Department of Economics and Statistics "Cognetti de Martiis"
Campus Luigi Einaudi, Lungo Dora Siena 100/A, 10153 Torino (Italy)
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DAVIDE LA TORRE, SIMONE MARSIGLIO, FRANKLIN MENDIVIL and FABIO PRIVILEGGI

UNIVERSITÀ DEGLI STUDI DI TORINO

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Generalized Fractal Transforms with Condensation: a Macroeconomic-Epidemiological Application

Davide La Torre[1], Simone Marsiglio[2], Franklin Mendivil[3], and Fabio Privileggi[4]

[1]*SKEMA Business School and Université Côte d'Azur, Sophia Antipolis, France.*
Email: davide.latorre@skema.edu.
[2]*University of Pisa, Department of Economics and Management, Pisa, Italy.*
Email: simone.marsiglio@unipi.it.
[3]*Department of Mathematics and Statistics, Acadia University, Wolfville, Canada.*
Email: franklin.mendivil@acadiau.ca.
[4]*Department of Economics and Statistics "Cognetti de Martiis", University of Turin, Torino, Italy.*
Email: fabio.privileggi@unito.it.

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Abstract

We establish novel results on generalized fractal operators with condensation and apply them in the analysis of a macroeconomic-epidemiological model characterized by deep uncertainty under the assumption that it is impossible to quantify with certainty the exact number of current and future infectives. The setting is simple: the level of prevalence of a communicable disease determines the size of the healthy labor force, affecting output and consumption; health policy is publicly funded via income taxation but the availability of resources is endogenously determined since depending on disease prevalence. Since the high degree of uncertainty is reflected also in the policymakers' choice of the policy tools to limit the spread of the disease, we investigate how the peculiarities of different policymakers (a short-sighted vs far-sighted approach) affect the asymptotic invariant distribution of macroeconomic activity. Specifically, we exploit the condensation term of the fractal operator to characterize the consequence of short-sighted policies. Through numerical simulations we find that, as we would expect, far-sighted policies lead to asymptotic invariant probability distributions concentrating more mass on high levels of aggregate consumption together with small numbers of infectives, while the invariant distribution reached through short-sighted policies, besides concentrating more mass on low levels of aggregate consumption together with large numbers of infectives, exhibits an additional layer of (uniform) uncertainty generated by the condensation term.

1 Introduction

In this paper we enrich the theory on generalized fractal operators by establishing new results that incorporate a condensation term into such operators for the case in which they transform probability distributions. The notion of condensation was introduced by Barnsley (1989) for the classical notion of Iterated Function Systems (IFS) and then extended by Kunze and others to the case of generalized fractal transforms (see, *e.g.*, Kunze et al., 2012). Our results, by establishing existence and uniqueness of a fixed point for fractal transforms acting on probability densities and cumulative distributions when a condensation term is included, allow for the application of deterministic operators that recursively generate dynamics for purely random objects—densities or cumulative distributions—in macroeconomic models. Such an approach seems to be especially suited in situations where the economic variables under study are characterized by a diffuse uncertainty that prevents them to be treated as standard random variables, so that tackling directly the probability distribution associated to them may turn out to be more appropriate. Epidemics provide an intuitive example of such a situation.

The recent coronavirus epidemic has revealed the potential dramatic effects of infectious diseases on macroeconomic outcomes. From the first case of COVID-19 reported in China in late 2019, in a matter of few months the epidemic has reached a pandemic status in March 2020, and the entire world is still today (early 2021) understanding how to cope with its devastating economic consequences which, because of its effects on workers and firms, range from a large number of jobs losses to a substantial reduction in GDP (Dong et al., 2020; La Torre et al., 2021a). This unexpected shock has severely hit all worldwide economies and no single country has been spared by the disease outbreak, giving rise to a growing interest in understanding the mutual relation between epidemics and macroeconomics. Borrowing from previous works on economic epidemiology which mainly have a microeconomic focus (Anderson et al, 2010; Gersovitz and Hammer, 2004; Goldman and Lightwood, 2002; Philipson, 2000), and more specifically from those on macroeconomic epidemiology (Chakraborty et al., 2010; Goenka and Liu, 2012; Goenka et al., 2014; La Torre et al., 2020), several studies analyze how different types of public health policies, including preventive measures, prophylactic treatment, social distancing, lockdowns, restrictions on individuals' mobility, affect both the disease and economic dynamics (Acemoglu et al., 2020; Alvarez et al., 2020; Eichenbaum et al., 2020; La Torre et al., 2021a). These works obtain quite a wide range of conclusions regarding the optimal intensity and duration of the different policy measures, highlighting that because of the large degree of uncertainty characterizing the evolution of epidemics it is very difficult to derive definitive conclusions. Indeed, epidemiological parameters including the recovery and the infectivity rates, along with the number of individuals already exposed to the disease and of those effectively infectives at different moments in time can only be roughly estimated and thus it is not possible to perform an accurate model's calibration (Acemoglu et al., 2020; La Torre et al., 2021a). Starting from this result, that is, the high uncertainty in disease dynamics, our paper aims to develop a simple macroeconomic-epidemiological framework in which health and macroeconomic outcomes are strictly related and quantified not by numbers but by probability densities.

The fact that uncertainty plays an essential role in driving macroeconomic dynamics and thus needs to be taken into account in the determination of macroeconomic policy has been known for long (Brock and Mirman, 1972; Rodrik, 1991; Olson and Roy, 2005; Baker et al., 2016). However, in order to properly design public policy it is important to recognize how different types of uncertainty affect macroeconomic outcomes, overcoming the simplistic scenario-based analysis typically employed in macroeconomics. Indeed, a standard assumption in macroeconomic theory is that the realization of a shock determines the specific value taken by some variable with a specific probability. This kind of approach does not allow to account for the high degree of uncertainty associated with parameter values and for how policymakers may account for such a parameter uncertainty (Brainard, 1967; Brock and Durlauf, 2006; Hansen and Sargent, 2007; Born and Pfeifer, 2014). In particular, different policymakers may respond to such an uncertainty by adopting a different combinations of policy tools or different levels of policy instruments, bringing the effects of uncertainty to be reflected in the implemented policy measures, giving rise to "deep uncertainty". Deep uncertainty may involve the inability to identify the appropriate models or to quantify the relevant parameters to characterize a system's dynamics, the probability distributions to represent uncertainty about the model's parameters, and/or the desirability of alternative possible outcomes (Walker et al., 2013; Marchau et al., 2019). Several studies have focused on a special case of deep uncertainty represented by ambiguity, which refers to the uncertainty about the model's parameters, analyzing its implications on macroeconomic policy in the context of short run economic fluctuations (Karantounias, 2013; Caprioli, 2015; Hollmayr and Matthes, 2015) and long run economic growth (Cozzi and Giordani, 2011; La Torre et al., 2021b). Building on La Torre et al.'s (2021b) approach based on iteration function systems on density functions, we develop a generalized fractal transforms with condensation framework, in which in their response to an epidemic outbreak different types of policymakers (short-sighted vs far-sighted) may implement different policy measures which in turn yield uncertainty at aggregate level about the effective level of disease prevalence and thus the effective level of economic activity. Unlike La Torre et al. (2021b), in this model we shall focus on the condensation term, on which the original mathematical results of Sections 3 and 4 are based, as the parameter generating some degree of "deep uncertainty". This setting allows us to discuss the implications of deep uncertainty on the epidemiological-macroeconomic steady state outcome.

Our work is closely linked to the literature on IFS generating stochastic dynamics converging to invariant probabilities possibly supported on fractal sets in macroeconomic models, which, in most cases, are one- or multi-sector growth models. The randomness characterizing such models is most commonly, but not exclusively, assumed to be originated by exogenous shocks on the productivity level (Montrucchio and Privileggi, 1999; Mitra et al., 2003; Mitra and Privileggi, 2004, 2006, 2009; La Torre et al., 2011, 2015, 2018b); there exist also few works in which shocks affect other variables, such as the pollution stock (Privileggi and Marsiglio, 2013; La Torre et al., 2018a; Marsiglio and Privileggi, 2021). To the best of our knowledge, none of these works has considered an epidemiological framework and how epidemic and macroeconomic dynamics may mutually affect each other, while in most of them uncertainty is described by a finite number of events, each occurring with a known probability, without considering the implications of deep uncertainty on steady state outcomes. The only work in which uncertainty is modeled as a form of ambiguity and thus it is taken into account in policymakers' decisions according to their degree of ambiguity aversion is La Torre et al.'s (2021b). Unlike them, who assume that some specific parameter values are not precisely known, we consider a situation in which the information about the value of a main variable (*i.e.*, the number of infectives) is not available and thus uncertainty at aggregate level affects the dynamic evolution of the key macroeconomic variables. Moreover, we introduce a *condensation term* summarizing the spread uncertainty related on any aspect of the epidemic in a scenario in which purposely—*i.e.*, as a policy choice—no research activities are being carried out to gather such information.

Specifically, we analyze a very simple macroeconomic-epidemiological model in which the level of prevalence of a communicable disease determines the size of the healthy labor force, affecting output and consumption. We focus on a simple epidemic management program in which health policy is entirely publicly funded via income taxation but the availability of resources happens to be endogenously determined as they depend on disease prevalence. The model is characterized by deep uncertainty as the number of infective individuals is not known with precision and thus epidemiological and macroeconomic outcomes seem to be more appropriately analyzed in terms of their density functions. Such a level of uncertainty is reflected also in the policymakers' choice of the policy tools to employ during the epidemic management program. As different policymakers implement different policy measures, the effective level of disease prevalence and thus the effective level of economic activity is highly uncertain, and thus we can analyze how different types of policymaking (short-sighted vs. far-sighted) approaches affect the asymptotic invariant distributions of macroeconomic activity, quantified both by consumption and by the number of infectives. By means of a numerical simulation under a specific parametrization we show that, if labor is sufficiently productive, far-sighted policies lead to high consumption levels and low numbers of infectives in the long-run, while, short-sighted, plainly redistributive policies asymptotically yield low consumption levels together with high numbers of infectives. The novelty of our approach is that such outcomes are described in terms of asymptotic invariant probability densities concentrating more mass on higher consumption levels and on lower numbers of infectives in the former case, while in the latter case the opposite occurs, with long-run invariant densities of consumption levels and infective numbers concentrating more mass on lower consumption levels and on higher numbers of infectives respectively. Moreover, a constant condensation term associated to the latter scenario lets the asymptotic densities in this case look flatter than in the former scenario; this is because a further layer of deep uncertainty is being added by the condensation.

The paper is organized as follows. Section 2 discusses the mathematical tools that we will employ in our analysis, presenting the theories of generalized fractal transforms, of Iterated Function Systems on Maps (IFSM) and the notion of condensation. Sections 3 and 4 contain our original mathematical results: they extend the theory of IFSM with condensation to the case of density functions and cumulative distributions respectively. Section 5 discusses our macroeconomic-epidemiological application and presents some numerical simulations. Section 6 as usual concludes and proposes directions for future research.

