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ONE-DIMENSIONAL DISSIPATIVE BOLTZMANN EQUATION: MEASURE SOLUTIONS, COOLING RATE, AND SELF-SIMILAR PROFILE*

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Abstract. This manuscript investigates the following aspects of the one-dimensional dissipative Boltzmann equation associated to a variable hard-spheres kernel: (1) we show the optimal cooling rate of the model by a careful study of the system satisfied by the solution's moments, (2) we give existence and uniqueness of measure solutions, and (3) we prove the existence of a nontrivial self-similar profile, i.e., homogeneous cooling state, after appropriate scaling of the equation. The latter issue is based on compactness tools in the set of Borel measures. More specifically, we apply a dynamical fixed point theorem on a suitable stable set, for the model dynamics, of Borel measures.

Key words. Boltzmann equation, self-similar solution, measure solutions, dynamical fixed point

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1. Introduction.

In this document we study a standard one-dimensional (1-D) dissipative Boltzmann equation associated to "variable hard potentials" interaction kernels. The model is given by

$$\begin{aligned} \partial_t f(t,x) &= \mathcal{Q}(f,f)(t,x), \qquad (t,x)\in[0,\infty)\times\mathbb{R}\,,\\ f(0,x) &= f_0(x), \end{aligned} \tag{1.1}$$

where the dissipative Boltzmann operator $\mathcal{Q}=\mathcal{Q}_\gamma$ is defined as

$$\mathcal{Q}(f,f)(x) := \int_{\mathbb{R}} f\,(x-ay)\,f\,(x+by)\;|y|^\gamma \mathrm{d}y - f(x)\int_{\mathbb{R}} f(x+y)\,|y|^\gamma \mathrm{d}y\,. \tag{1.2}$$

The parameters of the model satisfy $\gamma>0$, $a\in(0,1)$, $b=1-a$ and will be *fixed throughout the paper*. Such a model can be seen as a generalization of the one introduced by Ben-Nam and Krapivsky [8] for $\gamma=0$ and happens to have many applications in physics, biology, and economics; see, for instance, the process presented in [5, 8] with applications to biology.

The case $\gamma=0$—usually referred to as the Maxwellian interaction case—is by now well understood [18] and we will focus our efforts on extending several of the results known for that case ($\gamma=0$) to the more general model (1.1). Let us recall that, generally speaking, the analysis of Boltzmann-like models with Maxwellian interaction essentially renders explicit formulas that allow for a very precise analysis [12, 14, 13, 32, 18, 8] because (1) moments solve closed ODEs and (2) Fourier transform techniques

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are relatively simple to implement. In [8] a Brownian thermalization is added to the equation which permits a study of stationary solutions of (1.1) for $\gamma = 0$. Here we are more interested in generalizing the works of [12, 14, 13, 32, 18] that deal with the so-called self-similar profile which in the particular case of Mawellian interactions is unique and explicit. Such a self-similar profile is the unique stationary solution of the *self-similar equation* associated to (1.1) and, by means of a suitable Fourier metric, it is possible to show (in the Maxwellian case) exponential convergence of the (time dependent) self-similar solution to this stationary profile [14, 13, 18]. In particular, this means that solutions of (1.1) with $\gamma = 0$ approach exponentially fast the "back rescaled" self-similar profile as $t \to +\infty$.

Self-similarity is a general feature of dissipative collision-like equations. Indeed, since the kinetic energy is continuously decreasing, solutions to (1.1) converge as $t \to \infty$ toward a Dirac mass. As a consequence, one expects that a suitable time-velocity scale depending on the rate of dissipation of energy may render a better setup for the analysis. For this reason, we expect that several of the results occurring for Maxwellian interactions remain valid for (1.1) with $\gamma > 0$. The organization of the document is as follows: We finish this introductory material with a general setup of the problem, including notation, scaling, relevant comments, and the statement of the main results. In section 2 the Cauchy problem is studied. The framework will be the space of probability Borel measures. Such a framework is the natural one for (1.1) as it is for kinetic models in general. In 1-D problems, however, we will discover that it is essential to work in this space in contrast to higher dimensional models, such as viscoelastic Boltzmann models in the plane or the space where one can avoid it and work in smaller spaces such as Lebesgue's spaces [11, 35, 2, 29]. This last fact proves to be a major difficulty in the analysis of the model. The Cauchy problem is then based on a careful study of a priori estimates for the moments of solutions of (1.1) and standard fixed point theory. In section 3 we find the optimal rate of dissipation (commonly referred to as Haff's law) which follows from a careful study of a lower bound for the moments using a technique introduced in [2, 3]. In section 4 we prove the existence of a nontrivial self-similar profile which is based, again, on the theory of moments and the use of a novel dynamical fixed point result on a compact stable set of Borel probability measures [6]. The key remaining argument is, then, to prove that the self-similar profile—which a priori is a measure—is actually an L^1-function. Needless to say, such a stable set is engineered out of the moment analysis of sections 2 and 3. In section 5, numerical simulations are presented that illustrate the previous quantitative study of (1.8) as well as some peculiar features of the self-similar profile G. The simulations are based upon a discontinuous Galerkin (DG) scheme. The paper ends with some perspectives and open problems related to (1.1), in particular, its link to a recent kinetic model for rods alignment [5, 9].

1.1. Self-similar equation and the long time asymptotic. The weak formulation of the collision operator $\mathcal{Q}$ reads

$$\int_{\mathbb{R}} \mathcal{Q}\big(f,f\big)(x)\psi(x)\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(y)\big|x-y\big|^{\gamma}\Big(\psi(ax+by)+\psi(bx+ay)-\psi(x)-\psi(y)\Big)\mathrm{d}x\mathrm{d}y \tag{1.3}$$

for any suitable test function ψ. In particular, plugging successively $\psi(x) = 1$ and $\psi(x) = x$ into (1.3) shows that (1.1) conserves mass and momentum. Namely, for any reasonable solution $f(t,x)$ to (1.1), one has

(1.4)
$$\int_{\mathbb{R}} f(t,x)\mathrm{d}x = \int_{\mathbb{R}} f_0(x)\mathrm{d}x \qquad \text{and} \qquad \int_{\mathbb{R}} x f(t,x) dx = \int_{\mathbb{R}} x f_0(x)\mathrm{d}x \qquad \forall t \geqslant 0\,.$$

However, the second-order moment is not conserved: indeed, plugging now $\psi(x) = |x|^2$ into (1.3) one sees that

$$\int_{\mathbb{R}} \mathcal{Q}(f,f)(x)\,|x|^2\mathrm{d}x = -a\,b \int_{\mathbb{R}^2} f(x)f(y)\,|x-y|^{2+\gamma}\mathrm{d}x\mathrm{d}y$$

since $|ax+by|^2 + |ay+bx|^2 - |x|^2 - |y|^2 = -2a\,b\,|x-y|^2$ for any $(x,y) \in \mathbb{R}^2$. Therefore, for any nonnegative solution $f(t,x)$ to (1.1), we get that the kinetic energy is nonincreasing,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} f(t,x)\,|x|^2\mathrm{d}x = -a\,b\int_{\mathbb{R}^2} f(t,x)\,f(t,y)\,|x-y|^{2+\gamma}\mathrm{d}x\mathrm{d}y \leqslant 0\,. \tag{1.5}$$

This is enough to prove that a nontrivial stationary solution to problem (1.1) exists. Indeed, for any $x_0 \in \mathbb{R}$, the Dirac mass δ_{x_0} is a steady (measure) solution to (1.1). For this reason one expects the large-time behavior of the system to be described by self-similar solutions. In order to capture such a self-similar behavior, it is customary to introduce the rescaling

$$V(t)\,g(s(t),\xi) = f(t,x)\,, \quad \xi = V(t)\,x, \tag{1.6}$$

where $V(t)$ and $s(t)$ are strictly increasing functions of time satisfying $V(0) = 1$, $s(0) = 0$, and $\lim_{t\to\infty} s(t) = \infty$. Under such a scaling, one computes the *self-similar* equation as

$$\begin{aligned}\partial_t f(t,x) &= \Big(\dot V(t)g(s,\xi) + V(t)\dot s(t)\partial_s g(s,\xi) + \xi\dot V(t)\partial_\xi g(s,\xi)\Big)\Big|_{s=s(t)\,,\,\xi=V(t)x}\\ &= \Big(\dot V(t)\partial_\xi\left(\xi\, g(s,\xi)\right) + V(t)\dot s(t)\partial_s g(s,\xi)\Big)\Big|_{s=s(t)\,,\,\xi=V(t)x},\end{aligned}$$

while the interaction operator turns into

$$\mathcal{Q}\big(f,f\big)(t,x) = V^{1-\gamma}(t)\mathcal{Q}\big(g,g\big)(s(t),V(t)x)\,. \tag{1.7}$$

Consequently $f = f(t,x)$ is a solution to (1.1) if and only if $g = g(s,\xi)$ satisfies

$$\dot s(t)V^{\gamma}(t)\partial_s g(s,\xi) + \frac{\dot V(t)}{V^{1-\gamma}(t)}\partial_\xi\big(\xi\,g\big)(s,\xi) = \mathcal{Q}\big(g,g\big)(s,\xi).$$

Choosing

$$V(t) = (1+c\gamma\,t)^{\frac{1}{\gamma}} \qquad \text{and} \qquad s(t) = \frac{1}{c\gamma}\log(1+c\,\gamma\,t)\,, \qquad c>0,$$

it follows that g solves

$$\partial_s g(s,\xi) + c\,\partial_\xi\left(\xi g(s,\xi)\right) = \mathcal{Q}\big(g,g\big)(s,\xi)\,. \tag{1.8}$$

Thus, the argument of understanding the long time asymptotic of (1.1) is simple: if there exists a unique steady solution G to (1.8), then such a steady state G should attract any solution to (1.8) and, back scaling to the original variables,

$$f(t,x) \simeq V(t)G(V(t)x) \text{ as } t\to\infty$$

in some suitable topology. We give in this paper a first step toward a satisfactory answer to this problem; more specifically, we address here two main questions:

Question 1. Determine the optimal convergence rate of solutions to (1.1) toward the Dirac mass centered in the center of mass $\overline{x}_0 := \int_{\mathbb{R}} x f_0(x)\mathrm{d}x$. The determination of this optimal convergence rate is achieved by identifying the *optimal rate* of convergence of the moments $M_k(t)$ of $f(t,x)$ defined as

$$M_k(t) := \int_{\mathbb{R}} |x - \overline{x}_0|^k f(t,x)\mathrm{d}x, \qquad k \geqslant 0.$$

Question 2. Prove the existence of a "physical" steady solution $G \in L^1_{\max(\gamma,2)}(\mathbb{R})$ to (1.8), that is, a function satisfying

$$c\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\, G(\xi)\right) = \mathcal{Q}(G,G)(\xi)\,, \qquad \xi \in \mathbb{R}\,, \tag{1.9}$$

in a weak sense (where $c > 0$ is arbitrary and, for simplicity, can be chosen as $c = 1$). Note that (1.9) has at least two solutions for any $\gamma > 0$, the trivial one and the Dirac measure at zero. None of them is a relevant steady solution since both have energy zero, which is a feature not satisfied by the dynamical evolution of (1.8) (provided the initial measure is neither the trivial measure nor the Dirac measure).

Similar questions have already been addressed for the 3-D Boltzmann equation for granular gases with different type of forcing terms [29, 24, 10]. For the inelastic Boltzmann equation in $\mathbb{R}^3$, the answer to Question 1 is known as Haff's law, proven in [29, 2, 3] for the interesting case of hard-spheres interactions (essentially the case $\gamma = 1$). The method we adopt here is inspired by the last two references since it appears to be the most natural to the equation. Concerning Question 2, the existence and uniqueness (the latter in a weak inelastic regime) of solutions to (1.9) has been established rigorously for hard-spheres interactions in [29, 30]. In [29, 24, 10], and in [6, 21] in the context of coagulation problems, the strategy to prove the existence of solutions to a problem similar to (1.9) is achieved through the careful study of the associated evolution equation ((1.8) in our context) and an application of the following dynamic version of the *Tykhonov fixed point theorem* (see [6, Appendix A] for a proof).

