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THE CAUCHY PROBLEM FOR A CLASS OF MARKOV-TYPE SEMIGROUPS

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ABSTRACT: We study the Cauchy problem for a class of Markov-type semigroups (not strongly continuous in general) in the space of all real, uniformly continuous and bounded functions on a separable metric space. In this class there are many transition Markov semigroups corresponding to stochastic differential equations in infinite dimensions as the heat semigroup and the one of Ornstein-Uhlenbeck. We define appropriate notions of solution and give existence and uniqueness theorems. Additional regularity results about the Cauchy problem associated with the Ornstein-Uhlenbeck semigroup are also proved.

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1. INTRODUCTION

Recently a new class of semigroups of bounded linear operators on $\mathcal{C}_b(E)$ (not strongly continuous in general) has been studied in Priola [17] ($\mathcal{C}_b(E)$ denotes the Banach space of all real, uniformly continuous and bounded functions on a separable metric space E, endowed with the sup norm). We call these semigroups π-semigroups, they are a development of weakly continuous semigroups introduced by Cerrai [4] (see also Cerrai and Gozzi [5]). A π-semigroup P_t is characterized by the following assumptions: (i) for any $f \in \mathcal{C}_b(E)$, $x \in E$, the map $[0,\infty[\to \mathbb{R}$, $t \to P_t f(x)$ is continuous; (ii) for any bounded sequence $(f_n) \subset \mathcal{C}_b(E)$ such that f_n converges pointwise to $f \in \mathcal{C}_b(E)$ (we briefly say that $f_n \xrightarrow{\pi} f$ as $n \to \infty$), we have $P_t f_n \xrightarrow{\pi} P_t f$ as $n \to \infty$, $t \geq 0$; (iii) $\|P_t\|_{\mathcal{L}(\mathcal{C}_b(E))} \leq M\, e^{\omega t}$, $t \geq 0$.

π- Semigroups are introduced in order to study semigroups of kernels in infinite dimensions. They arise as transition semigroups of Markov Processes (see formula (2.7)) corresponding to the solutions of Stochastic Differential Equations and representing the solutions of PDE's with infinitely many variables (our main references

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are: Cannarsa and Da Prato [2], [3], Daletskii and Fomin [6], Da Prato and Zabczyk [9], Gross [13]). In Priola [17] it has been verified that the heat and the Ornstein-Uhlenbeck semigroups on $C_b(H)$ as well as the semigroup associated with a Dirichlet problem in a half space of H (see Priola [16]) are π-semigroups (here H stands for a real separable Hilbert space). The main properties of π-semigroups are recalled without proof in §2, for more details see Priola [17].

This paper is devoted to the following Cauchy problem (see §3)

$$\begin{cases} \partial_t u(t,x) = \mathcal{A}u(t,x) + F(t,x), & t \in]0,T],\ x \in E, \\ u(0,x) = f(x), & x \in E, \end{cases} \tag{1.1}$$

where $\mathcal{A}$ is the generator of a π-semigroup (see Definition 2.3), $F : [0,T] \times E \to \mathbb{R}$ and $f \in C_b(E)$. We define appropriate notions of solution for (1.1), by introducing classical, strict and strong solutions. Existence and uniqueness theorems for these solutions are proved. In particular we are able to give conditions for uniqueness of the classical solution and show that this solution can be represented by the variation of constants formula (see Theorem 3.5). This leads to a new uniqueness result (see Theorem 3.6) on strong solutions of Cauchy problems for a large class of Ornstein-Uhlenbeck operators in $C_b(H)$, studied by Cerrai and Gozzi [5]. The statements of our results are quite natural extensions of the classical ones, considered in the theory of C_0 semigroups (see for instance §4 of Pazy [15]). However the proofs are involved and require new arguments.

In the last section we deal with the Cauchy problem (1.1) associated with the Ornstein-Uhlenbeck semigroup in $C_b(H)$. In this case a "natural" restriction of $\mathcal{A}$ is given by the differential operator $\mathcal{U}_0$, defined as follows:

$$\mathcal{U}_0 f(x) = \frac{1}{2}\mathrm{Tr}\,[MD^2 f(x)] + < A^* Df(x), x >, \quad x \in H, \tag{1.2}$$

where M is a self-adjoint, non negative, bounded linear operator on H, A generates a C_0 semigroup on H (A^* denotes the adjoint of A) and f is suitably regular (see Definition 4.2). We present a new method to approximate the strong solution by means of a sequence of strict solutions for which the operator $\mathcal{U}_0$ is well defined. This is useful to give a meaning to the Ito formula that is needed in applications like the study of second order Hamilton-Jacobi equations, arising from control theory (see for instance Gozzi [12]). We extend the results of Cerrai and Gozzi [5] (see Theorems 4.3 and 4.4).

2. BASIC DEFINITIONS

In this section we briefly recall basic concepts on π-semigroups (see Priola [17] for more details).

Let (E,d) be a separable metric space, with metric d, we denote by $C_b(E)$ the set of all real, uniformly continuous and bounded functions on E. We consider $C_b(E)$ as a Banach space endowed with the sup norm $\|f\|_0 = \sup_{x\in E}|f(x)|$, $f \in C_b(E)$.

A sequence $(f_n) \subset C_b(E)$ is said to be π- *convergent* to a map f and we shall write $f_n \xrightarrow{\pi} f$ as $n \to \infty$ if the following conditions hold:

$$\begin{aligned} &(a)\quad f \in C_b(E), \quad \sup_{n\geq 1}\|f_n\|_0 < \infty; \\ &(b)\quad \lim_{n\to\infty} f_n(x) = f(x), \quad x \in E. \end{aligned} \tag{2.1}$$

Similarly, let J be a real interval and $\hat{t} \in J$. Let $F : J\backslash\{\hat{t}\} \to C_b(E)$, we say that $F(t) \xrightarrow{\pi} f$, as $t \to \hat{t}$, if for any sequence $(t_n) \subset J\backslash\{\hat{t}\}$ that converges to $\hat{t}$, we have that $F(t_n) \xrightarrow{\pi} f$ as $n \to \infty$. Notice that the previous condition implies that there exists a neighborhood U of $\hat{t}$ such that $\sup_{t\in U\backslash\{\hat{t}\}}\|F(t)\|_0 < \infty$. Now we introduce π-semigroups on $C_b(E)$.

Definition 2.1. Let P_t, $t \geq 0$ be a semigroup of bounded linear operators on $C_b(E)$, namely $P_{t+s} = P_t P_s$, $P_0 = I$ for $t, s \geq 0$. We say that P_t is a *π-semigroup* on $C_b(E)$ if the following conditions hold ([1]):

$$\begin{aligned} &(i)\ \text{there exist } M \geq 1 \text{ and } \omega \geq 0, \text{ such that } \|P_t\|_{\mathcal{L}(C_b(E))} \leq M e^{\omega t}, \quad t \geq 0; \\ &(ii)\ \text{for any } x \in E,\ f \in C_b(E), \text{ the map } [0,\infty[\to \mathbb{R},\ t \mapsto P_t f(x) \\ &\qquad \text{is continuous;} \\ &(iii)\ \text{for any } (f_n) \subset C_b(E),\ f_n \xrightarrow{\pi} f \text{ implies that } P_t f_n \xrightarrow{\pi} P_t f \\ &\qquad \text{as } n \to \infty,\ t \geq 0. \end{aligned} \tag{2.2}$$

If P_t is a π-semigroup, we define the *type* of P_t as the real number

$\omega = \inf\{\alpha \geq 0$ such that there exists $M_\alpha \geq 1$, $\|P_t\|_{\mathcal{L}} \leq M_\alpha e^{\alpha t}$, $t \geq 0\}$.

Let now $\mathcal{S} = \{S_i\}_{i\in I}$ be a non trivial covering of E, i.e. $S_i \subset E$, $i \in I$, $E = \cup_{i\in I} S_i$ and assume that there exists $S_{\bar{i}} \in \mathcal{S}$ that is infinite. In the sequel we also consider π-semigroups P_t that satisfy the following additional condition:

$$\lim_{t\to 0^+}\sup_{x\in S}|P_t f(x) - f(x)| = 0, \quad f \in C_b(E),\ S \in \mathcal{S}. \tag{2.3}$$

The next lemma is basic for the treatment of π-semigroups on $C_b(E)$.

[1] Let $(X, \|\cdot\|_X)$ be a real Banach space, we denote by $(\mathcal{L}(X), \|\cdot\|_{\mathcal{L}})$ the Banach space of all linear and continuous operators on X, endowed with the usual norm $\|T\|_{\mathcal{L}} = \sup_{\|x\|_X\leq 1}\|Tx\|_X$, $T \in \mathcal{L}(X)$.

Lemma 2.2. *Let I be an interval of $\mathbb{R}$ and μ be a Borel finite measure on I. Consider a function $F : I \times E \to \mathbb{R}$ that satisfies the following conditions:*
(i) $F(\cdot, x)$ is a Borel mapping for any $x \in E$;
(ii) $F(t, \cdot)$ is a uniformly continuous mapping, for $t \in I$;
(iii) there exists $g : I \to \mathbb{R}$, μ- integrable such that $|F(t,x)| \leq g(t)$, $x \in E$, $t \in I$.
Then the map

$$h : E \to \mathbb{R}, \ h(x) = \int_I F(t,x)\,\mu(dt), \quad x \in E$$

is uniformly continuous and bounded.

Definition 2.3. Let P_t be a π-semigroup on $C_b(E)$ we set $\Delta_h = \frac{P_h - I}{h}$, $h > 0$ and define its *infinitesimal generator* $\mathcal{A}$ as follows

$$\begin{cases} D(\mathcal{A}) = \{f \in C_b(E) \text{ such that } \exists g \in C_b(E)\ ,\ \Delta_h f \xrightarrow{\pi} g \text{ as } h \to 0^+\} \\ \mathcal{A}f(x) \overset{def}{=} \lim_{h \to 0^+} \Delta_h f(x), \quad f \in D(\mathcal{A}),\ x \in E. \end{cases} \tag{2.4}$$

Let now $\mathcal{L} : D(\mathcal{L}) \subset C_b(E) \to C_b(E)$ be a linear operator, we say that $\mathcal{L}$ is a *π-closed operator* if for any $(f_n) \subset D(\mathcal{L})$, the following condition holds:

$$f_n \xrightarrow{\pi} f \text{ and } \mathcal{L}f_n \xrightarrow{\pi} g \text{ as } n \to \infty \ \Rightarrow\ f \in D(\mathcal{L}) \text{ and } \mathcal{L}f = g. \tag{2.5}$$

Finally a subset C of $C_b(E)$ is said to be *π-dense* in $C_b(E)$ if for any $f \in C_b(E)$, there exists $(f_n) \subset C$, such that $f_n \xrightarrow{\pi} f$ as $n \to \infty$.

Some properties of $\mathcal{A}$ are stated below. Note that $\mathcal{A}$ has not dense domain in general.

Proposition 2.4. *Let $\mathcal{A}$ be the generator of a π-semigroup P_t of type ω on $C_b(E)$, then for any $f \in D(\mathcal{A})$, it holds:*
(i) $P_t f \in D(\mathcal{A})$ and $\mathcal{A}P_t f = P_t \mathcal{A} f$, $t \geq 0$;
(ii) for any $x \in E$, the map: $[0, \infty[\to \mathbb{R}$, $t \mapsto P_t f(x)$ is continuously differentiable and one has $\frac{d}{dt} P_t f(x) = P_t \mathcal{A} f(x)$, $t \geq 0$;
(iii) $D(\mathcal{A})$ is π- dense in $C_b(E)$;
(iv) $\mathcal{A}$ is a π- closed operator on $C_b(E)$.

Let P_t be a π-semigroup on $C_b(E)$ such that $\|P_t\|_{\mathcal{L}(C_b(E))} \leq M e^{\alpha t}$, $t \geq 0$ with $\alpha \in \mathbb{R}$ and $M \geq 1$. Consider the following operators

$$F_\lambda f(x) = \int_0^\infty e^{-\lambda u} P_u f(x)\,du, \quad f \in C_b(E),\ x \in E,\ \lambda > \alpha. \tag{2.6}$$

By Lemma 2.2, we deduce that each $F_\lambda : C_b(E) \to C_b(E)$, $\lambda > \alpha$. Moreover $(F_\lambda)_{\lambda > \alpha}$ is a family of bounded linear operators on $C_b(E)$. Now we review the characterization of the resolvent operator $R(\lambda, \mathcal{A})$, where $\mathcal{A}$ is the generator of a π-semigroup.

