# The Peano School:
# Logic, Epistemology and Didactics

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# Giuseppe Peano and his School: Axiomatics, Symbolism and Rigor

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Peano's axioms for arithmetic, published in 1889, are ubiquitously cited in writings on modern axiomatics, and his *Formulario* is often quoted as the precursor of Russell's *Principia Mathematica*. Yet, a comprehensive historical and philosophical evaluation of the contributions of the Peano School to mathematics, logic, and the foundation of mathematics remains to be made. In line with increased interest in the philosophy of mathematics for the investigation of mathematical practices, this thematic issue adds some contributions to a possible reconstruction of the philosophical views of the Peano School. These derive from logical, mathematical, linguistic, and educational works[1], and also interactions with contemporary scholars in Italy and abroad (Cantor, Dedekind, Frege, Russell, Hilbert, Bernays, Wilson, Amaldi, Enriques, Veronese, Vivanti and Bettazzi).

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1. The published and unpublished writings of Peano are collected in [Roero 2008]. An English anthology of Peano's texts is [Peano 1973]. For a rich literature on Peano and other members of his school see in particular [Luciano 2017]. A complete list of Padoa's writings can be found in [Cantù 2007]. The publications of Vailati and a rich literature on his life and works are listed in the introduction to [Arrighi, Cantù *et al.* 2009]. On Pieri see in particular the references quoted in [Marchisotto & Smith 2007]. On Burali-Forti see the references added to [Burali-Forti 1919].

# 1 The Peano School

It is debatable whether the group can be classified as a "scientific school" with an exhaustive list of all its members. The category of a mathematical research school, explored in its distinctions and national features by David Rowe [Rowe 2002], has recently been opposed to the category of a mathematical tradition. According to José Ferreirós a mathematical research school is "a group led normally by only one mathematician, localized within a single institutional setting and which counts on a significant supply of advanced students", whereas a mathematical tradition "implies that one can find a common research orientation in different actors that do not share a common institutional site, but are linked by traceable influences on each other" [Ferreirós 1999, xxii–xxiii]. To settle the question whether the Peano group should be considered as a research school or as a mathematical tradition, we first need to deconstruct several clichés from the literature and clarify the nature of Peano's leadership, the circulation of knowledge within the Peano School, and the role of other collective enterprises beside the *Formulario* (e.g., the *Rivista di Matematica*, the journal *Schola et Vita*, the *Dizionario*, as well as other contemporary articles and teaching materials). Original contributions have recently been based on the exploitation of new archival sources. These include the discovery of new previously unmentioned collaborators, the distinction of different levels of decision-making in Peano's editing process for the *Formulario*, and new insights into the original contributions of each member to shared knowledge in the group [Luciano 2017].

# 2 Philosophical interest

The Peano School is generally considered to be a phenomenon that suddenly appeared in all its splendour at the Paris congress of 1900 and then was extinguished like a firework that leaves a vivid but indefinite memory. Given the long-lasting impression made on Russell and other participants in the 1900 Paris Conferences in Mathematics, Philosophy and Psychology by the contributions of Peano, Burali-Forti, Padoa, Pieri, and Vailati, literature on the subject has often sought to find reasons to explain a general loss of philosophical interest in the Peano school in the first half of the 20th century. General explanations abound thereon and include: the non-academic nature of the group; the multiform topics of interest ranging from mathematical analysis to geometry, from linguistics to universal languages, and from philosophical pragmatism to logicism [Roero 2010], [Skof 2011], [Kennedy 2002]; the fact that scarce attention has been given to the transformation of mathematics and to the development of set theory after 1910 [Quine 1987]; a general belief that Peano was not really interested in the theory of inferential reasoning, or

in the metalogical and metamathematical investigation of the properties of axiomatic theories [van Heijenoort 1967].

Other philosophical explanations have also been suggested: Peano's utilitarian approach to logic [Grattan-Guinness 2000] and symbolic notation [Bellucci, Moktefi *et al.* 2018]; the lack of a shared and explicit epistemological framework for relevant logical and methodological issues such as functions [Luciano 2017], [Cantù 2021], logical identities [Cantù 2007], definitions by abstraction [Mancosu 2018], and questions of purity [Arana & Mancosu 2012]; a subdivision of labour that led to Giovanni Vailati in Italy [Arrighi, Cantù *et al.* 2009] and Louis Couturat in France [Luciano & Roero 2005] becoming the chief philosophical spokesmen of the group; the belief that Peano's presentation of arithmetical axioms had less interesting philosophical implications with respect to logicism and structuralism than that of Dedekind [Ferreirós 2005]; the interest of Peano's collaborators in pedagogical and political issues [Giacardi 2006], [Luciano 2012].

The topic is reconsidered in a new light in this special issue, as the authors discuss the relationship between Dedekind's and Peano's axioms (Kahle, this volume), or the absence of the universal quantifier among the primitive symbols of Peano's *Formulario* and its relation to the use of free variables (von Plato, this volume). Other subjects covered are the peculiarities of Peano's symbolic notation (Schlimm, this volume) and differences with respect to Frege's (Betran-San Millán, this volume), the lack of recognition of Pieri's pedagogical remarks in Italy (Marchisotto & Millán Gasca, this volume), the early association in the USA with Russell's point of view (Lolli, this volume), the interaction between Peano's auxiliary international language project and the internationalization movement at the beginning of the century (Aray, this volume), and finally the limits of Peano's proof of the impossibility of infinitesimals (Freguglia, this volume).

# 3   Logic and epistemology

Some of the usual explanations have lost a degree of effectiveness, because of new specific results, and also because of the interdisciplinary and practical turn suggested by the intertwining of logic and epistemology. In this context, the latter is taken to mean both the analysis of scientific knowledge and the critique of scientific theories, as in neo-Latin languages. This perspective constituted the red thread of an international project (PICS INTEREPISTEME 2018-2020) co-funded by the French National Center for Scientific Research and the Vienna Circle Institute and co-directed by Paola Cantù and Georg Schiemer in collaboration with Erika Luciano at the University of Turin. The objective was to compare three distinct collaborative and interdisciplinary epistemologies developed by the members of the Peano School, the editorial board of the *Revue de métaphysique et de morale*, and the Vienna Circle.

The project showed various points of connection between collaborative and interdisciplinary approaches and educational and political aims, such as the vulgarization of scientific knowledge, and the criticism of disciplinary and national boundaries. However, it focused on the origins and development of non-mainstream philosophical views that cannot be reduced to logicism or structuralism, and investigated the underestimated influence of Leibniz's philosophy [Luciano 2006], [Cantù 2014], 19th century positivism, empiricism, and neo-criticism on these standard views in philosophy of mathematics [Cantù & Schiemer forthcoming].

The specificity of the School's research programme was only partially received because of a misunderstanding of the deep relation between education, linguistics, and axiomatics, and also a simplistic association of Peano's ideas with Russell's philosophy. This tendency emerged in van Heijenoort's remarks on the lack of inference rules and metatheoretical investigations, or in the quick tendency to classify Peano as a logicist but in fact was already evident in the early reception of Peano in the USA. Gabriele Lolli shows how the works of the Peano's School had already been discussed by Edwin B. Wilson in 1904 in a review of two pieces of writing by Bertrand Russell, which contributed to the two conceptions being eventually combined as "the Peano-Russell point of view".

The ability to discriminate subtle differences between the positions of Russell, Frege and Peano characterized a fine reader of Peano's work: Kurt Gödel. The philosophical notebooks (*Max Phil*) reveal a deep understanding of differences on the notions of function and definite description [Crocco, Van Atten *et al.* 2017], [Cantù 2016a]. The summary of the *Formulario* to be found in one of his *Excerptenhefte* shows the analysis of the rules of inferences used in deductive chains. The accurate summary of Peano's *Arithmetices Principia* written in Gabelsberger shorthand on a loose sheet of paper when Gödel was preparing the article on Russell's logic (early 1943) has been edited by Jan von Plato for this special issue. It clearly shows that Gödel read not only the *Formulario* [Peano 1895], but also the *Arithmetices Principia* [Peano 1889], focusing his comments on the formal character of proofs.

# 4   The implicit philosophy within Peano School's practices

The attention paid to mathematical practices has shown that Peano had a strong impact on the writings of Frege, Russell, Carnap and Gödel, and also developed a proper philosophical view that emerged from the logical investigation of definitions, the logical interpretation of the symbols of a formal language, the distinction between relations and functions, and the difference between primitive and derived terms or propositions in an axiomatic system. Peano's philosophical views is distinct from both logicism and structuralism

and emerges as a result of a joint investigation of logic, language and mathematics, considered both as theoretical and didactic practices. The interest in definitions and the analysis of language had significant effects on Peano's semantics, which differs from what is usually described as conceptualist (or as a three-level: words/ concepts/ objects) semantics because symbols refer to concepts only through the mediation of language. In the same way as dictionary entries that only get meaning when inserted into a given linguistic context, the symbols' meaning can only be determined through a preliminary substitution with linguistic sentences in each of which the symbols refer to the concepts expressed by the corresponding words in ordinary mathematical language [Cantù 2021].

This volume constitutes a further decisive step towards the reconstruction of Peano's philosophical views from a detailed analysis of logical, mathematical, pedagogical and also linguistic practices. The essays gathered here focus on the works of Giuseppe Peano, Alessandro Padoa and Mario Pieri, but the same method could be fruitfully applied to other members of the school, such as Giovanni Vailati [Arrighi, Cantù *et al.* 2009], Cesare Burali-Forti and Alessandro Padoa. The contributions of Peano and other members of the school are also evaluated by comparison with contemporaries (Richard Dedekind, Gottlob Frege, Bertrand Russell, David Hilbert and Paul Bernays), resulting in a historically accurate analysis of some subtle but fundamental differences between their respective projects which aimed to present, analyze or ground mathematics as a rigorous, deductive science.

Three examples will be briefly mentioned in this introduction: axiomatics, linguistic symbolization and rigour. Different terms are often used to characterize the school's foundational enterprise: symbolization, formalization, axiomatization, reduction. A deep investigation of the Peano school's practices might help disentangle some of the differences between these fundamental notions, and shed new light on different ways to conceive generality, ideography, metatheoretical inquiries, and the role of notation, intuition, and rigor.

# 5   Axiomatics

General philosophical and historical reconstructions of the development of logic in the early 20th century have accustomed us to think of Peano as one of the fathers of modern axiomatics because of his contribution to the formulation of the axioms of arithmetic, which still bear his name. Yet, a detailed analysis of the connections between logical, linguistic, mathematical and pedagogic writings of the Peano School might help re-evaluate his contributions to logic and philosophy of mathematics and discover a specific approach to axiomatics. An axiomatization is a particular kind of presentation of a theory, in which the logical and the mathematical content is specified by the respective axioms. Reinhard Kahle's contribution traces a history of the formulation of the

properties of real numbers, considered as *Sätze* by Dedekind and explicitly
formulated as axioms by Peano, acquiring a non-logical nature in Hilbert's
works and a first-order formulation in Bernay's contributions. This is a
historically fruitful example of how the investigation of different uses and
presentations of the same mathematical properties of numbers can reveal very
different conceptions of the axiomatization of arithmetic.

Yet, axiomatics cannot be reduced to the investigation of the axiomatic
formulation of single theories. It is a back-and-forth process between the
syntactic, semantic and pragmatic-linguistic levels and their goals: 1) to make
the implicit assumptions of a theory explicit (e.g., by stating all the hypotheses
necessary to prove a given theorem); 2) to investigate the tacit assumptions
of a theory, considering what happens if they are not implicitly assumed (e.g.,
by testing the possibility of creating non-standard models of a theory); 3) to
define the scope and goals of a research programme or discipline [Woodger
1959]. Linguistic analysis is a pillar of Peano's approach, and cannot be
dissociated from epistemological goals, such as the search for good order
and the minimal number of concepts, and the questioning of the relation
between mathematical practices and a rigourous mathematical language. Far
from being exclusively aimed at the construction of axiomatic systems or
the investigation of deductive inferences, the symbolization of logic rests on
questions very similar to those that have developed in the social sciences: the
need to distinguish the simple from the complex, the first for us from the first
for itself, a canonical form from deviant forms, the definition of a term from
the formation of a concept, the pragmatic consequences of a hypothesis from
its theoretical role [Cantù 2020].

Axiomatics relies heavily on complex practices of symbolization and
formalization, which have a social and interactive nature, and that should
be studied in their different components: phases of redaction (study of the
pertaining bibliography, construction of a hypothetic-deductive order of the
collected results, codification in symbols), division of the tasks among group
members, circulation of knowledge within the group in a hierarchical or peer
context, and the construction of domains of shared knowledge, that do not
need to be mirrored in the final version of publications [Luciano 2017].

# 6   Symbolization and language

Symbolization is a process that associates symbols to words, but symbols
can play the role of schematic letters, as in Hilbert's formalization, i.e., as
terms having a merely formal sense that allows for a variety of interpretations,
or have a substantive role, as terms whose meanings have to be conveyed
by elucidation [Klev 2011]. Bertran-San Millán's contribution explains how
Frege used the symbols of arithmetic as canonical names, i.e., as symbols
with a specific and fixed meaning, so that mathematical letters always have

a specific domain, determined by the intended application. Peano shared a similar substantive understanding of mathematical symbols in his early writings but moved towards a view of undefined symbols as uninterpreted non-logical constants devoid of meaning, when he investigated metatheoretical questions concerning the independence of the axioms with Padoa.

The comparative investigation of logic, linguistics and notational practices offer further insights into the particular version of ideography that is developed in the *Formulario*. In this, symbols mean ideas but are first introduced as names for terms of an interpreted mathematical language having those ideas as meaning, and then also considered as schematic variables that might receive different interpretations by substitution of different linguistic terms. The relation between mathematical symbols, words of mathematical language, mathematical concepts and mathematical objects has a complex history that a comparative investigation of Peano's contributions to logics, mathematics, linguistics, and symbolic notation might help unravel.

The symbolization of mathematics is often discussed in the light of a reduction of mathematics to logic or as a translation that preserves the relevant mathematical meaning. However, it cannot be fully understood without a detailed investigation of the design principles and the didactical and practical constraints that accompany the search for technical symbols in a new notation. The formalization is a way to distinguish the logical form from the non-logical content but can also be conceived as a method of conceptual analysis that identifies the relevant logical and mathematical ideas. As Dirk Schlimm shows in his contribution, this analysis might be used to determine the primitive terms and propositions and to check the adequacy of the analysis itself, thereby evaluating whether definitions are correct, and proofs are rigorous.

The distinction between symbolization and formalization is often difficult to trace, but the investigation of definitions and of metatheoretical issues of independence between axioms and a careful investigation of the interactions with linguistics might be of help. There are several aspects of Peano's approach to the *Interlingua* that relate it to mathematical logic: in both cases a language in use (mathematical language and Latin) is taken as a starting point for the development of a universally understandable language (logical symbolism, Interlingua); secondly the two enterprises are based on collaborative networks; both are grounded in the legacy of Leibniz' *characteristica universalis*; finally they are combined in the last edition of the *Formulario*, written in "*Latino sine flexione*" and symbolic language [Cantù 2016b]. In her contribution, Başak Aray highlights another similarity: the connection between algebra and grammar developed in the *Formulario*, and suggests that the symbolization developed in Peano's mathematical practice guided the design of his proposal for an international auxiliary language: the *Latino sine flexione*.

This algebraic understanding of grammar more effectively explains how logic and language are both presented in an equational form, and generality is expressed using free variables instead of assuming a universal quantifier

as a primitive logical term. According to Jan von Plato this is the reason why Peano's axiomatic systems, like Schröder's algebraic logic, lacked some principles of reasoning with the quantifiers even though they contained other rules of inferences.

# 7 Mathematical education and rigour

The interrelation between mathematical education and conceptual analysis offers further hints to understanding the main traits of the Peano school's epistemology, the importance of rigour in scientific knowledge and education and also the interpretation of axiomatics as a metatheoretical investigation based on a variety of alternative conceptual analyses leading to different axiomatic presentations and definitions of the mathematical concepts. The attention to the pedagogic component in the Peano School shows that the sociological singularity of this research team (the only non academic-based group in the international panorama) corresponds to a unique educational project on rigour. This project was deeply intertwined with the mathematical, philosophical, logical and linguistic views of the group and had non-negligible effects on the evolution of mathematical teaching in Italy, and beyond. Rigour is not an accessory or external element that can be imposed on mathematical teaching. It is instead a result of the development of rational mathematics and the evolution of all sciences towards the structure of axiomatic-deductive systems. Rigour is not primarily a foundational problem, although conversely, the foundational enterprise is intertwined with didactic concerns [Luciano 2020].

Rigour is a distinctive feature of the Peano School's style and also an essential topic in the Italian debate on mathematical pedagogic theory and teaching practice at the turn of the century. Peano's crusade in defence of rigour is both a distinctive mark of his axiomatics and a feature of the School's linguistic, mathematical and educational research programmes. It was neither a negation of the importance of experimental methods in the early stages of mathematical education [Luciano 2020] nor a simplistic negation of mathematical intuition which was banished from the proofs of a theory but remained decisive in the choice of axioms [Rizza 2009]. It was instead a didactical objective developed through exchanges with school teachers and their associations, the publication of new textbooks, and participation in educational Governmental Committees [Giacardi 2006].

In their contribution, Elena Marchisotto & Ana Millán Gasca illustrate Pieri's belief that an integration of sensible and rational intuition can deeply renew the teaching of geometry while also deploying a profound heuristic value. The analysis of Pieri's axiomatization of geometry exemplifies the almost symbiotic relation between axiomatics and pedagogy that is typical

of the Peano School as well as the partial and complex reception of this idea in works by Italian contemporary mathematicians, like Enriques and Amaldi.

And yet, the very same idea of rigour gave rise to famous debates, whose philosophical objectives were sometimes obscured by putting forth educational motivations or formal demonstrations. The famous debate with Segre on rigour and intuition was both a manifesto of Peano's style and an implicit criticism of Veronese's hyperspaces and the expression of the rivalry with the geometrical Italian School [Luciano 2020].

Similarly, Freguglia claims that the famous proof of the impossibility of infinitesimals was not developed just to complete and rectify an untenable proof by Cantor—undergoing the similar mistake of presupposing an axiom that is equivalent to the Archimedean axiom and therefore incompatible with the existence of infinitesimals. It was also an implicit criticism of Veronese's theory of a geometrical non-Archimedean continuum and provided an opportunity to host a scientific discussion of the topic in the newly founded *Rivista di Matematica*.

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# Frege, Peano and the Interplay between Logic and Mathematics

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**Résumé :** Dans les études historiques contemporaines, les contributions de Peano sont généralement envisagées dans le cadre de la tradition logiciste initiée par Frege. Dans cet article, je vais d'abord démontrer que Frege et Peano ont développé de manière indépendante des approches semblables visant à s'appuyer sur la logique pour exprimer rigoureusement des lois mathématiques et les prouver. Ensuite, je soutiendrai cependant que Peano a également utilisé sa logique mathématique d'une manière qui anticipait la formalisation des théories mathématiques, laquelle est incompatible avec la conception de la logique défendue par Frege.

**Abstract:** In contemporary historical studies, Peano is usually included in the logical tradition pioneered by Frege. In this paper, I shall first demonstrate that Frege and Peano independently developed a similar way of using logic for the rigorous expression and proof of mathematical laws. However, I shall then suggest that Peano also used his mathematical logic in such a way that anticipated a formalisation of mathematical theories which was incompatible with Frege's conception of logic.

## 1 Introduction

Even by the early twentieth-century, in Jourdain's Preface to the English translation of Couturat's *L'Algèbre de la Logique* [Jourdain 1914, viii], Frege and Peano had been presented as members of the same logical tradition. The alleged proximity of the views of Frege and Peano, purportedly synthesised by Russell, has been retained in the contemporary historiography of logic and has

become a commonplace.[1] Frege's and Peano's conceptions of logic stand out in opposition to the algebra of logic tradition.

In this paper I shall question Peano's inclusion in the Frege-Russell tradition on the basis that Peano develops a specific application of logic to mathematics that is incompatible with Frege's view. First, I shall argue that Frege intends to use the logical system developed in his mature works not only to show that arithmetic can be reduced to logic but also as a tool for the rigorous expression and proof of mathematical laws. Second, I shall propose that although Peano devises a reformulation of mathematical theories by means of logic similar to Frege's, in addition, Peano and the members of the so-called Peano school develop a new understanding of the resulting expressions of this reformulation that anticipates a contemporary notion of formalisation which Frege cannot accept.[2] In sum, I shall investigate Frege's and Peano's views on the application of logic to mathematical theories and the formalisation of the latter, and conclude that they develop accounts that are, in significant respects, irreconcilable.

This paper is in two parts. First, I shall discuss Frege's views on the application of logic to arithmetic. This involves his logicist project but also, and crucially, his proposal to apply the formal resources of logic to a reformulation of mathematical theories. Second, I shall study, on the one hand, Peano's aim of creating an ideography by means of the combination of logical and mathematical symbols and, on the other, the development by the members of Peano's school of a new understanding of the expressions of such an ideography in the context of proofs of independence.

# 2   Frege's reduction and symbolisation

## 2.1   Logicism and the reduction of arithmetic

For a significant stretch of his career, Frege understood the relationship between arithmetic and logic as the *reduction* of the former to the latter. The purpose of Frege's logicist project is to demonstrate that arithmetic is a logical theory. In *Grundlagen der Arithmetik* [Frege 1884], Frege considers

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1. Van Heijenoort develops Jourdain's dichotomy of two logical traditions in terms of the "logic as language" tradition and the "logic as calculus" tradition [van Heijenoort 1967*b*]. This paper became very influential and established a conceptual framework for the history of modern logic.

2. In the context of this paper, I understand by "*formalisation*" the replacement of a set of sentences expressed in a language $L$ (usually, natural language) with a corresponding set of sentences expressed in a *formal* language $L'$ (typically, that of first-order logic), which preserves the logical form of the sentences of $L$ and expresses it using logical symbols, but substitutes non-logical constants (which are uninterpreted) for the non-logical terms of $L$. On the notion of formal language, see [Church 1956, 2–68].

the logicist project from a philosophical point of view and tries to informally justify that the reduction can be carried out. He then attempts a formal proof of the reduction of arithmetic to logic in *Grundgesetze der Arithmetik* [Frege 1893, 1903, hereinafter, *Grundgesetze*].

One of the objectives of the logicist project is the explicit definition of the basic notions of arithmetic by means of the logical symbols. This requires the development of a logical language with enough expressive power. In order to achieve this goal, in *Grundgesetze* Frege profoundly modifies the concept-script—the logical system he had first presented in *Begriffsschrift, eine der arithmetischen Formelsprache des reinen Denkens* [Frege 1879a, hereinafter, *Begriffsschrift*]. Among other things, in *Grundgesetze* he rigidly regiments quantification and incorporates the notion of value-range in the language by means of a function symbol, "$\dot{\epsilon}\varphi(\epsilon)$".

Frege's logicist project also aims to prove that all arithmetical laws are logical laws, i.e., to prove that the laws of arithmetic can be derived in the calculus of the concept-script from logical laws and explicit definitions. Such a proof involves a modification of the semantical status of some of the components of arithmetical propositions; the letters occurring in them then go on to express generality over the domain of logical objects and, accordingly, the quantifiers cease to range exclusively over natural numbers. Therefore, after the reduction of arithmetic to the concept-script, arithmetical laws can still be interpreted judgements, although they would go on to express purely logical facts.

Moreover, after the explicit definition of the basic notions of arithmetic, for Frege there is no need to keep the symbols that represent them in the process of proving arithmetical laws by logical means. The proofs and judgements of the first volume of *Grundgesetze* do not contain arithmetical symbols, but the primitive symbols of the concept-script, letters and symbols that Frege introduces by means of definitions, such as "$0$" and "$1$"—which refer to the cardinal numbers 0 and 1, respectively.

All in all, the reduction of a theory to another is significantly different from the formalisation of a theory. A formalisation requires a formal language or, at least, a symbolic language that contains non-logical constants. Since non-logical constants are uninterpreted, the resulting formulas of a formalisation do not preserve the meaning of the formalised sentences; only the syntactic status of the symbols of the formalised theory is kept. In contrast, a reduction does not require a formal language; in fact, it can only be performed by means of an interpreted language, since the original meaning of both the primitive symbols and the laws of the reduced theory have to be maintained in essence. In fact, the basic terms of the theory by means of which the reduction is performed are substantive, in the sense that they refer to the specific entities the theory is about.[3] This enables the provision of explicit definitions of the
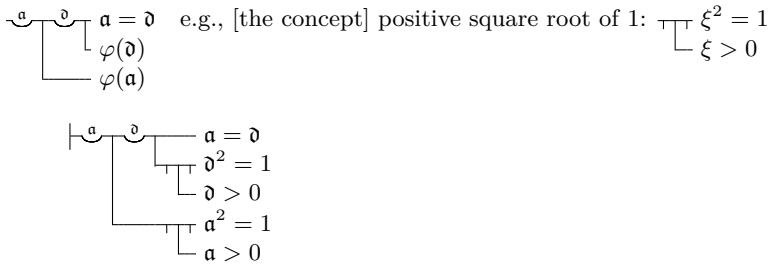
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3. I take the notion of substantive basic terms and their role in the reduction of a mathematical theory from [Klev 2011].

basic notions of the reduced theory and the preservation of their properties. For instance, in Frege's reduction of arithmetic to the concept-script, the cardinal numbers are defined as logical objects, but at the same time they retain their mathematical properties.

## 2.2    Early and late applications of logic

As is well known, Frege's logicist project ended abruptly with the discovery of the inconsistency of the concept-script presented in *Grundgesetze*. After 1902, Frege was forced to modify his views on the relation between logic and arithmetic. The best witness to Frege's post-logicist understanding of the relationship between the concept-script and mathematics can be found in the student notes Carnap wrote while attending some of Frege's courses in Jena between 1910 and 1913 [Frege 1996]. In the first of these courses, *Begriffsschrift I* (which took place in the winter semester of 1910-1911), Frege presents the main components of the language of the concept-script—as they are described in *Grundgesetze*, but without mentioning the symbols for value-ranges or for the function $\backslash\xi$. He thus obtains a higher-order logical language. Frege then shows, with examples, how its syntax could be naturally adapted to the expressions of arithmetic. This process consists in connecting atomic expressions of number theory, such as "$a > 0$" or "$(a - b) + b = a$", using the logical symbols of the concept-script. The combination of atomic expressions of number theory and logical symbols also involves the incorporation of the letters of the concept-script—by means of which generality is expressed—into the aforementioned atomic expressions. For instance:[4]

> If we want to express that at most *one* object falls under a concept, we write:

$$\underset{\mathfrak{a}}{} \underset{\mathfrak{d}}{} \left[ \begin{array}{l} \mathfrak{a} = \mathfrak{d} \\ \varphi(\mathfrak{d}) \\ \varphi(\mathfrak{a}) \end{array} \right. \quad \text{e.g., [the concept] positive square root of 1:} \quad \left[ \begin{array}{l} \xi^2 = 1 \\ \xi > 0 \end{array} \right.$$

$$\vdash \underset{\mathfrak{a}}{} \underset{\mathfrak{d}}{} \left[ \begin{array}{l} \mathfrak{a} = \mathfrak{d} \\ \left[ \begin{array}{l} \mathfrak{d}^2 = 1 \\ \mathfrak{d} > 0 \end{array} \right. \\ \left[ \begin{array}{l} \mathfrak{a}^2 = 1 \\ \mathfrak{a} > 0 \end{array} \right. \end{array} \right.$$

> [Frege 1996, 17; 77]

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In the second course, *Begriffsschrift II* (which took place in the summer semester of 1913), Frege first presents the logical fragment of the calculus of the *Grundgesetze* concept-script: he introduces the basic laws and some of its inference rules, but omits basic laws (V) and (VI), which involve value-ranges.[5] Frege then exemplifies how the calculus of the concept-script can be applied to prove two theorems of analysis. These proofs are detailed reconstructions of mathematical proofs using the formal tools provided by the concept-script. First, Frege reformulates the theorem he wants to justify using a combination of logical and mathematical symbols. Second, he lists and reformulates in the explained way the propositions of analysis that are needed in the proof as premises. Third, the logical principles that are required in the proof are incorporated as premises by means of substitutions, in such a way that simple formulas belonging to the language of the concept-script, such as "$M_\beta(f(\beta))$" or "$f(a)$" (which, strictly speaking, should be considered terms) are replaced with expressions of analysis. With all these components, Frege conducts the proof in a similar way as he had done in *Grundgesetze*: he renders explicit all the logical principles and formal steps involved, using the inference rules available.

Frege's methodology and goals in these courses coincides with the application of the concept-script he devised during the immediate years after the publication of *Begriffsschrift*, in the papers "Anwendungen der Begriffsschrift", "Booles rechnende Logik und die Begriffsschrift" and "Über den Zweck der Begriffsschrift" [Frege 1879*b*, 1880-1881, 1882]. In these papers he is explicit about the aim of such a combination of the concept-script with a scientific theory: Frege strongly associates it with the rigorous expression of the laws and proofs of such a theory. He rejects the perspective of producing what he calls an "abstract logic", i.e., a symbolism isolated from the expression of specific meaning. As he says in "Über den Zweck der Begriffsschrift", in which he compares the 1879 concept-script with Boolean logic, "I did not wish to present an abstract logic in formulas, but to express a content through written symbols in a more precise and perspicuous way than is possible with words" [Frege 1882, 97; 90–91]. Also in this paper Frege offers a general overview of how he intends to apply his concept-script to arithmetic:

> Now I have attempted to supplement the formula language of arithmetic with symbols for the logical relations in order to

---

5. There is no mention of basic law (IV) in the student notes. However, this basic law belongs to the propositional fragment of the concept-script and is completely unrelated to the notion of value-range. In the notes, right before basic law (III) is introduced by Frege, several pages are empty—which indicates that Carnap missed some lectures or failed to take notes in them. Either Frege mentioned basic law (IV) during the course and Carnap did not record it or Frege considered that this basic law was unnecessary for his purposes in this course and did not mention it.

produce—at first just for arithmetic—a concept-script[6] of the
kind I have presented as desirable. This does not rule out the
application of my symbols to other fields. The logical relations
occur everywhere, and the symbols for particular contents can be
so chosen that they fit the framework of the concept-script. [Frege
1882, 113–114; 89]

Frege's view in this passage coincides with the use of the concept-script
described in the 1910-1913 courses—that of a formal structure that could be
combined with the atomic expressions of mathematical theories in such a way
that the meaning of the laws of these theories could be expressed in a precise
way and their proofs could be conducted with the standards of rigour of the
concept-script.[7]

## 2.3   Frege's symbolisation

The application of the concept-script which Frege proposes both in his 1879-
1882 papers and in the post-*Grundgesetze* courses departs from a formalisation.
For the sake of clarity, I shall refer to Frege's proposed application of the
concept-script to a scientific theory as "*symbolisation*", although he never used
this term in this sense.

Frege wants to preserve the symbols of arithmetic and use them as
canonical names, i.e., as symbols with a specific and fixed meaning. Even
quantification is restricted in this application; in the examples in the student
notes, all letters are supposed to range over real numbers, since the numerical
operations and relations are only defined for them:

And we use:

$$
\begin{array}{l}
a = b \\
a > b \\
b > a \\
c > a \\
d > b
\end{array}
$$

This [$c > a$ and $d > b$] is supposed to mean that $a$ and $b$ are
real numbers, since it is only for them that $>$ is supposed to be
defined. [Frege 1996, 26; 100]

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6. For the sake of terminological homogeneity, I have replaced "concep-
tual notation" with "concept-script" as the English counterpart of the German
"*Begriffsschrift*" in this quote, taken from Bynum's translation of [Frege 1882].

7. Frege's position regarding the application of the concept-script to the rigoriza-
tion of mathematical theories is related to his project of creating a *lingua characterica*.
This latter notion can be connected to Leibniz's ideal of a scientific language. The
choice of the term "concept-script" [*Begriffsschrift*] is also related to this project.
On Frege's notion of *lingua characterica* and its relation to the concept-script, see
[Bertran-San Millán 2020a]. See also [Patzig 1969], [Kluge 1977], [Peckhaus 2004]
and [Korte 2010].

Note that Frege shows no difficulty in restricting the domain of the letters or the applicability of arithmetical relations. In this context, if the letters "*a*" and "*c*" were to have a domain wider than the set of the real numbers, then it would not be possible to determine the meaning of an expression such as "*c* > *a*", since the relation > is, as Frege acknowledges, only defined for real numbers as arguments.

The symbols of number theory used in this application are not employed by Frege to express abstract properties and relations. They are not, therefore, seen as uninterpreted non-logical constants devoid of meaning. Only the letters have a specific domain, determined by the intended application.

By means of a symbolisation, Frege aims to overcome the lack of precision posed by the use of natural language in the definition of the derived concepts of mathematical theories and in their proofs. These theories do not have the expressive means necessary for the symbolic representation of the logical relations that form complex sentences. At the same time, most derived notions are defined in complex sentences. As a consequence, the derived notions, if defined at all, have to be defined using natural language, by means of which it is not possible to attain the level of exactness and rigour Frege requires for mathematics. Likewise, in "Booles rechnende Logik und die Begriffsschrift", while comparing his concept-script with Boolean logic, Frege states the following:

> [The concept-script] is in a position to represent the formation of the concepts actually needed in science, in contrast to the relatively sterile multiplicative and additive combinations we find in Boole. [Frege 1880-1881, 52; 46]

By 1910-1913 Frege's remark on the poor expressive capabilities of Boolean logic had to be qualified. By then, the proponents of the algebra of logic had overcome all the expressive shortcomings they had faced in 1880. However, in the 1910–1913 courses, Frege retains unmodified his claim that the language of mathematical theories needs to be complemented with the formal resources of the concept-script if their new concepts are to be defined with an adequate standard of rigour. In these courses, Frege even appeals to the same examples as those of "Booles rechnende Logik und die Begriffsschrift" [Frege 1880-1881]—namely, the definition of the notion of the continuity of a function—in order to show the fruitfulness of his symbolisation in the processes of concept formation.

In fact, by applying the formal resources of the concept-script to a scientific theory such as analysis, Frege shows a lack of interest in logic as a subject matter. He takes great pains to carefully show how a proof of a theorem of analysis can be performed using the formal resources of the concept-

script,[8] but there is absolutely no evidence in the 1879-1883 papers or in the 1910-1913 courses to show that by symbolising analysis or arithmetic in the way he does, he intends to answer metatheoretical questions such as the completeness or consistency of these theories, or the independence of their axioms. The focus is put on precision and rigour. In "Booles rechnende Logik und die Begriffsschrift', after a full symbolisation of the proof of an arithmetical theorem [Frege 1880-1881, 30–36; 27–32]—analogous to those performed in *Begriffsschrift II* [Frege 1996, 25–37; 98–119]—Frege lists the demands fulfilled by such a symbolisation: a complete and clear specification of all the principles necessary for the derivation of the theorem; a warrant that the proof contains no appeal to intuition; and, finally, the certainty that there are no formal steps missing in the proof, since all of them have been rendered explicit [Frege 1880-1881, 36; 32].

# 3   Peano's symbolisation and formalisation

## 3.1   Peano's ideography

One of the most prominent elements in Peano's development of his logic, which he calls "mathematical logic", is the construction of a logical symbolism that can be used as a tool for the rigorous expression of the laws of scientific theories as well as for helping making explicit the logical principles involved in their proofs. Even in Peano's first uses of his logical symbolism, rigour in the derivation of theorems and a precise characterisation of scientific terms are already established as the main goals of this reformulation of scientific theories. In his seminal *Arithmetices principia nova methodo exposita* [Peano 1889*a*, hereinafter, *Arithmetices principia*], Peano expresses himself thus:[9]

> With this notation every proposition assumes the form and precision equations enjoy in algebra, and from propositions so written others may be deduced, by a process which resembles the solution of algebraic equations. That is the chief reason for writing this paper.
>
> [...] Those arithmetical signs which may be expressed using others along with signs of logic represent the ideas we can define.

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8. More than a third of the pages that correspond to Carnap's notes on *Begriffsschrift II* are devoted to the proof of a single theorem. See [Frege 1996, 29–37; 103–119].

9. In *Principii di Geometria logicamente esposti* [Peano 1889*b*, hereinafter, *Principii di Geometria*], Peano discusses several axioms of Pasch's axiomatisation of geometry—which are formulated in natural language—and contrasts them with his own axioms—which are symbolised [Peano 1889*b*, 84–85]. In this discussion he highlights the ambiguities involved in the expression of mathematical laws by means of natural language.

> Thus I have defined every sign, if you except the four which are contained in the explanations of § 1. [Peano 1889*a*, 21; 102]

As we shall see, Peano's intended use of mathematical logic can be seen as what I called a symbolisation. In this sense, he shares Frege's view on the combination of logic and scientific theories for the construction of a symbolised reformulation of these theories.[10]

Still, Peano's understanding of a symbolisation cannot be reduced to a mere rewriting of sentences in the natural language by means of which the laws of scientific theories are expressed. He produces what he calls "ideography" and thus connects it—just as Frege had done via Trendelenburg's use of the term "*Begriffsschrift*" [Trendelenburg 1856]—with Leibniz's scientific ideal of a *characteristica universalis*, which is not intended to express uttered sounds but to represent the structure of concepts.[11]

In *Notations de logique mathématique*, Peano reflects on the creation of an ideography. He describes a similar process to Frege's symbolisation [Peano 1894*b*]. As a first step, Peano proposes to extract the logical form of the sentences of a given theory and to express it using the logical symbols. He then suggests an analysis of the terms of the theory, by means of which its primitive terms can be located and their connection with the other terms can be discovered. This last step makes it clear that Peano does not intend to perform a mere rewriting of the theory.[12] In his words:

> Any theory can be reduced to symbols, for every spoken language, and every writing, is a symbolism, or a series of signs that represent ideas. In order to apply the signs we have explained, we can take the propositions of the theory in question, written in ordinary language, and replace the word *is* with the signs $\varepsilon$, $=$, $\supset$, as the case may be, and [put] instead of *and*, *or*,

---

10. Since Frege first developed his approach to the symbolisation of mathematical theories as early as 1879, it might be asked whether he influenced Peano's notion of symbolisation using mathematical logic. It is very unlikely. Peano's first symbolisation was presented in *Arithmetices principia*, published in 1889, while—as Nidditch [Nidditch 1963, 105] states—he first refers to Frege in [Peano 1891*a*, 101, note 5; 155, fn. 5]. Prior to 1891, Peano had published other papers in which he symbolised mathematical theories: [Peano 1889*b*, 1890*a,b*]. More importantly, Frege articulates in full the application of the concept-script to logic in [Frege 1880-1881]; however, he attempted three times but failed to publish this paper, so probably Peano never had access to it. The extant correspondence between Frege and Peano started in 1891—no trace of any previous letter can be found—and there Frege neither mentions any of his 1879-1882 papers nor explains how he conceives the application of the concept-script to mathematical theories.

11. As Barnes argues, the term "*Begriffsschrift*" can be translated as "ideography" [Barnes 2002]. On the relation between Peano and Leibniz's scientific ideal, see [Cantù 2014].

12. See also [Peano 1896–1897, 203, 191], where Peano distinguishes between a symbolisation—by means of an ideography—and a mere rewriting.

>    ... the signs ∩, ∪, ...; and that *cum granu salis*, because we saw
>    for instance that, depending on the position, the conjunction *and*
>    is represented by means of ∩ or ∪.
>
>      After this first transformation, the propositions are expressed
>    in a few words, linked by the logical signs ∩, ∪, =, ⊃, etc.; and
>    if it has been well done, the words that remain are devoid of any
>    grammatical form; for all the relations of grammar are expressed
>    by means of the signs of logic. These words represent the proper
>    ideas of the theory being studied. Then the ideas represented by
>    these words are analysed, the composed ideas are decomposed into
>    the simple parts, and only, after a long series of reductions and
>    transformations, one obtains a small group of words, which can
>    be considered as minimum, by means of which, combined with the
>    signs of logic, all the ideas and propositions of the science under
>    study can be expressed. [Peano 1894*b*, 164, my translation]

With this ideography, i.e., with the combination of mathematical logic and
the primitive terms of the language of scientific theories, Peano can eliminate
all traces of natural language in the formulation of these theories. Since their
primitive terms are preserved, the original meaning of the expressions of these
theories is also kept.

Peano refers to his symbolisation of scientific theories as a reduction.
However, he does not intend to define the primitive notions of a theory in
terms of another (in this case, mathematical logic), nor prove that the axioms
of the former are, in fact, theorems of the latter. In this sense, Peano's notion
of reduction does not correspond to the characterisation of Frege's reduction
of arithmetic to the concept-script provided in Section 2.1.[13]

Peano focusses on the ideographic reformulations of mathematical theories.
With the axiomatic method in mind, he produces several symbolic axiomati-
sations of arithmetic and geometry.[14]  The resulting theories are constituted
by two separate groups of axioms: a set of logical principles (which usually

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13. Neither did Frege see Peano's symbolisation as a reduction, see [Frege 1897,
365–366; 237]. Although there has been a debate in the literature, some consensus
has arisen over the thesis that Peano did not endorse Frege's logicist project. Most
historical studies plainly deny that Peano was a logicist (see [Kennedy 1963, 264],
[Segre 1995], [Lolli 2011]), while others also emphasise his rejection of philosophical
discussions see [Geymonat 1955]. See also [Grattan-Guinness 2000, 247–249].

14. Having presented the first formulation of his mathematical logic in *Arithmetices
principia*, during the early 1890s Peano provides multiple examples of the ideographic
formulations of mathematical theories: analysis, see, for instance [Peano 1890*b*,
1892*b*]; geometry, see [Peano 1889*b*, 1894*a*]; arithmetic, see, for instance, [Peano
1891*d*]; and even Euclid's *Elements*, see [Peano 1890*a*, 1891*b*, 1892*a*]. Peano's major
ideographic endeavour is his collective project of a *Formulaire de mathématiques*,
which was published in several volumes and revised in subsequent editions. On this
project, see [Borga, Freguglia *et al.* 1985, 163–170] and [Roero 2011].

includes principles of the logic of classes) and a set of mathematical axioms.[15] The clear separation of the logical and the mathematical constituents of the theory is shared with Frege.

Moreover, Peano's presentation of the language of a symbolised mathematical theory also preserves this distinction. Peano consistently provides specific lists for logical (and class-theoretical) and mathematical symbols and treats the latter as substantive, as canonical names.[16] In this regard, later in 1897 he expresses the convenience of preserving the symbols of arithmetic:

> The symbols of Algebra allow us to express some propositions:
>
> $$2 + 3 = 5, \qquad 5 < 7, \ldots$$
>
> We keep these symbols; sometimes we even generalise their meaning; but when we encounter ideas that cannot be expressed by the symbols of Algebra, we introduce new symbols. For instance, we want to express the proposition
>
> $$7 \text{ is a prime number};$$
>
> we already have a symbol to indicate the subject 7; we introduce a symbol Np to signify "prime number"; and a symbol $\varepsilon$ to indicate "is a"; then the stated proposition is transformed into
>
> $$7 \,\varepsilon\, \text{Np}.$$

[Peano 1897, 241, my translation]

Peano is aware that the expression of mathematical principles and the definition of derived notions require, besides the use of logical symbols and symbols of the calculus of classes, the enlargement of the set of primitive symbols. For instance, he introduces "N" and "Np" to refer to the class of natural numbers and the class of prime numbers, respectively. These new symbols should also be taken as canonical names, although they do not belong to the language of arithmetic *stricto sensu*. After all, Peano includes them in the list of primitive symbols of his symbolisation of arithmetic and assumes that they express basic properties of numbers that have been left undefined.

