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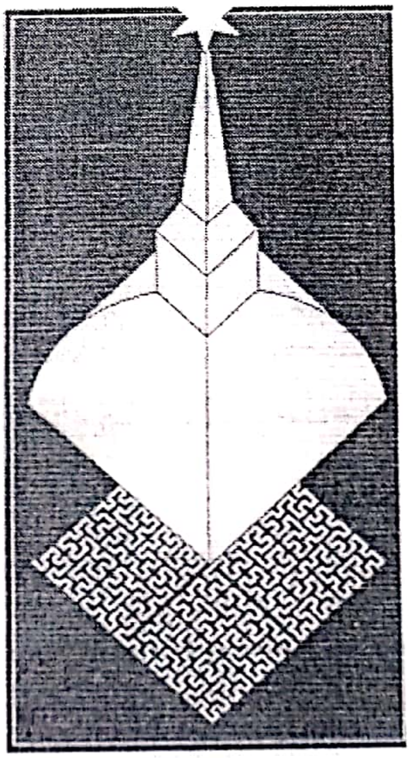

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The fundamental solution for a degenerate parabolic Dirichlet problem

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Abstract. – *We study a homogeneous parabolic Dirichlet problem involving a possibly degenerate Ornstein-Uhlenbeck operator in a half space H_+ of $\mathbf{R}^n$. We find an explicit formula for the fundamental solution. Under the Hörmander condition of hypoellipticity, we prove a global regularity result in spaces of continuous and bounded functions. We extend our explicit formula to the infinite dimensional setting.*

1. – Introduction and Preliminaries.

We consider a parabolic Dirichlet problem in a half space of $\mathbf{R}^n$ involving the possibly degenerate Ornstein-Uhlenbeck differential operator

$$\mathcal{U}\phi = \frac{1}{2}\mathrm{Tr}(QD^2\phi) + \langle Bx, D\phi(x)\rangle = \frac{1}{2}\sum_{i,j=1}^{n} Q_{ij}D_{ij}\phi(x) + \sum_{i,j=1}^{n} B_{ij}x_j D_i\phi(x), \tag{1.1}$$

$x \in \mathbf{R}^n$, where $B = B_{ij}$ is a non zero matrix on $\mathbf{R}^n$ and $Q = Q_{ij}$ is a non negative symmetric matrix on $\mathbf{R}^n$. Let $v_0 \in \mathbf{R}^n$, $|v_0| = 1$, and define

$$H_+ = \{x \in \mathbf{R}^n \text{ such that } \langle x, v_0\rangle > 0,\ x \in \mathbf{R}^n\}, \tag{1.2}$$

$\partial H_+ = \{x \in \mathbf{R}^n \text{ such that } \langle x, v_0\rangle = 0\}$, $\overline{H}_+ = H_+ \cup \partial H_+$. We are concerned with the problem

$$\begin{cases} \partial_t u(t,x) = \mathcal{U}u(t,x), & x \in H_+, \quad t > 0, \\ u(z,t) = 0, & z \in \partial H_+, \quad t > 0, \\ u(0,x) = f(x), & x \in H_+, \end{cases} \tag{1.3}$$

where $u : [0, \infty) \times H_+ \to \mathbf{R}$ and $f \in U\mathcal{C}_b(H_+)$, the space of all uniformly continuous and bounded functions on H_+, endowed with the supremum norm.

Under some assumptions on the coefficients Q and B, see Hypothesis 1, we prove that there exists a unique classical solution u of (1.3) and further $u(t,x) = P_t f(x)$, $t \geq 0$, $x \in H_+$, where P_t is a transition semigroup of contractions on $U\mathcal{C}_b(H_+)$. By using analytic methods, mainly semigroup theory, we show that the transition measures, which determine P_t, have a density with respect to the Lebesgue measure and provide an explicit formula for such a density (i.e. we compute the fundamental solution of (1.3)). This density involves Gaussian kernels, see Proposition 2.3. We use techniques similar to the ones contained in [16] and [18]. Moreover, assuming the Hörmander condition on hypoellipticity of $\mathcal{U}$, see (iii) in Hypothesis 1, some regularity properties of P_t are established, see Proposition 2.4 and Theorem 2.7. In particular we obtain that $P_t f(\cdot)$ is infinitely differentiable on H_+ with the partial derivatives of any order (in the x -variable) which are bounded on the whole of H_+. This global regularity result shows that the semigroup P_t has a regularizing effect (in particular it is strong Feller).

We point out that by taking the Laplace transform of $P_{(\cdot)} f(x)$, see (2.13), it is possible to investigate a corresponding elliptic Dirichlet problem on H_+, see (2.12). We think that the Schauder estimates given in [14] for the elliptic equation $\lambda\phi - \mathcal{U}\phi$ on $\mathbf{R}^n$, $\lambda > 0$, could be extended to this elliptic Dirichlet problem.

