Precise Determination of the Cabibbo-Kobayashi-Maskawa Element $V_{cb}$

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We extract the magnitude of the Cabibbo-Kobayashi-Maskawa matrix element $V_{cb}$ and the most relevant parameters of the heavy quark expansion from data of inclusive semileptonic $B$ decays. Our calculation includes the recently computed $O(\alpha_s\Lambda_{QCD}/m_b^2)$ corrections and a careful estimate of the residual theoretical uncertainty. Using a recent determination of the charm quark mass, we obtain $|V_{cb}| = (42.21 \pm 0.78) \times 10^{-3}$ and $m_b^{\text{lat}}(1\text{ GeV}) = (4.553 \pm 0.020)\text{ GeV}$.

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Introduction.—The Cabibbo-Kobayashi-Maskawa (CKM) mechanism of quark flavor violation is one of the main components of the Standard Model (SM) of fundamental interactions [1,2]. It accommodates very well all of the observed $CP$ violation, as well as the flavor changing phenomena studied at kaon experiments, the $B$ factories, and at high-energy colliders like the LHC (see [3,4] for recent reviews). The $3 \times 3$ unitary CKM matrix, which parametrizes flavor violation in this context, has only four independent parameters. While they are strongly constrained by present data, any improvement would be welcome as it would sharpen our tools for future tests of the SM.

In particular, more precise measurements of $|V_{cb}|$, the CKM element controlling charged current $b \leftrightarrow c$ transitions, would crucially help the search for new physics in rare decays, which requires accurate SM predictions. Indeed, the present $\sim 2\%$ error on this single CKM element represents the dominant uncertainty on the SM prediction of important flavor-changing neutral current decays such as $B_s \to \mu^+\mu^-$ [5], $K^+ \to \pi^+\nu\bar{\nu}$, and $K_L \to \pi^0\nu\bar{\nu}$ [6], as well as of the $CP$ violation parameter $\varepsilon_K$ [7].

Direct information on $|V_{cb}|$ can be obtained from inclusive and exclusive semileptonic $B$ decays to charmed hadrons, which are subject to different theoretical and experimental systematics. In the first case, the operator product expansion (OPE) allows us to describe the relevant nonperturbative physics in terms of a small number of parameters that can be extracted from experiment. In the case of the exclusive decays $B \to D^{(*)}\ell\nu$, the form factors have to be computed by nonperturbative methods, e.g., lattice QCD. The most precise recent results of each method are

$$|V_{cb}| = (42.42 \pm 0.86) \times 10^{-3}$$

from a global fit to inclusive semileptonic moments [8], and

$$|V_{cb}| = (39.04 \pm 0.49_{\text{exp}} \pm 0.53_{\text{lat}} \pm 0.19_{\text{QED}}) \times 10^{-3}$$

from an unquenched lattice QCD calculation of the zero recoil form factor of $B \to D^\ast\ell\nu$ by the Fermilab-MILC Collaboration [9]. They disagree by $3\sigma$, which remains a long-standing tension. There also exist less precise determinations of $|V_{cb}|$ based on heavy quark sum rules and the decay $B \to D\ell\nu$ (see [4] for a review).

It is also possible to determine $|V_{cb}|$ indirectly, using the CKM unitarity relations together with $CP$ violation and flavor data, excluding the above direct information: SM analyses by the UTfit and CKMFitter collaborations give $(42.05 \pm 0.65) \times 10^{-3}$ [10] and $(41.4^{+4.5}_{-3.4}) \times 10^{-3}$ [11], which are both closer to the inclusive value of Eq. (1).

In principle, the lingering discrepancy between the values of $|V_{cb}|$ extracted from inclusive decays and from $B \to D^\ast\ell\nu$ could be ascribed to physics beyond the SM, as the $B \to D^*$ transition is sensitive only to the axial-vector component of the $V-A$ charged weak current. However, the new physics effect should be sizable (8\%) and would require new interactions ruled out by electroweak constraints on the effective $Zb\bar{b}$ vertex [12]. The most likely explanation of the discrepancy between Eqs. (1) and (2) is therefore a problem in the theoretical or experimental analyses of semileptonic decays.

In this Letter we focus on the inclusive extraction of $|V_{cb}|$, including all contributions of $O(\alpha_s\Lambda_{QCD}/m_b^2)$, whose calculation has recently been completed [13–15], and we discuss how this improvement affects the results.

