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Giuseppe Peano and his School: Axiomatics, Symbolism and Rigor

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Peano's axioms for arithmetic, published in 1889, are ubiquitously cited in writings on modern axiomatics, and his *Formulario* is often quoted as the precursor of Russell's *Principia Mathematica*. Yet, a comprehensive historical and philosophical evaluation of the contributions of the Peano School to mathematics, logic, and the foundation of mathematics remains to be made. In line with increased interest in the philosophy of mathematics for the investigation of mathematical practices, this thematic issue adds some contributions to a possible reconstruction of the philosophical views of the Peano School. These derive from logical, mathematical, linguistic, and educational works[1], and also interactions with contemporary scholars in Italy and abroad (Cantor, Dedekind, Frege, Russell, Hilbert, Bernays, Wilson, Amaldi, Enriques, Veronese, Vivanti and Bettazzi).

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1. The published and unpublished writings of Peano are collected in [Roero 2008]. An English anthology of Peano's texts is [Peano 1973]. For a rich literature on Peano and other members of his school see in particular [Luciano 2017]. A complete list of Padoa's writings can be found in [Cantù 2007]. The publications of Vailati and a rich literature on his life and works are listed in the introduction to [Arrighi, Cantù *et al.* 2009]. On Pieri see in particular the references quoted in [Marchisotto & Smith 2007]. On Burali-Forti see the references added to [Burali-Forti 1919].

1 The Peano School

It is debatable whether the group can be classified as a "scientific school" with an exhaustive list of all its members. The category of a mathematical research school, explored in its distinctions and national features by David Rowe [Rowe 2002], has recently been opposed to the category of a mathematical tradition. According to José Ferreirós a mathematical research school is "a group led normally by only one mathematician, localized within a single institutional setting and which counts on a significant supply of advanced students", whereas a mathematical tradition "implies that one can find a common research orientation in different actors that do not share a common institutional site, but are linked by traceable influences on each other" [Ferreirós 1999, xxii–xxiii]. To settle the question whether the Peano group should be considered as a research school or as a mathematical tradition, we first need to deconstruct several clichés from the literature and clarify the nature of Peano's leadership, the circulation of knowledge within the Peano School, and the role of other collective enterprises beside the *Formulario* (e.g., the *Rivista di Matematica*, the journal *Schola et Vita*, the *Dizionario*, as well as other contemporary articles and teaching materials). Original contributions have recently been based on the exploitation of new archival sources. These include the discovery of new previously unmentioned collaborators, the distinction of different levels of decision-making in Peano's editing process for the *Formulario*, and new insights into the original contributions of each member to shared knowledge in the group [Luciano 2017].

2 Philosophical interest

The Peano School is generally considered to be a phenomenon that suddenly appeared in all its splendour at the Paris congress of 1900 and then was extinguished like a firework that leaves a vivid but indefinite memory. Given the long-lasting impression made on Russell and other participants in the 1900 Paris Conferences in Mathematics, Philosophy and Psychology by the contributions of Peano, Burali-Forti, Padoa, Pieri, and Vailati, literature on the subject has often sought to find reasons to explain a general loss of philosophical interest in the Peano school in the first half of the 20th century. General explanations abound thereon and include: the non-academic nature of the group; the multiform topics of interest ranging from mathematical analysis to geometry, from linguistics to universal languages, and from philosophical pragmatism to logicism [Roero 2010], [Skof 2011], [Kennedy 2002]; the fact that scarce attention has been given to the transformation of mathematics and to the development of set theory after 1910 [Quine 1987]; a general belief that Peano was not really interested in the theory of inferential reasoning, or in the metalogical and metamathematical investigation of the properties of axiomatic theories [van Heijenoort 1967].

Other philosophical explanations have also been suggested: Peano's utilitarian approach to logic [Grattan-Guinness 2000] and symbolic notation [Bellucci, Moktefi *et al.* 2018]; the lack of a shared and explicit epistemological framework for relevant logical and methodological issues such as functions [Luciano 2017], [Cantù 2021], logical identities [Cantù 2007], definitions by abstraction [Mancosu 2018], and questions of purity [Arana & Mancosu 2012]; a subdivision of labour that led to Giovanni Vailati in Italy [Arrighi, Cantù *et al.* 2009] and Louis Couturat in France [Luciano & Roero 2005] becoming the chief philosophical spokesmen of the group; the belief that Peano's presentation of arithmetical axioms had less interesting philosophical implications with respect to logicism and structuralism than that of Dedekind [Ferreirós 2005]; the interest of Peano's collaborators in pedagogical and political issues [Giacardi 2006], [Luciano 2012].