Concerning possibly degenerate elliptic and parabolic equations on $\mathbf{R}^n$, with unbounded coefficients, after the pioneering works in [1], [2], some optimal regularity results have been obtained recently by using semigroups theory, probabilistic methods and computing explicit formulas for the solutions (we only mention [6], [7], [14], [5], [15], [18]). These global regularity results are also motivated by applications to stochastic differential equations, see for instance [8], [11], [20], and to financial mathematics, see [3].

Let us introduce the following linear operators Q_t,

$$(1.4) \qquad Q_t = \int_0^t e^{sB} Q e^{sB^*} ds\,, \qquad e^{tB} = \sum_{k \geq 0} \frac{(tB)^k}{k!}\,, \qquad t \geq 0\,.$$

Throughtout the paper we make the following assumptions:

Hypothesis 1

(i) v_0 is an eigenvector of B^*, the adjoint of B, i.e. $B^* v_0 = b v_0$, $b \in \mathbf{R}$;
(ii) $Q v_0$ is an eigenvector of B;
(iii) $\det Q_t > 0$, $t > 0$.

These assumptions are all invariant under any linear transformation of coordinates in $\mathbf{R}^n$. It seems a hard problem finding an explicit formula for the fundamental solution of (1.3) avoiding (i) and (ii) in Hypothesis 1. In Example 2.8 we consider two matrices Q, B and a unitary vector v_0 which satisfy Hypothesis 1.

Condition (iii) in Hypothesis 1 is equivalent to the fact that the operator $\mathcal{U}$ is hypoelliptic, see [9]. So if $f \in \mathcal{C}^\infty(\Omega)$ and ψ is a distributional solution of $\mathcal{U}\psi = f$, then $\psi \in \mathcal{C}^\infty(\Omega)$, for any open set $\Omega \subseteq \mathbf{R}^n$. The hypoellipticity of $\mathcal{U}$ may be expressed in other equivalent ways, which we briefly review.

(a) *The kernel of Q does not contain subspaces which are invariant for* B^* (Hörmander pointed out that this condition is equivalent to the hypoellepticity of $\mathcal{U}$).

(b) *rank* $\{\mathcal{L}(X_1, \ldots, X_n, Y)(x)\} = n$, $x \in \mathbf{R}^n$, *where* $\mathcal{L}(X_1, \ldots, X_n, Y)$ *denotes the Lie algebra generated by the first order differential operators* $X_j = \sum_{k=1}^{n} Q_{jk} D_k$, $j = 1, \ldots n$, $Y = \langle Bx, D\cdot\rangle$ (this is the celebrated Hörmander condition on hypoellipticity).

(c) *rank* $[Q^{1/2}, BQ^{1/2}, \ldots, B^{n-1}Q^{1/2}] = n$, *where* $[Q^{1/2}, BQ^{1/2}, \ldots, B^{n-1}Q^{1/2}]$ *is the matrix obtained «attaching» the matrices* $Q^{1/2}, \ldots, B^{n-1}Q^{1/2}$ (this a condition is called the Kalman rank condition and it is well known in control theory, see [21]).

The equivalence between (a), (b) and (iii) of Hypothesis 1 can be found in [13]. For the equivalence of (c), we refer to Chapter 1 in [21].

Now we fix notations and give some preliminaries. Let us choose once and for all an orthonormal basis (e_k), $k = 1, \ldots n$, of $\mathbf{R}^n$, such that $v_0 = e_1$. By means of the basis (e_k), we always identify H_+ with the canonical open half space

(1.5) $$\mathbf{R}^n_+ = \{x = (x_1, x'), x_1 > 0, x' \in \mathbf{R}^{n-1}\}, \quad x_k = \langle x, e_k\rangle, \quad k = 1, \ldots n,$$

and ∂H_+ with $\mathbf{R}^{n-1}$. We also define $\mathbf{R}^n_- = \{x = (x_1, x'), x_1 < 0, x' \in \mathbf{R}^{n-1}\}$. Let Ω be an open subset of $\mathbf{R}^n$ and denote by $|\cdot|$ the Euclidean norm of any $\mathbf{R}^k$, $k \geq 1$. The space $U\mathcal{C}_b(\Omega, \mathbf{R}^k)$ stands for the Banach space of all uniformly continuous and bounded functions $f : \Omega \to \mathbf{R}^k$, endowed with the sup norm: $\|f\|_0 = \sup_{x \in \Omega} |f(x)|$, $f \in U\mathcal{C}_b(\Omega, \mathbf{R}^k)$. We set $U\mathcal{C}_b(\Omega) = U\mathcal{C}_b(\Omega, \mathbf{R})$. Note that the uniform continuity of f allows to consider values of f on $\partial\Omega$ and implies that $U\mathcal{C}_b(\Omega) = U\mathcal{C}_b(\overline{\Omega})$. The space $U\mathcal{C}^k_b(\Omega)$, $k \in \mathbf{Z}_+$, is the set of all k-times differentiable functions f, whose partial derivatives, $D_\alpha f$, $\alpha \in \mathbf{Z}^n_+$, $|\alpha| = \alpha_1 + \ldots \alpha_n \leq k$, are uniformly continuous and bounded on Ω up to the order k. It is a Banach space endowed with the norm $\|f\|_k = \|f\|_0 + \sum_{|\alpha| \leq k} \|D_\alpha f\|_0$, $f \in U\mathcal{C}^k_b(\Omega)$.