Theorem 1.1 (dynamic fixed point theorem). *Let $\mathcal{Y}$ be a locally convex topological vector space and $\mathcal{Z}$ a nonempty convex and compact subset of $\mathcal{Y}$. If $(\mathcal{F}_t)_{t\geq 0}$ is a continuous semigroup on $\mathcal{Z}$ such that $\mathcal{Z}$ is invariant under the action of $\mathcal{F}_t$ (that is, $\mathcal{F}_t z \in \mathcal{Z}$ for any $z \in \mathcal{Z}$ and $t \geq 0$), then there exists $z_o \in \mathcal{Z}$ which is stationary under the action of $\mathcal{F}_t$ (that is, $\mathcal{F}_t z_o = z_o$ for any $t \geq 0$).*

In the aforementioned references, the natural approach consists in applying Theorem 1.1 to $\mathcal{Y} = L^1$ endowed with its weak topology and consider for the subset $\mathcal{Z}$ a convex set which includes an upper bound for some of the moments and some L^p-norm, with $p > 1$, which yield the desired compactness in $\mathcal{Y}$. As a consequence, with such approach a crucial point in the analysis is to determine uniform L^p-norm bounds for the self-similar evolution problem. This last particular issue, if true, appears to be quite difficult to prove in the model (1.8), mainly because of the lack of angular averaging in the 1-D interaction operator $\mathcal{Q}$ as opposed to higher dimensional interaction operators. This problem is reminiscent of related 1-D interaction operators associated to coagulation-fragmentation problems, for instance, in Smoluchowski equation, for which propagation of L^p-norms is hard to establish; see [26] for details. Having this in mind, it appears to us more natural to work with measure solutions and considering then $\mathcal{Y}$ as a suitable space of real Borel measures endowed with the weak-$\star$ topology for which the compactness will be easier to establish. Of course, the main difficulty will then be to determine that the fixed point provided by Theorem 1.1 is not the Dirac measure at zero (the trivial solution is easily discarded by mass conservation). In fact, we will prove that this steady state is a L^1 function (see Theorem 1.5). Let us introduce some notation before entering in more details.

1.2. Notation. Let us introduce the set $\mathcal{M}_s(\mathbb{R})$ as the Banach space of real Borel measures on $\mathbb{R}$ with finite total variation of order s endowed with the norm $\|\cdot\|_s$ defined as

$$\|\mu\|_s := \int_{\mathbb{R}} \langle x\rangle^s \,|\mu\,|(\mathrm{d}x) < \infty \qquad \text{with} \quad \langle x\rangle := \left(1+|x|^2\right)^{\frac{1}{2}} \qquad \forall x\in\mathbb{R}\,,$$

where the positive Borel measure $|\mu|$ is the total variation of μ. We also set

$$\mathcal{M}_s^+(\mathbb{R}) = \{\mu\in\mathcal{M}_s(\mathbb{R})\,;\,\mu\geqslant 0\}$$

and denote by $\mathcal{P}(\mathbb{R})$ the set of probability measures over $\mathbb{R}$. For any $k\geqslant 0$, define the set

$$\mathcal{P}_k(\mathbb{R}) = \left\{\mu\in\mathcal{P}(\mathbb{R})\,;\,\int_{\mathbb{R}}|x|^k\mu(\mathrm{d}x)<\infty\right\}.$$

For any $\mu\in\mathcal{P}_k(\mathbb{R})$ and any $0\leqslant p\leqslant k$, let us introduce the p-moment

$$M_p\left(\mu\right) := \int_{\mathbb{R}}|x|^p\mu(\mathrm{d}x).$$

If μ is absolutely continuous with respect to the Lebesgue measure with density f, i.e., $\mu(\mathrm{d}x)=f(x)\mathrm{d}x$, we simply denote

$$M_p(f) = M_p(\mu) = \int_{\mathbb{R}}|x|^p f(x)\mathrm{d}x \qquad \text{for any} \quad p\geqslant 0.$$

We also define, for any $k\geqslant 1$, the set

$$\mathcal{P}_k^0(\mathbb{R}) = \left\{\mu\in\mathcal{P}_k(\mathbb{R})\,;\,\int_{\mathbb{R}}x\mu(\mathrm{d}x)=0\right\}.$$

In the same way, we set $L_k^1(\mathbb{R}) = L^1(\mathbb{R})\cap\mathcal{M}_k(\mathbb{R})$ for any $k\geqslant 0$. Moreover, we introduce the set L^∞_{-s} $(s\geqslant 0)$ of locally bounded Borel functions φ such that

$$\|\varphi\|_{L^\infty_{-s}} := \sup_{x\in\mathbb{R}}|\varphi(x)\,|\,\langle x\rangle^{-s} < \infty\,.$$

For any $p\geqslant 1$ and $\mu,\nu\in\mathcal{P}_p(\mathbb{R})$, we recall the definition of the Wasserstein distance of order p, $W_p(\mu,\nu)$ between μ and ν by

$$W_p(\mu,\nu) = \left(\inf_{\pi\in\Pi(\mu,\nu)}\int_{\mathbb{R}^2}|x-y|^p\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}},$$

where $\Pi(\mu,\nu)$ denotes the set of all joint probability measures π on $\mathbb{R}^2$ whose marginals are μ and ν. For the peculiar case $p=1$, we shall address the *first-order Wasserstein distance* $W_1(\mu,\nu)$ as the *Kantorovich–Rubinstein distance*, denoted d_{KR}, i.e.,

$d_{\mathrm{KR}}(\mu,\nu) = W_1(\mu,\nu)$. We refer to [36, section 7] and [37, Chapter 6] for more details on Wasserstein distances. The Kantorovich–Rubinstein duality asserts that

$$d_{\mathrm{KR}}(\mu,\nu) = \sup_{\varphi\in \mathrm{Lip}_1(\mathbb{R})} \int_{\mathbb{R}} \varphi(x)(\mu-\nu)(\mathrm{d}x),$$

where $\mathrm{Lip}_1(\mathbb{R})$ denotes the set of Lipschitz functions φ such that

$$\|\varphi\|_{\mathrm{Lip}(\mathbb{R})} = \sup_{x\neq y} \frac{|\varphi(x)-\varphi(y)|}{|x-y|} \leqslant 1\,.$$

For a given $T>0$ and a given $k \geqslant 0$, we shall indicate as $\mathcal{C}_{\mathrm{weak}}([0,T],\mathcal{P}_k(\mathbb{R}))$ the set of continuous mappings from $[0,T]$ to $\mathcal{P}_k(\mathbb{R})$, where the latter is endowed with the weak-$\star$ topology.

1.3. Collision operator and definition of measure solutions. We extend the definition (1.3) to nonnegative Borel measures; namely, given $\mu,\nu \in \mathcal{M}^+_\gamma(\mathbb{R})$, let

$$\langle \mathcal{Q}(\mu,\nu)\,;\,\varphi\rangle := \frac{1}{2}\int_{\mathbb{R}^2} \big|x-y\big|^\gamma\, \Delta\varphi(x,y)\mu(\mathrm{d}x)\nu(\mathrm{d}y) \tag{1.10}$$

for any test function $\varphi \in \mathcal{C}(\mathbb{R})\cap L^\infty(\mathbb{R})$, where

$$\Delta\varphi(x,y) := \varphi\big(ax+by\big) + \varphi\big(bx+ay\big) - \varphi\big(x\big) - \varphi\big(y\big), \qquad (x,y)\in\mathbb{R}^2\,.$$

A natural definition of measure solutions to (1.1) is the following; see [27].

Definition 1.2. *Let $\gamma>0$, $\gamma_* := \max(\gamma,2)$, $\mu_0 \in \mathcal{M}^+_{\gamma_*}(\mathbb{R})$, and $(\mu_t)_{t\geqslant 0} \subset \mathcal{M}^+_{\gamma_*}(\mathbb{R})$ be given. We say that $(\mu_t)_{t\geqslant0}$ is a measure weak solution to* (1.1) *associated to the initial datum μ_0 if it satisfies*

1. $\sup_{t\geqslant 0}\|\mu_t\|_{\gamma_*} \leqslant \|\mu_0\|_{\gamma_*}$,

$$\int_{\mathbb{R}} \mu_t(\mathrm{d}x) = \int_{\mathbb{R}}\mu_0(\mathrm{d}x) \qquad \text{and} \qquad \int_{\mathbb{R}} x\mu_t(\mathrm{d}x) = \int_{\mathbb{R}} x\mu_0(\mathrm{d}x) \qquad \forall t>0; \tag{1.11}$$

2. *for any test function $\varphi \in \mathcal{C}_b(\mathbb{R}) := \mathcal{C}(\mathbb{R})\cap L^\infty(\mathbb{R})$, the following hold:*
   (i) *the mapping $t\mapsto \langle\mathcal{Q}(\mu_t,\mu_t)\,;\,\varphi\rangle$ belongs to $\mathcal{C}\big([0,\infty)\big)$,*
   (ii) *for any $t\geqslant 0$ it holds that*

$$\int_{\mathbb{R}} \varphi(x)\mu_t(\mathrm{d}x) = \int_{\mathbb{R}}\varphi(x)\mu_0(\mathrm{d}x) + \int_0^t \langle \mathcal{Q}\big(\mu_\tau,\mu_\tau\big)\,;\,\varphi\rangle\,\mathrm{d}\tau\,. \tag{1.12}$$

Notice that if $\mu_t \in \mathcal{M}^+_{\gamma_*}(\mathbb{R})$ for any $t\geqslant 0$, then

$$\int_{\mathbb{R}^2} \big|x-y\big|^\gamma \big|\Delta\varphi(x,y)\big|\,\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y) \leqslant 4\|\varphi\|_\infty \|\mu_t\|_\gamma^2 < \infty$$

for any $\varphi\in\mathcal{C}_b(\mathbb{R})$ and $t\geqslant 0$. This shows that $\langle\mathcal{Q}(\mu_t,\mu_t)\,;\,\varphi\rangle$ is well defined for any $t\geqslant0$ and any $\varphi\in\mathcal{C}_b(\mathbb{R})$. Similarly, the notion of measure solution to (1.9) is given in the following statement.

Definition 1.3. *A measure $\boldsymbol{\mu} \in \mathcal{P}^0_{\max(\gamma,2)}(\mathbb{R})$ is a solution to* (1.9) *if*

$$-\int_{\mathbb{R}} \xi\phi'(\xi)\boldsymbol{\mu}(\mathrm{d}\xi) = \frac{1}{2}\int_{\mathbb{R}^2} |\xi-\eta|^\gamma \big(\phi\left(a\xi+b\eta\right) + \phi\left(a\eta+b\xi\right) - \phi(\xi) - \phi(\eta)\big)\boldsymbol{\mu}(\mathrm{d}\xi)\,\boldsymbol{\mu}(\mathrm{d}\eta) \tag{1.13}$$

for any $\phi\in\mathcal{C}^1_b(\mathbb{R})$, where ϕ' stands for the derivative of ϕ.

Notice that, by assuming $\|\boldsymbol{\mu}\|_0 = 1$, we naturally discard the trivial solution $G=0$ to (1.9).

1.4. Strategy and main results. Thanks to the conservative properties (1.11), we shall assume in what follows, and without any loss of generality, that the initial datum $\mu_0 \in \mathcal{M}^+_{\gamma_*}(\mathbb{R})$ is such that

$$\int_{\mathbb{R}} \mu_0(\mathrm{d}x) = 1 \qquad \text{and} \qquad \int_{\mathbb{R}} x\mu_0(\mathrm{d}x) = 0\,,$$

i.e., $\mu_0 \in \mathcal{P}^0_{\gamma_*}(\mathbb{R})$. This implies that any weak measure solution $(\mu_t)_{t\geqslant 0}$ to (1.1) associated to μ_0 is such that

$$\mu_t \in \mathcal{P}^0_{\gamma_*}(\mathbb{R}) \qquad \forall t \geqslant 0\,.$$

THEOREM 1.4. *Fix $\gamma > 0$, $\gamma_* = \max(\gamma, 2)$ and let $\mu_0 \in \mathcal{P}^0_{\gamma_*}(\mathbb{R})$ be given an initial datum. Then, there exists a measure weak solution $(\mu_t)_{t\geqslant 0}$ to* (1.1) *associated to μ_0 in the sense of Definition* 1.2 *satisfying*

$$\int_{\mathbb{R}} \mu_t(\mathrm{d}x) = \int_{\mathbb{R}} \mu_0(\mathrm{d}x) = 1\,, \qquad \int_{\mathbb{R}} x\mu_t(\mathrm{d}x) = \int_{\mathbb{R}} x\mu_0(\mathrm{d}x) = 0\,,$$
$$\text{and} \quad \int_{\mathbb{R}} |x|^{\gamma_*}\mu_t(\mathrm{d}x) \leqslant \int_{\mathbb{R}} |x|^{\gamma_*}\mu_0(\mathrm{d}x) \qquad \forall t \geqslant 0\,.$$

Moreover, such a solution enjoys the following instantaneous appearance of higher-order moments: for all $t_0 > 0$,

$$\sup_{t\geqslant t_0} \int_{\mathbb{R}} |x|^s \mu_t(\mathrm{d}x) < \infty \qquad \forall s > \gamma_*.$$

If additionally there exists $\varepsilon > 0$ such that

$$\int_{\mathbb{R}} \exp(\varepsilon|x|^{\gamma})\mu_0(\mathrm{d}x) < \infty\,, \tag{1.14}$$

then such a measure weak solution is unique. Furthermore, if μ_0 is absolutely continuous with respect to the Lebesgue measure, i.e., $\mu_0(\mathrm{d}x) = f_0(x)\mathrm{d}x$ with $f_0 \in L^1_{\gamma_*}(\mathbb{R})$, then μ_t is absolutely continuous with respect to the Lebesgue measure for any $t \geqslant 0$. That is, there exists $(f_t)_{t\geqslant 0} \subset L^1_{\gamma_*}(\mathbb{R})$ such that $\mu_t(\mathrm{d}x) = f_t(x)\mathrm{d}x$ for any $t \geqslant 0$.