Proposition 2.5. *Let P_t be a π-semigroup with generator $\mathcal{A}$ such that $\|P_t\|_{\mathcal{L}(C_b(E))} \leq M\ e^{\alpha t}$, $t \geq 0$ with $\alpha \in \mathbb{R}$ and $M \geq 1$. Consider the operators $(F_\lambda)_{\lambda > \alpha}$ defined in (2.6). Then it holds for any $\lambda > \alpha$:*
(i) there exists $R(\lambda, \mathcal{A}) = F_\lambda$;
(ii) we have $\|R(\lambda, \mathcal{A})^n\|_{\mathcal{L}(C_b(E))} \leq \frac{M}{(\lambda - \alpha)^n}$, $n \geq 1$.

Now we define the important class of transition π-semigroups. Let T_t be a π-semigroup on $C_b(E)$. We say that T_t is a *transition (Markov) π-semigroup* if in addition

$$T_t f(x) = \int_E f(y)\,p(t,x,dy), \quad f \in C_b(E),\ x \in E,\ t > 0. \tag{2.7}$$

Here $p(t, x, B)$, $t > 0$, B Borel set of E and $x \in E$, denotes a *transition (Markov) function* on E (see for instance Dynkin [11] for details).

Remark 2.6. We point out that the choice of the space $C_b(E)$ is done for convenience. Indeed, according to §6.4 in Priola [17], the theory of π-semigroups can be developed also in more general functions spaces as $\mathcal{BC}(E)$ (the Banach space of all real continuous and bounded functions on E, endowed with the sup norm). In particular all results of this section and of the next one can be extended to $\mathcal{BC}(E)$. Note that many transition Markov semigroups, even the heat semigroup, are not strongly continuous on $\mathcal{BC}(\mathbb{R})$.

3. CAUCHY PROBLEM FOR π-SEMIGROUPS

Let P_t be a π-semigroup of type ω on $C_b(E)$ and let $\mathcal{A}$ be its generator. This section is devoted to study the following initial value problem for a fixed $T > 0$:

$$\begin{cases} \partial_t u(t,x) = \mathcal{A}u(t,x) + F(t,x), & t \in]0,T],\ x \in E, \\ u(0,x) = f(x), & x \in E, \end{cases} \tag{3.1}$$

where $f \in C_b(E)$ and F satisfies suitable assumptions. We set $\mathcal{A}u(t,x) = \mathcal{A}u(t,\cdot)(x)$, $x \in E$, $t \in]0,T]$. In the sequel we will use indifferently the symbols ∂_1 and ∂_t to denote the partial derivative with respect to the time variable. In order to treat (3.1) we introduce appropriate functions spaces.

Definition 3.1. Let I be an interval of $\mathbb{R}$ and $G : I \times E \to \mathbb{R}$ be a map, we say that:
(i) $G \in C_\pi(I; C_b(E))$ if $G(t, \cdot) \in C_b(E)$ for any $t \in I$, $G(\cdot, x) : I \to \mathbb{R}$ is continuous for any $x \in E$ and

$$\|G\|_0 = \sup_{t \in I} \sup_{x \in E} |G(t,x)| < \infty.$$

(ii) $G \in C^1_\pi(I; C_b(E))$ if $G \in C_\pi(I; C_b(E))$, $G(\cdot, x) : I \to \mathbb{R}$ is continuously differentiable for any $x \in E$ and $\partial_t G \in C_\pi(I; C_b(E))$.

(iii) $G \in \mathcal{C}_\pi(I;D(\mathcal{A}))$ if $G \in \mathcal{C}_\pi(I;\mathcal{C}_b(E))$, $G(t,\cdot) \in D(\mathcal{A})$ for any $t \in I$ and the map $\mathcal{A}G$, $\mathcal{A}G(t,x) = \mathcal{A}[G(t,\cdot)](x)$, $t \in I$, $x \in E$, belongs to $\mathcal{C}_\pi(I;\mathcal{C}_b(E))$.

Let us remark that for any $T > 0$, if $G \in \mathcal{C}^1_\pi(]0,T[,\mathcal{C}_b(E)) \cap \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$ then it is easy to verify that G is continuous on $[0,T] \times E$. Now we define the notion of π_I-convergence. Let $(G_n) \subset \mathcal{C}_\pi(I,\mathcal{C}_b(E))$, we say that G_n is *π_I-convergent* to $G \in \mathcal{C}_\pi(I,\mathcal{C}_b(E))$ and we shall write $G_n \xrightarrow{\pi_I} G$ as $n \to \infty$, if the following conditions hold:

$$\lim_{n\to\infty} G_n(t,x) = G(t,x) \quad t \in I,\ x \in E \quad \text{and} \quad \sup_{n\geq 1} \|G_n\|_0 < \infty. \tag{3.2}$$

We simply write π_T instead of $\pi_{[0,T]}$, for any $T > 0$.

Having in mind the standard theory of Cauchy problems associated with strongly continuous semigroups (see Pazy [15] and also Lunardi [14]), we make precise the notions of solution of (3.1).

Definition 3.2. Consider the problem (3.1) with the initial datum $f \in \mathcal{C}_b(E)$ and $F \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$. Then

(a) a map $u \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{A}))$ that satisfies (3.1) is said to be a *strict solution* of (3.1);
(b) a map $u \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$ is said to be a *strong solution* of (3.1), if there exists a sequence $(u_n) \subset \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{A}))$ such that

$$\begin{cases} u_n \xrightarrow{\pi_T} u, \quad \partial_t u_n - \mathcal{A}u_n \xrightarrow{\pi_T} F \ \text{ as } n \to \infty, \\ u_n(0,\cdot) \xrightarrow{\pi} f \ \text{ as } n \to \infty. \end{cases} \tag{3.3}$$

Let now $F \in \mathcal{C}_\pi(]0,T];\mathcal{C}_b(E))$. Then
(c) a map $u \in \mathcal{C}^1_\pi(]0,T];\mathcal{C}_b(E)) \cap \mathcal{C}_\pi(]0,T];D(\mathcal{A})) \cap \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$ that satisfies (3.1) is said to be a *classical solution* of (3.1).

Clearly any strict solution of (3.1) is also a classical solution. We stress that for any $g \in D(\mathcal{A})$, the map $u(t,x) = P_t g(x)$ is a strict solution of (3.1) with the initial datum $f = g$ and $F = 0$. This follows readily by Proposition 2.4. We are going to prove a uniqueness result about classical solutions. We need two preliminary lemmas that will be frequently used in the sequel.

Lemma 3.3. *Let I be an interval of $\mathbb{R}$, μ be a Borel finite measure on I and (X,d) be a separable metric space. Consider the functions $G,\ G_n : I \times X \to \mathbb{R}$, $n \geq 1$ that satisfy the following conditions:*
(i) $G_n(\cdot,x)$ is a Borel mapping for any $x \in X$, $n \geq 1$;

(ii) $G_n(t,\cdot)$ is a continuous mapping, for $n \geq 1$, $t \in I$;
(iii) there exists $g : I \to \mathbb{R}$, μ- integrable such that $|G_n(t,x)| \leq g(t)$, $n \geq 1$, $x \in X$, $t \in I$;
(iv) $\lim_{n\to\infty} \sup_{x\in X} |G_n(t,x) - G(t,x)| = 0$, for $t \in I$.
Then we have

$$\lim_{n\to\infty} \sup_{x\in X} \int_I |G_n(t,x) - G(t,x)|\,\mu(dt) = 0.$$

Proof. First let us notice that by assumptions (ii) and (iii), applying the Dominated Convergence Theorem, we find that the map $G(t,\cdot) : X \to \mathbb{R}$ is continuous and bounded for $t \in I$. Fix a countable dense subset D in X. We get

$$\sup_{x\in X} |G_n(t,x) - G(t,x)| = \sup_{x\in D} |G_n(t,x) - G(t,x)|, \quad n \geq 1,$$

since $|G_n(t,\cdot) - G(t,\cdot)|$ is continuous from X into $\mathbb{R}$, for $n \geq 1$, $t \in I$. Moreover for any $n \geq 1$, the map: $I \to \mathbb{R}$, $t \mapsto \sup_{x\in D} |G_n(t,x) - G(t,x)|$ is Borel and further $\sup_{x\in D} |G_n(t,x) - G(t,x)| \leq 2g(t)$, $n \geq 1$, $t \in I$. Thus we have

$$\sup_{x\in X} \int_I |G_n(t,x) - G(t,x)|\,\mu(dt) \leq \int_I \sup_{x\in D} |G_n(t,x) - G(t,x)|\,\mu(dt). \tag{3.4}$$

Letting $n \to \infty$ in the right-hand side of (3.4), by the Dominated Convergence Theorem, we get the assertion. □

Lemma 3.4. *Let $\mathcal{A}$ be the generator of a π-semigroup P_t on $\mathcal{C}_b(E)$ of type ω. For any $f \in \mathcal{C}_b(E)$ we have that $\lambda R(\lambda,\mathcal{A})f \xrightarrow{\pi} f$ as $\lambda \to \infty$.*
Moreover if P_t satisfies condition (2.3) of Definition 2.1 with respect to a non trivial covering $\mathcal{S}$ of E, then we have

$$\lim_{\lambda\to\infty} \sup_{x\in S} |\lambda R(\lambda,\mathcal{A})f(x) - f(x)| = 0, \quad f \in \mathcal{C}_b(E),\ S \in \mathcal{S}. \tag{3.5}$$

Proof. Fix $f \in \mathcal{C}_b(E)$, by Proposition 2.5, we have $\|\lambda R(\lambda,\mathcal{A})f\|_0 \leq 2M\|f\|_0$, $\lambda > 2\omega$. Then, by changing variable, we get

$$|\lambda R(\lambda,\mathcal{A})f(x) - f(x)| \leq \int_0^\infty e^{-w} |P_{\frac{w}{\lambda}}f(x) - f(x)|dw, \quad x \in E,\ \lambda > \omega. \tag{3.6}$$

Now the first assertion follows by the Dominated Convergence Theorem. If in addition P_t satisfies (2.3) then the second statement is proved invoking Lemma 3.3. □

Theorem 3.5. *Consider the initial value problem (3.1) and suppose that $f \in \mathcal{C}_b(E)$ and $F \in \mathcal{C}_\pi(]0,T];\mathcal{C}_b(E))$. Then the problem has at most one classical solution. Further if it has a classical solution u, this solution is given by*

$$u(t,x) = P_t f(x) + \int_0^t P_{t-s}F(s,x)ds, \quad t \in [0,T],\ x \in E. \tag{3.7}$$

Proof. First let us notice that for any $x \in E$, $t > 0$, the map:

$$]0,t] \to \mathbb{R}, \quad s \mapsto P_{t-s}F(s,x) = P_{t-s}[F(s,\cdot)](x)$$

in general is not continuous on $]0,t]$ (an example is given in Priola [17, Examples 5.1.5]). However we are going to show that it is a Borel and bounded map and so the integral in (3.7) is meaningful in the Lebesgue sense.

Consider the map: $\phi :]0,t]\times]0,t] \to \mathbb{R}$, $\phi(p,q) = P_{t-p}F(q,x)$, $p,q \in]0,t]$.
We claim that ϕ is separately continuous in each variable. The continuity with respect to p (with q fixed) is clear. As for the continuity with respect to q, remark that for any $q \in]0,t]$, $F(q+h,\cdot) \xrightarrow{\pi} F(q,\cdot)$ as $h \to 0$ and so $\lim_{h\to 0} P_{t-p}F(q+h,x) = P_{t-p}F(q,x)$. Thus ϕ is a Borel map on $]0,t]\times]0,t]$ and consequently $s \mapsto \phi(s,s)$ is a Borel map on $]0,t]$.

Moreover since $\|P_{t-s}F(s,\cdot)\|_0 \le Me^{\omega t}\|F\|_0$, $s \in]0,t]$, the map $s \mapsto P_{t-s}F(s,x)$ is also bounded. Applying Lemma 2.2, we get that the integral in (3.7) defines a function that belongs to $\mathcal{C}_b(E)$, for any $t \in [0,T]$.