We set $U\mathcal{C}^\infty_b(\Omega) = \bigcap_{k \geq 0} U\mathcal{C}^k_b(\Omega)$. If $A \subset \mathbf{R}^n$, $\mathcal{B}_b(A)$ denotes the Banach space of all real, bounded and Borel functions on A, endowed with the sup norm. We finally introduce the space

(1.6) $$U\mathcal{C}(\mathbf{R}^n_+)_0 = \{f \in U\mathcal{C}_b(\mathbf{R}^n_+),\ f(0, x') = 0,\ x' \in \mathbf{R}^{n-1}\}.$$

Let M be a symmetric non negative matrix on $\mathbf{R}^n$, we denote by $N(x, M)$, $x \in \mathbf{R}^n$, the *Gaussian measure* on $\mathbf{R}^n$ with mean $x \in \mathbf{R}^n$ and covariance operator M; it has density $\frac{1}{\sqrt{(2\pi)^n \det(M)}} e^{-\frac{1}{2}\langle M^{-1}(x-y), x-y\rangle}$, with respect to the Lebesgue measure dy. Let B be any nonzero matrix on $\mathbf{R}^n$ and Q be a non negative symmetric matrix on $\mathbf{R}^n$, we introduce the *Ornstein-Uhlenbeck semigroup* U_t, associated with Q and B, as follows

(1.7) $$U_t f(x) = \int_{\mathbf{R}^n} f(e^{tB}x + y)\, N(0, Q_t)\, dy, \quad f \in \mathcal{B}_b(\mathbf{R}^n), \quad x \in \mathbf{R}^n, \quad t > 0,$$

$U_0 = Id_{\mathcal{B}_b(\mathbf{R}^n)}$. It is well known that $U_t \in \mathcal{L}(UC_b(\mathbf{R}^n))$, $t \geq 0$, and $U_{t+s} = U_t U_s$, $t, s \geq 0$ (if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two real Banach spaces, $\mathcal{L}(X, Y)$ stands for the Banach space of all bounded linear operators from X into Y, endowed with the norm: $\|T\|_{\mathcal{L}(X, Y)} = \sup_{\|x\|_X \leq 1} \|Tx\|_Y$, $T \in \mathcal{L}(X, Y)$; we define also $\mathcal{L}(X) = \mathcal{L}(X, X)$). In [12], Kolmogorov showed that $u(t, x) = U_t g(x)$ is the «solution» of the parabolic problem

$$\begin{cases} \partial_t u(t, x) = \mathcal{U} u(t, x), & t > 0, \\ u(0, x) = g(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.8}$$

for a large class of initial data g. Note that, when no confusion can arise, we will use the same symbol $\mathcal{U}$ to denote the Ornstein-Uhlenbeck differential operator, see (1.1), acting on functions defined on $\mathbf{R}^n$ or on $\mathbf{R}^n_+$.

2. – An explicit formula for the solution.

In this section we will compute the fundamental solution for (1.3). First let us consider some simple consequences of Hypothesis 1. The first one is that $Q_{11} > 0$. Indeed we have

$$0 < \langle Q_t e_1, e_1 \rangle = \int_0^t \langle Q e^{sB^*} e_1, e^{sB^*} e_1 \rangle \, ds = \int_0^t e^{2sb} \, ds \langle Q e_1, e_1 \rangle,$$

since $B^* e_1 = b e_1$. Moreover, by the identity $\langle BQe_1, e_1 \rangle = \langle Qe_1, B^* e_1 \rangle$, we derive that $BQe_1 = be_1$. The explicit formula for the solution u of (1.3) is based on the following lemma (see [16] for a similar lemma in the infinite dimensional setting).

LEMMA 2.1. – *Assume that Hypothesis 1 holds and define* $\phi_1 \colon \mathbf{R}^n \to \mathbf{R}^n$ *as follows:* $\phi_1(x) = x - \frac{2Qe_1}{q} x_1$, $q = \langle Qe_1, e_1 \rangle = Q_{11}$. *Then it holds:*

(i) $\phi_1 Q \phi_1^* = Q$;

(ii) $\phi_1(\mathbf{R}^n_+) \subset \mathbf{R}^n_-$, $\phi_1(0, x') = (0, x')$, $x' \in \mathbf{R}^{n-1}$;

(iii) $\phi_1^2 = Id_{\mathbf{R}^n}$, $B\phi_1 = \phi_1 B$.