We prove Theorem 1.4 following a strategy introduced in [27] and [23] for the case of a Boltzmann equation with hard potentials (with or without cut-off). The program consists essentially of the following steps: (1) Establish a priori estimates for measure weak solutions to (1.1) concerning the creation and propagation of algebraic moments, and (2) construct measure weak solutions to (1.1) by approximation of L^1-solutions. Step (1) helps prove that such an approximating sequence converges in the weak-$\star$ topology. The final step is (3) for the uniqueness of measure weak solution, it seems difficult to adapt the strategy of [27] for which the conservation of energy played a crucial role. For this reason, we rather follow the approach of [23], which requires the strong confining assumption (1.14) and is based upon suitable log-Lipschitz estimates for the Kantorovich–Rubinstein distance between two solutions of (1.1). It is likely that assumption (1.14) can be relaxed. Notice that even though such an assumption is a restriction for the Cauchy theory, it provides valuable information on the exponential tail of the self-similar profile we aim to construct.

As far as Question 1 is concerned, we establish the optimal decay of the moments of the solutions to (1.1) by a suitable comparison of ODEs. Such techniques are natural and have been applied to the study of Haff's law for the 3-D granular Boltzmann equation in [2]. The main difficulty is to provide optimal lower bounds for the moments; see Propositions 3.3 and 3.4. Essentially, we obtain that

$$M_k(t) = \int_{\mathbb{R}} |x|^k \, \mu_t(\mathrm{d}x) \approx C_k \, t^{-\frac{k}{\gamma}} \qquad \text{as} \ \ t \to \infty;$$

see Theorem 3.5 for a more precise statement. Such a decay immediately translates into convergence of μ_t toward δ_0 in the Wasserstein topology.

Regarding Question 2, once the fixed point Theorem 1.1 is at hand, the key step is to engineer a suitable *stable* compact set $\mathcal{Z}$. Compactness is easily achieved in $\mathcal{Y} = \mathcal{P}_{\max(\gamma,2)}(\mathbb{R})$ endowed with weak-$\star$ topolgy; only uniform boundedness of some moments suffices. However, $\mathcal{Y}$ must overrule the possibility that the fixed point will be a plain Dirac mass located at zero. This is closely related to the sharp lower bound found for the moments in Question 1. In such a situation, a series of simple observations on the regularity of the solution to (1.13) proves that such a steady state is actually an L^1-function. Namely, one of the most important steps in our strategy is the following observation.

THEOREM 1.5. *Any steady measure solution* $\boldsymbol{\mu} \in \mathcal{P}^0_{\max(\gamma,2)}(\mathbb{R})$ *to* (1.9) *such that*

$$\mathbf{m}_\gamma := \int_{\mathbb{R}} |\xi|^\gamma \boldsymbol{\mu}(\mathrm{d}\xi) > 0 \tag{1.15}$$

is absolutely continuous with respect to the Lebesgue measure over $\mathbb{R}$*, i.e., there exists some nonnegative* $G \in L^1_{\max(\gamma,2)}(\mathbb{R})$ *such that*

$$\boldsymbol{\mu}(\mathrm{d}\xi) = G(\xi)\mathrm{d}\xi \,.$$

In other words, any solution to (1.9) lying in $\mathcal{P}^0_{\max(\gamma,2)}(\mathbb{R})$ different from a Dirac mass must be a regular measure. This leads to our main result.

THEOREM 1.6. *For any* $\gamma > 0$*, there exists* $G \in L^1_{\max(\gamma,2)}(\mathbb{R})$ *which is a steady solution to* (1.9) *in the weak sense.*

2. Cauchy theory.

We are first concerned with the Cauchy theory for problem (1.1) and we begin with studying a priori estimates for weak measure solutions to (1.1). Let us fix $\gamma > 0$ and set $\gamma_* = \max(\gamma, 2)$.

2.1. A priori estimates on moments.

We first state the following general properties of weak measure solutions to (1.1) which have sufficient bounded moments.

PROPOSITION 2.1. *Let* $\mu_0 \in \mathcal{M}^+_{\gamma_*}(\mathbb{R})$ *and let* $(\mu_t)_{t \geqslant 0}$ *be any weak measure solution to* (1.1) *associated to* μ_0*. Given* $k \geqslant 0$*, assume there exists* $p > k + \gamma$ *such that*

$$\sup_{\delta < t < T} \|\mu_t\|_p < \infty \qquad \forall \, T > \delta > 0 \,. \tag{2.1}$$

Then, the following hold:

1. *For any* $\varphi \in L^\infty_{-k}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$*, the mapping* $t \geqslant 0 \mapsto \langle \mathcal{Q}(\mu_t, \mu_t) \, ; \, \varphi \rangle$ *is continuous in* $(0, \infty)$*.*
2. *For any* $\varphi \in L^\infty_{-k}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ *it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \varphi(x) \mu_t(\mathrm{d}x) = \langle \mathcal{Q}(\mu_t, \mu_t) \, ; \, \varphi \rangle \qquad \forall \, t \geqslant 0 \,. \tag{2.2}$$

The proof of Proposition 2.1 will need the following preliminary lemma (see [27, Proposition 2.2] for a complete proof).

LEMMA 2.2. *Let* $(\mu^n)_{n\in\mathbb{N}}$ *be a sequence from* $\mathcal{M}^+_{\gamma_*}(\mathbb{R})$ *that converges weakly-*$\star$ *to some* $\mu \in \mathcal{M}^+_{\gamma_*}(\mathbb{R})$. *We assume that for some* $p > 0$ *it holds that*

$$\sup_{n\in\mathbb{N}} \|\mu^n\|_p < \infty\,.$$

Then, for any $\psi \in \mathcal{C}(\mathbb{R}^2)$ *satisfying* $\lim_{|x|+|y|\to+\infty} \frac{\psi(x,y)}{\langle x\rangle^p+\langle y\rangle^p} = 0$, *one has*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^2} \psi(x,y)\,\mu^n(\mathrm{d}x)\,\mu^n(\mathrm{d}y) = \int_{\mathbb{R}^2} \psi(x,y)\,\mu(\mathrm{d}x)\,\mu(\mathrm{d}y).$$

Proof of Proposition 2.1. Let $k \geqslant 0$ and $\varphi \in L^\infty_{-k}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ be given. Choose $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\|\chi\|_\infty \leqslant 1$ with $\chi(x) = 0$ if $|x| \geqslant 2$ and $\chi(x) = 1$ for $|x| \leqslant 1$ and set for any $n \in \mathbb{N}_*$, $\varphi_n(x) = \varphi(x)\chi\left(\frac{x}{n}\right)$. It follows that $\varphi_n \in \mathcal{C}_c(\mathbb{R}) \subset \mathcal{C}_b(\mathbb{R})$ for any n with $\varphi_n(x) \to \varphi(x)$ for any $x \in \mathbb{R}$ as $n \to \infty$. Consequently, $\Delta\varphi_n(x,y) \to \Delta\varphi(x,y)$ for any $(x,y) \in \mathbb{R}^2$ as $n \to \infty$. Now, since $\varphi_n \in \mathcal{C}_b(\mathbb{R})$ for any $n \in \mathbb{N}$, one deduces from (1.12) that

$$\int_{\mathbb{R}} \varphi_n(x)\mu_{t_2}(\mathrm{d}x) = \int_{\mathbb{R}} \varphi_n(x)\mu_{t_1}(\mathrm{d}x) + \int_{t_1}^{t_2} \langle \mathcal{Q}(\mu_\tau,\mu_\tau)\,;\,\varphi_n\rangle\,\mathrm{d}\tau \quad \forall\, t_2 > t_1 > 0, \quad \forall\, n \geqslant 1\,.$$

Notice that $\|\varphi_n\|_{L^\infty_{-k}} \leqslant \|\varphi\|_{L^\infty_{-k}} < \infty$. Thus, there is $C > 0$ such that for any $n \in \mathbb{N}_*$, any $(x,y) \in \mathbb{R}^2$,

$$|\Delta\varphi(x,y)| \leqslant C\left(\langle x\rangle^k + \langle y\rangle^k\right) \qquad \text{and} \qquad |\Delta\varphi_n(x,y)| \leqslant C\left(\langle x\rangle^k + \langle y\rangle^k\right)\,, \tag{2.3}$$

where the constant C depends only on φ (and a). Using the dominated convergence theorem together with (2.1), one deduces that

$$\begin{aligned}\int_{t_1}^{t_2} \langle \mathcal{Q}(\mu_\tau,\mu_\tau)\,;\,\varphi_n\rangle\,\mathrm{d}\tau &= \frac{1}{2}\int_{t_1}^{t_2}\mathrm{d}\tau \int_{\mathbb{R}^2} |x-y|^\gamma \Delta\varphi_n(x,y)\mu_\tau(\mathrm{d}x)\mu_\tau(\mathrm{d}y)\\ &\underset{n\to\infty}{\longrightarrow} \frac{1}{2}\int_{t_1}^{t_2}\mathrm{d}\tau \int_{\mathbb{R}^2} |x-y|^\gamma \Delta\varphi(x,y)\mu_\tau(\mathrm{d}x)\mu_\tau(\mathrm{d}y) = \int_{t_1}^{t_2} \langle \mathcal{Q}(\mu_\tau,\mu_\tau)\,;\,\varphi\rangle\,\mathrm{d}\tau,\end{aligned}$$

so, the identity

$$\int_{\mathbb{R}} \varphi(x)\mu_{t_2}(\mathrm{d}x) = \int_{\mathbb{R}} \varphi(x)\mu_{t_1}(\mathrm{d}x) + \int_{t_1}^{t_2} \langle \mathcal{Q}(\mu_\tau,\mu_\tau)\,;\,\varphi\rangle\,\mathrm{d}\tau \qquad \forall\, t_2 > t_1 > 0\,, \tag{2.4}$$

holds true. It follows from (2.3) that

$$\Big|\langle \mathcal{Q}(\mu_\tau,\mu_\tau)\,;\,\varphi\rangle\Big| \leqslant C\int_{\mathbb{R}^2} |x-y|^\gamma \left(\langle x\rangle^k + \langle y\rangle^k\right)\mu_\tau(\mathrm{d}x)\mu_\tau(\mathrm{d}y) \leqslant 2C\|\mu_\tau\|^2_{k+\gamma} \quad \forall \tau > 0.$$

Combining this with (2.4), one sees that, for any $T > \delta > 0$,

$$\left|\int_{\mathbb{R}} \varphi(x)\mu_{t_2}(\mathrm{d}x) - \int_{\mathbb{R}} \varphi(x)\mu_{t_1}(\mathrm{d}x)\right| \leqslant 2\,C\,|t_2 - t_1| \sup_{\delta\leqslant\tau\leqslant T} \|\mu_\tau\|^2_{k+\gamma} \qquad \forall\, t_1,\, t_2 \in [\delta, T]\,.$$

In particular, under assumption (2.1), the mapping $t \mapsto \int_{\mathbb{R}} \varphi(x)\mu_t(\mathrm{d}x)$ is continuous over $(0,\infty)$. This shows that, for any $t > 0$ and sequence $(t_n)_n \subset [t/2, 3t/2]$ with

$\lim_n t_n = t$, the sequence $(\mu_{t_n})_n$ converges weakly-$\star$ toward μ_t. In addition, since $\varphi \in L^\infty_{-k}(\mathbb{R})$ and $p > k+\gamma$ one concludes that

$$\lim_{|x|+|y|\to\infty} \frac{|x-y|^\gamma \Delta\varphi(x,y)}{\langle x\rangle^p + \langle y\rangle^p} = 0\,.$$

Thus, it readily follows from Lemma 2.2 that

$$\lim_n \int_{\mathbb{R}^2} |x-y|^\gamma \Delta\varphi(x,y)\mu_{t_n}(\mathrm{d}x)\mu_{t_n}(\mathrm{d}y) = \int_{\mathbb{R}^2} |x-y|^\gamma\, \Delta\varphi(x,y)\, \mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)\,. \tag{2.5}$$

Henceforth, the mapping $t \mapsto \langle \mathcal{Q}(\mu_t,\mu_t),\varphi\rangle$ is continuous over $(0,\infty)$ proving point (1). Point (2) then follows directly from (2.4). ∎

Moments estimates of the collision operator are given by the following.