Let now $u(t,x)$ be a classical solution of (3.1). Fix $t \in]0,T]$ and $z \in E$ and consider the map: $[0,t] \to \mathbb{R}$, $s \mapsto P_{t-s}u(s,z)$. In general this map is not differentiable (an example is provided in Priola [17, Examples 5.1.5]), so we can not proceed as in the theory of strongly continuous semigroups (see for instance Corollary 4.2.2 of Pazy [15]). However we are going to prove that setting $R(\lambda,\mathcal{A}) = R(\lambda)$, for a fixed $\lambda > \omega$, the mapping:

$$\eta : [0,t] \to \mathbb{R}, \quad \eta(s) \stackrel{def}{=} R(\lambda)\,P_{t-s}u(s,z) = R(\lambda)\,[P_{t-s}u(s,\cdot)](z) \qquad (3.8)$$

is continuous on $[0,t]$ and differentiable on $]0,t]$, having a bounded derivative. From this fact we will deduce (3.7). We have, by changing variable,

$$\begin{aligned} \eta(s) &= R(\lambda)\,P_{t-s}u(s,z) = \int_0^\infty e^{-\lambda w}\,P_w(P_{t-s}u(s,z))\,dw \\ &= \int_0^\infty e^{-\lambda w}\,P_{w+t-s}u(s,z)\,dw = e^{\lambda t}e^{-\lambda s}\int_{t-s}^\infty e^{-\lambda v}\,P_v u(s,z)\,dv \\ &= g(s,s,s), \qquad \text{where} \quad g : [0,t]^3 \to \mathbb{R}, \\ g(r_1,r_2,r_3) &= e^{\lambda t}e^{-\lambda r_1}\int_{t-r_3}^\infty e^{-\lambda v}\,P_v u(r_2,z)\,dv, \quad r_i \in [0,t],\ i = 1,2,3. \end{aligned} \qquad (3.9)$$

Next computations are devoted to study differentiability properties of g in order to obtain that η is differentiable in $]0,t]$ (the continuity of η in $s = 0$ will follow by similar arguments).

Claim 1. $\partial_1 g : [0,t]^3 \to \mathbb{R}$ is continuous.
One can verify that $\partial_1 g$ is continuous in each variable, uniformly with respect to the other ones. To this end we only remark that $u(r_2+h,\cdot) \xrightarrow{\pi} u(r_2,\cdot)$ as $h \to 0$, since $u \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$, and so $\lim_{h\to 0} P_v u(r_2+h,z) = P_v u(r_2,z)$, $v \ge 0$.

Claim 2. There exists $\partial_2 g$ on $[0,t]\times]0,t]\times[0,t]$ and it is a continuous function on $[0,t]\times]0,t]\times[0,t]$.

Since $u \in \mathcal{C}^1_\pi(]0,T];\mathcal{C}_b(E))$, we have $\left\|\frac{u(s+h,\cdot)-u(s,\cdot)}{h}\right\|_0 \le \|\partial_1 u\|_0$, $s \in [0,t]$, h small enough. Hence, using that P_t preserves π-convergence, we obtain that there exists $\partial_s P_v u(s,z) = P_v \partial_s u(s,z)$, $s \in]0,t]$, $v \ge 0$. Moreover, since $|P_v\partial_s u(s,z)| \le M\,\|\partial_1 u\|_0 e^{\omega v}$, $s \in]0,t]$, $v \ge 0$, we can differentiate with respect to r_2 in the last integral of (3.9) and obtain the assertion arguing as in claim 1.

Claim 3. There exists $\partial_3 g$ on $[0,t]^3$ and further $\partial_3 g : [0,t]^3 \to \mathbb{R}$ is continuous.

Indeed we have $\partial_3 g(r_1,r_2,r_3) = e^{-\lambda(r_1-r_3)}\,P_{t-r_3}u(r_2,z)$, for r_1, r_2, $r_3 \in [0,t]$. Now the continuity of $\partial_3 g$ shall follow by proving that the map:

$$[0,t]\times[0,t] \to \mathbb{R}, \quad (r_2,r_3) \mapsto P_{t-r_3}u(r_2,z), \qquad (3.10)$$

is continuous. To see this fact, first observe that the map is separately continuous in each variable on $[0,t]^2$. Then consider the estimate $\left|\partial_{r_3}(P_{t-r_3}u(r_2,z))\right| \le M\,e^{\omega t}\|\mathcal{A}u\|_0$, $(r_2,r_3) \in [0,t]^2$. Thus also claim 3 is proved.

We revert to the map $\eta(s) = g(s,s,s)$ defined in (3.8). By claim 1, claim 2 and claim 3, we derive that η is continuously differentiable on $]0,t]$ and that

$$\frac{d}{ds}\eta(s) = \partial_1 g(s,s,s) + \partial_2 g(s,s,s) + \partial_3 g(s,s,s), \quad s \in]0,t]. \qquad (3.11)$$

Remark that one can prove the continuity of η in $s = 0$, proceeding in three steps as before. By (3.11) we get, for any $s \in]0,t]$,

$$\begin{aligned} &\frac{d}{ds}R(\lambda)P_{t-s}u(s,z) = -\lambda e^{\lambda t}e^{-\lambda s}\int_{t-s}^\infty e^{-\lambda v}\,P_v u(s,z)\,dv \\ &+ P_{t-s}u(s,z) + e^{\lambda t}e^{-\lambda s}\int_{t-s}^\infty e^{-\lambda v}\,P_v\partial_s u(s,z)\,dv \\ &= -\lambda R(\lambda)P_{t-s}u(s,z) + P_{t-s}u(s,z) + R(\lambda)P_{t-s}\partial_s u(s,z). \end{aligned} \qquad (3.12)$$

From this formula, using that u is a solution of the initial value problem (3.1) and the identity: $\mathcal{A}R(\lambda,\mathcal{A}) = \lambda R(\lambda,\mathcal{A}) - I$, we obtain, for any $s \in]0,t]$,

$$\begin{aligned} &\frac{d}{ds}R(\lambda)P_{t-s}u(s,z) = -\lambda R(\lambda)P_{t-s}u(s,z) + P_{t-s}u(s,z) \\ &+ R(\lambda)\mathcal{A}P_{t-s}u(s,z) + R(\lambda)P_{t-s}F(s,z) = R(\lambda)P_{t-s}F(s,z). \end{aligned} \qquad (3.13)$$

Then for any $\epsilon > 0$, $R(\lambda)u(t,z) - R(\lambda)P_{t-\epsilon}u(\epsilon,z) = \int_\epsilon^t R(\lambda)P_{t-s}F(s,z)ds$.
Since the map $s \mapsto R(\lambda)P_{t-s}u(s,z)$ is continuous on $[0,t]$ and the map $s \mapsto R(\lambda)P_{t-s}F(s,z)$ is bounded on $]0,t]$, letting $\epsilon \to 0^+$ in the last formula, we obtain $R(\lambda)u(t,z) - R(\lambda)P_t f(z) = \int_0^t R(\lambda)P_{t-s}F(s,z)ds$. Multiplying both sides of this formula for λ we find

$$\lambda R(\lambda)\,[u(t,z) - P_t f(z)] = \int_0^t \lambda R(\lambda)P_{t-s}F(s,z)ds, \quad \lambda > \omega. \qquad (3.14)$$

We claim that letting $\lambda \to \infty$ in (3.14), we get formula (3.7). Indeed, invoking Lemma 3.4, the left-hand side of (3.14) tends to $u(t,z) - P_t f(z)$ as $\lambda \to \infty$. Let us consider the right-hand side. For any fixed $s \in]0,t]$, by Lemma 3.4, we have

$$\lim_{\lambda\to\infty} \lambda R(\lambda) P_{t-s} F(s,z) = P_{t-s} F(s,z).$$

Further $|\lambda R(\lambda) P_{t-s} F(s,z)| \le \lambda \|R(\lambda)\|_{\mathcal{L}} \|P_{t-s} F(s,\cdot)\|_0 \le 2M e^{\omega t} \|F\|_0$, $s \in]0,t]$, $\lambda > 2\omega$. Applying the Dominated Convergence Theorem, the right-hand side of (3.14) tends to $\int_0^t P_{t-s} F(s,z) ds$ as $\lambda \to \infty$. This completes the proof. □

Now we prove a result of existence and uniqueness for strong solutions.

Theorem 3.6. *Consider the initial value problem (3.1) and suppose that $f \in \mathcal{C}_b(E)$ and $F \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$. Then there exists a unique strong solution u for (3.1) and further for any fixed $t \in [0,T]$, $x \in E$, we have*

$$u(t,x) = P_t f(x) + \int_0^t P_{t-s} F(s,x) ds. \tag{3.15}$$

Proof. Existence. First we verify that $u \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(E))$. It is enough to consider the term:

$$v(t,x) = \int_0^t P_{t-s} F(s,x) ds, \quad t \in [0,T],\ x \in E. \tag{3.16}$$

Clearly $\|v\|_0 \le MT e^{\omega T} \|F\|_0$. By Lemma 2.2, it follows that $v(t,\cdot) \in \mathcal{C}_b(E)$. Let us fix $x \in E$. It is straightforward to verify that $v(\cdot,x)$ is continuous on $[0,T]$..

Let $R(\lambda) = R(\lambda,\mathcal{A})$, $\lambda > \omega$, we consider the following approximations for any $n > \omega$,

$$u_n(t,x) = nR(n) P_t f(x) + \int_0^t nR(n) P_{t-s} F(s,x) ds, \quad x \in E,\ t \in [0,T]. \tag{3.17}$$

We check that for $n > \omega$, $u_n \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{A}))$.

Remark that $R(\lambda) P_t f = P_t R(\lambda) f$, $f \in \mathcal{C}_b(E)$, $\lambda > \omega$, $t \ge 0$ (it follows by Proposition 2.4). Since $P_t nR(n) f \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{A}))$, $n > \omega$, let us consider directly the more difficult term

$$v_n(t,x) = \int_0^t nR(n) P_{t-s} F(s,x) ds, \quad x \in E,\ t \in [0,T]. \tag{3.18}$$

We set $F_n(s,\cdot) = nR(n) F(s,\cdot)$, $s \in [0,T]$. Let us fix $n > \omega$ and $z \in E$ and establish the differentiability of $v_n(\cdot,z)$. Let $t \in]0,T[$, we start to prove the existence of the right derivative of $v_n(\cdot,z)$ in t. To this purpose we write for any $h > 0$, sufficiently small,

$$\frac{v_n(t+h,z) - v_n(t,z)}{h} = \frac{1}{h}\Big(\int_0^{t+h} P_{t+h-s} F_n(s,z) ds - \int_0^t P_{t+h-s} F_n(s,z) ds\Big)$$
$$+ \frac{1}{h}\Big(\int_0^t P_{t+h-s} F_n(s,z) ds - \int_0^t P_{t-s} F_n(s,z) ds\Big) = \Gamma_1(h) + \Gamma_2(h) \quad \text{where}$$

$$\Gamma_1(h) = \frac{1}{h}\int_t^{t+h} P_{t+h-s} F_n(s,z) ds, \quad \Gamma_2(h) = \int_0^t \Big(\frac{P_{t+h-s} - P_{t-s}}{h}\Big) F_n(s,z) ds. \tag{3.19}$$

As concerns Γ_2, taking into account that $\partial_t \Big(P_{t-s} F_n(s,z)\Big) = P_{t-s} \mathcal{A} F_n(s,z)$, $s \in [0,t[$ and $|\partial_t P_{t-s} F_n(s,z)| \le M \|\mathcal{A} nR(n)\|_{\mathcal{L}} \|F\|_0 e^{\omega T}$, applying the Dominated Convergence Theorem, we get $\lim_{h\to 0^+} \Gamma_2(h) = \int_0^t P_{t-s} \mathcal{A} F_n(s,z) ds$.