2 Mathematical Preliminaries

In this section we recall the main mathematical techniques that will be used in the sequel of this paper and mainly focused on the notion of condensation. We first recall the definition of Generalized Fractal Transform as this provides a general framework which includes all fractal operators. We then present three different subsections dedicated to the notions of Iterated Function Systems with Condensation, Iterated Function System with Probabilities and Condensation, and finally Iterated Function Systems on Mappings with Condensation. This section introduces some classical mathematical preliminaries in fractal theory that will be used in the following sections to introduce the original part of our paper.

2.1 Generalized Fractal Transforms

Let (X, d) be a metric space. A Generalized Fractal Transform (GFT) is an operator $T : X \to X$ whose action on an element $u \in X$ to get the element $v \in X$, $v = Tu$, is described by the following procedure: starting from u, it first produces a set of N spatially-contracted copies of u which are modified by means of a suitable range-mapping and then it recombines them using an appropriate operator (Barnsley, 1989; Kunze et al., 2012). A crucial property within the theory of GFT is the contractivity of T under appropriate conditions. Banach's fixed point theorem, in fact, the contractivity hypothesis guarantees the existence of a unique fixed point $\bar{u} = T\bar{u}$ that is a global attractor for X.

Definition 1 (Contraction mapping; Banach, 1922)**.** *Let $T : X \to X$ be a mapping on a complete metric space (X, d). Then T is said to be contractive if there exists a constant $c \in [0, 1)$ such that $d(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$. The smallest such $c \in [0, 1)$ for which the above inequality holds true is the* contraction factor *of T.*

The following result, known as Banach's Fixed Point Theorem, is perhaps the most famous theorem regarding contraction maps on metric spaces and certainly central to fractal-based methods.

Theorem 1 (Banach's Fixed Point Theorem, 1922)**.** *Let $T : X \to X$ be a contraction mapping on X with contraction factor $c \in [0, 1)$ mapping on X. Then,*

1. *There exists a unique element $\bar{x} \in X$, the fixed point of T, for which $T\bar{x} = \bar{x}$.*
2. *Given any $x_0 \in X$, if we form the iteration sequence $x_{n+1} = T(x_n)$, then $x_n \to \bar{x}$, i.e., $d(x_n, \bar{x}) \to 0$ as $n \to \infty$. In other words, the fixed point $\bar{x}$ is globally attractive.*

Theorem 1 states that, under the contractivity condition, there exists a unique fixed point of T, to which any orbit in X converges.

2.2 Iterated Function Systems with Condensation

Given a compact metric space (X, d), we denote by $\mathcal{H}(X)$ the set of all nonempty compact subsets of X. The distance between two sets $A, B \in \mathcal{H}(X)$ is defined by means of the classical Hausdorff metric h defined as follows:

$$h(A, B) = \max\left\{\max_{x\in A}\min_{y\in B} d(x, y), \max_{x\in B}\min_{y\in A} d(x, y)\right\}.$$

It can be proved (see, for instance, Barnsley, 1989) that $(\mathcal{H}(X), h)$ is a complete metric space. A set $\mathbf{w}$ of contraction mappings on X is defined to be an N-map *Iterated Function System* (IFS) on X (see Barnsley, 1989; Hutchinson, 1981; Kunze et al., 2012). Each element of $\mathbf{w}$ is a contraction map $w_i : X \to X$, $i = 1, \ldots, N$, with contraction factors $c_i \in [0, 1)$. Associated with an N-map IFS is the following set-valued mapping $\hat{w}$ on the space $\mathcal{H}(X)$ of nonempty compact subsets of X:

$$\hat{w}(A) := \bigcup_{i=1}^{N} w_i(A), \qquad A \in \mathcal{H}(X).$$

Theorem 2 (Hutchinson, 1981)**.** *For $A, B \in \mathcal{H}(X)$,*

$$h(\hat{w}(A), \hat{w}(B)) \leq ch(A, B) \qquad \text{where } c = \max_{1\leq i\leq N} c_i < 1.$$

Corollary 1 (Hutchinson, 1981)**.** *There exists a unique set $\hat{A} \in \mathcal{H}(X)$, the* attractor *of the IFS $\mathbf{w}$, such that*

$$\hat{A} = \hat{w}\left(\hat{A}\right) = \bigcup_{i=1}^{N} w_i\left(\hat{A}\right).$$

Moreover, for any $B \in \mathcal{H}(X)$, $h\left(\hat{A}, \hat{w}^n B\right) \to 0$ as $n \to \infty$.

The notion of condensation term was introduced by Barnsley and coworkers in Barnsley (1989) for the classical notion of IFS and then extended by Kunze and others to the case of generalized fractal transforms.[1] In Kunze et al. (2012) several examples of generalized fractal transforms with condensation term are presented as well as their applications to fractal image processing and inverse problems. In particular it is shown that the condensation term arises quite naturally when analyzing transformed fractal operators on the set of frequency-expanded images via Fourier transforms.

Given a subset $\Gamma \subset X$, an *IFS operator with condensation set* Γ is defined as:

$$\hat{w}_\Gamma (A) := \left(\bigcup_{i=1}^{N} w_i (A) \right) \bigcup \Gamma, \qquad A \in \mathcal{H}(X).$$

The operator $\hat{w}_\Gamma (A)$ satisfies the same properties of a classical IFS operator, as well summarized in the following results.

Corollary 2 (Kunze et al.,2012)**.** *For* $A, B \in \mathcal{H}(X)$*,*

$$h\left(\hat{w}_\Gamma (A), \hat{w}_\Gamma (B)\right) \leq c h (A, B) \qquad \text{where } c = \max_{1 \leq i \leq N} c_i < 1.$$

Corollary 3 (Kunze et al., 2012)**.** *There exists a unique set* $\hat{A}_\Gamma \in \mathcal{H}(X)$*, the* attractor *of the IFS* $\hat{w}_\Gamma$*, such that*

$$\hat{A}_\Gamma = \hat{w}_\Gamma \left(\hat{A}_\Gamma\right) = \left(\bigcup_{i=1}^{N} w_i \left(\hat{A}_\Gamma\right) \right) \bigcup \Gamma.$$

Moreover, for any $B \in \mathcal{H}(X)$*,* $h\left(\hat{A}_\Gamma, \hat{w}^n B\right) \to 0$ *as* $n \to \infty$*.*

2.3 Iterated Function Systems with (Constant) Probabilities and Condensation

An N-map *Iterated Function System on (constant) Probabilities (IFSP)* $(\mathbf{w}, \mathbf{p})$ is an N-map IFS $\mathbf{w}$ with associated probabilities $\mathbf{p} = \{p_1, \ldots, p_N\}$, $\sum_{i=1}^N p_i = 1$. Let (X, d) be a compact metric space and let $\mathcal{M}(X)$ denote the set of probability measures on (Borel subsets of) X. The distance between two probability measures μ, ν in $\mathcal{M}(X)$ is determined by means of the Monge-Kantorovich distance which is defined as follows:

$$d_{MK}(\mu, \nu) = \sup_{f \in Lip_1(X)} \left[\int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\nu \right].$$

where $\mu, \nu \in \mathcal{M}(X)$, and $Lip_1(X) = \{f : X \to \mathbb{R} \mid |f(x) - f(y)| \leq d(x, y)\}$. It can be proved (Hutchinson, 1981; Barnsley, 1989) that the metric space $(\mathcal{M}(X), d_{MK})$ is complete.

The *Markov operator* associated with an N-map IFSP is a mapping $M : \mathcal{M} \to \mathcal{M}$, is defined as follows: For any $\mu \in \mathcal{M}(X)$, $\nu = M\mu$, and any measurable set $S \subset X$,

$$\nu(S) = (M\mu)(S) = \sum_{i=1}^{N} p_i \mu\left(w_i^{-1}(S)\right).$$

Theorem 3 (Hutchinson, 1981)**.** *For* $\mu, \nu \in \mathcal{M}(X)$*,*

$$d_{MK}(M\mu, M\nu) \leq c d_{MK}(\mu, \nu).$$

[1] see Kunze et al. (2012) and the references therein.

Theorem 4 (Hutchinson, 1981)**.** *There exists a unique measure* $\bar{\mu} \in \mathcal{M}$*, the* invariant measure *of the IFSP* $(\mathbf{w}, \mathbf{p})$*, such that*

$$\bar{\mu}(S) = (M\bar{\mu})(S) = \sum_{i=1}^{N} p_i \bar{\mu}\left(w_i^{-1}(S)\right)$$

for any measurable set $S \subset X$*. Moreover, for any* $\nu \in \mathcal{M}(X)$*,* $d_{MK}(\bar{\mu}, M^n \nu) \to 0$ *as* $n \to \infty$*.*

Theorem 5 (Hutchinson, 1981)**.** *The support of the invariant measure* $\bar{\mu}$ *of an* N*-map IFSP* $(\mathbf{w}, \mathbf{p})$ *is the attractor* A *of the IFS* $\mathbf{w}$*, i.e.,*

$$\operatorname{supp} \bar{\mu} = A.$$

An approximation of the attractor of an IFSP could be determined by implementing the following random dynamical system, known as Chaos Game: Starting from $x_0 \in X$, let us determine $x_{t+1} = w_\sigma(x_t)$ where σ is chosen in the set $\{1, \ldots, N\}$ with probabilities $p_1, \ldots, p_N$. It can be proved[2] that the orbit of this random dynamical system is dense in the attractor $\hat{A}$ of the IFS $\mathbf{w}$.

Given a probability $\gamma \in \mathcal{M}$ and a trade-off parameter $\xi \in [0, 1]$, let us define an N-map IFSP *with Condensation* a mapping $M_\gamma : \mathcal{M} \to \mathcal{M}$, defined as follows:

$$\nu = M_{\gamma,\xi}\mu = \xi \sum_{i=1}^{N} p_i \mu \circ w_i^{-1} + (1 - \xi)\gamma.$$

The following corollaries present the extension of the previous results to the case of IFSP with condensation.

Corollary 4 (Kunze et al., 2012)**.** *For* $\mu, \nu \in \mathcal{M}(X)$*,*

$$d_{MK}(M_{\gamma,\xi}\mu, M_{\gamma,\xi}\nu) \leq c\, d_{MK}(\mu, \nu).$$

Corollary 5 (Hutchinson, 1981)**.** *There exists a unique measure* $\bar{\mu}_{\gamma,\xi} \in \mathcal{M}$*, the* invariant measure*, such that*

$$\bar{\mu}_{\gamma,\xi} = M\bar{\mu}_{\gamma,\xi} = \xi \sum_{i=1}^{N} p_i \bar{\mu}_{\gamma,\xi} \circ w_i^{-1} + (1 - \xi)\gamma$$

Moreover, for any $\nu \in \mathcal{M}(X)$*,* $d_{MK}\left(\bar{\mu}_{\gamma,\xi}, M_{\gamma,\xi}^n \nu\right) \to 0$ *as* $n \to \infty$*.*

2.4 Iterated Function Systems on Mappings with Condensation

This section focuses on notion of IFSM (see Forte and Vrscay, 1995, for more details). The definition of IFSM extend the one of IFS to the case of space of functions (Kunze et al., 2012) and it can be used to generate integrable "fractal" functions.