Proposition 2.3. *Let $\mu \in \mathcal{P}^0_{k+\gamma}(\mathbb{R})$ with $k \geqslant 2$. Then*

$$\langle \mathcal{Q}(\mu,\mu)\,;\,|\cdot|^k\rangle \leqslant -\frac{1}{2}\left(1-a^k-b^k\right) M_{k+\gamma}(\mu) \leqslant 0\,. \tag{2.6a}$$

In particular,

$$\langle \mathcal{Q}(\mu,\mu)\,;\,|\cdot|^k\rangle \leqslant -\frac{1}{2}\left(1-a^k-b^k\right) M_k\,(\mu)^{1+\frac{\gamma}{k}}\;, \tag{2.6b}$$

and, if $k>2$,

$$\langle \mathcal{Q}(\mu,\mu)\,;\,|\cdot|^k\rangle \leqslant -\frac{1}{2}\left(1-a^k-b^k\right) M_2(\mu)^{-\frac{\gamma}{k-2}}\, M_k(\mu)^{1+\frac{\gamma}{k-2}}. \tag{2.6c}$$

Proof. We apply the weak form (1.10) to $\varphi(x) = |x|^k$. Using the elementary inequality

$$|x|^k + |y|^k - |ax+by|^k - |ay+bx|^k \geqslant \left(1-a^k-b^k\right)\,|x-y|^k \tag{2.7}$$

valid for any $(x,y)\in\mathbb{R}^2$ and $k\geqslant 2$ (with equality sign whenever $k=2$), and noticing that $1-a^k-b^k$ is nonnegative for any $k\geqslant 2$ and any $a\in(0,1)$ we have

$$\langle \mathcal{Q}(\mu,\mu)\,;\,|\cdot|^k\rangle \leqslant -\frac{1}{2}\left(1-a^k-b^k\right)\int_{\mathbb{R}^2} |x-y|^{\gamma+k}\mu(\mathrm{d}x)\mu(\mathrm{d}y)\,. \tag{2.8}$$

Since $\mu\in\mathcal{P}^0_{k+\gamma}(\mathbb{R})$ and the mapping $[0,\infty)\ni r\mapsto r^{\gamma+k}$ is convex, one deduces from Jensen's inequality that

$$\int_{\mathbb{R}^2} |x-y|^{\gamma+k}\mu(\mathrm{d}x)\mu(\mathrm{d}y) \geqslant \int_{\mathbb{R}} |x|^{k+\gamma}\mu(\mathrm{d}x)\,,$$

from which inequality (2.8) yields (2.6a). Using again Jensen's inequality we get also that $M_{k+\gamma}(\mu) \geqslant M_k(\mu)^{1+\frac{\gamma}{k}}$ which proves (2.6b). Finally, according to the Hölder inequality

$$M_{k+\gamma}(\mu) \geqslant M_k(\mu)^{1+\frac{\gamma}{k-2}}\, M_2(\mu)^{-\frac{\gamma}{k-2}} \qquad \forall\, k>2\,,$$

and (2.6c) is deduced from (2.6a). ∎

The assumption on the measure μ in the above proposition is stronger than the one made on the initial datum μ_0 in our main result—Theorem 1.4. It will serve as a tool toward a priori estimates on solutions to (1.1) in Proposition 2.4. Hereafter, (2.9) will result in the construction of the stable set $\Omega_{K,\delta}$ in Theorem A.1 while (2.10) is the main tool for appearance of higher-order moments.

PROPOSITION 2.4. *Let $\mu_0 \in \mathcal{P}^0_{\gamma_*}(\mathbb{R})$ be a given initial datum and $(\mu_t)_{t\geqslant 0}$ be a measure weak solution to* (1.1) *associated to μ_0. Then, the following holds:*

1. *If $(\mu_t)_{t\geqslant 0} \subset \mathcal{P}^0_p(\mathbb{R})$ with $p > k+\gamma$ for some $k \geqslant 2$, then $t \geqslant 0 \mapsto M_k(\mu_t)$ is decreasing with*

$$M_k(\mu_t) \leqslant M_k(\mu_0)\left(1+\frac{\gamma}{2k}\big(1-a^k-b^k\big)M_k(\mu_0)^{\frac{\gamma}{k}}\,t\right)^{-\frac{k}{\gamma}} \qquad \forall\, t \geqslant 0\,. \tag{2.9}$$

In particular, $\sup_{t\geqslant 0} M_k(\mu_t) < \infty$.

2. *If for some $k > 2$ and $p > k+\gamma$, $\sup_{t\geqslant\delta}\|\mu_t\|_p < \infty$ for any $\delta > 0$, then*

$$M_k(\mu_t) \leqslant C_k(\gamma)\,\|\mu_0\|_{\gamma_*}\,\min\left\{t^{-\frac{k-2}{\gamma}}, t^{-\frac{k}{\gamma}}\right\} \qquad \forall\, t > 0\,, \tag{2.10}$$

where $C_k(\gamma) > 0$ is a positive constant depending only on $k > 2$, γ, and a.

Proof. For the proof of (2.9), apply (2.2) with $\varphi(x) = |x|^k$ and note that using (2.6b)

$$\frac{\mathrm{d}}{\mathrm{d}t}M_k(\mu_t) \leqslant -\frac{1}{2}\left(1-a^k-b^k\right)\,M_k(\mu_t)^{1+\frac{\gamma}{k}} \qquad \forall t \geqslant 0\,.$$

This directly implies point (1) of Proposition 2.4. To prove estimate (2.10) for short time observe that applying (2.2) to $\varphi(x) = |x|^k$ and using (2.6c), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}M_k(\mu_t) \leqslant -\frac{1}{2}\left(1-a^k-b^k\right)M_2(\mu_t)^{-\frac{\gamma}{k-2}}\,M_k(\mu_t)^{1+\frac{\gamma}{k-2}} \qquad \forall\, t > 0\,,$$

and, since by the definition of measure weak solution it holds that $M_2(\mu_t) \leqslant \|\mu_0\|_{\gamma_*}$ for any $t \geqslant 0$, one deduces that

$$\frac{\mathrm{d}}{\mathrm{d}t}M_k(\mu_t) \leqslant -\frac{1}{2}\left(1-a^k-b^k\right)\|\mu_0\|_{\gamma_*}^{-\frac{\gamma}{k-2}}\,M_k(\mu_t)^{1+\frac{\gamma}{k-2}} \qquad \forall\, t > 0\,.$$

This inequality leads to estimate (2.10) with constant $C_k(\gamma) = \left(\frac{\gamma}{2(k-2)}(1-a^k-b^k)\right)^{-\frac{k-2}{\gamma}}$. The long time decay follows applying (2.9) for $t \geqslant \delta$. ∎

Remark 2.5. It is not difficult to prove that the conclusions of the previous two propositions hold for general measure $\mu_0 \in \mathcal{M}^+_{\gamma_*}(\mathbb{R})$ with

$$\|\mu_0\|_0 = \varrho \neq 0 \ \text{ and } \int_{\mathbb{R}} x\mu_0(\mathrm{d}x) = 0\,.$$

In such a case, the constant $C_k(\gamma)$ depends also continuously on ϱ.

Introduce the class $\mathcal{P}_{\exp,\gamma}(\mathbb{R})$ of probability measures with exponential tails of order γ,

$$\mathcal{P}_{\exp,\gamma}(\mathbb{R}) = \left\{\mu \in \mathcal{P}(\mathbb{R})\ ;\, \exists\,\varepsilon > 0 \text{ such that } \int_{\mathbb{R}} \exp(\varepsilon|x|^\gamma)\mu(\mathrm{d}x) < \infty\right\}. \tag{2.11}$$

We have the following.

THEOREM 2.6. *Let $\mu_0 \in \mathcal{P}^0_{\gamma_*}(\mathbb{R})$ be an initial datum and $(\mu_t)_t$ be a measure weak solution to* (1.1) *associated to μ_0. If $\mu_0 \in \mathcal{P}_{\exp,\gamma}(\mathbb{R})$, then there exists $\alpha > 0$ and $C > 0$ such that*

$$\sup_{t\geqslant 0}\int_{\mathbb{R}} \exp(\alpha|x|^{\gamma})\mu_t(\mathrm{d}x) \leqslant C\,. \tag{2.12}$$

Proof. Since $\mu_0 \in \mathcal{P}_{\exp,\gamma}(\mathbb{R})$, there exists $\alpha > 0$ and $C_0 > 0$ such that

$$\int_{\mathbb{R}} \exp(\alpha|x|^{\gamma})\mu_0(\mathrm{d}x) \leqslant C_0\,.$$

Let us denote by p_0 the integer such that $\gamma\, p_0 \geqslant \gamma_*$ and $\gamma\,(p_0-1) < \gamma_*$. Thus, for $0 \leqslant p \leqslant p_0 - 1$ and $t \geqslant 0$,

$$M_{\gamma p}(\mu_t) \leqslant \|\mu_t\|_{\gamma_*} \leqslant \|\mu_0\|_{\gamma_*}\,.$$

For $p \geqslant p_0$ and $t \geqslant 0$, one deduces from (2.6a) that

$$M_{\gamma p}(\mu_t) \leqslant M_{\gamma p}(\mu_0)\,.$$

Thus, for any $t \geqslant 0$ and any $n > p_0$

$$\sum_{p=0}^{n} M_{\gamma p}(\mu_t)\frac{\alpha^p}{p!} \leqslant \|\mu_0\|_{\gamma_*}\sum_{p=0}^{p_0-1}\frac{\alpha^p}{p!} + \sum_{p=p_0}^{n} M_{\gamma p}(\mu_0)\frac{\alpha^p}{p!} \leqslant \|\mu_0\|_{\gamma_*}\exp(\alpha) + C_0\,.$$

Letting $n \to +\infty$ we get (2.12). ∎

2.2. Cauchy theory. The main ingredients of the proof of Theorem 1.4 are Propositions 2.7 and 2.10. Namely, by studying first the Cauchy problem for L^1 initial data and then introducing a suitable approximation we can construct weak measure solutions to (1.1) leading to the following result of existence of solutions.

PROPOSITION 2.7. *For any $\mu_0 \in \mathcal{P}^0_{\gamma_*}(\mathbb{R})$, $\mu_0 \neq \delta_0$, there exists a measure weak solution $(\mu_t)_{t\geqslant 0} \subset \mathcal{P}^0_{\gamma_*}(\mathbb{R})$ associated to μ_0 and such that*

$$\sup_{t\geqslant 0}\|\mu_t\|_{\gamma_*} \leqslant \|\mu_0\|_{\gamma_*} \qquad \textit{and} \qquad \sup_{t\geqslant t_0}\|\mu_t\|_s < \infty \qquad \forall\, t_0 > 0,\ s > \gamma_*\,.$$

Remark 2.8. The instantaneous appearance of higher-order moments for solution to Boltzmann-like equations associated to so-called hard potentials (corresponding to $\gamma > 0$) is a well-documented feature which can be traced back to [38] (see also [1] for the appearance of exponential moments). It has already been observed for measure solutions like the ones considered here in [27].

Proof. The proof follows the approach of [27, section 4] and it relies on an existence Theorem in an L^1-framework borrowing ideas from [16] (see Theorem A.1 in Appendix A). We only sketch here the main steps.