By the end of the nineteenth century, geometry lacked a symbolic language like that of arithmetic. In this sense, Peano's symbolisation of geometry could

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15. This claim should be qualified if the earliest formulations of Peano's mathematical logic are considered. Peano does not axiomatise the logical component of his axiomatisation of arithmetic presented in *Arithmetices principia*. He first offers an axiomatic presentation of the calculus of propositions in [Peano 1891*c*]. Moreover, in *Principii di Geometria* the logical principles are not explicit. In this work Peano only includes three axioms that involve equality [Peano 1889*b*, 61].

16. See, for instance, his presentation in *Arithmetices principia* [Peano 1889*a*, 23; 103–104]. On Peano's view regarding the substantivity of the primitive notions of geometry and arithmetic, see [Borga, Freguglia *et al.* 1985, 51–54, 88–94, 109–110].

not preserve established geometrical symbols that were already in use: the list of primitive symbols of geometry had to be created anew. In *Principii di Geometria*, adopting "1" and " $\varepsilon$ " as primitive symbols, and using the symbols of the language of his mathematical logic introduced in *Arithmetices principia*, Peano offers a symbolisation of geometry and presents the theory axiomatically. He thus employs a mixture of logical symbols, arithmetical symbols and symbols of the calculus of classes and assigns the latter two a geometrical meaning (sometimes preserving, for certain applications, their original meaning). For instance, "1" is used to refer to the class of points and " $\varepsilon$ " to the relation between a point and a segment [Peano 1889*b*, 59–61]. However, at the same time, Peano would express that the objects *a* and *b* are points by " $a, b\,\varepsilon\,1$ ", where " $\varepsilon$ " is used as the symbol for membership.

## 3.2   Formal understanding of symbolised theories

Almost simultaneously to his work on the symbolisation of mathematical theories, Peano developed a new understanding of symbolised expressions that was intimately connected with the evaluation of the independence of the axioms of these theories. As we shall see below, this new understanding of symbolised mathematical axioms in the context of proofs of independence anticipates in significant ways a formalisation.

Peano does not explain in detail the nature of this new understanding of symbolised mathematical laws. However, some members of the Peano school offer lengthy accounts that are related to their explanation of the resolution of metamathematical questions such as the independence of the axioms or the primitive notions of mathematical theories. These accounts can shed light on Peano's position.

A fundamental element for the understanding of a symbolised mathematical theory in Peano's school is the stratification of the components of this theory. In *Arithmetices principia*, Peano distinguishes between the axioms and the theorems of arithmetic, and also between its defined and undefined symbols:

> Those arithmetical signs which may be expressed by using others along with signs of logic represent the ideas that we can define. Thus I have defined every sign, if you except the four which are contained in the explanations of § 1 [N, 1, +1, =]. If, as I believe, these cannot be reduced further, then the ideas expressed by them may not be defined by ideas already supposed to be known.
>
> Propositions which are deduced from others by the operations of logic are *theorems*; those for which this is not true I have called *axioms*. There are nine *axioms* here (§ 1), and they express fundamental properties of the undefined signs.
>
> [Peano 1889*a*, 21; 102]

In "Formole di Logica Matematica" [Peano 1891*c*, 102–104] Peano rephrases this double distinction in terms of primitive and derived propositions and symbols. Primitive propositions, or axioms, are left unproved and primitive symbols are not defined. By means of definitions in terms of primitive symbols all derived notions can be obtained, and theorems (i.e., derived propositions) are the result of derivations that start from primitive propositions and definitions. This idea refines Peano's view on the process of the creation of an ideography.

From this conceptual framework, Padoa characterises in "Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque" [Padoa 1901] what he calls a "deductive theory". The components of a deductive theory are expressed in a language constituted by a system of primitive symbols (which Padoa calls "undefined symbols"), while the theory is determined by a system of primitive propositions ("unproved propositions" in Padoa's terminology). The deductive approach is defined by the disentanglement of these systems of symbols and propositions from their original meaning:

> [D]uring the period of *elaboration* of any deductive theory we choose the *ideas* to be represented by the undefined symbols and the *facts* to be stated by the unproved propositions; but, when we begin to *formulate* the theory, we can imagine that the undefined symbols are *completely devoid of meaning* and that the unproved propositions (instead of stating *facts*, that is, *relations* between the *ideas* represented by the undefined symbols) are simply *conditions* imposed upon undefined symbols.
>
> Then, the *system* of *ideas* that we have initially chosen is simply *one interpretation* of the *system* of *undefined symbols*; but from the deductive point of view this interpretation can be ignored by the reader, who is free to replace it in his mind by *another interpretation* that satisfies the conditions stated by the *unproved propositions*. And since these propositions, from the deductive point of view, do not state *facts*, but *conditions*, we cannot consider them true *postulates*. [Padoa 1901, 318; 120-121]

This way of understanding a theory is thus not meant to preserve its content and express it in a rigorous way, as is the case in the symbolisation that Peano himself or Frege developed. On the one hand, primitive symbols are detached from their original meaning and are effectively seen as non-logical constants, that is, as uninterpreted symbols, whereas on the other hand, primitive propositions cease to be seen as expressing true facts; they express the conditions that an interpretation must hold in order to satisfy them.[17]

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17. The fact that Peano and Padoa talk about interpretations and consider specific domains and interpretations for non-logical symbols does not mean that they anticipate the contemporary notion of model. On the differences between the notion

This process entails that the propositions of the formalised theory only express abstract relations between unspecified objects, properties and relations. In this sense, the development of a formalised theory (i.e., in Padoa's terminology, a deductive theory) involves only a deductive relation between primitive and derived propositions:

> [F]or what is necessary to the logical development of a deductive theory is not *the empirical knowledge of properties of things*, but *the formal knowledge of relations between symbols.* [Padoa 1901, 319; 121]

The distinction between primitive and derived symbols, and between axioms and theorems, guarantees that by merely providing an interpretation of the primitive symbols that satisfies the axioms, the whole theory is satisfied. All relations between primitive and derived symbols are made explicit through definitions and, similarly, all theorems are deduced from axioms and definitions.[18]

Each of Peano, Padoa and Pieri insist upon putting the notion of deduction at the centre of their accounts of the formalisation of mathematical theories. However, they never characterise precisely this notion. In their works, deduction remains an informal notion that is not formally defined. Peano does offer several specifications of logical principles in his presentation of the mathematical logic, but all things considered he fails to provide a full characterisation of the notion of deduction: crucially, a complete system of inference rules cannot be found in Peano's presentations of mathematical logic.[19]

The hierarchic structure of a mathematical theory proposed by the members of Peano's school also involves some methodological principles that would determine their work on metamathematical questions. Since a deductive theory is built from a system of primitive propositions and a system of primitive symbols, the independence of these propositions and the

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of interpretation common to Peano and Padoa as opposed to the contemporary notion of model see [Mancosu, Zach *et al.* 2009, 323–324].

18. As Blanchette states, the stratification of a mathematical language in terms of primitive and derived symbols is instrumental to the understanding of reinterpretation as a method for proving independence, and can be seen as a distinctive feature of late nineteenth-century approaches to the independence of the axioms of geometry [Blanchette 2017, 47]. The pioneering work of Peano's school in this regard should not be underestimated, especially because of the fact that it predates by a decade Hilbert's work on this field.

19. Peano's metatheoretical questions are, to a great extent, intuitively answered. Although he does not consider the notion of soundness, his results in metamathematics presuppose that the calculi he uses are sound. For a discussion on the claim that Peano does not adopt a fully deductive approach to logic, see [Bertran-San Millán 2020*b*]. See also [Goldfarb 1980]. For a critical approach to this claim, see [von Plato 2017, 50–57].

irreducibility of these symbols is understood as a methodological goal. As Pieri puts in in "Sur la Géométrie envisagée comme un système purement logique":

> As far as possible, primitive ideas should be *irreducible* to one another, so that none of them can be explicitly defined by means of others; and, similarly, the postulates should be *independent* of each other, so that none can be deduced from the others. [Pieri 1901, 380, my translation]

It is thus understandable that, right after providing an axiomatisation of a mathematical theory, Peano studies the independence of their axioms.[20]

# 4   Concluding remarks

In this paper I have focused upon the views of Frege and Peano on the application of logical symbolism and the methods of logic to mathematical theories, and concluded that they disagreed as regards substantial aspects.

Their varying views on the formalisation of mathematical theories are rooted in a deep disagreement regarding their goals. For a significant part of his career, Frege aimed at showing that arithmetic could be reduced to logic. Before this project was fully articulated and after it had failed, he intended to use logic as a formal structure appropriate to supplement the language of arithmetic. Peano never attempted to reduce arithmetic to logic, but he also devised—independently of Frege—a symbolisation of mathematical theories with the assistance of logic. However, Peano also aimed at answering metatheoretical questions such as the independence of the axioms of a mathematical theory, and he developed an alternative understanding of symbolised expressions to fulfil this aim.

## Acknowledgements

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20. Peano's independence arguments in geometry can be found in [Peano 1889*b*, 1894*a*]. Most of Peano's proofs of independence have the axioms of arithmetic as their object. See [Peano 1889*a*, 1891*d*, 1897, 1898, 1899, 1901].

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# Logic as Calculus and Logic as Language: Too Suggestive to be Truthful?

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**Résumé :** Le présent article porte sur le rôle inférentiel des quantificateurs chez Frege, Peano et Russell. Nous abordons ici deux aspects qui caractérisent la logique mathématique à ses débuts : le progressif perfectionnement des principes de raisonnement à l'aide des quantificateurs d'une part, la prétendue impossibilité conceptuelle de poser des questions de type métathéorique d'autre part, telle qu'elle est incarnée par le célèbre dicton de Jean van Heijenoort sur « la logique comme calcul et la logique comme langage ».

**Abstract:** The paper focuses on the inferential role of quantifiers in Frege, Peano and Russell. Two aspects of the early years of mathematical logic are discussed: the gradual perfection of the principles of reasoning with quantifiers, and the presumed conceptual impossibility of posing metatheoretical questions, as embodied in Jean van Heijenoort's well-known dictum about "logic as calculus and logic as language."

## 1 Introduction

Modern logic is usually counted to begin with the times of George Boole, around 1850, with an initial phase of algebraic logic. It was gradually replaced by the logic of Frege and Peano, conceived in 1879 and 1889, and received through the mediation of Russell in the first decade of the 20th century. In this essay, I shall discuss two aspects of these pioneer years: the principles of reasoning with the quantifiers, and the awareness about metatheoretical questions. Jean van Heijenoort, in a widely read essay "Logic as calculus and logic as language" [van Heijenoort 1967*b*], claims that such questions were a conceptual impossibility for the early pioneers of modern logic; a suggestive thesis that turns out to be in great part a myth not supported by any close reading of the sources.

# 2    Quantifiers in Frege, Peano, and Russell

The alternatives for the formal treatment of foundational questions in mathe-
matics at the end of the 19th century were—excepting Frege whose work was
not understood by others—Schröder's algebraic logic and Peano's axiomatic
systems, both lacking essential principles of reasoning with the quantifiers.

The very first appearance of generality and existence were in Aristotle's
syllogistic: *every A is B* and *some A is B*. These quantifiers were, though,
not variable-binding operations and therefore did not play any clear role in
ancient mathematics [see von Plato 2017, chap. 1].

Traditional modes of inference with existence and universality include that
one can take instances of a universal and conclude existence from an instance.
When mathematical proofs got formalised, these were written as axioms, with
$t$ a term that names a constant:

I $\forall x A(x) \supset A(t)$     II $A(t) \supset \exists x A(x)$

To infer the other way around, to generality in I, the traditional pattern has
been:

III Assume a counterexample, $\exists x \neg A(x)$, and if this turns out
impossible, conclude $\forall x A(x)$.

For this pattern to be useful, one needs to take instances of $\exists x \neg A(x)$, a step
that was never made formally precise before 1920.

A principle that relates to II is:

IV Assume the contrary of $\exists x A(x)$ for all cases, namely $\forall x \neg A(x)$,
take instances by axiom I, and if something impossible turns out,
conclude $\exists x A(x)$.

Whereas in II existence is inferred from an instance, in IV it is proved
*indirectly*.

Principles III and IV are based on the intuitive equivalences between
$\neg \exists x \neg A(x)$ and $\forall x A(x)$, and $\neg \forall x \neg A(x)$ and $\exists x A(x)$. The former, in
particular, has been very prominent, already endorsed by Aristotle (namely
that generality means the lack of a counterexample).

No systematic theory of inferences with the quantifiers appeared before
Frege discovered such in 1879, but it remained unrecognised for some 25 years.
Peano, for example, had axiom II, and had even invented the symbol $\exists$,
but expressed generality by free variables, typically the inductive step in
arithmetic in which one proves for a property of numbers the implication
$A(n) \supset A(n+1)$. If $n$ is "arbitrary" and if $A(0)$ holds, the free-variable
formula $A(x)$ can be inferred. Then any substitution instance can be taken,
analogously to axiom I, but Peano recognised no general logical pattern here,

contrary to Frege, and his formal systems of mathematics remained severely incomplete. This feature was, in the end, responsible for the failure of the Peano school. Peano himself dismissed Frege's discovery by writing in 1895 in a reply to a letter of Frege's that his approach with just three instead of Frege's five basic signs "corresponds to a more profound analysis."

Frege explains in his *Begriffsschrift* [Frege 1879] the truth of a universal proposition by stating that each of its instances is true. Then comes the "illuminating observation" that one can infer to generality if one has proved an *arbitrary instance*, the latter meaning that a proof of $A(a)$ has been given for a "Latin letter." The condition is that nothing on which $A(a)$ depends, given as an antecedent $B$ in an implication, must contain the "Latin letter." Frege expressed the universal quantifier by a "German letter" (fraktur) and a notch in the "judgment stroke." In modern notation, $B \supset \forall x A(x)$ can be inferred from $B \supset A(a)$ when a purely syntactic criterion on the condition $B$ is fulfilled. Today this is seen as the greatest of logical discoveries:

Russell's 1903 book *The Principles of Mathematics* contains an appendix "The logical and arithmetical doctrines of Frege," with Frege clearly singled out as the one to explicitly recognise the difference between "any and every," and the need of rules for handling the latter:

> He has a special symbol for assertion, and he is able to assert for all values of $x$ a propositional function not stating an implication, which Peano's symbolism will not do. He also distinguishes, by the use of Latin and German letters, respectively, between any proposition of a certain propositional function and all such propositions. [Russell 1903, 519]

These were afterthoughts. No explicit logical notation is used in *The Principles of Mathematics* proper, but the treatment is based on Peano's work. Russell thinks he can get along with a single primitive notion in logic, what he called the "formal implication" rendered as "$\varphi x$ implies $\psi x$ for all values of $x$" [Russell 1903, 11]. Peano had used the notation $\varphi x \supset_x \psi x$ for such an implication with a free variable, typically an eigenvariable in an inductive step from $x$ to its successor $x'$. That was the nature of Peano's arithmetic: there was no universal quantifier in his formalism. It is easy for us today to understand that this is the systematic reason for the failure of his attempt to formalize mathematics and for the demise of his school, but such insights come from hindsight: Russell, for example, tells in the preface to his book that he had seen Frege's *Grundgesetze der Arithmetik* [Frege 1893] but added that he "failed to grasp its importance or to understand its contents," the reason being "the great difficulty of his symbolism" [Russell 1903, xvi].

A somewhat neglected paper of 1906, "The theory of implication" [Russell 1906], is Russell's first contribution to the deductive machinery of logic. Published four years before the *Principia*, it shows clearly the origins of Russell's formal system of proof. In this work, he uses negation and implication

as primitives, with a handsome axiomatization as a result. As shown in my *Formal Machinery* [von Plato 2017, section 5.1], the structure of derivations in Russell is *identical* to Peano's, with two formal rules of inference: to take instances of axioms, and to apply the rule of detachement.

Van Heijenoort, who edited the book that contains the first English translation of the main part of Peano's 1889 work [Peano 1889], instead of figuring out what Peano's notation for derivations means, claims in his introduction that there is "a grave defect. The formulas are simply listed, not derived; and they could not be derived, because no rules of inference are given [...] he does not have any rule that would play the role of the rule of detachment" [van Heijenoort 1967*a*, 84]. Had he not seen the forms $a \supset b$ and $a \,.\, a \supset b \,:\supset\, b$ in Peano's derivations, the typographical display of steps of axiom instances and implication eliminations with the conclusion $b$ standing out at right, and the rigorous rule of combining the antecedent of each two-premiss derivation step from previously concluded formulas? Had he not seen the identical structure in Russell's 1906 article? Van Heijenoort's unfortunate assessment has undermined the view of Peano's contribution for a long time, when instead Peano's derivations are constructed purely formally, with a notation as explicit as one can desire, by the application of axiom instances and implication eliminations.

Russell in his 1906 article goes beyond Peano by introducing the notation $(x)C(x)$ for universal quantification, presumably the first such notation in place of Frege's notch in the assertion sign, alongside the $\Pi_x$ notation in Schröder's algebraic logic. The notation for universal quantification allows to express the rule of generalization as:

> If $C(y)$ is true whatever $y$ may be, then $C(x)$ is true for all values of $x$. [Russell 1906, 195, slightly modified]

Russell's formal notation is $\vdash C(y) \supset\, \vdash (x)C(x)$, i.e., from the derivability of $C(y)$ "whatever $y$ may be," $(x)C(x)$ can be inferred.

The universal quantifier makes its next appearance in Russell's famous 1908 paper on the theory of types. Its section II is titled "All and any" and contains:

> The distinction between all and any is, therefore, necessary to deductive reasoning and occurs throughout in mathematics, though, so far as I know, its importance remained unnoticed until Frege pointed it out. [Russell 1908, 2289]

Mathematical reasoning proceeds through *any*:

> In any chain of mathematical reasoning, the objects whose properties are being investigated are the arguments to *any* value of a propositional function. [*ibid.*, 227]

Still, reasoning with just free variables would not do, for bound variables are needed in definitions (Russell's terminology for free and bound variables is "real" and "apparent"). Remarkably, his example is from mathematics proper:

> We call $f(x)$ continuous for $x = a$ if, for every positive number $\sigma \ldots$ there exists a positive number $\varepsilon \ldots$ such that, for all values of $\delta$ which are numerically less than $\varepsilon$, the difference $f(a + \delta) - f(a)$ is numerically less than $\sigma$. [*ibid.*]

He goes on to explain that $f$ appears in the definition in the *any*-mode, as an arbitrary function, and that $\sigma, \varepsilon$, and $\delta$ instead are just "apparent variables" without which the definition could not be made.

# 3   First-order logic

If we take the classical propositional calculus and add to it the rules and axioms for universality, a complete classical first-order theory of quantification emerges. An understanding of this matter was slow in coming: The study of Russell's logic in Göttingen, from around 1917 on, led in some eight years to the first impeccable formulation of the axioms and rules of first-order logic. The first publication of all the axioms and rules, with universality and existence both taken as primitive notions, was in Hilbert and Ackermann's *Grundzüge der theoretischen Logik* of 1928 [Hilbert & Ackermann 1928]—a book actually written by Paul Bernays—in which it is acknowledged that the axiomatisation of quantificational logic was found by Bernays.

Existence was a primitive notion in Hilbert-Ackermann for the somewhat casual reason that the book emphasised prenex normal form in which both quantifiers appear only as alternating strings at the head of formulas, followed by a propositional part as in the example $\forall x \exists y \forall z (A(x, y) \& A(y, z) \supset A(x, z))$. There is another, subtler reason not revealed by Hilbert-Ackermann, namely, the impossibility to define $\exists$ in terms of $\forall$ in intuitionistic logic. When first-order intuitionistic logic was definitively axiomatised by Heyting in 1930, he received a letter of congratulations from Bernays: The latter wrote that after Brouwer's visit to Göttingen in 1925, he had figured out the axiomatics of intuitionistic logic so that classical logic results if the law of double negation is added. (The letter is found in Troelstra 1990.)

The rule of inference for existence is to instantiate $\exists x A(x)$ by a freshly chosen *arbitrary* variable $y$, in the form of an *assumption $A(y)$*, where arbitrariness has the same meaning as with universal generalisation. Whenever a consequence $C$ of $A(y)$ and possible other assumptions is reached, the assumption $A(y)$ can be deleted if the *eigenvariable of existential instantiation $y$* has no occurrences in the other assumptions nor $C$. In typical cases, $\exists x A(x)$ is itself an existential assumption, and it replaces $A(y)$ by the step that is classified as "existence elimination" in terms of Gentzen's natural deduction.

It is instructive to see, in the light of hindsight, how logicians failed in getting the principles of inference with the quantifiers right. One who got them right, but did not formalise the quantificational part, was Skolem in his long paper of 1920 [Skolem 1920]: When an existential *assumption* is put into use, that happens by taking an instance with "new letters." In Skolem's case, existence elimination was signalled by the use of Greek letters, only for this purpose.

In Kolmogorov's 1925 paper on intuitionistic logic [Kolmogorov 1925], there is a clear awareness that the principles of inference with generality cannot be just axioms, but there has to be a rule that "cannot be expressed symbolically," just as there has to be at least one rule of inference in axiomatic propositional logic.

Clamorous misses with the principles of quantificational logic include Rudolf Carnap's little treatise *Abriss der Logistik* of 1929 [Carnap 1929], meant as a concise presentation of the logic of Russell's *Principia*: The most central component of Russell's logic is absent from Carnap's presentation. Yet another famous logician who never saw the necessity of a rule of generalisation is Alfred Tarski: Again, one searches in vain for this rule in his path-breaking treatise on the concept of truth in formalised languages [Tarski 1935]. C. H. Langford's 1930 review of the Hilbert-Ackermann book is yet another witness of how difficult it was to even entertain the idea of a formal system of rules for the quantifiers, a review full of nonsense but still published in the *Bulletin of the American Mathematical Society* [Langford 1930].[1] Last in this line comes Ludwig Wittgenstein who tried to do truth-tables even with quantified formulas.

Kurt Gödel had studied the *Principia* in the summer of 1928. Next he found the completeness problem, clearly spelt out in Hilbert-Ackermann [Hilbert & Ackermann 1928] that he also studied carefully. Gödel, clearly, would have no completeness theorem for first-order logic had he missed the crucial rule of generalisation. It is instructive to locate the exact point in which it is used in his 1930 proof [Gödel 1930]. The step is rather well hidden in the presentation that proceeds in terms of satisfiability. At one point, Gödel moves to provability of a free-variable formula, then universally quantified "by 3," the number given for the rule of generalisation.

Not long after Gödel, young Gerhard Gentzen gave the definitive formulation of the axioms and rules for quantifiers, in his system of natural deduction. His approach has a most remarkable property, never seen before, that the principles of reasoning with each connective and quantifier are *pure*. By pure is here meant: formulated independently of the other connectives. In particular, the principles for universal and existential quantification are fixed once and for all, and whether a logic is classical or intuitionistic, say, is decided on another

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1. This blunder was repaired by Barkley Rosser's review of the second edition of 1937, with a closely identical title.

level. Previously it was thought that quantificational inferences would be the "dubitable" part of classical logic that could lead to contradictions.

A particular feature of natural deduction is the rule of existence elimination that has two premisses independent of each other: 1. The existential formula $\exists x A(x)$, either assumed or derived, and 2. A derivation of some consequence $C$ from the assumption $A(y)$ in which $y$ is the arbitrary eigenvariable, i.e., one that does not occur anywhere but in this subderivation, and not in C. These two independent premisses are displayed in a two-dimensional scheme:

$$\frac{\exists x A(x) \quad \begin{array}{c} A(y) \\ \vdots \\ C \end{array}}{C}$$

Even implication elimination and some other rules with independent premisses, say $A$ and $B$ in the introduction of a conjunction $A\&B$ are thus displayed, which opens the possibility to analyse the structure of derivations in first-order logic to the full for Gentzen, with well-known results about the normalization of derivations and the subformula property of such normal derivations [see von Plato 2008].

On a more general level, Gentzen's systematisation of quantificational logic answers all questions about the meaning of quantifiers and the role of free-variable formulas, by dividing such questions into two: 1. The meaning when universality and existence are asserted, given through the conditions for their introduction. 2. The meaning when they are assumed, given through the elimination rules that show how such assumptions are put into use. Free-variable formulas are governed by the restrictions in universal introduction and existence elimination. The contrast with axiomatic logic is great, for there free-variable formulas are interpreted as equal in meaning to their universal closures.

# 4  "The universality of logic"

One of van Heijenoort's famous theses about Frege, and by implication about Russell and to a lesser extent Peano, concerns "the universality of logic." That notion is meant to illuminate an important aspect of Frege's and Russell's logic, as in the widely read essay "Logic as calculus and logic as language:"

> Another important consequence of the universality of logic is that nothing can be, or has to be, said outside of the system. And, in fact, Frege never raises any metasystematic question (consistency, independence of axioms, completeness). [van Heijenoort 1967*b*, 13]

Van Heijenoort wrote such things under the authority of Gödel who in his paper on the completeness of predicate logic had stated that with a system such as the *Principia Mathematica*, the question of completeness "*at once* arises."

Frege's *Begriffsschrift* made little compromise in the direction of the reader. In the long preface to the *Grundgesetze* some fifteen years later, Frege was more forthcoming when explaining formulas in proofs:

> Each of these formulas is a complete sentence together with all the conditions that are necessary for it to hold. This completeness, one that will not tolerate assumptions that could be added tacitly, seems to me to be indispensable for the rigorous carrying through of proofs. [Frege 1893, v–vi]

Completeness is used here in two senses, first the grammatical completeness of a sentence, then the complete display of the conditions under which a formula holds. This display is further explained as follows:

> It cannot be required that all things be proved because that is impossible; but one can require that all sentences that are used without proof are expressly stated to be such, so that one can see clearly on what the whole edifice depends. One must, also, try to diminish the number of these ground laws, by proving all that is provable [*was beweisbar ist*]. Moreover, and here is where I go beyond Euclid, I require that all forms of inference and consequence that come to be used are listed in advance. [Frege 1893, v–vi]

The condition of mutual independence of the axioms is expressed by a notion of *provability*.

Frege identified the classical propositional calculus as defined through the primitive notions of conditional and negation, and the semantical criterion of correctness of formulas and inferences through two truth values formulated in terms of *admission* and *denial*, rather than simply truth and falsity, a step from an abstract notion of logical truth to semantical principles of reasoning.

Frege does not give any explicit axiom system, but the list of deductive dependences at the end of the *Begriffsschrift* identifies one. The question of its completeness seems to be an aspect Frege did not question but simply believed in, by his explanation of "the derivation of the more composite judgments from simpler ones" [Frege 1879, 25]:

> In this way, one arrives at a small number of laws that contain, if one adds those contained in the rules, the content [*Inhalt*] of all the others, even if in an undeveloped way.

Consistency of at least the propositional axiom system is immediate, because Frege shows how to prove that his axioms must be "admitted" by semantical

criteria and that such admission is maintained by the only rule of inference, implication elimination.

Peano's "Sul concetto di numero" [Peano 1891], an article in his newly established *Rivista di Matematica*, relates his 1889 axiomatization of natural numbers to the axioms of Dedekind [Dedekind 1888]. The latter made him realise that the axioms can be taken in an abstract way, as follows:

> These propositions express the necessary and sufficient conditions for the objects of a system to correspond univocally to the series of the *N*; and they can be enunciated also as follows:
>
> 1. The name 1 is given to a particular object of the system.
>
> 2. Let an operation be defined for which there corresponds to every object *a* of the system another, *a*+, even that in the system.
>
> 3. And that two objects, the correspondents of which are equal, be equal.
>
> 4. The object called 1 shall not be the correspondent of any.
>
> 5. And finally that it be the class common to all classes *s* that contain the individual 1 and that, when they contain an individual, they also contain its correspondent.
>
> It is easy to see that these conditions are independent. [Peano 1891, 93]

Browsing further in the *Rivista*, one finds in volume VI of 1899 Peano's notes on the *Formulario* project with the following passage:

> The composition of my work of the year 1889 was still independent of the mentioned script of Dedekind; I had, before the printing, the moral proof of the independence of the primitive propositions from which I began, those with the substantial coincidence with the definitions of Dedekind. Later I succeeded in proving the independence. [Peano 1899, 85]

In the 1889 booklet on the foundations of geometry, we find a similar admission that the independence of the geometric axioms is a "moral certainty."

Peano's last exposition of his arithmetic was in the fifth edition of the *Formulario Mathematico* of 1908, written in his own invented language "Latino sine Flexione." On page 15, he explains:

> We prove that a system of primitive propositions is mutually independent, in an absolute way, if we adduce, for each proposition, an interpretation of the system of primitive ideas that satisfies each primitive proposition except the one considered. [Peano 1908, 15]

Such proofs of independence of the Peano axioms are given on page 27 of the *Formulario*—so much for the presumed absence of metatheoretical questions in Peano.

With Russell, van Heijenoort takes an equally cavalier attitude as with Frege:

> Questions about the system are as absent from *Principia mathematica* as they are from Frege's work. Semantic notions are unknown. [van Heijenoort 1967*a*, 14]

The notion of derivability that Russell expresses by the Fregean turnstile ⊢ is in van Heijenoort's reading a synonym for "is true." Russell is, indeed, confused about propositions, assertions, truth, and derivability, in a way Frege would never be.

Russell's 1906 article "The theory of implication" formulates a requirement of completeness in the following way:

> Every deductive system must contain among its premises as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction.
>
> If it is our purpose to make all our assumptions explicit, and to effect the deduction of all our other propositions from these assumptions, it is obvious that the first assumptions we need are those that are required to make deduction possible. [Russell 1906, 159]

Russell takes it for granted that a complete system of deduction exists, but he is unable to express the matter in precise terms. A bit later, the formulation is:

> Now in order that one proposition may be inferred from another, it is necessary that the two should have that relation which makes the one a consequence of the other. When a proposition $q$ is a consequence of a proposition $p$, we say that $p$ *implies* $q$. Thus deduction depends upon the relation of *implication*, and every deductive system must contain among its premises as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction. [Russell 1906, 159]

Consequence is here clearly a notion outside the formal system of deduction. It is the notion of implication defined by the truth value semantics of classical propositional logic. The requirement that "all our other propositions" be derivable is a clear condition of completeness. As to the mutual independence of the axioms, Russell writes:

> In the present article, certain propositions concerning implication will be stated as premises, and it will be shown that they are sufficient for all common forms of inference. It will not be shown

> that they are all necessary, and it is probable that the number of them might be diminished. [Russell 1906, 159]

Here it is presumed that some purported axioms could be in fact derivable theorems. Russell summarises the discussion by:

> All that is affirmed concerning the premisses is (1) that they are true, (2) that they are sufficient for the theory of deduction, (3) that I do not know how to diminish their number. But with regard to (2), there must always be some element of doubt, since it is hard to be sure that one never uses some principle unconsciously. The habit of being rigidly guided by formal symbolic rules is a safeguard against unconscious assumptions; but even this safeguard is not always adequate. [Russell 1906, 159–160]

Van Heijenoort's thesis that truth and provability in Russell are the same can be hardly maintained, for the reason that the unproved axioms must be true.

Russell's discussion covers the problem of consistency in propositional logic: by (1), the axioms are true, and the only rule of inference is one Russell sometimes formulated as: "inference consists in the dropping of true antecedents in implications," namely, if we have that $p$ and $p \supset q$ are true, then even $q$ must be true. Nothing more is needed for a consistency proof of an axiomatic system of classical propositional logic.

Later in his 1985 article on Herbrand, van Heijenoort writes:

> The only question of completeness that may arise is, to use an expression of Herbrand's, an *experimental* question. As many theorems as possible are derived in the system. Can we exhaust the intuitive modes of reasoning actually used in science? [van Heijenoort 1985, 14]

Let's grant to van Heijenoort this point, but the ones about consistency and independence don't hold water, they are unsustainable. A counterexample to consistency would be a derivable formula that is not true, say $A \,\&\, \neg A$ or $0 \neq 1$. A counterexample to independence is given by a purported axiom that turns out to be derivable, the latter a purely syntactic notion. A direct method for proofs of independence is the one of Peano in which an abstract calculus gets different interpretations. A counterexample to completeness is a formula that is true but not derivable.

Metatheoretical notions such as consistency, independence, and completeness are meaningful enough when it is understood what would show such properties not to hold. Indeed, Paul Bernays found out in his *Habilitationsschrift* of 1918 that one of Russell's axioms in the *Principia* turns out to be derivable from the rest of the axioms, a result published in Bernays [Bernays 1926]. Who can deny, then, that a syntactic method for proofs of underivability in axiomatic logic was an impossibility in the times of Frege and

Russell? Had someone shown to Frege or Russell an implication that is valid by the criterion of two truth values, but underivable from the axioms listed in the *Begriffsschrift* or *Principia Mathematica*, these two authors would have declared their axiomatisation *incomplete*, something van Heijenoort would have to regard a conceptual impossibility. Frege, Peano, and Russell would have naturally taken incompleteness as a failure that can be repaired by the addition of some principle that had remained hidden. They all clearly believed that deductive arguments can be perfected.

Van Heijenoort's pet idea of the "universality of logic" gives a suggestive and incisive view of the early phase of mathematical logic, but it is mostly a fantasy when confronted with the sources.

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# Peano's Reception in the USA. Wilson's Review of Russell's Principles

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**Résumé :** Dans une recension des *Principles* de Russell datant de 1904, Edwin B. Wilson accorde une attention particulière aux travaux de Peano et de ses collaborateurs. Son but était de mieux les faire connaître aux USA, où leurs œuvres « étaient malheureusement peu diffusées et n'étaient en outre pas spécialement appréciées ». La reconnaissance dont Peano bénéficiait aux yeux de Russell est amplifiée par Wilson, ce dernier estimant que la logique de Peano est bien plus qu'un nouvel « outil logique », tout en considérant le logicien italien comme un précurseur du logicisme. Wilson s'oppose ainsi au jugement dépréciatif dont Peano faisait l'objet de la part de Poincaré. Dans le domaine de la géométrie en particulier, Wilson revendique diverses avancées accomplies par l'école de Peano par rapport à Hilbert en ce qui concerne la philosophie de la méthode axiomatique.

**Abstract:** In a review of Russell's *Principles* from 1904, Edwin B. Wilson pays great attention to Peano's work and that of his collaborators. His purpose was to make this work known in the USA where it "unfortunately is very little known and still less appreciated". Wilson expands Russell's well-known acknowledgement of Peano's influence on his own development, seeing in Peano's logic more than a new "mathematical tool", describing Peano as a kind of proto-logicist, and defending him from Poincaré's criticisms. Especially in geometry, he vindicates several priority issues for Peano's school with respect to Hilbert in the philosophy of the axiomatic method.

## 1  Introduction

In 1904 Edwin Bidwell Wilson (1879-1964) wrote a review in the *Bulletin of AMS* of two books by Bertrand Russell (1872-1970), *The Principles of*

*Mathematics* [Russell 1903] and a French translation of the older *An Essay on the Foundations of Geometry* [Russell 1897, 1901*a*]. Wilson's review of these books [Wilson 1904] is interesting per se, as a description of the natural or naïve motivation of a logicistic view of mathematics, and we will briefly also dwell on this part. However one has the impression that the review had been primarily conceived as an opportunity to talk about Peano and his school, and to describe his work as a prelude to Russell's. In fact, while we do not know, we doubt that at that period any of Peano's paper had been translated into English and thus made known to the American scientific public.

In [Wilson 1904, 74, fn.*], Wilson refers readers to another review by Couturat of the same 1903 book by Russell which is mainly dedicated to Cantor and set theory: "So large is the work of Russell, that Couturat's review and our own supplement rather than overlap one another". However Ivor Grattan-Guiness (1941-2014) glosses both reviews together noting that [Couturat 1904] "concentrated on Cantor [...] having written elsewhere on the Peanists", while Wilson "dwelt on the Peanists, whose work [according to Wilson] 'is very little known and still less appreciated' in the USA (p. 76), referring to their four lectures at Paris [1900 International Congress of Philosophy]" [Grattan-Guinness 2000, 330]. So there seems to be an agreement with Grattan-Guiness in seeing [Wilson 1904] as focused on the Peanists.

Wilson had begun his career in mathematics with a Ph. D. at Yale in 1901 and the same year wrote a book on *Vector analysis* based on Josiah Willard Gibbs' (1839-1903) lectures. He then studied in Paris in 1902-1903, and for a while was interested in the foundations of geometry (he criticized "so-called" Hilbert's foundations in 1903). Next he began teaching at Yale and later at MIT in the department of physics. He was inspired by Gibbs to work in mathematical physics, first in mechanics and the theory of relativity, then in aeronautics and aerodynamics and later in statistics with applications in many fields. He wrote the first American advanced calculus text. His most demanding logical contribution, apart from a few reviews, is a discussion of categoricity, which we shall come back to later.

His time in Paris clearly brought him into contact with the lively discussions on foundations that were beginning in Europe. While he was in Paris, he possibly met Louis Couturat (1868-1914) who was devoting himself to promoting Russell's work. Wilson begins his review by explaining what is "the problem of the ultimate foundation of mathematics": throughout history,

> [...] what has been accepted [in pure mathematics] as sure and accurate in one generation has frequently required fundamental revision in the next. Euclid and his pupils could doubtless have complained of the lack of rigour and logical precision in his predecessors just as forcibly as some modern pupils of Weierstrass

berate their scientific ancestors and companions.
[Wilson 1904, 74[1]]

He mentions Euler, Cauchy, Laplace, and Dirichlet, and some of their inaccuracies or errors, which nonetheless did not hinder their work.

> We notice that the advance toward our present rigour has been made step by step by great men who, however, were no greater— one might almost say no more careful—than their fellows working in apparent unconsciousness of the impending trouble and perhaps even incredulous at first as to its reality. When will this revision stop? And whereunto will it finally lead? This is the problem of the ultimate foundation of mathematics. (75)

This is a problem that cannot be easily disposed of by ignoring it:

> the delicacy of the question is such that even the greatest mathematicians and philosophers of to-day have made what seem to be substantial slips of judgement and have shown on occasion an astounding ignorance of the essence of the problem they were discussing. (75)

At times this could have been caused by the failings of individual intuition in dealing with matters that are still unsettled, "but all too frequently it has been the result of a wholly unpardonable disregard of the work already accomplished by others" (75).

After this criticism of his fellow-mathematicians, Wilson enters the subject by quoting the first section of the first chapter of *The Principles*, containing Russell's famous definition of pure mathematics as the class of all propositions, with variables, of the form "$p$ implies $q$", with a few qualifications (the logical constants allowed in $p$ and $q$ being: implication, to be an element of, such that, the notion of relation, and truth, plus "such further notions as may be involved in the general notion of propositions of the above form" (75)).[2]

# 2   Peano & Co.'s encomium

Russell's was "probably the first attempt to give a complete definition of mathematics solely in terms of the laws of thought and the other necessary paraphernalia of the thinking mind"; it was made possible by "two things: first, the more careful discrimination of what *pure* mathematics is; second, the extraordinary development of logic since Boole removed it from the trammels of medieval scholasticism" (76). But

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1. Hereinafter, page numbers in the text, $(n)$, without further reference, indicate the source [Wilson 1904].

2. Russell's presentation of symbolic logic as in the second chapter of the book was still in its infancy.

> He to whom the presently highly developed state of the foun-
> dations of mathematics is chiefly due is Peano—one whose work
> unfortunately is very little known and still less appreciated in this
> country. (76)

Although Leibniz achieved a lot and his work in recent years has been
made known by L. Couturat,[3] George Boole (1815-1964) freed us from
Aristotelianism, and C. S. Peirce (1839-1914) and Ernst Schröder (1841-1902)
carried the technique of logic much farther,[4]

> [this notwithstanding] they had never accomplished that intimate
> formal relation between logic and all mathematics which was the
> necessary precursor to a yet more intimate philosophic relation
> and which has been brought about by Peano aided by a large
> school of pupils and fellow-workers. The advance has been made
> largely by introducing into symbolic logic such a simplification
> of notation as to relieve it of its unwieldiness and to allow its
> development into a powerful instrument without which one can
> hardly hope to get the best results in the treacherous though
> treasure-laden fields of the foundations of mathematics. (76–77)

Wilson is however aware that not everyone concurs with his view and
that Peano even had his detractors. Henri Poincaré (1854-1912) in particular
(in his review of Hilbert's *Grundlagen der Geometrie*) "spurns this [Peano's]
*pasigraphy*, characterizing it as disastrous in teaching, hurtful to mental
development, and deadening for investigators, nipping their originality in the
bud" (77).[5] Even accepting the first statements, to which Wilson will return
in the conclusions,

> we had best be cautious in accepting such sweeping statements as
> the last, even from so great an authority—especially in view of the
> fact that, equipped with this pasigraphy, the Italian investigators,
> Peano and his pupil Pieri,† with some rights of priority, had given
> a more fundamental *logical*‡ treatment of the subject on which
> Poincaré was writing than is to be found in the work he was
> praising so highly [the *Grundlagen*]. (77–78)

Footnote † (77) is simply is a reference to [Pieri 1898] and [Pieri 1899].
Footnote ‡ (77) instead is a long note in which Wilson disputes some of

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3. Wilson here quotes [Couturat 1901].

4. According to Wilson the importance of the logic of relations was emphasized
by Peirce in 1880-1884, but only now begins to show its utility. He does not mention
here, but later in the review, (80, fn.*), that the first exercise Russell did to master
Peano's machinery was to treat the theory of relations symbolically, in [Russell 1901*b*].
Wilson excuses Peano and his followers for neglecting the importance of this subject
because they were "so busy [...] with other important questions".

5. Wilson is quoting (77), from [Poincaré 1903, 5].

Poincaré's judgements. His own assessments in this footnote amount to a vindication of several priority issues for Peano's school.

Since Poincaré praised the "long step in advance" made by David Hilbert (1862-1943) in the philosophy of mathematics by regarding "his geometric elements as *mere things"*, Wilson reminds us that Peano had already taken the same stance [Peano 1889] and that Giovanni Vailati (1863-1909) had expressed the same idea in [Vailati 1891] and [Vailati 1892]. He also notes that by 1897 Peano and his students had gone further by envisaging the postulate that points are classes, with a twofold advance: first of all the necessity of a postulate, secondly the use of the term "class", which permitted a further reduction of mathematical reasoning to logic.

As for attributing the idea of the independence of the axioms to Hilbert, in 1894 Peano had already stated the problem and given proofs of independence for certain axioms with simple systems of elements. "By 1899 the idea and method were both five years old at least" (77, fn.[‡]).

Of course, in Hilbert's work "there still remains [...] matter enough for the amplest praise":

> The archimedean axiom, the theorems of Pascal and Desargues, the analysis of segments and areas, and a host of things are treated either for the first time or in a new way, and with consummate skill. We should say that it was in the techniques rather than in the philosophy of geometry that Hilbert created an epoch. (77, fn.[‡])

What Wilson is implying with this strong assertion is that the philosophy of the axiomatic method was already well established and widespread. This was in fact the subject of Vailati's two papers and he could have mentioned others geometricians, such as Moritz Pasch (1843-1930) for example.

Turning to arithmetic and algebra, Wilson is on slippery ground either because of his lesser competence, or for his propensity to take the claims of Alessandro Padoa (1868-1937) and Cesare Burali-Forti (1861-1931) at face value. Indeed he probably only read these in the summaries of their Paris 1900 papers without studying them, otherwise he would have been doubtful about or mystified by Padoa's alleged feat.