PROOF. – Remark that $\phi_1^*(x) = x - \frac{2e_1}{q} \langle Qe_1, x \rangle$. Now (i) follows since

$$\phi_1 Qx = Qx - \frac{2Qe_1 \langle Qx, e_1 \rangle}{q} = Q\phi_1^* x, \quad x \in \mathbf{R}^n.$$

The assertion (ii) is easily verified (take into account that $\phi_1^* e_1 = -e_1$). As for (iii), we only remark that

$$B\phi_1 x = Bx - \frac{2BQe_1}{q} x_1 = Bx - \frac{2bQe_1}{q} x_1 = \phi_1 Bx\,, \quad x \in \mathbf{R}^n,$$

since $BQe_1 = bQe_1$ and $B^* e_1 = be_1$. ■

Note that if e_1 is an eigenvector of Q, one has $\phi_1(x_1, x') = (-x_1, x')$ and we find the standard reflection. By means of ϕ_1, we introduce an extension operator $E : \mathcal{B}_b(\overline{\mathbf{R}^n_+}) \to \mathcal{B}_b(\mathbf{R}^n)$,

(2.1) $$Ef(x) \stackrel{\text{def}}{=} = \begin{cases} f(x) & \text{if } x \in \overline{\mathbf{R}^n_+} \\ -f(\phi_1(x)) & \text{if } x \in \mathbf{R}^n_- . \end{cases}$$

For any $f \in \mathcal{B}_b(\overline{\mathbf{R}^n_+})$. Clearly $Ef \in \mathcal{B}_b(\mathbf{R}^n)$ and E is an isometry. Denoting by Rf the restriction of $f \in \mathcal{B}_b(\mathbf{R}^n)$ to $\overline{\mathbf{R}^n_+}$ and using the semigroup U_t, see (1.7), let us define the following family of operators: P_t: $\mathcal{B}_b(\overline{\mathbf{R}^n_+}) \to \mathcal{B}_b(\overline{\mathbf{R}^n_+})$,

(2.2) $$P_t \stackrel{\text{def}}{=} RU_t E\,, \quad t \geq 0\,, \quad P_t f(x) = \int_{\mathbf{R}^n} Ef(y)\, N(e^{tB}x, Q_t)\, dy\,,$$

$f \in \mathcal{B}_b(\overline{\mathbf{R}^n_+})$, $x \in \overline{\mathbf{R}^n_+}$ (note that $P_0 = Id$). It is clear that $P_t \in \mathcal{L}(\mathcal{B}_b(\overline{\mathbf{R}^n_+}))$, $t \geq 0$; moreover one has

LEMMA 2.2. – *The family of operators P_t, see (2.2), is a semigroup of contractions on $\mathcal{B}_b(\overline{\mathbf{R}^n_+})$.*

PROOF. – The proof of the semigroup property of P_t is based on the formula:

(2.3) $$ER(U_t Ef)(x) = U_t Ef(x)\,, \quad t \geq 0\,, \quad f \in \mathcal{B}_b(\overline{\mathbf{R}^n_+})\,, \quad x \in \mathbf{R}^n.$$

It is enough to check (2.3) when $x \in \mathbf{R}^n_-$. By Lemma 2.1 we get that $e^{tB}\phi_1 = \phi_1 e^{tB}$ and $\phi_1 Q_t \phi_1^* = \int_0^t \phi_1 e^{sB} Q e^{sB^*} \phi_1^*\, ds = Q_t$, $t \geq 0$. Using these formulas and Lemma 2.1, we compute, by changing variable in the integrals,

$$ER(U_t Ef)(x) = -\int_{\mathbf{R}^n} Ef(y)\, N(e^{tB}\phi_1 x, Q_t)\, dy =$$

$$-\int_{\mathbf{R}^n} Ef(\phi_1 y)\, N(e^{tB}x, \phi_1 Q_t \phi_1^*)\, dy = -\int_{\mathbf{R}^n} Ef(\phi_1 y)\, N(e^{tB}x, Q_t)\, dy =$$

$$\int_{\mathbf{R}^n} Ef(y)\, N(e^{tB}x, Q_t)\, dy = U_t Ef(x)\,, \quad x \in \mathbf{R}^n_-$$

(note that $-Ef(\phi_1(y)) = Ef(y)$, for any $y \notin \partial\mathbf{R}^n_+$ and $N(e^{tB}x, Q_t)$ $(\partial\mathbf{R}^n_+) = 0$). Thus (2.3) is established. Now the semigroup property follows since

$$P_t P_s f = (RU_t E)(RU_s E) f = RU_{t+s} Ef = U_{t+s} f, \qquad f \in \mathcal{B}_b(\overline{\mathbf{R}^n_+}),\ t,\ s \geq 0 .$$

This completes the proof. ■

Next we derive an integral representation formula for the semigroup P_t.