First, since $\mu_0 \in \mathcal{P}^0_{\gamma_*}(\mathbb{R})$ is not the Dirac mass centered at 0, the temperature $T_0 := \int_{\mathbb{R}} x^2\mu_0(\mathrm{d}x)$ is positive and one can define a sequence $(F_0^n)_n)$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}}\varphi(x)F_0^n(x)\mathrm{d}x = \int_{\mathbb{R}}\varphi(x)\mu_0(\mathrm{d}x) \qquad \forall\,\varphi \in L^\infty_{-\gamma_*}(\mathbb{R})\cap\mathcal{C}(\mathbb{R}) \tag{2.13}$$

with $F_0^n \in \bigcap_{s\geqslant 0} L^1_s(\mathbb{R})$ for any $n \geqslant 1$ (notice that, as in [27], F_0^n is some slight modification of the Mehler transform of μ_0). Then, according to Theorem A.1, for any $n \geqslant 0$, there exists a family $(F_t^n)_{t\geqslant 0} \subset L^1_{\gamma_*}(\mathbb{R})$ such that $(\mu_t^n)_{t\geqslant 0}$ is a weak measure solution to (1.1) associated to μ_0^n, where

$$\mu_t^n(\mathrm{d}x) = F_t^n(x)\mathrm{d}x \qquad \forall\, t \geqslant 0\,.$$

Then, noticing that

$$\|\mu_t^n\|_\gamma \leqslant \|\mu_t^n\|_{\gamma_*} \leqslant \|\mu_0^n\|_{\gamma_*} \qquad \forall\, n \geqslant 1,$$

one easily checks that

$$\big|\langle \mathcal{Q}(\mu_t^n, \mu_t^n)\,;\,\varphi\rangle\big| \leqslant 2\|\varphi\|_\infty \|\mu_0^n\|^2_{\gamma_*} \qquad \forall\, t \geqslant 0,\, n \geqslant 1,$$

from which one deduces as in [27] that there exists $C = C(\mu_0) > 0$ (depending only on $\|\mu_0\|_{\gamma_*}$) such that, for any $t_2 \geqslant t_1 \geqslant 0$,

(2.14)
$$\sup_{n\geqslant 1}\left|\int_{\mathbb{R}} \varphi(x)\mu^n_{t_1}(\mathrm{d}x) - \int_{\mathbb{R}} \varphi(x)\mu^n_{t_2}(\mathrm{d}x)\right| \leqslant C(\mu_0)\,\|\varphi\|_\infty\,\big|t_2 - t_1\big| \qquad \forall\, \varphi \in \mathcal{C}_b(\mathbb{R})\,.$$

Moreover, on the basis of the a priori estimates (2.10) (see also Remark 2.5),

$$M_k(F_t^n) \leqslant C_k(\gamma, \|\mu_0^n\|_0)\|\mu_0^n\|_{\gamma_*}\, t^{-\frac{k-2}{\gamma}} \qquad \forall\, k > 2\,.$$

Since $\lim_{n\to\infty} \|\mu_0^n\|_0 = 1$ and $\lim_{n\to\infty} \|\mu_0^n\|_{\gamma_*} = \|\mu_0\|_{\gamma_*}$ according to (2.13), we deduce that, for any $k > 2$, there exists some positive constant $\mathbf{C}_k$ depending only on k, γ, and $M_{\gamma_*}(\mu_0)$ such that

$$\sup_{n\geqslant 1} M_k(F_t^n) \leqslant \mathbf{C}_k t^{-\frac{k-2}{\gamma}} \qquad \forall\, k > 2\,.$$

From this, we conclude as in [27] that there exists a subsequence (still denoted by) $(\mu_t^n)_{t\geqslant 0}$ and a family $(\mu_t)_{t\geqslant 0} \subset \mathcal{M}^+_{\gamma_*}(\mathbb{R})$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(x)\mu_t^n(\mathrm{d}x) = \int_{\mathbb{R}} \varphi(x)\mu_t(\mathrm{d}x) \qquad \forall\, \varphi \in \mathcal{C}_c(\mathbb{R}), \qquad t \geqslant 0\,, \tag{2.15}$$

$$\|\mu_t\|_{\gamma_*} \leqslant \|\mu_0\|_{\gamma_*}\,, \qquad M_k(\mu_t) \leqslant \mathbf{C}_k t^{-\frac{k-2}{\gamma}} \qquad \forall\, t > 0,\, k > 2\,,$$ ◻

and (2.14) still holds for the limit μ_t (which implies that, for any $\varphi \in \mathcal{C}_b(\mathbb{R})$, the mapping $t \in [0,\infty) \mapsto \int_{\mathbb{R}} \varphi(x)\mu_t(\mathrm{d}x)$ is continuous). To prove that $(\mu_t)_{t\geqslant 0}$ is a measure weak solution to (1.1) associated to μ_0 in the sense of Definition 1.2, one argues exactly as in [27, section 4].

Remark 2.9. Arguing as in [27], it is not difficult to prove that any weak measure solution to (1.1) is in fact a strong solution in the sense of [27].

To achieve the proof of Theorem 1.4, it remains only to prove the uniqueness of the solution. It seems difficult here to adapt the strategy of [27] for which the conservation of energy played a crucial role. For this reason, we rather follow the approach of [23], which requires the exponential tail estimate (1.14). The main step toward uniqueness is the following *log-Lipschitz estimate* for the Kantorovich–Rubinstein distance.

PROPOSITION 2.10. *Let μ_0 and ν_0 be two probability measures in $\mathcal{P}^0_{\gamma_*}(\mathbb{R})$ satisfying* (1.14)*, i.e., there exists $\varepsilon > 0$ such that*

$$\int_{\mathbb{R}} \exp\left(\varepsilon\,|x|^\gamma\right)\mu_0(\mathrm{d}x) + \int_{\mathbb{R}} \exp\left(\varepsilon\,|x|^\gamma\right)\nu_0(\mathrm{d}x) < \infty\,.$$

Let then $(\mu_t)_{t\geqslant 0}$ and $(\nu_t)_{t\geqslant 0}$ be two measure weak solutions to (1.1) *associated respectively to the initial data μ_0 and ν_0. There exists $K_\varepsilon > 0$ such that, for any $T > 0$,*

$$d_{\mathrm{KR}}(\mu_t,\nu_t) \leqslant d_{\mathrm{KR}}(\mu_0,\nu_0) + K_\varepsilon\, C_T(\varepsilon)\int_0^t d_{\mathrm{KR}}(\mu_s,\nu_s)\Big(1+\big|\log d_{\mathrm{KR}}(\mu_s,\nu_s)\big|\Big)\mathrm{d}s \qquad \forall\, t\in[0,T]\,, \tag{2.16}$$

where $C_T(\varepsilon) = \sup_{t\in[0,T]}\int_{\mathbb{R}}\exp(\varepsilon\,|x|^\gamma)(\mu_t+\nu_t)(\mathrm{d}x) < \infty$.

The proof of Proposition 2.10 can be found in Appendix A.

Proof of Theorem 1.4. Given Proposition 2.7, to prove Theorem 1.4 it suffices to show the uniqueness of measure weak solutions to (1.1). Let μ_0 be a probability measure in $\mathcal{P}^0_{\gamma_*}(\mathbb{R})$ satisfying (1.14) and let $(\mu_t)_{t\geqslant 0}$ and $(\nu_t)_{t\geqslant 0}$ be two measure weak solutions to (1.1) associated to μ_0. From Proposition 2.10, given $T > 0$ there exists a finite positive constant K_T such that

$$d_{\mathrm{KR}}(\mu_t,\nu_t) \leqslant K_T\int_0^t \mathbf{\Phi}(d_{\mathrm{KR}}(\mu_s,\nu_s))\mathrm{d}s \qquad \forall t\in[0,T]$$

with $\mathbf{\Phi}(r) = r(1+|\log r|)$ for any $r > 0$. Since $\mathbf{\Phi}$ satisfies the so-called Osgood condition

$$\int_0^1 \frac{\mathrm{d}r}{\mathbf{\Phi}(r)} = \infty\,, \tag{2.17}$$

a nonlinear version of the Gronwall lemma (see, for instance, [7, Lemma 3.4, p. 125]) asserts that $d_{\mathrm{KR}}(\mu_t,\nu_t) = 0$ for any $t\in[0,T]$. Since $T > 0$ is arbitrary, this proves the uniqueness. □

The existence and uniqueness of a weak measure solution to (1.1) allows us to define a semiflow $(\mathcal{S}_t)_{t\geqslant 0}$ on $\mathcal{P}_{\exp,\gamma}(\mathbb{R})$ (recall (2.11)). Namely, Theorem 1.4 together with Theorem 2.6 asserts that for any $\mu_0\in\mathcal{P}_{\exp,\gamma}(\mathbb{R})\cap\mathcal{P}^0(\mathbb{R})$, there exists a unique weak measure solution $(\mu_t)_{t\geqslant 0}$ to (1.1) with $\mu_t\in\mathcal{P}_{\exp,\gamma}(\mathbb{R})$ for any $t\geqslant 0$ and we shall denote

$$\mu_t := \mathcal{S}_t(\mu_0) \qquad \forall t\geqslant 0\,.$$

Then, the semiflow $\mathcal{S}_t$ is a well-defined nonlinear mapping from $\mathcal{P}_{\exp,\gamma}(\mathbb{R})\cap\mathcal{P}^0(\mathbb{R})$ into itself. Moreover, by definition of weak solution, the mapping $t\mapsto\mathcal{S}_t(\mu_0)$ belongs to $\mathcal{C}_{\mathrm{weak}}([0,\infty),\mathcal{P}_2(\mathbb{R}))$. One has the following weak continuity result for the semiflow.

PROPOSITION 2.11. *The semiflow $(\mathcal{S}_t)_{t\geqslant 0}$ is weakly continuous on $\mathcal{P}_1(\mathbb{R})$ in the following sense. Let $(\mu_n)_n\in\mathcal{P}^0_{\gamma_*}(\mathbb{R})$ be a sequence such that there exists $\varepsilon > 0$ satisfying*

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}}\exp(\varepsilon|x|^\gamma)\mu_n(\mathrm{d}x) < \infty\,. \tag{2.18}$$

If $(\mu_n)_n$ converges to some $\mu\in\mathcal{P}^0_{\gamma_*}(\mathbb{R})$ in the weak-⋆ topology, then for any $t\geqslant 0$,

$$\mathcal{S}_t(\mu_n)\longrightarrow\mathcal{S}_t(\mu) \qquad \textit{in the weak-}\star\textit{ topology as } n\to\infty\,.$$

Proof. The proof is based upon the stability result established in Proposition 2.10. Namely, because $(\mu_n)_n$ converges in the weak-⋆ topology to $\mu\in\mathcal{P}^0_{\gamma_*}(\mathbb{R})$, one deduces from (2.18) that

$$\int_{\mathbb{R}}\exp(\varepsilon|x|^\gamma)\mu(\mathrm{d}x) < \infty,$$

which means that all μ_n and μ share the same exponential tail estimate with *some common* $\varepsilon > 0$. Then, for any $T > 0$ one deduces from (2.16) that

$$d_{\mathrm{KR}}(\mathcal{S}_t(\mu_n)\,,\mathcal{S}_t(\mu)) \leqslant d_{\mathrm{KR}}(\mu_n\,,\,\mu) \\ + K(\varepsilon)\,C_T(\varepsilon)\int_0^t \mathbf{\Phi}\left(d_{\mathrm{KR}}(\mathcal{S}_s(\mu_n)\,,\,\mathcal{S}_s(\mu)\right)\mathrm{d}s \qquad \forall\, t \in [0,T],\ n \in \mathbb{N}\,,$$

for some universal positive constant $K(\varepsilon)$ and

$$C_T(\varepsilon) = \sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\int_{\mathbb{R}} \exp(\varepsilon|x|^\gamma)\,(\mathcal{S}_t(\mu_n)+\mathcal{S}_t(\mu))\,(\mathrm{d}x) < \infty$$

according to Theorem 2.6. Here above, $\mathbf{\Phi}(r) = r\,(1+|\log r|)$ satisfies the Osgood condition (2.17); thus, using again a nonlinear version of the Gronwall lemma [7, Lemma 3.4], we deduce from this estimate that

$$\mathbf{\Psi}\big(d_{\mathrm{KR}}\,(\mu_n\,,\,\mu)\big) - \mathbf{\Psi}\big(d_{\mathrm{KR}}\,(\mathcal{S}_t(\mu_n)\,,\,\mathcal{S}_t(\mu))\big) \leqslant K(\varepsilon)\,C_T(\varepsilon)\,t \qquad \forall\, t\in[0,T],\ n\in\mathbb{N},$$

where

$$\mathbf{\Psi}(x) = \int_x^1 \frac{\mathrm{d}r}{\mathbf{\Phi}(r)} \qquad \forall\, x \geqslant 0\,.$$

Taking now the limit $n \to \infty$, since d_{KR} metrizes the weak-$\star$ topology of $\mathcal{P}_1(\mathbb{R})$ it follows that $\lim_{n\to\infty} d_{\mathrm{KR}}(\mu_n,\mu) = 0$. Furthermore, recalling that $\mathbf{\Psi}(0) = \infty$ one concludes that

$$\lim_{n\to\infty} \mathbf{\Psi}\big(d_{\mathrm{KR}}\big(\mathcal{S}_t(\mu_n)\,,\,\mathcal{S}_t(\mu)\big)\big) = \infty\,.$$

That is, $\lim_{n\to\infty} d_{\mathrm{KR}}\big(\mathcal{S}_t(\mu_n)\,,\,\mathcal{S}_t(\mu)\big) = 0$, which proves the result. ∎

3. Optimal decay of the moments. We now prove that the upper bounds obtained for the moments of solutions to (1.1) in Proposition 2.4 are actually optimal.