In his communication at the Congresses,[6] Padoa had explained very precisely how to analyse the formal structure of a deductive theory. He had in a sense defined and set out the canon of the axiomatic method which Wilson largely draws from in section 4 of the review dedicated to the fundamental concepts of the axiomatic method. One of the principles was that

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6. Padoa repeated his communication at both congresses, preceding it at the philosophy congress with a "logical introduction to any deductive theory". An English translation of this introduction is included in [van Heijenoort 1967, 118–123].

> Pour **démontrer** la compatibilité d'un système de postulats,
> *il faut trouver* une interprétation des symboles non définis, qui
> vérifie simultanément tous les postulats. [Padoa 1903, 87]

On that occasion, Padoa had also addressed the problem of consistency posed by Hilbert in 1899 for the number system (in a conference that became [Hilbert 1900*b*]) and inserted as problem n. 2 in his list of problems, [Hilbert 1900*a*]. Padoa was probably disappointed by the silence that had greeted his contention when he claimed explicitly in 1903 that Hilbert's problem n. 2 was completely solved, and that actually "Le problème n. 2 [de M. Hilbert] n'était qu'une causerie" [Padoa 1903, 86].

What he had done was to define an interpretation saying that "**entier** signifie nombre entier relatif" and "**suc**$x$ signifie $1 + x$". This is not very different from Poincaré's assertion (p. 8 of his review) that we know that axioms are non-contradictory "since geometry exists", to which Wilson implicitly takes issue (78, fn.‡).

Wilson however accepts that "a solution [of problem n. 2] has long since been proposed in the article here referred to" (78, fn.*), the reference being [Padoa 1903]. He admits nevertheless in a cryptic way that "There are those, however, who hold that Padoa has gone so far as to overshoot the mark" (78, fn.*). In the same footnote, Wilson regrets having overlooked the fact that a solution to the consistency problem along the lines proposed by Hilbert "seems logically impossible". What Hilbert asked for in 1900 was direct proof of the consistency of the axioms of arithmetic through "an appropriate modification of known methods of proof", applying only "the known inference methods of the theory of irrational numbers" [Hilbert 1900*a*, § 41]. However Wilson acknowledges that "Hilbert has again taken up the matter much more searchingly than in 1900" (78, fn.*). He was also informed that Hilbert had proposed a new approach at the Heidelberg congress in August 1904 and regretted that he had not seen the text.

It is consistent, however, with the attitude of Peano's collaborators towards building models that the interpretation be made with the "ideas" we already have. In 1906, Peano defined a model for his arithmetic axioms using some work which had had a different aim (we shall mention this later), but remarking at the same time that "proof that the system of axioms for arithmetic, or for geometry, do not involve contradictions is not, to my mind, necessary. For we do not create axioms arbitrarily, we rather assume the simplest propositions we find to be as axioms, whether explicitly written or implicit, in every treatise of arithmetic or geometry" [Peano 1906, 365].

What Wilson could not be aware of is that there were also reservations about Padoa's attitude inside Peano's school but these were only to be expressed in the immediately ensuing years. Mario Pieri (1860-1913) indirectly disagreed with Padoa by affirming at the end of his 1904 paper that "it is vain to seek a direct and absolute proof of the compatibility of the arithmetical axioms in the field of Arithmetic itself" [Pieri 1904, 331, my translation]. Pieri

on the contrary looked at Hilbert's effort to give a logical proof of *compatibility* with interest although he remained conscious of the difficulties of avoiding the notion of number.

In 1906 Pieri came to believe in the possibility of a logical proof of compatibility with a logic that includes the concept of class:

> je me propose précisément d'etablir la *compatibilité des axiomes arithmétiques de* R. Dedekind et G. Peano *dans un domaine Δ de Logique pure* [...] en raisonnant dans les limites de la Logique des classes. [Pieri 1906, 203]

Pieri started from Burali-Forti's proof that finite classes are a model of Peano's axioms but he had to modify this by substituting II below to a Burali-Forti axiom that A. N. Whitehead (1861-1947) had pointed out as erroneous, and [Poincaré 1908, 209] had unfairly laughed at.[7] Pieri's proof succeeds by relying on the logical axioms of the *Formulario* and the two "principes suivants:

I. *Il y a au moins une classe infinie* (Le Tout est une classe infinie);

II. *Étant donnnée une classe infinie, dont les élements sont à leur tour des classes, la classe formée par tous les élements de celles-ci est elle-même infinie.*" [Pieri 1906, 207]

Pieri maintains that he can include these two principles without scruple among the logical axioms "car je n'y vois qu'une *détermination* convenable des concepts de *classe* et *représentation*" [Pieri 1906, 207].

But, independently of a discussion of this conviction,

> je crois avoir établi que le concept de nombre entier, avec ses propriétés fondamentales (y compris le *principe d'induction*) peut être construit sur la Logique des classes de M. Peano, au moyen des propositions I et II. [Pieri 1906, 207]

Wilson would have appreciated Pieri's attempt which was the only piece of true logicism which came from Peano's school. For now, he continues to speak highly of the Italians' contributions as known to him:

> Anyone who is acquainted with the articles presented to the Philosophical Congress at Paris in 1900 by Peano, Burali-Forti, Padoa and Pieri,[8] cannot be convinced that these authors had become deadened, and the artificiality of their system is by no means so certain as it might be. (78)

Again at the end of the review in the bibliographic suggestions for further reading, Wilson suggests that these papers should be read.

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7. For more details on this comedy of errors, see e.g., [Lolli 2012, x and xx-xxi]. [Burali-Forti 1896] is relevant also for the history of the axiom of choice, see [Moore 1982, 129].

8. [See references, for all the four of them identifiable by the year 1900.]

> Since then, our author, Russell, has simplified and improved the
> older work of C. S. Peirce on the theory of relations, adapting it
> to the system of Peano, and has produced a coherent treatment
> of the great problems underlying mathematics. (78)

To him those papers "show the point at which the Italian school had arrived in 1900. It is since that time that most of Russell's technical work has appeared" (93).

Wilson fails to perceive that the handing of the baton from Peano to Russell did not really constitute the launching of a project to take matters further but that there was a hiatus, a different conception. Peano was no logicist. He wanted to express existing mathematics in a rigourous, compressed and complete way. He obviously had to deal with arithmetic, geometry, algebra, real and complex numbers, calculus, but did not want to define mathematical entities ("numbers cannot be defined" he said in [Peano 1891]) and once the *Formulario* was on the right track he had only to make it grow, up to [Peano 1908]. As for foundations, "[h]ic difficultas maxime ex sermonis ambiguitate oritur", difficulties come from the language ambiguities. The solution is the same as that suggested by Leibniz, namely to assign signs to the simplest ideas from which all others are composed by logical operators.

That date in 1900 also marks the point at which the Italian school practically ceased to be effectively involved. Wilson could probably not have had the historical perspective to see the school had run its course but contemporary mathematicians felt that no further contribution could come from it. Apart from the work of Pieri mentioned above which concerns questions of set theory which were in any case alien to Peano's vision, the only breath of life in the following years is limited to [Peano 1906] and his defence of non predicative definitions against Poincaré.

In his 1906 paper on the Cantor-Bernstein theorem Peano answered Poincaré's question as to whether the theorem could be proved without recourse to the natural numbers. After a general discussion of the impossibility of eliminating all mathematical ideas from such proof, otherwise only the logical signs would remain—showing clearly on this occasion that he is no logicist—Peano remarks that in this case one may avoid numbers.

The incriminated definition occurring in the known proofs was of the form

$$Z(u) = \bigcup \{g^n(u) : n \in \mathbb{N}\},$$

$g$ being a bijection, which Peano substituted by

$$Z(u) = \bigcap \{v : u \subseteq v \ . \ g``v \subseteq v\}.$$

Then "suppose that only logic, and not arithmetic, is known, so that the symbols $0$, $N_0$ and $+$ are meaningless"; if $u$ is not empty, and one of its elements is indicated with $0$, if we let $N_0 = Z(\{0\})$ and $x+ = g(x)$,

> [and] I read 0, $N_0$, + as in arithmetic [...] we deduce theorems
> identical with the axioms of arithmetic. [Peano 1906, 364–365,
> my translation],

just to add immediately however that there is no necessity for a model.

In any case, in 1908 Wilson still did not have the sensation that Peano's
work was finished and that Russell had taken on a new way of working and
could thus repeat that "reading modern Italian is a necessary condition", to
become symbolic logicians [Wilson 1908a, 188]. And perhaps this was not
enough and Latin was also required as Wilson warns elsewhere "if I understand
his inflexionless latin" [Wilson 1908b, 437].

# 3   Russell's logicism

Now Wilson turns to Russell, although he will have the opportunity to come
back to Peano with pertinent remarks.

In section 3 of the review, *Reason*, Wilson briefly discusses how mathe-
matics had needed to push its foundations back until they rested solely on
logic. Mathematics as well as any reasoning obviously presupposes a mind
capable of ratiocinative processes. It is usually assumed that if we are careful
enough there is no need for formulating and learning the laws of thought before
beginning to reason or even that a formulation of those laws is impossible.[9]
However the review began by talking of errors. Where do errors creep in? It
is interesting to read the ideas of someone who was still near the beginning
of the movement of arithmetization and introduction of epsilon proofs almost
to the point of having experienced its bewildering novelties in his lifetime and
who saw it as something still meaningful for foundations.

> Where then do the errors creep in? An examination of some
> typical cases shows that it is generally through lack of a sufficiently
> careful definition of the terms. [...] In mathematics it is
> the absence of precise definition which brings in the erroneous
> statements concerning differentiation, continuity and infinity, with
> a host of others. The perception of this difficulty was the origin
> of the principle of arithmetization and of epsilon proofs. (79)

Finally, after one has mastered the principles of modern analysis, there is
no need for the actual presence of epsilon in the proofs.

> Nevertheless it is a satisfaction to have this formal method to
> fall back on whenever challenged by one's own hesitancy or by
> that of others. In like manner, who has not at times during some

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9. He refers to "[J.-M. C.] Duhamel, *Des méthodes dans les sciences [de
Raisonnement]*, [Gauthier-Villars], Paris, 1875, vol. 1, p. 17".

> long complicated or indirect logical demonstration felt the least
> bit uncertain; who would not be glad to have at his hand some
> formal method such as Peano's, based upon certain rudimentary
> propositions and concepts?
>
> In truth, it is a matter of more consequence than is sometimes
> thought, to have clearly in mind those processes which are
> definitely to be admitted as logical. [...]
>
> The question then becomes of fundamental importance: What
> is at the bottom of our logic? (79)

"We constantly use *propositions*, passing from certain propositions as
hypotheses to certain others as conclusion" (80); looking for other principles
"we come upon *classes* or sets of objects represented in ordinary speech by
common nouns"; finally we perceive that the "*relations* are of the utmost
importance" ("one of the lasting services of Russell" (80, fn.*).

> The complete logical calculus, as now used, is a combination of
> these three types. (80)

> We may grant, then, that logic is *necessary* to mathematics. It is
> affirmed to be *sufficient.* This in reality is the remarkable content
> of the definition [of pure mathematics] given by the author. (80)

But the affirmation of its sufficiency fully justifies and even renders imperative
a critical examination of its principles that the simple necessity might never
force us to. The number of logical premises which are sufficient to establish the
calculus in all generality necessary for mathematics is small. However there
are a certain number of elementary ideas such as implication, the notions of
proposition, class, and relation that must be known.

> It is the discussion of these questions which are of a philosophical
> rather than mathematical nature that fills the first Part of Rusell's
> Principles. (80)

Notwithstanding the novelty, and the many philosophical and mathematical
difficulties, Wilson has no doubt that "to a large extent the author is successful
in his attempt" (80).

# 4   Some methodological notions

The next section, *Some notions*, is dedicated to elucidating certain technical
notions to avoid the inconvenience of having to include definitions throughout
the rest of the speech.

"Axiom" is a word that is best abandoned in pure mathematics. Its more
familiar meaning, as "self-evident truth", has no place in pure mathematics,
which is "a formal subject over which formal and not material implication

reigns.†"[10] The proper word to use for statements we posit in mathematics would seem to be "postulate".

> What self-evident truths can there be concerning objects which are not dependent on any definite interpretation but are merely marks to be operated upon in accordance with the rules of formal logic? Postulates, however, may be laid down at will so long as they are not contradictory. It is the postulates which give the objects their intellectual though not physical existence. (81)

"Definition" for philosophers stands for "a process of analysis and exemplification which brings before the mind a real consciousness of the object defined" (81–82). Mathematical definition is the attribution of a name to some object whose existence has been established or postulated. "It is the process of replacing a set of statements by a single name and is resorted to solely for convenience" (82). Footnote* (82), quotes [Peano 1900] to the effect that it might be better to exclude a large number of definitions.

All definitions are nominal. But there are three styles of definition that can be illustrated in connection with the theory of integers.[11] One possibility for defining the class of integers is to find a class, whose elements are classes or propositions, among which there is an element analogous to zero (e.g., the null class), and it is possible to define operations with the properties we use for integers. We could say that this class is the class of integers. It would be a particular but satisfactory definition of integers. We would also be sure that there would be no contradictions in our system of integers unless there were a contradiction in our logic. Another possibility would be to write a suitable set of postulates with the appropriate symbols. "In order to prove the noncontradictoriness of our system of postulates and indefinables, that is, the existence of our elements" (83),[12] we should build some system which provided an interpretation of the indefinables and of the postulates, and in the end we should resort to our logic and thus there would not be much difference from the previous case. The definition would be more general in that the integers would not be a particular set but any set satisfying the postulates. Thirdly we could define, as Russell does, numbers by means of the property of having the same number in case a one-to-one relation exists between the elements. Also this kind of definition becomes a nominal one thanks to Russell's work on relations.

Finally, since the use of postulates is so common, although in principle avoidable in pure mathematics, Wilson expresses "a few words" on consistency, independence, irreducibility and completeness. After mentioning Padoa's

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10. In † it is admitted that in logic "we should incline to use the word axiom [...] for we are dealing with the actual (mental) world and not with a system of marks. The basis of rationality must go deeper than a mere set of marks and postulates".

11. Although implicit, the reference here is to [Burali-Forti 1901].

12. Notice how Wilson independently states Hilbert's famous thesis on the connection between consistency and existence.

method for proving the redundancy of a symbol in terms of others, he observes that "Huntington[‡] seems to have been the first to bring to effective use the idea of completeness" (84); with this "we have arrived at the limit of present ideas concerning the interrelations of the notions at the base of mathematics as defined by postulates" (84). These ideas were well known and discussed in the US, paving the way in fact for the clarification of the concepts of completeness and categoricity. "Completeness" was the translation of Hilbert's *Vollständigkeit* and it was used in the sense of completeness in its semantical form, meaning that every sentence of the language of an axiomatic theory was either a logical consequence of the axioms or was incompatible with them, or as deductive completeness. It was also used for what came to be called categoricity, together with other words.

Edward Vermylie Huntington (1874-1952) in [Huntington 1902] had provided a set of postulates for the continuous magnitudes that he believed to be "complete", meaning that postulates were mutually independent, non-contradictory and "sufficient". By this he meant that "there exists essentially only one" model. "Only one" is intended through the modifier "essentially" to mean the same thing as our "up to isomorphisms", although the word was not available.

In 1904 Oscar Veblen (1880-1960) followed advice from John Dewey (1859-1952) to suggest the term "categoricity" to substitute the ambiguous "completeness" [*Vollständigkeit*]. Veblen maintained he had the right to apply the undefined terms point and order to any class of objects satisfying the axioms but that he also aimed to show that "there exists *essentially one* such class" [Veblen 1904, 346]. Completeness, which he stated in semantical terms, would follow.

In 1905 Huntington borrowed Veblen's term "categorical" for "sufficient", conceiving it in the sense that any proposition in primitive language is either deducible from the postulates or contradictory to them (deductive completeness). "We have to admit however that our command of the processes of logical deduction is not yet, and probably never will be sufficiently complete to justify this assertion" [Huntington 1905, 210†].

Wilson must have known Veblen's proposal since he uses the term "categoricity". He had the opportunity to come back to categoricity in 1908 by intervening in the debate on Zermelo's axiom of choice which he saw rather as a logical axiom. He is thus led to discuss logic in general. First he observes that

> It is not always desirable and indeed not always possible to obtain a set of postulates which shall be categorical: for it may well happen that the systems to be determined are such that not even a one to one correspondence between their elements is available, to say nothing of the preservation of the interpretation of the symbols [*S*]. [Wilson 1908b, 434]

Then he recalls that

> Huntington gives a subsidiary definition or explanation of the idea
> of categoricity wherein he asserts that if a set [*P*] of postulates on
> the undefined symbols [*S*] is categorical, then every proposition
> concerning [*S*] must be deducible from the postulates [*P*] or be in
> contradiction with them. [Wilson 1908b, 434]

In a system determined categorically, every proposition phrased in terms of
[*S*] is either compatible or incompatible with [*P*].

> What, however, does the word deducible mean? The meaning
> is entirely relative to the system of logic which is available for
> drawing conclusions from the set of primitive propositions [*P*].
> Some may consider that the human mind has instinctively at its
> disposal all valid methods of deduction. This is a tremendous
> postulate, and one entirely devoid of other than sentimental value.
> In fact, if it leads to the abandoning of the research for valid
> methods of deduction, it is dangerous and worse than useless. It
> is an essential of the modern attitude in logic that the deducer
> should state distinctly his form of inference. Hence deducible
> cannot be regarded as equivalent to compatible.
> It is clear that in an ideal perfection of logic compatibility and
> deducibility would be equivalent for categorically defined systems.
> That state of perfection appears at present to be very remote.
> [Wilson 1908b, 436]

# 5    *The Principles*

The next two sections, 5 *Numbers* (85) and 6 *Geometry and Mechanics* (87),
are entirely devoted to the mathematical development of the *Principles*. The
definitions of numbers, of cardinals and of ordinals are obviously discussed,
"with the guidance of the principle of abstraction" (85) (meaning: equivalence
classes). Finite and infinite are neatly distinguished. According to Wilson, an
advantage of Russell's method, with contributions by Alfred North Whitehead
(1861-1947), "is that by the use of logical addition the numerical addition of a
finite or infinite number of finite or infinite cardinals may be and indeed (if we
invoke the principle of abstraction) should be defined in such a manner that
the order in which the numbers are added plays no part" (85). To Wilson,

> This is a great victory for common sense and must appeal to
> everyone as a vindication of the school-child in his inherent notion
> that he has the same number of marbles whether he has five in
> one pocket and three in another or three in two pockets and two in
> a third, no matter which of his pockets these be. The principle of
> commutation and association of the terms in addition is entirely
> done away with, except in so far as mechanical difficulties prevent

us from writing simultaneously a number of terms and the signs
of addition connecting them. (85)

Of course, "there still remain difficulties to solve", but "there is no reason
why he [the author] should not find adherents who will take up the work and
attempt the solution in a spirit of hearty cooperation" (87).

As for real numbers,

> [t]here is a school of creationists who, when they find that certain
> infinite processes lead to no rational limit nor yet to a number
> which becomes infinite, postulate the existence of a limit and thus
> obtain the irrational numbers. The author does not consider an
> *ipse dixit* like this to be a sufficiently good theorem of existence.
> He therefore considers infinite sets of rationals and by means of
> them he forms a set of things which he calls real numbers. A
> real number is neither a rational nor an irrational; it is a certain
> infinite set of rationals. (87)

Wilson appreciates the "very satisfactory account of the philosophy of the
infinite and of the continuous".

The comments on geometry are interesting in that Wilson sees Russell's
presentation as a good example of how to avoid the postulates method. He
skips however the details of the definition of geometry as "the study of series
[successions] of two or more dimensions" (87) so that readers of the review
cannot grasp it unless they resort to Russell's book. The simplest example is
a succession of reals. The anomalous definition is partly due to the fact that
"Mathematical geometry has long since been divested of all spatial relations
between its elements" (87). On the other hand, as those who define geometry
by postulates are forced to show the existence of their elements by having
recourse to systems of numbers the question is quite pertinent:

> Why not begin with a purely nominal definition like the above
> and avoid the trouble of proofs of existence, of independence, and
> of irreducibility? (88)

Wilson confronts the new approach with the previous one of [Russell 1897]
whereby geometry depended on mechanics, and shows the path leading to its
foundations pursued by the author as deeply as to the logical basis before
returning to rebuilding it through the solution of many logical issues.

# 6   Conclusions

One conclusive remark on logic, in section 7 *Conclusions* (90), is the warning
that there are many systems of logic at present. Since a few of the topics which
need a logical treatment are very complex like infinity and the continua, "it

might not be regarded as surprising if some points were found to stand out permanently, so that logicians will permanently disagree" (90). In fact Wilson is aware that there is a logical difficulty in the very logical system developed by Russell. Frege's repair work is discussed in the appendix to the book under review. In (90, fn.*), Wilson recalls Hilbert's attempt in the Heidelberg address (quoted before) to recast the principles of logic and arithmetic so as "to render them sufficient for mathematical reasoning":

> We certainly hope he has succeeded in doing so to the satisfaction
> of both mathematicians and philosophers. (90, fn.*)

Meanwile

> [...] [i]t is dangerous to accept the naïve point of view of those who
> claim that a certain piece of reasoning depends on the operation
> of logic alone but who fail to state what those operations are. (90)

From the pedagogical standpoint, Wilson entirely accepts Poincaré's warning that "pure logic alone [...] is harmful to the earlier development of the mind" (91). Hence, instead of troubling students with elaborate deductions of the properties of arithmetic operations, it would be better to let them appreciate the ideas of finite and infinite cardinals and ordinals, of compactness and continuity, the different kinds of infinity. A clear-cut "physical conception" that numbers possess order and may be associated with the points of a line is both necessary and sufficient for ordinary rigourous analysis.

> An inadequate vague idea regarded as a useful working hypothesis
> seems, on the whole, productive of more good and less harm than
> an inadequate definite idea regarded as final. (91)

As for mathematics, "we have learned that many of the objects which have been thought of as individual must be regarded as classes" (91). According to Wilson, "we cannot define Euclidean space, but we can define the class of all Euclidean spaces". In (91, fn.†), he notes that this apparently lowers the importance of the concept of completeness (discussed before in relation to Huntington):

> For it appears as if the one-to-one correspondence between the
> different Euclidean spaces were really of minor significance. This
> is but another instance of the fact that the elements themselves
> are unimportant—that it is the abstraction from them which is
> most fundamental. (91, fn.†)

The idea of completeness is however a step forward toward a fuller description of the systems dealt with. Wilson would certainly be happy with the many later concepts of model theory.

A note of optimism in conclusion. Since during the construction of the mathematical objects "we have introduced no new indefinables, no new

postulates, no processes other than those of logic, there is no possibility of our arriving at contradictions except through the failure of our logical system to be logical; and behind this we cannot go" (91). The existence of the classes which we have dealt with remained to be shown and this was done by Russell, starting from below—the null class, the finite cardinals, the class of finite cardinals, and so on.

The promised second volume will contain "actual chains of deduction leading from the premises of logic through arithmetic to geometry" (92). Wilson could hardly surmise that this logic would have to wait till 1908.

Finally: "For those who wish sooner to get at the Peano-Russell point of view", a short bibliography is added which is "very incomplete" but useful to trace the development of the idea. Beyond the papers quoted in the footnotes, it comprises Peano's two 1889 seminal works, "the starting point of the whole movement", the four Italian Paris contributions again, a paper by Whitehead on cardinal numbers and one by Burali-Forti on a general theory of numbers. *Logica matematica* by Burali-Forti in the series of Manuali Hoepli may serve as a textbook.

> The *Formulaire de mathématiques* edited by Peano, is rather hard to begin on. The *Rivista de matematica* [...] also edited by Peano, furnishes much easy and instructive reading matter. (92)

As for our conclusions, it seems to us that when talking of the "Peano-Russell point of view", Wilson's assessment is even too generous, but after all fair, when he ascribes to Peano "an intimate formal relation between logic and all mathematics [...] precursor to a yet more intimate philosophical relation" (76), left to be revealed by Russell. It is not the case that "he [Peano] was always more precise than anyone else, and that he invariably got the better of any arguments upon which he embarked [at Paris 1900 Congress]" as Russell said. In Russell's exhalted words, Peano's logic was "a new mathematical technique", as stated in [Russell 1967, 144–145]. Peano's logic used classes, and, although he himself probably lost interest in its development, certain of his followers at least [Pieri 1906] paid attention to the necessity of axioms for classes and [Burali-Forti 1916a,b] later gave a definition of ordered $n$-tuples.

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# Dedekinds *Sätze* und Peanos *Axiomata*

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**Résumé :** Dans cet article nous entendons comprendre comment l'arithmétique de Peano a atteint la place qu'elle occupe aujourd'hui en mathématiques. Nous comparons tout d'abord les approches de Peano et de Dedekind, avant de mettre en avant le rôle de Hilbert et Bernays dans les développements ultérieurs de l'arithmétique de Peano.

**Abstract:** In this paper, we discuss the question how Peano's Arithmetic reached the place it occupies today in Mathematics. We compare Peano's approach with Dedekind's account of the subject. Then we highlight the role of Hilbert and Bernays in subsequent developments.

> Propositiones, quae logicae operationibus a caeteris deducuntur, sunt *theoremata*; quae vero non, *axiomata* vocavi.
>
> [Peano 1889, iv]

## 1 Einleitung

Die *Peano-Arithmetik* ist die heute allgemein anerkannte axiomatische Grundlage der Arithmetik. Dabei handelt es sich allerdings nicht um die Originalformulierung von Peano [Peano 1889], sondern um eine Variante, die vor allem den metamathematischen Resultaten von Kurt Gödel Rechnung trägt. Modern gesprochen arbeitet Peano bei der Formulierung des Induktionsaxioms in einer *Logik zweiter Stufe*. Da eine solche Logik aber, wie wir aus Gödels Resultaten wissen, nicht (rekursiv) axiomatisierbar ist, ist es für den mathematischen Beweisbegriff geboten, sich auf die Logik erster Stufe zu beschränken. Damit müssen aber zumindest die elementaren zahlentheoretischen Funktionen der Addition und Multiplikation axiomatisch

hinzugenommen werden (während sie in Peanos Originalansatz noch definitorisch eingeführt werden konnten).

In diesem Artikel wollen wir der Frage nachgehen, wie die Peano-Arithmetik ihre aktuelle Form erhielt und wie sie ihren Platz in der heutigen Mathematik gewinnen konnte. Auf der Basis der historischen Entwicklung zeichnen wir dabei die theoretisch notwendig gewordenen Modifikationen nach und betonen die Abgrenzung zu Dedekind, der kurz vor Peano in der Schrift *Was sind und was sollen die Zahlen?* [Dedekind 1888] eine Begründung der Arithmetik vorgelegt hat, die Peano erst kurz vor der Veröffentlichung seiner Axiome vorlag.[1]

Diese Verbindung zu Dedekind wird von einigen Autoren dazu benutzt, statt von den *Peano-Axiomen* von den *Dedekind-Peano-Axiomen* zu sprechen.[2] Tatsächlich muß man Dedekind zugestehen, daß er die *mathematischen* Sachverhalte, die den Axiomen zugrundeliegen, wohl sehr viel tiefer erkannt hatte als Peano. Wir wollen aber dafür argumentieren, daß es Peanos Verdienst ist, die Auszeichnung der charakterisierenden Eigenschaften der natürlichen Zahlen *in Axiomen* herausgestellt zu haben, während Dedekind dezidiert diese „Axiome" auf tiefliegende logische Prinzipien zurückführen wollte, so daß jene bei ihm zu *Sätzen* wurden.

Insofern wäre es wohl historisch angebracht, von den *Dedekind-Peano-Eigenschaften* der natürlichen Zahlen zu sprechen. Wenn man aber die spezifische Auszeichnung von *Axiomen* vor Augen hat, muß man die Bezeichnung *Peano-Axiome* nicht kritisieren. Dabei ist es nicht von Belang, ob man *Axiom* im traditionellen Sinne als *intuitiv wahre Aussage* versteht oder doch nur im modernen Sinne als *unbewiesene, an den Anfang einer Untersuchung gestellte, nicht hinterfragte Voraussetzung*; beide Lesarten sind mit Peanos Darstellung verträglich.

Eine eingehende Untersuchung der Geschichte der mathematischen Induktion findet man bei [Felgner 2012]. Wir setzen hier erst bei Dedekind und Peano an und betrachten vor allem die Unterschiede zwischen ihren Ansätzen. Anschließend beleuchten wir das weitere Schicksal der Peano-Axiome, insbesondere bei Hilbert und Bernays. Letztere können wohl dafür verantwortlich gemacht werden, daß die Peano-Arithmetik ihren heutigen ausgezeichneten Status gewinnen konnte.

---

1. Für die retrospektive Frage, wie Peano zu seinen Axiomen kam, können wir auf die detaillierte Ausarbeitung von Segre verweisen [Segre 1994], die sich bei der Beurteilung des Verhältnisses von Peano zu Dedekind wesentlich mit unserer Sichtweise deckt.

2. Siehe z. B. bei [Sieg & Morris 2018, B.2], aber auch schon, als „PEANO-DEDEKIND", bei [Weyl 1928, 86]. Tapp schreibt: „In der historisch informierten Literatur hat sich daher auch schon eingebürgert, diese Axiome nach Dedekind und Peano zu benennen" [Tapp 2013, 88].

# 2 Dedekind (zum ersten)

Richard Dedekind (1831–1916) veröffentlichte 1888 seine berühmte Schrift *Was sind und was sollen die Zahlen?*. Dort wird in § 6 die „Reihe der natürlichen Zahlen" als „einfach unendliches System" eingeführt. Wir wollen hier nicht näher auf die spezifische Terminologie von Dedekind eingehen, an die sich unmittelbar die Frage nach seiner Eingruppierung in die verschiedenen modernen mathematikphilosophischen Positionen anschließen würde.[3] Der Begriff „System" steht aber im Prinzip für das, was wir heute „Menge" nennen.[4] Es gibt noch weitere, auch notationelle Eigenheiten, die sich nicht weiter durchgesetzt haben. So entspricht „3" der heutigen Teilmengenrelation „$\subseteq$" und durch die durch den Index 0 bezeichnete Funktion erhält man aus der Menge $A$ die „Kette des Systems $A$" [Dedekind 1888, § 44].[5] Schließlich ist $\varphi$ die Nachfolgerfunktion, die aber nicht nur auf einzelnen Zahlen, sondern allgemein auf Mengen operiert. Die Einführung der natürlichen Zahlen wird dann wie folgt gegeben:

> 71. *Erklärung.* Ein System $N$ heißt *einfach unendlich*, wenn es eine solche ähnliche Abbildung $\varphi$ von $N$ in sich selbst gibt, daß $N$ als Kette (44) eines Elementes erscheint, welches nicht in $\varphi(N)$ enthalten ist. Wir nennen dies Element, das wir im folgenden durch das Symbol 1 bezeichnen wollen, das *Grundelement* von $N$ und sagen zugleich, das einfach unendliche System $N$ sei durch diese Abbildung $\varphi$ *geordnet*. Behalten wir die früheren bequemen Bezeichnungen für die Bilder und Ketten bei (§ 4), so besteht mithin das Wesen eines einfach unendlichen Systems $N$ in der Existenz einer Abbildung $\varphi$ von $N$ und eines Elements 1, die den folgenden Bedingungen $\alpha$, $\beta$, $\gamma$, $\delta$ genügen:
>
> $\alpha$. $N'$ 3 $N$.
>
> $\beta$. $N = 1_0$.
>
> $\gamma$. Das Element 1 ist nicht in $N'$ enthalten.
>
> $\delta$. Die Abbildung $\varphi$ ist ähnlich.

---

3. Siehe dazu das Ende von § 7 unten, insbesondere Fußnote 22.

4. Dedekind erwähnt zwar Bolzano und Cantor kurz in der Vorrede in Bezug auf seine Endlichkeitsdefinition, bedient sich aber nicht deren Namen *Menge* oder *Mannigfaltigkeit*.

5. Der Begriff der *Kette* wird gut in Zermelos Würdigung der Dedekindschen Arbeit erläutert, die er für Landaus Nachruf auf Dedekind verfaßt hat:

> Bei jeder Abbildung eines Systems $S$ auf einen (echten oder unechten) Teil $S'$ von $S$ gibt es „Ketten", d. h. Teilsysteme $K$, welche gleichfalls in sich selbst abgebildet werden, und jeder beliebige Teil $A$ von $S$ läßt sich zu einer kleinsten Kette $A_0$ ergänzen, welche als Durchschnitt aller $A$ umfassenden Ketten einfach als „die Kette von $A$" bezeichnet wird. [Zermelo 2010, 298]

Die Rolle von $\varphi$ als Nachfolgerfunktion und die damit einhergehende Einführung der $'$-Notation[6] wird wenig später wie folgt erläutert:

> [U]nter $a, b \ldots m, n \ldots$ [werden] stets Elemente von $N$, also Zahlen, unter $A, B, C \ldots$ Teile [modern gesprochen: Teilmengen] von $N$, unter $a', b' \ldots m', n' \ldots A', B', C' \ldots$ die entsprechenden Bilder verstanden [...], welche durch die ordnende Abbildung $\varphi$ erzeugt werden und stets wieder Element oder Teile von $N$ sind; das Bild $n'$ einer Zahl $n$ wird auch die auf $n$ *folgende* Zahl genannt.

Damit kann Dedekind nun den folgenden Satz *beweisen*:

> 80. *Satz der vollständigen Induktion* (Schluß von $n$ auf $n'$). Um zu beweisen, daß ein Satz für alle Zahlen $n$ einer Kette $m_0$ gilt, genügt zu zeigen,
>
> > $\rho$. daß er für $n = m$ gilt, und
> >
> > $\sigma$. daß aus der Gültigkeit des Satzes für eine Zahl $n$ der Kette $m_0$ stets seine Gültigkeit auch für die folgende Zahl $n'$ folgt.

# 3  Peano (zum ersten)

Giuseppe Peano (1858–1932) veröffentlichte [Peano 1889] seine Schrift *Arithmetices principia nova methodo exposita*. Abbildung 1 zeigt, wie die Axiome für die Arithmetik in §1 vorgestellt werden.

Zwei Jahre später legte Peano eine modifizierte Version vor (die in Abbildung 2 wiedergegeben ist und in der u.a. die sich auf die Gleichheit beziehenden Axiome nicht mehr zur Arithmetik gerechnet werden) [Peano 1891, 84].[7]

Gegenüber Dedekind fällt bei Peanos Darstellungen sofort auf, daß dieser sich einer formalen Notation bedient, die der heute geläufigen schon sehr nahe kommt. Diese Einführung formaler Notation war eine der Hauptmotivationen von Peano, und seine Verdienste um die moderne logico-mathematische Notation sind allgemein bekannt und müssen hier nicht gesondert gewürdigt werden.[8]

Den für uns wichtigen sachlichen Unterschied — die Einführung der arithmetischen Eigenschaften statt ihrer Herleitung aus übergeordneten Prinzipien — spricht Peano selbst aus:

---

6. Diese Notation hat sich bis heute gehalten; siehe dazu auch Fußnote 19.

7. Für eingehende Beschreibung der Entwicklung von Peanos Zahlbegriff, auch über die natürlichen Zahlen hinaus, siehe [Kennedy 1974b].

8. Siehe z. B. [Grattan-Guinness 2011], der weiter auf [Grattan-Guinness 2000, Kap. 2, 4 und 5] verweist, wobei das Kapitel 5 *Peano: the Formulary of Mathematics* besonders hervorzuheben ist.

## § 1. De numeris et de additione.

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### Explicationes.

Signo N significatur *numerus (integer positivus).*

> » 1 » *unitas.*
>
> » $a + 1$ » *sequens a,* sive *a plus* 1.
>
> » $=$ » *est aequalis.* Hoc ut novum signum conside-
>     randum est, etsi logicae signi figuram habeat.

### Axiomata.

1.  $1 \,\epsilon\, N.$
2.  $a \,\epsilon\, N \,.\, \supset \,.\, a = a.$
3.  $a, b, c \,\epsilon\, N \,.\, \supset \,:\, a = b \,.\, = \,.\, b = a.$
4.  $a, b \,\epsilon\, N \,.\, \supset \,\therefore\, a = b \,.\, b = c \,:\, \supset \,.\, a = c.$
5.  $a = b \,.\, b \,\epsilon\, N \,:\, \supset \,.\, a \,\epsilon\, N.$
6.  $a \,\epsilon\, N \,.\, \supset \,.\, a + 1 \,\epsilon\, N.$
7.  $a, b \,\epsilon\, N \,.\, \supset \,:\, a = b \,.\, = \,.\, a + 1 = b + 1.$
8.  $a \,\epsilon\, N \,.\, \supset \,.\, a + 1 - = 1.$
9.  $k \,\epsilon\, K \,.\, \therefore\, 1 \,\epsilon\, k \,.\, \therefore\, x \,\epsilon\, N \,.\, x \,\epsilon\, k \,:\, \supset_x \,.\, x + 1 \,\epsilon\, k \,::\, \supset \,.\, N \supset k.$

Abbildung 1

Fra quanto precede, e quanto dice il DEDEKIND, vi ha una contraddizione apparente, che conviene subito rilevare. Qui non si difinisce il numero, ma se ne enunciano le proprietà fondamentali. Invece il DEDEKIND definisce il numero, e precisamente chiama numero ciò che soddisfa alle condizioni predette. Evidentemente le due cose coincidono.[9] [Peano 1891, 88]

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9. „Wie Dedekind bemerkt, enthält das Vorhergehende einen scheinbaren [termi-nologischen] Widerspruch, den man sofort feststellen sollte. Hier wird nicht die Zahl definiert, sondern es werden ihre grundlegenden Eigenschaften formuliert. Dedekind hingegen definiert die Zahl; genauer [gesagt] nennt er Zahl, was die vorgenannten Bedingungen erfüllt. Offensichtlich stimmen die beiden Dinge überein." [Übersetzung von Peter Schuster].
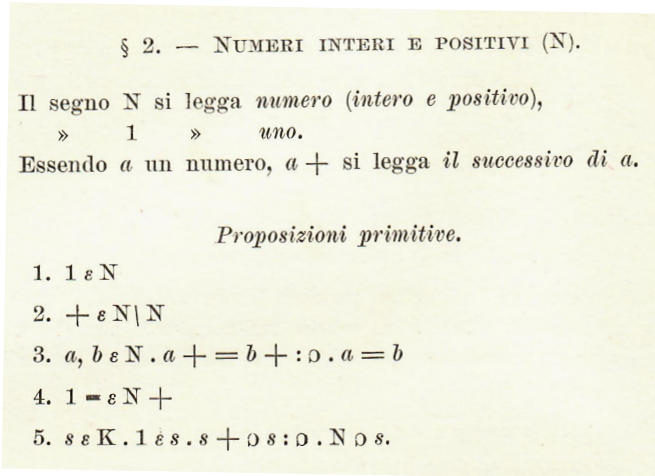
§ 2. — NUMERI INTERI E POSITIVI (N).

Il segno N si legga *numero* (*intero e positivo*),
&raquo;     1     &raquo;     *uno*.
Essendo $a$ un numero, $a +$ si legga *il successivo di a*.

*Proposizioni primitive.*

1. $1 \, \varepsilon \, \mathrm{N}$
2. $+ \, \varepsilon \, \mathrm{N} \setminus \mathrm{N}$
3. $a, b \, \varepsilon \, \mathrm{N} \, . \, a + = b + : \supset . \, a = b$
4. $1 - \varepsilon \, \mathrm{N} +$
5. $s \, \varepsilon \, \mathrm{K} \, . \, 1 \, \dot\varepsilon \, s \, . \, s + \supset s : \supset . \, \mathrm{N} \supset s.$

Abbildung 2

# 4 Peano und Dedekind

In der Literatur ist es eine wiederkehrende Diskussion, ob:[10]

– Peano durch „[d]as Studium des Dedekindschen Essays auf eine Liste von ein paar Axiomen" geführt wurde,[11] oder:

– Peano „unabhängig und in völliger Unkenntnis der Dedekindschen Veröffentlichung zu seiner Analysis der natürlichen Zahlen gekommen war".[12]

Die Priorität von Dedekind ist unbestritten, und Peano schreibt selbst, daß ihm Dedekinds Arbeit vorlag.[13]

Ob Peano seine Axiome aus dem Studium der Dedekindschen Arbeit gewann oder schon vorher unabhängig gefunden hatte, ist wohl auch deshalb kontrovers, weil er sich selbst dazu widersprüchlich geäußert hat. Eine Bemerkung von Peano kann leicht als Argument für die erste Sichtweise gelesen werden (auch wenn sie eine unabhängige Entdeckung nicht ausschließt):

---

10. Siehe auch [Segre 1994, 292, für weitere Zitate insbesondere Fußnote 21]; die Liste ließe sich inzwischen weiter verlängern.

11. So [Felgner 2012, 41]. In ähnlicher Weise auch [van Heijenoort 1967, 83], u.a.

12. So sein Biograph [Kennedy 1974a, 15], aber auch [von Plato 2019] u.a.

13. „Utilius quoque mihi fuit recens scriptum: R. DEDEKIND, *Was sind und was sollen die Zahlen;* Braunschweig, 1888, in quo quaestiones, quae ad numerorum fundamenta pertinent, acute examinantur" [Peano 1889, v].

> Le proposizioni primitive che precedono sono dovute al Dedekind, [Dedekind 1888, n. 71]; [...].[14] [Peano 1891, 86]

Später hat Peano explizit die zweite Auffassung zu Protokoll gegeben:

> La composizione del mio lavoro a. 1889 fu ancora indipendente dallo scritto menzionato del Dedekind ; prima della stampa, ebbi la prova morale dell'indipendenza delle proposizioni primitive da cui io partivo, nella loro coincidenza sostanziale colle definitioni del Dedekind.[15] [Peano, 1896–1899  1898, 243]

Für eine unabhängige Entdeckung der charakterisierenden Eigenschaften der natürlichen Zahlen seitens Peano spricht aus unserer Sicht, daß sich seine Axiome nicht unmittelbar mit denen von Dedekind identifizieren lassen. Die notwendige „Übersetzung" bedarf eines gewissen Aufwandes.[16] Hätte Peano diesen Aufwand vor der Niederschrift seiner Arbeit betrieben, denn hätte er sich wahrscheinlich auch des komfortablen Rekursionstheorems von Dedekind bedient.

Letztlich ist es müßig, ohne zusätztlich Quellen über den genauen historischen Hergang zu spekulieren.[17] Und für die uns interessierende Frage des Unterschieds, ob die charakterisierenden Eigenschaften der natürlichen Zahlen als Axiome oder Sätze aufgefaßt wurden, spielt es keine Rolle, ob Peano bei der Formulierung direkt auf Dedekind zurückgriff oder nicht; der Unterschied der Auffassungen besteht unabhängig davon.

# 5    Dedekind (zum zweiten)

Bei Dedekind werden die charakteristischen Eigenschaften der natürlichen Zahlen nicht axiomatisch gegeben, sondern ergeben sich definitorisch innerhalb seiner informellen Mengenlehre. Tatsächlich verfolgt er ein *logizistisches Programm* für die Arithmetik, wobei die — nicht formal eingeführte — Mengenlehre als Teil der Logik betrachtet wird. Der Dedekindsche Logizismus wird von ihm in den Eröffnungsworten des Vorworts eindeutig zum Ausdruck gebracht:

---

14. „Die vorhergehenden grundlegenden Sätze sind Dedekind zuzuschreiben [...]". [Übersetzung von Peter Schuster.]