Let us recall that $L^p([0, 1])$, with $p \geq 1$, is the space of p-integrable functions and that this space is complete when it is equipped with the distance d_p induced by the classical p-norm. Ingredients of an N-map IFSM on $L^p([0, 1])$ are:

1. a set of N contractive mappings $w = \{w_1, w_2, \ldots, w_N\}$, $w_i : [0, 1] \to [0, 1]$, most often affine in form:

$$w_i(x) = s_i x + a_i, \qquad 0 \leq |s_i| < 1, \qquad i = 1, 2, \ldots, N; \tag{1}$$

[2] See Kunze et al. (2012) for more details.

2. a set of associated functions—the greyscale maps—$\phi = \{\phi_1, \phi_2, \dots, \phi_N\}$, $\phi_i : \mathbb{R} \to \mathbb{R}$. Affine maps are usually employed:

$$\phi_i (y) = \alpha_i y + \beta_i. \tag{2}$$

Associated with the N-map IFSM (w, ϕ) is the *fractal transform* operator T, the action of which on a function $u \in L^p ([0,1])$ is given by:

$$(Tu)(x) = \sum_{i=1}^{N}{}' \phi_i \left(u \left(w_i^{-1} (x)\right)\right), \tag{3}$$

where the prime means that the sum operates only on those terms for which w_i^{-1} is defined. The following result in Proposition 1 states that T is a Lipschitz map on $L^p ([0,1])$.

Proposition 1. *[Forte and Vrscay, 1995] For any $p \geq 1$ we have that $T : L^p ([0,1]) \to L^p ([0,1])$ and for any $u, v \in L^p ([0,1])$ we have:*

$$d_p (Tu, Tv) \leq C d_p (u, v)$$

where:

$$C = \sum_{i=1}^{N} s_i^{\frac{1}{p}} |\alpha_i| .$$

Corollary 6. *Suppose that $C = \sum_{i=1}^{N} s_i^{\frac{1}{p}} |\alpha_i| < 1$. Then T has a unique fixed point $\bar{u} \in L^p ([0,1])$ and, for any $u_0 \in L^p ([0,1])$, the orbit generated $u_{n+1} = Tu_n$ converges to $\bar{u}$ whenever $n \to +\infty$.*

The above corollary states that if $\sum_{i=1}^{N} s_i^{\frac{1}{p}} |\alpha_i| < 1$ then the IFSM operator is a contraction on $L^p ([0,1])$ and hence it has a unique fixed point $\bar{u}$ that is attracting any orbit $T^n u_0$ generated starting from any point $u_0 \in L^p ([0,1])$. Notice that if $\bar{u} \in L^p ([0,1])$, $p \geq 1$, then $\bar{u} \in L^q ([0,1])$ for any $1 \leq q \leq p$.

We now recall the definition of IFSM with condensation (Kunze et al., 2012). Given a fixed function $\theta \in L^p ([0,1])$, let us construct the following IFSM operator with condensation θ:

$$(T_\theta u)(x) = \sum_{i=1}^{N}{}' \alpha_i u \left(w_i^{-1} (x)\right) + \theta (x) \tag{4}$$

Let us notice that the operator T_θ collapses to the case of classical IFSM operator whenever the condensation term θ is given by

$$\theta (x) = \sum_{i=1}^{N} \beta_i I_{w_i([0,1])} (x)$$

where $I_{w_i([0,1])} (x)$ are the indicator functions of the sets $w_i ([0,1])$. The following result in Proposition 2 states that T_θ is a Lipschitz map on $L^p ([0,1])$.

Proposition 2 (Kunze et al., 2012)**.** *For any $p \geq 1$ and fixed $\theta \in L^p ([0,1])$ we have that $T_\theta : L^p ([0,1]) \to L^p ([0,1])$ and for any $u, v \in L^p ([0,1])$ we have:*

$$d_p (T_\theta u, T_\theta v) \leq C d_p (u, v)$$

where:

$$C = \sum_{i=1}^{N} s_i^{\frac{1}{p}} |\alpha_i| .$$

It is worth noting that we can give an explicit formula for the fixed point $\bar{u}$ for the operator T in (4). To see this, we note that T is an affine operator with $T(u) = Au + \theta$ and so

$$\bar{u} = (I - A)^{-1} \theta = \theta + A\theta + A^2\theta + \cdots$$

(where the series converges since A is contractive). A nice way to think about this is that θ provides the details of $\bar{u}$ on the largest scale, then $A\theta$ refines this on the next smaller scale, then $A^2\theta$ refines this by filling in even finer details, and so on.

3 IFSM with Condensation on Densities

We are now ready to show that, under certain hypotheses, an IFSM operator with condensation is a contraction with respect to the usual norm introduced into the space of density functions.

Definition 2. *For any $p \geq 1$, the space of* density functions U^p *is defined as follows:*

$$U^p = \left\{ u : [0,1] \to \mathbb{R}, u \in L^p([0,1]),\ u(x) \geq 0\ \forall x \in [0,1],\ \int_{[0,1]} u(x)\,\mathrm{dx} = 1 \right\},$$

where dx *denotes the Lebesgue measure on* $[0,1]$.

Let us notice that $U^p \subseteq U^q$ for any $1 \leq q \leq p$. Now we show that under certain conditions the IFSM operator with condensation T_θ earlier defined is a contraction mapping on U^p. It is trivial to prove that $U^p \subset L^p([0,1])$ as defined earlier.

Proposition 3. *The space U^p is complete with respect to the usual d_p metric.*

Proof. The proof of this result follows from the following two facts: if f_n is a converging sequence of (*a.e.*) positive functions in L^p to f then there exists a subsequence that is *a.e.* pointwise converging to f and this implies the positivity of f. Furthermore, if f_n has integral over $[0,1]$ equal to 1 then the L^p limit also possesses this property. ■

Proposition 4. *Suppose that the following conditions are satisfied:*

i) $\alpha_i \geq 0$ *for all* $i = 1...N$,

ii) $\theta(x) \geq 0$ *for a.e.* $x \in [0,1]$,

iii) $\int_0^1 \theta(x)\,dx \in [0,1)$,

iv) $\sum_{i=1}^N s_i\alpha_i + \int_0^1 \theta(x)\,dx = 1$,

v) $\sum_{i=1}^N s_i^{\frac{1}{p}}\alpha_i < 1$.

Then the operator T_θ defined as:

$$(T_\theta u)(x) = \sum_{i=1}^{N}{}' \alpha_i u\left(w_i^{-1}(x)\right) + \theta(x), \tag{5}$$

maps U^p into itself. T_θ is also a contraction over U^p. This implies that T_θ has a unique fixed point $\bar{u}_\theta$ that is also a global attractor for any sequence taking the form:

$$u_{n+1} = T_\theta u_n$$

for any initial condition $u_0 \in U^p$.

Proof. The only property that needs to be proved is that T_θ maps U^p into itself. From the hypotheses on α_i and θ, it follows that $T_\theta u$ is positive whenever u is positive. To show that the integral is one, let us do some computations:

$$\begin{aligned}\int_{[0,1]} (Tu)(x)\,\mathrm{dx} &= \int_{[0,1]} \sum_{i=1}^{N}{}' \alpha_i u\left(w_i^{-1}(x)\right)\mathrm{dx} + \int_{[0,1]} \theta(x)\,dx \\ &= \sum_{i=1}^{N} \int_{[0,1]} {}'\alpha_i u\left(w_i^{-1}(x)\right)\mathrm{dx} + \int_{[0,1]} \theta(x)\,dx \\ &= \sum_{i=1}^{N} \int_{w_i([0,1])} \alpha_i u\left(w_i^{-1}(x)\right)\mathrm{dx} + \int_{[0,1]} \theta(x)\,dx \\ &= \sum_{i=1}^{N} s_i \int_{[0,1]} \alpha_i u(x)\,\mathrm{dx} + \int_{[0,1]} \theta(x)\,dx \\ &= \sum_{i=1}^{N} s_i\alpha_i \int_{[0,1]} u(x)\,\mathrm{dx} + \int_{[0,1]} \theta(x)\,dx = 1\end{aligned}$$

■

Proposition 4 states that the operator T_θ maps U^p into itself and the fixed point equation $T_\theta u_\theta = u_\theta$ has a unique solution that is attracting any orbit $T^n u_0$ for any $u_0 \in U^p$. In the sequel we will suppose, for simplicity, $p = 2$ and we denote U^2 by U. All the results can be easily extended to the case $p \neq 2$.

4 IFSM with Condensation on Cumulative Distributions

In this section we extend the previous analysis to the case of cumulative distribution functions and we show that, under certain hypotheses, an IFSM operator with condensation is a contraction on the space of cumulative distribution function endowed with the $d_{\sup}$ metric.

Definition 3. *The space of* cumulative distribution functions D *is defined as follows:*

$$D = \{F : [0,1] \to [0,1]\,,\ F(0) \in [0,1],\ F(1) = 1,\ F \textit{ is non-decreasing, } F \textit{ is right continuous}\}\,.$$

Notice that we allow for the possibility of $F(0) > 0$ to have a point mass at $x = 0$.

Proposition 5. *The space D is complete with respect to the d_∞ metric.*

Proof. The proof of this result follows from the notion of uniform convergence induced by the $d_{\sup}$ metric. ■

In order for our IFSM operator T to map D into itself we need a few simple conditions. Letting $C = \sum_i \alpha_i$, we require that

1. $\alpha_i \geq 0$ for $i = 1, 2, \ldots, n$ and $C \leq 1$;
2. $\frac{1}{C}\theta \in D$;
3. each mapping $w_i : [0,1] \to [0,1]$ be non-decreasing.

Notice that we don't require that w_i is contractive or even continuous.

For each $w_i : [0,1] \to [0,1]$, define its *extended inverse* $\omega_i^{-1} : [0,1] \to [0,1]$ by

$$\omega_i^{-1}(y) = \begin{cases} 0, & \text{if } y < w_i(0); \\ \sup\{x : w_i(x) \le y\}, & \text{if } y \in [w_i(0), w_i(1)]; \\ 1, & \text{if } y > w_i(1). \end{cases}$$

It is straightforward to show that ω_i^{-1} is non-decreasing since w_i is and also that ω_i^{-1} is right-continuous. This means that $F \circ \omega_i^{-1} \in D$ whenever $F \in D$.

Now we show that the following IFSM operator with condensation T_θ is a contraction on the space of cumulative distribution functions. Let us define for any $F \in D$ the operator TF as follows (recall we have the condition that $\left(\sum_i \alpha_i\right)^{-1} \theta \in D$):

$$T_\theta F(x) = \sum_{i=1}^{N} \alpha_i F\left(\omega_i^{-1}(x)\right) + \theta(x), \qquad x \in [0,1] \tag{6}$$

Proposition 6. *Suppose that $\sum_i \alpha_i < 1$, then the operator T_θ defined as:*

$$T_\theta F(x) = \sum_{i=1}^{N} \alpha_i F\left(\omega_i^{-1}(x)\right) + \theta(x), \qquad x \in [0,1]$$

maps D into itself and it is also a contraction over D. This implies that T_θ has a unique fixed point $\bar{F}_\theta$ that is also a global attractor for any sequence taking the form:

$$F_{n+1} = T_\theta F_n$$

for any initial condition $F_0 \in D$.