3.1. Lower bounds for moments. We begin with the case $\gamma \in (0,1]$.

PROPOSITION 3.1. *Fix $\gamma \in (0,1]$ and let $\mu_0 \in \mathcal{P}_2^0(\mathbb{R})$ be an initial datum and $(\mu_t)_{t\geqslant 0}$ be a measure weak solution to* (1.1) *associated to μ_0. Then, there exists $\mathcal{K}_\gamma > 0$ depending only on a and γ such that*

$$M_\gamma(\mu_t) \geqslant \frac{M_\gamma(\mu_0)}{1+\mathcal{K}_\gamma M_\gamma(\mu_0)t} \qquad \forall\, t \geqslant 0\,. \tag{3.1}$$

Thus,

$$M_2(\mu_t) \geqslant M_\gamma(\mu_0)^{\frac{2}{\gamma}}\,(1+\mathcal{K}_\gamma\,M_\gamma(\mu_0)t)^{-\frac{2}{\gamma}} \qquad \forall\, t\geqslant 0\,. \tag{3.2}$$

Proof. For $\gamma \in (0,1]$ the following elementary inequality holds:

$$B_\gamma(x,y) = \Big(\,|ax+by|^\gamma + |ay+bx|^\gamma - |x|^\gamma - |y|^\gamma\Big)|x-y|^\gamma \geqslant -C_\gamma|x|^\gamma\,|y|^\gamma \quad \forall\, x,y\in\mathbb{R} \tag{3.3}$$

for some positive constant $C_\gamma > 0$ explicit depending only on a and γ. Using (2.2) and the weak form (1.10) with $\varphi(x) = |x|^\gamma \in L^\infty_{-2}(\mathbb{R})\cap\mathcal{C}(\mathbb{R})$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}M_\gamma(\mu_t) = \frac{1}{2}\int_{\mathbb{R}^2} B_\gamma(x,y)\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y) \geqslant -\frac{C_\gamma}{2}\int_{\mathbb{R}^2} |x|^\gamma\,|y|^\gamma \mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)$$

from which it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}M_\gamma(\mu_t) \geqslant -\frac{C_\gamma}{2}M_\gamma(\mu_t)^2 \qquad \forall\, t \geqslant 0\,.$$

This inequality, after integration, yields (3.1) with $\mathcal{K}_\gamma = \frac{C_\gamma}{2}$. Inequality (3.2) follows from the fact that according to the Hölder inequality

$$M_\gamma(\mu_t) \leqslant M_2(\mu_t)^{\frac{\gamma}{2}}\,\|\mu_t\|_0^{\frac{2-\gamma}{2}}\,,$$

while $\|\mu_t\|_0 = \|\mu_0\|_0 = 1$ for any $t \geqslant 0$. ∎

The remaining case $\gamma > 1$ is more involved. In this case, inequality (3.3) no longer holds and we need the following result.

LEMMA 3.2. *Fix $\gamma > 1$. For any $p > 1$ and $\mu \in \mathcal{P}^0_{p+\gamma}(\mathbb{R})$ it holds that*

$$\langle \mathcal{Q}(\mu\,,\,\mu)\,;\,|\cdot|^p\rangle \leqslant -\big(1-\beta_p(a)\big)M_{p+\gamma}(\mu) + \mathcal{R}_p(\mu), \tag{3.4}$$

where $\beta_{2p}(a) = \max\{a^{p-1}, b^{p-1}\}$ and

$$\mathcal{R}_p(\mu) = 2^{\gamma-1}\beta_p(a)\sum_{k=1}^{\left[\frac{p+1}{2}\right]}\left(\begin{array}{c} p \\ k \end{array}\right)\Big(M_{k+\gamma}(\mu)\,M_{p-k}(\mu) + M_k(\mu)\,M_{p-k+\gamma}(\mu)\Big)\,.$$

Moreover, for $k \in (0,1]$

$$-\,\langle \mathcal{Q}(\mu\,,\,\mu)\,;\,|\cdot|^k\rangle \leqslant C_{\gamma,k}\,M_\gamma(\mu)\,M_k(\mu)\,. \tag{3.5}$$

The constant $C_{\gamma,k}$ depends only on γ and k.

Proof. Let us begin with inequality (3.4). We first notice the following elementary inequality, valid for any $p > 0$:

$$|ax+by|^p + |ay+bx|^p \leqslant \beta_p(a)\left(x^2+y^2\right)^{\frac{p}{2}} \leqslant \beta_p(a)\,(|x|+|y|)^p \qquad \forall (x,y) \in \mathbb{R}^2\,, \tag{3.6}$$

where $k \mapsto \beta_k(a)$ is decreasing with $\beta_2(a) = 1$ and $\lim_{k\to\infty}\beta_k(a) = 0$ (recall that $\max\{a,b\} < 1$). We use then the following useful result given in [15, Lemma 2] for estimation of the binomial for fractional powers. For any $p \geqslant 1$, if k_p denotes the integer part of $\frac{p+1}{2}$, the following inequality holds for any $u, v \in \mathbb{R}_+$:

$$\big(u+v\big)^p - u^p - v^p \leqslant \sum_{k=1}^{k_p}\left(\begin{array}{c} p \\ k \end{array}\right)\big(u^k v^{p-k} + u^{p-k}v^k\big)\,, \tag{3.7}$$

where the binomial coefficients are defined as

$$\left(\begin{array}{c} p \\ k \end{array}\right) = \left\{\begin{array}{ll} \frac{p(p-1)\cdots(p-k+1)}{k!} & \text{for } \ k \geqslant 1, \\ 1 & \text{for } \ k = 0\,. \end{array}\right.$$

Therefore, for any $p > 1$, one gets

$$|ax+by|^p+|ay+bx|^p-|x|^p-|y|^p \leqslant -\big(1-\beta_p(a)\big)\big(|x|^p+|y|^p\big) + \beta_p(a)\sum_{k=1}^{k_p}\binom{p}{k}\big(|x|^k\,|y|^{p-k}+|x|^{p-k}\,|y|^k\big).$$

Therefore, using (1.10) we get

$$\langle \mathcal{Q}(\mu\,,\,\mu)\,;\,|\cdot|^p\rangle \leqslant -\tfrac{1}{2}\big(1-\beta_p(a)\big)\int_{\mathbb{R}^2}\left(|x|^p+|y|^p\right)\big|x-y\big|^\gamma\mu(\mathrm{d}x)\mu(\mathrm{d}y) + \frac{1}{2}\beta_p(a)\sum_{k=1}^{k_p}\binom{p}{k}\int_{\mathbb{R}^2}|x-y|^\gamma\left(|x|^k\,|y|^{p-k}+|x|^{p-k}\,|y|^k\right)\mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

Furthermore, since $\gamma>1$ using Jensen's inequality and the fact that $\mu\in\mathcal{P}^0(\mathbb{R})$, one obtains

$$\int_{\mathbb{R}^2}\left(|x|^p+|y|^p\right)\big|x-y\big|^\gamma\mu(\mathrm{d}x)\mu(\mathrm{d}y) \geqslant 2\,M_{p+\gamma}(\mu).$$

Consequently,

$$\langle \mathcal{Q}(\mu\,,\,\mu)\,;\,|\cdot|^p\rangle \leqslant -\big(1-\beta_p(a)\big)M_{p+\gamma}(\mu) + \frac{1}{2}\,\beta_p(a)\sum_{k=1}^{k_p}\binom{p}{k}\int_{\mathbb{R}^2}|x-y|^\gamma\left(|x|^k\,|y|^{p-k}+|x|^{p-k}\,|y|^k\right)\mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

For the remainder term, one uses the inequality $|x-y|^\gamma \leqslant 2^{\gamma-1}\big(|x|^\gamma+|y|^\gamma\big)$ to get that

$$\int_{\mathbb{R}^2}|x-y|^\gamma\left(|x|^k\,|y|^{p-k}+|x|^{p-k}\,|y|^k\right)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \leqslant 2^\gamma\big(M_{k+\gamma}(\mu)\,M_{p-k}(\mu)+M_k(\mu)\,M_{p-k+\gamma}(\mu)\big)\,,$$

which proves (3.4). The proof of (3.5) relies on inequality (3.3). Indeed, for any μ and any $k\in(0,1)$ one has

$$\langle \mathcal{Q}(\mu\,,\,\mu)\,;\,|\cdot|^k\rangle = \frac{1}{2}\int_{\mathbb{R}^2}\Big(|ax+by|^k+|ay+bx|^k-|x|^k-|y|^k\Big)|x-y|^\gamma\,\mu(\mathrm{d}x)\mu(\mathrm{d}y)\,.$$

Now, by virtue of (3.3) with $k\in(0,1)$ instead of γ

$$\Big(|ax+by|^k+|ay+bx|^k-|x|^k-|y|^k\Big)|x-y|^k \geqslant -C_k|x|^k\,|y|^k \qquad \forall\,x,y\in\mathbb{R}$$

for some positive constant $C_k>0$ explicit and depending only on a and k. Therefore,

$$-\langle \mathcal{Q}(\mu\,,\,\mu)\,;\,|\cdot|^k\rangle \leqslant \frac{C_k}{2}\int_{\mathbb{R}^2}|x|^k\,|y|^k\,|x-y|^{\gamma-k}\mu(\mathrm{d}x)\mu(\mathrm{d}y)\,.$$

Using that $|x-y|^{\gamma-k}\leqslant \max\{1,2^{\gamma-k-1}\}\left(|x|^{\gamma-k}+|y|^{\gamma-k}\right)$ we get

$$-\langle \mathcal{Q}(\mu\,,\,\mu)\,;\,|\cdot|^k\rangle \leqslant \frac{1}{2}\max\{1,2^{\gamma-k-1}\}\,C_k\int_{\mathbb{R}^2}|x|^k\,|y|^k\left(|x|^{\gamma-k}+|y|^{\gamma-k}\right)\mu(\mathrm{d}x)\mu(\mathrm{d}y) = \max\{1,2^{\gamma-k-1}\}\,C_kM_\gamma(\mu)M_k(\mu),$$

which gives the proof with $C_{\gamma,k}=\max\{1,2^{\gamma-k-1}\}\,C_k$. ∎

As a consequence, we have the following proposition.

PROPOSITION 3.3. *For any $\gamma > 1$ and $s \in (0,1]$ it follows that*

$$M_{s+\gamma}(\mu_t) \leqslant A_s\, M_s(\mu_t)^{1+\frac{\gamma}{s}}, \qquad t \geqslant 0\,, \tag{3.8}$$

with A_s any constant such that

$$A_s \geqslant \max\left\{\tilde{C}_{\gamma,s},\, \frac{M_{s+\gamma}(\mu_0)}{M_s^{1+\frac{\gamma}{s}}(\mu_0)}\right\}\,, \tag{3.9}$$

and $\tilde{C}_{\gamma,s}$ is a constant depending only on γ and s.

Proof. Define $X(t) := M_{s+\gamma}(\mu_t) - A\, M_s(\mu_t)^{1+\frac{\gamma}{s}}$, where the constant A will be conveniently chosen later on. Since $\gamma + s > 1$ and $s \in (0,1]$ we can use Lemma 3.2 to conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}X(t) &= \frac{\mathrm{d}}{\mathrm{d}t}M_{s+\gamma}(\mu_t) - A\left(1+\frac{\gamma}{s}\right) M_s(\mu_t)^{\frac{\gamma}{s}}\frac{\mathrm{d}}{\mathrm{d}t}M_s(\mu_t)\\ &\leqslant -\left(1-\beta_{s+\gamma}(a)\right) M_{s+2\gamma}(\mu_t) + \mathcal{R}_{s+\gamma}(\mu_t) + A\, C_{\gamma,s}\left(1+\frac{\gamma}{s}\right) M_s(\mu_t)^{1+\frac{\gamma}{s}}\, M_\gamma(\mu_t)\,.\end{aligned} \tag{3.10}$$