The topic is reconsidered in a new light in this special issue, as the authors discuss the relationship between Dedekind's and Peano's axioms (Kahle, this volume), or the absence of the universal quantifier among the primitive symbols of Peano's *Formulario* and its relation to the use of free variables (von Plato, this volume). Other subjects covered are the peculiarities of Peano's symbolic notation (Schlimm, this volume) and differences with respect to Frege's (Betran-San Millán, this volume), the lack of recognition of Pieri's pedagogical remarks in Italy (Marchisotto & Millán Gasca, this volume), the early association in the USA with Russell's point of view (Lolli, this volume), the interaction between Peano's auxiliary international language project and the internationalization movement at the beginning of the century (Aray, this volume), and finally the limits of Peano's proof of the impossibility of infinitesimals (Freguglia, this volume).

3 Logic and epistemology

Some of the usual explanations have lost a degree of effectiveness, because of new specific results, and also because of the interdisciplinary and practical turn suggested by the intertwining of logic and epistemology. In this context, the latter is taken to mean both the analysis of scientific knowledge and the critique of scientific theories, as in neo-Latin languages. This perspective constituted the red thread of an international project (PICS INTEREPISTEME 2018-2020) co-funded by the French National Center for Scientific Research and the Vienna Circle Institute and co-directed by Paola Cantù and Georg Schiemer in collaboration with Erika Luciano at the University of Turin. The objective was to compare three distinct collaborative and interdisciplinary epistemologies developed by the members of the Peano School, the editorial board of the *Revue de métaphysique et de morale*, and the Vienna Circle.

The project showed various points of connection between collaborative and interdisciplinary approaches and educational and political aims, such as the vulgarization of scientific knowledge, and the criticism of disciplinary and national boundaries. However, it focused on the origins and development of non-mainstream philosophical views that cannot be reduced to logicism or structuralism, and investigated the underestimated influence of Leibniz's philosophy [Luciano 2006], [Cantù 2014], 19th century positivism, empiricism, and neo-criticism on these standard views in philosophy of mathematics [Cantù & Schiemer forthcoming].

The specificity of the School's research programme was only partially received because of a misunderstanding of the deep relation between education, linguistics, and axiomatics, and also a simplistic association of Peano's ideas with Russell's philosophy. This tendency emerged in van Heijenoort's remarks on the lack of inference rules and metatheoretical investigations, or in the quick tendency to classify Peano as a logicist but in fact was already evident in the early reception of Peano in the USA. Gabriele Lolli shows how the works of the Peano's School had already been discussed by Edwin B. Wilson in 1904 in a review of two pieces of writing by Bertrand Russell, which contributed to the two conceptions being eventually combined as "the Peano-Russell point of view".

The ability to discriminate subtle differences between the positions of Russell, Frege and Peano characterized a fine reader of Peano's work: Kurt Gödel. The philosophical notebooks (*Max Phil*) reveal a deep understanding of differences on the notions of function and definite description [Crocco, Van Atten *et al.* 2017], [Cantù 2016a]. The summary of the *Formulario* to be found in one of his *Excerptenhefte* shows the analysis of the rules of inferences used in deductive chains. The accurate summary of Peano's *Arithmetices Principia* written in Gabelsberger shorthand on a loose sheet of paper when Gödel was preparing the article on Russell's logic (early 1943) has been edited by Jan von Plato for this special issue. It clearly shows that Gödel read not only the *Formulario* [Peano 1895], but also the *Arithmetices Principia* [Peano 1889], focusing his comments on the formal character of proofs.

4 The implicit philosophy within Peano School's practices

The attention paid to mathematical practices has shown that Peano had a strong impact on the writings of Frege, Russell, Carnap and Gödel, and also developed a proper philosophical view that emerged from the logical investigation of definitions, the logical interpretation of the symbols of a formal language, the distinction between relations and functions, and the difference between primitive and derived terms or propositions in an axiomatic system. Peano's philosophical views is distinct from both logicism and structuralism and emerges as a result of a joint investigation of logic, language and mathematics, considered both as theoretical and didactic practices. The interest in definitions and the analysis of language had significant effects on Peano's semantics, which differs from what is usually described as conceptualist (or as a three-level: words/ concepts/ objects) semantics because symbols refer to concepts only through the mediation of language. In the same way as dictionary entries that only get meaning when inserted into a given linguistic context, the symbols' meaning can only be determined through a preliminary substitution with linguistic sentences in each of which the symbols refer to the concepts expressed by the corresponding words in ordinary mathematical language [Cantù 2021].