Proof. The only property that needs to be proved is that T_θ is a contraction on D. In fact, from the hypotheses on α_i and θ, it follows that T_θ maps D into itself. To prove contractivity, let us compute:

$$d_{\sup}(T_\theta F_1, T_\theta F_2) = \sup_{x\in[0,1]} |T_\theta F_1(x) - T_\theta F_2(x)| = \sup_{x\in[0,1]} \left| \sum_{i=1}^{N} \alpha_{i=1}^{N} \left(F_1\left(\omega_i^{-1}(x)\right) - F_2\left(\omega_i^{-1}(x)\right)\right) \right|$$

$$\le \sum_{i=1}^{N} \alpha_i \sup_{x\in[0,1]} \left| \left(F_1\left(\omega_i^{-1}(x)\right) - F_2\left(\omega_i^{-1}(x)\right)\right) \right| \le \left(\sum_{i=1}^{N} \alpha_i\right) d_{\sup}(F_1, F_2).$$

■

Proposition 6 states that the operator T_θ maps D into itself and the fixed point equation $T_\theta F_\theta = F_\theta$ has a unique solution that is attracting any orbit $T_\theta^n F_0$ for any $F_0 \in D$.

5 A Macroeconomic-Epidemiological Application

We now consider a very simple epidemiological-macroeconomic model describing the dynamics of an infectious disease which affects economic production and how alternative policies may affect epidemiological and macroeconomic outcomes, entirely captured by the level of consumption which, since proportional to income, depends on disease prevalence. Such an example is certainly too stylized to provide a true insight on how an epidemic outbreak may be handled by the public authorities in order to minimize its socio-economic impact, but its purpose is to show how the mathematical approach described in the previous sections can be meaningful in macroeconomic applications as its structure contains the main, if minimal, traits that characterize deterministic operators that transform probability distributions rather than numbers. Specifically, we study the dynamics that are defined by parameters which are purely deterministic but act directly on the probability distribution, assumed to be a density, of consumption over time rather than on the consumption itself. The uncertainty that in our basic model lets consumption be a random variable with some probability density associated to it, originates from the intrinsic randomness related to the outbreak of any epidemic, which, at any time, makes it impossible to predict with certainty how many people in the population are currently infected and with even less certainty how many will be infected in the future periods.

5.1 A Simple Epidemic Dynamic

Independently of the specific epidemiological setup considered, during the early phase on an epidemic we can describe the evolution of the infectives (*i.e.*, individuals who have already contracted a disease and can transmit it to others via social contacts) through a simple linear equation (La Torre et al. 2021a). Specifically, the epidemic dynamics is fully characterized by one parameter measuring the net infectives growth rate, quantifying the infectivity rate adjusted for recovery and the effects of different policy measures. For the sake of simplicity and without loss of generality we assume that the population size is normalized to 1, so that the level and share of infectives perfectly coincide. We consider two alternative scenarios: an active policy, which we will index with $i = 1$, in which the policymaker relies on a broad range of economic and health measures to limit and control the spread of the epidemics (like lockdown, social distancing, prophylactic intervention, vaccination, travel bans, etc.), and a *laissez-faire* situation in which no policy is being taken at all, labelled with $i = 2$. To each scenario $i = 1, 2$ we associate an affine map defining the infectives dynamics according to

$$I_{t+1} = \begin{cases} sI_t & \text{if } i = 1 \\ (1-s)\, I_t + (1-s) & \text{if } i = 2, \end{cases} \tag{7}$$

where $0 < s < 1$ denotes the net infectives growth rate while $0 < (1-s) < 1$ represents the gross infectives growth rate. Therefore, an active policy reduces the disease incidence and thus also disease prevalence over time. Apart from the infectives growth term, in the laissez-faire scenario the disease dynamics is also increased by the additive constant[3] $a_2 = (1-s)$ representing a further spread of infections independent of public health policy (due to new infections associated with social contacts with individuals outside the economy's borders, as for example to business travels or commuting). Under an active policy we assume that interactions with other economies are limited to the extent that this term is null, and so such an additive constant does not show up in the first map.

Therefore, the first map in (7), describing the active policy scenario, can be interpreted as an affine map in which the additive constant $a_1 = 0$ represents the effect of economic policies aimed at limiting social contacts outside the economy's borders; it reduces the overall number of infected people by transforming any number $I_t \in [0, 1]$ into a number I_{t+1} in the sub-interval $[0, s] \subset [0, 1]$. The second map in (7), describing the laissez-faire scenario, increases the overall

[3]For simplicity we assume this additive constant to be equal to the gross infectives growth rate in order to keep the population normalized to 1 at all moments in time.

number of infected people by transforming any value $I_t \in [0,1]$ into a value I_{t+1} in the sub-interval $[1-s,1] \subset [0,1]$. We can think of the latter map as the one describing the onset of an epidemics in $t=0$, in which no active policies take place because the epidemic has not burst yet. The key assumption here is that the (unique) parameter s is well known to policymakers, or, equivalently, that they are aware of an emergency plan envisaging exactly how a potential epidemic can spread and what options are available to contain it.

5.2 The Macroeconomic Setting

By maintaining a balanced budget at any moment in time, policymakers finance active containment policies through income taxation, leading thus to a diversion of resources away from other alternative uses (*i.e.*, consumption). Similar to (La Torre et al. 2020), output is produced according to a linear production function employing only (healthy) labor as input, $Y_t = A(1-I_t)$, where $A > 0$ denotes the labor productivity, while agents consume entirely their disposable income: $C_t = (1-\tau) Y_t = (1-\tau) A (1-I_t)$, where $0 < \tau < 1$ denotes the tax rate. At time t the total tax revenue, τY_t, can be either employed in the active policy in which the policymaker puts in place actions to limit and control the spread of the epidemics or it can be directly transferred to the whole population as a lump-sum transfer to sustain income (for simplicity, covering both healthy workers employed in production and the sick and unemployed, a "helicopter money" type of intervention). Hence, assuming that a fraction γ, with $0 < \gamma < 1$, of the tax revenue τY_t is devoted to the active policy $i=1$ while a fraction $1-\gamma$ of the tax revenue τY_t is devoted to the direct income assistance, consumption turns out to be given by:

$$C_t = (1-\tau) Y_t + (1-\gamma) \tau Y_t = [1-\tau+(1-\gamma)\tau] Y_t = (1-\gamma\tau) A (1-I_t). \quad (8)$$

We emphasize that all the economic parameters introduced so far, the labor productivity A, the tax rate τ, and the coefficient γ that distributes the tax revenues between the two policies available, are well known and controlled by policymakers.

To further simplify the model, with regard to the choice on parameter γ we consider only the two alternative extreme policies, corresponding to the two scenarios $i=1,2$ for the spread of the epidemics envisaged by (7), in which either all tax revenues are employed in active policies to contain disease prevalence or all of them are employed to sustain income through lump-sum transfers to the population, without any active policy aimed at disease containment. That is, we assume that $\gamma=1$ for the $i=1$ active policy, while $\gamma=0$ for the $i=2$ laissez-faire scenario. Hence, at each time t consumption is given by:

$$C_t = \begin{cases} (1-\tau) Y_t & \text{if } i=1 \\ (1-\tau) Y_t + \tau Y_t = Y_t & \text{if } i=2, \end{cases}$$

or equivalently:

$$C_t = \begin{cases} (1-\tau) A (1-I_t) & \text{if } i=1 \\ A(1-I_t) & \text{if } i=2, \end{cases} \quad (9)$$

while, as $C_t = (1-\tau) A - (1-\tau) A I_t$ when $i=1$ and $C_t = A - A I_t$ when $i=2$, we have

$$I_t = \begin{cases} 1 - \frac{1}{(1-\tau)A} C_t & \text{if } i=1 \\ 1 - \frac{1}{A} C_t & \text{if } i=2, \end{cases} \quad (10)$$

which, by using (7) and (10), leads to the following consumption dynamics:

$$\begin{aligned} C_{t+1} &= (1-\tau) A (1-I_{t+1}) = (1-\tau) A (1-sI_t) = (1-\tau) A \left[1-s+\frac{s}{(1-\tau)A} C_t\right] \\ &= sC_t + (1-\tau)(1-s) A \qquad \text{if } i=1, \end{aligned}$$

$$C_{t+1} = A(1 - I_{t+1}) = A[1 - (1-s)I_t - (1-s)] = A\left[s - (1-s)\left(1 - \frac{1}{A}C_t\right)\right]$$
$$= A\left(2s - 1 + \frac{1-s}{A}C_t\right) = (1-s)C_t + (2s-1)A \qquad \text{if } i = 2,$$

that is,

$$C_{t+1} = \begin{cases} w_1(C_t) = sC_t + (1-\tau)(1-s)A & \text{if } i = 1 \\ w_2(C_t) = (1-s)C_t + (2s-1)A & \text{if } i = 2. \end{cases} \tag{11}$$

Note that this construction holds under the simplifying assumption that the active policy leading to the infectives dynamics $I_{t+1} = sI_t$ in the scenario $i = 1$ requires funds equal to the whole tax revenue τY_t, regardless on how much it is. In other words, we assume that, whenever the whole amount τY_t is employed in containment policies, they always manage to reach the target $a_1 = 0$ associated to the first map in (7); this may be possible because in order to limit the spread of infections independent of health policy, border closure is an economic costless intervention independent of τY_t.

According to (9), besides depending on the tax rate τ, consumption C_t depends directly on the number of infected workers, I_t. As the recent coronavirus epidemic has shown, there is a large degree of uncertainty associated with the actual number of infectives, both in the current and in the future periods. Therefore, it seems sensible to treat the number of infectives at time t, I_t, as a random variable affected by the uncertainty characterizing the epidemic trend or the future (desired or undesired) effects of containment policies. Thus, in the sequel we assume that I_t is a random variable with an associated density function v_t, to which, according to the one-to-one correspondence between C_t and I_t defined by (9) and (10), implies that also consumption C_t is a random variable with associated some density u_t.