Let us turn to Γ_1. Changing variable, first $t + h - s = w$ and then $rh = w$, we get

$$\Gamma_1(h) = \frac{1}{h}\int_0^h P_w F_n(t+h-w,z) dw = \int_0^1 P_{rh} F_n(t+h-rh,z) dr. \tag{3.20}$$

Now one has $\lim_{h\to 0^+} P_{rh} F_n(t+h-rh,z) = F_n(t,z)$, $r \in [0,1]$. Indeed let us consider the map $\phi : [0,t] \times [0,t] \to \mathbb{R}$, $\phi(u,v) = P_u F_n(v,z)$. ϕ is separately continuous in each variable and further there exists the partial derivative $\partial_u \phi$ on $[0,t] \times [0,t]$. Since $\partial_u \phi$ is bounded on $[0,t] \times [0,t]$, we can easily conclude that ϕ is continuous on $[0,t] \times [0,t]$ and so the previous formula holds.

Applying the Dominated Convergence Theorem in (3.20) we deduce that $\lim_{h\to 0^+} \Gamma_1(h) = F_n(t,z)$. Thus there exists the right derivative of $v_n(\cdot,z)$ on $]0,T[$ and is given by

$$\frac{d^+}{dt} v_n(t,z) = \int_0^t P_{t-s} \mathcal{A} nR(n) F(s,z) ds + nR(n) F(t,z). \tag{3.21}$$

Let us remark that the right-hand side of (3.21), in the variable t, is a continuous function on $[0,T]$. Hence for a well known lemma of Real Analysis we deduce that $v_n(\cdot,z)$ is differentiable on $[0,T]$, with the derivative given by (3.21). To prove that $v_n \in \mathcal{C}^1_\pi([0,T];\mathcal{C}_b(E))$, it remains to verify that $\partial_t v_n(t,\cdot) \in \mathcal{C}_b(E)$, $t \in [0,T]$, $n > \omega$. But this fact follows invoking Lemma 2.2.

Now we check that $v_n \in \mathcal{C}_\pi([0,T];D(\mathcal{A}))$. To this end we remark that

$$P_h(v_n(t,\cdot))(x) = \int_0^t P_h P_{t-s} F_n(s,x) ds, \quad t \in [0,T],\ n > \omega,\ x \in E. \tag{3.22}$$

Indeed thanks to the continuity of the map $s \mapsto nP_{t-s} R(n) F(s,x)$ on $[0,t]$ (see the proof of Theorem 3.5), $\int_0^t P_{t-s} F_n(s,\cdot)$ is a π-limit in $\mathcal{C}_b(E)$ of a sequence of Riemann sums. Now by (3.22), we obtain easily that $v_n(t,\cdot) \in D(\mathcal{A})$ and it holds

$$\mathcal{A} v_n(t,x) = \lim_{h\to 0^+} \Big(\frac{P_h - I}{h}\Big) v_n(t,\cdot)(x) = \int_0^t \mathcal{A} P_{t-s} F_n(s,x) ds. \tag{3.23}$$

It follows that $\mathcal{A} v_n(\cdot,x)$ is continuous on $[0,T]$. Further $\mathcal{A} v_n(t,\cdot) \in \mathcal{C}_b(E)$, $t \in [0,T]$, $n > \omega$, thanks to Lemma 2.2. Hence $v_n \in \mathcal{C}_\pi([0,T];D(\mathcal{A}))$.

Finally we prove that $u_n(0,\cdot) \xrightarrow{\pi} f$, $u_n \xrightarrow{\pi_T} u$, $\partial_t u_n - \mathcal{A}u_n \xrightarrow{\pi_T} F$ as $n \to \infty$.
First we have readly that $\|u_n\|_0 \le 2M\, e^{\omega T}(\|f\|_0 + T\|F\|_0)$, $n > 2\omega$.
Then applying Lemma 3.4 and the Dominated Convergence Theorem we obtain

$$\begin{aligned} \lim_{n\to\infty} u_n(0,x) &= \lim_{n\to\infty} nR(n)f(x) = f(x) \\ \lim_{n\to\infty} u_n(t,x) &= P_t f(x) + \int_0^t P_{t-s}F(s,x)ds = u(t,x), \\ & \qquad t \in [0,T],\ x \in E. \end{aligned} \tag{3.24}$$

Using (3.21) and (3.23) we have that $\partial_t u_n(t,x) - \mathcal{A}u_n(t,x) = nR(n)F(t,x), t \in [0,T]$, $x \in E$. Then it is clear that $nR(n)F \xrightarrow{\pi_T} F$ as $n \to \infty$. The existence of a strong solution is proved.

Uniqueness. Suppose that w is a strong solution of the Cauchy problem (3.1) and let (w_n) be a sequence of approximating strict solutions for w.

Setting $G_n \stackrel{def}{=} \partial_t w_n - \mathcal{A}w_n$ and $g_n \stackrel{def}{=} w_n(0,\cdot)$, for any $n \ge 1$, w_n is the strict solution of the following initial value problem

$$\begin{cases} \partial_t w_n(t,x) = \mathcal{A}w_n(t,x) + G_n(t,x), & t \in]0,T],\ x \in E, \\ w_n(0,x) = g_n(x), & x \in E. \end{cases} \tag{3.25}$$

Applying Theorem 3.5 we obtain that

$$w_n(t,x) = P_t g_n(x) + \int_0^t P_{t-s}G_n(s,x)ds, \quad x \in E,\ t \in [0,T]. \tag{3.26}$$

By our assumptions, $g_n \xrightarrow{\pi} f$ and so $P_q g_n \xrightarrow{\pi} P_q f$ as $n \to \infty$ for any $q \ge 0$. Moreover $G_n \xrightarrow{\pi_T} F$ and hence, by the Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_0^t P_{t-s}G_n(s,x)ds = \int_0^t P_{t-s}F(s,x)ds, \quad x \in E,\ t \in [0,T].$$

Since $w_n \xrightarrow{\pi_T} u$ as $n \to \infty$, it follows that $u(t,x) = \lim_{n\to\infty} w_n(t,x) = P_t f(x) + \int_0^t P_{t-s}F(s,x)ds$, $t \in [0,T]$, $x \in E$. The proof is complete. □

The next two results show that if one imposes additional conditions on f and F in problem (3.1), then the strong solution (3.15) becomes a strict solution.

Theorem 3.7. *Consider the initial value problem (3.1) and suppose that $f \in D(\mathcal{A})$ and $F \in C^1_\pi([0,T];C_b(E))$. Then the strong solution u of (3.1) is a strict solution.*

Proof. Of course $f \in D(\mathcal{A})$ is a necessary condition in order to obtain that there exists a strict solution for (3.1). We write for any $t \in [0,T]$, $x \in E$.

$$u(t,x) = P_t f(x) + \int_0^t P_{t-s}F(s,x)ds = P_t f(x) + v(t,x). \tag{3.27}$$

It is easy to verify that $P_t f \in C^1_\pi([0,T];C_b(E)) \cap C_\pi([0,T];D(\mathcal{A}))$. Therefore we deal with the map v. We already know that $v \in C_\pi([0,T];C_b(E))$. We will deduce differentiability for v by considering the approximating mappings

$$v_n(t,x) = \int_0^t nR(n)P_{t-s}F(s,x)ds = \int_0^t P_q F_n(t-q,x)dq, \tag{3.28}$$

where $R(n)$ stands for $R(n,\mathcal{A})$, $n > \omega$, $F_n(t,\cdot) = nR(n)F(t,\cdot)$, $t \in [0,T]$. Thanks to Theorem 3.6, we already know that $v_n \in C^1_\pi([0,T];C_b(E))$, $n > \omega$.

We need a suitable representation for $\partial_t v_n$ based on the existence of $\partial_t F$ (compare with (3.19) and (3.21)). To this purpose we fix $t \in]0,T[$, $x \in E$ and write for any $h > 0$, small enough,

$$\frac{v_n(t+h,x) - v_n(t,x)}{h} = \Gamma_1(h) + \Gamma_2(h) \text{ where } \Gamma_1(h) = \frac{1}{h}\int_t^{t+h} P_q F_n(t+h-q,x)dq,$$

$$\Gamma_2(h) = \int_0^t P_q\Big(\frac{F_n(t+h-q,\cdot) - F_n(t-q,\cdot)}{h}\Big)(x)\,dq. \tag{3.29}$$

As for Γ_2, since $|\partial_t F_n(s,x)| \le 2\|\partial_t F\|_0$, $s \in [0,T]$, $n > 2\omega$, we have $h^{-1}F(s+h,\cdot) - F(s,\cdot) \xrightarrow{\pi} \partial_1 F(s,\cdot)$ as $h \to 0^+$, for any $s \in [0,T]$. Applying the Dominated Convergence Theorem we obtain that

$$\lim_{h\to 0^+} \Gamma_2(h) = \int_0^t P_q nR(n)\partial_1 F(t-q,x)dq. \tag{3.30}$$

Let us turn to Γ_1. We can argue similarly to (3.20) in order to obtain that $\lim_{h\to 0^+} \Gamma_1(h) = P_t F_n(0,x)$. Using the above computations, we find for any $r \in [0,T]$,

$$\partial_1 v_n(r,x) = \int_0^r P_{r-s}nR(n)\partial_1 F(s,x)\,ds + nR(n)P_r F(0,x). \tag{3.31}$$

Now we can show that $v(\cdot,x)$ is differentiable on $[0,T]$. Indeed we already know that $\lim_{n\to\infty} v_n(r,x) = v(r,x)$, $r \in [0,T]$. Further it is clear that

$$\lim_{n\to\infty} \partial_1 v_n(r,x) = \int_0^r P_{r-s}\partial_1 F(s,x)\,ds + P_r F(0,x) = w(r,x),\ r \in [0,T].$$

By (3.31) it follows $\|\partial_1 v_n\|_0 \le 2M\, e^{\omega T}\big(T\|\partial_1 F\|_0 + \|F\|_0\big)$, $n > 2\omega$.
Taking into account this estimate and the fact that $w(\cdot,x)$ is continuous on $[0,T]$, we find that $v(\cdot,x)$ is differentiable on $[0,T]$ and further that

$$\partial_1 v(r,x) = \int_0^r P_{r-s}\partial_1 F(s,x)\,ds + P_r F(0,x), \quad r \in [0,T]. \tag{3.32}$$

From this formula we deduce that $v \in C^1_\pi([0,T];C_b(E))$.

It remains to check that $u \in C_\pi([0,T];D(\mathcal{A}))$ and u satisfies the problem (3.1). To this purpose setting $u_n(t,x) = v_n(t,x) + nR(n)P_t f(t,x)$, $t \in [0,T]$, $x \in E$, we obtain $\mathcal{A}u_n(t,x) = \partial_1 u_n(t,x) - nR(n)F(t,x)$, $t \in [0,T]$, $x \in E$, $n > \omega$.
By the previous computations, $\partial_1 u_n(t,\cdot) \xrightarrow{\pi} \partial_1 u(t,\cdot)$ as $n \to \infty$ so that

$$\mathcal{A}u_n(t,\cdot) \xrightarrow{\pi} \partial_1 u(t,\cdot) - F(t,\cdot) \text{ as } n \to \infty, \quad t \in [0,T].$$

Since $\mathcal{A}$ is a $\pi-$ closed operator we find that $u(t,\cdot) \in D(\mathcal{A})$ and moreover $\mathcal{A}u(t,x) = \partial_1 u(t,x) - F(t,x)$, $x \in E$, $t \in [0,T]$. The proof is complete. □

Theorem 3.8. *Consider the initial value problem (3.1) and suppose that* $f \in D(\mathcal{A})$ *and* $F \in \mathcal{C}_\pi([0,T]; D(\mathcal{A}))$. *Then the strong solution u of (3.1) is a strict solution.*

Proof. We argue as in the proof of Theorem 3.7, with the same notations. This way we find readly $\partial_1 v(r,x) = \int_0^r P_{r-s}\mathcal{A}F(s,x)\,ds + F(r,x), \quad r \in [0,T]$ and the assertion follows. □

4. STRONG APPROXIMATION RESULTS

Let Ω be an open set of a real separable Hilbert space H (with inner product $< \cdot,\cdot >$ and norm $|\cdot|$). In several applications, given a transition π-semigroup with generator $\mathcal{A}$, representing the solutions of a PDE in $\mathcal{C}_b(\Omega)$, there exists a "natural" subspace of $D(\mathcal{A})$ where $\mathcal{A}$ can be represented as a "concrete" differential operator. We will denote this restriction of $\mathcal{A}$ by $\mathcal{A}_0$.