PROPOSITION 2.3. – *For any* $f \in \mathcal{B}_b(\overline{\mathbf{R}^n_+})$, $t > 0$, *one has:*

$$\text{(2.4)} \qquad P_t f(x) = \int_{\mathbf{R}^n_+} f(y) \left(1 - \exp\left[-\frac{2e^{tb}}{q_t} x_1 y_1\right]\right) N(e^{tB}x, Q_t)(dy), \quad x = (x_1, x') \in \mathbf{R}^n_+,$$

where $B^* e_1 = b e_1$, $BQe_1 = bQe_1$, $q_t = \langle Q_t e_1, e_1 \rangle = Q_{11} \int_0^t e^{2sb}\, ds$.

PROOF. – First we obtain, by changing variable,

$$\text{(2.5)} \qquad P_t f(x) = \int_{\mathbf{R}^n} Ef(e^{tB}x + y)\, N(0, Q_t)\, dy =$$

$$\int_{\mathbf{R}^n_+} f(y) [N(e^{tB}x, Q_t) - N(\phi_1 e^{tB}x, Q_t)]\, dy, \qquad x \in \mathbf{R}^n_+,\ t > 0 .$$

Now it is useful to compute the Radon-Nikodym derivative of the Gaussian measure $N(\phi_1 e^{tB}x, Q_t)$ with respect to $N(e^{tB}x, Q_t)$, $t > 0$. We obtain (remind that $\phi_1(x) - x = -\frac{2x_1}{q} Qe_1$)

$$\frac{dN(\phi_1 e^{tB}x, Q_t)}{dN(e^{tB}x, Q_t)}(y) = \exp\left[-\frac{1}{2} |Q_t^{-1/2}(\phi_1 e^{tB}x - e^{tB}x)|^2 + \right.$$

$$\left. \langle Q_t^{-1/2}(y - e^{tB}x),\ Q_t^{-1/2}(\phi_1 e^{tB}x - e^{tB}x)\rangle \right] =$$

$$\exp\left[-\frac{2e^{2tb}x_1^2}{q^2} \langle Q_t^{-1} Qe_1, e_1 \rangle - \frac{2e^{tb}x_1}{q} \langle Q_t^{-1}(y - e^{tB}x), Qe_1 \rangle\right],$$

$y \in \mathbf{R}^n$, a.e.. Since $Q_t e_1 = \int_0^t e^{2sb}\, ds\ Qe_1$, we finally get, for any $t > 0$,

$$\frac{dN(\phi_1 e^{tB}x, Q_t)}{dN(e^{tB}x, Q_t)}(y) = \exp\left[-\frac{2e^{tb}}{q \int_0^t e^{2sb}\, ds} x_1 y_1\right],$$

where $y \in \mathbf{R}^n$, - a.e.. Now we infer easily the assertion. ■

Let us notice that by (2.5) one also deduces, by using the map $I_{\mathbf{R}_+}$ ($I_{\mathbf{R}_+}(y)=1$, $y\in\mathbf{R}_+$ and $I_{\mathbf{R}_+}(y)=0$, $y\notin\mathbf{R}_+$),

$$(2.6)\qquad P_t f(x)=\int_{\mathbf{R}^n} I_{\mathbf{R}_+}(\langle e^{tB}x+y,\, e_1\rangle)\, f(e^{tB}x+y)\, N(0,\,Q_t)\, dy -$$

$$\int_{\mathbf{R}^n} I_{\mathbf{R}_+}(\langle \phi_1 e^{tB}x+y,\, e_1\rangle)\, f(\phi_1 e^{tB}x+y)\, N(0,\,Q_t)\, dy$$

$$=\int_{y_1>-e^{tb}x_1} f(e^{tB}x+y)\,N(0,Q_t)\,dy-\int_{y_1>e^{tb}x_1} f(e^{tB}\phi_1 x+y)\,N(0,Q_t)\,dy\,,$$

The restriction of P_t to $UC_b(\mathbf{R}^n_+)$, still denoted by P_t, turns out to be a semigroup of bounded linear operators on $UC_b(\mathbf{R}^n_+)$. This is a consequence of the following result.

PROPOSITION 2.4. – *The following statements hold:*

(i) $P_t(\mathcal{B}_b(\overline{\mathbf{R}^n_+}))\subset UC_b^\infty(\mathbf{R}^n_+)\cap UC(\mathbf{R}^n_+)_0$, $t>0$;

(ii) let $f\in UC_b(\mathbf{R}^n_+)$. *For any ball* $C\subseteq\mathbf{R}^n_+$, *one has:*

$$\lim_{s\to 0}\ \sup_{x\in C}|P_{t+s}f(x)-P_tf(x)|=0\,,\qquad t\geq 0\,;$$

(iii) for any $f\in UC_b(\mathbf{R}^n_+)$, $x\in\mathbf{R}^n_+$, *the map:* $[0,\,\infty)\to\mathbf{R}$, $t\to P_tf(x)$ *is differentiable on* $(0,\,\infty)$; *further we have* $\partial_t P_tf(x)=\mathcal{U}P_tf(x)$, $t>0$.