5.3 From Numeric Variables to Densities

In general, if x_t is a random variable depending on the underlying probability space $X = [0,1]$ with density u_t, and evolving over time according to $x_{t+1} = w_i(x_t) = s_i x_t + a_i$ then, denoting by $w_i([0,1])$ the image set of w_i, for any $\delta_1 \leq \delta_2$ such that $[\delta_1, \delta_2] \subseteq w_i([0,1])$,

$$\int_{\delta_1}^{\delta_2} u_{t+1}(y)\,\mathrm{d}y = \Pr(\delta_1 \leq x_{t+1} \leq \delta_2) = \Pr(\delta_1 \leq s_i x_t + a_i \leq \delta_2)$$
$$= \Pr(\delta_1 - a_i \leq s_i x_t \leq \delta_2 - a_i) = \Pr\left(\frac{\delta_1 - a_i}{s_i} \leq x_t \leq \frac{\delta_2 - a_i}{s_i}\right)$$
$$= \int_{\frac{\delta_1 - a_i}{s_i}}^{\frac{\delta_2 - a_i}{s_i}} u_t(y)\,\mathrm{d}y\,. \tag{12}$$

By setting

$$y = w_i^{-1}(z) = \frac{z - a_i}{s_i} \qquad \Longleftrightarrow \qquad z = w(y) = s_i y + a_i,$$

the integral in (12) boils down to:

$$\int_{\delta_1}^{\delta_2} u_{t+1}(y)\,\mathrm{d}y = \int_{\frac{\delta_1 - a_i}{s_i}}^{\frac{\delta_2 - a_i}{s_i}} u_t(y)\,\mathrm{d}y = \int_{w^{-1}(\delta_1)}^{w^{-1}(\delta_2)} u_t(y)\,\mathrm{d}y = \int_{\delta_1}^{\delta_2} u_t\left[w^{-1}(z)\right]\left(w^{-1}\right)'(z)\,\mathrm{d}z$$
$$= \frac{1}{s_i}\int_{\delta_1}^{\delta_2} u_t\left[w^{-1}(z)\right]\mathrm{d}z\,.$$

As this is true for any pair δ_1, δ_2 such that $\delta_1 \leq \delta_2$ and $[\delta_1, \delta_2] \subseteq w_i([0,1])$, we can summarize the temporal evolution of the density u_t of x_t by means of the following operator $T^* : U^2 \to U^2$ defined as:

$$u_{t+1} = T^* u_t = \frac{1}{s_i} u_t \circ w^{-1}. \tag{13}$$

Whenever N maps of the form $x_{t+1} = w_i(x_t) = s_i x_t + a_i$, for $i = 1, \ldots, N$, are considered, such a construction can be generalized either to the operator $T : L^2([0,1]) \to L^2([0,1])$ defined in (3) according to

$$u_{t+1} = T u_t = \sum_{i=1}^{N}{}' \frac{1}{s_i} u_t \circ w_i^{-1},$$

where the prime means that the sum operates only on those terms for which w_i^{-1} belong to $[0,1]$, or to the operator $T_\theta^* : L^2([0,1]) \to L^2([0,1])$ defined in (4) as

$$u_{t+1} = T_\theta^* u_t = \sum_{i=1}^{N}{}' \frac{1}{s_i} u_t \circ w_i^{-1} + \theta. \tag{14}$$

whenever a condensation term $\theta \in L^2([0,1])$ is being included. Clearly, as, according to (13), each single term $\frac{1}{s_i} u_t \circ w_i^{-1}$ in (14) defines a density over $[0,1]$ such that $\int_0^1 \frac{1}{s_i} u_t\left(w_i^{-1}(x)\right) \mathrm{dx} = 1$, necessarily $\int_0^1 \left[\sum_{i=1}^{N}{}' \frac{1}{s_i} u_t\left(w_i^{-1}(x)\right) + \theta(x)\right] \mathrm{dx} > 1$ and thus the whole term $T_\theta^* u_t$ cannot be a density itself. In order to build an operator that maps densities into densities, $T_\theta : U^2 \to U^2$, we must introduce constants ω_i and consider a condensation term θ so that the weights defined as $\alpha_i = \frac{\omega_i}{s_i}$, for $i = 1...N$, together with θ satisfy conditions (i)–(v) of Proposition 4. Under these assumptions Proposition 4 guarantees that the operator defined as

$$T_\theta u = \sum_{i=1}^{N}{}' \frac{\omega_i}{s_i} u \circ w_i^{-1} + \theta \tag{15}$$

maps U^2 into itself and converges to a unique fixed point $\bar{u}_\theta \in U^2$.

5.4 An IFSM Operator with Condensation on Densities

For the specific model we are discussing here we set $N = 2$ and consider the IFS (11) defined by the two maps w_1 and w_2 transforming C_t into C_{t+1} under the two alternative policy scenarios $i = 1$ (active policy) and $i = 2$ (laissez-faire). Rather than applying such a dynamic to the numeric variable consumption, C_t, under the assumption that consumption is highly uncertain due to its dependence on the number of infected workers I_t according to (9), we consider the probability density u_t associated to C_t at time t and study its time evolution by defining the following operator $T_\theta : U^2 \to U^2$:

$$u_{t+1} = T_\theta u_t = \frac{\omega_1}{s_1} u_t \circ w_1^{-1} + \frac{\omega_2}{s_2}\left(u_t \circ w_2^{-1} + \theta\right), \tag{16}$$

in which the second term, associated to the map w_2 describing the evolution of consumption under laissez-faire, includes the exogenous condensation term $\theta(C)$ having the purpose of modeling the uncertainty specifically related to the laissez-faire scenario, in which neither policies nor research to gather information on how the epidemics spreads are undertaken; it has the effect of diffusing the probability distribution over the space of all possible consumption levels by adding a (possibly uniform) positive probability to all such values, thus adding a dispersed uncertainty component to this specific scenario. Hence, operator T_θ in (16) can be written as

$$T_\theta u_t = \frac{\omega_1}{s_1} u_t \circ w_1^{-1} + \frac{\omega_2}{s_2} u_t \circ w_2^{-1} + \frac{\omega_2}{s_2}\theta,$$

resembling the formulation in (15) in which the condensation function $\frac{\omega_2}{s_2}\theta(C)$ appears to be exogenous with respect to both scenarios $i = 1, 2$ considered. In what follows we shall assume that both weights $\alpha_1 = \frac{\omega_1}{s_1}$ and $\alpha_2 = \frac{\omega_2}{s_2}$ together with the condensation term $\alpha_2\theta(C) = \frac{\omega_2}{s_2}\theta(C)$ satisfy conditions (i)–(v) of Proposition 4, so that the sequence of densities u_t associated to consumption at each time t generated by successive iterations of operator T_θ in (16) converges to a unique time-invariant density $\bar{u}_\theta$.

Before discussing the interpretation of the constants ω_1 and ω_2 in (16), we further restrict the assumptions on the model's parameters by assuming that labor productivity satisfies $A > 1$ and by setting

$$s = (1 - s) = \frac{1}{2} \qquad \text{and} \qquad \tau = 1 - \frac{1}{A}. \tag{17}$$

Under conditions (17) the dynamics defined by (11) become

$$C_{t+1} = \begin{cases} w_1(C_t) = \frac{1}{2}C_t + \frac{1}{2} & \text{if } i = 1 \\ w_2(C_t) = \frac{1}{2}C_t & \text{if } i = 2, \end{cases} \tag{18}$$

having the properties that the invariant (trapping) region for consumption is the interval $[0, 1]$, *i.e.*, $w_1([0,1]) \cup w_2([0,1]) = [0,1]$, and that the images of the maps w_1 and w_2 intersect on the unique middle point $\frac{1}{2}$, *i.e.*, $w_1([0,1]) \cap w_2([0,1]) = \left\{\frac{1}{2}\right\}$, thus satisfying the so called *almost no-overlap property*. Moreover, if $A > 1$ condition $0 < \tau < 1$ certainly holds. Note that for the parameterization in (17) the dynamics of consumption defined by (18) becomes the same as those of the infectives in (7), only with the maps switched between the two scenarios because more infections correspond to less consumption and viceversa. This relationship holds because we, crucially, assume that $A > 1$; in other words, the technology available in production lets labor to be sufficiently productive so to generate a substantial output increase which is worth the tax revenues τY_t employed in the active policy scenario aimed at increasing the number of healthy workers. Such an assumption determines the property that in (18) the map w_1 lies all above the map w_2, thus associating larger consumption to a population of less infected workers.[4]

In this specific model, provided that conditions (i)–(v) of Proposition 4 are satisfied (namely, $\omega_i \geq 0$ for $i = 1, 2$, $\theta(C) \geq 0$ for *a.e.* $C \in [0,1]$, $\int_0^1 \theta(C)\,\mathrm{dC} \in [0,1)$, $\omega_1 + \omega_2 + 2\omega_2 \int_0^1 \theta(C)\,\mathrm{dC} = 1$ and $\sqrt{2}(\omega_1 + \omega_2) < 1$), the constants ω_1 and ω_2 to be associated to each policy scenario, $i = 1$

[4]We investigated different values for parameters s, A and τ for the dynamics defined by (11) and realized that, in order to have non negative consumption and a tax rate satisfying $0 < \tau < 1$, the range of values for the net infectives growth rate parameter s happens to be quite narrow: $\frac{1}{2} \leq s < \frac{3}{5}$. Whenever $\frac{1}{2} < s < \frac{3}{5}$ two different possibilities occur: a situation similar to the case discussed in the text in which $w_1 > w_2$ whenever $0 < \tau < \frac{2-3s}{1-s}$, so that active policies financed by taxation have a positive effect on output and consumption, and a situation in which $w_1 < w_2$ whenever $\frac{2-3s}{1-s} < \tau < 1$, when active policies financed by taxation turn out to depress output and consumption while lassaize-faire together with lump-sum transfers to the population yields higher aggregate consumption. In both situations the invariant consumption set is a proper subinterval of $[0, 1]$ and the maps w_1, w_2 have overlapping images. While such variants of the model may provide interesting insights from the economic perspective, they are beyond the scope of the example discussed in this section, which has the only purpose of illustrating the approach described in the previous sections based on the deterministic operator T_θ defined in (5).

(active policy) and $i = 2$ (laissez-faire) respectively, in operator (16) have the role of considering how different policymakers (governments) may attribute relatively different importance to the two alternative policies $i = 1, 2$. Whenever a government is elected, it has a strong pressure from its electors to adopt the latter policy, which envisages direct monetary transfers to the population, which are actually unproductive and can only sustain income, rather than an indirect (and more efficient) welfare effect that occurs through the technology in the productive sector ($A > 1$), but the latter policy requires a reduction in consumption in the short term due to taxation (not compensated by the direct lump-sum transfers), so that myopic electors prefer the laissez-faire policy $i = 2$ which, besides increasing disease prevalence, also reduces consumption in the long-run through a missing opportunity due to under-capacity in the productive sector. In short, as in this simple model the issuance of public debt is not allowed, the $i = 1$ active policy, besides reducing the number of infectives, may be seen as a long-run investment policy to be confronted with a myopic $i = 2$ policy based exclusively on short-run income assistance. From this perspective, different relative values for the pairs ω_1 and ω_2 in operator (16) may denote different types of governments: $\omega_1 < \omega_2$ characterizes a "short-sighted" government, while $\omega_1 > \omega_2$ characterizes a "far-sighted" government. Such a construction allows for a wide range of choices for the ω_1, ω_2 parameters' values and for the condensation term $\theta(C)$ to study how different types of government lead to different asymptotic invariant distributions (densities) for consumption.