The calculation.—Let us briefly review the calculation of the quantities that enter the inclusive analysis. The OPE allows us to write sufficiently inclusive quantities (typically the width and the first few moments of kinematic distributions) as double series in $\alpha_s$ and $\Lambda_{QCD}/m_b$. The expansion in powers of the heavy quark mass starts at $O(1/m_b^2)$ [16–19] and involves the $B$-meson expectation values of local
TABLE I. Coefficients of (3) for $m_b^{\text{kin}}$ (1 GeV) = 4.55 GeV and with the charm mass in the kinetic scheme, $m_b^{\text{kin}}$(1 GeV) = 1.091 GeV (first row), and in the $\overline{\text{MS}}$ scheme, $m_b$(3 GeV) = 0.986 GeV (second row) and $m_b$(2 GeV) = 1.091 GeV (third row).

<table>
<thead>
<tr>
<th>$a^{(1)}$</th>
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<th>$a^{(2)}$</th>
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operators of growing dimension. These nonperturbative parameters can be constrained from the measured values of the normalized moments of the lepton energy and invariant hadronic mass distributions in $B \to X_c \ell \nu$ decays:

$$
\langle E^2_\ell \rangle = \frac{1}{\Gamma_{E > E_{\text{cut}}} \int_{E_{\ell} > E_{\text{cut}}} \frac{d\Gamma}{dE_{\ell}} \int_{E_{\ell} > E_{\text{cut}}} \frac{dE_{\ell}}{dE_{\ell}},
$$

$$
\langle m^2_X \rangle = \frac{1}{\Gamma_{E > E_{\text{cut}}} \int_{E_{\ell} > E_{\text{cut}}} \frac{d\Gamma}{dE_{\ell}} \int_{E_{\ell} > E_{\text{cut}}} \frac{dE_{\ell}}{dE_{\ell}},
$$

where $E_{\ell}$ is the lepton energy, $m_X^2$ the invariant hadronic squared mass, and $E_{\text{cut}}$ an experimental threshold on the lepton energy applied by some of the experiments. Since the physical information of moments of the same type is highly correlated, for $n > 1$ it is better to employ central moments, computed relative to $\langle E_{\ell} \rangle$ and $\langle m^2_X \rangle$. The information on the nonperturbative parameters obtained from a fit to the moments enables us to extract $|V_{cb}|$ from the total semileptonic width [20–22].

The expansion for the total semileptonic width is

$$
\Gamma_{\text{sl}} = \Gamma_0 \left[ 1 + a^{(1)} \frac{\alpha_s(m_b)}{\pi} + a^{(2, \beta_0)} \frac{\alpha_s^2}{\pi} + a^{(2)} \frac{\alpha_s^2}{\pi} \right. \\
\left. + \left( \frac{1}{2} + p^{(1)} \frac{\alpha_s}{\pi} \frac{m_b^2}{m_b^2} + \left( g^{(0)} + g^{(1)} \frac{\alpha_s}{\pi} \right) \frac{m_b^2}{m_b^2} \right) \right].
$$

where $\Gamma_0 = A_{\text{ew}} |V_{cb}|^2 |G_F^2 m_b^2 (1 - 8 \rho + 8 \rho^3 - \rho^4 - 12 \rho^2 \ln \rho) / 192 \pi^3$ is the tree-level free quark decay width, $\rho = m^2_\ell / m_b^2$, and $A_{\text{ew}} = 1.014$ the leading electroweak correction. We have split the $\alpha_s^2 / \pi$ coefficient into a Brodsky-Lepage-Mackenzie piece proportional to $\beta_0 = 9$ (with three massless active quark flavors) and a remainder. The expansions for the moments have the same structure. The parameters $g^2, \mu_G, \rho_D, \rho_L$ are the $B$-meson expectation values of the relevant dimension 5 and 6 local operators.

In Eq. (3) and in the calculation of all the moments, we have included the complete one- and two-loop perturbative corrections [23–28], as well as $1/m_b^{2,3}$ power corrections [16–18,29]. We neglect contributions of order $1/m_b^4$ and $1/m_b^6$ [30], which appear to lead to a very small shift in $|V_{cb}|$, but we include for the first time the perturbative corrections to the leading power-suppressed contributions [13–15] to the width (see also [31] for the limit $m_c \to 0$) and to all the moments [32].

The coefficients $a^{(1)}, g^{(0)}, g^{(1)}$, $d^{(0)}$ in Eq. (3) are functions of $\rho$ and of various unphysical scales, such as the one of $\alpha_s$. They are given in Table I for specific values of the quark masses. We use the kinetic scheme [33] with a cutoff at 1 GeV for $m_b$ and the OPE parameters and three different options for the charm mass.