Given the initial value problem (3.1), it is useful for various applications (see for instance Gozzi [12]), to approximate any strong solution of (3.1), by means of a sequence u_n of strict solutions such that $u_n(t,\cdot) \in D(\mathcal{A}_0)$, $t \in [0,T]$, $n \geq 1$.

This problem has been investigated by Cerrai and Gozzi [5] for some classes of semigroups. We present here a different and more general approach. To be definite, in this section we only consider the Ornstein-Uhlenbeck operator on $\mathcal{C}_b(H)$. This is a prototype for illustrating our method that can be applied in several situations. In Priola [17] we have considered the approximation of strong solutions associated with the heat semigroup on $\mathcal{C}_b(H)$ and also with the semigroup corresponding to an infinite dimensional Dirichlet problem in a half space of H.

$\mathcal{L}_1(H)$ stands for the subspace of $\mathcal{L}(H)$ of all trace class (or nuclear) operators. $\mathcal{L}_1(H)$ is a Banach space endowed with the norm $\|T\|_{\mathcal{L}_1(H)} = \text{Tr}\,(\sqrt{T^*T})$, $T \in \mathcal{L}_1(H)$ (Tr (T) and T^* denote respectively the trace and the adjoint operator of T).

Let M be a self-adjoint, non negative, bounded linear operator on H. Let A be the generator of a strongly continuous semigroup S_t on H. We suppose that there exist $\omega < 0$ and $C > 0$ such that $\|S_t\|_{\mathcal{L}(H)} \leq C\,e^{\omega t}$, $t \geq 0$. This assumption is not restrictive. Indeed by standards arguments, it is possible to adapt all the proofs to the general case of $\omega \in \mathbb{R}$. In addition we assume that for each $t \geq 0$, the bounded linear operators $M(t)$, $M(t)x = \int_0^t S_u M S_u^* x\,du$, $x \in H$, are trace class. We are dealing with the following initial value problem:

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\text{Tr}\,[MD^2u(t,x)] + < Ax, Du(t,x) > + F(t,x), & x \in D(A),\ t \in]0,T] \\ u(0,x) = f(x), & x \in H, \end{cases} \tag{4.1}$$

where $f \in \mathcal{C}_b(H)$ and $F \in \mathcal{C}_\pi([0,T]; \mathcal{C}_b(H))$. Under our assumptions, there exist the Gaussian measures $\mathcal{N}(S_t x, M(t))$, $t \geq 0$, $x \in H$, with covariance operator $M(t)$ and mean x (see Da Prato and Zabczyk [9] for more details). The Ornstein-Uhlenbeck semigroup on $\mathcal{C}_b(H)$ associated with S_t and M is defined as follows:

$$U_t f(x) = \int_H f(S_t x + y)\,\mathcal{N}(0, M(t))\,dy, \quad f \in \mathcal{C}_b(H),\ x \in H,\ t > 0. \tag{4.2}$$

This semigroup has been intensively studied, under various assumptions (cf. Cannarsa and Da Prato [2], Cerrai [4], Cerrai and Gozzi [5], Da Prato and Lunardi [8], Priola [17], Zambotti [18]). Unless $S_t = I$, for any $t \geq 0$, U_t is not strongly continuous even in $\mathcal{C}_b(\mathbb{R})$ (see §6 of Cerrai [4]). However U_t turns out to be a transition π-semigroup on $\mathcal{C}_b(H)$. Actually a stronger assertion holds.

Proposition 4.1. *For any compact subset* K *of* H, $f \in \mathcal{C}_b(H)$ *one has*

$$\lim_{h\to 0}\sup_{x\in K} |U_{t+h}f(x) - U_t f(x)| = 0, \quad t \geq 0. \tag{4.3}$$

This result was proved by Cerrai [4] in case when S_t is a semigroup of negative type (this hypothesis can be removed, with few changes in Cerrai's proof). For a simpler proof we refer to Proposition 3.3.1 of Priola [17].

$\mathcal{U}$ will denote the generator of U_t. Following Cerrai and Gozzi [5], we use the next notations.

A sequence $(f_n) \subset \mathcal{C}_b(H)$ *is said to be* $\mathcal{K}$*-convergent to* $f \in \mathcal{C}_b(H)$ *and we shall write* $f_n \xrightarrow{\mathcal{K}} f$ *as* $n \to \infty$, *if for any compact set* $K \subset H$, *we have that*

$$\sup_{n\geq 1}\|f_n\|_0 < \infty, \quad \text{and} \quad \lim_{n\to\infty}\sup_{x\in K}|f_n(x) - f(x)| = 0. \tag{4.4}$$

Similarly let $(G_n) \subset \mathcal{C}_\pi([0,T], \mathcal{C}_b(H))$, *we say that* G_n *is* $\mathcal{K}_T$ *- convergent to a map* $G \in \mathcal{C}_\pi([0,T], \mathcal{C}_b(H))$ *and we shall write* $G_n \xrightarrow{\mathcal{K}_T} G$ *as* $n \to \infty$, *if* $\sup_{n\geq 1}\|G_n\|_0 < \infty$ *and moreover for any compact set* $K \subset H$ *one has*

$$\lim_{n\to\infty}\sup_{x\in K,\, t\in[0,T]} |G_n(t,x) - G(t,x)| = 0. \tag{4.5}$$

$\mathcal{C}_b^k(H)$, $k \geq 1$ denotes the subspace of $\mathcal{C}_b(H)$ of all functions having uniformly continuous and bounded Fréchet derivatives up to the order k. This is a Banach space endowed with the norm: $\|f\|_k = \|f\|_0 + \sum_{j=1}^k \|D^j f\|_0$, $f \in \mathcal{C}_b^k(H)$ (D^j stands for the Frechét derivative of order $j \geq 1$).

We will define a natural restriction of $\mathcal{U}$. To this purpose for any $f \in \mathcal{C}_b(H)$, following Ahmed et al [1], we define the map

$$f_A : D(A) \subset H \to \mathbb{R}, \quad f_A(x) = f \circ A(x) = f(Ax),\ x \in D(A). \tag{4.6}$$

We shall write $f_A \in C_b(H)$ if f_A has a uniformly continuous extension to the whole of H. This extension, that is unique, will be again denoted by f_A. Notice that if $f \in C_b^1(H)$ and in addition $f_A \in C_b^1(H)$ it is easy to verify that $Df(x) \in D(A^*)$, $x \in H$ and $A^*Df \in C_b(H,H)$. Now we define the Banach space $(\tilde{C}_b^2(H), \|\cdot\|_{\tilde{2}})$ as follows:

$$\tilde{C}_b^2(H) \stackrel{def}{=} \{f \in C_b^2(H), \text{ such that } D^2 f \in C_b(H, \mathcal{L}_1(H))\,(^2)\},$$

$\|f\|_{\tilde{2}} = \|f\|_2 + \sup_{x\in H} \|D^2 f(x)\|_{\mathcal{L}_1(H)}, \quad f \in \tilde{C}_b^2(H).$

Definition 4.2. We define a linear operator $\mathcal{U}_0 : D(\mathcal{U}_0) \subset C_b(H) \to C_b(H)$,

$$\begin{cases} D(\mathcal{U}_0) = \{f \in \tilde{C}_b^2(H) \text{ such that } f_A \in C_b^1(H) \\ \text{and the map } x \mapsto < A^*Df(x), x > \text{ belongs to } C_b(H)\}. \\ \mathcal{U}_0 f(x) \stackrel{def}{=} \frac{1}{2}\mathrm{Tr}\,[MD^2 f(x)] + < A^*Df(x), x >, \ f \in D(\mathcal{U}_0), \ x \in H. \end{cases} \tag{4.7}$$

$\mathcal{U}_0$ is not a core in the usual meaning. However in Cerrai and Gozzi [5] it is shown that $\mathcal{U}$ is the $\mathcal{K}$-closure of $\mathcal{U}_0$, namely $\mathcal{U}_0$ is a restriction of $\mathcal{U}$ and moreover for any $f \in D(\mathcal{U})$ there exists a sequence $(f_n) \subset D(\mathcal{U}_0)$ such that $f_n \xrightarrow{\mathcal{K}} f$, $\mathcal{U}_0 f_n \xrightarrow{\mathcal{K}} \mathcal{U}f$ as $n \to \infty$. We are ready to state our main results which improve Theorem 5.8 of Cerrai and Gozzi [5].

Theorem 4.3. *Consider the initial value problem (4.1) with $f \in C_b(H)$ and $F \in C_\pi([0,T]; C_b(H))$. Let u be the strong solution, namely*

$$u(t,x) = U_t f(x) + \int_0^t U_{t-s} F(s,x)ds, \quad t \in [0,T], \ x \in H. \tag{4.8}$$

Then there exists a sequence $(u_n) \subset C_\pi^1([0,T]; C_b(H)) \cap C_\pi([0,T]; D(\mathcal{U}))$ such that:

(i) $u_n(t,\cdot) \in D(\mathcal{U}_0)$, $t \in [0,T]$, $n \geq 1$.

(ii) $\sup_{n\geq 1} \left(\|u_n\|_0 + \|\partial_t u_n - \mathcal{U}_0 u_n\|_0\right) < \infty$ and
$u_n(t,\cdot) \xrightarrow{\mathcal{K}} u(t,\cdot)$, $\partial_t u_n(t,\cdot) - \mathcal{U}_0 u_n(t,\cdot) \xrightarrow{\mathcal{K}} F(t,\cdot)$ *as* $n \to \infty$, $t \in [0,T]$.

Theorem 4.4. *In (4.1) suppose that $f \in C_b(H)$, $F \in C_\pi([0,T]; C_b(H))$ and F is continuous on $[0,T] \times H$. Let u be the strong solution. Then there exists a sequence $(u_n) \subset C_\pi^1([0,T]; C_b(H)) \cap C_\pi([0,T]; D(\mathcal{U}))$ such that:*

(i) $u_n(t,\cdot) \in D(\mathcal{U}_0)$, $t \in [0,T]$, $n \geq 1$.

(ii) $u_n \xrightarrow{\mathcal{K}_T} u$, $\partial_t u_n - \mathcal{U}_0 u_n \xrightarrow{\mathcal{K}_T} F$ as $n \to \infty$.

[2] Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces, we denote by $C_b(E,F)$ the Banach space of all uniformly continuous and bounded functions from E into F, endowed with the usual sup norm: $\|f\|_0 = \sup_{x\in E} \|f(x)\|_F$.

Of course (ii) of Theorem 4.3 implies $u_n \xrightarrow{\pi_T} u$, $\partial_t u_n - \mathcal{U}_0 u_n \xrightarrow{\pi_T} F$ as $n \to \infty$. The remainder of this section is devoted to the proof of the previous two results.

The first step is defining a suitable functions space related to $\mathcal{U}_0$.

$$\tilde{C}_A^2(H) \stackrel{def}{=} \{f \in \tilde{C}_b^2(H), \text{ such that } f_A \in \tilde{C}_b^2(H)\}. \tag{4.9}$$

This space was introduced in Ahmed et al [1]. It is easy to prove that $\tilde{C}_A^2(H)$ is a Banach space, endowed with the norm: $\|f\|_{\tilde{2},A} = \|f\|_{\tilde{2}} + \|f_A\|_{\tilde{2}}$, $f \in \tilde{C}_A^2(H)$.

We need a technical lemma.

Lemma 4.5.
(a) Let $G : [0,T] \times H \to \mathbb{R}$ be a map satisfying:

(i) $G(\cdot, x)$ is a Borel map for any $x \in H$;

(ii) $G(s,\cdot) \in \tilde{C}_b^2(H)$, $s \in [0,T]$;

(iii) $\|G(s,\cdot)\|_{\tilde{C}_b^2(H)} \leq g(s)$, $s \in [0,T]$, where $g \in L^1([0,T])$.

Then for any fixed $t \in [0,T]$ the map $\phi : H \to \mathbb{R}$,

$$\phi(x) = \int_0^t G(s,x)\,ds, \quad x \in H \qquad \text{belongs to } \tilde{C}_b^2(H).$$

(b) Suppose that $J : [0,T] \times H \to \mathbb{R}$ is a map satisfying assumptions (i), (ii), (iii) with the space $\tilde{C}_b^2(H)$ replaced by $\tilde{C}_A^2(H)$. Then the map $\psi(\cdot) = \int_0^t J(s,\cdot)\,ds$ belongs to $\tilde{C}_A^2(H)$ for any $t \in [0,T]$.