Note that this model is based on the construction of a truly deterministic operator—T_θ in (16), which depends on the parameters $s, A, \tau, \omega_1, \omega_2$ that, together with the condensation term $\theta(C)$, under our assumptions are all determined with certainty—that transforms density functions into density functions. Specifically, we have explicitly chosen not to assume a convex linear combination of tax revenues being devoted to both policies (transfers together with disease containment policy), that is, we have ruled out any value $0 < \gamma < 1$ in the definition of disposable income according to (8) and considered only the extreme alternative policies ($\gamma = 1$ when $i = 1$ and $\gamma = 0$ when $i = 2$) in the deterministic model, only to mix them up by means of the weights $\alpha_1 = \frac{\omega_1}{s} = 2\omega_1$ and $\alpha_2 = \frac{\omega_2}{s} = 2\omega_2$ through the (deterministic) operator T_θ in (16).

5.5 Numerical Simulations

In the sequel we shall assume that the condensation term describes a uniform noise exogenously added to the consumption density only in the laissez-faire scenario, *i.e.*, $\theta(C) \equiv \theta$. The interpretation of such a type of condensation term is that, because in the laissez-faire scenario no research effort to gather information on how the epidemics spreads—and thus on how consumption is being affected by the number of infected workers—is being undertaken, the intrinsic uncertainty on the distribution of infectives is increased by a further uniform component resembling some "veil of ignorance".

Hence, taking into account all assumptions introduced so far, the operator T_θ with condensation defined in (16) becomes:

$$T_\theta u_t = 2\omega_1 u_t \circ w_1^{-1} + 2\omega_2 u_t \circ w_2^{-1} + 2\omega_2\theta, \tag{19}$$

with coefficients ω_1, ω_2 and θ that must satisfy conditions (i)–(v) of Proposition 4; specifically:

$$\omega_1, \omega_2, \theta \geq 0, \qquad \omega_1 + \omega_2 + 2\omega_2\theta = 1, \qquad \omega_1 + \omega_2 < \frac{1}{\sqrt{2}}. \tag{20}$$

From the second condition we get

$$\omega_2 = \frac{1 - \omega_1}{1 + 2\theta}, \tag{21}$$

which, when replaced into the third condition, easily yields

$$\omega_1 < \frac{1+2\theta-\sqrt{2}}{\sqrt{2}}.$$

The last inequality, together with the first condition in (20), requires that $1+2\theta-\sqrt{2}>0$, that is,

$$\theta > \frac{1}{\sqrt{2}} - \frac{1}{2} \simeq 0.21.$$

We set $\theta = \frac{1}{4} = 0.25$ and consider two pairs of values for ω_1 and ω_2 that satisfy (21): $\omega_1 = \frac{5}{6}, \omega_2 = \frac{1}{9}$ compatible with a "far-sighted" government, and $\omega_1 = \frac{1}{10}, \omega_2 = \frac{3}{5}$ compatible with a "short-sighted" government. As far as the initial density of consumption before the epidemic outbreak is concerned, we assume that $u_0(C) = 3C^2$, which is increasing and thus representing a probability distribution that concentrates most of the mass on higher values of consumption. Conversely, we conjecture that the probability distribution of infectives before the epidemic outbreak is decreasing, thus concentrating most of the mass on lower values of disease prevalence; specifically, for the infectives at time $t=0$, I_0, we assume a density $v_0(I) = 3(I-1)^2$, which is symmetric with respect to u_0.

To study the evolution over time of the density v_t associated to the number of infectives we consider the same operator T_θ as defined in (19) in which the two maps w_1 and w_2 in (18) are exchanged, that is, it is defined according to

$$I_{t+1} = \begin{cases} w_1(I_t) = \frac{1}{2}I_t & \text{if } i=1 \\ w_2(I_t) = \frac{1}{2}I_t + \frac{1}{2} & \text{if } i=2, \end{cases} \tag{22}$$

while the condensation term corresponding to the laissez-faire scenario $i=2$ remains the same as that considered for consumption: $\theta(I) = 2\omega_2\theta$.

A modified version of the algorithm[5] used in La Torre et al. (2021b) allows for plotting the density functions u_t of consumption (as well as the densities v_t of infections) obtained through successive iterations of operator T_θ defined in (19) starting from the initial density $u_0(C) = 3C^2$ [or using the same operator T_θ on the dynamics of infection given by (22) starting from the initial density $v_0(I) = 3(I-1)^2$]. For the coefficients $\omega_1 = \frac{5}{6}$ and $\omega_2 = \frac{1}{9}$ describing the behavior of a "far-sighted" government, Figure 1 plots the first 7 iterations of operator T_θ applied to the density on consumption according to (18) starting from the initial density $u_0(C) = 3C^2$, while Figure 2 plots the evolution of the corresponding cumulative distribution functions F_t associated to the densities u_t reported in Figure 1. The latter plots can be interpreted as the evolution of cumulative distributions generated by the operator $T_\theta F$ defined in (6) of Section 4 for the special case in which the initial probability distribution is defined by means of a density.

[5]The detailed code is available upon request.

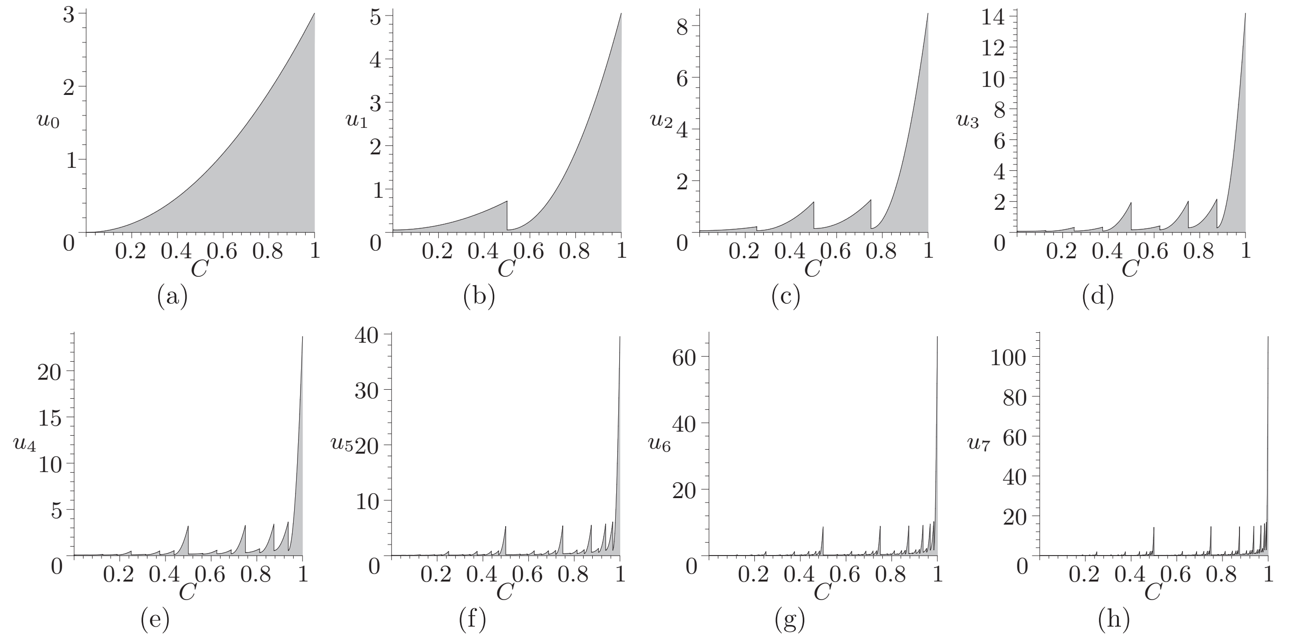

FIGURE 1: First 7 iterations of operator T_θ defined in (19) applied to the density of consumption for $\omega_1 = \frac{5}{6}$ and $\omega_2 = \frac{1}{9}$ starting from $u_0\,(C) = 3C^2$.

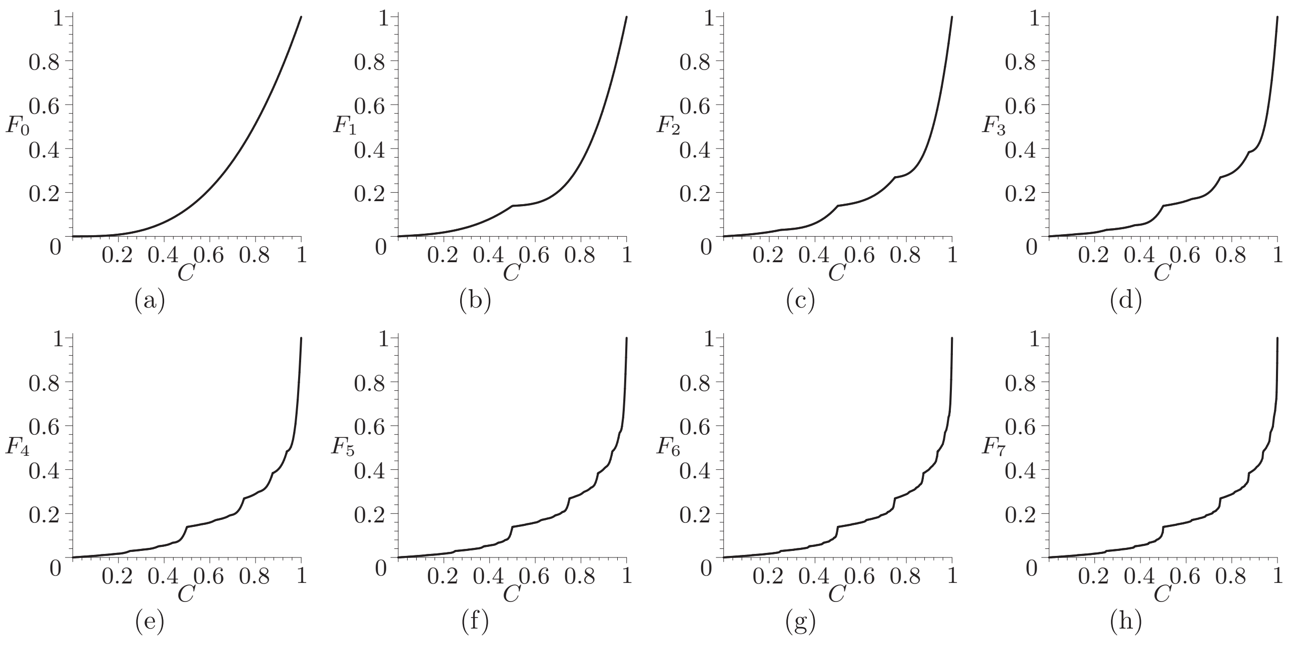

FIGURE 2: Cumulative distribution functions associated to the densities u_t in Figure 1.