The global fit.—The available measurements of the semileptonic moments [4] and the recent, precise determinations of the heavy quark masses significantly constrain the parameters entering Eq. (3), making a determination of $|V_{cb}|$ whose uncertainty is dominated by our ignorance of higher order effects possible. Duality violation effects can be constrained a posteriori, by checking whether the OPE predictions fit the experimental data, but this again depends on precise OPE predictions.

We perform a fit to the semileptonic data listed in Table I of Ref. [8] with $\alpha_s(4.6 \text{ GeV}) = 0.22$ and employ a few additional inputs. Since the moments are mostly sensitive to $\approx m_b - 0.8 m_c$, it is essential to include information on at least one of the heavy quark masses.

| $m_b^{\text{kin}}$ (1 GeV) | $m_b$(3 GeV) | $\mu_G^2$ | $\rho_D^3$ | $\rho_G^3$ | $\rho_L^3$ | BR$_{c \ell \nu}$ | $10^3|V_{cb}|$
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TABLE II. Results of the global fit in our default scenario. All parameters are expressed in GeV at the appropriate power and all, except $m_c$, in the kinetic scheme at $\mu = 1$ GeV. The first and second rows give central values and uncertainties; the correlation matrix follows.
Because of its smaller absolute uncertainty, $m_c$ is preferable. Among recent $m_c$ determinations [34–36], we choose $\bar{m}_c(3 \text{ GeV}) = 0.986(13) \text{ GeV}$ [34], although we will discuss the inclusion of $m_b$ determinations as well. We also include a loose bound on the chromomagnetic expectation value from the $B$ hyperfine splitting, $\mu_C^2(4.66 \text{GeV}) = 0.35(7) \text{GeV}^2$. Finally, as all observables depend very weakly on $\mu^2_{LS}$, we use the heavy quark sum rule constraint $\rho^3_{LS} = -0.15(10) \text{ GeV}^3$.

As should be clear from the above discussion on higher orders in the OPE, the estimate of theoretical errors and of their correlation is crucial. We follow the strategy of [8,20] for theoretical uncertainties, updating it because of the new corrections that we include. In particular, we assign an irreducible uncertainty of 8 MeV to $m_{c,b}$, and vary $\alpha_s(m_b)$ by $\pm 0.018$, $\mu_C^2$ and $\mu_C^3$ by $\pm 7\%$, and $\rho_D^3$ and $\rho^3_{LS}$ by $\pm 30\%$. This implies a total theoretical uncertainty between 2.0% and 2.6% in the semileptonic width, depending on the scheme. For the theory correlations we adopt scenario D of Ref. [8]; i.e., we assume no correlation between different central moments and a correlation between the same moment measured at different $E_{\text{cut}}$, depending on the proximity of the cuts and their magnitude. In the extraction of $|V_{cb}|$ we use the latest isospin average $\tau_B = 1.579(5) \text{ ps}$ [37].

In Table II we show the results of the fit and the correlation matrix among the fitted parameters. With respect to the default fit of Ref. [8], $|V_{cb}|$ is reduced by 0.5% [see Eq. (1)], $m_b^{\text{kin}}$ is increased by about 10 MeV, and $\mu_C^2$ and $\rho_D^3$ are both shifted upward by about 10%. As the method and inputs are the same as in Ref. [8], except for the value of $t_B$ which is only reflected in a tiny $+0.1\%$ shift in $|V_{cb}|$, the difference can be mostly attributed to the new corrections. Because of smaller theoretical errors, the final uncertainties are slightly reduced. The $\chi^2$/d.o.f. is very good, about 0.4.

It is interesting to compare the $b$ mass extracted from the fit with other recent determinations, generally expressed in terms of $\bar{m}_b(\bar{m}_b)$ in the MS scheme. This is shown in Fig. 1, after converting $m_b^{\text{kin}}$ into $\bar{m}_b(\bar{m}_b)$. The scheme conversion implies an additional $\sim 30 \text{ MeV}$ uncertainty [28], enlarging the final error to 37 MeV, because it is known only through $O(\alpha_s^2)$. Our result, $\bar{m}_b(\bar{m}_b) = 4.183(37) \text{ GeV}$, agrees well with those reported in the figure. The combination $m_b^{\text{kin}}(1 \text{ GeV}) - 0.85m_c(3 \text{ GeV})$ is best determined to $3.714 \pm 0.018 \text{ GeV}$.