15. „Die Erstellung meiner Arbeit von 1889 war noch unabhängig von der von Dedekind erwähnten Schrift; vor Drucklegung hatte ich den moralischen Beweis, daß die grundlegenden Sätze, von denen ich ausging, unabhängig waren, in ihrer wesentlichen Übereinstimmung mit Dedekinds Definitionen."

16. Tapp gibt eine solche in zwei Zwischenschritten, die aus unserer Sicht — und entgegen der Intention des Autors — die *Un*ähnlichkeit der beiden Formulierungen deutlich hervortreten lassen [Tapp 2013, 87].

17. Segre bezeichnet die Frage nach der Priorität als „hoary and unimportant" [Segre 1994, 292].

> Was beweisbar ist, soll in den Wissenschaften nicht ohne Beweis
> geglaubt werden. So einleuchtend diese Forderung erscheint, so
> ist sie doch, wie ich glaube, selbst bei der Begründung der
> einfachsten Wissenschaft, nämlich desjenigen Teils der Logik,
> welcher die Lehre von den Zahlen behandelt, auch nach den
> neuesten Darstellungen noch keineswegs als erfüllt anzusehen.
> [I]ch [nenne] die Arithmetik (Algebra, Analysis) nur einen Teil
> der Logik [...]. [Dedekind 1932, 335]

Die Idee eines rein axiomatischen Aufbaus der natürlichen Zahlen, wie er von
Peano gegeben wird, ist mit diesem Anliegen unverträglich.

Wie gesagt, kommt der Logizismus Dedekinds in einem mengenthe-
oretischen Gewand daher. Wir müssen hier nicht der Frage nachgehen, ob
Dedekinds Mengenlehre tatsächlich einen rein logischen Charakter hat; Hilbert
wird dies später verneinen (siehe unten § 9). Als viel schlimmer hat sich heraus-
gestellt, daß diese Mengenlehre im Prinzip den bekannten mengentheoretischen
Paradoxien, wie z. B. Russells Paradoxon, ausgesetzt ist. Dedekind ist sich
dieser Problematik in den 1890er Jahren bewußt geworden und hat lange mit
der Herausgabe der dritten Auflage gezögert; als er dieser 1911 schließlich
zustimmte, schrieb er im Vorwort:

> Als ich vor etwa acht Jahren aufgefordert wurde, die damals
> schon vergriffene zweite Auflage dieser Schrift durch eine dritte
> zu ersetzen, trug ich Bedenken, darauf einzugehen, weil inzwi-
> schen sich Zweifel an der Sicherheit wichtiger Grundlagen meiner
> Auffassung geltend gemacht hatten. Die Bedeutung und teilweise
> Berechtigung dieser Zweifel verkenne ich auch heute nicht. Aber
> mein Vertrauen in die innere Harmonie unserer Logik ist dadurch
> nicht erschüttert; ich glaube, daß eine strenge Untersuchung
> der Schöpferkraft des Geistes, aus bestimmten Elementen ein
> neues Bestimmtes, ihr System zu erschaffen, das notwendig
> von jedem dieser Elemente verschieden ist, gewiß dazu führen
> wird, die Grundlagen meiner Schrift einwandfrei zu gestalten.
> [Dedekind 1932, 343]

Aus moderner Perspektive läßt sich dazu sagen, daß Dedekind insoweit Recht
hat, als sich die „Grundlagen" — vor allem aber auch die mathemati-
schen Resultate — seiner Schrift innerhalb einer axiomatischen Mengenlehre,
wie z. B. der von Zermelo, einwandfrei entwickeln lassen. Die logizistische
Grundüberzeugung Dedekinds, nach der die Logik alleine (wie sie auch immer
genau abgegrenzt sein sollte) als Grundlage seiner Resultate dienen kann, ist
aber nicht mehr zu retten.

# 6   Peano (zum zweiten)

Peanos Ansatz, die Arithmetik auf — modern gesprochen — *nicht-logische Axiome* aufzubauen, ist grundlagentheoretisch unproblematisch und hat sich, der Idee nach, heute durchgesetzt.

Allerdings darf man nicht übersehen, daß auch Peanos Axiomatisierung, so wie sie von ihm 1889 und 1891 gegeben wurde, eine problematische Mengenlehre zugrunde liegt. Axiom 9 (1889) bzw. Axiom 5 (1891) benutzt, so wie es formuliert ist, mit $K$ offensichtlich eine Allmenge. Da Peano keine formalen Mengenbildungsoperationen angibt, kann man ihm wohl nicht direkt einen Widerspruch durch die Formalisierung des Cantorschen Paradoxons über die Menge aller Mengen nachweisen; aber eine genaue Abgrenzung der Formelklasse, für die vollständige Induktion zur Verfügung steht, bleibt zumindest offen. Wie auch immer die genaue Abgrenzung der Eigenschaften auszusehen hat, sie erlaubt zumindest *zweitstufige* Eigenschaften, so daß sich die arithmetischen Operationen der Addition und Multiplikation definieren lassen.[18]

Im Gegensatz zu Dedekind war sich Peano aber des nicht-logischen Charakters seiner Axiome bewußt, was durch die Einführung der Symbole $N$, 1, und $a+1$ (bzw. $a+$)[19] in den *Explicationes* (1889) bzw. den vorangestellten Sätzen (1891) deutlich wird. Diese sind nicht, wie bei Dedekind, durch Definitionen in einer übergeordneten Theorie gegeben. Die Notwendigkeit einer solchen Hinzunahme nicht-logischer Zeichen spricht Peano explizit aus:

---

18.  Dabei kann sich Peano auf Graßmann stützen:

> In arithmeticae demonstrationibus usus sum libro: H. GRASSMANN, *Lehrbuch der Arithmetik*, Berlin 1891. [Peano 1889, v]

19.  Das Problem der Bezeichnung der Nachfolgerfunktion und deren Kollision mit der (zweistelligen) Addition wurde von Bernays zum Ausdruck gebracht:

> Der Fortschreitungsprozeß pflegt in der Mathematik durch   „+1" angegeben zu werden. Diese Bezeichnungsweise hat jedoch den Mangel, daß der begriffliche Unterschied zwischen der Auffassung von „$a + 1$" als der auf $a$ folgenden Zahl und andererseits der Summe von $a$ und 1 nicht zur Darstellung gelangt. [Hilbert & Bernays 1934, 219, Fußnote 1]

Bernays bedient sich daher der auch schon bei Dedekind verwendeten Schreibweise $a'$ für den Nachfolger von $a$.

Leonard Nelson zum Beispiel hat den begrifflichen Unterschied bei der unterschiedlichen Verwendung des + Zeichens nicht gesehen:

> Hieran scheitert z. B. auch der Versuch, das oben genannte GRASSMANNsche Axiom als Definition der Addition aufzufassen. [... wir müßten,] um den Sinn des + Zeichens auf der linken Seite der Gleichung durch die rechte Seite zu erklären, bereits die Bedeutung kennen [. . . ], die es auf der rechten Seite hat [...]. [Nelson 1905-1906, 43]

> Puto vero his tantum logicae signis propositiones cuiuslibet scientiae exprimi posse, *dummodo adiungantur signa quae entia huius scientias representant.* [Peano 1889, v, unsere Hervorhebung]

# 7 Dedekind (zum dritten)

Wenn wir bis hierher die Verdienste Peanos für die Herausarbeitung der Axiome der Arithmetik als nicht-logische Komponenten gegenüber Dedekind hervorgehoben haben, müssen wir aber noch hinzufügen, daß die *mathematische* Analyse der Grundlagen der Arithmetik bei Dedekind sehr viel tiefer geht als bei Peano.

Zuerst hat Dedekind ein allgemeines Rekursionsschema zur Einführung von Funktionen bewiesen, das bei Peano fehlt [Dedekind 1888, Satz 126].[20]

Darüberhinaus beweist Dedekind die *Kategorizität* seiner Charakterisierung:

> 132. *Satz.* Alle einfach unendlichen Systeme sind der Zahlenreihe *N* und folglich (nach 33) auch einander ähnlich. [Dedekind 1932, 376]

Dieses Resultat kann vielleicht als das mathematisch tiefste der Dedekindschen Abhandlung betrachtet werden. Es hat so keine Entsprechung bei Peano (der dazu überhaupt erst eine Theorie der Semantik seines Axiomensystems hätte entwickeln müssen) und sein Fehlen hat zu einem aus unserer Sicht tiefgreifenden Mißverständnis des axiomatischen Ansatzes bei Russell geführt (siehe unten §8).

Schließlich hatte Dedekind aber auch die Notwendigkeit der zweitstufigen Formulierung der vollständigen Induktion erfaßt, wie aus den Ausführungen in seinem Brief an Keferstein deutlich wird.[21] Dabei hatte er insbesondere schon die Konstruktion von heute sogenannten *Nichtstandardmodellen* der (erststufigen) Peano-Arithmetik vor Augen [siehe Kahle 2017]. Bei Peano finden sich keine Anhaltspunkte, daß er bei der Formulierung seiner Axiome vergleichbare Analysen vorgenommen haben könnte.

Felgner rekonstruiert Dedekinds Vorgehensweise in algebraischer Begrifflichkeit:

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20. Van Heijenhoort schreibt dazu in seiner Einführung zur englischen Übersetzung von [Peano 1889]:

> [Peano] proves for addition a theorem [...] and a similar theorem [...] for multiplication; but these theorems are far from having the same effect as Dedekind's Theorem 126. [van Heijenoort 1967, 84]

21. Dieser Brief findet sich in englischer Übersetzung in der Textsammlung von van Heijenhoort [van Heijenoort 1967]; eine kritische Edition wurde von Tapp [Tapp 2017] besorgt.

> [Die Grundidee des Beweises von Dedekind] kommt aus der
> Algebra. Wenn $\Phi$ eine Eigenschaft ist, die der Zahl 0 zukommt
> und mit einer Zahl $n$ auch ihrem Nachfolger $n + 1$, dann ist die
> Menge $E = \{n; n \in N \,\&\, \Phi(n)\}$ unter der Nachfolger-Operation
> abgeschlossen und ist also (algebraisch gesehen) eine Substruktur
> der Halbgruppe $\langle \mathbb{N}, \nu \rangle$, wenn $\nu$ die 1-stellige Nachfolger-Funktion
> ist: $\nu(x) = x + 1$. Um $E = \mathbb{N}$ zu beweisen, muß man also
> „nur" zeigen, daß $E = \mathbb{N}$ überhaupt keine echten Substrukturen,
> die die 0 enthalten, besitzt. (Aus algebraischer Sicht ist dies
> genau die Aussage des Prinzips der vollständigen Induktion.)
> [Felgner 2012, 38f]

Vor diesem Hintergrund ist man geneigt, Dedekinds Analyse weniger dem Logizismus zuzurechen, als vielmehr dem *Strukturalismus*. Der Strukturalismus, wie er in der Mathematik vor allem durch Bourbaki propagiert wurde, hat seine Grundlagen ohne Zweifel in Dedekinds algebraischen Arbeiten, die später von Emmy Noether weiter entwickelt wurden. Insbesondere Sieg weist darauf hin, daß auch schon Dedekinds Behandlung der Arithmetik in diesem Kontext gesehen werden muß [Sieg & Morris 2018].[22]

# 8 Russell, Couturat, nochmal Russell und auch Gödel

Peanos axiomatischer Ansatz wurde natürlich von ihm selbst und seiner Schule weitergeführt.[23]

Wie sich die Peano-Arithmetik zur heutigen Standardtheorie entwickeln konnte, scheint aber eine vergleichsweise unklare Geschichte zu sein.

Oft wurden Peanos Axiome als eine Art *Lemmata* behandelt, die zwar die charakteristischen Eigenschaften der natürlichen Zahlen durchaus erschöpfend zusammenstellen, aber eben nicht den Charakter von eigenständigen Axiomen haben, sondern letztlich für passend definierte logische Ausdrücke *bewiesen* werden sollen. Dieses Verständnis entspricht im wesentlichen dem von Dedekind, allerdings wird — Russell folgend — häufig auf Freges Definition der

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22. Es ließe sich hier noch eine eingehende Debatte um das Verständnis von Dedekinds Ansatz anschließen, auf die wir verzichten müssen. Einen guten Überblick zum aktuellen Stand der Diskussion gibt [Reck 2017]. Für die Beziehung von Dedekind zu Peano können wir noch spezifisch auf [Ferreirós in Vorbereitung] verweisen, der auch die beiden zentralen Artikel in einen historischen Vergleich zueinander gesetzt hat [Ferreirós 2005].

23. Dabei kam Peano auch z. B. zur Überzeugung, daß es besser sei, die Zahlenreihe mit 0 statt mit 1 beginnen zu lassen. Als Beispiel für weitergehende Arbeiten können wir auch auf [Padoa 1902] verweisen.

Zahlen zurückgegriffen.[24] Bei diesen *logizistischen* Ansätzen sehen die Autoren insbesondere die Mengenlehre, sofern sie von ihr Gebrauch machen, als einen Teil der Logik an.

Eine erste derartige Behandlung der Peano-Axiome findet sich in Russells *The Principles of Mathematics* [Russell 1903]. Dort wird Peanos Axiomatik in Kapitel XIV äußerst kritisch behandelt. Russell gibt zwar die Peano-Axiome an, hat aber das erklärte Ziel, die nicht-logischen Zeichen durch logisch definierte zu ersetzen. Unter Benutzung der Zahldefinition von Frege will er gezeigt haben, daß für diese Zahlen die „Axiome" erfüllt sind und kommt zur Konklusion:

> There is, therefore, from the mathematical standpoint, no need whatever of new indefinables or indemonstrables in the whole of Arithmetic and Analysis. [Russell 1903, § 123]

Der Titel unserer Arbeit könnte aber von Russell übernommen sein, wenn er schreibt:

> Dedekind *proves* mathematical induction, while Peano regards it as an axiom. [Russell 1903, § 241]

Im anschließenden Satz bescheinigt er deshalb Dedekind "an apparent superiority". Heute sehen wir das genau umgekehrt.

Couturat hatte es sich in seinem Buch *Les Principes des mathématiques* [Couturat 1905] zur Aufgabe gemacht, eine Darstellung von Russells *Principles* zu geben, die dann um „eine Analyse der meisten Arbeiten der mathematischen Zeitgenossen bezüglich derselben Frage" [Couturat 1908, viii] erweitert wurde. In Kapitel II, § B werden Peanos Axiome zur Einführung der natürlichen Zahlen als eine „sog. *Definition mittels Postulaten*" vorgestellt. Doch der Status dieser Art der Definition wird im Zusammenhang mit der von [Padoa 1902] vorgenommenen Reduktion der nicht-logischen Konstanten ausdrücklich in Zweifel gezogen:

> Diese Vervollkommnung der Grundlagen der Arithmetik hat gleichwohl keine so grundlegende Bedeutung, wie man glauben könnte. Sie hätte sie sicherlich, wenn man kein anderes Mittel zur Begründung der Arithmetik besäße als ihre Stützung durch irgend welche Grundbegriffe und Postulate; sie hat jene Bedeutung nicht mehr, sobald man das eine oder andere System von Postulaten aus einer ausdrücklichen Definition der ganzen Zahl herleiten kann, wie dies H. Russell gezeigt hat. Denn dann sind weder die Grundbegriffe wahrhaftig undefinierbar, noch die Grundsätze unbeweisbar, und infolgedessen hat es keine so große Bedeutung mehr, daß ihre Anzahl auf ein Minimum reduziert werde. [Couturat 1908, 60]

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24. Eine eingehende Behandlung der Einflüsse Freges kann hier nicht gegeben werden, wir können aber auf [Gillies 1982] verweisen.

Anschließend wird die Russell zugerechnete und auf Frege zurückgehende
Definition der Zahlen in mengentheoretischer Sprache gegeben mit der
Konklusion:

> Diese Definition setzt, wie man sieht, den Sinn fest, den man den
> drei undefinierbaren Symbolen beizulegen hat und zwar in rein
> logischen Ausdrücken. Nunmehr hört die Definition der ganzen
> Zahlen auf, an drei von den logischen Konstanten unabhängige
> Grundbegriffe gebunden zu sein; sie reduziert sich auf eine
> Nominaldefinition, die keine anderen undefinierbaren Begriffe au-
> ßer den logischen Konstanten enthält. Dadurch ist die Anknüpfung
> der Arithmetik an die Logik vollendet ohne *Hinzunahme* irgend
> eines neuen Grundbegriffes. [Couturat 1908, 62]

In der *Introduction to Mathematical Philosophy* von 1919 hat Russell Peano
eine positivere Darstellung zukommen lassen als noch 1903. Bei der Einführung
der natürlichen Zahlen schreibt er:

> Having reduced all traditional pure mathematics to the theory of
> the natural numbers, the next step in logical analysis was to reduce
> this theory itself to the smallest set of premises and undefined
> terms from which it could be derived. This work was accomplished
> by Peano. [Russell 1919, 5]

Nach einer eingehenden Behandlung der Peanoschen Axiome bleibt er aber
bei seiner kritischen Haltung, daß die nicht-logischen Symbole noch auf der
Grundlage des Fregeschen Zahlbegriffs durch definierte „logische" Ausdrücke
ersetzt werden können, so daß die Axiome zu beweisbaren Sätzen werden:

> It is time now to turn to the considerations which make it necessa-
> ry to advance beyond the standpoint of Peano, who represents the
> last perfection of the "arithmetisation" of mathematics, to that of
> Frege, who first succeeded in "logicising" mathematics, *i.e.*, in
> reducing to logic the arithmetical notions which his predecessors
> had shown to be sufficient for mathematics. [Russell 1919, 6f.]

Diese Sichtweise konnte sich sogar noch bis zu Kurt Gödel halten. Als
er 1931 seine epochale Arbeit zur prinzipiellen Unvollständigkeit rekursiver
Axiomatisierungen der Arithmetik veröffentlichte, bezog er sich formal auf
Whitehead und Russells *Principia Mathematica* [Whitehead & Russell 1910-
1913] aber mit Rückgriff auf Peano:

> $P$ ist im wesentlichen das System, welches man erhält, wenn man
> die Peanoschen Axiome mit der Logik der $PM$[16] überbaut. [Gödel
> 1931, 176]

Die Fußnote 16 zur Logik der *Principia Mathematica* stellt dann aber klar,
daß auch Gödel eine logizistische Einführung der Zahlen für ausreichend hält:

> Die Hinzufügung der Peanoschen Axiome [...] dien[t] lediglich zur
> Vereinfachung des Beweises und [ist] prinzipiell entbehrlich.

Bei Russell (und mit ihm auch Couturat) ist noch interessant, daß er
Peano dafür kritisiert, daß dessen Axiome unterschiedliche Interpretationen
zulassen.[25] Wenn Russell eine solche Kritik auch auf Dedekind ausdehnt
[Russell 1903, chap. XXX], zeigt das in erster Linie, daß er die tiefere
Bedeutung von Dedekinds Kategorizitätsresultat nicht adäquat erfaßt hat.

Hier kann man durchaus ein grundsätzliches philosophisches Problem im
Verständnis der natürlichen Zahlen sehen, das bis heute nachwirkt. Carl Siegel
hatte im Vorwort zu seiner deutschen Übersetzung von [Couturat 1905] zur
Unterscheidung von Arithmetik (als einer auf die Logik zu gründende Theorie)
zur Geometrie (als einer nicht-logischer Elemente bedürftigen Theorie)
bemerkt:

> Die Axiome der Arithmetik lassen sich von den rein logischen
> Grundsätzen herleiten, die der Geometrie dagegen nicht. Die
> Tatsache, daß es nur *eine* Arithmetik, aber *mehrere* (logisch gleich
> mögliche) Geometrien gibt, beweist das eben Gesagte schlagend.
> [Couturat 1908, vi]

Unser modernes Verständnis einer Axiomatik läßt zumindest im Prinzip
immer unterschiedliche Interpretationen zu. Dieses Verständnis wurde ganz
wesentlich von Hilbert geprägt, der es von der Geometrie auf beliebige
Axiomensysteme ausgedehnt hat. Und dabei läßt sich unmittelbar auf Peanos
Axiomatisierung zurückgreifen, auch wenn das bei Hilbert nicht explizit
herausgestellt ist.

# 9   Hilbert

David Hilbert (1862–1943) hat zusammen mit seiner Schule in den 1920er
Jahren die mathematische Logik im wesentlichen so herausgearbeitet, wie wir
sie heute kennen.[26] Ausgehend von seiner bahnbrechenden Neuaxiomatisierung

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25. 1919 hat er das in den folgenden Worten ausgedrückt:

> In Peano's system there is nothing to enable us to distinguish
> between these different interpretations of his primitive ideas. [...]
> This point, that „0“ and „number“ and „successor“ cannot be
> defined by means of Peano's five axioms, but must be independently
> understood, is important. We want our numbers not merely to verify
> mathematical formulæ, but to apply in the right way to common objects.
> [Russell 1919, 9]

Besonders der letzte Satz steht der späteren Hilbertschen Auffassung entgegen, die
sich in gewissem Sinne in der Mathematik durchgesetzt hat.

26. Im Prinzip kamen später lediglich die Beschränkung auf Logik erster Stufe, vor
allem von Skolem gefordert, hinzu [siehe Moore 1988] sowie der scharf abgegrenzte

der euklidischen Geometrie hat er nach der Reduktion dieser auf die Arithmetik — und wir müssen hinzusetzen: Arithmetik zweiter Stufe, was für Hilbert seinerzeit eine natürliche Auffassung war — erkannt, daß eben diese Arithmetik einer Axiomatisierung bedarf. Eine solche Axiomatisierung gibt zu neuen grundlagentheoretischen Fragestellungen Anlaß. Die wichtigste dieser Fragestellungen war das *zweite Hilbertsche Problem* in der berühmten Liste von Problemen, die Hilbert auf dem Internationalen Mathematikerkongreß in Paris vorgestellt hat; weitere Fragestellungen wurden in dem programmatischen Vortrag *Axiomatisches Denken* [Hilbert 1918] aufgeworfen, den Hilbert 1917 vor der Schweizerischen Mathematischen Gesellschaft hielt. Die Durchdringung des hinter diesen Fragestellungen liegenden logischen und mathematischen Feldes, das nicht zuletzt durch die Hilbertsche Beweistheorie abgegrenzt werden kann, hat schließlich zu unserem modernen Verständnis von mathematischer Logik geführt.[27]

Das zweite Hilbertsche Problem trägt den Titel *Die Widerspruchslosigkeit der arithmetischen Axiome*, wobei es aber ausdrücklich um die Konsistenz einer Axiomatik der *reellen* Zahlen geht (da, wie gesagt, Hilbert die Arithmetik zweitstufig aufgefaßt hat). Eine solche Axiomatik hatte Hilbert im Jahr zuvor in seinem Aufsatz *Über den Zahlbegriff* [Hilbert 1900] gegeben. Dabei ging es ihm neben der konkreten Axiomatik auch darum, die axiomatische Methode als solche (und in ihrem modernen Verständnis) der von ihm sogenannten genetischen Methode[28] gegenüberzustellen. Er schreibt:

> Meine Meinung ist diese: *Trotz des hohen pädagogischen und heuristischen Wertes der genetischen Methode verdient doch zur endgültigen Darstellung und völligen logischen Sicherung des Inhaltes unserer Erkenntnis die axiomatische Methode den Vorzug.* [Hilbert 1900, 181]

Dem Beispiel seiner Axiomatisierung der Geometrie folgend beginnt er anschließend die Auflistung seiner Axiome mit den Worten:

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Begriff einer Semantik, eingeführt von Tarski. Bei Hilbert selbst darf man die heute übliche Beschränkung auf Logik erster Stufe aber noch nicht voraussetzen, siehe [Kahle 2019].

27. Man muß sich darüber im Klaren sein, daß Hilbert eine wechselvolle Entwicklung durchgemacht hat, bevor er zu der schließlich in [Hilbert & Bernays 1934, 1939] niedergelegten Form der Beweistheorie kam. Hier ist nicht der Platz, diese Entwicklungen detaillierter nachzuzeichnen, wir können aber auf entsprechenden Arbeiten von Sieg verweisen [Sieg 1999, 2013].

28. „Ausgehend von dem Begriff der Zahl 1, denkt man sich gewöhnlich durch den Prozeß des Zählens zunächst die weiteren ganzen positiven Zahlen $2, 3, 4, \ldots$ entstanden und ihre Rechengesetze entwickelt; sodann gelangt man durch die Forderung der allgemeinen Ausführung der Substraktion zur negativen Zahl; [usw. bis zu den reellen Zahlen als Schnitt oder Fundamentalreihe]. Wir können diese Methode der Einführung des Zahlbegriffs die *genetische Methode* nennen, weil der allgemeinste Begriff der reellen Zahl durch sukzessive Erweiterung des einfachen Zahlbegriffs *erzeugt* wird" [Hilbert 1900, 180].

> Wir denken ein System von Dingen; wir nennen diese Dinge
> Zahlen und bezeichnen sie mit $a, b, c, \ldots$ . Wir denken diese
> Zahlen in gewissen gegenseitigen Beziehungen, deren genaue und
> vollständige Beschreibung durch die folgenden Axiome geschieht.
> [Hilbert 1900, 181]

Jetzt folgen nicht die Peanoschen Axiome für die Arithmetik der natürlichen
Zahlen, sondern eine Axiomatik für die reellen Zahlen. Mit den reellen
Zahlen hatte Hilbert einen sehr viel problematischeren Gegenstandsbereich
vor sich, als wenn man nur die natürlichen Zahlen betrachtet. Insbesondere
ergibt sich dabei schon die Frage nach dem Status aktual unendlicher
Mengen; zudem führt Hilbert auch sein umstrittenes *Axiom der Vollständigkeit*
ein. Allerdings bedurfte es der Resultate von Kurt Gödel, um formal zu
zeigen, daß die reellen Zahlen — in einer erststufigen Axiomatisierung —
prinzipiell nicht auf die natürlichen Zahlen zurückführbar sind, bzw. daß
eine zweitstufige „Axiomatisierung" bereits für den rein logischen Teil nicht
mehr rekursiv sein kann.

Wenn man vom Vollständigkeitsaxiom absieht, ist die Charakteristik
der Hilbertschen Axiomatik aber kaum von der Peanos zu unterscheiden.
Insbesondere erlauben Hilberts Ausführungen keinesfalls, ihn hier in eine Reihe
mit Dedekind, Frege, Russell und Couturat zu stellen, soweit man deren
logizistischen Anspruch, die Zahlen durch logische Definitionen einzuführen,
herausstellt.

In seinem Beitrag zum Internationalen Mathematikerkongreß 1904 in
Heidelberg (*Über die Grundlagen der Logik und der Arithmetik*, [Hilbert
1905]) hat Hilbert eine erste rudimentäre Idee seiner später entwickelten
Beweistheorie gegeben. Dabei werden Frege und Dedekind kurz behandelt[29]
und auch Cantor, dessen Mengenlehre aber auch nicht ausreichende Sicherheit
gewährleiste.[30] Der Logizismus, wie er sich 1904 darstellte, wird schließlich
explizit verworfen, wenn Hilbert schreibt:

> Man bezeichnet wohl die Arithmetik als einen Teil der Logik
> und setzt meist bei der Begründung der Arithmetik die herge-
> brachten logischen Grundbegriffe voraus. Allein bei aufmerksa-
> mer Betrachtung werden wir gewahr, daß bei der hergebrach-
> ten Darstellung der Logik gewisse arithmetische Grundbegriffe,
> z. B. der Begriff der Menge, zum Teil auch der Begriff der

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29. Freges Ansatz wird dabei wegen der Russellschen Antinomie verworfen; ebenso
Dedekind „wegen des unvermeidlichen Widerspruchs des [bei ihm] zur Verwendung
kommenden Begriffs der Gesamtheit aller Dinge" [Hilbert 1905, 249].

30. „G. CANTOR hat den genannten Widerspruch [bei Dedekind] empfunden
und diesem Empfinden dadurch Ausdruck verliehen, daß er ‚konsistente' und
‚nichtkonsistente' Mengen unterscheidet. Indem er aber, meiner Meinung nach, für
diese Unterscheidung kein scharfes Kriterium aufstellt, muß ich seine Auffassung
über diesen Punkt als eine solche bezeichnen, die dem *subjektiven* Ermessen noch
Spielraum läßt und daher keine objektive Sicherheit gewährt" [Hilbert 1905, 249].

Zahl, insbesondere der Anzahl, bereits zur Verwendung kommen.
[Hilbert 1905, 250]

Hilbert hat den 1904 skizzierten Ansatz zur Beweistheorie erst einmal beiseitegelegt (wohl auf Grund der Kritik Poincaré's an der vermeintlichen Zirkularität einer beweistheoretischen Begründung der Induktion).[31] Stattdessen trat er nach der Veröffentlichung der *Principia Mathematica* von Whitehead und Russell [Whitehead & Russell 1910-1913] selbst in eine „logizistische Phase" ein.[32] Dabei wurden die *Principia Mathematica* einer eingehenden Prüfung unterzogen, insbesondere in der Dissertation seines Doktoranden Heinrich Behmann. Letztlich sah er aber auch Whitehead und Russells Ansatz als gescheitert an und drückte das 1920 in einer seiner Vorlesungen deutlich aus [Hilbert 1920, 33]:[33]

> Das Ziel, die Mengenlehre und damit die gebräuchlichen Methoden der Analysis auf die Logik zurückzuführen, ist heute nicht erreicht und ist vielleicht überhaupt nicht erreichbar.

Dennoch — und das durchaus überraschend — findet sich im Lehrbuch von Hilbert und Ackermann *Grundlagen der theoretischen Logik* eine Einführung des Zahlbegriffs auf der Grundlage der Fregeschen Definition und eine eindeutige Übernahme des Russellschen Standpunkts von 1903/1919, wobei lediglich das Unendlichkeitsaxiom bemängelt wird:

> Von besonderem Interesse ist auch, wie unter Zugrundelegung der logischen Einführung des Anzahlbegriffs und allerdings wesentlicher Benutzung des genannten Axioms der Unendlichkeit die zahlentheoretischen Axiome zu logischen, beweisbaren Sätzen werden. [Hilbert & Ackermann 1928, 88]

In einer Fußnote wird dabei auf die deutsche Übersetzung von Russells *Einführung in die mathematische Logik* [Russell 1923] verwiesen. Die zahlentheoretischen Axiome werden von Hilbert und Ackermann aber überhaupt

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31. Hier sollten wir bemerken, daß es neben der logizistischen und der (später so charakterisierten) formalistischen Auffassung der Induktion im Rahmen einer Axiomatik noch die auf (Kronecker und) Poincaré zurückgehende und durch Brouwer und Weyl aufgegriffene heute *intuitionistisch* zu nennende Auffassung der Induktion als Prinzip des „inhaltlichen *mathematischen* Denkens" [Weyl 1928, 86] gibt. Diese Auffassung steht der Hilbertschen nicht notwendig entgegen; sie wird von ihm in der Beweistheorie sogar ausdrücklich als Grundlage des *metamathematischen* Schließens herangezogen. Eine eingehende Auseinandersetzung mit diesem Problemkreis, der keinen unmittelbaren Bezug zu Dedekind und Peano mehr hat, geht aber über den Rahmen dieser Arbeit hinaus.

32. Siehe [Kahle 2013] und auch z. B. [Tapp 2013, § 4.1.4 Hilberts Logizismus].

33. Das Zitat findet sich veröffentlicht in [Hilbert 2013, 363]. In einem Vortrag, den Hilbert 1927 in Hamburg hielt, gibt er seinen inzwischen verabsolutierten Anti-Logizismus wie folgt wieder:

> Die Mathematik wie jede andere Wissenschaft kann nie durch Logik allein begründet werden. [Hilbert 1928, 65]

nicht angegeben und dementsprechend fehlt auch jeglicher Hinweis auf Peano.[34]

Tatsächlich finden sich in Hilberts Veröffentlichungen[35] keine expliziten Referenzen auf Peanos Axiomatisierung der natürlichen Zahlen.[36] Es gibt aber einen — für uns zentralen — Verweis auf Peano in einer unveröffentlichten Mitschrift zu der 1917 von Hilbert gehaltenen Vorlesung über *Mengenlehre*:

> Peano [konnte] die bei Dedekind verborgenen Axiome ans Licht ziehen und an die Spitze seiner Begründung der Theorie der ganzen Zahlen stellen. [...]
>
> Die ganzen Zahlen nehmen also tatsächlich in der Mathematik eine Sonderstellung ein; und wenn wir die Axiome derselben noch weiter begründen wollen durch Zurückführung auf die Gesetze der Logik selbst, so stehen wir vor einem der schwierigsten Probleme der Mathematik überhaupt.
>
> Wenn wir uns also jetzt auf den Peanoschen Standpunkt stellen, d. h. wenn wir Axiome der Arithmetik aufstellen, aber d. h. auf eine weitere Zurückführung derselben verzichten und die gewöhnlichen Gesetze der Logik ungeprüft übernehmen, so müssen wir uns bewusst sein, dass wir dadurch die Schwierigkeiten einer ersten philosophisch-erkenntnistheoretischen Begründung nicht überwunden, sondern nur kurz abgeschnitten haben. [Hilbert 1917, 145f.], zitiert nach [Hilbert 2013, 33f., Fußnoten 5 und 6]

Hilbert kritisiert hier im Jahr 1917 Peanos Standpunkt dahingehend, daß die „philosophisch-erkenntnistheoretischen Begründung" lediglich „abgeschnitten" worden sei. Nachdem sich Hilbert aber spätestens seit 1920 zu einer Aufgabe des logizistischen Programms genötigt sah, ist es genau dieses „Abschneiden" der — überzogenen oder zumindest nicht erfüllbaren — philosophischen Begründungsansprüche, das der Axiomatik in der Mathematik zu ihrem durchgreifenden Erfolg verholfen hat.

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34. Erst in der vierten Auflage von [Hilbert & Ackermann 1958] hat Ackermann die Peano-Axiome explizit diskutiert, dann aber schon mit dem erststufigen Induktionsschema (womit sie als Beispiel einer nicht mehr endlich axiomatisierten Theorie dienen). Auch wird auf die Notwendigkeit hingewiesen, „noch rekursive Definitionen für zahlentheoretische Funktionen" hinzunehmen zu müssen. Die Fregesche Anzahldefinition und der Hinweis, daß sich damit „die zahlentheoretischen Axiome [...] logisch beweisen lassen", wurden aber beibehalten [Hilbert & Ackermann 1958, 152].

35. Das unten noch zu nennende zweibändige Werk *Grundlagen der Mathematik* [Hilbert & Bernays 1934, 1939] muß man natürlich Bernays alleine zurechnen.

36. Man kann dafür beispielsweise den Fundstellen des Namens PEANO im Namensverzeichnis von [Hilbert 2013] nachgehen.

# 10   Bernays

Wenn wir Hilbert als Wegbereiter der heute dominierenden, anti-logizistischen Auffassung der Peanoschen Axiome, die diese unbewiesen an den Anfang der Arithmetik stellt, betrachten dürfen, so liegt das nicht an einer einzelnen Bemerkung in einem unveröffentlichen Vorlesungsmanuskript. Er verdankt diese Stellung wohl vor allem den Ausführungen von Paul Bernays (1888–1977), durch den das moderne Verständnis in dem zusammen mit Hilbert herausgegebenen zweibändigen Werk *Grundlagen der Mathematik* [Hilbert & Bernays 1934, 1939] detailliert entwickelt und bekannt gemacht wurde.

In §6 des ersten Bandes, das den Titel *Widerspruchsfreiheit unendlicher Individuenbereiche. Anfänge der Zahlentheorie* trägt, verweist Bernays schon im Titel von Abschnitt a) 3. auf Peano: *Übergang zu Axiomen ohne gebundene Variablen unter Verschärfung der Existenzaxiome: das Symbol* 0*; Ziffern im neuen Sinne;* PEANO*sche Axiome; Zusammenstellung der erhaltenen Axiome.* Die (technisch sehr detaillierte) Darstellung von Bernays steht, wie aus dem Titel des Paragraphen hervorgeht, unter der Zielsetzung, Widerspruchsfreiheitsbeweise zu führen. Und diese sollen, auch nach der Veröffentlichung von Gödels Unvollständigkeitssätzen, im Rahmen der Hilbertschen Beweistheorie geführt werden und dementsprechend *nicht* durch logizistische Interpretationen der Zahlentheorie erbracht werden. Wenn man sich die anti-logizistischen Bemerkungen von Hilbert von 1904 in Erinnerung ruft und wenn man die Erkenntnisse, insbesondere die Gödelschen Resultate, der sich in den folgenden 30 Jahren rasant entwickelnden mathematischen Logik vor Augen hält, würden sowohl eine mengentheoretische Interpretation als auch die Benutzung zweitstufiger Logik nicht mehr den grundlagentheoretischen Ansprüchen für einen Widerspruchsfreiheitsbeweis genügen. Insofern kann Bernays gar nicht anders, als eine Axiomatik mit Hilfe von nicht-logischen Zeichen, wie Peano sie gegeben hatte, an den Anfang zu stellen. Er tut dies zuerst nur für die Peano-Axiome ohne Induktion [Hilbert & Bernays 1934, 219]. Anschließend wird die vollständige Induktion separat eingeführt und detailliert erörtert. Dabei wird klar herausgestellt, daß sie „von Aussagen der Theorie handelt" [Hilbert & Bernays 1934, 265], was in Bernays' Kontext an dieser Stelle eine Beschränkung auf Formeln der Logik erster Stufe beinhaltet. Bernays legt dann in §7 *Die rekursiven Definitionen* dar, daß die Peanoschen Axiome für 0 und Nachfolger „keineswegs ausreich[en], um die üblichen Begriffsbildungen der Zahlentheorie zur Darstellung zu bringen" [Hilbert & Bernays 1934, 286]. Es folgt eine Zwischenstation, wo die Rekursionsgleichungen für die Addition zu den Peano-Axiomen mit erststufiger Induktion hinzugenommen werden und das Ergebnis von Presburger zur Vollständigkeit dieser Theorie ausgeführt wird. Mit dem Axiomensystem (*Z*) haben wir schließlich die moderne Darstellung der erststufigen Peano-Arithmetik vor uns, wie sie — unter Auslassung der

beiden ersten Gleichheitsaxiome — heute allgemein bekannt ist [Hilbert & Bernays 1934, 371]:

$$
\left\{
\begin{aligned}
& a = a, \\
& a = b \rightarrow (A(a) \rightarrow A(b)), \\
& a' \neq 0, \\
& a + 0 = a, \\
& a + b' = (a + b)', \\
& a \cdot 0 = 0, \\
& a \cdot b' = a \cdot b + a, \\
& A(0) \,\&\, (x)\,(A(x) \rightarrow A(x')) \rightarrow A(a).
\end{aligned}
\right. \qquad (Z)
$$

Zu dieser einerseits im Induktionsschema gegenüber der Originalformulierung Peanos beschränkten, andererseits um die Addition und Multiplikation erweiterten Axiomatisierung bemerkt Bernays:

> Außerdem hat [die in diesem Kapitel diskutierte Methode für Widerspruchsfreiheitsbeweise] auch den Nachweis ermöglicht, daß das Axiomensystem Peanos bei Zugrundelegung des Prädikatenkalkuls und der Gleichheitsaxiome noch nicht zum Aufbau der Zahlentheorie ausreicht, daß vielmehr die Hinzufügung der Rekursionsgleichungen für die Addition und Multiplikation zu diesem Axiomensystem eine wesentliche Erweiterung bedeutet, durch welche erst der Reichtum der zahlentheoretischen Beziehungen zustande kommt. [Hilbert & Bernays 1934, 373]

Wir können nicht sicher sein, ob Bernays der erste war, der die erststufige Peano-Arithmetik unter Einschluß der definierenden Axiome von Addition und Multiplikation herausgearbeitet hat.[37] Die Geschichte der Rekursion ist lang und verwickelt und wird von von Plato geschildert [von Plato 2017]. Man kann dabei z. B. Skolem herausstellen, der bedeutende Beiträge zur Entwicklung der elementaren Rekursionstheorie geliefert hatte und zudem als entschiedener

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37. Von Plato hat im Bernays-Nachlaß, der an der ETH Zürich aufbewahrt wird, ein stenographisches Manuskript zu Bernays' Vortrag *Die Rekursion als Grundlage der Zahlentheorie*, den er am 21. Februar 1928 in Göttingen gehalten hat, ausfindig machen können. Die Transkription dieses Manuskripts, die mir von Maria Hämeen-Anttila und Jan von Plato freundlicherweise angeboten wurde, zeigt, daß sich Bernays schon zu dieser Zeit eingehend mit der hier zur Diskussion stehenden Fragestellung beschäftigte, auch wenn er damals — aus der Perspektive der Rekursion heraus — noch nicht die in *Z* zusammengestellten Axiome behandelte.

Verfechter der (ausschließlichen) Verwendung von Logik erster Stufe in der Formalisierung der Mathematik gilt. Aber seine Arbeiten zur Rekursivität von Addition und Multiplikation schließen nicht die Induktion als formales Axiom bzw. Axiomenschema ein. Die (erststufige) Peano-Arithmetik ist in der Form, wie wir sie heute kennen, allem Anschein nach von Bernays in seinem System *Z* zum ersten Mal niedergelegt worden.

# 11 Konklusion

Zusammenfassend würden wir die Situation bezüglich der historischen Beiträge zur Axiomatisierung der Arithmetik wie folgt beschreiben:

1. DEDEKIND hatte von einem mathematischen Standpunkt aus die charakterisierenden Eigenschaften der natürlichen Zahlen am besten erfaßt; seine Behandlung leidet aber darunter, daß er sie im Rahmen eines logizistischen Programms durchführt, die diese Eigenschaften nicht axiomatisch an den Anfang der Theorie stellt, sondern sie als *Sätze* aus einer übergeordneten (und später als problematisch erkannten) Mengenlehre, die er als einen Teil der Logik auffaßt, herzuleiten versucht.

2. PEANO erreicht zwar nicht die mathematische Tiefe von Dedekind bei der Behandlung der Arithmetik. Die Isolierung der charakterisierenden Eigenschaften der natürlichen Zahlen *als Axiome* stellt aber einen konzeptionellen Gewinn dar, der allerdings erst durch die weitere Entwicklung der mathematischen Logik erkennbar wurde.

3. HILBERT hatte richtig erkannt, daß eine logizistische Begründung der arithmetischen Axiome letztlich unergiebig ist, womit neben dem Ansatz von Dedekind auch die Ansätze von Frege und Russell zum Scheitern verurteilt sind. Das liegt nicht zuletzt daran, daß sich die Mengenlehre nicht, wie von Dedekind und anderen vorausgesetzt, als Teil der Logik erweisen ließ. Vor diesem Hintergrund läßt sich — in Einklang mit Hilberts allgemein propagierter *axiomatischen Methode* — der Status der Peano-Axiome als *nicht-logische Axiome* erkennen.

4. BERNAYS, im Wissen um die den logischen Formalismen durch die Gödelschen Resultate gesetzten Grenzen, arbeitete schließlich die *erststufige* Peano-Arithmetik heraus, wie wir sie heute kennen und schätzen.

## Danksagung

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# Peano on Symbolization, Design Principles for Notations, and the Dot Notation

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**Résumé :** Peano a été l'une des forces motrices dans le développement du formalisme mathématique actuel. Dans cet article, nous étudions son approche particulière de la conception notationnelle et présentons quelques caractéristiques originales de ses notations. Pour motiver l'approche de Peano, nous présentons d'abord sa vision de la logique comme méthode d'analyse et son désir d'un symbolisme rigoureux et concis pour représenter les idées mathématiques. Sur la base à la fois de sa pratique et de ses réflexions explicites sur les notations, nous discutons des principes qui ont guidé Peano dans l'introduction de nouveaux symboles, le choix des caractères et la mise en forme des formules. Enfin, nous examinons de plus près, d'un point de vue systématique et historique, l'une des innovations les plus marquantes de Peano, à savoir l'usage de points pour regrouper des sous-formules.