Let us observe that for such a choice of s and γ, one has $\left[\frac{s+\gamma+1}{2}\right] < s+\gamma$ and for $1 \leqslant k \leqslant \left[\frac{s+\gamma+1}{2}\right]$, a simple use of Hölder and Young's inequalities leads to, for any $\epsilon > 0$,

$$\begin{aligned}M_{k+\gamma}(\mu_t)\, M_{s+\gamma-k}(\mu_t) &\leqslant M_{s+2\gamma}(\mu_t)^{\frac{k+\gamma}{s+2\gamma}}\, M_{s+\gamma}(\mu_t)^{\frac{s+\gamma-k}{s+\gamma}}\\ &\leqslant \epsilon\, M_{s+2\gamma}(\mu_t) + K_{s,\gamma,\epsilon}\, M_{s+\gamma}(\mu_t)^{\frac{s+2\gamma}{s+\gamma}}\end{aligned}$$

with $K_{s,\gamma,\epsilon} > 0$ a constant depending only on s, γ, and ϵ. Similar interpolation holds for the terms of the form $M_k(\mu_t)\, M_{s+2\gamma-k}(\mu_t)$ appearing in $\mathcal{R}_{s+\gamma}(\mu_t)$. As a consequence, for any $\epsilon > 0$, we have

$$\mathcal{R}_{s+\gamma}(\mu_t) \leqslant \epsilon\, K_{s,\gamma}\, M_{s+2\gamma}(\mu_t) + K_{s,\gamma}\, K_{s,\gamma,\epsilon}\, M_{s+\gamma}(\mu_t)^{\frac{s+2\gamma}{s+\gamma}}\,, \tag{3.11}$$

where $K_{s,\gamma} > 0$ is a constant depending only on s and γ. Choosing $\epsilon = \epsilon(\gamma) > 0$ such that

$$2\, K_{s,\gamma}\, \epsilon \leqslant\; 1 - \beta_{s+\gamma}(a)$$

we obtain from estimates (3.10) and (3.11) that there are constants $K_1 = K_1(s,\gamma)$ and $K_2 = K_2(s,\gamma)$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}X(t) &\leqslant -\frac{1}{2}\left(1-\beta_{s+\gamma}(a)\right) M_{s+2\gamma}(\mu_t)\\ &\quad + A\, K_1\, M_s(\mu_t)^{1+\frac{\gamma}{s}+\frac{s}{\gamma}}\, M_{s+\gamma}(\mu_t)^{1-\frac{s}{\gamma}} + K_2\, M_{s+\gamma}(\mu_t)^{\frac{s+2\gamma}{s+\gamma}}\,,\end{aligned} \tag{3.12}$$

where we also used the interpolation

$$M_\gamma(\mu_t) \leqslant M_s(\mu_t)^{\frac{s}{\gamma}}\, M_{s+\gamma}(\mu_t)^{1-\frac{s}{\gamma}}\,.$$

As a final step, notice that

$$\frac{M_{s+\gamma}(\mu_t)^2}{M_s(\mu_t)} \leqslant M_{s+2\gamma}(\mu_t)\,. \tag{3.13}$$

Therefore, including (3.13) in (3.12) one finally concludes that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}X(t) \leqslant -\frac{1}{2}\,(1-\beta_{s+\gamma}(a))\,M_{s+\gamma}(\mu_t)^2\,M_s(\mu_t)^{-1}\\ + A\,K_1\,M_s(\mu_t)^{1+\frac{\gamma}{s}+\frac{s}{\gamma}}\,M_{s+\gamma}(\mu_t)^{1-\frac{s}{\gamma}} + K_2\,M_{s+\gamma}(\mu_t)^{\frac{s+2\gamma}{s+\gamma}}\,.\end{aligned} \tag{3.14}$$

Now, choosing $A>0$ such that $X(0)<0$, if there exists $t_0>0$ for which $X(t_0)=0$, then estimate (3.14) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t_0) \leqslant \Big(-\frac{1}{2}\,(1-\beta_{s+\gamma}(a))\,A^2 + K_1\,A^{2-\frac{s}{\gamma}} + K_2\,A^{1+\frac{\gamma}{s+\gamma}}\Big)M_s(\mu_{t_0})^{1+\frac{2\gamma}{s}}\,. \tag{3.15}$$

Then, choosing $A=A(\gamma,s)$ sufficiently large such that the term in parentheses in (3.15) is negative we conclude that $X'(t_0)\leqslant 0$. This shows that, for such a choice of A, $X(t)\leqslant 0$ for any $t\geqslant 0$. □

Using Proposition 3.3 the desired lower bound is obtained.

Proposition 3.4. *For any $\gamma>1$ and $s\in(0,1]$ one has*

$$M_s(\mu_t) \geqslant \frac{M_s(\mu_0)}{\left(1+C_{\gamma,s}\,A_s^{1-\frac{s}{\gamma}}\,\frac{\gamma}{s}\,M_s(\mu_0)^{\frac{\gamma}{s}}\,t\right)^{\frac{s}{\gamma}}}\,, \tag{3.16}$$

where $C_{\gamma,s}$ depends only on γ and s and A_s is given by (3.9). *Moreover, it holds that*

$$M_k(\mu_t) \geqslant \frac{M_s(\mu_0)^{\frac{k}{s}}}{\left(1+C_{\gamma,s}\,A_s^{1-\frac{s}{\gamma}}\,\frac{\gamma}{s}\,M_s(\mu_0)^{\frac{\gamma}{s}}\,t\right)^{\frac{k}{\gamma}}} \quad \forall\, k>1,\ s\in(0,1], \ \text{ and }\ t\geqslant 0\,. \tag{3.17}$$

Proof. Using Lemma 3.2 and inequality (3.8), since $s\in(0,1)$ one gets

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}M_s(\mu_t) &= -\langle \mathcal{Q}(\mu_t,\mu_t)\,;\,|\cdot|^s\rangle \leqslant C_{\gamma,s}\,M_s(\mu_t)\,M_\gamma(\mu_t)\\ &\leqslant C_{\gamma,s}\,M_s(\mu_t)^{1+\frac{s}{\gamma}}\,M_{s+\gamma}(\mu_t)^{1-\frac{s}{\gamma}} \leqslant C_{\gamma,s}\,A_s^{1-\frac{s}{\gamma}}\,M_s(\mu_t)^{1+\frac{\gamma}{s}}\,,\end{aligned}$$

where, for the second estimate, we used the inequality

$$M_\gamma(\mu_t) \leqslant M_s^{\frac{s}{\gamma}}(\mu_t)\,M_{s+\gamma}^{1-\frac{s}{\gamma}}(\mu_t).$$

Integration of this differential inequality leads to

$$M_s(\mu_t) \geqslant \frac{M_s(\mu_0)}{\left(1+C_{\gamma,s}\,A_s^{1-\frac{s}{\gamma}}\,\frac{\gamma}{s}\,M_s(\mu_0)^{\frac{\gamma}{s}}\,t\right)^{\frac{s}{\gamma}}}\,.$$

The second part of the result follows from

$$M_s(\mu_t) \leqslant M_k^{\frac{s}{k}}(\mu_t)\,M_0(\mu_t)^{1-\frac{s}{k}} = M_k^{\frac{s}{k}}(\mu_t),$$

valid for any $k>1$. □

We can now completely characterize the decay of any moments of the weak measure solution $(\mu_t)_{t\geqslant 0}$ associated to μ_0.

THEOREM 3.5. *Let $k \geqslant 2$ and $\mu_0 \in \mathcal{P}^0_{k+\gamma}(\mathbb{R})$ be given. Denote by $(\mu_t)_{t\geqslant 0}$ a measure weak solution to* (1.1) *associated to the initial datum μ_0. Then, we have the following:*

(1) *When $\gamma \in (0,1]$, there exists some universal constant $\mathcal{K}_\gamma > 0$ (not depending on μ_0) such that for any $p \geqslant \gamma$*

$$\frac{M_\gamma\,(\mu_0)^{\frac{p}{\gamma}}}{\Big(1+\mathcal{K}_\gamma\, M_\gamma(\mu_0)\, t\Big)^{\frac{p}{\gamma}}} \leqslant M_p\,(\mu_t) \qquad \forall\, t \geqslant 0\,, \tag{3.18a}$$

and for any $p \in [2,k]$

$$M_p\,(\mu_t) \leqslant \frac{M_p\,(\mu_0)}{\Big(1+\frac{\gamma}{2p}\,(1-a^p-b^p)\,M_p\,(\mu_0)^{\frac{\gamma}{p}}\;t\Big)^{\frac{p}{\gamma}}} \qquad \forall\, t \geqslant 0\,. \tag{3.18b}$$

In particular, if $\mu_0 \in \mathcal{P}_{\exp,\gamma}(\mathbb{R})$, then (3.18b) *holds true for any $p \geqslant 2$.*

(2) *When $\gamma > 1$, for any $p > 1$,*

$$\frac{M_1\,(\mu_0)^p}{\Big(1+\gamma\, C_{\gamma,1}\, A_1^{1-\frac{1}{\gamma}}\, M_1(\mu_0)^\gamma\, t\Big)^{\frac{p}{\gamma}}} \leqslant M_p\,(\mu_t) \qquad \forall\, t \geqslant 0\,, \tag{3.19a}$$

and for any $p \in [2,k]$

$$M_p\,(\mu_t) \leqslant \frac{M_p\,(\mu_0)}{\Big(1+\frac{\gamma}{2p}\,(1-a^p-b^p)\,M_p\,(\mu_0)^{\frac{\gamma}{p}}\;t\Big)^{\frac{p}{\gamma}}} \qquad \forall\, t \geqslant 0\,, \tag{3.19b}$$

where $C_{\gamma,1}$ and A_1 are defined in Proposition 3.4.

Proof. The upper bounds in (3.18b) and (3.19b) have been established in Proposition 2.4. For the lower bound, whenever $\gamma \in (0,1)$, one simply uses Jensen's inequality to get

$$M_p(\mu_t) \geqslant M_\gamma(\mu_t)^{\frac{p}{\gamma}} \qquad \forall\, t \geqslant 0, \;\; p \geqslant \gamma\,,$$

and then conclude thanks to (3.1). For $\gamma > 1$, the lower bound is just (3.17) with $s = 1$. ∎

COROLLARY 3.6. *Fix $\gamma > 0$ and let $\mu_0 \in \mathcal{P}^0(\mathbb{R}) \cap \mathcal{P}_{\exp,\gamma}(\mathbb{R})$ be an initial datum. Then, for any $p \geqslant 1$, the unique measure weak solution $(\mu_t)_{t\geqslant 0}$ is converging as $t \to \infty$ toward δ_0 in the weak-$\star$ topology of $\mathcal{P}_p(\mathbb{R})$ with the explicit rate*

$$W_p(\mu_t,\delta_0) \propto \big(1+C\,t\big)^{-\frac{1}{\gamma}} \quad \textit{as}\;\; t \to \infty$$

for some positive constant C depending on μ_0.

Proof. The result is a direct consequence of Theorem 3.5 since

$$W_p(\mu_t,\delta_0)^p = \int_{\mathbb{R}} |x|^p \mu_t(\mathrm{d}x) = M_p(\mu_t), \qquad t \geqslant 0\,,$$

and W_p metrizes $\mathcal{P}_p(\mathbb{R})$ (see [37, Theorem 6.9]). ∎

3.2. Consequences on the rescaled problem. Assume $\mu_0 \in \mathcal{P}^0_{\gamma_*}(\mathbb{R})$ satisfying (1.14) and let $(\mu_t)_{t\geqslant 0}$ denote the unique weak measure solution to (1.1) associated to μ_0. Recall from section 1 the definitions of the rescaling functions

$$V(t) := (1 + c\,\gamma\, t)^{\frac{1}{\gamma}} \quad \text{and} \quad s(t) := \frac{1}{c\,\gamma}\log(1 + c\,\gamma\, t), \qquad (c > 0).$$

For simplicity, in what follows, we shall assume $c = 1$. The inverse mappings are defined as

$$\mathcal{V}(s) = \exp(s), \quad t(s) = \frac{\exp(\gamma\, s) - 1}{\gamma},$$

i.e.,

$$s(t) = s \Longleftrightarrow t(s) = t \quad \text{and} \quad V\big(t(s)\big) = \mathcal{V}(s) \quad \forall\, t, s \geqslant 0\,.$$

In this way we may define, for any $s \geqslant 0$, the measure $\boldsymbol{\nu}_s$ as the image of $\mu_{t(s)}$ under the transformation $x \mapsto \mathcal{V}(s)x$,

$$\boldsymbol{\nu}_s(\mathrm{d}\xi) = \big(\mathcal{V}(s)\#\mu_{t(s)}\big)(\mathrm{d}\xi), \qquad \forall\, s \geqslant 0\,,$$

where # stands for the push-forward operation on measures,

$$\int_{\mathbb{R}} \phi(\xi)\boldsymbol{\nu}_s(\mathrm{d}\xi) = \int_{\mathbb{R}} \phi(\mathcal{V}(s)x)\mu_{t(s)}(\mathrm{d}x) \qquad \forall\, \phi \in \mathcal{C}_b(\mathbb{R}),\ s \geqslant 0\,. \tag{3.20}$$

Notice that whenever μ_t is absolutely continuous with respect to the Lebesgue measure over $\mathbb{R}$ with $\mu_t(\mathrm{d}x) = f(t,x)\mathrm{d}x$, the measure $\boldsymbol{\nu}_s$ is also absolutely continuous with respect to the Lebesgue measure over $\mathbb{R}$ with $\boldsymbol{\nu}_s(\mathrm{d}\xi) = g(s,\xi)\mathrm{d}\xi$, where

$$g(s,\xi) = \mathcal{V}(s)^{-1}\, f\big(t(s), \mathcal{V}(s)^{-1}\xi\big) \qquad \forall\, \xi \in \mathbb{R},\ s \geqslant 0\,,$$

which is nothing but (1.6). Such a definition of $\boldsymbol{\nu}_s$ allows us to define the semiflow $(\mathcal{F}_s)_{s\geqslant 0}$ which given any initial datum $\mu_0 \in \mathcal{P}^0(\mathbb{R})$ satisfying (1.14) associates

$$\mathcal{F}_s(\mu_0) = \boldsymbol{\nu}_s = \mathcal{V}(s)\#\mathcal{S}_{t(s)}(\mu_0) \qquad \forall\, s \geqslant 0\,,$$

where $(\mathcal{S}_t)_{t\geqslant 0}$ is the semiflow associated to (1.1). Notice that the semiflow $(\mathcal{S}_t)_{t\geqslant 0}$ satisfies the following scaling property.