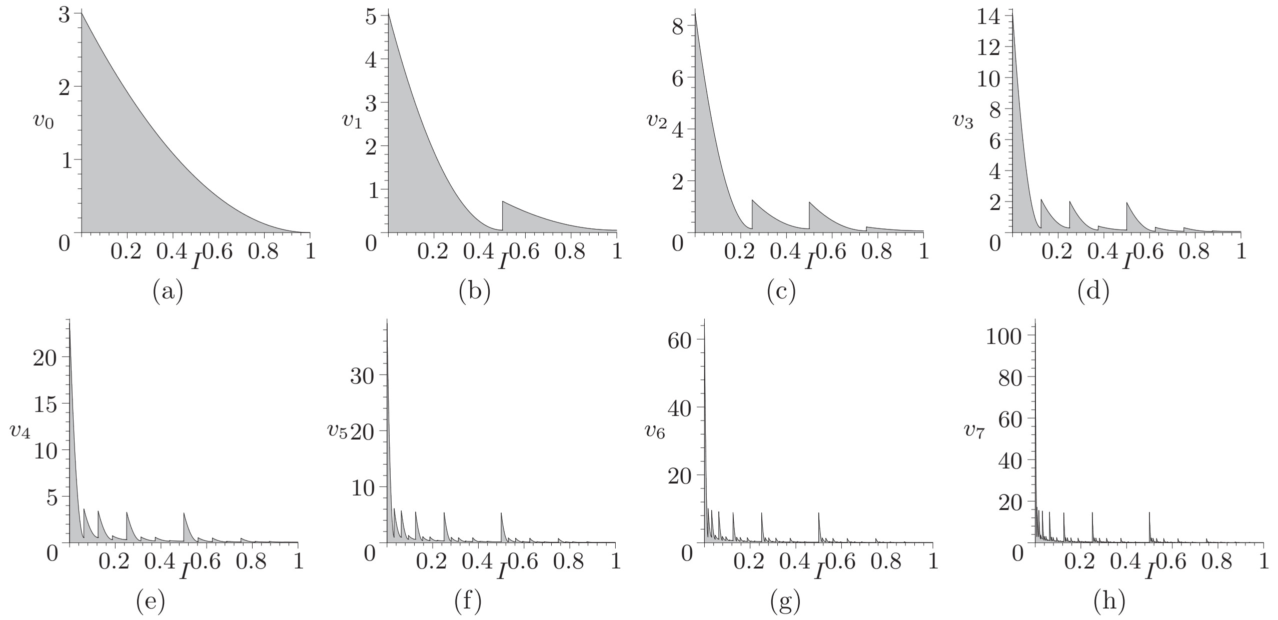

FIGURE 3: First 7 iterations of operator T_θ defined in (19) applied to the density of infected people for $\omega_1 = \frac{5}{6}$ and $\omega_2 = \frac{1}{9}$ starting from $v_0(I) = 3(I-1)^2$.

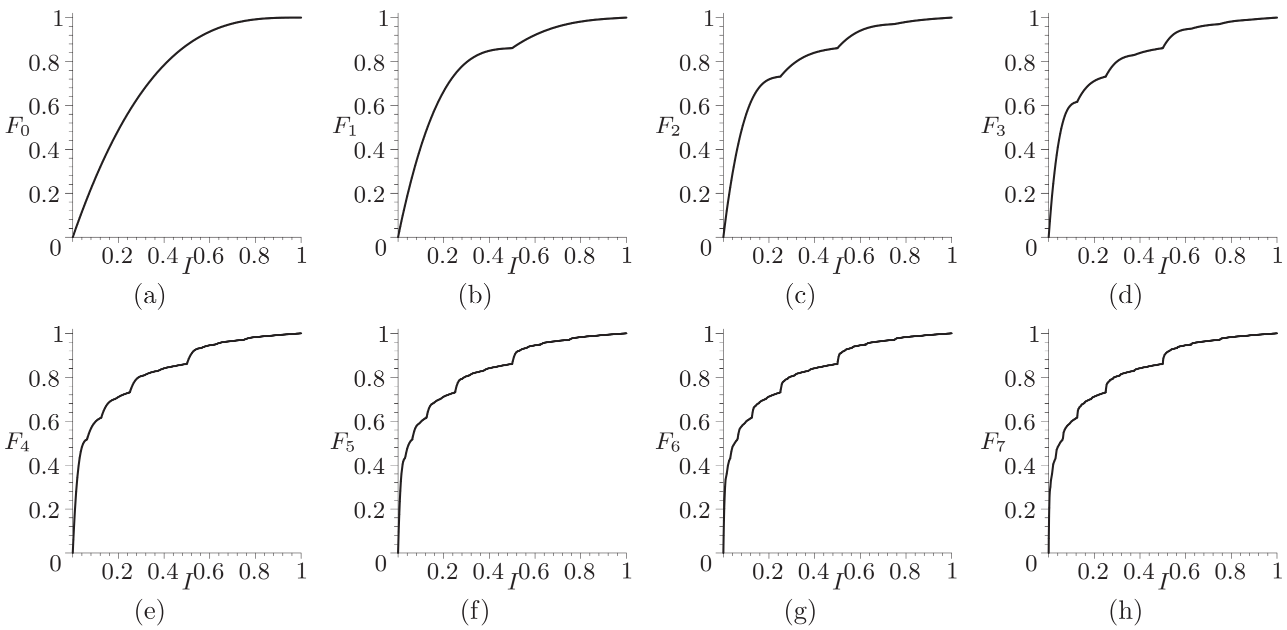

FIGURE 4: Cumulative distribution functions associated to the densities v_t in Figure 3.

Figure 3 plots the first 7 iterations of operator T_θ applied to the density on the number of infectives starting from the initial density $v_0(I) = 3(I-1)^2$ for the same values $\omega_1 = \frac{5}{6}$ and $\omega_2 = \frac{1}{9}$, while Figure 4 plots the evolution of the corresponding cumulative distribution functions F_t associated to the densities v_t reported in Figure 3. Clearly, as the number of infectives evolves according to the dynamics (22) described by the same maps w_1 and w_2 as in (18) only exchanged in their order, both densities and cumulative distributions in Figures 3–4 appear to be perfectly symmetric with respect to those in Figures 1–2, so that higher probability values on larger consumption correspond to lower probability values on larger numbers of infectives and viceversa. Note that, as Proposition 4 establishes uniqueness of the asymptotic invariant density, the densities v_t pictured in Figure 3 (as well as the cumulative distributions reported in Figure 4) would converge to the same fixed point accumulating most of the mass on healthy (non-infective) workers as time elapses also if the initial density v_0 were increasing, *e.g.*, of the form $v_0(I) = 3I^2$, that is, also if the implementation of active policies in scenario $i = 1$, corresponding to a coefficient $\omega_1 > \omega_2$, would start in a depressed economy characterized by a large number of infectives. The shape of such an invariant density (cumulative distribution) turns out to be very close to its approximation provided by Figure 3(h) (Figure 4(h)).

Figures 5–8 report the same plots as in Figures 1–4 but for the coefficients' values $\omega_1 = \frac{1}{10}$ and $\omega_2 = \frac{3}{5}$ describing the behavior of a "short-sighted" government in which the laissez-faire scenario dominates. As we would expect, the densities evolution turns out to be all reversed with respect to the former active policy scenario, as the density on consumption accumulates more on lower consumption levels while the density on the number of infectives concentrates on higher levels of prevalence as time elapses. There is, however, an important difference with respect to the former case: both Figures 5(h) and 7(h) report densities that manifestly exhibit a larger noise uniformly spread on all consumption amounts and infectives with respect to their counterparts in Figures 1(h) and 3(h). Such a higher degree of uncertainty is due to the larger weight put on the condensation term, $2\omega_2\theta$, representing a more diffuse uncertainty characterizing the laissez-faire scenario in which less information on the epidemic are available.

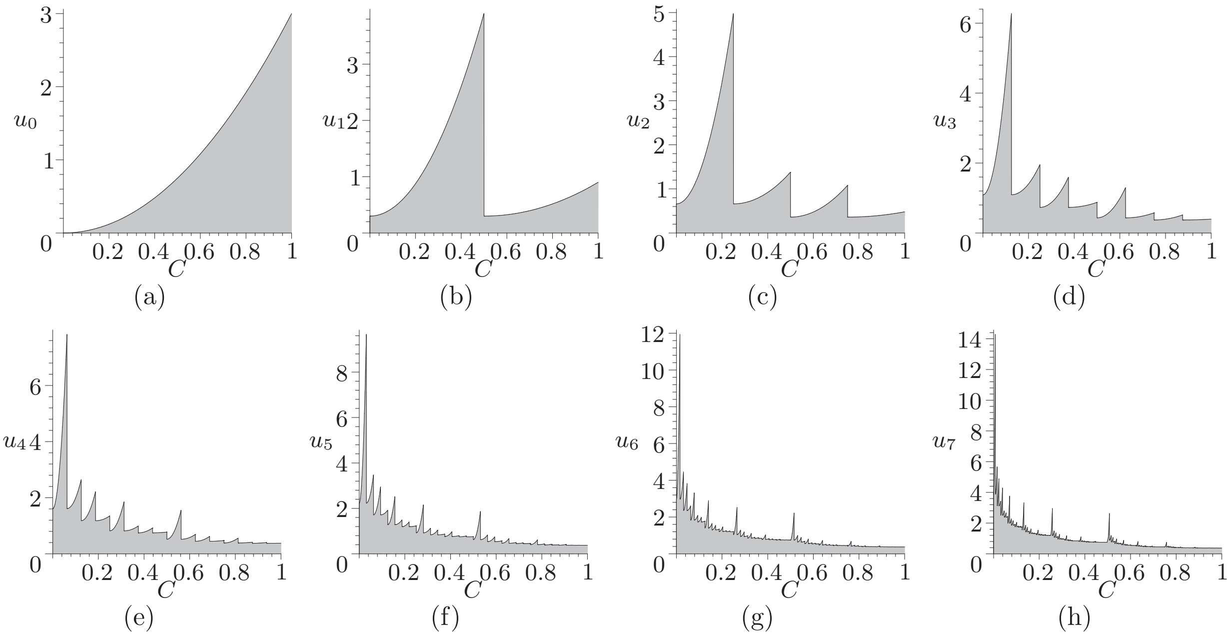

FIGURE 5: First 7 iterations of operator T_θ defined in (19) applied to the density of consumption for $\omega_1 = \frac{1}{10}$ and $\omega_2 = \frac{3}{5}$ starting from $u_0(C) = 3C^2$.