PROOF. – (i) Let $f\in UC_b(\mathbf{R}^n_+)$. To prove that $P_tf\in UC_b^k(\mathbf{R}^n_+)$ for $k\geq 1$, one differentiates the Gaussian kernel in formula (2.2) with respect to the x-variable $k+1$-times and one shows that all the partial derivatives up to the order $k+1$ are bounded on $\mathbf{R}^n_+$. Similar computations are given in [7]. In particular here we deal with the first partial derivatives.

$$D_kP_tf(x)=\int_{\mathbf{R}^n} Ef(e^{tB}x+y)\langle e^{tB^*}Q_t^{-1}y,\, e_k\rangle\, N(0,\,Q_t)\, dy=$$

$$\int_{\mathbf{R}^n} Ef(e^{tB}x+Q_t^{1/2}y)\langle e^{tB^*}Q_t^{-1/2}y,\, e_k\rangle\, N(0,\,Id)\, dy\,;$$

it follows that $|D_kP_tf(x)|\leq\|Q_t^{-1/2}e^{tB}\|\,\|f\|_0$, $k=1,\,\ldots n$, $t>0$, $x\in\mathbf{R}^n_+$.

One can prove that $\|Q_t^{-1/2}e^{tB}\|\leq C\max\left(1,\,\frac{1}{t^{1/2+m}}\right)$, see [19] and the proof of Theorem 3.4 in [18]; here $m\leq n-1$ is the smallest integer such that rank $[Q^{1/2},\, BQ^{1/2},\,\ldots,\, B^mQ^{1/2}]=n$ (compare with condition (c) in Section 1). Using the previous estimate we can conclude that

$$(2.7)\qquad \|D_kP_tf\|_0\leq C\max\left(1,\,\frac{1}{t^{1/2+m}}\right)\|f\|_0,\quad k=1,\,\ldots n,\ t>0\,.$$

(ii) First we change variable in (2.2),

$$P_t f(x) = \int_{\mathbf{R}^n} Ef(e^{tB}x + Q_t^{1/2}y)\, N(0, I)\, dy, \quad x \in \mathbf{R}^n_+,\ t > 0. \tag{2.8}$$

Then to deduce the assertion one can proceed similarly to the proof of Proposition 4.2 in [17] (we only remark that Ef is continuous on $\mathbf{R}^n$ apart from a set of Lebesgue measure 0).

(iii) Note that one can differentiate $P_t f(x)$, with respect to t, in (2.2), using that the density of $N(e^{tB}x, Q_t)$ with respect to the Lebesgue measure dy is the fundamental solution of (1.8). ■

Some comments on the semigroup P_t are in order. By (ii) of Proposition 2.4, we deduce that P_t is not strongly continuous on $UC_b(\mathbf{R}^n_+)$. Moreover, following [18], one can show that P_t is not analytic on $UC_b(\mathbf{R}^n_+)$ (the same happens for the Ornstein-Uhlenbeck semigroup on $UC_b(\mathbf{R}^n)$, see [4] and [7]). Thanks to Proposition 2.4, the semigroup P_t belongs to the class of π-semigroups, see [17]. We review below some basic concepts from the theory of π-semigroups.

DEFINITION 2.5. – Let Ω be any open set of $\mathbf{R}^n$. A sequence $(f_n) \subseteq UC_b(\Omega)$ is said to be *π-convergent* to a map f, and we shall write $f_n \xrightarrow{\pi} f$ as $n \to \infty$, if the following conditions hold:

$$(a)\quad f \in UC_b(\Omega),\ \sup_{n \geq 1} \|f_n\|_0 < \infty; \qquad (b)\quad \lim_{n \to \infty} f_n(x) = f(x), \quad x \in \Omega. \tag{2.9}$$

A semigroup of linear contractions S_t on $UC_b(\Omega)$ is said to be a *π-semigroup* of contractions if it satisfies:

(i) for any $f \in UC_b(\Omega)$, $x \in \Omega$, the map $[0, \infty[\to \mathbf{R}$, $t \to S_t f(x)$ is continuous;

(ii) for any $(f_n) \subset UC_b(\Omega)$, $f_n \xrightarrow{\pi} f$ implies that $S_t f_n \xrightarrow{\pi} S_t f$ as $n \to \infty$, $t \geq 0$.