This volume constitutes a further decisive step towards the reconstruction of Peano's philosophical views from a detailed analysis of logical, mathematical, pedagogical and also linguistic practices. The essays gathered here focus on the works of Giuseppe Peano, Alessandro Padoa and Mario Pieri, but the same method could be fruitfully applied to other members of the school, such as Giovanni Vailati [Arrighi, Cantù *et al.* 2009], Cesare Burali-Forti and Alessandro Padoa. The contributions of Peano and other members of the school are also evaluated by comparison with contemporaries (Richard Dedekind, Gottlob Frege, Bertrand Russell, David Hilbert and Paul Bernays), resulting in a historically accurate analysis of some subtle but fundamental differences between their respective projects which aimed to present, analyze or ground mathematics as a rigorous, deductive science.

Three examples will be briefly mentioned in this introduction: axiomatics, linguistic symbolization and rigour. Different terms are often used to characterize the school's foundational enterprise: symbolization, formalization, axiomatization, reduction. A deep investigation of the Peano school's practices might help disentangle some of the differences between these fundamental notions, and shed new light on different ways to conceive generality, ideography, metatheoretical inquiries, and the role of notation, intuition, and rigor.

5 Axiomatics

General philosophical and historical reconstructions of the development of logic in the early 20th century have accustomed us to think of Peano as one of the fathers of modern axiomatics because of his contribution to the formulation of the axioms of arithmetic, which still bear his name. Yet, a detailed analysis of the connections between logical, linguistic, mathematical and pedagogic writings of the Peano School might help re-evaluate his contributions to logic and philosophy of mathematics and discover a specific approach to axiomatics. An axiomatization is a particular kind of presentation of a theory, in which the logical and the mathematical content is specified by the respective axioms. Reinhard Kahle's contribution traces a history of the formulation of the properties of real numbers, considered as *Sätze* by Dedekind and explicitly formulated as axioms by Peano, acquiring a non-logical nature in Hilbert's works and a first-order formulation in Bernay's contributions. This is a historically fruitful example of how the investigation of different uses and presentations of the same mathematical properties of numbers can reveal very different conceptions of the axiomatization of arithmetic.

Yet, axiomatics cannot be reduced to the investigation of the axiomatic formulation of single theories. It is a back-and-forth process between the syntactic, semantic and pragmatic-linguistic levels and their goals: 1) to make the implicit assumptions of a theory explicit (e.g., by stating all the hypotheses necessary to prove a given theorem); 2) to investigate the tacit assumptions of a theory, considering what happens if they are not implicitly assumed (e.g., by testing the possibility of creating non-standard models of a theory); 3) to define the scope and goals of a research programme or discipline [Woodger 1959]. Linguistic analysis is a pillar of Peano's approach, and cannot be dissociated from epistemological goals, such as the search for good order and the minimal number of concepts, and the questioning of the relation between mathematical practices and a rigourous mathematical language. Far from being exclusively aimed at the construction of axiomatic systems or the investigation of deductive inferences, the symbolization of logic rests on questions very similar to those that have developed in the social sciences: the need to distinguish the simple from the complex, the first for us from the first for itself, a canonical form from deviant forms, the definition of a term from the formation of a concept, the pragmatic consequences of a hypothesis from its theoretical role [Cantù 2020].

Axiomatics relies heavily on complex practices of symbolization and formalization, which have a social and interactive nature, and that should be studied in their different components: phases of redaction (study of the pertaining bibliography, construction of a hypothetic-deductive order of the collected results, codification in symbols), division of the tasks among group members, circulation of knowledge within the group in a hierarchical or peer context, and the construction of domains of shared knowledge, that do not need to be mirrored in the final version of publications [Luciano 2017].

6 Symbolization and language

Symbolization is a process that associates symbols to words, but symbols can play the role of schematic letters, as in Hilbert's formalization, i.e., as terms having a merely formal sense that allows for a variety of interpretations, or have a substantive role, as terms whose meanings have to be conveyed by elucidation [Klev 2011]. Bertran-San Millán's contribution explains how Frege used the symbols of arithmetic as canonical names, i.e., as symbols with a specific and fixed meaning, so that mathematical letters always have a specific domain, determined by the intended application. Peano shared a similar substantive understanding of mathematical symbols in his early writings but moved towards a view of undefined symbols as uninterpreted non-logical constants devoid of meaning, when he investigated metatheoretical questions concerning the independence of the axioms with Padoa.

The comparative investigation of logic, linguistics and notational practices offer further insights into the particular version of ideography that is developed in the *Formulario*. In this, symbols mean ideas but are first introduced as names for terms of an interpreted mathematical language having those ideas as meaning, and then also considered as schematic variables that might receive different interpretations by substitution of different linguistic terms. The relation between mathematical symbols, words of mathematical language, mathematical concepts and mathematical objects has a complex history that a comparative investigation of Peano's contributions to logics, mathematics, linguistics, and symbolic notation might help unravel.