**Abstract:** Peano was one of the driving forces behind the development of the current mathematical formalism. In this paper, we study his particular approach to notational design and present some original features of his notations. To explain the motivations underlying Peano's approach, we first present his view of logic as a method of analysis and his desire for a rigorous and concise symbolism to represent mathematical ideas. On the basis of both his practice and his explicit reflections on notations, we discuss the principles that guided Peano's introduction of new symbols, the choice of characters, and the layout of formulas. Finally, we take a closer look, from a systematic and historical perspective, at one of Peano's most striking innovations, his use of dots for the grouping of subformulas.

# 1   Introduction

One of the concerns of philosophers of mathematics is to clarify the principles and methods that drive the development of mathematics. In this paper, we shall take a closer look at Giuseppe Peano's (1858-1932) general views about logic as a method of analysis of mathematical ideas and his more practical concerns regarding the presentation of the results of such analyses.[1]  As we shall see, these notions are subtly intertwined and motivated by his general aims of striving for rigor and conciseness.

   Like other mathematicians and logicians in the 19th century, Peano attributed the lack of satisfying solutions to many questions in the foundations of mathematics to the ambiguities of ordinary language [Peano 1889*a*, III]. In the use of symbolic languages to represent and analyze mathematical ideas and their logical relations, Peano envisaged a way of avoiding such ambiguities.  However, Peano also realized that certain restrictions had to be imposed on these symbolisms.  For example: to avoid ambiguities, each symbol should have a unique and precise meaning; to avoid errors, the symbols themselves, although arbitrary in principle, should be such that the cognitive effort necessary for their use is reduced to a minimum.  Accordingly, Peano considered the development of an appropriate symbolic language, which he called "symbolic writing" (*scrittura simbolica*) or "ideography" [Peano 1896-1897, 202], to be a crucial task for the advancement of mathematics.  As a consequence, in addition to formulating his famous axiomatization of arithmetic, Peano also originated many innovations in mathematical symbolism, including the dot notation in logic.

   The early development of Peano's logical notation can be easily retraced by considering his publications from 1887 to 1889.[2]  Before 1888, Peano's publications (e.g., [Genocchi 1884] and [Peano 1887]) do not contain any specific notations for logic.  In the latter, Grassmann is mentioned in the Preface, but no logicians are.  Logical notation appears for the first time in Peano's *Calcolo Geometrico secondo l'Ausdehnungslehre di H. Grassmann* [Peano 1888], whose preface is dated February 1, 1888, and which begins with a short chapter on "The operations of deductive logic", based on Schröder's *Der Operationskreis des Logikkalkuls* [Schröder 1877]. However, Peano replaces all of Schröder's symbols "in order to forestall any possible confusion between the symbols of logic and those of mathematics" [Peano 1888, X] (quoted from [Peano 2000, xiv]).  A year later, Peano published his famous work on arithmetic, *Arithmetices Principia nova methodo exposita* [Peano 1889*a*],

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   1. For background on Peano's life and works, see [Kennedy 2002]; for discussions of his philosophy and works, see [Kennedy 1963] and [Skof 2011]. In the following text, all translations are by the author (DS) unless a reference to a published translation is given.

   2. For a more detailed discussion, including the background of Peano's development, see [Bottazzini 1985].

in Latin, which begins with a chapter on logical notations in which he also introduces the difference between set membership (symbolized by "$\varepsilon$") and inclusion (replacing the symbol "$<$" that he used earlier with "$\supset$"). Again, Peano also replaces some of the logical symbols, but, more importantly, he introduces the dot notation for the grouping of subexpressions. From then on, this notation was employed in most of Peano's publications, beginning with *I Principii di Geometria logicamente esposti* [Peano 1889b] published later in the same year, as well as articles dedicated explicitly to mathematical logic, such as [Peano 1891b] and [Peano 1891a], and the various editions of the *Formulario* [Peano 1895a, 1897, 1901, 1903, 1905].

How these historical developments are intertwined with Peano's general views about methodology in mathematics is the main topic of this paper. In the following, we begin by discussing Peano's general views on symbolization and his view of logic as a method of analysis (Section 2). In Section 3, we relate these views to Peano's considerations for the design of notations. In particular, we present in detail the principles that guide the introduction of new symbols, the choice of characters, and the layout of mathematical formulas. In the third part of the paper (Section 4), Peano's use of dots for the grouping of subformulas is explained and discussed in the context of its historical development. This particular notation is one of the most striking of Peano's innovations and has been widely popularized by its use in Whitehead and Russell's *Principia Mathematica* [Whitehead & Russell 1910-1913], but it has hardly received any attention in the literature.[3]

# 2   Logic as method of analysis

## 2.1   Peano and Frege on logic

Let us begin by comparing and contrasting Peano's general attitude toward logic with Frege's, given that the latter has been studied extensively and is thus widely known. Both share the desire to secure rigorous reasoning in mathematics with the use of a symbolic language with clearly defined, unique meanings [Peano 1890a, 186]. However, their conceptions of rigor differ with regard to the level of explicitness of the analysis of logical reasoning. Frege, on the one hand, wanted to avoid any appeal to intuition in mathematical inferences and thus emphasized his use of formal rules of inference. On the other hand, possibly due to the fact that his main influences in logic came from the algebraic tradition of Boole and Schröder, Peano's paradigm of deduction was that of reasoning with algebraic equations [Peano 1889a, III] and [Peano 1889b, 28–29]. His lack of explicit inference rules was criticized by van Heijenoort as "a grave defect" [van Heijenoort 1967, 84]. In practice,

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3. For example, a discussion of the dot notation is conspicuously missing in [Kennedy 2002].

however, Peano's derivations can be construed formally as being based on instances of axioms, the substitution of equalities, and *modus ponens* [von Plato 2017, 55–56]. In short, Peano's system is not a "formal system" in the modern sense, i.e., with a recursively defined language and explicit rules of inference, but, according to von Plato, it could be fairly straightforwardly constructed as one.[4]

Another aspect in which Frege and Peano differ is their attitude toward an investigation into the fundamental principles of logic. While Frege put his theory on a firm axiomatic foundation, Peano did not, although he had done so for arithmetic and geometry, and remarked that "it would be an interesting study" [Peano 1889*b*, 29]. Unlike Boole and Schröder, both Frege and Peano intended their logical formalisms to be applied to mathematics and not be used in isolation, merely for the efficient solution of logical problems. Peano writes:

> I understand how important theoretical studies of logic are; but, given the immensity of such studies, I prefer directing my forces toward application. [Letter from Peano to Couturat, 1 June 1899] (reprinted in [Roero 2011, 87])

Nevertheless, with regard to the aim of applying logic to mathematics, Frege and Peano differed: for Frege, it was a theoretical exercise aimed at clarifying concepts and securing the foundations of mathematics; for Peano, it was a practical matter of actually doing mathematics in a new way. Because of this emphasis on practical use, Peano concentrated his efforts on developing a convenient formalism for the analysis and concise representation of mathematical ideas.

## 2.2   The *Formulario* project and concise notations

Soon after completing his axiomatizations and symbolic presentations of arithmetic [Peano 1889*a*] and geometry [Peano 1889*b*], Peano envisaged an impressive collaborative project, aimed at publishing a collection of important mathematical results expressed in a symbolic language. The first edition of the *Formulaire de mathématiques*, or *Formulario Mathematico*, as it was later called, appeared in 1895; an Introduction, in which Peano presented his logical notation, had already been published one year earlier [Peano 1894]. Four different editions of the *Formulario* were subsequently published in 1897, 1901, 1903, and 1905, each of which was the result of substantial revisions of the one preceding it. Peano also showed great historical awareness by often listing a theorem together with a reference to where it first occurred. The idea for this project is put forward in print for the first time in 1891 as the concluding note of a paper on the concept of number. Before ending the paper by inviting

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4. Deviating from the modern usage, we shall thus refer to Peano's symbolic language as a formalization.

suggestions for theorems to be included in the collection, Peano motivates the project as follows:

> It would also be very useful to collect all the known propositions referring to certain parts of mathematics, and to publish these collections. Limiting ourselves to arithmetic, I do not believe there would be any difficulty in expressing them in logical symbols. Then, besides acquiring precision, they would also be concise, so much so, probably, that the propositions referring to certain subjects in mathematics could be contained in a number of pages not greater than that required for the bibliography. [Peano 1891*c*]; [Peano 1957-1959, III, 109] (quoted from [Kennedy 2002, 63])

Given the sheer volume of the project, a concise form of representation was indispensable. Thus, while Peano writes that "the fundamental utility of the logical symbols is rigor and precision" [Peano 1908, X], he also emphasizes the importance of symbolization for reducing the length of presentations, because in some cases they would be impossible otherwise:

> It turns out that symbolic writing is about ten times shorter than in ordinary language. A publication of the ample present *Formulario* in ordinary language would be almost impossible in practice, as would be the publication of logarithmic tables in ordinary language or using Roman numerals. [Peano 1908, IX][5]

We note that, for Peano, one of the main practical requirements for the design of a notation is the reduction of the length of individual formulas, rather than the number of different signs that are employed. These two desiderata are frequently in tension with each other, as the comparison between binary and decimal place-value notations illustrates: the former uses only two signs instead of ten but results in longer expressions.[6] Given the aim of reducing the length of expressions, Peano's interest in reducing the use of parentheses should not come as a big surprise. We shall return to this in Section 4, when discussing the development of Peano's dot notation.

## 2.3 Formalization as method of analysis

It is clear from the announcement of the *Formulario* quoted above that the use of a symbolism was an integral part of the project from the beginning, since it allows for both precision and conciseness. Moreover, the process of formalization itself is a method of conceptual analysis that begins with the

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5. In fact, Roman numerals for natural numbers (without subtractive notation) are on average 2.6 times longer than Indo-Arabic numerals [Schlimm & Neth 2008, 2101].

6. E.g., "10010011" vs. "147".

following two steps:[7] (1) The identification of the fundamental mathematical ideas, and (2) the representation of these ideas by primitive signs of the symbolism. The first step requires a precise and unambiguous identification of the underlying ideas:

> The reduction of a new theory into symbols requires a profound analysis of the ideas that occur in this branch. Imprecise ideas cannot be represented by symbols. [Peano 1895*a*, iv] (quoted from [Kennedy 2002, 67])

As a consequence, the more thoroughly ideas have been analyzed and expressed in ordinary language (Step 1), the easier it is to translate them into a symbolism (Step 2). Peano writes:

> The transformation into symbols of propositions and proofs expressed in the ordinary form [...] is a very easy thing when treating propositions of the more accurate authors, who have already analyzed their ideas. It is enough to substitute, in the works of these authors, for the words of ordinary language, their equivalent symbols. Other authors present greater difficulty. For them one must completely analyze their ideas and then translate into symbols. Not rarely it is the case that a pompously stated proposition is only a logical identity or a preceding proposition, or a form without substance. [Peano 1891*c*]; [Peano 1957-1959, III, 109] (quoted from [Kennedy 2002, 63])

However, it is not only the use of ambiguous or pompous language that obscures the ideas to be uncovered by logical analysis, but also the ideas' fundamental character and the fact that they do not necessarily correspond to basic expressions in ordinary language. As Peano explains in a textbook of arithmetic and algebra written for use in secondary schools, the logical symbols "$\supset$", "$\varepsilon$", and "$\exists$", which stand for derivation, membership, and existence,

> represent simple ideas and it is precisely their simplicity that prevented them for a long time to be isolated and stripped from the complexity with which they present themselves both in ordinary language and the language of science. [Peano 1902, III]

The surface structure of language can mislead even skilled logicians, such as Schröder. His use of a single sign to denote the ideas represented by "$\supset$" and "$\varepsilon$" is criticized by Peano as a major flaw that prevents Schröder's symbolism from being a proper ideography [Peano 1898*a*, 97–98].

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7. The view of logic as analysis was also clearly formulated by Peirce, who wrote: "In logic, our great object is to analyze all the operations of reason and reduce them to their ultimate elements; and to make a calculus of reasoning is a subsidiary object" [Peirce 1880, 21]. This work is referred to in [Peano 1889*a*, IV]. However, in contrast to Peano, Peirce was also interested in theoretical investigations of logic itself.

That Peano indeed considered formalization as a method of analysis can also be seen from the subtitle of Peano's work on the axiomatization of natural numbers, which reads "nova methodo exposita" ("presented by a new method", [van Heijenoort 1967, 83]), and in [Peano 1896-1897, 202], where he speaks of the "analytic instrument"[8] that has been applied by himself and others.

## 2.4   Formalization as a method for checking an analysis

The utility of a symbolic language is not exhausted once mathematical ideas are expressed in it; the formalization itself can be used to check the adequacy of the analysis. Thus, a third step is added to the method of analysis: (3) Further study of the symbolic expressions to determine consequences and possible simplifications. For this, Peano suggests the following:

> After having written a formula in symbols, it is useful to apply several logical transformations to it. It can thus be seen if it is possible to reduce it to a simpler form, and one can easily recognize if the formula has not been well written. This is because the notations of logic are not just a shorthand way of writing mathematical propositions; they are a powerful instrument for analyzing propositions and theories. [Peano 1895*a*, vi] (adapted from [Kennedy 2002, 68])

With regard to the analysis of theories, i.e., sets of propositions and not just individual ones, a formalization can also be used to impose a logical order on the propositions (i.e., present some as axioms and others as theorems) and to check the definitions. Peano writes:

> It is always difficult to order the propositions of a theory. One can order them according to the signs employed for writing them. This rule yields, in general, good results. [Peano 1895*a*, vi][9]

Once a theory is symbolized, i.e., the primitive ideas are determined and expressed by primitive symbols, the propositions ordered, and symbols for complex ideas introduced by definitions, one can easily verify that all symbols used in the definiens have been properly introduced. This can be done "in a mechanical way", because only the symbols need to be considered and no recourse to the original ideas is necessary. Peano explains:

> The ideography makes evident, in a mechanical way, that definitions are correct and that demonstrations are rigorous.

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8. Kennedy translates "strumento analitico" as "analytic method" [Kennedy 1973, 190].

9. See [Peano 1897, 28] for an application of this suggestion; see [Cantù 2014] for a discussion of Peano's views on the order of the primitive ideas of a science.

> For example, it is a fundamental rule of definitions, that the
> defined symbol must be expressed by previous symbols. Thus, if
> we consider for example the definition of prime number on p. 58,
> we see that it is expressed by the symbols $-$, $1$, $+$, $\times$, $N_i$, which
> were introduced on pages 10, 29, 29, 32, 37, and that several of
> among these symbols are defined by previous symbols, and so
> on, until we reach a decomposition into primitive ideas that are
> determined by primitive propositions. [Peano 1908, X]

## 2.5   Depth, uniqueness, and arbitrariness of analysis

### 2.5.1   Depth and uniqueness of analysis

So far, we have learned that formalization yields a "profound" and "complete"
analysis of mathematical ideas, but how do we know when this process is
complete? In the following passage from a letter to Felix Klein, Peano explains
the aim of mathematical logic and mentions an additional goal with regard to
the outcome of logical analysis:

> It is the aim of mathematical logic to analyze the ideas and forms
> of reasoning that occur especially in the mathematical sciences.
> The analysis of the ideas allows to find the fundamental ideas,
> with which all other ideas are expressed, and the relations between
> various ideas, i.e., the logical identities, that are those forms of
> reasoning. This analysis also leads us to indicate the simplest
> ideas with conventional symbols, which, when appropriately
> combined, represent composite ideas. This yields a symbolism or
> symbolic writing that represents all propositions with the smallest
> number of signs. [Letter from Peano to Felix Klein, 19 September
> 1894] (reprinted in [Peano 1990, 124])

Thus, a successful analysis yields a minimal set of fundamental, simple ideas
that are represented by symbols, such that through the combinations of these
symbols all complex ideas can also be expressed.[10] A small number of symbols
is therefore a hallmark of a formalism, because it indicates the depth of the
analysis. Indeed, at the beginning of many of his publications, Peano proudly
emphasizes the small number of primitive symbols being used and, in his
discussion of Frege's work, he takes the diminution of the number of primitive
symbols as indicative of a more thorough and deeper analysis. In his review
of the first volume of Frege's *Grundgesetze* [Peano 1895b], Peano compares
his own with Frege's formalism, acknowledging that many of their ideas are
analogous. However, among other criticisms, Peano points out the lack of a

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10. See also [Peano 1894, 173] for a similar formulation; in [Peano 1890b], he
sets out to "find the minimum of signs and conventions necessary to express the
25 propositions of the Fifth Book of Euclid".

symbol for set membership in Frege's *Begriffsschrift* as a defect and, since Peano's notation is allegedly built on fewer primitives than Frege's, Peano regards his own analysis as "more penetrating" [Dudman 1971, 30].[11]

With regard to the outcome of different symbolic analyses, Peano makes the following general remark:

> Now if, independently of each other, there arise two systems both capable of representing and analysing the propositions of a theory, one will have to be able to present an absolute formal difference between them; but there will have to exist at bottom a substantial analogy; and if the two systems are equally developed, the relation between them will have to be that of identity. For mathematical logic does not consist of a set of arbitrary conventions, variable according to the author's fancy. It consists rather of the analysis of ideas and propositions into those that are primitive and those that are derivative. And this analysis is unique. [Peano 1895*b*, 123] (quoted from [Dudman 1971, 28])

This claim about the uniqueness of logical analysis, which is repeated again at the end of Peano's review, fits together with the earlier claim about the minimality of the set of simple ideas. In what sense, however, different analyses could result in unique, "substantially analogous" systems is left unclear. Based on the minimality and uniqueness claims, the passages quoted above could be interpreted as expressing some kind of realist view, according to which the structure of the symbolism mirrors the (true) logical structure of the ideas.

### 2.5.2   Arbitrariness of analysis

The realist interpretation of the representations of mathematical ideas and propositions offered at the end of the previous paragraph is called into question by the intertranslatability of various logical connectives, which Peano discusses in the same text [Peano 1895*b*]. As he is well aware, in propositional logic either implication or disjunction can be taken as primitive (together with negation) and the other as defined. Moreover, in other places, Peano is quite explicit about the difficulties involved in determining which ideas and propositions should be taken as fundamental: he notes that the distinction between primitive and derived ideas is "somewhat arbitrary" or "a little bit arbitrary" on numerous occasions[12] and that "each author can begin with the group that they find most satisfying" [Peano 1898*a*, 100]. To choose

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11. Frege vehemently disagreed: "I do not regard the mere counting of primitive symbols as sufficient to substantiate a judgment about the profundity of analysis toward fundamentals" [Dudman 1971, 35]. See also Peano's review of Schröder's formalism, in which he points out that the latter is based on 15 primitive ideas, whereas his own system is built on 8 [Peano 1898*a*].

12. See, e.g., [Peano 1889*b*, 25], [Peano 1891*a*, 25], [Peano 1894, 50–51], and [Peano 1897, 27].

between alternative sets of primitives, Peano frequently invokes a notion of "simplicity".[13]    However, this notion is left unspecified, and he notes that "there is arbitrariness in the assessment of simplicity" [Peano 1894, 51].

On the first page of the Preface to the first edition of the *Formulario*, Peano states the independence of mathematics from particular representations even more forcefully:

> The notations are a bit arbitrary, but the propositions are absolute truths, independent of the notations used. [Peano 1895a, III]

On the basis of these considerations, Peano's attitude has been frequently characterized as "strictly instrumental" with regard to the role of logic [Segre 1994, 286] and "instrumentalist" with regard to notations [Bellucci, Moktefi *et al.* 2018, 3].

### 2.5.3   Possible resolution of the tension between uniqueness and arbitrariness

The tension between Peano's claims about the uniqueness of an analysis, which leads to a minimal set of primitives, and his conviction about a certain insurmountable arbitrariness regarding the choice of primitives can be resolved by taking a careful look at what Peano says in the following passage, in the context of whether "point" and "segment", or "point" and "ray", should be chosen as primitives in geometry (as we have seen above, an analogous situation arises in logic):

> It is clear that not all entities can be defined, but it is important in every science to reduce the number of the undefined entities to a minimum. [...] The reduction of the undefined entities to a minimal number can be somewhat arbitrary; so, if by means of $a$ and $b$ we can define $c$, and by means of $a$ and $c$ we can define $b$, our choice between $a$, $b$ and $a$, $c$ as an irreducible system remains arbitrary. [Peano 1889b, 25] ([Peano 1957-1959, II, 78])

Here, both minimality and arbitrariness are considered: the number of primitives should be minimal, but among the possible minimal sets, it is arbitrary which one is chosen. Moreover, if one of these sets is taken as primitive and the other entities are defined in terms of this set, and if the axioms are chosen appropriately, the same theorems will follow. The theories, then, understood as sets of theorems, are indeed identical and the analysis unique, as Peano claimed in his review of Frege's *Grundgesetze* (see the second quote in Section 2.5.1).[14]    This interpretation is also

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13. See, e.g., [Peano 1891a, 25], [Peano 1894, 50–51], and [Peano 1897, 27].

14. This interpretation is also consistent with Peano's remark on the identity of domains on the basis of them satisfying the same propositions [Kennedy 1973, 225].

compatible with all examples that Peano mentions in his discussions of arbitrary choices of primitives.[15]

A prominent case in which the minimality of the set of primitives is frequently given up is propositional logic itself. As Peirce noticed in the 1880s and Sheffer rediscovered some 30 years later [Sheffer 1913], a single binary connective (either Peirce's arrow or the Sheffer stroke) can be used to define all other propositional connectives. Accordingly, we would expect Peano to consider this analysis of propositional logic to be deeper and more profound than his own.[16] Although it is known that Sheffer visited Peano in 1911 and that they corresponded in 1921, I am not aware of any reactions by Peano to Sheffer's discovery.[17]

We have established so far that, given different minimal sets of primitives, Peano saw no theoretical reasons to prefer one over the others.[18] However, for any actual presentation, such a choice has to be made, and for this, practical reasons come into play, even ones that push toward giving up the minimality of the set of primitives.

### 2.5.4   Practical considerations against minimality

The demand for a notation to be concise was discussed in Section 2.2 in connection with Peano's *Formulario* project. This suggests adopting a set of primitives that is not minimal, as they allow for shorter expressions without having to define derived symbols. Other reasons given by Peano for dropping the requirement of minimality of the set of primitives in logic are related to the readability of formulas and their connection to expressions in ordinary language. For example, after noting that $a \supset b$ is equivalent to $a - b = \Lambda$, so that the sign "$\supset$" could be omitted from the list of primitives, Peano notes:

> We shall keep it, however, for greater variety and for analogy with the common form of expressing the thought. [Peano 1891*b*, 6], [Peano 1957-1959, II, 98] (quoted from [Kennedy 1973, 160])

In sum, it appears that Peano's criteria for deciding which ideas are to be taken as primitive are guided more by considerations of practicality and convenience of use than by some kind of epistemological or metaphysical considerations.

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15. See the references in Footnote 12.

16. Such was indeed the reaction of Russell; see the Introduction to the second edition of *Principia Mathematica* [Whitehead & Russell 1925].

17. I thank Juliet Floyd for this information. The extant correspondence between Peano and Sheffer, which is held at the Harvard University Archives, does not mention Sheffer's innovation.

18. This is in contrast to Frege, who invoked a notion of simplicity of content to choose the conditional as a primitive connective in his system; see [Bellucci, Moktefi *et al.* 2018, 6–7] and [Schlimm 2018, 71–73].

# 3   Design principles for characters and layout

After having discussed Peano's general outlook on logic and formalization, we now take a closer look at his approach to mathematical notations. We have seen above that, for Peano, a symbolism that represents the result of an analysis should represent the basic concepts of a domain of inquiry by individual symbols and more complex concepts by symbols that are defined from them. We have also seen that there are some difficulties in identifying the primitive concepts, but that is not our primary concern here. Rather, it is the question of how to represent them, once we have settled on them.

In general, we can consider a notation to consist of a set of characters (also referred to as signs or symbols)[19], structural rules that determine well-formed expressions, and an interpretation that assigns meanings to (at least some of) the characters and expressions. Although the choice of characters is arbitrary from a theoretical point of view, Peano did formulate some design principles explicitly, while others can be extrapolated from his practice.

As we shall see, the general aims of rigor and conciseness that motivated Peano's use of a symbolic language in the first place also underlie his choice of characters (Section 3.1). In addition, Peano tried to design his notations in such a way that they reduce the cognitive effort necessary for their use, e.g., by linking their shapes to their meanings and by using a layout that facilitates their readability (Section 3.2). Presumably, this would reduce errors and mistakes when using the notation. Finally, Peano also considered factors that influence the horizontal and vertical arrangement of the notation on the printed page (Section 3.3).

## 3.1   Conciseness and reduction of ambiguity

### 3.1.1   Uniqueness of meanings and new symbols to avoid ambiguities

In [Peano 1888], where he presents the logical calculus of Schröder, Peano replaces each of the five basic symbols employed by Schröder with his own:

> It seemed useful to substitute the symbols $\cap$, $\cup$, $-$, $\bigcirc$, $\bullet$ for the logical symbols $\times$, $+$, $A_i$, 0, 1 used by Schröder, in order to forestall any possible confusion between the symbols of logic and those of mathematics (a thing otherwise advised by Schröder himself). [Peano 1888, X] (quoted from [Peano 2000, xiv])

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19. To be clear, we mean here character types or symbol types, not tokens, but this distinction plays no particular role in our discussion.

Because Peano intends to use his logical symbolism in conjunction with the usual mathematical notations, he must introduce new characters, which are not already used in other mathematical domains, to avoid ambiguities. The use of standard arithmetical symbols in logic does not pose a problem for Schröder, since he presents logic as a self-standing theory that is not used in conjunction with other theories, such as ordinary arithmetic.

Peano took the symbols "∩" and "∪" possibly from Grassmann [1844, 5], who uses them for his more abstract theory of magnitudes; the circle and filled circle do not have any obvious previous uses, but the symbol for negation (or set complement) looks very similar to the minus sign. This seems to have bothered Peano as he later recommends the following:

> In the manuscript, it is best to give the sign for 'not' the form ∼, so as not to confuse it with − (minus). [Peano 1895$a$, VI] (quoted from [Kennedy 2002, 68])

This comment comes from the beginning of the first edition of the *Formulario*, where Peano included a list of remarks and rules to facilitate future collaborations. The above considerations about the introduction of new symbols are encapsulated for the general case in the third item on the list:

> Every time a new theory is translated into symbols, new signs will be introduced to indicate the new ideas, or the new combinations of preceding ideas, that are met in this theory. [Peano 1895$a$, III–IV] (quoted from [Kennedy 2002, 67])[20]

Behind this principle lies the more fundamental principle that each symbol that stands for an idea should have only a single fixed meaning. An example for application of this design principle is Peano's preference for writing the decimal point:

> I prefer the English notation $1 \cdot 23$ and $\cdot 45$ to $1, 23$ and $0, 45$ for writing decimal fractions, because the comma has too many meanings. [Peano 1916] (translated from [Peano 1957-1959, I, 448])

An explicit formulation of the principle that each symbol must be assigned a unique meaning is given in the following passage, where Peano extends it also to those symbols that are used for grouping subexpressions, such as parentheses:

> If one wants mathematical formulas to say everything without the need of verbal additions, one cannot give two values to the same sign. The assignment of another function to parentheses, other than the grouping of more signs, is like trying to make a decimal arithmetic in which the numbers 6 and 9 are represented by the same sign. [Peano 1912, 377]

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20. See also [Peano 1895$b$] for a similar formulation [Dudman 1971, 28].

Despite his convictions, Peano himself did not always adhere to this principle. For example, he interpreted the symbol "⊃" as both deduction and material conditional, for which he was criticized by Frege [1896, 372–374] (quoted from [Frege 1969, 8–9]).

### 3.1.2   Simplifications of frequently used expressions

Peano not only uses symbols to represent the primitive ideas of a discipline, but also allows for the introduction of new symbols within a theory through definitions. However, he suggests to restrict such additions to the following situations:

> A new notation will be introduced by means of a definition when it brings a notable simplification. A new notation will not be formed when the same ideas can already be simply represented by the preceding notations. [...]
>
> A new notation will be introduced only if the simplification that it brings will be used in the propositions following. Definitions alone do not make a theory. [Peano 1895$a$, IV] (quoted from [Kennedy 2002, 68])

An early example for the application of this principle is Peano's introduction of a symbol to express "every $A$ is $B$" in [Peano 1888, 3]. After noting that this can be expressed with the primitive symbols of his theory as $A\overline{B} = \bigcirc$, he continues:

> Even though the preceding proposition for indicating that proposition is already quite simple, for greater convenience we will nevertheless also indicate it by the expression $A < B$ or $B > A$ [...]. [Peano 1888, 3] (quoted from [Peano 2000, 2])

This symbol is indeed used very frequently in the further development and it considerably shortens the expression introducing only one symbol in addition to the variables, instead of three ("$\overline{\phantom{x}}$", "$=$", "$\bigcirc$"). The notion of simplicity appealed to in this principle thus refers to reducing the length of expressions.[21]

Another example for the application of this principle is Peano's introduction of expressions containing an existential quantifier "Ǝ$a$", which he motivates as follows:

> The proposition $a \sim = \Lambda$, where $a$ is a class, thus signifies "some $a$ exist". Since this relation occurs rather often, some workers in this field hold it useful to indicate it by a single notation, instead of the group $\sim = \Lambda$. [Peano 1896-1897] (quoted from [Kennedy 1973, 203])

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21. That expressions can be expressed with fewer symbols is one of Peano's two meanings of "simplicity", according to [Bellucci, Moktefi *et al.* 2018, 3]; the other concerns the number of primitive logical symbols used in a theory.

Again, a single symbol replaces three, and Peano explicitly mentions the frequent use of this expression. We can thus summarize Peano's principle as: symbols that stand for derived ideas or relations should shorten expressions and be used frequently. The particular shape of the new symbol, "Ǝ", was chosen on the basis of semantical considerations, to which we turn next.

## 3.2   Semantical considerations

### 3.2.1   Iconicity and mnemonics

Even when taking only a cursory glance at Peano's works, one cannot miss his use of mnemonics when introducing new characters, although he does not discuss it as an explicit principle. To illustrate this practice, let us look at one of the most famous symbols introduced by Peano, the "horseshoe". The symbol for "proves" (or deduces) and "contains" was changed several times in Peano's writings. With an implicit analogy to the less-than relation in algebra, it was first introduced as $a < b$ in [Peano 1888, 3] for the calculus of classes. In later writings, Peano formulated this analogy explicitly:

> Segner in 1740 and Lambert in 1765 used $a < b$ and $a > b$, respectively; because the relation corresponds to the sign $<$ or $>$, or better to $\leqq$ or $\overline{\gtrless}$, of algebra, depending on whether with the class one considers the number of individuals that constitute it, or the number of ideas that determine it. [Peano 1900, 10]

Thus, Peano deliberately chose a symbol that bears some connection to the represented relation. This connection, whereby the intended interpretation is suggested by the particular shape of the symbol, is often called "iconic", following terminology introduced by [Peirce 1885, 181].[22]

   Possibly because the less-than symbol is also used in algebra, thus violating the principle that new symbols should be introduced for new ideas (Section 3.1.1), Peano quickly replaced it with an inverted capital letter "C" a year later, writing $a \supset b$ [Peano 1889a, viii] and interpreting it as a relation both between classes and propositions.[23] It is described as "the reversed initial letter of the word *contains* [*contiene*]" in [Peano 1889b, 6],[24] whereas the symbol

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22. Peano himself called such notations "figurative": referring to $a^{-} b$, $a^{\vdash} b$, $a^{\dashv} b$, $a^{\dashv\vdash} b$, to indicate whether the endpoints of an interval $a$ and $b$ are excluded or included, he notes that "this figurative notation is very convenient and fairly widespread" [Peano 1916-1917, 455]. That the use of iconic mathematical symbols has indeed cognitive advantages has recently been shown in [Wege, Batchelor *et al.* 2020].

23. In the original publication the symbol C does not appear aligned on the baseline as the text, but somewhat lower, as in: $a \supset b$.

24. Similarly, with the French word "*contient*" in [Peano 1890a, 183]; [Peano 1900, 316] refers to Gergonne for using "C" as the initial of "contains"; Quine [1987, 5] refers to [Gergonne 1816-1817].

"C" is described as the first letter of the word "consequence [*consequenza*]" in [Peano 1891*b*, 100, footnote 5]. In [Peano 1894], the symbol remains an inversed capital "C", but in a smaller font, such that it appears as $a \supset b$. Finally, the "$\supset$" symbol appears in Peano's writings in 1898, e.g., [Peano 1898*a*], and is described as "a deformation of Ɔ, the reversed first letter of the word "contains" [*contient*]" in [Peano 1900, 10]. Thus, Peano used the first letter of a word that expresses the meaning of the relation as the sign that represents it. The reversal was probably done to avoid confusing the symbol with the name of a variable.

Other examples of Peano's use of mnemonics to guide the choice of symbols are: "P" for propositions, "Th" for theorems, "M" for maximum, and "D" for divides [Peano 1889*a*, VI]. In some cases Peano chose the first letter of a word in a different language than Italian or French, such as "V" for *verum* (the Latin word for true),[25] or "$\varepsilon$" and "$\iota$", which are the first letters of the Greek words for "is" (ἐστί) and "equal" (ἴσος) [Peano 1894, 7 and 38].

### 3.2.2   Inverted symbols for inverse relations and operations

Another principle for the choice of characters that Peano frequently employs, and that was already hinted at above, is the introduction of an inverted symbol to express the contrary or inverse of the meaning of the original symbol. This practice is referred to by Quine [1987, 18] as "Peano's strategy of notational inversion".

For example, after introducing the symbol "V" for *verum*, Peano replaced his earlier symbol for absurdity, "○", with "Λ" [Peano 1889*a*, VIII]. What is unusual in this case is that the "V" itself is not used in the further development of the theory, thus violating the principle identified above, according to which only symbols that are actually used should be introduced (Section 3.1.2). The desire for providing a set of symmetric symbols is likely to have motivated him to do so. This becomes clear in a later publication, where, after listing the symbols $\varepsilon$, C, Ɔ, $=$, ∪, ∩, $-$, V, Λ, which allow for the expression of all logical relations, he remarks:

> The signs C and V are mentioned here for the sake of symmetry, but they have no practical utility. [Peano 1894, 7]

In [Peano 1891*b*, 159], Peano explicitly refuses the use of "V", explaining:

> We shall not introduce the sign V, which corresponds by duality to Λ, because we do not need it.

In other works again, the inverted "V" is introduced without even mentioning the upright letter at all; e.g., in [Peano 1889*b*, 6], where Peano simply explains that "Λ" is the first letter of the word *vero* ("true" in Italian), and in

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25. In works written in Italian or French, it is motivated by the words *vero* and *vrai*.

[Peano 1894, 7], where it is introduced as the first letter of the French word for "true", *vrai.*

Although Peano does not give reasons for his frequent choice of inverted symbols, their use arguably reduces the cognitive effort of learning the meaning of new symbols. For example, given the meanings of "M" and "D" as maximum and divisibility, the meanings of "Ɯ" and "Ɑ" as minimum and multiple can easily be inferred [Peano 1889$a$, VI].[26]

Another reason for simply inverting symbols lies in the fact that the printing types are readily available. For example, while Peano uses square brackets as "symbols for inversion", e.g., to write $[x\varepsilon]$ for class abstraction in [Peano 1889$a$, XIV], he changes this to $\overline{x\varepsilon}$ in [Peano 1894, 20] without giving any reasons, possibly to shorten the expression. But in the German translation of [Peano 1896-1897], which appeared as an appendix in [Genocchi 1899], we find the added footnote: "Instead of $\overline{x\varepsilon}p_x$ one can also write $x\, ɜ\, p_x$ for easier printing" [Peano 1990, 18].[27] Here, the inversion of symbols is explicitly motivated by typographical considerations, a topic we turn to next.

## 3.3   Horizontal and vertical arrangement

### 3.3.1   Symbol size and spacing for easier readability

In addition to the choice of characters themselves, Peano's concerns extended also to their arrangement on the page, e.g., their spacing. For the second edition of the *Formulario*, he explicitly designed his symbolism for easier readability:

> In providing this material we tried to combine the clarity of the formulas with the ease of composition. For example, we fixed the length of the signs
>
> $$= \quad > \quad \supset \quad + \quad - \quad \times \quad / \quad \wedge \quad \surd$$
>
> in proportion to the numbers
>
> $$10 \quad 10 \quad 10 \quad 8 \quad 8 \quad 6 \quad 6 \quad 4 \quad 4$$
>
> that measure them in typographical points; these dimensions help to naturally read the formulas according to the common conventions regarding the omission of parentheses. [Peano 1898$b$, 233]

What lies behind this remark is the idea that symbols that are closer together are more readily seen as belonging together, such that the spacing around a

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26. Other pairs of symbols that Peano uses are "f" and "Ⅎ" [Peano 1894, 27–29] and "↑" and "↓" [Peano 1894, 39–40]; the symbol for exponentiation "⌐" is described as "the reversed sign for radicals" [Peano 1905, 34].

27. On the use of overlining, see also [Peano 1900, 8].

symbol can support the correct interpretation of its binding strength. Recent work by Landy and Goldstone [2007] has empirically validated this claim with regard to the reading and writing of algebraic equations. Notice in the above quotation how the width of the symbols ("the length of the signs") correlates with their usual binding strength: the less space a symbol occupies, the stronger it binds.[28] In his later reflections on mathematical typography, Peano elaborates on how the spacing can support the correct reading of formulas:

> The spacing of the formulas does not present typographical difficulties; it can facilitate the reading. The formulas $a+b \times c$ and $a + b\times c$ suggest the readings $(a + b) \times c$ and $a + (b \times c)$, where the former is contrary to and the latter in conformity with the conventions in algebra. The spacing $a + b \times c$ has become standard in typography. The reading will be easier if the sign $\times$ is smaller than $+$. [Peano 1915, 403]

In the last remark, Peano not only suggests to tighten the spacing but to actually make the sign smaller to indicate a stronger binding, which accords with the common writing of $a \times b$ as $a \cdot b$ or simply $ab$. By carefully selecting the size of the characters and the spacing between them, Peano wanted to ensure that the way his formulas appeared on the paper would facilitate their readability.

### 3.3.2  Printing costs and typographical convenience

Due to his leading position in the *Formulario* project, Peano was more involved with the practical matters of printing than most other mathematicians. In particular, this included being concerned about the cost of publishing mathematical works. In general, whenever a notation requires types that are not readily available by the typesetter or yields expressions that exceed the height of a line, its production becomes more costly. Peano frequently alluded to the cost of printing when discussing notational design, e.g., noting that the typographical realization of the usual notation for fractions, where the numerator appears above a line and the denominator below it, as in "$\frac{a}{b}$", costs three times of that of writing them in a single line, as "$a/b$" [Peano 1912, 377].[29] Accordingly, Peano notes that from the second edition onward of the *Formulario*, he introduced only notations that could be printed within a single line.

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28. This idea is formulated somewhat cryptically in the introduction of the third edition of the *Formulario* as: "To make them stand out better, we will give the signs different dimensions, helping ourselves with typographic spaces" [Peano 1901, 3].

29. The production process and pricing is explained in [Peano 1915, 281]: multiple fractions in a line are five times as expensive as a single line, and with nested fractions the cost increase is "dizzying" (*vertiginosamente*).

In an article dedicated exclusively to typographical issues, Peano offers suggestions for writing formulas without extending the height of a line so as to keep the publishing costs down and to increase the readability of mathematical texts, such as: avoiding large parentheses, large integral and sum signs, stacked symbols such as "$\dot{x}$" and "$\binom{m}{n}$", and radical signs with a vinculum [Peano 1915]. Here is an example of how such considerations affected Peano's notations: Peano did not use overlining in his 1889 books to write the inverse of a function $f$ as $\overline{f}$, as Dedekind did in his axiomatic presentation of arithmetic that was published a year earlier in 1888, but rather square brackets, because of "typographical convenience" [Peano 1890$a$, 187].[30]

# 4 Peano's dot notation

We now turn to one of the most striking innovations in Peano's notation, the use of dots to indicate the grouping of subformulas. In the 1894 introduction to the *Formulario*, the dot notation is described as being "equivalent" to the use of parentheses and vincula. To illustrate this point, Peano presents the following three representations,

$$ab \, . \, cd : e \, . \, fg \, \therefore \, hk \, . \, l, \qquad \Big\{ [(ab)(cd)][e(fg)] \Big\}[(hk)l], \qquad \underline{ab \, cd \, efg \, hkl},$$

and justifies his choice of using the dots for the grouping of propositions by a brief remark that parentheses render formulas "very complicated" [Peano 1894, 11]. At other occasions, Peano notes that "a convenient system of dots" achieves the same as parentheses, but "with greater simplicity" [Peano 1891$b$, 155], and that "parentheses would be absolutely bulky and cumbersome [*absolument encombrantes*]" [Peano 1897, 22][31]. As we shall see presently, the notions of simplicity and convenience that Peano attributes to the dot notation are closely related to considerations about notations discussed earlier: conciseness (Section 2.2) and vertical arrangement (Section 3.3.2). Compared to those with parentheses, expressions that are grouped using dots are shorter, while at the same time requiring no extra vertical space, as vincula do.

Before addressing the development of the dot notation in Peano's writings in Section 4.2, let me first briefly explain how it works and use syntax trees to illustrate its relation to the usual linear notation.[32]

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30. See also the example discussed at the end of the section 3.2.2.

31. This is repeated in [Peano 1900, 1901].

32. Syntax trees can be seen as a canonical notation for propositional logic and were also employed in [Schlimm 2018] to shed light on Frege's *Begriffsschrift* notation.

## 4.1   Syntax trees and the dot notation

The main idea behind the dot notation is that, instead of *aggregating* elements that belong together by enclosing them within a pair of parentheses, groups of dots are used to *separate* two parts of an expression by marking the position at which the formula is divided. To understand the way in which a formula is partitioned by the dot notation, it is illustrative to consider the grouping of a string of symbols. The following are two examples discussed by Peano when introducing the dot notation for the first time [Peano 1889*a*, 104]. The groupings that are effected by parentheses in the expressions

$$(ab)(cd) \quad \text{and} \quad (((ab)(cd))((ef)(gh)))k$$

are represented in the dot notation by

$$ab \,.\, cd \qquad \text{and} \qquad ab \,.\, cd : ef \,.\, gh \,\therefore\, k$$

Notice that in the first example, all four parentheses are replaced by a single dot, while the 14 symbols for parentheses in the second example are replaced by four groups of dots, for a total of seven dots. This economy of symbols is the main reason explicitly stated by Peano for using dots instead of parentheses.[33] In order to deepen our intuitive understanding of the dot notation, it is instructive to look at the following syntax trees, which display the structure of the above groupings in a perspicuous fashion:
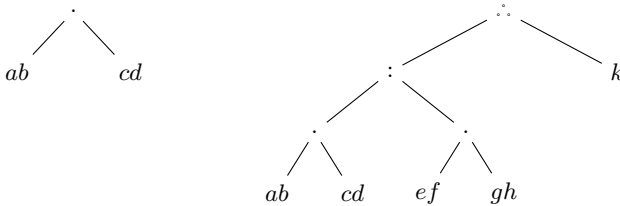


This representation illustrates how the number of dots in Peano's notation corresponds to the level in the syntax tree; more precisely, the number of dots of a node indicates (or is determined by) the length of the longest path from it to a leaf.