We define a *generator* $\mathcal{B}$ for S_t as follows (we set $\Delta_h = \dfrac{S_h - I}{h}$, $h > 0$):

$$\begin{cases} D(\mathcal{B}) = \{f \in UC_b(\Omega) \text{ such that } \exists g \in UC_b(\Omega),\ \Delta_h f \xrightarrow{\pi} g \text{ as } h \to 0^+\}, \\ \mathcal{B}f(x) \stackrel{\text{def}}{=} \lim_{h \to 0^+} \Delta_h f(x), \quad f \in D(\mathcal{B}),\ x \in \Omega, \end{cases}$$

where $\Delta_h f \xrightarrow{\pi} g$ means that for any sequence (t_n) that converges to 0, we have $\Delta_{t_n} f \xrightarrow{\pi} f$ as $n \to \infty$ (this implies that $\sup_{h>0} \|\Delta_h f\|_0 < \infty$). In [17] it is proved that $\frac{d}{dt} S_t g(x) = \mathcal{B} S_t g(x) = S_t \mathcal{B} g(x)$, for any $g \in D(\mathcal{B})$, $x \in \Omega$, $t \geq 0$. Moreover $\mathcal{B}$ is π-closed, i.e. for any $(f_n) \subset D(\mathcal{B})$, one has $f_n \xrightarrow{\pi} f$ and $\mathcal{B} f_n \xrightarrow{\pi} g$ as $n \to \infty \Rightarrow f \in D(\mathcal{B})$ and $\mathcal{B} f = g$. (this condition implies in particular that $\mathcal{B}$ is a closed operator on

$U\mathcal{C}_b(\Omega)$). The resolvent operator of $\mathcal{B}$ is given by

$$(2.10)\qquad R(\lambda, \mathcal{B})\, f(x) = (\lambda - \mathcal{B})^{-1} f(x) = \int_0^\infty e^{-\lambda t} S_t f(x)\, dt\,, \qquad f \in U\mathcal{C}_b(\Omega), x \in \Omega, \lambda > 0\,.$$

We will denote by $\mathcal{G}$ the *infinitesimal generator* of the semigroup P_t on $U\mathcal{C}_b(\mathbf{R}^n_+)$. By (2.10) and (ii) of Proposition 2.4 one deduces that $D(\mathcal{G}) \subset U\mathcal{C}(\mathbf{R}^n_+)_0$. ■

Proceeding as in the proof of Theorem 3.4 in [18], one can prove the next result which gives a characterization of the generator $\mathcal{G}$ of P_t.

THEOREM 2.6. – *Let* $D_0 = \{g \in U\mathcal{C}(\mathbf{R}^n_+)_0 \cap U\mathcal{C}^2_b(\mathbf{R}^n_+)$ *such that* $\mathcal{U}g \in U\mathcal{C}_b(\mathbf{R}^n_+)\}$. *The following statements hold:*

(*i*) $D_0 \subset D(\mathcal{G})$ *and* $\mathcal{G}f = \mathcal{U}f$, $f \in D_0$;

(*ii*) $\psi \in D(\mathcal{G}) \Leftrightarrow$ *there exists* $(\psi_n) \subset D_0$ *such that* $\psi_n \xrightarrow{\pi} \psi$, $\mathcal{U}\psi_n \xrightarrow{\pi} \mathcal{U}\psi$.

(*iii*) (2.11) $\begin{cases} D(\mathcal{G}) = \{\phi \in U\mathcal{C}(\mathbf{R}^n_+)_0 \text{ such that } \mathcal{U}\phi \in U\mathcal{C}_b(\mathbf{R}^n_+)\}\,, \\ \mathcal{G}\phi = \mathcal{U}\phi\,,\ \phi \in D(\mathcal{G}), \text{where } \mathcal{U}\phi \text{ is in the sense of distributions}. \end{cases}$

Let us revert to (1.3). A bounded map $u : [0, T] \times \overline{\mathbf{R}^n_+} \to \mathbf{R}$, $T > 0$, is said to be a *classical solution* of (1.3) in $[0, T) \times \overline{\mathbf{R}^n_+}$ if it satisfies:

(i) u is continuous on $[0, T] \times \overline{\mathbf{R}^n_+} \setminus \{t = 0\} \times \partial\mathbf{R}^n_+$; (ii) u has the partial derivatives $\partial_t u$, $\partial_{x_k} u$, $\partial_{x_j x_k} u$, $j, k = 1, \dots n$, on $(0, T) \times \mathbf{R}^n_+$ and solves (1.3); (iii) the map $u(t, \cdot)$ is twice differentiable with continuity on $\mathbf{R}^n_+$, $t \in (0, T)$.

The uniqueness of classical solutions for (1.3) follows by a maximum principle for parabolic problems, see 8.1.12 and 8.1.17 in [10]. As concerns existence, using Proposition 2.4 and Definition 2.5, we get the following result.