Proof. (a) Fix any $t \in]0,T]$. First let us observe that by Lemma 2.2, the mapping ϕ belongs to $C_b(H)$. First we verify that $\phi \in C_b^1(H)$. By the Dominated Convergence Theorem, it follows easily that ϕ is Gâteaux differentiable on H with the Gâteaux derivative that is given by

$$< D\phi(x), v > = \int_0^t < D_x G(s,x), v > ds, \quad x \in H, \ v \in H. \tag{4.10}$$

Notice that $s \mapsto < D_x G(s,x), v >$ is Borel for any $x \in H$, $v \in H$ and so the map: $[0,t] \to H$, $s \mapsto D_x G(s,x)$. Now we check that $D\phi \in C_b(H,H)$, so that $D\phi$ is a Fréchet derivative and $\phi \in C_b^1(H)$. The boundedness of $D\phi$ is evident, let us prove the uniform continuity. For any $v \in H$ such that $\|v\|_H = 1$, $x, z \in H$, we have

$$| < D\phi(x) - D\phi(z), v > | \leq \int_0^t \|D_x G(s,x) - D_x G(s,z)\|_H\,ds \leq 2\int_0^t g(s)ds. \tag{4.11}$$

It follows $\|D\phi(x) - D\phi(z)\|_H \leq \int_0^t \|D_x G(s,x) - D_x G(s,z)\|_H\,ds$, $x, z \in H$.

To prove the uniform continuity of $D\phi$ it is enough to verify that for any sequence $(z_n) \subset H$ such that $z_n \to 0$, it holds:

$$\lim_{n\to\infty} \sup_{x\in H} \|D\phi(x+z_n) - D\phi(x)\|_H = 0. \tag{4.12}$$

Fix a countable dense subset L of H. For any $s \in [0,t]$, we have:

$$\sup_{x\in H} \|D_xG(s,x+z_n) - D_xG(s,x)\|_H = \sup_{x\in L} \|D_xG(s,x+z_n) - D_xG(s,x)\|_H.$$

Notice that the maps $\gamma_n : s \mapsto \sup_{x\in L} \|D_xG(s,x+z_n) - D_xG(s,x)\|_H$ are Borel, for any $n \geq 1$. Further we have $\gamma_n(s) \leq 2g(s)$, $s \in [0,T]$, $n \geq 1$. Hence we can write

$$\sup_{x\in H} \|D\phi(x+z_n) - D\phi(x)\|_H \leq \int_0^t \sup_{x\in L} \|D_xG(s,x+z_n) - D_xG(s,x)\|_H \, ds.$$

Letting $n \to \infty$, by the Dominated Convergence Theorem, we obtain (4.12) and the uniform continuity of $D\phi$ is proved.

To verify the other properties of ϕ one proceeds similarly. Notice that the measurability of the map $s \mapsto \|D_x^2G(s,z)\|_{\mathcal{L}_1(H)}$, for $z \in H$, follows from measurability of the maps: $[0,t] \to \mathbb{R}$, $s \mapsto < D_x^2G(s,z)u, v >$ for $u, v \in H$ and by the following two facts: (1) an operator $T \in \mathcal{L}(H)$ belongs to $\mathcal{L}_1(H)$ if and only if it holds (see Lemma 14, p. 1098 of Dunford and Schwartz [10]):

$$\sup\{|\mathrm{Tr}\,(NT)|,\ N \in \mathcal{L}(H), \text{ of finite rank and } \|N\|_{\mathcal{L}(H)} \leq 1\} = C < \infty,$$

further if $T \in \mathcal{L}_1(H)$ then $\|T\|_{\mathcal{L}_1(H)} = C$; (2) the subspace of $\mathcal{L}(H)$ of all finite rank operators is separable.

(b) Let us remark that the map $W(s,x) \overset{def}{=} J(s,Ax)$, $s \in [0,T]$, $x \in H$, has the following properties: $W(\cdot,x)$ is Borel for any $x \in H$, $W(s,\cdot) \in \tilde{C}_b^2(H)$ and $\|W(s,\cdot)\|_{\tilde{C}_b^2(H)} \leq g(s)$, $s \in [0,T]$. Hence applying (a), we get that ψ and $\psi_A \in \tilde{C}_b^2(H)$. Thus $\psi \in \tilde{C}_A^2(H)$ and the proof is complete. □

A connection between $\tilde{C}_A^2(H)$ and $\mathcal{U}_0$ is given next.

Proposition 4.6. *The following statements hold:*

$$\begin{array}{ll} (i) & U_t \in \mathcal{L}(\tilde{C}_A^2(H)) \text{ and } \|U_tf\|_{\tilde{2},A} \leq \|f\|_{\tilde{2},A},\ t\geq 0,\ f \in \tilde{C}_A^2(H); \\ (ii) & D(\mathcal{U}_0) \supset \bigcup_{\lambda>0} R(\lambda,\mathcal{U})\,(\tilde{C}_A^2(H)). \end{array} \tag{4.13}$$

Proof. (i) Let $f \in \tilde{C}_A^2(H)$ and $t > 0$, we prove that $U_tf \in \tilde{C}_A^2(H)$.

It is not difficult to verify that $U_tf \in \tilde{C}_b^2(H)$. We only remark that for the second Fréchet derivative of U_tf one has:

$$D^2U_tf(x) = \int_H S_t^* D^2f(S_tx+y)\, S_t \ \mathcal{N}(0,M(t))\, dy,$$

where the integral is in the Bochner sense, $\mathcal{L}_1(H)$-valued, since $D^2f \in C_b(H,\mathcal{L}_1(H))$. This way it is clear that $D^2U_tf \in C_b(H,\mathcal{L}_1(H))$ and further that it holds: $\|U_tf\|_{\tilde{2}} \leq$

$\|f\|_{\tilde{2}}$, $t \geq 0$. Now we deal with the map $(U_tf)_A : D(A) \subset H \to H$. For any $x \in D(A)$, one has by changing variable:

$$[U_tf](Ax) = \int_H f_A(S_tx+z)\ \mathcal{N}(0, A^{-1}M(t)(A^{-1})^*)\, dz. \tag{4.14}$$

Arguing as for U_tf we obtain that $(U_tf)_A \in \tilde{C}_b^2(H)$ and $\|(U_tf)_A\|_{\tilde{2}} \leq \|f_A\|_{\tilde{2}}$, $t \geq 0$. The assertion (i) is proved.

(ii) Fix $\hat{f} \in \tilde{C}_A^2(H)$ and $\lambda > 0$, we have to prove that $\psi = R(\lambda,\mathcal{U})\hat{f} = R(\lambda)\hat{f} \in D(\mathcal{U}_0)$.

By assertion (i) and proceeding as in Lemma 4.5, we get that $\psi \in \tilde{C}_b^2(H)$ and $\psi_A \in C_b^1(H)$. We introduce the following notation. For any $g \in \tilde{C}_b^2(H)$ such that $g_A \in C_b^1(H)$, we define the maps T_1g and T_2g, as follows

$$T_1g(x) = \frac{1}{2}\mathrm{Tr}\,(MD^2g(x)), \quad T_2g(x) = < x, A^*Dg(x) >, \ x \in H. \tag{4.15}$$

It is clear that $T_1g \in C_b(H)$ and further that T_2g is continuous on H. According to this notation, to prove that $\psi \in D(\mathcal{U}_0)$, it remains to check that $T_2\psi \in C_b(H)$.

We use the following formula that can be proved as in Chapter 9 of Da Prato and Zabczyk [9] (see also Lemma 5.6 of Cerrai and Gozzi [5]).

$$\frac{d}{dt}U_t\hat{f}(x) = \begin{cases} T_1U_t\hat{f}(x) + T_2U_t\hat{f}(x), & t > 0,\ x \in H \\ T_1\hat{f}(x) + T_2\hat{f}(x), & t = 0,\ x \in H. \end{cases} \tag{4.16}$$

Now we argue similarly to Theorem 5.1 of Zambotti [18]. By (4.16), integrating by parts, we get, for any $x \in H$,

$$T_1\psi(x) + T_2\psi(x) = -\hat{f}(x) + \lambda\int_0^\infty e^{-\lambda t}U_t\hat{f}(x)dt = -\hat{f}(x) + \lambda\psi(x).$$

It follows that $T_2\psi = -T_1\psi - \hat{f} + \lambda\psi$ and so $T_2\psi \in C_b(H)$. Hence $\psi \in D(\mathcal{U}_0)$ and the proof is complete. □

The next step is concerned with the approximation of functions belonging to $C_b(H)$ by maps which lie in $\tilde{C}_A^2(H)$. To this purpose we introduce a family of approximation operators on $C_b(H)$.

Fix an orthonormal basis $\{e_k\}_{k\geq 1}$ of H and negative numbers μ_k such that $-\sum_{k=1}^\infty \frac{1}{\mu_k} < \infty$. For any $x \in H$, set $x_k =< x, e_k >$, $k \geq 1$. Then define the following linear operator $\tilde{B} : D(\tilde{B}) \subset H \to H$, where $D(\tilde{B}) = \{x \in H, \text{ such that } \sum_{k=1}^\infty \mu_k^2x_k^2 < \infty\}$, $\tilde{B}x \overset{def}{=} \sum_{k=1}^\infty \mu_kx_k\,e_k$.

Now we consider $Q_t : H \to H$, $t \geq 0$, $Q_tx = \int_0^t e^{s\tilde{B}}e^{s\tilde{B}^*}x\,ds = \frac{1}{2}\Big(e^{2t\tilde{B}}\tilde{B}^{-1}x - \tilde{B}^{-1}x\Big)$, $x \in H$. It is easy to verify that Q_t is a self-adjoint, one to one, positive and trace class operator on H, for any $t > 0$. Finally we define

$$\tilde{Z}_t f(x) = \int_H f(e^{t\tilde{B}}x + y)\mathcal{N}(0,Q_t)\,dy, \quad f \in \mathcal{C}_b(H),\ x \in H,\ t \geq 0. \qquad (4.17)$$

We call $\tilde{Z}_t$ the *O-U approximations* on $\mathcal{C}_b(H)$. Notice that formula (4.17) is a special case of (4.2). Moreover

$$\begin{aligned}
&(a) \text{ for any compact set } K \subset H,\ \lim_{h\to 0}\sup_{x\in K}|\tilde{Z}_{t+h}f(x) - \tilde{Z}_t f(x)| = 0,\ f \in \mathcal{C}_b(H),\ t \geq 0;\\
&(b)\ \tilde{Z}_t(\mathcal{C}_b(H)) \subset \tilde{\mathcal{C}}_b^2(H) \text{ and } \tilde{Z}_t : \mathcal{C}_b(H) \to \tilde{\mathcal{C}}_b^2(H) \text{ is continuous},\ t > 0.
\end{aligned} \qquad (4.18)$$

Assertion (a) is valid even for U_t (see Proposition 4.1). Statement (b) can be proved in the same way as Theorem 2.5 of Da Prato [7]. Now we recall a definition that will be frequently used.

Let M be a family of probability Borel measures on H. M is said to be tight if for any $\epsilon > 0$ there exists a compact subset C_ϵ of H such that $\sup_{\mu\in M}\mu(H\backslash C_\epsilon) < \epsilon$.

Lemma 4.7. *Set* $L_n = nR(n,A)$, $n \geq 1$. *One has:*

(i) for any $g \in \tilde{\mathcal{C}}_b^2(H)$, $g \circ L_n \in \tilde{\mathcal{C}}_A^2(H)$, $n \geq 1$;

(ii) for any $f \in \mathcal{C}_b(H)$, $\left(\tilde{U}_{\frac{1}{n}}f\right)\circ L_n \in \tilde{\mathcal{C}}_A^2(H)$, $n \geq 1$ *and further* $\left(\tilde{Z}_{\frac{1}{n}}f\right)\circ L_n \xrightarrow{\mathcal{K}} f$ *as* $n \to \infty$.