Table III shows the results when the fit is performed with $m_c$, in a different scheme or at a different scale with respect to our default fit of Table II. The results are remarkably consistent and very close to the default fit, with the only partial exception of $m_b$, which becomes $1\sigma$ higher when $\bar{m}_c(2 \text{ GeV})$ is used as input. Table III also reports the results of a fit with an additional constraint on $m_b$. Even the currently most precise $m_b$ determinations are spoiled by the uncertainty due to the scheme conversion to $m_b^{\text{kin}}$. Because of this, and because of the large range of $m_b$ values given in the literature, we prefer to avoid using a $m_b$ constraint in our default fit.

Overall, the fit results depend little on the scale of $\alpha_s$. This is shown in Fig. 2 for the default fit. $|V_{cb}|$ and $m_b^{\text{kin}}$ increase by less than 0.5% if we perform the whole analysis using $\alpha_s(m_b/2)$, while $\mu_C^2$ and, in general, the OPE parameters are slightly more sensitive. A similar behavior is observed for the fits in Table III. Figure 3 shows instead the $\mu_b^{\text{kin}}$ dependence of $|V_{cb}|$ in case (a), keeping the scales of $m_b$ and $m_c$ distinct. In all cases, the scheme and scale

![Image](061802-3)

**TABLE III.** Results of the fit in different scenarios: (a) with $m_c$ in the kinetic scheme, $m_b^{\text{kin}} = 1.091(20) \text{ GeV}$ from [34]; (b) in the MS scheme at a lower scale, with $m_c(2 \text{ GeV}) = 1.091(14) \text{ GeV}$ from [34]; (c) same as our default fit, with an additional constraint $m_b^{\text{kin}} = 4.533(32) \text{ GeV}$, derived from [34].

| $m_b^{\text{kin}}$ | $m_c$ | $\mu_C^2$ | $\rho_D^3$ | $\mu_C^3$ | $\rho_D^3$ | $\rho^3_{LS}$ | $BR_{c\tau}$ | $10^3|V_{cb}|$ |
|------------------|-------|------------|------------|------------|------------|-------------|-------------|-------------|
| (a) 4.561        | 1.092 | 0.464      | 0.175      | 0.333      | -0.146     | 10.66       | 42.04       |
| (b) 0.021        | 0.020 | 0.067      | 0.040      | 0.061      | 0.096      | 0.16        | 0.67        |
| (c) 4.576        | 1.092 | 0.466      | 0.174      | 0.332      | -0.146     | 10.66       | 42.01       |
| (c) 0.020        | 0.014 | 0.068      | 0.039      | 0.061      | 0.096      | 0.16        | 0.68        |
| (c) 4.548        | 0.985 | 0.467      | 0.168      | 0.321      | -0.146     | 10.66       | 42.31       |
| (c) 0.017        | 0.012 | 0.068      | 0.038      | 0.058      | 0.096      | 0.16        | 0.76        |
dependence confirms the size of theoretical errors employed in our analysis.

Finally, we update the value of the semileptonic phase space ratio $C$,

$$C = \frac{|V_{ub}|^2 \Gamma[B \rightarrow X_c e\bar{\nu}]}{|V_{cb}| \Gamma[B \rightarrow X_u e\bar{\nu}]}$$

which is often used in the calculation of the branching ratio of radiative and rare semileptonic $B$ decays, see [14] for details. Using the default fit and $\mu_{WA} = m_b/2$, we find $C = 0.574 \pm 0.008 \pm 0.014$, where the first uncertainty comes from the parameters determined in the fit and the second from unknown higher orders, estimated as explained above. Since the ratio $C$ receives large perturbative corrections when it is expressed in terms of $m_c (3 \text{ GeV})$ [8], we believe that using $m_c (2 \text{ GeV})$ leads to a more reliable estimate. Including the $m_b^{\text{kin}}$ mass constraint derived from [34] as well, we find

$$C = 0.568 \pm 0.007 \pm 0.010,$$

slightly higher but with a smaller error than the corresponding value in [8].

**Conclusion.**—In summary, we have improved the inclusive determination of $|V_{cb}|$ through the inclusion of the complete $O(\alpha_s^2 \Lambda_{\text{QCD}}^2/m_b^2)$ effects. Our final value,

$$|V_{cb}| = (42.21 \pm 0.78) \times 10^{-3},$$

is compatible with previous analyses, but its uncertainty is slightly reduced thanks to the smaller theoretical errors. Equation (5) still differs at the 2.9σ level from Eq. (2). We find no sign of inconsistency in the inclusive analysis, and we adopt a conservative estimate of theory errors. The latter could be further reduced by a calculation of $O(\alpha_s^2 \Lambda_{\text{QCD}}^2/m_b^2)$ contributions, as well as by a better understanding of higher power corrections (see [44]).

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The calculation in the presence of experimental cuts is not trivial; details will be given in a future publication.