To employ the dot notation in logical formulas, we place a group of dots adjacent to a connective (to the left, right, or both) to separate a subformula from the rest of the expression. Consider, for example, the formula

$$(p \cup q) \supset ((p \cup (q \supset r)) \cap (p \cup r)) \tag{1}$$

---

33. These comparisons in terms of the number of symbols used raise the question of whether to count either the individual dots or each group of dots (e.g., . , : , ∴ , :: ) as symbols. We shall consider the latter as symbols because groups of dots are never broken up and always used as a single unit.

The use of syntax trees allows us to obtain the corresponding formula in the dot notation very easily. We first draw the syntax tree for the formula and label its edges as follows: if the longest path from the following node to a leaf has length $n$, mark the edge with a group of $n$ dots. For the formula shown above, this yields:



$$(2)$$
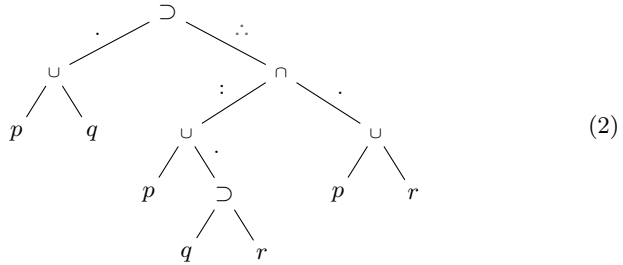
We can also think of arriving at these labels by starting from the leaves and labeling each edge with an increasing number of dots, while moving upward toward the root, starting with zero. If the edges below a node are labeled with different numbers of dots, say $n$ and $m$, then the edge immediately above this node is $\max(n, m) + 1$. In other words, the label of an edge above a connective is one more than the greatest label of the edges that are immediately below it. For example, since the edges that extend downward from the "∩" symbol are labeled with groups of 1 and 2 dots, the edge above it must be labeled with 3 dots. If a syntax tree is annotated in this way, the labels contain information about the nesting of the subformulas: we immediately notice that the connective that has the greatest label on one of its downward edges is the main connective; the same also holds for each subtree.

Finally, to represent a formula in the linear dot notation, we parse its syntax tree in the usual (infix) way and write the groups of dots before a connective, if they appear on the left downward edge of the corresponding connective, and after it, if they appear on the right downward edge. In our example, this yields:

$$p \cup q \mathbin{.} \supset \therefore p \cup \mathbin{.} q \supset r : \cap \mathbin{.} p \cup r \qquad (3)$$

When comparing this formula to its representation with parentheses (1), we see that a group of dots to the right of a connective corresponds to one or more opening parentheses and that a group to the left corresponds to one or more closing parentheses. This close connection between parentheses and the dot notation can also be illustrated by using labeled parentheses. Using a numeric label above a parenthesis to indicate its depth of nesting, we obtain for (1):

$$\overset{1}{(}\, p \cup q \,\overset{1}{)} \supset \overset{32}{((}\, p \cup \overset{1}{(}\, q \supset r \,\overset{12}{))} \cap \overset{1}{(}\, p \cup r \,\overset{13}{))} \qquad (4)$$

To arrive at the dot notation starting from this representation, we first have to discard some of the parentheses that are redundant: all outer parentheses, both at the beginning and the end of the formula, are omitted; if two or more parentheses occur consecutively, we only keep the one with the greatest label and discard all others. After these modifications, formula (4) becomes:

$$p \cup q \overset{1}{)} \supset \overset{3}{(} p \cup \overset{1}{(} q \supset r \overset{2}{)} \cap \overset{1}{(} p \cup r \qquad (5)$$

Now, simply replacing any labeled parenthesis with a group of as many dots as are indicated by the label yields the dot representation (3) of the formula. If the dot notation is introduced without reference to syntax trees, one often speaks of the *scope* of a group of dots, i.e., the subformula that is determined by that group.[34] The scope extends to the left of the group if the dots are to the left of a connective, and to the right if the dots are to the right of a connective. By looking at the syntax tree, it is easy to see that the scope of a group of dots is a subtree, i.e., it extends beyond all groups that consist of a smaller number of dots.

Because we always need two parentheses to enclose a subformula, but only one group of dots to separate a subformula, the dot notation uses fewer symbols. In our example, Formula (1), which contains 6 connectives, has 10 parentheses, omitting outer parentheses as is convention, but its representation in the dot notation (3) needs only 5 groups of dots. Because of this, the dot representation is more concise, a fact that is frequently used to argue in its favor.

## 4.2   Peano's use of the dot notation

Peano introduced the dot notation in his 1889 booklet on arithmetic, in which he presented his famous axiomatization of the natural numbers [Peano 1889a].[35] His explanation for the notation is surprisingly short; he apparently expected his readers to have no difficulties in using it.[36] Peano writes:

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34. See, e.g., the introduction of dots in *Principia Mathematica* [Whitehead & Russell 1910-1913, I, 9–11].

35. Shortly afterwards, Peano published a logical exposition of geometry in which the dot notation is introduced with a very similar wording [Peano 1889b, 7].

36. In [Peano 1891b], he remarks that dots are already used in analysis, where "one writes *d.uv* and *du.v* instead of *d(uv)* and *(du)v*" and notes some analogy to a notation used by Leibniz, referring to [Leibniz 1855, 276] and [Leibniz 1863, 55]. In these passages, Leibniz discusses the use of vincula and parentheses to group subexpressions, and he also uses groups of commas for this purpose, but without much explanation; in one example he also uses a combination of a comma with a dot. Whether Peano's dot notation was actually inspired by Leibniz or whether he found the passage from Leibniz only after having developed his own notation remains unclear.

We shall generally write signs on a single line. To show the order
in which they should be taken, we use *parentheses*, as in algebra,
or *dots*, ., :, ∴, ::, and so on.

To understand a formula divided by dots we first take together
the signs that are not separated by any dot, next those separated
by one dot, then those separated by two dots, and so on. [Peano
1889*a*, VII] (quoted from [van Heijenoort 1967, 86])

This explanation is followed by the two examples presented at the beginning
of Section 4.1, and by a brief remark that the dots can be omitted if different
punctuations do not change the meaning of a formula (e.g., both "*ab . c*" and
"*a . bc*" can be rendered simply as "*abc*' if the operation is associative) and
if no ambiguities arise. Because groups of dots have also been used in other
mathematical contexts (e.g., for multiplication and division), Peano warns:

To avoid the danger of ambiguity we never use . or : as signs
for arithmetic operations. [Peano 1889*a*, VII] (adapted from [van
Heijenoort 1967, 87])[37]

Despite the fact that dots make formulas shorter, Peano allows for both
dots and parentheses to be used within the same formula, with the convention
that parentheses bind stronger than dots [Peano 1889*a*, VII] ([van Heijenoort
1967, 87]).[38] In general, both dots and parentheses occur in the same formula
to syntactically mark semantical differences between logical and arithmetical
expressions, e.g., in Definition 18 [Peano 1889*a*, 2]:

$$a, b \, \epsilon \, \mathrm{N} \, . \, \supset . \, a + (b + 1) = (a + b) + 1.$$

From this example, one might be tempted to surmise that Peano uses
parentheses for the grouping of mathematical (or algebraic) expressions and
dots for their logical grouping. While this is indeed mostly the case and he
praises the use of dots for avoiding "the confusion with parentheses in algebraic
formulas" [Peano 1891*b*, 155], Peano's usage is not completely consistent in
this regard, sometimes relying simply on good judgement. For example, in
[Peano 1889*a*, IX], he uses dots in

23.   $a \cup b \, . = \, \therefore \, - \, : - \, a \, . - \, b$

but parentheses in the very similar formula

25.   $-(a \cap b) = (-a)(-b)$

---

37. See also [Peano 1894, 13]: "When introducing the dots to separate the parts of
a proposition, one must discontinue their use for indicating multiplication $a . b$, which
will be written $ab$ or $a \times b$, and division $a : b$, which will be written $a/b$."

38. The sentence in question is missing in Kennedy's translation [Kennedy 1973,
104].

The two previous formulas also illustrate that, in addition to the symbol for conjunction, "∩", Peano also uses juxtaposition to indicate conjunction. Thus, the dot between "$-a$" and "$-b$" in Proposition 23, above, separates these two expressions, such that they are rendered as "$(-a)(-b)$" using parentheses (cf. Proposition 25, above). Citing the conciseness of the resulting expressions as motivation, Peano explains:[39]

> The sign ∩ is read *and*. Let $a$ and $b$ be propositions; then $a \cap b$ is the simultaneous affirmation of the propositions $a$ and $b$. For the sake of brevity, we ordinarily write $ab$ instead of $a \cap b$. [Peano 1889a, VII] (quoted from [van Heijenoort 1967, 87])

In rare cases, Peano even uses both juxtaposition and ∩ within the same formula, as in [Peano 1894, 12]:

$$a \supset . b \supset c : d \supset e \cup f \therefore \supset : h \cap k \supset l . \supset . m \supset n$$

Here, on the one hand, the conjunction of "$a \supset . b \supset c$" and "$d \supset e \cup f$" is indicated by the first occurrence of ":", which separates the two juxtaposed expressions. The conjunction of $h$ and $k$, on the other hand, is expressed by "∩".

To determine the number of dots in a group that separates two conjuncts, it is helpful to consider again the representation of formulas in terms of syntax trees. As an example, consider Formula (3) and its corresponding syntax tree (2) on p. 115, above. In this case the conjunction symbol ("∩") is explicit and has a group of two dots on the left and a single dot on the right. To express this formula using juxtaposition for conjunction, we simply have to replace the symbol "∩" by the group next to it with the greatest number of dots and omit the other group. Replacing ": ∩ ." with ":", this results in the formula

$$p \cup q . \supset \therefore p \cup . q \supset r : p \cup r$$

Note that, while Whitehead and Russell explicitly introduced a single dot as the symbol for conjunction in *Principia Mathematica* [Whitehead & Russell 1910-1913, I, 6], for Peano both dots and parentheses are *exclusively* marks for grouping.[40] In the third edition of the *Formulario*, Peano is unequivocal about the use of parentheses for this unique purpose:

> The symbols of the *Formulario* always have the same meaning. Using parentheses to group the parts of a formula prevents us from using them in another meaning. We will be able to denote by $(a)$ neither a power of $a$, as does Girard 1629 [...], nor the

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39. This is also consistent with the considerations about readability and spacing (Section 3.3.1).

40. Presumably due to Russell's use, Peano's notation has been misinterpreted in the literature as using dots for conjunction; see, e.g., [Kneale & Kneale 1984, 521] and [Lolli 2011, 57, footnote 27].

> integral part of $a$, nor the absolute value of $a$, nor a function of $a$. In general, a single letter will never be enclosed in parentheses, because it is not grouped. [Peano 1901, 3][41]

Analogously, as Peano uses dots as marks for grouping, the expression "$a . b$" would not be in accord with his usage, given that each symbol must have a unique meaning and that a single letter cannot be grouped. Accordingly, Peano omits dots on the side of a connective that has only a single variable in its scope, as in $a \supset . a \cup b$ [Peano 1889$a$, IX, Prop. 26].

Peano's use of the dot notation is systematic but not rigid. It is evident from his formulas that he also employs implicit conventions regarding the binding strength of operations that are familiar from algebra, e.g., that juxtaposition binds stronger than any binary connective and that logical connectives bind stronger than the equality symbol. If this were not the case, Propositions 11, 14, and 27 in [Peano 1889$a$, IX],

$$ab \supset a, \qquad aa = a, \qquad \text{and} \qquad a \cup b = b \cup a$$

would have to be written as

$$ab . \supset a, \qquad aa . = a, \qquad \text{and} \qquad a \cup b . = . b \cup a$$

In later publications, Peano discusses some of his binding conventions explicitly, e.g., [Peano 1894, 12–13], but also mentions additional ones to further reduce the number of dots needed in a group. Presumably, however, this practice would not pose serious difficulties for a reader familiar with the typical binding conventions used in algebra.

In general, using more dots than are necessary in a group does not alter the structure of a formula, as long as the number of dots in groups with more dots are also increased accordingly. Thus, superfluous dots may be introduced in a formula and some later authors will systematically do so, presumably to enhance readability.[42]  Also, larger groups of dots stand out more and are thus easier to see at a glance, which allows readers to faster identify the main connective in a formula, because it is flanked by the greatest number of dots. However, Peano does not add dots in a systematic fashion, but only occasionally, e.g., in Proposition 2 [Peano 1889$a$, VIII], possibly due to considerations of symmetry:

$$a \supset b . b \supset c : \supset : a \supset c. \tag{6}$$

In fact, this proposition is rendered without the superfluous dot on the right side of the main implication symbol as Proposition 6 in [Peano 1891$b$, 156]:

$$a \supset b . b \supset c : \supset . a \supset c. \tag{7}$$

41. See also [Peano 1897, 54], where Peano insists on writing $fx$ instead of $f(x)$.

42. For example, Landini adds dots such that each connective has always the same number of dots on each side, "for easier reading" [Landini 2012, ix].

As we have seen earlier, Peano's symbolic language is not a formal language in the modern sense, i.e., based on an explicit, recursively defined grammar. Similarly, his dot notation is not defined rigorously, its usage is not entirely uniform, and several conventions are left implicit. This practice is consistent with common practice in algebra and with Peano's general attitude of being more interested in actually using the logical formalism for the representation of mathematics than in giving a rigorous presentation of the symbolism itself. Moreover, his general aims of attaining rigor and clarity while striving at the same time for conciseness can also be seen at play in this use of the dot notation.

# 5   Conclusion

Although some of Peano's views were shared by other influential logicians at the time, such as Peirce and Frege, his particular outlook and the associated enterprise of the *Formulario* remain unique. Perhaps because of the collaborative nature of the latter, Peano was more explicit than most other mathematicians about the principles that guide the introduction of new symbols, their shapes, and their layout. Moreover, while Frege maintained that "the convenience of the typesetter is not the highest Good" [Frege 1896], Peano was willing to take such practical matters into consideration. Despite his efforts, Peano's symbolism was not widely adopted outside of Italy in the late 19th century. Felix Klein wrote to Pieri in 1897:

> My general experience indicates that articles which are written us-
> ing this symbolism, at least in Germany, find practically no read-
> ers and moreover meet with immediate rejection. [Marchisotto &
> Smith 2007, 365; see also 383]

Nevertheless, with time, many of Peano's symbols did eventually enter the mathematical canon, and, through its adoption by Whitehead and Russell in their groundbreaking *Principia Mathematica* [Whitehead & Russell 1910-1913], Peano's dot notation also became very popular in logical works in the first half of the 20th century, though it has by now almost completely faded into the background.[43]

The above discussions have given us insight into Peano's views on logic and his motivations for the development of a logical symbolism as a methodological tool for the *analysis* of mathematical ideas and as an indispensable practical tool for the *presentation* of mathematical theories. Accordingly, two main normative ideals underlie Peano's symbolizations: attaining *rigor and clarity*, mainly through the avoidance of ambiguities, which is primarily achieved by ensuring the uniqueness of meanings as well a judicious choice of notation,

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43. A more detailed account of the use of the dot notation in early 20th century logic is in preparation by the author.

and *conciseness*. In addition, practical considerations, such as reducing the effort to learn and memorize the meanings of the notation, enhancing the clarity of the presentation and the ease with which it can be read, and, finally, reducing the printing costs, all guided Peano's design of notations. All of these considerations also support Peano's most conspicuous notational innovation, the dot notation for grouping subexpressions.

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# Logic and Axiomatics
# in the Making of Latino sine Flexione

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**Résumé :** Cette contribution examine l'arrière-plan scientifique de Latino sine Flexione (LSF), une langue auxiliaire internationale élaborée par Peano. Le LSF s'insère dans le cadre d'un mouvement linguistique plus vaste résultant des nouvelles technologies, lesquelles accélérèrent la mondialisation. La science constitue une force motrice dans le développement d'une langue auxiliaire internationale, étant donné qu'elle favorise les contacts internationaux et qu'elle fournit des données et des méthodes permettant de construire une telle langue. Avec le LSF, Peano entreprit de réaliser une partie du rêve leibnizien d'une langue universelle, dont une version simplifiée et provisoire du latin représenterait la première étape. Le LSF fut conçu à partir des fragments de Leibniz rassemblés par Couturat. En éliminant les traits conventionnels du latin standard, Peano entreprit de le réduire à son expression logique. Inspiré par des préoccupations similaires à celles qui furent à l'origine du symbolisme du *Formulario*, il chercha à mettre sur pied une langue simple, réduite à un noyau logique commun à toutes les langues, et de ce fait adaptée à l'usage international. Pour ce faire, Peano procéda de la manière suivante : il élimina les inflexions de tous les mots et il établit une « algèbre de la grammaire » régissant les règles de formation des mots. La simplicité, la non-redondance et la calculabilité sont les valeurs-clés du LSF inspirées de la pratique mathématique de Peano.

**Abstract:** This contribution examines the scientific background of Latino sine Flexione (LSF), an international auxiliary language constructed by Peano. LSF is part of a larger linguistic movement resulting from new technologies that accelerated globalisation. Science is a major driving force behind the international auxiliary language movement, both for creating an increased need for international contacts and for lending its data and methods to language construction. With LSF, Peano attempted to realize part of Leibniz's dream of a universal language, of which a temporary simplified form of Latin would become the first step. LSF was designed following Leibniz's fragments

compiled by Couturat. By eliminating conventional features from standard Latin, Peano attempted to reduce it to its logical expression. Inspired by the same concerns that motivated the symbolism of *Formulario*, he aimed for a simple language that owed its fit for international use to its being stripped down to the logical core shared by all languages. To achieve this, Peano proceeded by eliminating inflections from all words and establishing an "algebra of grammar" that governed the rules of word-formation. Simplicity, non-redundancy and computability are key values of LSF inspired from Peano's mathematical practice.

# 1   Introduction

This essay is dedicated to Giuseppe Peano's work in interlinguistics, or language construction for international communication. Peano's reflections on mathematical symbolism and his pursuit of major Leibnizian ideas resulted in *Formulario* and the axiomatic theory of natural numbers that earned him his reputation in the history of mathematics and also in the construction of an international language called Latino sine Flexione (LSF), or Interlingua. This linguistic project occupied most of his later years, ensuring him an important place in the history of international auxiliary languages. In the following, we examine the philosophical background of LSF. We start with a short overview of the historical context that led to the emergence of constructed languages for international communication. Peano's involvement in the movement was boosted by his reading of Leibniz's newly discovered manuscripts and his contact with Louis Couturat. Following Leibniz's idea of a characteristica universalis, Peano separated logic from convention in language and planned LSF as a language free from conventions reigning in natural languages. The influence of Peano's mathematical thinking on LSF can be seen in its axiomatic properties of simplicity and non-redundancy, as well as in its algebraic modification of Latin.

# 2   The question of language in the early 20th century

International auxiliary languages (also called interlingua by Peano, or inter-language) are defined as languages constructed for communication between people with different native tongues. They appeared towards the end of the 19th century when national independence movements and rivalries grew simultaneously with cross-border commercial, administrative and scientific

contacts in Europe. The international auxiliary language (IAL) movement can be considered an intellectual product of "the first wave of globalization" that lasted from 1870 to 1914. In economic history, this period is marked by the rapid development of transportation and telecommunication technologies and a resulting increase in the circulation of goods and persons. This situation gave rise to a growing number of international bodies and an associated trend toward standardization across national borders. The International Telegraph Union was established in 1865, the Universal Postal Union in 1874 and the International Bureau of Weights and Measures in 1875. The Olympic Games were initiated in 1896 and the Nobel Prizes in 1901. From 1867 onwards world fairs began to be organized regularly. As well as the 2nd International of the workers' movement, transnational political structures were founded such as the International Federation of Free Thought (1880), the International Sionist Organization (1897), the International Bureau for Masonic Affairs (1902) and the first international positivist congress which met in 1908. Among other notable structures set up during the first wave of globalization are the Hague conferences in 1899 and 1907, the international peace congresses and several international scientific bodies [Rasmussen 2001].

IALs (or interlanguages) were suggested by many as a viable alternative to any of the existing languages belonging to a particular nation because of the increased need to cooperate in a multilingual Europe of emerging and competing national sensitivities. War resisters were among the leading supporters of such a seemingly neutral solution to the language problem. A constructed language that was nobody's property would, they expected, contribute to mutual understanding of peoples and prevent enmities. Despite general scepticism at its reception, IAL was a project with considerable backing from a good deal of scientists, including linguists. The leading interlanguages like Esperanto and its derivative Ido figured in the agendas of scientific organizations [Gordin 2015] and were far from being marginal amateur creations. IAL was discussed alongside other possible solutions to the perceived language problem in Europe such as an existing language (and possibly more than one) or the revival of a dead language (Latin). But IAL advocates advanced strong arguments against these. Firstly, the use of the language of a dominant nation would grant unfair privilege to its native speakers, whereas a neutral language would put all interlocutors on an equal footing. Therefore, a language that would be learned by everyone was thought to be an antidote to nationalism and an incentive to build a peaceful global community. Secondly because an interlanguage would be built specifically for the purpose of international communication, it would be a more efficient instrument to accomplish that function. To succeed, an interlanguage would have to be designed with the ease of learning and its expressive potential primarily in mind. The movement started with the creation of Volapük by Johann Martin Schleyer, a German priest, in 1879. Volapük was the subject of considerable but short-lived interest. In 1887, Ludwik Lejzer Zamenhof, a Russian-Jewish ophtalmologist in the multi-ethnic and multilingual city of

Bialystok, published Esperanto, an interlanguage whose community outlived its creator and which has remained active to this day. Esperanto was followed by alternatives, including those emerging from reform proposals, such as Ido, devised by the French logician and Leibniz scholar Louis Couturat to be an improvement of Esperanto. It is in this context that Peano created LSF as a Latin-based isolating language for international communication. He used this language in the 5th edition of *Formulario* and published some other mathematical papers using it.[1]

Peano maintained strong ties with the IAL movement and he had in-depth knowledge of its most important projects. In 1906 (3 years after publishing LSF), Peano attended the World Esperanto Congress in Geneva. In 1907, he took part in the meeting of the Delegation for the adoption of an international auxiliary language, set up under the leadership of Couturat following the International Congress of Philosophy (1900). This event marks the beginning of Peano's active interest in language construction which constituted his main focus in his later life. In 1908, he was elected member and director of the Akademi internasional de lingu universal (the Volapük Academy). The following year he renamed it Academia pro Interlingua, made the membership open to all and declared that each academician was free to use his own form of interlanguage—a major change from Schleyer's monopolistic attitude with Volapük which arguably alienated its supporters in the long term and hampered its adoption. Peano praised Volapük for its morphological regularity (which constitutes the main reason of its success, according to Peano) but also criticized it for its lack of internationality as its lexicon was mainly based on shortened Germanic roots:

> Each affix has a fixed meaning; the affix is added to the root without reduction. This constitutes the biggest superiority of Volapük over natural languages and explains its rapid diffusion in 1887. But the author gives the affixes values of Germanic affixes; and there is no univocal correspondence between affixes, prepositions and grammatical elements from different languages.[2]
> [Peano 1912, 479]

After taking over the directorship of the Academy, Peano soon turned it into a democratic platform for experimenting with IAL and then an organ of diffusion for LSF/Interlingua.

In a sense, one might consider the existence of IALs as a testimony to the capacities of modern science. Not only were they inspired by new

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1. See [Kennedy 1980] for a full bibliography.
2. "Omni affixo habe sensu constante ; affixo es addito ad radice sine reductione. Hoc constitue magno superioritate de Volapük super linguas naturale, et explica suo diffusione rapido in 1887. Sed auctore da ad affixos de Volapük valore de affixos de germanico ; et non existe correspondentia univoco inter affixos, praepositiones et elementos grammaticale de linguas differente." [All translations are mine, unless otherwise indicated.]

technologies and used data from comparative linguistics, but, to some extent, they also aimed at increasing scientific literacy and sustaining a common world culture around science. For Peano, the *de facto* internationality of scientific terminology shows the path to the internationalism desired in other areas. For this reason, a scientific vocabulary common to all (or most) European languages forms the core of LSF/Interlingua vocabulary. Internationalization through the progressive expansion of scientific vocabulary to other contexts also has a secondary pedagogic effect of familiarizing the lay public with the language of science. For Peano, Interlingua would be intelligible to scientifically literate Europeans with virtually no effort required. Those not familiar with scientific jargon could acquire this knowledge through learning and using Interlingua which is a positive side effect of this linguistic project:

> Every cultured person who knows either Latin vocabulary or the scientific vocabulary of a European language understands Interlingua without studying it. Through Interlingua a less cultured person acquires Latin vocabulary which is living within his own language and becomes cultured.[3] [Peano 1927, 501]

Although the major influence came from mathematics and linguistics, scientists from other disciplines engaged in the IAL activism in the first half of the 20th century, notably Richard Lorenz, Leopold Pfaundler and Wilhelm Ostwald, who contributed with Couturat and Jespersen to a volume entitled *International Language and Science* in 1910 [Couturat, Jespersen *et al.* 1910]. A decade later, the question of an international language for science figured in the agenda of the British Association for the Advancement of Science (BAAS). The possibility of an international auxiliary language for scientific publications was discussed at the 1921 meeting of the International Research Council in Brussels. A committee was appointed by the BAAS "to investigate and report to it [council] the present status and possible outlook of the general problem of an international auxiliary language" [British Association for the Advancement of Science 1921, 390]. The question is discussed in the 89th report of the BAAS, with a detailed examination of concurring solutions to the problem of international communication, namely Latin, English, Esperanto and Ido. The question of an IAL is introduced in terms of finding ways to ensure peace. The committee's assessment of the respective merits and weaknesses of all the above options led to its preference for a constructed language such as Esperanto and Ido ("Esperanto and Ido are suitable: but the Committee is not prepared to decide between them" [British Association for the Advancement of Science 1921, 401]) to be adopted as the IAL:

> From the evidence laid before it, the Committee (Professor Ripman dissenting) has come to the conclusion that a language

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3. "Omne homine culto, que cognosce aut vocabulario latino, aut vocabulario scientifico de unu lingua de Europa, intellige Interlingua, sine studio. Homine minus culto disce, in Interlingua, vocabulos latino vivente in suo lingua, et fi culto."

of the type of Esperanto and Ido should be adopted as the
International Auxiliary Language; and also, that, whatever lan-
guage be adopted, it should be placed under scientific control.
[British Association for the Advancement of Science 1921, 401]

The report mentions LSF as a constructed language rather than as a way
of internationalizing Latin, then dismisses it due to the proven success of
Esperanto/Ido that was already in stable use.

For the BAAS committee, a world unified by unprecedented developments
in transportation and telecommunication technologies needs another linguistic
technology to complement this ongoing globalization process:

It is a truism that modern science has revolutionised the material
conditions of our existence and that, in particular, the devel-
opment of means of inter-communication—railway, steamship,
telegraph—has added to the amenities of life; but, unfortunately,
opportunities for strife have increased almost *pari passu* and
what is now required is some means of attaining greater mutual
knowledge as an insurance against future conflicts and misunder-
standings. Experimental science has forged the wheels of civilised
life; can humanistic science provide a lubricant to make them
run more smoothly? [British Association for the Advancement of
Science 1921, 390]

Indeed, arguments from technology and historical development are prominent
in the IAL movement, which often portrayed itself as a logical product of its
times.[4] IAL advocates praised technological developments and highlighted
the contrast between the material unification of the world and subsisting
national divisions. For instance, Couturat and Léau insist on the necessity
for language to catch up with the advancement and standardization reigning
in other products of human civilization.[5] This distance between perceived
levels of material and intellectual advancement is also used by Lorenz for the

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4. Gordin notes that "[t]he common trope [in the movement] was to draw
inspiration from contemporary innovations in communications and transportation
technologies and the standardisations that followed in their wake" [Gordin 2015,
114].

5. "Its necessity results even more obviously from the development of means of
communication: what good is being able to commute abroad in a few hours if one can
neither understand the inhabitants nor make oneself understood by them? What good
is being able to telegraph from a continent to another and make a phone call from a
country to another if the two interlocutors do not have a common language to write
or converse in?" ["Sa nécessité résulte encore plus évidemment du développement des
moyens de communication : à quoi bon pouvoir se transporter en quelques heures dans
un pays étranger, si l'on ne peut ni comprendre ses habitants ni se faire comprendre
d'eux ? À quoi bon pouvoir télégraphier d'un continent à l'autre, et téléphoner d'un
pays à l'autre, si les deux correspondants n'ont pas de langue commune dans laquelle
ils puissent écrire ou converser ?"] [Couturat & Leau 1903, ix].

defence of IAL.[6] Likewise, Pfaundler situates the IAL in the continuity of the modern standardization process.[7] Through comparisons with successful accomplishments in other areas, IAL's advocates intended to mitigate prejudice against it while also putting the movement into a wider historical perspective to help it gain legitimacy.

# 3  Leibniz, the precursor to Interlingua

Peano's introduction to the second tome of *Formulaire de mathématiques* and his explanations elsewhere make clear that the symbolism used in *Formulario* originates in the Leibnizian idea of an ideographic writing [Peano 1896a,b, 1897]. But it is after starting his exchanges with Couturat that Peano turned to Leibniz for a solution to the problem of international communication which was much debated in his day. In 1899, Peano sent his assistant Giovanni Vacca to Hannover to study Leibniz's unpublished manuscripts. The following year, Vacca met Couturat at the first International congress of philosophy (Paris). At the time, Couturat was already working on the *Logic of Leibniz*, a monograph in which he introduced Leibniz's thoughts on universal language [Couturat 1901]. After his contact with Vacca, Couturat started an editorial project of compiling Leibniz's unpublished writings which was to be completed in 1903.[8] Peano's quotations of Leibniz in his first article on LSF [Peano 1903] are taken from this compilation by Couturat. Later, Peano collaborated with Couturat on the publication of a mathematical dictionary in 1910, in Ido, German, English, French and Italian. After fruitful contacts lasting for years, Peano and Couturat fell out, seemingly due to rivalry about IAL.[9]

As noted by Couturat [Couturat 1901], in addition to his influential characteristica universalis, the need to construct a rational grammar for the universal-language-to-come brought Leibniz to search for a provisory auxiliary idiom that would "serve as an intermediary between living languages and the

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6. "We boast of our international intercourse. The civilised world has extended to new nations and has embraced whole regions of the earth and yet, in spite of the magnificent means of material communication, nothing of a similar nature has been done for the purpose of uniting minds together in an equally practical manner" [Lorenz 1910].

7. "The introduction of a common system of weights and measures was also declared to be impossible at one time, nevertheless it has since been carried out in science. The construction of a system of telegraph wires connecting the whole civilised world and a telegraph alphabet common to all nations was declared seventy years ago to be an impossibility. Now it is ancient history" [Pfaundler 1910].

8. In the preface, Couturat mentions Vacca who drew his attention to the unpublished manuscripts of Leibniz and credits Peano's school for initiating his interest in the logic of Leibniz [Couturat 1903, i].

9. See [Roero 1999], [Luciano & Roero 2005] and [Luciano 2012] for Peano's correspondence with Couturat. For Couturat's exchange with Russell on Peano, see [Russell 2001] and [Garvia 2015].

future rational language". Leibniz chose Latin as the basis of this auxiliary language. While *characteristica universalis* inspired Peano the symbolism of *Formulario*, the side project of an international language was realized through LSF. Like his predecessor, Peano turned to Latin to solve the problem of international communication because it had been the international language of science until the end of the 18th century and he considered that an interlingua based on Latin would benefit from this historical and cultural basis. Peano situates LSF in the continuity of Latin—not the high Latin of scientists but the Vulgar Latin, where cases were simplified. For Peano, the common vocabulary of European languages is also "a living document about the history of civilization". Updating Latin for contemporary use would help preserve the cultural heritage of Europe by building a common identity beyond national divisions. However Peano wanted to adapt Latin to contemporary use by "rationalizing" its grammar following the guidelines set up by Leibniz. Peano's recipe for a successful IAL is to combine international elements of vocabulary with a "minimal" grammar that leaves words identical throughout different propositional contexts. An isolating grammar helps the reader in the immediate identification of words, which are to be selected from international ones for a maximal efficiency:

> Experience proves that, by using international vocabulary and simple or no grammar, many authors write in a language that cultured people understand with almost no study.[10]
> [Peano 1930, 515]

The result was to be Latino sine Flexione.

In his first exposé of LSF, Peano lists the rules of the language with corresponding quotations from Leibniz [Peano 1903]. The text starts in Latin but Peano adopts each rule from the moment it has been stated, so that the paper ends in LSF after the incorporation of successive rules. The first rule states the defining feature of LSF that makes it different from Latin—the absence of affixes and the resulting invariability of words:

> The noun case can always be eliminated by substitution of some particle in another place.[11] (Leibniz qtd. by [Peano 1903, 439])

Following Leibniz, Peano prefers the use of standardized prepositions over variation in word endings:

> We indicate genitive with *of*, dative with *to*, ablative with *from, out of*,...[12] [Peano 1903, 440]

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10. "Experientia proba quod per usu de vocabulos internationale, et grammatica simplice aut nullo, numeroso auctore scribe in lingua que homine culto intellige sine studio, aut quasi."

11. "Nominum casus semper eliminari possunt substitutis in eorum locum particulis quibusdam."

12. "Indicamus genitive cum de, dative cum ad, ablative cum ab, ex, ..."

The adoption of the SVO word order common to most European languages makes the accusative unnecessary—as long as this regular word order is respected, object and subject of a verb are sufficiently well distinguished even in the absence of an accusative marker on the object. Substantives do not vary according to the case and they will have an inflexible form generally corresponding to the ablative. This is the main principle of LSF. The second rule of LSF excludes grammatical gender:

> The distinction of gender has nothing to do with rational grammar.[13] (Leibniz qtd. by [Peano 1903, 440])

Substantives are genderless but *mas* (male) and *femina* (female) can be used with them to emphasize gender when needed. Singular and plural can be indicated by *uno* and *plure* respectively but substantives are not marked by number ("the plural seems useless in a rational language" (Leibniz qtd. by [Peano 1903, 440])).[14] Conjugation is eliminated using a similar method:

> Persons of verbs can be invariable, it suffices to change *I, you, he*, etc.[15] (Leibniz qtd. by [Peano 1903, 441])

Like the substantives, verbs are inflexible: the person is indicated by the subject *(me, te, nos…)*, the tense by adverbs of time such as *heri* [yesterday], *in passato* [in the past], *nunc* [now], *cras* [tomorrow], *in future* [in the future], etc. Participles are expressed without changing the verb ending: *laudante – qui lauda* [who praises], *laudando – dum laudo* [while praising], *laudato – qui aliquo laudo* [whom someone praises], *laudaturo – qui lauda in future* [who will praise], etc. In vocabulary building, the guiding principle is again to leave each word inflexible: *hortulo – parvo horto, Romano – de Roma, Chartaceo – ex charta [paper], animose – cum animo, amabilo – qui aliquo pote ama*, etc.

# 4    Logic and convention in language

For Peano, who followed the ideas in Leibniz's linguistic writings, rationalizing Latin means to remove grammatical "conventions", avoid redundancy and ambiguity, be regular and economical. LSF or Interlingua is an attempt to make Latin rational by eliminating grammatical difficulties that lack any demonstrable logical function. In designing LSF, Peano aspired to develop a language "without grammar" (in his own words) in which sentences would be formed merely by juxtaposition of vocabulary and all words would keep the same form in all contexts, as they are found in the dictionary. This led Peano to turn away from agglutinative languages and choose an isolating structure instead, like English, Chinese, or the language of mathematics ("Chinese has

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13. "Discrimen generis nihil pertinet ad grammaticam rationale."
14. "Videtur pluralis inutilis in lingua rationali."
15. "Personae verborum possunt esse invariabiles, sufficit variari ego, tu, ille, etc."

no grammar. Mathematical formulas, such as $2 + 3 = 5$, are propositions without grammar.")[16] In an analytic language, parts-of-speech do not affect the form of words through modifications such as declension or conjugation. As indicated by the name LSF, the language's main novelty is the absence of inflexions. Peano's dismissal of parts-of-speech is the result of the distinction he made between universal logic and a myriad of differing, conventional grammars. To prove his point, Peano illustrates how cases can be expanded with, with an example from his native Italian language:

> Italian: *Io scrivo. Tu leggi. Noi abbiamo una lingua e due orecchi. La lingua internazionale ieri era un'utopia, domani sarà la verità.*
>
> Italian without flexions: *Io scrivere. Tu leggere. Noi avere uno lingua e dua orecchio. Lingua internazionale ieri essere utopia, domani essere verità.* [Peano 1927, 492]

This experiment in translation shows that inflections can be disposed of without affecting the intelligibility of the text:

> Such a language is as clear as a language with grammar. Therefore gender, number, articles, person, mode, verbal tense, etc. are useless.[17]

Incidentally the absolute distinction of parts-of-speech is only found in inflecting languages:

> Distinction of parts-of-speech 'substantive, adjective, verb, adverb, preposition' is relative to inflecting languages and have no logical value; therefore it is of interest to linguists only. All the resulting grammatical nomenclature is without value.[18]

Therefore, parts-of-speech do not correspond to "real" categories which need to be faithfully represented in an ideography (or, for instance, an ideographically-inspired language like LSF).

The opposition of linguistics to logic was a grounding idea in the emergence of symbolic logic, to which Peano contributed greatly. Peano's view of logic as distinct from grammar follows the Aristotelian distinction between formal and real definitions. According to that, to find out whether a property of a noun is formal or real, we can substitute it with another noun with the same meaning. If the resulting sentence conserves the same truth value, then it

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16. "Lingua de Sina (China) non habe grammatica. Formulas de mathematica, quale $2 + 3 = 5$, es propositione sine grammatica" [Peano 1927, 493].

17. "Tale lingua es tam claro quam lingua cum grammatica. Resulta inutile genere, numero, articulo, persona, modo, tempore de verbo, etc." [Peano 1927, 492].

18. "Distinctione de partes de oratione 'substantivo, adjectivo, verbo, adverbio, praepositione', es relativo ad lingua cum flexiones ; et habe nullo valore logico ; hoc es ultra noto ad linguistas. Toto nomenclatura de grammatica resulta sine valore" [Peano 1927, 493].

expresses a "real" property possessed by its corresponding object; otherwise, it expresses a "formal" property belonging to the name only. For example, in the proposition "*homo es rationale*" ["man is rational"], "man", which has the same meaning as "*homo*" can replace the latter without affecting the truth value of the proposition. By contrast, "*homo* es bisyllabo" ("*homo* is disyllabic") states a property of the noun *homo*, not of the man. Referring to Max Müller (*The Science of Thought* [Müller 1887]) and Michel Bréal (*Essai de Sémantique* [Bréal 1899]), Peano points out the relativity of parts-of-speech on an interlinguistic scale. Müller said that Aristotle's categories correspond to the categories of Greek grammar (for instance, they are not relevant to Semitic languages). In English, too, the same word can be used as a verb, subjective or adjective (Peano's example is "I ink a pen, I pen a word, I word a thing."). Therefore, substantive, verb, etc., are only *formal* properties of words, not *real* ones.

> The result of the fact that grammatical categories are relative to Greek and its affiliated languages and not to all languages is that this classification is formal. A property of a word is *real* (of the *thing*) if it is about the object or idea indicated by the word, it is *formal* (of the *form*) if it is about the word that indicates an idea.[19]

Following this distinction, Peano criticizes Esperanto for taking syntactic categories for granted. In Esperanto, word endings are standardized according to the part-of-speech (POS). All words are roots, to which a POS-marker is added: all substantives end with –o, adjectives with –a, adverbs with –e. Conjugation is also regular: the –i ending marks the infinitive, –as the present tense, –is the past tense, –os the future tense, –u the imperative, –us the conditional. Zamenhof built the entire Esperanto grammar around the parts-of-speech distinction. He eliminated all cases but retained the accusative for a flexible word order (ex: *Kato ĉasas muson = muson ĉasas kato*, "the cat chases the mouse"). In Esperanto, Peano appreciated the elimination of grammatical gender and of the personal marking by verbs. He considered this an improvement over natural languages. Yet, for him, Zamenhof did not go far enough in rationalization because he maintained parts-of-speech (although in a more systematic way than in natural languages). In Peano's view, as a merely formal property of words, POS-markers do not have a place in a rational language. Like Couturat, who criticized Esperanto's derivational system for failing to meet the logical criteria of univocity and reversibility [Couturat 1910], Peano saw objective (therefore, truly neutral) grounds for an IAL and a solid reason for its universal acceptance in universal logic. Both separated universal logic from conventional languages, associating the former

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19. "Ex facto que categorias grammaticale es relativo ad graeco, et linguas affine, et non ad omni lingua, resulta que isto classificatione es formale. Proprietate de vocabulo es reale, de re, si tracta de objecto aut idea indicato ab vocabulo, es formale, de forma, si tracta de forma de vocabulo, que indica idea" [Peano 1912, 459].

with the unity needed for a successful IAL and the latter with the diversity responsible for the language barriers dividing peoples of the world. Under Leibniz's influence, both privileged logic over linguistics and both insisted on making the IAL conform to the requirements of logic.

# 5   The influence of mathematics

Mathematicians first debated IAL at the International congress of philosophy in 1900. Couturat was in the organizing committee. He brought up the topic during the event, then pioneered the Delegation for the Adoption of an International Auxiliary Language that met following on from the congress. The Delegation was formed as a self-appointed dedicated body to settle the international language problem by engaging experts. The philosophy congress was followed by the congress of mathematicians, where Charles Méray suggested the adoption of Esperanto [Méray 1902, paper read by Léopold Léau]. Couturat, Léau, Charles-Ange Laisant and Alessandro Padoa (of Peano's school) were in favour of the proposal while Ernst Schröder[20] and Aleksandr Vasil'ev were against. Despite Couturat's efforts to recruit him to the cause of IAL, Bertrand Russell only expressed unenthusiastic interest in it. Couturat himself broke from the Esperanto movement after successive rejections of his reform proposal. He also ended his collaboration with Peano and both men continued to promote their own IALs.

Among mathematicians, Peano's biggest support came from Paul Mansion, who appreciated the mathematical principles of construction behind LSF and declared it to be "the real IAL of the future" ([Mansion 1904], cited by [Roero 1999, 12]). Indeed LSF's morphology had qualities that appealed to mathematicians because it had taken inspiration from axiomatics and algebra. Peano's search for simplicity, non-redundancy and computability in IAL attests to the influence that his axiomatisation of the system of natural numbers had on LSF, despite these two projects being clearly separated in his work.[21] In his attempt to purify Latin from redundancies proper to natural languages, Peano is led to eliminate inflections whenever their meaning can be clearly expressed by adjoined words. Moreover, Peano goes so far as to claim he had eliminated all grammar derived from Latin ("Grammar can be reduced to

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20. In 1897, Schröder had introduced a pasigraphy of his own making for scientific purposes only, in his talk in the International Congress of Mathematicians in Zürich. See [Gray 2008], [Peckhaus 2014] and [Schröder 1898].