Proof. (i) Fix $g \in \tilde{\mathcal{C}}_b^2(H)$ and $n \geq 1$. We set $\phi(x) = g \circ L_n(x) = g(L_n x)$, $x \in H$. It is clear that $\phi \in \tilde{\mathcal{C}}_b^2(H)$, let us verify that also $\phi_A \in \tilde{\mathcal{C}}_b^2(H)$. We have

$$\phi_A(x) = \phi(Ax) = g(L_n Ax) = g(AL_n x),\ x \in D(A).$$

Since $AL_n \in \mathcal{L}(H)$, the map ϕ_A can be extended to a uniformly continuous map defined on the whole of H. Moreover it is straightforward to verify that $\phi_A \in \tilde{\mathcal{C}}_b^2(H)$ and its second Fréchet derivative is given by $D^2\phi_A(x) = (AL_n)^* D^2 g(AL_n x)\, AL_n$, $x \in H$.

(ii) Let $f \in \mathcal{C}_b(H)$. By (b) of (4.18), we know that $\tilde{Z}_{\frac{1}{n}}f \in \tilde{\mathcal{C}}_b^2(H)$ for any $n \geq 1$. Thus by assertion (i), $(\tilde{Z}_{\frac{1}{n}}f)\circ L_n \in \tilde{\mathcal{C}}_A^2(H)$.

Now we prove the second part of (ii). Clearly $\|(\tilde{Z}_{\frac{1}{n}}f)\circ L_n\|_0 \leq \|f\|_0$ for $n \geq 1$. Fix a compact subset K of H. We use the following formula

$$\sup_{x\in K}|L_n x - x| \leq \sup_{x\in K}\int_0^\infty e^{-v}|S_{\frac{v}{n}}x - x|\,dv \longrightarrow 0 \text{ as } n \to \infty. \qquad (4.19)$$

In the last passage we have used Lemma 3.3 and the fact that S_t is a $\mathcal{C}_0$-semigroup on H. Now fix $\epsilon > 0$. By the uniform continuity of f, we can find $\delta > 0$ such that for any $x, z \in H$, $|x - z| \leq \delta$ implies that $|f(x) - f(z)| \leq \epsilon$. We can write

$$\begin{aligned}
\sup_{x\in K}|\tilde{Z}_{1/n}f(L_n x) - f(x)| &\leq \sup_{x\in K}\int_{|y|<\frac{\delta}{2}}|f(e^{\frac{1}{n}\tilde{B}}L_n x + y) - f(x)|\mathcal{N}(0,Q_{1/n})dy\\
&+ \frac{4}{\delta}\|f\|_0\int_H |y|\,\mathcal{N}(0,Q_{1/n})\,dy.
\end{aligned} \qquad (4.20)$$

By (4.19), we get $\sup_{x\in K}|e^{\frac{1}{n}\tilde{B}}L_n x - x| \leq \sup_{x\in K}\left(|L_n x - x| + |e^{\frac{1}{n}\tilde{B}}x - x|\right) \longrightarrow 0$ as $n \to \infty$. Using this fact we find by formula (4.20), for n large enough,

$$\sup_{x\in K}|\tilde{Z}_{1/n}f(L_n x) - f(x)| \leq \epsilon + \frac{4}{\delta}\|f\|_0\sqrt{\text{Tr } Q_{\frac{1}{n}}}.$$

Letting $n \to \infty$ we get $\lim_{n\to\infty}\sup_{x\in K}|\tilde{Z}_{1/n}f(L_n x) - f(x)| = 0$ and assertion (ii) is proved. The proof is complete. □

Proof of Theorem 4.3. Using the O-U approximations $\tilde{Z}_t$ (see (4.17)), we consider the following approximations, for any $n \geq 1$, $t \in [0,T]$, $x \in H$,

$$u_n(t,x) = nR(n)U_t\,[\tilde{Z}_{\frac{1}{n}}f \circ L_n](x) + \int_0^t U_{t-s}nR(n)\,[\tilde{Z}_{\frac{1}{n}}F \circ L_n](s,x)\,ds, \quad (4.21)$$

where $R(n) = R(n,\mathcal{U})$, $L_n = nR(n,A)$, $[\tilde{Z}_{\frac{1}{n}}F \circ L_n](s,x) = [\tilde{Z}_{\frac{1}{n}}F(s,\cdot)](L_n x)$.

We claim that the maps (u_n) satisfy our assertions. We define $f_n = \tilde{Z}_{\frac{1}{n}}f \circ L_n$ and $F_n = \tilde{Z}_{\frac{1}{n}}F \circ L_n$, $n \geq 1$. This way we can write for any $n \geq 1$, $t \in [0,T]$, $x \in H$,

$$u_n(t,x) = U_t nR(n)f_n + \int_0^t U_{t-s}nR(n)F_n(s,x)\,ds = U_t nR(n)f_n + v_n(t,x). \qquad (4.22)$$

Let us consider F_n. It is clear that $F_n \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(H))$, $n \geq 1$. Thus, arguing as in the proof of Theorem 3.6, we obtain that $u_n \in \mathcal{C}_\pi^1([0,T];\mathcal{C}_b(H)) \cap \mathcal{C}_\pi([0,T];D(\mathcal{U}))$. Moreover by (b) of (4.18) we deduce that $F_n(s,\cdot) \in \mathcal{C}_b^1(H)$ for $s \in [0,T]$, $n \geq 1$ and in addition the following estimate holds for any $n \geq 1$,

$$\sup_{s\in[0,T]}\|D_x F_n(s,\cdot)\|_0 = \sup_{s\in[0,T]}\|D_x[\tilde{Z}_{\frac{1}{n}}F(s,L_n\cdot)]\|_0 \leq \sup_{s\in[0,T]}c_n\|F(s,\cdot)\|_0 = c_n\|F\|_0.$$

It follows that the maps F_n are also continuous on $[0,T]\times H$.

(i) Fix $t \in [0,T]$. We prove that $u_n(t,\cdot) \in D(\mathcal{U}_0)$, $n \geq 1$.

First note that $f_n \in \tilde{\mathcal{C}}_A^2(H)$, $n \geq 1$ (see Lemma 4.7). Then by Proposition 4.6 we get that $U_t nR(n)f_n = nR(n)U_t f_n \in D(\mathcal{U}_0)$, $n \geq 1$. Let us consider the remainder term v_n. It is convenient to introduce the maps G_n,

$$G_n(s,x) = U_{t-s}F_n(s,x) = \int_H F_n(s, S_{t-s}x + y)\,\mathcal{N}(0, M(t-s))dy,$$

where $s \in [0,t]$, $x \in H$. In order to prove that $\int_0^t G_n(s,\cdot)ds \in \tilde{\mathcal{C}}_A^2(H)$, we apply Lemma 4.5. To this purpose first we show that for a fixed $x \in H$, $G_n(\cdot,x)$ is Borel on $[0,t]$. Consider the maps $\eta_n : [0,t]^2 \to \mathbb{R}$,

$$\eta_n(h,k) = \int_H F_n(h, S_{t-h}x + y)\,\mathcal{N}(0, M(t-k))dy, \quad h,\ k \in [0,t].$$

It is not difficult to verify, by considering standard properties on weak convergence of Gaussian measures, that η_n is Borel on $[0,t]^2$. Hence the map $G_n(s,x) = \eta_n(s,s)$ is Borel on $[0,t]$ for any $n \geq 1$. Now by using Proposition 4.6, Lemma 4.7 and (b) of (4.18) it is straightforward to obtain the following estimates, for suitable constants $c_n, d_n, \; n \geq 1$,

$$\begin{aligned}&\|G_n(s,\cdot)\|_{\tilde{2},A} = \|U_{t-s}\tilde{Z}_{\frac{1}{n}} F \circ L_n(s,\cdot)\|_{\tilde{2},A} \leq \|\tilde{Z}_{\frac{1}{n}} F \circ L_n(s,\cdot)\|_{\tilde{2},A} \\ &\leq c_n \|\tilde{Z}_{\frac{1}{n}} F(s,\cdot)\|_{\tilde{2}} \leq c_n d_n \|F(s,\cdot)\|_0 \leq c_n d_n \|F\|_0, \quad s \in [0,t].\end{aligned}$$

Using Lemma 4.5, we get $\int_0^t G_n(s,\cdot)ds \in \tilde{C}^2_A(H)$, $n \geq 1$. Then by the Fubini Theorem one has

$$\int_0^t nR(n)G_n(s,x)\,ds = nR(n)\Big(\int_0^t U_{t-s}F_n(s,\cdot)ds\Big)(x), \; x \in H, \, n \geq 1.$$

It follows that $v_n(t,\cdot) \in D(\mathcal{U}_0)$, $n \geq 1$ and the proof of (i) is complete.
(ii) By Lemma 4.7, we know that $f_n \xrightarrow{\mathcal{K}} f$ as $n \to \infty$. Now we show that

$$nR(n)f_n \xrightarrow{\mathcal{K}} f \;\text{ as }\; n \to \infty. \tag{4.23}$$

Fix a compact subset K of H. Since $\|nR(n)f_n\|_0 \leq \|\tilde{Z}_{1/n} f \circ L_n\|_0 \leq \|f\|_0, \; n \geq 1$, to verify (4.23) it is enough to prove that

$$\lim_{n\to\infty} \sup_{x\in K} |nR(n)f_n(x) - f(x)| = 0. \tag{4.24}$$

Let $n \geq 1$, we consider

$$\begin{aligned}&\sup_{x\in K} |nR(n)f_n(x) - f(x)| \leq \\ &\sup_{x\in K} |nR(n)f_n(x) - nR(n)f(x)| + \sup_{x\in K} |nR(n)f(x) - f(x)| \\ &= \Gamma^1(n) + \Gamma^2(n).\end{aligned} \tag{4.25}$$

Combining Proposition 4.1 with Lemma 3.4 ($\mathcal{S}$ is the family of all compact subsets of H), we have that $\lim_{n\to\infty} \Gamma^2(n) = 0$. Let us consider the remainder term.

$$\Gamma^1(n) \leq \sup_{x\in K} \int_0^\infty e^{-v}\, |U_{\frac{v}{n}} f_n(x) - U_{\frac{v}{n}} f(x)|\, dv. \tag{4.26}$$

Now we prove that for any $v > 0$, it holds:

$$\lim_{n\to\infty} \sup_{x\in K} |U_{v/n}(f_n - f)(x)| = 0. \tag{4.27}$$

Notice that $\|U_{v/n}(f_n - f)\|_0 \leq 2\|f\|_0, \; n \geq 1, \, v \geq 0$. Hence once (4.27) is verified, we can conclude, by Lemma 3.3, that $\lim_{n\to\infty} \Gamma^1(n) = 0$ and formula (4.24) follows.

Let us check (4.27). First it is not hard to verify that the family of measures $\{\mathcal{N}(S_t x, M(t))\}$, where $x \in K$, $t \in [0,T]$, is tight for any $T > 0$ (see Lemma 6.3 of Cerrai [4] for details). Fix $v > 0$, for any $\epsilon > 0$ there exists a compact set $C_\epsilon \subset H$ such that $\mathcal{N}(S_{\frac{v}{n}} x, M(\frac{v}{n}))\,(H \backslash C_\epsilon) < \epsilon$ for any $n \geq 1$, $x \in K$. Thus we have

$$\begin{aligned}&\sup_{x\in K} |U_{\frac{v}{n}}(f_n - f)(x)| \\ &\leq \sup_{x\in K} \int_{C_\epsilon} |f_n(y) - f(y)|\,\mathcal{N}(S_{\frac{v}{n}} x, M(v/n))dy \; + 2\epsilon\|f\|_0.\end{aligned} \tag{4.28}$$

For f_n $\mathcal{K}$- converges to f, we can choose n_0 such that $\sup_{x\in C_\epsilon} |f_n(x) - f(x)| < \epsilon$, for any $n \geq n_0$. We obtain $\;|U_{\frac{v}{n}}(f_n - f)(x)| \leq \epsilon\,[1 + 2\|f\|_0], \;\; n \geq n_0$.
Thus (4.27) is established and (4.23) follows. By formula (4.23), using the tightness of $\{\mathcal{N}(S_t x, M(t))\}_{x\in K, t\in[0,T]}$ as in (4.28), one deduces: $\;U_t nR(n)\, f_n \xrightarrow{\mathcal{K}} U_t f\;$ as $n \to \infty$, for any $t \geq 0$. By this fact, applying Lemma 3.3, it follows

$$\int_0^t U_{t-s} nR(n)F_n(s,\cdot)ds \xrightarrow{\mathcal{K}} \int_0^t U_{t-s}F(s,\cdot)ds \;\text{ as }\; n \to \infty, \;\; t \in [0,T].$$

We have proved that $u_n(t,\cdot) \xrightarrow{\mathcal{K}} u(t,\cdot)$ as $n \to \infty$. Now consider that $\partial_t u_n - \mathcal{U}u_n = nR(n)F_n$ and further $\;\sup_{n\geq 1} \big(\|u_n\|_0 + \|nR(n)F_n\|_0\big) \leq \|f\|_0 + \big(1+T\big)\|F\|_0$.
Finally by (4.23) we get, for any $t \in [0,T]$, $\;nR(n)F_n(t,\cdot) \xrightarrow{\mathcal{K}} F(t,\cdot)\;$ as $n \to \infty$.
The proof is complete. □

To prove the second theorem we need a preliminary result.