The symbolization of mathematics is often discussed in the light of a reduction of mathematics to logic or as a translation that preserves the relevant mathematical meaning. However, it cannot be fully understood without a detailed investigation of the design principles and the didactical and practical constraints that accompany the search for technical symbols in a new notation. The formalization is a way to distinguish the logical form from the non-logical content but can also be conceived as a method of conceptual analysis that identifies the relevant logical and mathematical ideas. As Dirk Schlimm shows in his contribution, this analysis might be used to determine the primitive terms and propositions and to check the adequacy of the analysis itself, thereby evaluating whether definitions are correct, and proofs are rigourous.

The distinction between symbolization and formalization is often difficult to trace, but the investigation of definitions and of metatheoretical issues of independence between axioms and a careful investigation of the interactions with linguistics might be of help. There are several aspects of Peano's approach to the *Interlingua* that relate it to mathematical logic: in both cases a language in use (mathematical language and Latin) is taken as a starting point for the development of a universally understandable language (logical symbolism, Interlingua); secondly the two enterprises are based on collaborative networks; both are grounded in the legacy of Leibniz' *characteristica universalis*; finally they are combined in the last edition of the *Formulario*, written in "*Latino sine flexione*" and symbolic language [Cantù 2016b]. In her contribution, Başak Aray highlights another similarity: the connection between algebra and grammar developed in the *Formulario*, and suggests that the symbolization developed in Peano's mathematical practice guided the design of his proposal for an international auxiliary language: the *Latino sine flexione*.

This algebraic understanding of grammar more effectively explains how logic and language are both presented in an equational form, and generality is expressed using free variables instead of assuming a universal quantifier as a primitive logical term. According to Jan von Plato this is the reason why Peano's axiomatic systems, like Schröder's algebraic logic, lacked some principles of reasoning with the quantifiers even though they contained other rules of inferences.

7 Mathematical education and rigour

The interrelation between mathematical education and conceptual analysis offers further hints to understanding the main traits of the Peano school's epistemology, the importance of rigour in scientific knowledge and education and also the interpretation of axiomatics as a metatheoretical investigation based on a variety of alternative conceptual analyses leading to different axiomatic presentations and definitions of the mathematical concepts. The attention to the pedagogic component in the Peano School shows that the sociological singularity of this research team (the only non academic-based group in the international panorama) corresponds to a unique educational project on rigour. This project was deeply intertwined with the mathematical, philosophical, logical and linguistic views of the group and had non-negligible effects on the evolution of mathematical teaching in Italy, and beyond. Rigour is not an accessory or external element that can be imposed on mathematical teaching. It is instead a result of the development of rational mathematics and the evolution of all sciences towards the structure of axiomatic-deductive systems. Rigour is not primarily a foundational problem, although conversely, the foundational enterprise is intertwined with didactic concerns [Luciano 2020].

Rigour is a distinctive feature of the Peano School's style and also an essential topic in the Italian debate on mathematical pedagogic theory and teaching practice at the turn of the century. Peano's crusade in defence of rigour is both a distinctive mark of his axiomatics and a feature of the School's linguistic, mathematical and educational research programmes. It was neither a negation of the importance of experimental methods in the early stages of mathematical education [Luciano 2020] nor a simplistic negation of mathematical intuition which was banished from the proofs of a theory but remained decisive in the choice of axioms [Rizza 2009]. It was instead a didactical objective developed through exchanges with school teachers and their associations, the publication of new textbooks, and participation in educational Governmental Committees [Giacardi 2006].

In their contribution, Elena Marchisotto & Ana Millán Gasca illustrate Pieri's belief that an integration of sensible and rational intuition can deeply renew the teaching of geometry while also deploying a profound heuristic value. The analysis of Pieri's axiomatization of geometry exemplifies the almost symbiotic relation between axiomatics and pedagogy that is typical of the Peano School as well as the partial and complex reception of this idea in works by Italian contemporary mathematicians, like Enriques and Amaldi.

And yet, the very same idea of rigour gave rise to famous debates, whose philosophical objectives were sometimes obscured by putting forth educational motivations or formal demonstrations. The famous debate with Segre on rigour and intuition was both a manifesto of Peano's style and an implicit criticism of Veronese's hyperspaces and the expression of the rivalry with the geometrical Italian School [Luciano 2020].

Similarly, Freguglia claims that the famous proof of the impossibility of infinitesimals was not developed just to complete and rectify an untenable proof by Cantor—undergoing the similar mistake of presupposing an axiom that is equivalent to the Archimedean axiom and therefore incompatible with the existence of infinitesimals. It was also an implicit criticism of Veronese's theory of a geometrical non-Archimedean continuum and provided an opportunity to host a scientific discussion of the topic in the newly founded *Rivista di Matematica*.

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