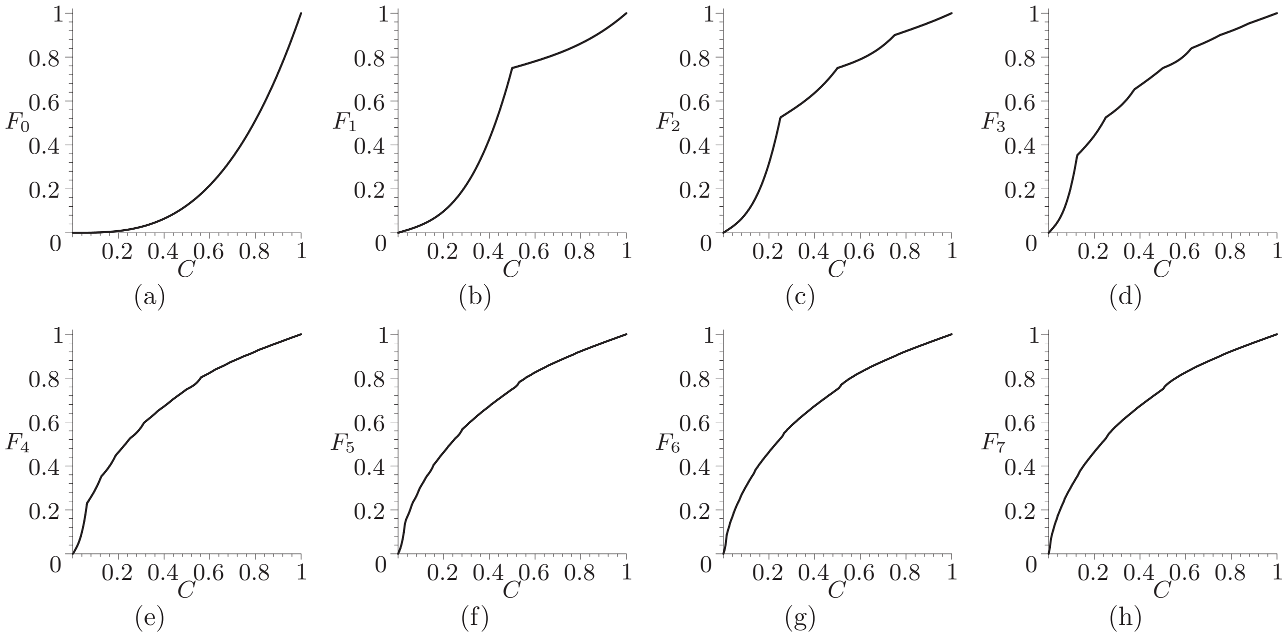

FIGURE 6: Cumulative distribution functions associated to the densities u_t in Figure 5.

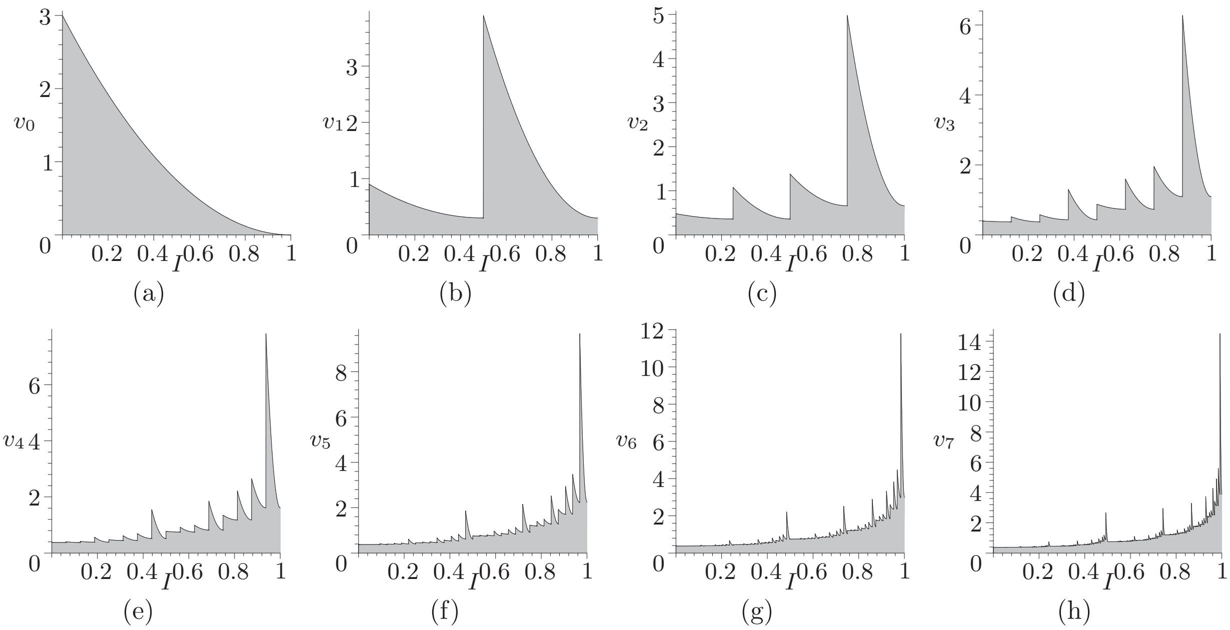

FIGURE 7: First 7 iterations of operator T_θ defined in (19) applied to the density of infected people for $\omega_1 = \frac{1}{10}$ and $\omega_2 = \frac{3}{5}$ starting from $v_0(I) = 3(I-1)^2$.

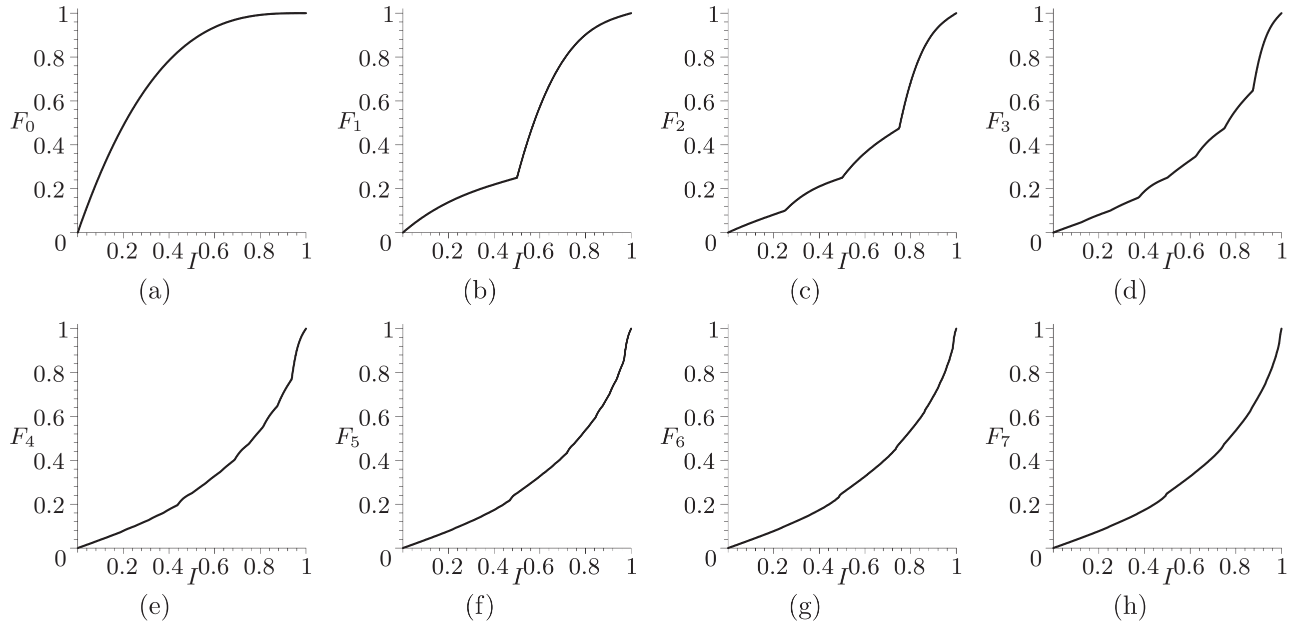

FIGURE 8: Cumulative distribution functions associated to the densities v_t in Figure 7.

To summarize, our numerical simulations show that, under the assumption that production is sufficiently productive in labor ($A > 1$), far-sighted active policies generate a virtuous circle capable of obtaining both higher consumption levels and lower numbers of infectives in the long-run. The peculiarity of our analysis implies that such goals are described in terms of invariant densities concentrating more mass on higher consumption levels and on lower numbers of infectives, as reported in Figures 1(h) and 3(h) respectively. The price to pay for such an approach is lower consumption levels in the short-run due to higher taxes charged on consumers' income, which may trigger high pressure on the government to rather pursue an unproductive short-sighted redistribution policy based exclusively on income transfers. If such a pressure turns out to be successful and only redistributive policies with no investment in broad active treatment of the epidemic are implemented, the resulting long-run densities of consumption levels and infective numbers would exhibit shapes that are symmetrical with respect to those in Figures 1(h) and 3(h), concentrating more mass on lower consumption levels and on higher numbers of infectives, as shown in Figures 5(h) and 7(h) respectively. Additionally, the inclusion in the analysis of a constant condensation term associated to the latter scenario lets the asymptotic densities in this case be flatter than in the former scenario, as this term adds a further layer of (deep) uncertainty to the stochastic steady state of the economy.

It is easy to imagine how different parametrizations may lead to completely different outcomes, starting form different values for parameters s, A and τ as mentioned in Footnote 4, and continuing with parameters ω_1 and ω_2 together with the condensation term θ, which further reinforces the deep uncertainty already characterizing the model. In other words, this method of analysis paves the way for the study of a broad range of models.

6 Conclusion

From the recent coronavirus epidemic experience, it has grown a wide consensus on the fact that infectious diseases may have dramatic implications for macroeconomic outcomes and so more research is needed to understand the possible mutual epidemic-macroeconomic links and how public policy may be used to limit the spread of such diseases. Several papers discuss the large degree of uncertainty which surrounds the information about the effective level of disease prevalence and thus how difficult obtaining accurate policy prescriptions might be. In order to take this into account we develop a simple macroeconomic-epidemiological framework in which health and macroeconomic outcomes are strictly related and characterized by deep uncertainty as the number of infectives is not known with precision and thus health and economic outcomes need to be analyzed in terms of density functions. Specifically, in our setting the level of prevalence determines the size of the healthy labor force, affecting output and consumption, and so the availability of resources to finance public policy, which is funded via income taxation. The high degree of uncertainty is reflected also in the policymakers' choice of the policy tools to employ in order to mitigate the socio-economic effects of the disease. As different policymakers implement different policy measures, the effective level of disease prevalence and thus the effective level of economic activity is highly uncertain, and thus we can analyze how different types of policymaking (short-sighted vs. far-sighted) approaches affect the asymptotic invariant distributions of macroeconomic activity, quantified by consumption, together with the spread of infectives. Through numerical simulations we show that short-sighted policies (far-sighted policies) lead to asymptotic invariant probability distributions concentrating more mass on low (high) levels of consumption together with large (small) numbers of infectives, and exhibit an additional layer of (uniform) uncertainty generated by the condensation term.

Our model is based on some simplistic assumptions which limit our ability to provide insightful policy recommendations. In particular, the epidemiological framework describes the early epidemic stage and thus cannot be applied to later stages of an epidemic dynamics; the macroeconomic setting is very simple as it abstracts from capital accumulation and thus does not allow us to analyze how health policy may impact saving decisions and long run growth; the model is very aggregative in nature and thus it does not permit to analyze how social interactions at individual level may impact the disease dynamics. Moreover, besides abstracting from possibly optimal behavior by individuals, our analysis here lacks a whole synthesis on welfare considerations, as, although consumption levels and the number of infectives both contribute to total welfare, they are studied separately. It seems sensible to add welfare targets to be pursued by policymakers, which are clearly affected both by macroeconomic and health outcomes. Extending our baseline analysis in order to account for the above mentioned issues will help us to develop a comprehensive analysis of the macroeconomic-epidemiological links and to better understand the working mechanisms of alternative policy tools. This is left for future research.

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