THEOREM 2.7. – *Let us consider problem* (1.3). *Then it holds:*

(*i*) *for any datum* $f \in U\mathcal{C}_b(\mathbf{R}^n_+)$, *there exists a unique classical solution* $u(t, x) = P_t f(x)$, *see* (2.4); *moreover* $u(t, \cdot) \in U\mathcal{C}^\infty_b(\mathbf{R}^n_+)$, $t > 0$;

(*ii*) *for any datum* $f \in D(\mathcal{G})$, *see* (2.11), *the classical solution* u *has the partial derivative* $\partial_t u$ *which is bounded on* $[0, \infty) \times \overline{\mathbf{R}^n_+}$ *and continuous on* $[0, \infty) \times \overline{\mathbf{R}^n_+} \setminus \{t = 0\} \times \partial\mathbf{R}^n_+$; *moreover* $\partial_t u = P_t(\mathcal{U}f)$.

Let us consider the following *elliptic* Dirichlet problem:

$$(2.12)\qquad \begin{cases} \lambda\psi(x) - \mathcal{U}\psi(x) = f(x)\,, & x \in \mathbf{R}^n_+\,,\ \lambda > 0\,, \\ \psi(0, x') = 0\,, & x' \in \mathbf{R}^{n-1}\,, \end{cases}$$

According to Theorem 2.6, we write (2.12) as $\lambda\psi - \mathcal{G}\psi = f$. For any $f \in U\mathcal{C}_b(\mathbf{R}^n_+)$, there exists a unique «generalized solution» $\psi \in D(\mathcal{G})$. Moreover by (2.10), we have an explicit formula for the solution ψ, namely

$$\psi(x) = \int_0^{\infty} e^{-\lambda t} P_t f(x)\, dt\,, \qquad x \in \mathbf{R}^n_+\,,\ \lambda > 0\,, \tag{2.13}$$

where P_t is the semigroup introduced in (2.2). Thanks to (2.13), one can investigate regularity properties of ψ, compare with [7], [14], [5], [16].

EXAMPLE 2.8. – Let us consider $\mathbf{R}^3$ and take $v_0 = e_1 = (1, 0, 0)$. Define two matrices Q and B as follows

$$Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ b & a & 0 \end{pmatrix}, \qquad b,\ a \in \mathbf{R},\ a \neq 0\,.$$

It is easy to check that Q, B and v_0 satisfy Hypothesis 1.

3. – The infinite dimensional case.

Here we clarify that the semigroup P_t makes sense also in the infinite dimensional case. For more details and comments on this section we refer to Zabczyk [11]. Let H be a real separable Hilbert space (with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$). Let us fix $v_0 \in H$, $|v_0| = 1$, and define the half space $H_+ = \{x \in H, \langle x, v_0 \rangle > 0\}$.

Let Q be a symmetric, non negative, bounded linear operator on H. Let B be the generator of a strongly continuous semigroup e^{tB} on H. Assume that for each $t > 0$, the bounded linear operators Q_t, $Q_t x = \int_0^t e^{uB} Q e^{uB^*} x\, du$, $x \in H$, are *nuclear and positive definite* (this hypothesis corresponds to (iii) in Hypothesis 1). We are dealing with the problem (1.3) in which $\mathbf{R}^n$ is replaced by H and the differential operator $\mathcal{U}$ has the form:

$$\mathcal{U}\phi(x) = \frac{1}{2} \mathrm{Tr}\,[Q D^2 \phi(x)] + \langle x, B^* D\phi(x) \rangle\,, \qquad x \in H\,;$$

here «Tr» denotes the trace of a nuclear operator and «D» and «D^2», respectively, the first and second Fréchet derivatives. Let $U\mathcal{C}_b(H)$ be the space of all real uniformly continuous and bounded functions on H, endowed with the supremum norm. We can define similarly to (1.7) an infinite dimensional Ornstein-Uhlenbeck semigroup, i.e.

$$U_t f(x) = \int_H f(e^{tB} x + y)\, N(0, Q_t)\, dy\,, \qquad f \in \mathcal{B}_b(H),\ x \in H,\ t > 0,\ U_0 = Id\,,$$

where $N(0, Q_t)$ is the Gaussian measure on H with mean 0 and covariance operator Q_t (note that such a measure exists since Q_t is positive and nuclear, $t > 0$). Under Hypothesis 1, arguing as in Section 2 and using the techniques developed in [16], one can show that there exists a semigroup $P_t \in \mathcal{L}(UC_b(H))$, $t \geq 0$, given by

$$P_t f(x) = \int_H Ef(y)\, N(e^{tB}x, Q_t)\, dy =$$

$$\int_{H_+} f(y)\left(1 - \exp\left[-\frac{2e^{tb}}{q_t} x_1 y_1\right]\right) N(e^{tB}x, Q_t)(dy), \qquad x \in H_+, \quad f \in UC_b(H_+),$$

where the extension E is defined as in (2.1). The map $u(t, x) = P_t f(x)$, when f is «regular» enough, is the «classical solution» of the infinite dimensional parabolic Dirichlet problem, which generalizes (1.3). Of course several difficulties arise in studying the regularity properties of $P_t f$ in the infinite dimensional setting.

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