21. For instance, Peano associates simplicity with the practice of mathematicians: "Mathematicians generally prefer simpler forms; orators and poets prefer long and sonorous sentences." ["Mathematicos praefer in generale forma plus simplice; oratores et poetas praefer periodo longo et sonoro"] [Peano 1912, 466].

little or nothing").[22]  This means paraphrasing standard Latin sentences by replacing affixes with appropriate accompanying words or prepositions.[23]

The strategy of eliminating all "useless elements" from IAL indicates a functionalist view of language that dominated the understanding of interlanguage planners. It helped them counter-object to accusations of not respecting historical languages as they are and attempting to change them in an unnatural way, without regard for their spiritual identity. For Peano, we do not owe the Latin of Cicero and Horatio respect for its traditional grammar as it is already a dead language. Using a living language "without grammar" would produce a similar effect to walking around with uncustomary clothing in public but the fact that Latin is not in public use makes it legitimate to modify it following the technical needs of international communication such as simplicity [Peano 1927, 493]. If the adjunction of fixed words in their form as found in dictionaries suffices to produce an intelligible output (and this is the case, as Peano shows with examples of translation from Latin to LSF), then this method should be preferred for its greater simplicity and thus its suitability for its purpose—easy universal use.

To some extent, Peano takes the ideographic language of algebra as a model for LSF, even though these two symbolisms do not have the same purpose. The ideographic nature of algebraic symbols makes them suitable for use in calculations.

> Algebraic equations are much shorter than their expression in ordinary language, are simpler and clearer and may be used in calculations. This is because algebraic symbols represent ideas and not words. [Peano 1915, 228]

The Peanian ideography is intended to "establish a one-to-one correspondence between ideas and symbols, a correspondence which is not found in our ordinary language" [Peano 1897, 191].

> The ideas represented by our symbols are very simple ideas and do not have exactly the value of their corresponding terms in ordinary language, which represent more complex ideas. Thus the sign $\epsilon$ may be read "is a", or "est" in Latin but represents the idea obtained from the term "est" when abstraction is made from grammatical mood, tense and person. [Peano 1897, 193]

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22. "Grammatica pote es reducto ad pauco aut nihil" [Peano 1927, 484].

23. Clearly, what Peano means by grammar is the agglutinative features of Indo-European languages. Against this ethnocentric misunderstanding, Jespersen points that, technically, there is no language without grammar. Even Chinese, a model for Peano (like Leibniz and other language constructors inspired by its ideography, before him) for its analytic structure that contrasts with Indo-European languages, incorporates grammatical features through means other than desinences. [Jespersen 1928, 47–48], qtd. [Falk 1999, 64–65].

In LSF, eliminating inflections that cause word variations can be read as the linguistic counterpart of this ideographic attempt.[24]

Peano's algebraic conception of grammar for a rational language is most clearly seen in the derivation system of LSF. Talking about *Formulario*, Peano insists on the role that symbols play in rigorous reasoning and discovery of knowledge beyond their more modest function as a shorthand for more cumbersome expressions. The Leibnizian calculus ratiocinator was influential mainly in Peano's search for an appropriate symbolism for writing mathematical statements. In a similar way, Peano undertook to create an "algebra of grammar" [*algebra de grammatica*] to clean Latin from redundant suffixes and, to achieve this, established morphological equations such as the following. For example, by putting *que* before or *–nte* after a verb, one obtains an adjective. *Que*, like *–tore, –ace, –ido*, etc., turns a verb into an adjective; therefore, it belongs to the category $A - V$ [*adjectivo ex verbo*]. "$A - V = que = (stude)nte = (audi)tore = (rap)ace = (val)ido = (noc)ivo = (pend)ulo = (viv)o = (med)ico$". Followed by an adjective, "es" makes a verbal construction (therefore, it is classified as $V - A$). Given that $V - A$ *(es)* and $A - V$ *(–nte)* have opposite values, they cancel each other and can be eliminated altogether for the sake of simplicity: "es $(V - A)$ stude**nte** $(A - V)$" equals "$(V - A) +$ stude $+ (A - V)$", equals "stude".[25] The suffix *–tate* turns an adjective into a noun (*substantivo abstracto ex adjective*, $S - A$). The reverse $(A - S)$ is expressed by *que habe, cum, –ale, –ose*, etc. Using basic algebraic equations, Peano offers logically simpler alternatives to some words in standard Latin:

> Justitia = jus + –to + –itia = jus + $A - S + S - A$ = jus, jure.
> Porositate = poro + –oso + –itate = poro. [Peano 1912, 471]

Likewise, "habe $(V-S)$ ardore $(S-V)$" equals "arde" $((V-S) + \text{arde} + (S-V) = 0)$, "habe dolore" equals "dole", "habe fervore" equals "ferve", since "habe $+ (-\text{ore}) = 0$".[26] Together, such equations constitute "the algebra of grammar". Incidentally, according to the algebra of grammar, *ente* or *ont–* [being] has no real conceptual value ("This word is commonly used in philosophy. We can see its null value.").[27] In a similar way to Carnap's elimination of Heidegger's discourse on Being and Nothing by a logical analysis of its propositions, Peano arrives at an anti-metaphysical position against the concept of "being" through a logical analysis of derivations in Latin.

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24. "Leibniz's idea of a characteristic containing 'real' characters is not completely abandoned in Peano's perspective. It emerges with even more force in Peano's investigations into a universal language because the latino sine flexione was to be based on symbols (roots of Latin words) that should preserve the essential relation to the denoted concept, independently of grammatical variations" [Cantù 2014, 30].

25. $0 = (V - A) + (A - V) =$ es que = es –nte.

26. $0 = (V - N) + (N - V) =$ habe -ore.

27. "Isto vocabulo es de usu commune in philosophie. Nos pote vide suo valore nullo" [Peano 1912, 464].

# 6 Conclusion

Peano created LSF in an age of globalization that led many intellectuals to look for an alternative medium of international communication. His engagement with Leibniz's influential work in logic and his familiarity with ongoing trends in linguistics are the main influences behind LSF—a constructed language modelled on axiomatics with a lexical basis in the common cultural heritage of Europe as found in scientific terminology.

## Acknowledgments

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# Peano and the Debate on Infinitesimals

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**Résumé :** Le principal objectif de cet article consiste à mettre en évidence la thèse de Peano visant à rejeter les infinitésimaux. Dans un premier temps, nous nous concentrons brièvement sur le contexte culturel dans lequel sont apparues et se sont développées les considérations de Peano. Ensuite, nous examinons l'article de Peano de 1892.

**Abstract:** The main aim of this paper is to put Peano's opinion about the unacceptability of the actual infinitesimal notion into evidence. First we briefly focus on the cultural environment where Peano's considerations originated and developed. Then we examine Peano's article of 1892, "Dimostrazione dell'impossibilità di segmenti infinitesimi costanti" [Peano 1892].

## 1   Introduction[1]

Between the 19th and 20th centuries the possibility of theoretically accepting, or not, *actual* infinitesimals was much debated.[2] Mathematicians who were interested in the foundations of mathematics often had different opinions. At first we think it is useful to outline the distinction between *actual* infinite and *potential* infinite, according to the classical tradition. The *potential* infinite is conceived as something to which it is always possible to add a certain quantity, while the *actual* infinite is the possibility of instantly imagining a collection, a whole that has no end.[3]   So, for instance, in Euclidean geometry, a half-straight line is generated by applying the first and second Euclidean postulates. For Aristotle the actual infinity cannot be accepted,

2. This debate takes place in the context of the rigorous foundation of the analysis [i.e., see Lolli 2004, 82–86].

3. For other general information [see Bottazzini 2018].

and this conception prevailed until the nineteenth century.[4] Likewise, we can state concerning the infinitesimals. A potentially very small magnitude can be generated through the principle according to which given a magnitude there is always a smaller one (see Archimedes' postulate). On that subject, Paul du Bois-Reymond wrote:

> The infinitely small is a mathematical quantity and has all its properties in common with the finite. [...] A belief in the infinitely small does not triumph easily. [...] A majority of educated people will admit an infinite in space and time, and not just an "unboundedly large". But they will only believe in the infinitely small with difficulty, despite the fact that the infinitely small has the same right to existence as the infinitely large. [du Bois-Reymond 1877, 152]; [English trans. in En. Wikipedia]

Once the infinite number is accepted as an infinity, this number does not stay in **R**. The infinitesimal can be seen as its reciprocal. Infinitesimals and infinities naturally occur in any description of the infinity. E.g., a potential infinitesimal is $1/n_{|n=\infty}$. The inverse is an infinity: $1/(1/n) = n_{|n=\infty} = \infty$.

We will define below the notion of the actual infinitesimal.

To understand the notion of infinitesimal we must take into account the concept of real numbers. In fact real numbers are based on the Dedekind axiom and it is well known that:

> Archimedes Post. + Cantor Ax. ⇔ Dedekind Ax. [see Benedetti 1937, 28–36]

But Archimedes' Postulate[5] is not compatible with the concept of the actual infinitesimal. Hence, if one considers the Dedekind axiom then we are at odds with the acceptability of (actual) infinitesimals.

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4. About the actual infinite, Desargues had written: "So every straight line is intended to be stretched indefinitely both from one side and the other [...]" [see Taton 1981, 99].

5. Let us recall Archimedes' Postulate (axiom):

Let $a$ and $b$ be two magnitudes, if for example $a < b$ then $\exists n \in \mathbf{N}$ such that $na > b$

The Dedekind axiom (1872) says:

IF a segment AB of a straight line is divided into two parts in such a way that

– each point $x$ of segment $AB$ belongs to one of the two parts

– the point $A$ belongs to the first part and $B$ to the second

– any point of the first part precedes any point of the second in the order established in $AB$,

THEN there exists a point $M$ of the segment $AB$ (which may belong to one or the second part), such that each point of $AB$ which precedes $M$ belongs to the first part, and each point of $AB$ which follows $M$ belongs to the second part according to the assigned subdivision.

From Dedekind's axiom one can derive Archimedes' Postulate.

In this article, first we present the Cantorian attempt to show the unacceptability of the infinitesimal notion. Then we highlight the debate on non-Archimedean concepts in Italy. Finally, we analyze Peano's position.[6]

# 2 Cantor and the unacceptability of the infinitesimals

Cantor examines the unacceptability of the actual infinitesimals. We will refer to a letter written by Cantor to Benno Kerry, dated 4 February 1887 [see Ehrlich 2006, 29–34]. As we know, Cantor conceives the actual infinite which plays a crucial role in his theory of transfinite numbers. Without this approach, he could not have constructed this theory. It may therefore seem strange that he does not accept the actual infinitesimals. At first he proposes the unacceptability of $1/\omega, \omega$ being the first transfinite ordinal. Following [Ehrlich 2006], and with our modifications, we expose the claimed (but not right) Cantorian proof. One considers a set $Z$ of linear magnitudes (i.e., the extremes of straight segments of finite length as linear numbers), which we denote $\zeta_1, \zeta_2, \ldots$ These numbers must satisfy the following axioms:

**Ax. 1** : The set of $\zeta_1, \zeta_2, \ldots$ is a commutative semi-group for the addition, where the operation $\nu \times \zeta$, with $\nu \in \mathbf{N}$, is possible.

**Ax. 2** : $\zeta_1 + \zeta_2 + \zeta_3 + \ldots = s \in Z$.

If $1/\omega$ exists, it is an infinitesimal because it is an inverse of an infinite ($\omega$). We will have the magnitude $\zeta = 1/\omega$ and that is:

$$\zeta \times \omega = 1. \tag{1}$$

If $\zeta_1 = \zeta_2 = \ldots = \zeta_n = \ldots = 1/\omega$ we could write (1) as:

$$\zeta_1 + \zeta_2 + \ldots + \zeta_n + \ldots = 1 \text{ (the sum on the left has } \omega \text{ addends).} \tag{2}$$

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The Cantor axiom (1872) says:

IF two classes of a straight line segments are such that

– no segment of the first class is greater than a segment of the second

– if a small segment $\sigma$, as small as one wants is prefixed, then there is a segment of the first class and one of the second class whose difference is less than $\sigma$.

THEN a segment exists which is neither less than any segment of the first class nor greater than any of the second.

It is possible to give a postulate which is equivalent to Cantor's axiom. According to Veronese this postulate consists in the following assertion:

"Ip. VIII: If a segment, whose extremities always vary in opposite directions becomes indefinitely small, it contains a point out of the range of variability of its extremes" [see Veronese 1889, 150].

6. For further information see *infra* [Benci & Freguglia 2019, 2016], [Ehrlich 2006], [Borga, Freguglia *et al.* 1985] and [Gemignani 1993].

At this point, Cantor uses the following property:

If $s \in Z$, with $s < 1$ then, in virtue of (2), $\exists n \in \mathbf{N}$ such that

$$\zeta_1 + \zeta_2 + \ldots + \zeta_n > s. \tag{3}$$

Hence for equation (3), let $s = {}^3/_4$, then we will have:

$$\zeta_1 + \zeta_2 + \ldots + \zeta_n > {}^3/_4. \tag{4}$$

Because $\zeta_1 = \zeta_2 = \ldots = \zeta_n = \ldots = {}^1/_\omega$ then (4) can be worth another $\zeta_i n$-tuple, so:

$$\zeta_{n+1} + \zeta_{n+2} + \ldots + \zeta_{2n} > {}^3/_4. \tag{5}$$

Adding (4) and (5) we achieve:

$$\zeta_1 + \zeta_2 + \ldots + \zeta_n + \zeta_{n+1} + \zeta_{n+2} + \ldots + \zeta_{2n} > {}^3/_4 + {}^3/_4 = {}^3/_2 > 1. \tag{6}$$

Which is in contradiction with (2). Therefore if we consider ${}^1/_\omega$ we arrive at a contradiction. Actually Cantor uses the following axiom (see equation (3)):

[**Ax. 3** ]: If $\zeta_1 + \zeta_2 + \ldots + \zeta_n + \ldots = s$, then $\forall s' < s$ and $\exists n \in \mathbf{N}$ such that

$$\zeta_1 + \zeta_2 + \ldots + \zeta_n > s'. \tag{7}$$

But one can see that this axiom [Ax.3] is a variant of Archimedes' Postulate. Therefore one remains within Archimedean mathematics where it is impossible to give the definition of actual infinitesimals (see equation (9)). So this Cantorian proof is not tenable.

Cantor develops this theme and he reaches an attempt to prove the unacceptability in general of infinitesimals, independently from the case ${}^1/_\omega$, which is however an important case. A first draft of this general demonstration is presented by Cantor in a letter to Karl Weirstrass dated May 16, 1887 [see Ehrlich 2006, 41]. These Cantorian analyses were accounted for by some Italian mathematicians.

## 3    The debate in Italy

A very interesting debate concerning the infinitesimals took place in Italy in the pages of the *Rivista di Matematica*, whose director was Giuseppe Peano from 1891 onward. We find contributions by Giulio Vivanti [Vivanti 1891a,b], by Rodolfo Bettazzi [Bettazzi 1891, 1892] and by Peano himself [Peano 1892]. Peano had asked for an invitation to the discussion also for Giuseppe Veronese, but this last declined the invitation. However, Veronese wrote an article about Peano's proof in *Rendiconti del Circolo Matematico di Palermo* in 1892 [Veronese 1892].

Giulio Vivanti in his article of 1891 wrote:

It seems that the idea of the proof of Cantor is this. Asserting that $\zeta$ [infinitesimal] is a segment, it is equivalent to admitting that if we successively arrange a sufficiently large series of segments, all equal to $\zeta$, upon a straight line, we shall of necessity eventually cover the assigned finite segment in its entirety; next Cantor states (and here there is a gap in his explanation) that if this is not possible by means of a finite series of segments, it is impossible by means of an infinite series as well, however extended the series might be.[7] [Vivanti 1891a, 138]

But—according to Vivanti—in the case of infinite series (a case which we cannot exclude), if the segment is covered then it is assimilable to a *linear continuous* which is compliant with the Dedekind axiom. Therefore the infinitesimals cannot be accepted. Vivanti wrote another article [see Vivanti 1891b] in the same journal and in the same year. Rodolfo Bettazzi in his article "Osservazioni sopra l'articolo del Dr. G. Vivanti sull'infinitesimo attuale", also published in 1891 in *Rivista di matematica*, wrote:

It is preferable to derive the definition of actual infinitesimal from the words of the author [Vivanti] [...] "when repeated any finite number of times, it (the infinitesimal) never constitutes [...] any finite determined quantity" than from the same author's words [...] which are not well defined [...] according to which the infinitesimal would be obtained by means of division into infinitely many equal parts, since the word *repeat* is understood in the ordinary manner of multiplication.[8] [Bettazzi 1891, 175]

Bettazzi claims that admitting (see (9) below) $n\alpha < \beta \ \forall n \in \mathbf{N}$ (where $\alpha$ is an infinitesimal with respect to $\beta$, and $\beta$ is a finite magnitude) one never reaches the finite determined quantity $\beta$. But is it possible to arrive at $\beta$ by means of $n$ as *transfinite number*? For the *nature* of the linear magnitudes (segments), which Cantor even considers, "with a sufficiently large number of magnitudes $\alpha$ one can reach or exceed $\beta$". Therefore the infinitesimals could

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7. See English trans. in [Ehrlich 2006, 78]. : "Il concetto della dimostrazione di Cantor sembra essere questo. Dire che $\zeta$ è un segmento equivale ad ammettere che, disponendo successivamente sopra una retta una serie abbastanza grande di segmenti tutti eguali a $\zeta$, si debba di necessità arrivare a coprire per intero un segmento finito assegnato; ora Cantor stabilisce (e qui v'ha una lacuna nella sua esposizione) che, se ciò non è possibile mediante una serie finita di segmenti, non lo è neppure mediante una serie infinita, comunque estesa essa sia" [Vivanti 1891a, 138].

8. See English trans. in [Ehrlich 2006, 86]. "La definizione dell'infinitesimo attuale, meglio che dalle parole dell'autore [Vivanti] [...] per le quali l'infinitesimo si otterrebbe dalla divisione in infinite parti uguali, non ben definita, si ha dalle altre dello stesso autore, che fanno seguito a quelle citate: '[...] esso (l'infinitesimo), ripetuto un numero finito qualsiasi di volte, non forma giammai [...] una quantità finita determinata qualunque' purché si prenda la parola *ripetere* nell'ordinario significato della moltiplicazione" [Bettazzi 1891, 175].

not be accepted (see Peano's proof). It remains to be established whether the transfinite numbers exhaust the "sufficiently large infinite numbers" (as requested above). But that—as Bettazzi says—is not proven. Bettazzi wrote in 1892 another article "Sull'infinitesimo attuale" in the same journal [see Bettazzi 1892].

The debate based on Veronese's ideas had an international readership which then led Hilbert to contemplate in a logically correct way non-Archimedean geometry. Veronese had also had controversial exchanges with mathematicians such as W. Killing (1895), Cantor himself [see Veronese 1896], L. Schoenflies [see Veronese 1897, 1898] and H. Poincaré (1904)[9] to mention just a few. Briefly, Veronese's structure $\mathbf{V}$ of numbers (according to our reconstruction) is the following:

– $\mathbf{V}$ is an ordered field
– $\mathbf{R}$ (real numbers) $\subset \mathbf{V}$
– (*) Exist $\alpha \in \mathbf{V}$ such that $\alpha > n, \forall n \in \mathbf{N}$

$\alpha$ is an *infinite* number, but it is not unique; in fact:

$\alpha + 1, 5\alpha, \alpha^2$, etc., are also infinite numbers.

If $x \in \mathbf{V}$ and for every $k \in \mathbf{N}$ we have: $|x| < 1/k$ then $x$ is called *infinitesimal*.

Indeed actually Veronese proposes the replacement of Archimedes' postulate with its negation, that is, with the fact that there are magnitudes that cannot be compared [Veronese 1891]. Explicitly, Veronese gives the following principles.

**Principle III**. In $\mathbf{R}$ there is no minimal interval (magnitude) if the zero is excluded.[10]

**Principle IV**. If an interval $(XX')$ whose extremities always vary in opposite directions becomes indefinitely small, it always contains an element $Y$ of $\mathbf{V}$ distinct from $X$ and $X'$.[11]

This Principle IV establishes that in $\mathbf{V}$ an infinitesimal (interval) is different from zero. If $(XX')$ is considered as a set of points, this principle affirms $(XX') \neq \varnothing$.

Otto Stolz read [Veronese 1891] and it seems that he shares Veronese's ideas, even if he prefers the Cantor-Dedekind approach [see Stolz 1891, 16].

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9. Poincaré in his review of Hilbert's *Grundlagen* calls into question Veronese who answers in [Veronese 1905], see also [Veronese 1909].

10. [Veronese 1889, 610], English trans. in [Ehrlich 2006]: "**Princ. III**. Nel sistema $\mathbf{R}$ non vi è un intervallo (grandezza) minimo se si esclude lo zero."

11. [Veronese 1889, 612], (English trans. in [Ehrlich 2006]): "**Princ. IV**. Se l'intervallo $(XX')$ i cui estremi sono sempre variabili in verso opposto diventa indefinitamente piccolo, esso contiene sempre un elemento $Y$ di $\Sigma$ distinto da $X$ e $X'$" A comparison between this Principle and Veronese's Ip. VIII is interestnig, see footnote 5.

So it is possible, through (*) to define the notion of *actual* infinitesimal (see (9) below). Tullio Levi-Civita in [Levi-Civita 1892-1893], based on Veronese's teaching, introduces the concept of *monosemii* and A. Bindoni, another disciple of Veronese, shows in 1902 [see Bindoni 1902] Hilbert's geometrical field is contained in **V**.

# 4 Peano's "proof"

Peano published an article in 1892 in *Rivista di matematica* entitled "Dimostrazione dell'impossibilità di segmenti infinitesimi costanti" [Peano 1892]. He examines the problem of the unacceptability of actual infinitesimal segments. He begins by this premise:

> This problem [that is, on the acceptability or not of the actual infinitesimals], debated between Dr. Vivanti and Dr. Bettazzi on the *Rivista di matematica*, is very interesting especially in this period where on the hypotheses of their existence theories and printed volumes exist. Cantor replied negatively to hypotheses of their existence, but the proof that this illustrious mathematician gave is so concise that it was judged incomplete. The aim of this article is to develop this proof.[12] [Peano 1892, 59]

Our aim is to analyze this proof, regardless of Ehrlich. Peano considers a half-straight line of origin $o$, as a set $P$ of points. It seems clear that for Peano an *ended segment* is an open interval on the half-line, so:

$op \equiv \{x : o < x < p \text{ and } p \in P\}$

$o$ is called *origin* and $p$ *ending*,

while a *segment* is an ended segment $u$ that verifies the following properties:

- $u$ is a proper subset of the half-straight line,
- every point $y$ between $o$ and a point $x \in u$ is also a point of $u$,
- vice versa: every point $y \in u$ is between $o$ and some other point $x$ of $u$.

The set of ended segments is denoted by $S$, and that of segments by $s$. It is possible to establish the sum of ended segments and the multiple of an ended segment and when one ended segment is lower than another: "Let $u$ and $v$ be two segments, we say that $v$ is less than $u$, or that $u$ is greater than $v$, if $v$ is a

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12. "Questa questione, dibattutasi tra i dott. Vivanti e Bettazzi sulla *Rivista di matematica*, è assai interessante tanto più che negli ultimi tempi sull'ipotesi della loro esistenza si sono fatte teorie e stampati dei volumi. Ad essa rispose negativamente il Cantor; ma la dimostrazione che questo illustre matematico ne diede è così concisa che fu giudicata incompleta. Scopo della presente nota si è di sviluppare questa dimostrazione" [Peano 1892, 59].

segment ended at a point in $u$, that is the ending of $v$ is a point of $u$". Every ended segment is a segment, but not vice versa.[13]

$\infty u$, which is called by Peano *multiple of infinite order* of $u$, denotes the set of points which stay on some of segments $u$, $2u$, $3u$, etc. or on their *upper limit*. Peano shows that $\infty u$ is a segment but not an ended segment. At first he affirms that an ended segment $u$ is such that when it is added to itself, it becomes a double segment that exceeds $u$ itself, that is :

$$2u > u. \tag{8}$$

Of course the (8) can be iterated.

Then he proposes the well-known actual infinitesimal definition, so:

> The segment $u$ is an infinitesimal related to the segment $v$, and we will write $u \in \{v|\infty\}$, *if every multiple of $u$ is lower than $v$.* $\tag{9}$

So if $u$ is an infinitesimal relatively to $v$, $\infty u$, being the upper limit of the multiples of an infinitesimal, is less than $v$, i.e., $\infty u < v$. In fact, $\infty u$ cannot exceed $v$ because it is still a multiple of $u$ and all multiples of $u$, for the (9), must be less than $v$, even if $\infty u$ is the upper limit of all multiples. Besides, Peano explicitly assimilates $\infty$ to $\aleph_0$,[14] therefore the following operations make sense under the assumption that $u$ is an infinitesimal.

$$u, v \in S, u \in \{v|\infty\} \Rightarrow (\infty + 1)u = \infty u. \tag{10}$$

$$u, v \in S, u \in \{v|\infty\} \Rightarrow 2\infty u = \infty u. \tag{11}$$

> If $u$ is an infinitesimal in comparison to $v$, we have a class $\infty u$ that is a segment contained in $v$. Therefore if we add $\infty u$ to the segment $u$, we obtain the segment $(\infty+1)u$ and by adding $u$ again we have $(\infty + 2)u$, etc. We can also add $\infty u$ with itself and we obtain $2\infty u$. So in general, we can have all multiples of $\infty u$; we can multiply $\infty u$ by $\infty$ and we can have $\infty^2 u$, and so on.
>
> But all these various segments obtained by multiplying $u$ by the transfinite numbers of Cantor [$\aleph_0$ as $\infty$] are always equal to [$\infty u$].[15] [Peano 1892, 61]

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13. Otherwise we would be in the context of Dedekind's axiom.

14. In the sense that it has the same arithmetic behavior. In fact i.e.: $\infty + 1 = 1 + \infty = \infty$ as $\aleph_0 + 1 = 1 + \aleph_0 = \aleph_0$; $2\infty = \infty$ as $2\aleph_0 = \aleph_0 2 = \aleph_0$, etc. But on the contrary $\omega + 1 \neq 1 + \omega$, $2\omega = \omega$ and $\omega 2 \neq \omega$. Hence, Peano (for $\infty$) refers to Cantorian transfinite cardinal.

15. Peano says: "Risulta che, se $u$ è un infinitesimo rispetto a $v$, la classe $\infty u$ è un segmento contenuto in $v$. In conseguenza possiamo aggiungere ad $\infty u$ il segmento $u$, ottenendo il segmento $(\infty + 1)u$, a cui aggiungendo $u$ otteniamo $(\infty + 2)u$, ecc. Possiamo sommare $\infty u$ con se stesso ottenendo così $2\infty u$, ed in generale possiamo formare tutti i multipli di $\infty u$; possiamo moltiplicare $\infty u$ per $\infty$ ed ottenere $\infty^2 u$, e così via". "Ma tutti questi varii segmenti, che si ottengono moltiplicando $u$ per i numeri transfiniti di Cantor sono uguali tra loro".

In short, we have:

$$\forall M \text{ transfinite number: } Mu = \infty u$$
$$\forall n \ \in \mathbf{N} : nu < v \tag{12}$$
$$\infty u < v.$$

That is: $\forall M$ transfinite number: $Mu = \infty u < v$ and in particular:

$$2\infty u = \infty u < v. \tag{13}$$

In this way Peano shows $\infty u$ (with $u$ infinitesimal) is not an ended segment. Indeed—according to Peano's opinion—equation (13) is in contradiction with the "nature" of the ended segments and their calculus, i.e., with equation (8). Thus, it is impossible to obtain the infinitesimal "as an element of finite magnitude". But in fact equations (8) and (13) cannot be compared because (8) is in an Archimedean context and (13) is not. Hence, Peano's proof is not tenable.

If equation (9) is satisfied, then $u$ is an infinitesimal. But if $u$ is a finite segment (that is, if infinitesimal segments do not exist—as Peano says) then $\infty u$ represents the infinite half-straight line with origin $o$. As we said, Veronese wrote a brief article that appeared in 1892 on *Rendiconti del Circolo Matematico di Palermo* [see Veronese 1892] entitled "Osservazioni sopra una dimostrazione contro il segmento infinitesimo attuale". This referred to Peano's proof, arguing this same conclusion. Afterward he repeats his proposal given in [Veronese 1891].[16]

# 5   Some conclusions

Despite the fact that history does not deal with ifs and buts, we must observe the discussion on aversion to the infinitesimals, which could have been more opportunely based on well-argued explicit positions, instead of on untenable proofs which in fact need, in various forms, Archimedes' postulate. Indeed, the proofs of the actual infinitesimals that we have examined in a more or less explicit way use Archimedes' postulate. These proofs are unacceptable, even with recourse to the proto-physical concept of "nature" of segments and the related calculus. From a foundational point of view, Hilbert in [Hilbert 1899, in particular see § 12 chap. II] analyzes the possibility of non-Archimedean geometry. But, in general, from a point of view pertaining to content, many analyses and proposals of theories in which one considers infinitesimals are still lacking sufficient rigor, except for the Levi-Civita proposal [Levi-Civita 1892-1893] and research on non-Archimedean fields. Among the explicit opinions

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16. Among other of Veronese's remarks we see that he interprets, unlike us, the symbol $\infty$ (used by Peano) with $\omega$.

against the actual infinite and infinitesimals, we report those of Poincaré and of Russell. The first, wrote: "The actual infinity does not exist: the Cantorians have forgotten it and they have obtained the antinomies" [see Poincaré 1906, 316–317] and Russell affirms: "It follows that the infinitesimals, in order to explain the continuity must be considered unnecessary, erroneous and self-contradictory [...] we have shown that differential and integral calculus do not need infinitesimals" [see Russell 1948, 480, 510].

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# Mario Pieri's View of the Symbiotic Relationship between the Foundations and the Teaching of Elementary Geometry in the Context of the Early Twentieth Century Proposals for Pedagogical Reform

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**Résumé :** Dans cet article nous discutons les enjeux des essais de réforme de l'enseignement de la géométrie élémentaire au tournant du xixᵉ siècle en examinant la contribution de Mario Pieri, membre distingué des entourages de Peano et Segre à Turin. Le rapport symbiotique entre axiomatique et pédagogie et le rôle de l'intuition dans l'apprentissage sont deux aspects majeurs à cet égard. La notion d'intuition a été l'objet d'un grand nombre d'analyses qui ont une longue histoire, de la part de philosophes, mathématiciens, didacticiens des mathématiques, psychologues, historiens. Pour établir le contexte de la réflexion de Pieri, on cherche tout simplement à offrir un bref aperçu de quelques idées au sujet de son rôle pédagogique, dans l'instruction élémentaire et jusqu'à celle universitaire, qui peuvent avoir façonné les efforts pur l'amélioration de l'enseignement de la géométrie au secondaire au début du xxᵉ siècle. Pieri prend en compte la question de l'intuition dans plusieurs parmi ses travaux d'axiomatisation, et notamment dans ceux qui sont consacrés à la géométrie projective qui a été son intérêt principal du point de vue des fondements des mathématiques. On considère ici cependant surtout ses systèmes axiomatiques pour la géométrie élémentaire, où il adopte l'approche basée sur les transformations géométriques de Felix Klein. Notre objectif est de représenter la pensée de Pieri sur la façon d'intégrer deux types d'intuition,

dites sensible et rationnelle, dans les démarches pour l'amélioration de l'enseignement de la géométrie d'Euclide. Nous montrons comment les vues de Pieri sur l'intuition géométrique et la réforme pédagogique ont été soit ignorées, soit déformées dans plusieurs publications notables au tournant du xxᵉ siècle. En particulier, nous donnons à Pieri une voix en réponse aux commentaires spécifiques formulés au début des années 1900 par Federigo Enriques, Ugo Amaldi et Florian Cajori dans des publications largement diffusées inspirées de Klein.

**Abstract:** In this paper, we discuss a proposal for reform in the teaching of Euclidean geometry that reveals the symbiotic relationship between axiomatics and pedagogy. We examine the role of intuition in this kind of reform, as expressed by Mario Pieri, a prominent member of the Schools of Peano and Segre at the University of Turin. We are well aware of the centuries of attention paid to the notion of intuition by mathematicians, mathematics educators, philosophers, psychologists, historians, and others. To set a context for Pieri's proposal, we only seek to open a small window on views of the pedagogical role of intuition, from primary education to university study that may have informed early 20th century efforts to improve the teaching of geometry at the secondary school level. Pieri addressed the topic of intuition in many of his axiomatizations, including those in projective geometry which was his main area of concentration in foundations. We focus here primarily on his axiom systems for elementary geometry, which embraced the transformational approach of Felix Klein's vision for the subject. Our goal is to convey Pieri's thoughts on how to integrate two types of intuition, denoted as *sensible* and *rational*, in his endeavors to improve the teaching of the geometry of Euclid. We show how Pieri's views on geometric intuition and pedagogical reform were either ignored or misrepresented in several notable publications at the turn of the 20th century. In particular, we give Pieri a voice in response to specific comments made in the early 1900s by Federigo Enriques, Ugo Amaldi, and Florian Cajori in widely circulated publications inspired by Klein.

# 1    Introduction

In this paper, we examine perceptions of the role of intuition in certain early 20th century proposals for reform of Euclidean geometry. We begin with a discussion of *sensible* intuition and studies in early education that promoted the teaching of intuitive geometry. Our examination of pedagogical reforms emanating from considerations of *rational* intuition forms the basis for our discussion of the relationship between the foundations of geometry and its teaching, as expressed by Mario Pieri in the context of his membership of the Schools of Giuseppe Peano and Corrado Segre at the University of

Turin. We show how Pieri's ideas were ignored or misrepresented by certain notable scholars who recognized the close connection between mathematical formulation and its educational outcomes, and by others who harbored deep rooted prejudices against that connection. In this short paper, we limit our focus to comments about Pieri in three publications inspired by Felix Klein that were widely circulated in the early decades of the 1900s.[1]

## 2  Sensible intuition as an impetus for pedagogical reform: Intuitive geometry

Before the 19th century, "geometry" was essentially a synonym for Euclidean geometry. The *Elements* served as a prism through which to view the subject. Synthetic geometry, in the style of Euclid, dominated the mathematical curriculum, both as a foundation for and educational pathway to mathematics [Giusti 1993, 2], [Rowe 2018, 370].

> The teaching of synthetic geometry was considered gymnastics for the mind that could only be fostered by a system of propositions as those set by Euclid. [Barbin & Menghini 2014, 482]

The use of the *Elements* in the classroom throughout history, for its content and methodology, is complicated to describe, in part due to variations in when and how it was adopted in the educational spectrum.[2] There is no question however, that well before the turn to the 20th century, serious concerns had emerged about its efficacy as a basis for geometry textbooks. There were various reasons for this. Among them was the exclusion of algebraic methods in Euclid's closed system, criticized by those who advocated a more analytic approach to the subject. Another was the belief that the emphasis on a formal exposition of geometry neglects the psychological and emotional needs of students, neither capturing their interest nor encouraging them to think mathematically.

The quest for pedagogical reform drew from a wide spectrum of both mathematics, which included research on its foundations, and mathematics education, in studies that ran the gamut from primary school to university. Scores of textbooks written with the intention of improving the exposition

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1. For broader discussions of Pieri's pedagogy in the context of his tenure at the University of Turin, see *Quaderni di storia dell'Università di Torino*, 10 (2009-2011), https://www.omeka.unito.it/omeka/items/show/396; also [Marchisotto 2010], [Luciano 2012*a*,*b*, 2017], [Marchisotto, Rodríguez-Consuegra *et al.* 2020, § 10].

2. For the most part, academic geometry textbooks had begun to find a place in European secondary schools in the 18th century. For comprehensive discussions see [Karp & Schubring 2014, Section IV].

of Euclid would appear. Among them, editions of [Clairaut 1741-1852] and [Legendre 1752-1833] found their place in 19th century classrooms. That their authors were noted research mathematicians, Alexis Clairaut and Adrien-Marie Legendre, is indicative of a history of concern about pedagogical issues within the mathematical community. Applying strategies of practical geometry, Clairaut replaced Euclid's deductive proofs with geometric constructions and reasoning based on them.[3] Legendre instead embraced Euclid's deductive methodology, but infused it with arithmetic notation and appeals to intuition. His approach was metrical as he envisioned geometry as a "science that has as its object the measure of extent"[4] [Legendre 1752-1833, citation from 1823, 1, Definition VII]. Legendre's text, which was widely used in Italy until 1867, inspired a great number of others, many less rigorous than his, and replete with mistakes. Largely through advocacy of Luigi Cremona, the noted mathematician who devoted himself to the study of geometry and its teaching, an Italian translation of Euclid's *Elements* [Betti & Brioschi 1867] was adopted for the schools. In response to those who believed that the return to a more rigorous text would be too challenging for "less gifted" students, Cremona reminded the Italian public that in Germany there was an increasing number of geometry books designed to be "more accessible to even mediocre intellects" [Cremona 1873, vii], see [Giacardi 2012, Giacardi & Scoth 2014], [Millán Gasca 2011], [Israel 2017].

Concerted national and international efforts had emerged by the turn to the 20th century to address the reform of secondary school mathematics teaching. Notable among them was the *Commission internationale de l'enseignement mathématique* whose founding president was Klein. Intuition in geometry became a focus for discussion.[5] Certain proposals called for the teaching of "intuitive geometry", which promoted strategies that emphasize "informal reasoning", understood as argumentation based on observation for which justifications are not explicitly provided. At the heart of this method of teaching is the idea of *sensible* intuition, as a form of immediate knowing, possibly linked to data provided by the senses from the very presence of the object of knowledge, see [Betz 1933], [Hendrix 1936].

There were many who advocated the teaching of intuitive geometry well before secondary school.[6] The Swiss education reformer, Johann Pestalozzi, a teacher of the geometer Jacob Steiner, believed that the education of the *mind* (the *head*, in his terminology) included three integrated components, *word, number*, and *form* [Pestalozzi, Cook *et al.* 1894, 118]. The teaching

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3. Along with geometric constructions, practical geometry incorporated mensuration techniques and other problem-solving strategies rooted in arts and crafts. It was adopted in middle school education, see [Menghini 2015].

4. This translation is ours, as are all others unless otherwise indicated.

5. For example [Poincaré 1899]. By this time, various interpretations of "intuition" had emerged from philosophical investigations, see [Osbeck & Held 2014].

6. For a discussion of developments in the teaching of geometry to children, see [Millán Gasca 2015].

of geometry (*form*) based on drawing is propaedeutic to graphical aspects of writing and reading, and is therefore central to early education.

The use of diagrams was intimately linked to intuitive reasoning, see [Lorenat 2020]. Students were taught linear drawing to initiate them into the study of geometry, see [D'Enfert 2003]. The German educator, Friedrich Fröbel, who was another of Pestalozzi's students, created a collection of geometrical exercises of decomposition and ratio with 3-dimensional regular forms for his kindergarten students [Fröbel 1826]. William George Spencer[7] wrote *Inventional Geometry* [Spencer 1860], which consisted of Pestalozzian sequences of questions appealing to intuition, designed to familiarize the pupil with geometrical concepts.

Insights about cognition obtained by studying early childhood education suggested other pathways for learning. Texts such as *Geometry for Beginners* [Minchin 1898] by the Irish mathematician and physicist George M. Minchin addressed the roles of instinct and observation in learning. Writers like Mary Everest Boole, wife of logician George Boole, stressed the need to prepare the unconscious mind for the development of a scientific attitude in children by "restoring the vitality of geometric instinct" [Boole 1904, 68]. Édouard Séguin, who pioneered modern educational methods for teaching cognitively-impaired children, considered the motion of the finger or the pencil from one point to another point "avec rectitude et précision" [Séguin 1843, citation from 1897, 123] as the first law of understanding. His ideas to promote understanding by means of exercises that use rods of increasing length was later adopted by Maria Montessori in [Montessori 1909, citation from 1912, 327].

Mentioned here are only a few of the methodologies, residing in studies of early education, which served to inform efforts to reform the teaching of Euclid.[8] At the opposite end of the spectrum were proposals that were largely infused by research at the university level.

# 3 Rational intuition as an impetus for pedagogical reform: Demonstrative geometry

"Demonstrative geometry" has been described as the teaching of geometry with an emphasis on logical reasoning. It pays particular attention to rigor in remedying the logical lapses of the *Elements*. By the early 1900s, it enjoyed a prominent place in the United States curriculum. The American

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7. Spencer's son, Herbert, a noted educator and philosopher, republished the book and wrote an introduction. Many editions were adopted as textbooks to transition students gradually from concrete thinking to abstract thinking.

8. For example, Enriques referenced Fröbel in [Enriques 1900, 26]. Klein weighed in on Pestalozzi [Lorenat 2020, 81]. See [Giacardi 2012], [Roberts 2014], [Millán Gasca 2015].

mathematician, David Eugene Smith, considered one of the founders of the field of mathematics education, proposed that "as an introduction to mathematics, it is the right and privilege of every student to know what demonstrative geometry means since that is where many students first awaken to the significance of mathematics" [Smith 1919, 112–113]. Those who advocated on behalf of demonstrative geometry cited, among its benefits, the power in uniquely developing the habit of deductive thinking [Hart 1924, 172].

At the same time in Italy, the intense production of technical papers, expository essays, textbooks, and publications addressed to mathematics teachers revealed deep connections between logical research and pedagogy, see [Giacardi & Scoth 2014]. The Peano School at the University of Turin flourished as a center for research in analysis, logic, foundations, and teaching. Peano and members of his School championed the role of logic to achieve more rigor in geometry. Prominent among them was Pieri—who explicitly recognized the "pedagogical and didactic" power of purely logical methods, attributing to them "the only capability to expose known truths", with an "economy of labor" as compared with inductive reasoning from experience [Pieri 1906, 56–57(1980, 442–443)]. Yet, while Pieri's immersion in the Peano School at the University of Turin is primarily seen as the context in which to view his foundational works, it is important to recognize also the influence of Pieri's long and enduring membership in Segre's School of algebraic geometry there. Aldo Brigaglia has demonstrated how Pieri's research in the field of algebraic geometry, under the mentorship of Segre, did not conflict with his axiomatic research and in fact closely intersected with it, see [Brigaglia 2012].[9] Referencing Pieri's first axiomatization of projective geometry (three Notes published between 1894 and 1896), Brigaglia observed that Pieri made use of "the *teachings* of Peano to bring to fruition a scientific program developed by Segre" [Brigaglia 2012, 26].

Pieri's proposals for teaching geometry are rooted in the symbiotic relationship between axiomatics and pedagogy. It was almost exclusively in his axiomatizations of elementary geometry [Pieri 1900a, 1908] that he explicitly talked about teaching. He did address the idea of geometric intuition in earlier foundational works. For example, to underscore his intention to develop projective geometry with a *puramente deduttivo ed astratto* approach, Pieri explained that by *abstract*, he meant,

> [...] it disregards any *physical* interpretation of the premises, and therefore also their *evidence*, and *geometric intuitiveness*: unlike another direction (which I would call *physical-geometric*) according to which primitive entities and axioms want to be inferred from direct observation of the external world, and identified with the ideas that are acquired through experimental induction from

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9. Brigaglia included a deep discussion of rigor in the application of the axiomatic method that informs the research of Pieri vis-à-vis his mentors.

certain physical objects and facts (PASCH, PEANO ...). [Pieri 1896, 10(1980, 84)]

In his constructions of *abstract* geometry, primitive notions, arbitrarily chosen, are implicitly defined by the postulates,[10] and theorems are formally derived using Peano's mathematical logic. Geometry, so developed as a *hypothetical deductive* system, distinguishes itself from the classical *content-centered* Euclidean system where primitive concepts are based on verifiable evidence that is "transmitted to derived propositions through the definition of evidence and the demonstration of veracity" [Ingaliso 2011, 242].