Lemma 4.8. *Let* $F \in \mathcal{C}_\pi([0,T];\mathcal{C}_b(H))$ *and suppose that* F *is continuous on* $[0,T] \times H$. *Set* $L_n = nR(n,A)$, $n \geq 1$. *Then one has*

$$(\tilde{Z}_{\frac{1}{n}} F) \circ L_n \xrightarrow{\mathcal{K}_T} F \;\text{ as }\; n \to \infty,$$

where $(\tilde{Z}_{\frac{1}{n}} F) \circ L_n\,(t,x) = (\tilde{U}_{\frac{1}{n}} F(t,\cdot))\,(L_n x)$, $\; x \in H, \, n \geq 1, \, t \in [0,T]$.

Proof. We set $\;F_n = (\tilde{Z}_{\frac{1}{n}} F) \circ L_n$, $\; n \geq 1$ and argue by contradiction. If the thesis is not true there exist $\epsilon_0 > 0$, a compact subset K of H and a sequence $t_m \subset [0,T]$ such that:

$$\sup_{x\in K} |F_{n_m}(t_m, x) - F(t_m, x)| > \epsilon_0, \;\; m \geq 1. \tag{4.29}$$

There exists a subsequence (t_j) of (t_m) such that $t_j \to r \in [0,T]$ as $j \to \infty$. Thus we can write, setting $n_{m_j} = j$ for convenience,

$$\begin{aligned}&0 < \epsilon_0 < \sup_{x\in K} |F_j(t_j,x) - F(t_j,x)| \leq \Gamma^1_j + \Gamma^2_j + \Gamma^3_j, \;\; j \geq 1 \\ &\text{where } \Gamma^1_j = \sup_{x\in K} |F_j(t_j,x) - F_j(r,x)|, \\ &\Gamma^2_j = \sup_{x\in K} |F_j(r,x) - F(r,x)|, \quad \Gamma^3_j = \sup_{x\in K} |F(r,x) - F(t_j,x)|.\end{aligned} \tag{4.30}$$

Now we will obtain a contradiction by showing that $\lim_{j\to\infty} \Gamma_j^1 + \Gamma_j^2 + \Gamma_j^3 = 0$.

First remark that since F is uniformly continuous on $[0,T]\times K$, it follows that $\lim_{j\to\infty}\Gamma_j^3 = 0$. Then $\Gamma_j^2 = \sup_{x\in K} |(\tilde Z_{\frac{1}{j}} F(r,\cdot))(L_j x) - F(r,x)|$ tends to 0 as $j\to\infty$, by Lemma 4.7. It remains to consider Γ_j^1.

$$\Gamma_j^1 \le \sup_{x\in K}\int_H |F(t_j,y) - F(r,y)|\,\mathcal{N}(e^{\frac{1}{j}\tilde B}L_j x, Q_{1/j})dy.$$

We claim that once it is proved that the family of measures $\mu_{j,x} = \mathcal{N}(e^{\frac{1}{j}\tilde B}L_j x, Q_{1/j})$, $j\ge 1$, $x\in K$ is tight, it follows $\lim_{j\to\infty}\Gamma_j^1 = 0$.

Indeed suppose that $\{\mu_{j,x}\}$ is tight. Then for any $\epsilon > 0$ there exists a compact subset C_ϵ of H, such that $\mu_{j,x}(H\backslash C_\epsilon) < \epsilon$ for any $j\ge 1$, $x\in K$. Using this fact we have for any $j\ge 1$

$$\Gamma_j^1 \le \sup_{x\in K}\int_{C_\epsilon} |F(t_j,y) - F(r,y)|\,\mu_{j,x}(dy) + 2\epsilon\|F\|_0. \tag{4.31}$$

Taking into account that F is uniformly continuous on $[0,T]\times C_\epsilon$, we obtain that, for j sufficiently large, $\Gamma_j^1 \le \epsilon\,(1 + 2\|F\|_0)$ and the statement is proved.

Let us verify that $\{\mu_{j,x}\}_{j\ge1,x\in K}$ is tight. By the Prokhorov Theorem (see for instance Da Prato and Zabczyk [9]) it is enough to show that $\{\mu_{j,x}\}_{j\ge 1,x\in K}$ is weakly relatively compact. To this end take any sequence (x_m) in K. There exists a subsequence (x_j) such that $x_j \to z\in K$. We check that μ_{j,x_j} converges weakly to δ_z as $j\to\infty$. For any $g\in C_b(H)$, $j\ge 1$, one has

$$\left|\int_H g(y)\mathcal{N}(e^{\frac{1}{j}\tilde B}L_j x_j, Q_{1/j})dy - g(z)\right| \le \sup_{x\in K}\left|\tilde Z_{\frac{1}{j}} g(L_j x) - g(x)\right| + \left|g(x_j) - g(z)\right|.$$

Now letting $j\to\infty$, the right-hand side tends to 0, by (ii) of Lemma 4.7. Therefore $\mu_{j,x}$, $j\ge 1$, $x\in K$ is tight. This completes the proof. □

Proof of Theorem 4.4. We consider the same approximations u_n of Theorem 4.3, which are for any $n\ge 1$, $t\in[0,T]$, $x\in H$,

$$u_n(t,x) = U_t nR(n)f_n + \int_0^t U_{t-s}nR(n)F_n(s,x)\,ds = U_t nR(n)f_n + v_n(t,x) \tag{4.32}$$

and $f_n = \tilde Z_{\frac{1}{n}} f\circ L_n$, $F_n = \tilde Z_{\frac{1}{n}} F\circ L_n$.

We split up the proof into some steps. To prove that $u_n \xrightarrow{\mathcal{K}_T} u$, we verify separately that $U_t nR(n)f_n \xrightarrow{\mathcal{K}_T} U_t f$ and that $v_n \xrightarrow{\mathcal{K}_T} v$ as $n\to\infty$.

Claim 1. $U_t nR(n)f_n \xrightarrow{\mathcal{K}_T} U_t f$ as $n\to\infty$.

Fix a compact subset K of H. Since $\{\mathcal{N}(S_t x, M(t))\}_{x\in K, t\in[0,T]}$ is tight, for any $\epsilon > 0$ we can choose a compact set $C_\epsilon\subset H$ such that $\mathcal{N}(S_t x, M(t))\,(H\backslash C_\epsilon) < \epsilon$ for any $t\in[0,T]$, $x\in K$. Thus we have for any $n\ge 1$,

$$\begin{aligned}&\sup_{x\in K,t\in[0,T]} |U_t(nR(n)f_n - f)(x)|\\ &\le \sup_{x\in K,t\in[0,T]}\int_{C_\epsilon} |nR(n)f_n(y) - f(y)|\,\mathcal{N}(S_t x, M(t))dy + 2\epsilon\|f\|_0.\end{aligned} \tag{4.33}$$

Since, by (4.23), $nR(n)f_n \xrightarrow{\mathcal{K}} f$ as $n\to\infty$, using (4.33) we obtain easily claim 1.

Claim 2. $v_n \xrightarrow{\mathcal{K}_T} v$ as $n\to\infty$.

Fix a compact subset K of H, it is convenient to set $U_\xi = 0$ for any $\xi < 0$. This way one has, for any $n\ge 1$,

$$\begin{aligned}&\sup_{t\in[0,T],x\in K} |v_n(t,x) - v(t,x)| \le \sup_{t\in[0,T],x\in K} B(n,t,x), \quad \text{where}\\ &B(n,t,x) = \int_0^T \left|U_{t-s}nR(n)F_n(s,x) - U_{t-s}F(s,x)\right|ds,\quad t\in[0,T],\ x\in K.\end{aligned} \tag{4.34}$$

We want to prove that

$$\lim_{n\to\infty}\sup_{t\in[0,T],x\in K} B(n,t,x) = 0. \tag{4.35}$$

Arguing by contradiction, we assume that (4.35) is not true. This means that there exist $\epsilon_0 > 0$ and two sequences (t_j) in $[0,T]$ and $(n_j)\subset\mathbb{N}$, such that $\sup_{x\in K} B(n_j,t_j,x) > \epsilon_0$, $j\ge 1$. There exists a subsequence of (t_j), again denoted by (t_j), that converges to some $r\in[0,T]$. Setting $n_j = j$ for convenience, in order to obtain a contradiction we will prove that $\lim_{j\to\infty}\sup_{x\in K} B(j,t_j,x) = 0$. To this purpose consider that for any $s\in[0,T]$, $j\ge 1$, it holds

$$\begin{aligned}&\sup_{x\in K}\left|U_{t_j-s}\,jR(j)F_j(s,x) - U_{t_j-s}F(s,x)\right|\\ &\le \sup_{w\in[0,T]}\sup_{x\in K}\left|U_w jR(j)F_j(s,x) - U_w F(s,x)\right| \longrightarrow 0 \ \text{as}\ j\to\infty.\end{aligned} \tag{4.36}$$

In the last passage we have used claim 1 (with f_n and f replaced respectively by $F_j(s,\cdot)$ and $F(s,\cdot)$). By (4.36), applying Lemma 3.3, we obtain $\lim_{j\to\infty}\sup_{x\in K} B(j,t_j,x) = 0$. Thus (4.35) follows and claim 2 is proved.

Claim 3. $\partial_t u_n - \mathcal{A}u_n = nR(n)F_n \xrightarrow{\mathcal{K}_T} F$ as $n\to\infty$.

Fix a compact subset K of H and consider for any $n\ge 1$,

$$\begin{aligned}&\sup_{t\in[0,T],x\in K} |nR(n)F_n(t,x) - F(t,x)| \le \sup_{t\in[0,T],x\in K}\int_0^\infty e^{-v}|U_{\frac{v}{n}}F_n(t,x) - F(t,x)|\,dv\\ &\le \sup_{t\in[0,T],x\in K}\int_0^\infty e^{-v}\left(|U_{\frac{v}{n}}F_n(t,x) - U_{\frac{v}{n}}F(t,x)| + |U_{\frac{v}{n}}F(t,x) - F(t,x)|\right)dv.\end{aligned} \tag{4.37}$$

From the estimate (4.37), once we have proved that for any $v > 0$,

$$
\begin{aligned}
&(i)\ \lim_{n\to\infty}\ \sup_{t\in[0,T],x\in K} |U_{\frac{v}{n}}F_n(t,x) - U_{\frac{v}{n}}F(t,x)| = 0, \\
&(ii)\ \lim_{n\to\infty}\ \sup_{t\in[0,T],x\in K} |U_{\frac{v}{n}}F(t,x) - F(t,x)| = 0,
\end{aligned}
\tag{4.38}
$$

claim 3 follows by using Lemma 3.3. To verify assertion (i), consider that by Lemma 4.8 one has, for any compact subset C of H,

$$
\lim_{n\to\infty}\ \sup_{t\in[0,T],x\in C} |F_n(t,x) - F(t,x)| = 0. \tag{4.39}
$$

Fix $v > 0$ and consider that the family of measures $\{\mathcal{N}(S_{\frac{v}{n}}x, M(\frac{v}{n}))\}_{n\geq 1, x\in K}$ is tight. Hence arguing as in (4.31) and using (4.39) we get

$$
\sup_{t\in[0,T],x\in K} |U_{\frac{v}{n}}F_n(t,x) - U_{\frac{v}{n}}F(t,x)| \leq \epsilon(1 + 2\|F\|_0), \quad \text{for } n \text{ large enough.}
$$

Thus condition (i) of (4.38) is proved. To verify assertion (ii) of (4.38) we argue by contradiction as in the proof of Lemma 4.8. This completes the proof. □

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