdorff topology on E such that:*
(i) σ is weaker than the norm topology;
(ii) (C, σ), i.e. C endowed with σ, is compact.
Then $\mathcal{C}^1_\sigma(E)$ is dense in $\mathcal{C}_\sigma(E)$.

Proof As in proof of Theorem 2.2, we consider E has an abstract Wiener space (E, H, p_1), denote by O_t the heat semigroup on $\mathcal{C}_b(E)$. It follows, by easy computations, that if $g \in \mathcal{C}_\sigma(E)$ then $O_t g \in \mathcal{C}_\sigma(E)$ for any $t > 0$.

By Lemma 4.1, arguing as for formula (2.6), to prove the assertion it is enough to verify that for any $f \in \mathcal{C}^{0,1}_\sigma(E)$ then $O_t f \in \mathcal{C}^1_\sigma(E)$, for any $t > 0$. Thus fix $f \in \mathcal{C}^{0,1}_\sigma(E)$ and $t > 0$.

First we remark that, by Hypothesis (i), $f \in \mathcal{C}^{0,1}_b(E)$ and so, by formula (2.5), $O_t f \in \mathcal{C}^1_s(E)$. We denote by $DO_t f$ the Hadamard derivative of $O_t f$.

Let $q \in \Gamma_\sigma$ such that

$$|f(x) - f(y)| \le \mathrm{L}(f)\, q(x - y), \quad x, y \in E,$$

then easily we get

$$|O_t f(x) - O_t f(y)| \le \mathrm{L}(f)\, q(x - y), \quad x, y \in E.$$

[6]A map $f : (E, \sigma) \to \mathbb{R}$ is uniformly continuous if and only if for any $\epsilon > 0$, there exist $\delta > 0$ and $q \in \Gamma_\sigma$ such that for any $x, y \in E$, $q(x - y) \le \delta$ implies that $|f(x) - f(y)| \le \epsilon$

Define the maps:

$$\phi_s \ : \ (C,\sigma) \ \to \ \mathcal{C}_\sigma(E), \ \ s \in (0,1] \quad \text{such that:}$$

$$\phi_s(v) \ = \ \frac{O_t f(\cdot + sv) - O_t f(\cdot)}{s}, \quad s \in (0,1], \ v \in C. \tag{4.1}$$

It is possible to prove, taking into account formulas (2.9) and (2.3), that for any $v \in C$,

$$\lim_{s \to 0^+} \ \frac{O_t f(x+sv) - O_t f(x)}{s} \ = \ DO_t f(x)(v)$$

and this limit is uniform in $x \in E$. Consequently for any $v \in C$,

$$\lim_{s \to 0^+} \ \phi_s(v) \ = \ DO_t f(\cdot)(v) \quad \text{in } \mathcal{C}_\sigma(E). \tag{4.2}$$

Take any sequence $s_n \subset (0,1]$, such that $s_n \to 0$. By formula (4.2), for any $v \in C$, the sequence $(\phi_{s_n}(v))$ is relatively compact in $\mathcal{C}_\sigma(E)$.
Further (ϕ_{s_n}) is an equicontinuous sequence of maps in $\mathcal{C}_\sigma(C, \mathcal{C}_\sigma(E))$([7]), since it holds:

$$\|\phi_{s_n}(v) - \phi_{s_n}(v')\|_{\mathcal{C}_\sigma(E)} \ \leq \ \mathrm{L}(f) \ q(v-v'), \ \ v, v' \in C.$$

Therefore applying the Arzela - Ascoli theorem (as in proof of Lemma 2.1) we deduce that $\lim_{n\to\infty} \phi_{s_n}(v) \ = \ DO_t f(\cdot)(v)$, uniformly in $v \in C$. Consequently for arbitrariness of (s_n), we deduce that

$$\lim\nolimits_{s \to 0^+} \sup\nolimits_{v \in C} \ \|\phi_s(v) - DO_t f(\cdot)(v)\|_{\mathcal{C}_\sigma(E)}$$

$$= \ \lim_{s \to 0^+} \sup_{v \in C} \sup_{x \in E} \Big| \frac{O_t f(x+sv) - O_t f(x)}{s} \ - \ DO_t f(x)(v) \Big| \ = \ 0. \tag{4.3}$$

This formula gives in particular that $O_t f$ is Fréchet differentiable in E. Moreover from (4.3), we also obtain that

$$DO_t f(\cdot)(\cdot) \ : (C,\sigma) \times (E,\sigma) \to \mathbb{R} \text{ is uniformly continuous and bounded.}$$

This fact yields that $DO_t f$ is uniformly continuous and bounded from (E,σ) into E'; indeed we have:

$$\sup_{v \in C} \ |DO_t f(x)(v) - DO_t f(z)(v)| \ = \ \|DO_t f(x) - DO_t f(z)\|_{E'}, \ \ x, z \in E.$$

The proof is complete. ∎

Let E be a separable reflexive Banach space. On E we consider the weak topology $\sigma(E,E')$. It is well known that the unit closed ball C in E, is compact with respect to $\sigma(E,E')$. Thus by the above theorem we deduce the following result.

Corollary 4.3 *Let $(E, \|\cdot\|_E)$ be a separable reflexive Banach space and consider the weak topology $\sigma \ = \ \sigma(E,E')$ on E. Then $\mathcal{C}^1_\sigma(E)$ is dense in $\mathcal{C}_\sigma(E)$.*

[7] $\mathcal{C}_\sigma(C, \mathcal{C}_\sigma(E))$ denotes the Banach space of all $g : (C,\sigma) \to \mathcal{C}_\sigma(E)$ which are uniformly continuous and bounded, endowed with the sup norm.

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