Envisioning geometry as a hypothetical deductive science was, in Pieri's view, the optimal way to teach it. It would foster, in students, *rational* intuition, which he defined as "a perception of the logical relationship between principles and consequences" [Pieri 1906, 58(1980, 444)]. Pieri saw great benefits for students to develop such an intuition because,

> a mind educated with general ideas and supported by a reasonable faculty of abstraction becomes capable of perceiving also, beyond the abstract logical sense, the connection between the various propositions and their deductive roles, the concatenation of the parts and their relationships with the whole. [Pieri 1908, 447(1980, 557)]

Pieri emphasized that *rational* intuition is "purely logical and intellectual syntheses, which do not admit anything from *sensible* intuition" [Pieri 1906, 58(1980, 444)]. However, he *only* excluded *sensible* intuition as an instrument to justify deductive reasoning [Pieri 1906, 35(1980, 421)]. In this he agreed with Moritz Pasch[11] who viewed such a use as a "sign of a shortcoming of deductions" [Pasch 1882, 82].

Pieri believed that geometry "as a formal science, should [...] be able to stand and to be understood without ever appealing to [...] intuitive or physical representations".[12] However, he explicitly noted "the heuristic importance, and [...] the didactic value, of [...] concrete interpretation[s] of geometric entities" [Pieri 1906, 47(1980), 413]. As early as 1889, in his annotated translation of [von Staudt 1847], Pieri had advocated for concrete interpretations in works "of geometry aimed especially at youth". He proposed:

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10. Pieri's understanding of primitives and postulates reflected his selective endorsement of views expressed by Peano and Segre, see [Bottazzini 2001].

11. Pasch gave two "conditions" to define the concept of rigor that eschewed intuition [Pasch 1882, 16].

12. Pieri spoke eloquently about intuition in an address [Pieri 1906, 35(1980, 421)] delivered at the University of Catania, see [Ingaliso 2011], [Marchisotto, Rodríguez-Consuegra *et al.* 2020, §9.4.3]. For a discussion of rigor and axiomatics, see [Israel 1981].

> Only after having constructed the figure of a demonstration, by himself and without any preconceptions, do we believe that the reader can assume that he has mastered this. [von Staudt 1889, xxv]

The important message here is that while Pieri advocated abstraction in the deductive process of proof, he saw the need for the visualization of a concrete image and/or construction of a figure to claim comprehension of that process and its outcome.[13]

We extract from these and other comments in his papers, our understanding of Pieri's stance on the pedagogical role of intuition. It is the responsibility of the teacher to promote in students a *rational* intuition of geometry as an abstract science. Among the benefits for students is that they learn how to draw consequences from principles, using logic, in a way that enables them to understand these consequences in relation not only to the principles, but also to one another. Students are encouraged to determine if these consequences conform to their *sensible* intuition of them, not only to assess their comprehension of them, but after deducing them logically, to appreciate the process used to derive them.

The underlying rationale of Pieri's proposal for educational reform thus emerges. He sees no contradiction in promoting both the intuitive and logical aspects of geometry in its teaching.

# 4 Pieri's strategies for teaching demonstrative geometry: Hypothetical-deductive systems and transformations

Pieri saw mathematics developed as a hypothetical-deductive system, not only as a means to achieve rigor and precision, but also as an opportunity to simplify its teaching in ways that promote in students a deeper understanding and appreciation of the subject. He proposed that the combination of the abstract with a transformational approach, in an exposition which simultaneously develops two and three-dimensional geometry, would make the subject more accessible to students.

Pieri believed he had experienced a level of success in his quest for simplicity with respect to his axiomatizations of elementary geometry. He

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13. Observing that Pieri "had proposed construction as a yardstick for assessing geometric mastery of a proof", Lorenat characterized Pieri's "claim for learning geometry" as representing a "crucial" distinction between him and Pasch [Lorenat 2020, 82].

claimed that in his exposition of absolute[14] geometry, he had achieved "such a degree of *deductive simplicity* that educational systems can certainly also take advantage of it" [Pieri 1900a, 3(1980, 185)]. Still, he called for further research to effect an even greater simplification. Answering his own call several years later with an axiomatization of Euclidean geometry [Pieri 1908], Pieri wrote: "Born of such research is the present Essay, which (if I am not mistaken) reaches exactly that degree of simplicity and rigor that I had in mind at that time, and which to my eyes represents the maximum value of this type of investigation" [Pieri 1908, 345(1980, 455)].

What did Pieri mean by *deductive simplicity*? This question is difficult to answer because although Pieri repeatedly emphasized its importance, he did not provide a formal definition.

Victor Pambuccian has systematically explored the many different ways to define the simplicity of an axiom system, see [Pambuccian 1988]. Under the rubric of purely syntactic considerations, one criterion addresses the language in which the axiom system is expressed. In this context "simple language" means having primitives that are both few in number and of lowest possible arity. Others concern the axioms themselves, asking for the fewest number of quantifier alternations appearing in each axiom or seeking a minimization of the number of variables appearing in axioms containing the largest number of variables [Pambuccian 2009, 328]. Another class of simple axiomatizations is the quantifier free one, in languages that contain only operation symbols (but no predicate symbols), see [Pambuccian 2008]. Showing the independence of the axioms in a finite set has been stipulated as a requirement for simplicity, see [Mancosu, Zach *et al.* 2009, § 1.3]. For didactic considerations, axioms have been described as "simpler", when they are "more intuitive" [Lolli 2011, § 4.4].

We can perhaps use some of these measures as a lens through which to examine Pieri's claims for simplicity. With respect to Pambuccian's descriptions of criteria for syntactic simplicity, we note that Pieri relied almost exclusively on principles expressed by first-order sentences in both [Pieri 1900a, 1908]. In this, he distanced himself from Peano who, for example in [Peano 1889], did not distinguish first-order from second-order quantification. First-order logic would only emerge as a coherent framework for logical studies in the 1920s. In the evolution of thought related to this, what Pieri accomplished received little explicit attention until Alfred Tarski brought [Pieri 1908] into the discussion, beginning with an address to a 1927 mathematical congress in Lwów, see [Pambuccian 2002], [Marchisotto & Smith 2007, § 5.2], [Smith 2010],[Marchisotto, Rodríguez-Consuegra *et al.* 2020, § 10.4]. In more recent times, Pambuccian credited Pieri with accomplishing in [Pieri 1908] "the task of achieving the upmost simplicity of the language of Euclidean geometry" [Pambuccian 2009, 328].

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14. This axiomatization focuses on Euclidean geometry as taught then in elementary courses, except for the theorems dependent on the Euclidean parallel postulate, see [Marchisotto, Rodríguez-Consuegra *et al.* 2020, §§ 8, 9.3].

Pieri also addressed another of Pambuccian's simplicity criteria in his advocacy of the efforts of the Peano School to minimize the number of primitives in axiom systems. In both [Pieri 1900a, 1908], he reduced this number to two, respectively "point and motion", and "point and sphere (a single ternary equidistance relation)". When they recommended [Pieri 1900a] for publication, Enrico D'Ovidio and Segre observed,

> This is a most notable result; and it does not seem that others previously have achieved such simplicity [...] From the purely logical point of view the system of Pieri is fully satisfactory, and contains [...] a result of particular importance in the reduction made in the primitive notions. [D'Ovidio & Segre 1899, 761]

Although Pieri believed that the independence of the postulates is a condition that nearly approaches "ideal perfection" [Pieri 1901, 380(1980, 248)], he did not prove the independence of his postulates for elementary geometry. Nonetheless we can say that he spoke to the criterion for simplicity specified in Mancosu, Zach *et al.* [2009] when he proved the ordinal independence of his [Pieri 1898] postulates for projective geometry.

With respect to Lolli's criteria for simplicity, we note Pieri's deep-rooted belief that the hypothetical-deductive approach would lead students to new questions and deeper intuitions of the subject [Pieri 1900a, 177(1980, 187)]. Consider, for example, his strategy for developing the notion of line in [Pieri 1900a]. He began with a postulate stated solely in terms of the primitives, point and motion,

> For distinct points *A*, *B*, *C*: If there exists a non-identity motion that fixes *A* and *B*, it will also fix *C*. [Pieri 1900a, §1 Postulate VIII, 182(1980, 192)]

Pieri remarked, "This is a principle of great deductive capacity [...]. It is now given to us to produce and develop through it the notion of 'line' and to recognize some of its more notable properties." Observing that postulate VIII (and his definition of collinear points based upon it) cannot describe the geometry of hyperspace, Pieri explained that he chose to consider "only elementary geometry, seeking as much as possible to establish the principles in a manner more suitable to deductive simplicity" [Pieri 1900a, 182(1980, 192)]. To develop the notion of line, Pieri followed his usual practice of formulating definitions explicitly (in terms of primitives) or implicitly (in terms of postulates). He used postulate VIII to define collinearity in terms of motion: a point is collinear with two given points if there exists a non-identity motion that fixes all three points. A straight line on two given points is then the line of points collinear with them, a unique line that remains motionless when it rotates around those two points, see [Marchisotto 1992].

Ugo Amaldi observed that with [Pieri 1900a], Pieri had established "a rigorous logical structure" based on this definition of line [Amaldi 1912-1914, 42]. Amaldi's use of the term "rigorous" would be, for Pieri, an affirmation of

the simplicity he sought in developing geometry as "a study of a certain order of logical relations" [Pieri 1908, Preface 347(1980, 457)]. Indeed to that very description in [Pieri 1908], Pieri (in a footnote referencing [Halsted 1904, 189]) included a quotation from [Hilbert 1900] proposing that the most rigorous method is often the simplest and the easiest to comprehend.

These examples suggest that Pieri saw the rigorous application of axiomatic method as one which enabled an exposition of geometry that he could characterize as *deductively simple*.[15] To that end, using a minimal number of primitives, he "unfolded" geometry axiomatically, introducing postulates and definitions only as needed to derive theorems by applying the laws of logic. His proofs were rigorously executed and extraordinarily detailed, showing precisely which postulates, definitions, and previously-proved theorems justified his statements. In these ways, Pieri hoped to promote in students a mindfulness of the deductive role that collections of postulates played with respect to when and how theorems are proved. He went to great lengths to encourage deep thinking about geometric propositions, exposing how postulates, definitions and theorems relate to each other in relation to the geometric edifice being constructed.

Still, Pieri's quest for simplicity was not restricted to logical deduction. His vision for teaching geometry as an abstract science emerged not only from his embrace of deductive logic, but also from his pioneering use of Klein's transformational approach to geometry, see [Marchisotto, Rodríguez-Consuegra *et al.* 2020, §§ 10.2, 10.4]. He believed the integration of these elements could address the challenges of "reconciling the needs of schools with the ideals of the deductive method". In his review of a secondary school textbook written by Giuseppe Ingrami [Ingrami 1899], Pieri described such a reconciliation as an "undertaking if it ever comes to be", which will be a result of "the toil and exertion of many" [Pieri 1899, 181].

For Pieri, the transformational approach would facilitate the teaching of geometry as a hypothetical-deductive system in a way that encouraged students to appreciate the duality between the abstract nature of mathematical objects and their concrete representations. He noted that,

> The act of using the simplest motions, such as translations, rotations, symmetries, and so on, and their products, more broadly than usual in definitions and arguments confers on the system as a whole a certain ease of manner that is not devoid of clarity and effectiveness. [Pieri 1901, 384 (1980, 252)]

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15. Pambuccian (in a personal communication of June 2020) has suggested that since Pieri referenced deduction, his idea of simplicity can be interpreted in the arena of what is today known as Hilbert's "24th Problem". See [Pambuccian 2019], which analyzes the manner in which the simplicity of proofs could be defined using concrete examples from other works of elementary geometry.

In his axiomatization of absolute geometry [Pieri 1900a], Pieri followed [Peano 1894] in choosing direct motion as primitive.[16] The Italian philosopher Antonio Aliotta called this choice "intuitive" [Aliotta 1911, citation from 1914, 317]. Bertrand Russell noted how Pieri used it[17] to its best advantage,

> Pieri has shown, in an admirable memoir, how to deduce metrical geometry by taking point and motion as the only indefinables. [Earlier] we objected to the introduction of motion, as usually effected, on the ground that its definition presupposes metrical properties; but Pieri escapes this objection by not defining motion at all, except through the postulates [...]. The straight line is the class of points that are unchanged by a motion that leaves the two points fixed. The sphere, the plane, perpendicularity, order of points on a line, etc., are easily defined. This procedure is logically unimpeachable, and is probably the simplest possible for elementary Geometry. [Russell 1903, § 395]

Consider, for example, Pieri's treatment of perpendicularity in [Pieri 1900a, § 2 P19, 190–191 (1980, 200–201)]. He "easily" defined it in terms of the existence of a motion,

> Let $A, B, C$ be points such that $A$ is different from $B$ and $C$. We say $AC$ is perpendicular to $AB$ if and only if there exists a motion that leaves $A$ and $B$ invariant while transforming $C$ into a point of $AC$ different from $C$.

He next gave two ways to interpret this definition,

> Given non-collinear points $A$, $B$, $C$, in the rotation of their plane onto itself about $A$ and $B$ as hinges, $C$ falls back onto the line $AC$.
> There exists a proper (non-identity) motion that leaves the points $A$ and $B$ fixed, bringing the line $AC$ back onto itself.

After which, illustrating the melding of its transformational and abstract aspects, he set the discussion in the context of the scheme of the logical-deductive plan,

> Here orthogonality is introduced in the form of a relation among three given points and no other, thus restored to its primitive terms and divested of all that is superfluous (with respect to our system). Thus, it is in the nature of algebraic logic.

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16. For a broader discussion of the commonalities and differences between Peano and Pieri relative to their treatment of geometry and its transformations, see [Marchisotto 2011].

17. Despite the fact that no direct influence has been shown so far, Pieri's approach has proved to be extraordinarily fruitful. In particular, Johannes Hjelmslev's approach, continued by Arnold Schmidt, and finally by Friedrich Bachmann, created an axiomatization of absolute geometry based entirely on "motions", see [Pambuccian, Struve *et al.* 2017], [Marchisotto, Rodríguez-Consuegra *et al.* 2020, § 7.5].

Thus, Pieri saw the integration of abstract and transformational approaches as a compelling pedagogical strategy. To further enhance its effectiveness, he appealed, albeit not explicitly, to fusionism.[18] Riccardo de Paolis had proposed this method, which treats plane and solid geometry simultaneously, to improve its teaching. He noted, "[...] there exist many analogies between certain figures of the plane and certain figures of space" for which reason "in studying them separately we lose the knowledge of everything these analogies teach us and voluntarily fall into useless repetition" [De Paolis 1884, Appendix 2].

Cremona had suggested that stereo-metric considerations often suggest a way to simplify complicated proofs in plane geometry and make them more intuitive.[19] He recommended alternating theorems of plane and solid geometry in the high school teaching of projective geometry to sharpen the intellect of students and help them develop "geometric imagination" [Cremona 1873, xi]. Yet traditions in teaching and other methodological commitments (related to "purity of methods") impeded the adoption of such strategies for Euclidean geometry [Arana & Mancosu 2012, 303].

Pieri, however, was not dissuaded. Fusion served his didactic objectives. It helped him show, among other things, that a geometric property does not exist in isolation. Pieri believed that when a property

> appears at the same time in several hypotheses and in different propositions, [...] it comes to connect with other properties: then only the logical process of deduction is sufficient to recognize in these other properties the existence of new bonds and new connections. [Pieri 1906, 42(1980, 408)]

For example, in [Pieri 1900a, § 3P20, 197-98(1980, 207-08)], Pieri demonstrated how one property can be used to define the reflection point $B$ of a point $A$ across a line $r$ or across a plane $\Pi$: that the point for which the midpoint of $A$ and $B$ is the foot of the perpendicular from $A$ to $\Pi$. It was Pieri's practice to append to his formal definitions, comments intended to clarify their meaning. Such observations here appended to P20 demonstrate the confluence of his transformational and fusionistic approaches,

> To say that points $A$ and $B$ are symmetric to each other with respect to a line $r$ or with respect to a plane $\Pi$ will be like asserting that these points are both on a line that meets $r$ or $\Pi$ orthogonally at the midpoint of $AB$.
>
> By means of a line or a plane, then, there is determined a certain representation of points by points (of space into itself),

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18. See [Arana & Mancosu 2012] for a discussion that includes an exposition of historical, methodological and foundational aspects of fusionism.

19. When he was in Rome, Cremona studied with De Paolis. After reading [Pasch 1882], he sent to its author a copy of [De Paolis 1880-1881] on the foundations of projective geometry, see [Millán Gasca 2017].

such that each point $A$ of $r$ or of $\Pi$ is associated with the same point $B$.

This transformation or geometric correspondence is to be called axial symmetry with respect to $r$ in the one case, and planar symmetry with respect to $\Pi$ in the other.

Pieri next focused attention on the abstract deductive scheme by advocating the consultation of previous propositions (§ 1P11, § 1P20 and § 3P1) for an understanding of why a planar symmetry cannot be a direct motion (all the points of a certain plane $\Pi$ are fixed there, but not all the points that exist). He then showed that any arbitrary axial symmetry is a direct motion, which exists by virtue of § 3 P20. Pieri proved that given three noncollinear points, any motion that fixes two of them and transforms the third into its symmetric point with respect to the line joining those two points will also have to change any other point into its symmetric point with respect to that line. His proof of this statement (summarized here) illustrates the level of detail in his exposition, constructed to reveal precisely how deduction logically proceeds from definitions and previously proven propositions:

Given noncollinear points $A$, $B$, $C$, let $\mu$ be a motion that fixes $A$ and $B$ and transforms $C$ into its symmetric point with respect to $AB$. By § 1P5, § 1P20, § 1P22, § 1P24, § 2P17, etc., $\mu^2$ is a motion that fixes $A$, $B$, and $C$, hence by § 1P11 leaves every point invariant. Therefore, if $\mu(Z) = Z'$, the points $Z$ and $Z'$ are permuted with each other by $\mu$. Thus, by § 2P5-P7, etc., their midpoint $X$ is fixed and thus will have to lie on $AB$, since $\mu$, by hypothesis, is a (non-identity) motion. It follows by § 2P19, § 2P27, etc., that either $Z = Z' = X$, or $XZ$ is perpendicular to $AB$. Thus in each case, $Z'$ is the point symmetric to $Z$ with respect to $AB$, and $\mu$ is none other than the rotation of the plane $ABC$ onto itself about $A$ and $B$ as hinges (see § 2P11), etc. [Pieri 1900a, § 3P21, 198(1980, 208)]

We have provided these few examples to convey the essence of Pieri's proposals for reforming the teaching of Euclidean geometry and to illustrate how his views on intuition informed them. We next address how Pieri's ideas were interpreted by several of his contemporaries.

# 5    A chorus of voices: the reception of Pieri's pedagogical views

Pieri made his proposals for reform at a time when there was considerable support for the idea that research in foundations could help transform pedagogy. Those who endorsed building on the connections between foundations of

mathematics and its teaching debated about the paths to do so, the underlying epistemological ideas, the rudiments of student understanding, and more—producing a *chorus* of voices, interpretations, and projects. We show how Pieri was represented in three influential publications in the early decades of the 20th century that served as forums for this *chorus*.

We begin with the prestigious *Encyklopädie der mathematischen Wissenschaften*[20] [Meyer & Mohrmann 1907-1934], and the article, entitled "Prinzipien der Geometrie" written by Federigo Enriques [Enriques 1907]. In his reference to [Pieri 1900a, 173], where Pieri had described what he meant by hypothetical-deductive system, Enriques observed that "*M. Pieri* defined 'segment' using the concepts 'point' and 'motion'" and, to that end, "had developed a system of postulates". He continued,

> It should be noted that so far *Pieri* alone has formulated in a complete way the postulates. However, these postulates, mainly due to the fact that the primitive concepts of order (i.e., the attributes of the straight line regarding only its being a line) were deleted, come in an extremely complicated form and lose all clarity and intuitive certainty (*[sic]* anschauliche Gewißheit); however Pieri attaches no importance to this feature. [Enriques 1907, citation from 1907-1910 § 6, 33]

Enriques' assertion that Pieri "attaches no importance" to the "intuitive certainty" of his postulates is a misrepresentation. Indeed a year before Enrique's article appeared, in an address for the inauguration of academic year at the University of Catania, Pieri explicitly warned against denying the role of "ingenious intuition", in the logical process [Pieri 1906, 79–80(1980, 445–446)].

In his revision for the 1911 French edition of the encyclopedia, Enriques focused his comments about Pieri's postulates more directly on the notion of "evidence", saying:

> It should be noted that so far *M. Pieri* alone has formulated in a *complete* way, the postulates of which he makes use. It should be added, however, that these postulates come in an extremely complicated form and lose all obviousness (*[sic]* tout caractère d'évidence) relative to the intuition we have of them. This is mainly due to the fact that the primitive concepts of order (that is to say, the attributes of the line as a line) were removed by *M. Pieri*. Besides, *M. Pieri* attaches no importance to the greater

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20. The first volume of the first edition of this German encyclopedia appeared in 1899. The original project, initiated by Klein, Heinrich Weber and Franz Meyer, sought to compile and present a comprehensive review of the science of mathematics and its allied fields. It was considered a monumental and ambitious task which aroused great interest among contemporary mathematicians [Ore 1942, 653].

or lesser *evidence* of his premises. [Enriques 1911-1915, citation from 1911, § 12, 33]

It is likely that Enriques' italicized reference to "evidence" spoke to his belief that the postulates of geometry are just rigorous forms of the intuitive concept of physical space, and his recognition that this was a belief to which Pieri did not subscribe. However, saying that Pieri attached "no importance" to the evidence of his premises is an inaccurate characterization. Although he believed that the *source* of primitive ideas and postulates resides "in the domain of abstractions", Pieri also stressed that they "must find an image [...] exact and in accordance, if not perfect agreement, with every sort of objects and phenomena to which one would apply the system in whole or in part" [Pieri 1901, 379(1980, 247)].

Amaldi was more accurate in conveying Pieri's stance on intuition. Amaldi had collaborated with Enriques on a noted anthology [Enriques 1900] that had been compiled as a resource for making changes in higher mathematics teaching for the preparation of secondary school teachers.[21] While he did not cite Pieri in his essay for the first edition, "Sui concetti di retta e di piano" [Amaldi 1900, 33–64], he did reference [Pieri 1898, 1900a, 1908] in the expanded version of that essay for the second edition [Amaldi 1912-1914, 37–91]. Unlike Enriques, Amaldi understood and reported that Pieri made no appeal to intuition in *formulating* his postulates:

> Starting from the formal point of view of reducing the number of primitive ideas [...] leaving aside any other need for the postulates with respect to intuition, [...] first analyzing the principles of projective geometry and then those of elementary geometry, he showed that all geometry can be built with only two primitive ideas, those of *point* and *distance*. [Amaldi 1912-1914, 79]

Thus, Amaldi recognized the fact that Pieri excluded the consideration of intuition in composing his primitives (and the postulates that define them). However, Amaldi neglected to convey Pieri's views, from both foundational and pedagogical perspectives, on the role of intuition in interpreting them. Therefore, Amaldi had told only part of the story. Pieri had emphasized,

> Learning geometric facts is greatly helped by always having at the onset an image or intuitive representation of a "point" and of the "sphere through one point centered at another": that is, the habit of contemplating the real and concrete sense that usage provides for statements such as "*A*, *B*, *C* are points, and *C* is as distant from *A* as *B* is". [Pieri 1908, 447(1980, 557)]

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21. The spirit that led to the publication of the first edition was pervasive in Italy and beyond. For example, Julio Rey Pastor, who believed Enriques was among those mainly responsible for the influx of foundational research in secondary education, brought this same spirit to Spain and Latin America, after spending time in Italy in 1914, see [Millán Gasca 1990], [Giacardi 2012].

Pieri's words here call to mind those of a fellow member of the Peano School, Giovanni Vailati, who characterized [Pieri 1908] as a "step forward" in treating the subject from "*the most general possible viewpoint*—compatible with the concrete material to which it refers" [Arrighi 1997, letter 126, June 25, 1908]. Amaldi appeared to agree, observing that [Pieri 1908] provided "a complete analysis of the foundations of Euclidean geometry", and using the words "perspiciacious" and "simple" to characterize Pieri's system [Amaldi 1912-1914, 79]. Unfortunately, Amaldi's comments were removed in later editions of the anthology, while Enriques' mischaracterizations continued to be widely circulated for decades in the many editions and translations of the encyclopedia.

Enriques and Amaldi were prolific in the area of pedagogical research, authoring textbooks and publications devoted to teaching and the training of teachers. Theirs were strong voices in the *chorus*, seeking to promote insights drawn from recent foundational, logical, psychological and historical research to improve the teaching of elementary geometry, while remaining faithful to Euclid. There is little reason to believe that their characterizations of Pieri would have motivated those seeking improvements in pedagogy in the early decades of the 20th century to examine Pieri's work. In the United States, a widely circulated didactic publication may have done a similar disservice to Pieri, in the context of its reporting on the research of the Peano School.

In [Cajori 1910], Florian Cajori, a Swiss-American historian and mathematician published a survey of worldwide pedagogical practices. Drawing from Klein, who used the word "radical" in his references to the Peano School [Klein 1909, citation from 2016, 262], Cajori reported,

> A very remarkable school came into being in Italy, the purpose of which is to render geometry still more rigorous than in the Euclidean text. [...] the great school of Peano, which endeavors to eliminate all intuition [...] has influenced even elementary instruction and the teaching in technical schools. This recent Italian emphasis upon extreme rigor has led to deplorable results with the less gifted pupils, and a reaction appears to be setting in. [Cajori 1910, 192]

The charge of endeavoring to "eliminate all intuition" is certainly not one that applies to Pieri, nor, in fact, to others in the Peano School, see [Luciano 2012*b*].

Cajori's survey was reprinted in a 1912 Report issued by a Committee that had been formed under the joint auspices of the National Education Association and the American Federation of Teachers of the Mathematical and Natural Sciences.[22] Cajori's negative characterization of the Peano School thus

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22. This "National Committee of Fifteen on Geometry Syllabus" was composed of representatives from universities and secondary schools. Although it had not completed its work in 1910, this committee advocated the early publication of Cajori's article, noting: "this historical setting prepared by Professor Cajori should be in the

had the potential to dissuade many from considering the pedagogical proposals of its members.

# 6   Pieri, in conversation with Klein

Klein was a significant influence for the works we cited in section 5. He was very much a hero to Pieri although their relationship was complex, see [Marchisotto, Rodríguez-Consuegra *et al.* 2020, §10]. In this section, we reference an 1897 exchange of letters between Klein and Pieri. The words of the two mathematicians are particularly pertinent to our discussion about Pieri's pedagogy.

David E. Rowe observed that "Klein, like Poincaré, saw the burgeoning interest in abstract structures and axiomatics around the turn of the century as a potential threat to the lifeblood of mathematics", but that being said, one "would be mistaken to think that Klein had no appreciation for axiomatic thinking" [Rowe 1989, 198]. A letter of March 31, 1897 reveals that Klein had engaged with Pieri in a discussion of axiomatics and intuition, relative to teaching [Arrighi 1997, Letter 65].

Referring to the "purely deductive mode of representation", which he acknowledged "to be scientifically very important", Klein had asked Pieri "what significance is to be given to this in teaching beginners". Klein advised, "in teaching it is necessary to begin with intuition (in order then to gradually ascend to more abstract views)". In his April 9, 1897 response to Klein, Pieri wrote,

> To answer your question about the influence that these purely deductive research can have over the elementary teaching of mathematics (where intuition must have a most essential part) I will tell you that, in my opinion, many improvements in the strictly deductive sense, [...] would perhaps help to make these doctrines easier to understand than they are at present: since the greater abstraction would be compensated by the greater simplicity of the fundamental concepts. [Klein papers, Staats- und Universitätsbibliothek, Göttingen, 22F, 97–99]

In this exchange, we see once again that Pieri fully acknowledged the importance of intuition in the elementary teaching of mathematics. We also see, in the presence of this acknowledgement, Pieri's rationale for advocating an abstract approach.

Still Pieri could find no place for his pedagogical strategies, even among those like J. W. Young who admired his foundational works, see [Marchisotto &

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hands of mathematical teachers at once". The survey was reprinted in their Final Report, published in 1912 in [National Committee of Fifteen on Geometry Syllabus 1912].

Smith 2007, 135, 142–43]. In [Young 1911, 164], Young indicated he could not recommend Pieri's assumptions for use in the school because "the subject must be presented to a boy of fourteen years in a different way from that employed in presenting it to a mature mind [...]", it being "necessary, in the beginning, to make continued and insistent appeal to concrete geometric intuition". This was precisely Klein's view. But it does not contradict what Pieri himself suggested. For the teaching of geometry, Pieri wrote,

> [...] it will never be superfluous to appeal [...] to [...] empirical methods to emphasize and bring alive for the young all sorts of intuitive and experimental cognitions of various geometric objects.
> [Pieri 1908, 447, Note 2(1980, 557)]

Young had advocated starting with an informal treatment of geometry so that "the pupil could be led to see the advantages of the more formal methods that follow" [Young 1911, 163–164]. In our view, it is not clear why that formal treatment could not be along the lines of that proposed by Pieri. Indeed, Louis Couturat called [Pieri 1900a] "the most profound analysis of the principles of geometry" [Couturat 1905, 193].

# 7    Concluding remarks

Gino Loria, Pieri's colleague at the University of Turin, applauded Pieri's efforts to improve pedagogy. He called [Pieri 1900a] "a notable result, not just because of its simplicity and originality, but also because it seems directed toward those schools which would banish motion from pure geometry". Loria added, "The future will decide whether Pieri's ideas can lead to a useful reform of elementary instruction. What is without question, however, is that they deserve the attention of scholars and teachers" [Loria 1899, 426].

That did not happen. Pieri's proposals remained largely hidden from view. Enriques' publications with Amaldi, like those of Klein, which inspired them, had a great impact on pedagogical reform in the early decades of the 20th century. Cajori's historical survey of pedagogical practices enjoyed wide circulation. Pieri's ideas were absent or misstated in them. Sadly, the lack of dissemination and misrepresentation of Pieri's ideas are also evident in examining Peano's publication, the *Formulario*, see [Luciano 2017].

Pieri's pedagogical proposals did not garner attention in the transnational reform efforts of *his* era. For scholars and teachers of *this* era, we have endeavored to shed light on Pieri's ideas about how to integrate two types of intuition, denoted as *sensible* and *rational*, in efforts to improve the teaching of the geometry of Euclid. We have also wanted to reveal how certain representations of Pieri's ideas in the decade prior to his early death at age 52 in 1913, may have contributed to the obfuscation of what he truly proposed for such reform.

It is our hope that we have provided reasons why Pieri's ideas should have been heard in that *chorus* of voices, interpretations, and proposals being discussed at the turn to the 20th century. We trust we have also conveyed why today's mathematicians and mathematics educators would benefit from hearing his voice.

## Acknowledgements

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# Gödel's Reading of Peano's *Arithmetices Principia*

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**Résumé :** Pour préparer son article sur la logique mathématique de Russell (1944), Gödel a lu attentivement les *Arithmetices Principia* de Peano. Les six pages résumant l'œuvre péanienne qu'il a écrites en employant la sténographie de Gabelsberger contiennent une analyse remarquable de la structure formelle des preuves de Peano, en contradiction manifeste avec l'opinion commune selon laquelle le traité de Peano ne contiendrait aucun mécanisme déductif formel.

**Abstract:** In preparation for his article on Russell's mathematical logic (1944), Gödel read carefully Peano's *Arithmetices Principia*. His six pages of summary in the Gabelsberger shorthand contain a remarkable analysis of the formal structure of Peano's proofs which is diametrically opposed to the common view that Peano's treatise contained no formal deductive machinery.

## 1   Introduction

In early 1943, Gödel left behind an incredibly intensive period of formal work, the main aims of which had been to prove the independence of the axiom of choice and of the continuum hypothesis in set theory on the one hand, the consistency of analysis on the other, with possibly the cardinality of the continuum decided—his conjecture was that it is $\aleph_2$. All of this can be seen through a study of his 16 mathematical *Arbeitshefte*: *Heft* 4 was begun after he had arrived to Princeton for good in late March 1940, *Heft* 15, eleven notebooks later, has at page 57 the date 14.X.42, and *Heft* 16 is the last one, with notes for his planned article "Russell's mathematical logic" [Gödel 1944] from page 9 on. Most notebooks run toward a hundred pages in length.

There is a notebook of Gödel's from the early 1930s, called the *Altes Excerptenheft*, that contains summaries of his reading of works for the writing of his part of the planned joint book with Arend Heyting, the *Mathematische Grundlagenforschung* [Heyting 1934]. He was unable to finish in time, mainly because of his incredible meticulousness, but his texts for two chapters for the book have survived [cf. von Plato 2021*a*].

From page 95 on, the *Heft* of excerpts for the book is written in a different style, and it even mentions the planned Russell article by which this part is from the early 1940s. The summaries are mainly about the work of Russell and Frege. Peano is not among the sources covered, and even the earlier parts of the *Heft* contain just the *Formulaire* project of Peano [Peano 1895], and readings from Peano's journal *Rivista di Matematica*. There are no signs that he would have studied the *Arithmetices Principia* directly [Peano 1889].

Gödel wanted, I believe, to be well prepared for the task of writing the article on Russell's logic, and to take a deep look at the work of Peano who so much influenced Russell, he ordered a copy of the *Arithmetices Principia* from a library in Chicago in early 1943. There is a summary of his reading of Peano, with a beautiful first page in which he tried to reproduce the appearance of the title page of Peano's little treatise. His six-page summary is among a large collection of excerpts written on loose papers at different times, some clearly from the 1930s, others later, in reel 36 of the Gödel microfilm collection, frames 416–421, document number 050135.

One sorry little corner in the historiography of logic and foundations concerns Peano's formalization of the rules of logical inference. In Jean van Heijenoort's widely studied *From Frege to Gödel* [van Heijenoort 1967], the claim is that there is "a grave defect. The formulas are simply listed, not derived; and they could not be derived, because no rules of inference are given [...] he does not have any rule that would play the role of the rule of detachment" [van Heijenoort 1967, 84], [cf. also von Plato 2021*b*, this journal]. It is an assessment that can at best be taken as a sign of total blindness. Gödel had from early on seen that Peano has precise rules of inference that are, however, not made explicit. His 1932 summary of Peano's 1895 version of the *Formulaire des Mathématiques* is found in one of his *Excerptenhefte* where he lists Peano's propositional axioms, then writes:

> Rules: implication and substitution of equals (not formulated but used).[1]

Gödel's summary of the *Arithmetices Principia* leaves no space for doubt about the formal character of proofs in Peano; this will be evident from the text itself to follow. It is clearly at the center of his comments.

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1. Found on frame 515 left part, reel 20 of the Gödel microfilm collection, written in a mixture of German and Gabelsberger shorthand. Transcription in German is: Regeln: Implikation und Einsetzung für gleiche (nicht formuliert aber angewendet).

Peano's treatise is written in Latin, a language in which Gödel was a great expert. The text that follows maintains the Latin, but I have transcribed and translated into English Gödel's German remarks that are written in the Gabelsberger shorthand. Peano's treatise is readily available online; the translation in [van Heijenoort 1967] can be of help with the Latin passages.

Gödel's summary turns into a detailed commentary on Peano's system of proof in the middle of his page 5.

# 2 Gödels' summary of the *Arithmetices Principia*[2]

1.

Arithmetices Principia

Novo Methodo Exposita

A

*Ioseph Peano*

in R. Academia militari professore

Analysin infinitorum in R. Taurinensi Athenaeo docente.

LABOR ET HONOR[3]

Augustae Taurinorum

Ediderunt Fratres Bocca

Romae                               Florentiae

Via del Corso, 216–217          Via Cerretani, 8

1889

This is an exact transcription of the title page of the book that I had borrowed from the University of Chicago, 18. Jan – 4 Feb. 1943, number: QA 142.P35

2.

*Sign. Tabula p. VI*

*Praefatio*    P III – V    bad Latin: Hic difficultas,

arithmeticae applicationes  [instead of ad]  processus  -i

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2. Kurt Gödel Papers, Box 10a, Folder 39, item accessions 050135, on deposit with the Manuscripts Division, Department of Rare Books and Special Collections, Princeton University Library. Used with permission of Institute for Advanced Study. Unpublished Copyright Institute for Advanced Study. All rights reserved.

3. ⟦Design of a circular stamp of the publishing house.⟧

caput scripti = the main thing, namely the precise formalization, beyond that, some theorems formulated unclearly

| Citations: | Boole | Camb. & Dublin Math J. | 1848 |
|---|---|---|---|
| | Pierce | Americ. J. | III 15 |
| | | | VII 180 |
| | Mc Coll | Proc. Lond. Math. IX 9 X 16 | 1878 |
| | Schröder | *Lehrbuch der Arithmetik und Algebra* | 1873, |
| | | *Der Operationskreis des Logikkalküls* | 1877 |
| | Jevons | *Princ. of Sci.* | 1883 |

**Logicae notationes** p. VI –XVI

1. Signum   $\supset$   significat   "deducitur"

   $\subset$                    est consequentia

2. Defined          in general: $-aRb$ the same as $a-Rb$

   in particular, $a-=_x\Lambda$ means: there exists $a$

   Similarly          $aR\cup Sb =_{Df} aRb \vee aSb$,

   for example $<\cup= aRSb =_{Df} aRb \,.\, aSb$

   For example,    $a \supset -= b$ means $a \supset b \,.\, -(a = b)$

3.

3. classis sive entium aggregatio

   $a \supset b$ means $a$ is a part of $b$

4. Classes with one element are identified with individuals, i.e.:

   If $k \subseteq s$, then:

   $k\,\varepsilon\,s \equiv k$ is a unit class $\equiv (x,y)(x,y\,\varepsilon\,k \supset x = y \,.\, k \neq \Lambda)$

5. $[x\,\varepsilon\,]a$ means: solutiones vel radices conditionis $a$

   Similarly $[(x,y)\,\varepsilon\,]a =$ the class of pairs $(x,y)$. The pair appears as a basic concept.

6. $\mathtt{\partial}Rx$ means $R^{\mathbf{66}}x \quad \mathtt{\partial} =$ qui (quae)

   In particular, $\mathtt{\partial} < y \qquad \mathtt{\partial}Dx$, the divisor of $x$

   It follows that $x\,\varepsilon\,\mathtt{\partial}\,\alpha y \quad$ (the same denoted by $[\varepsilon\,<]y$ and then generally).[4]

   Especially also: $\mathtt{\partial}\,\varepsilon\,\alpha = \alpha$

7. $(x', y' \ldots)[x, y \ldots]\alpha =_{Df} Subst(\alpha^{x\ y\ldots}_{x'y'\ldots})$

8. $\varphi\,\varepsilon\,F^{\mathbf{6}}s. \quad \varphi$ is a function over $s$ (that is denoted by putting it ahead).[5]

   Therefore $a + . \ \varepsilon\, . \ F^{\mathbf{6}}N, +a\, . \ \varepsilon\, . \ N^{\mathbf{6}}F$[6]

9. $[\varphi]$ denotes the inverse. In case of ambiguity, the class is associated)

   *Square brackets in the above usage* mean *an inversion*, namely

   1. $[x\,\varepsilon](x\,\varepsilon\,a) = a, x\,\varepsilon\,([x\,\varepsilon]\alpha) = \alpha$

## 4.

2. Let $\alpha$ be a formula that contains the letter $x$ and $\varphi$ the sign for the function thus defined. Then we have $\alpha = \varphi x$, therefore $\varphi = \alpha[x]$ and $\varphi x' = \alpha[x]x'$. Analogously for $\alpha = x\varphi$

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4. That is probably an error in printing and should be: $[\varepsilon\,]< x$.
5. That the function $F$ is defined for $x$, is written as $F(x) = F(x)$.
6. ⟦The second comma should be inverted as in Peano, p. XIII.⟧

3.  From $dF(x) = f(x)dx$ follows    symbolically    $F(x) = [d]f(x)dx$

therefore        $F(x) = \int f(x)dx$

10. Functions applied on classes are defined in the known way.

11. The powers of a "functionis postsignum" $\varphi$ are written as follows: $\varphi n$

Therefore, if $+$ means the operation $+1$, then $a + b$ means automatically the correct one.

### Arithmetices Principia p. 1–20

There are nine axioms, namely five axioms of equality, reflexivity, symmetry, transitivity, invariance for $N$ and for $+$, and further $1\,\varepsilon\,N, N+1 \subseteq N, x+1 \neq 1$ for $x\,\varepsilon\,N$, complete induction, but the last axiom of equality contains
$a + 1 = b + 1 . = . a = b$

Example of a theorem:[4] $\ni D(a,b) = \ni DM \ni D(a,b)$, i.e., the divisors of $a$ and $b$ are the same as the divisors of the *Max* of the divisors of $a$ and $b$.

5.

§ 1 On numbers and addition

§ 2 Subtraction

§ 3 Maxima and minima (of sets of numbers) $M, W$ in relation to sets

§ 4 Multiplication

§ 5 Powers

§ 6 Division, $xDy = x$ is a divisor of $y$, analogously $yⲀx$

§ 7 Various number-theoretic theorems without proof. $x\pi y$ (relatively prime)

§ 8 Rational numbers (here also $\frac{p}{1} =_{Df} p$)

§ 9 Irrational numbers (as "upper limit of arbitrary sets of rational numbers") Add & Mult defined

§ 10 Sets of real numbers, especially treated are the operations, inside, outside, limit and different relations between these. Remark: Everywhere, only *positive* numbers are defined.

Abbreviation of proofs: No distinction is made between "because" and "and," i.e., the theorems cited are always written as conjunctions of conditions. Each proof has the form (in which the theorems cited are left out):

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4. ⟦Peano's theorem 36, p. 11.⟧

$P : \supset . Q . \supset . R . \supset . S . \supset . T$ etc $. \supset . Y$

and the theorem proved reads then $P : \supset . Q \supset Y$.[5] $P$ and $Q$, are, then arbitrary, and all of the previous theorems hold as conditions from $R$ on [so they can be used in the proof]. There stand next to the $P, Q, R$ the theorems cited in a conjunction [or obtained through substitution

6.

of their specializations] that are required for the derivation of the next theorem.[6]

Direct use is very often made of the substitution of equals for equals [on the basis of preceding theorems], in particular also $\lambda$-conversion, the distributivity of $x \, \varepsilon$, etc. Theses, hypotheses, numbers of preceding theorems (possibly with the indication of some substitution) are taken to be simply abbreviations for the formulas in question [these can also contain free variables]. A proof is, then, a chain of trivial implications that are constituted so that:

1·) There occur in the conditions for the proof a set of theorems earlier proved that can be simply put aside.

2·) Then all the members can be put aside so that only the first (or the first two) and the last remain. [Bound variables are often denoted the same as free ones, $x \, \varepsilon \, [x3]$.

The implications proved one after the other are simply the initial part of the entire chain, where the preceding implications separate more strongly than those that follow. This resembles much more the logic of assumptions than Russell.

A definition is, after Peano, a proposition of the form $V \supset . x = a$ in which $x$ is a sign (or a combination of signs) for which no meaning had been given so far and $a$ one for which a meaning had been already given.

With the definition of the rational numbers it is stated: $\frac{p}{q}$ est novum ens.

## Acknowledgement

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5.  $P, Q$ can also be absent in theorems without or with a condition.

6.  I.e., one obtains through a reference to this condition plain trivial implications, and one can strike over afterwards in the conditions that which has already been proved.

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