Lemma 3.7. *For any $\lambda > 0$, $t \geqslant 0$, $\mu_0 \in \mathcal{P}_{\exp,\gamma}(\mathbb{R})$,*

$$\mathcal{S}_t(\lambda\mu_0) = \lambda\mathcal{S}_{\lambda t}(\mu_0), \qquad \mathcal{S}_t(\lambda(\lambda\#\mu_0)) = \lambda(\lambda\#(\mathcal{S}_{\lambda^{\gamma+1}t}(\mu_0))).$$

From this, it is not difficult to prove that $(\mathcal{F}_s)_{s\geqslant 0}$ is indeed a semiflow on $\mathcal{P}_{\exp,\gamma}(\mathbb{R})$. The decay of the moments given by Theorem 3.5 readily translates into the following result.

Theorem 3.8. *Let $\gamma > 0$ and let $\mu_0 \in \mathcal{P}^0(\mathbb{R}) \cap \mathcal{P}_{\exp,\gamma}(\mathbb{R})$ be a given initial datum. Denote by $\mathcal{F}_s(\mu_0) = \boldsymbol{\nu}_s$ for any $s \geqslant 0$. Then,*

(1) *when $\gamma \in (0,1]$, there exists some universal constant $\mathcal{K}_\gamma > 0$ (not depending on μ_0) such that for any $p \geqslant \gamma$*

$$\min\left\{M_\gamma\,(\mu_0)^{\frac{p}{\gamma}}\ ;\ \left(\frac{\gamma}{\mathcal{K}_\gamma}\right)^{\frac{p}{\gamma}}\right\} \leqslant M_p\,(\boldsymbol{\nu}_s) \qquad \forall\, s \geqslant 0\,, \tag{3.21a}$$

and for any $p \geqslant 2$,

$$M_p(\boldsymbol{\nu}_s) \leqslant \max\left\{M_p(\mu_0)\,;\,\left(\frac{2p}{1-a^p-b^p}\right)^{\frac{p}{\gamma}}\right\} \qquad \forall\, s \geqslant 0\,; \tag{3.21b}$$

(2) *when* $\gamma > 1$, *for any* $p > 1$

$$\min\left\{M_1(\mu_0)^p\,;\,\left(\frac{1}{C_{\gamma,1}\,A_1^{1-\frac{1}{\gamma}}}\right)^{\frac{p}{\gamma}}\right\} \leqslant M_p(\boldsymbol{\nu}_s) \qquad \forall\, s \geqslant 0\,, \tag{3.22a}$$

and for any $p \geqslant 2$

$$M_p(\boldsymbol{\nu}_s) \leqslant \max\left\{M_p(\mu_0)\,;\,\left(\frac{2p}{1-a^p-b^p}\right)^{\frac{p}{\gamma}}\right\} \qquad \forall\, s \geqslant 0\,, \tag{3.22b}$$

where $C_{\gamma,1}$ *and* A_1 *are defined in Proposition* 3.4.

Proof. The proof follows simply from the fact that

$$M_p(\boldsymbol{\nu}_s) = \mathcal{V}(s)^p\,M_p(\mu_{t(s)}) \qquad \forall\, s \geqslant 0\,,$$

where $(\mu_t)_t$ is the weak measure solution to (1.1) associated to μ_0. Then, according to Theorem 3.5 and using that $\exp(p\,s) = \big(1+\gamma\,t(s)\big)^{p/\gamma}$, we see that for $\gamma \in (0,1]$ it holds that

$$\left(\frac{M_\gamma(\mu_0)(1+\gamma t(s))}{1+\mathcal{K}_\gamma\,M_\gamma(\mu_0)\,t(s)}\right)^{\frac{p}{\gamma}} \leqslant M_p(\boldsymbol{\nu}_s) \leqslant M_p(\mu_0)\left(\frac{1+\gamma t(s)}{1+\frac{\gamma}{2p}\,(1-a^p-b^p)\,M_p(\mu_0)^{\frac{\gamma}{p}}\;t(s)}\right)^{\frac{p}{\gamma}},$$

where the lower bound is valid for any $p \geqslant \gamma$ while the upper bound is valid for any $p \geqslant 2$. Since $\min(1,\frac{A}{B}) \leqslant \frac{1+At}{1+Bt} \leqslant \max(1,\frac{A}{B})$ for any $A, B, t \geqslant 0$, we get the conclusion. We proceed in the same way for $\gamma > 1$. ∎

An important consequence of the above decay is the following proposition.

Proposition 3.9. *Let* $\mu_0 \in \cap_{k\geqslant 0}\mathcal{P}_k^0(\mathbb{R})$ *be a given initial condition. Assume that*

$$M_p(\mu_0) \leqslant \mathbf{M}_p \qquad \forall\, p \geqslant \max(\gamma, 2)\,, \tag{3.23}$$

where

$$\mathbf{M}_p := \left(\frac{2p}{1-a^p-b^p}\right)^{\frac{p}{\gamma}} \qquad \forall\, p \geqslant \max(\gamma,2)\,.$$

Then, $\mu_0 \in \mathcal{P}_{\exp,\gamma}(\mathbb{R})$ *and there exists an explicit* $\alpha > 0$ *and* $C = C(\alpha) > 0$ *such that*

$$\sup_{s\geqslant 0}\int_{\mathbb{R}} \exp(\alpha|\xi|^\gamma)\,\boldsymbol{\nu}_s(\mathrm{d}\xi) \leqslant C\,,$$

where $\boldsymbol{\nu}_s = \mathcal{F}_s(\mu_0)$ *for any* $s \geqslant 0$.

Proof. Let us first prove that $\mu_0 \in \mathcal{P}_{\exp,\gamma}(\mathbb{R})$. Notice that for any $z > 0$

$$\int_{\mathbb{R}} \exp(z\,|\xi|^\gamma)\mu_0(\mathrm{d}\xi) = \sum_{p=0}^{\infty} M_{\gamma\,p}(\mu_0)\frac{z^p}{p!}\,.$$

Let us denote by p_0 the integer such that $\gamma\, p_0 \geqslant \max(\gamma, 2)$ and $\gamma\,(p_0-1) < \max(\gamma, 2)$. Using the Stirling formula together with the fact that $\lim_{p\to\infty}(1 - a^{\gamma\, p} - b^{\gamma\, p}) = 1$, one can check that there exists some explicit $\alpha > 0$ such that the series

$$\sum_{p=p_0}^{\infty} \mathbf{M}_{\gamma\, p}\frac{z^p}{p!}$$

converges for any $0 \leqslant z \leqslant \alpha$, which gives the result. Now, we may define $\boldsymbol{\nu}_s = \mathcal{F}_s(\mu_0)$ for any $s \geqslant 0$. As previously, for any $z > 0$

$$\int_{\mathbb{R}} \exp(z\,|\xi|^{\gamma})\boldsymbol{\nu}_s(\mathrm{d}\xi) = \sum_{p=0}^{\infty} M_{\gamma\, p}(\boldsymbol{\nu}_s)\frac{z^p}{p!}\,,$$

and we deduce from Theorem 3.8 that

$$M_p(\boldsymbol{\nu}_s) \leqslant \mathbf{M}_p \qquad \forall\, p \geqslant \max(\gamma, 2),\; s \geqslant 0\,.$$

The conclusion follows. ◻

Remark 3.10. We do not need to derive the equation satisfied by $\boldsymbol{\nu}_s$ since we are interested only in the fixed point of the semiflow. However, using the fact that $\mathcal{S}_t(\mu_0)$ actually provides a *strong solution* to (1.1), using the chain rule it follows that

(3.24)
$$\int_{\mathbb{R}} \phi(\xi)\boldsymbol{\nu}_s(\mathrm{d}\xi){=}\int_{\mathbb{R}} \phi(\xi)\mu_0(\mathrm{d}\xi)+\int_0^s \mathrm{d}\tau\int_{\mathbb{R}} \xi\phi'(\xi)\boldsymbol{\nu}_\tau(\mathrm{d}\xi)+\int_0^s \langle \mathcal{Q}\,(\boldsymbol{\nu}_\tau\,,\,\boldsymbol{\nu}_\tau)\,;\,\phi\rangle \mathrm{d}\tau \quad \forall\, s{\geqslant}0$$

for any $\phi \in \mathcal{C}_b^1(\mathbb{R})$ and where ϕ' stands for the derivative of ϕ.

The link between the solution to (1.13) and the semiflow $(\mathcal{F}_s)_{s\geqslant 0}$ is established by the following lemma.

LEMMA 3.11. *Any fixed point* $\boldsymbol{\mu} \in \mathcal{P}^0(\mathbb{R}) \cap \mathcal{P}_{\exp,\gamma}(\mathbb{R})$ *of the semiflow* $(\mathcal{F}_s)_{s\geqslant 0}$ *is a solution to* (1.13)*.*

Proof. Let $\boldsymbol{\mu}$ be a fixed point of the semiflow $(\mathcal{F}_s)_s$, that is, $\mathcal{F}_s(\boldsymbol{\mu}) = \boldsymbol{\mu}$ for any $s \geqslant 0$. Then, according to (3.20), for any $\phi \in \mathcal{C}_b(\mathbb{R})$

$$\int_{\mathbb{R}} \phi(\xi)\boldsymbol{\mu}(\mathrm{d}\xi) = \int_{\mathbb{R}} \phi(\mathcal{V}(s)x)\mu_{t(s)}(\mathrm{d}x) \qquad \forall\, s \geqslant 0\,,$$

where $\mu_{t(s)} = \mathcal{S}_{t(s)}(\boldsymbol{\mu})$. In particular, choosing $s = s(t)$

$$\int_{\mathbb{R}} \phi(\xi)\boldsymbol{\mu}(\mathrm{d}\xi) = \int_{\mathbb{R}} \phi(V(t)x)\mu_t(\mathrm{d}x) \qquad \forall\, t \geqslant 0\,.$$

Applying the above to $\phi\big(V(t)^{-1}\,\cdot\big)$ instead of ϕ, one obtains

$$\int_{\mathbb{R}} \phi\big(V(t)^{-1}\xi\big)\boldsymbol{\mu}(\mathrm{d}\xi) = \int_{\mathbb{R}} \phi(x)\mu_t(\mathrm{d}x) \qquad \forall\, t \geqslant 0\,.$$

Computing the derivative with respect to t and assuming $\phi \in \mathcal{C}_b^1(\mathbb{R})$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(V(t)^{-1}\big)\int_{\mathbb{R}} \xi\phi'\big(V(t)^{-1}\xi\big)\boldsymbol{\mu}(\mathrm{d}\xi) = \langle \mathcal{Q}(\mu_t, \mu_t)\,;\,\phi\rangle \qquad \forall\, t \geqslant 0\,.$$

Using the definition of $V(t)$ it finally follows that

$$-\big(1+\gamma t\big)^{-\frac{1}{\gamma}-1}\int_{\mathbb{R}} \xi\phi'\big(V(t)^{-1}\xi\big)\boldsymbol{\mu}(\mathrm{d}\xi) = \langle \mathcal{Q}(\mu_t, \mu_t)\,;\,\phi\rangle \qquad \forall\, t \geqslant 0\,.$$

Taking in particular $t = 0$ it follows that $\boldsymbol{\mu}$ satisfies (1.13). ◻

4. Existence of a steady solution to the rescaled problem.

4.1. Steady measure solutions are L^1 steady states. In this section we prove Theorem 1.5, that is, we prove that steady measure solutions to (1.9) are in fact L^1 functions provided no mass concentration happens at the origin. The argument is based on the next two propositions.

PROPOSITION 4.1. *Let $\boldsymbol{\mu} \in \mathcal{P}^0_{\max(\gamma,2)}(\mathbb{R})$ be a steady solution to* (1.9). *Then, there exists $H \in L^1(\mathbb{R})$ such that*

$$\xi\, \boldsymbol{\mu}(\mathrm{d}\xi) = H(\xi)\mathrm{d}\xi\,.$$

Proof. Introduce the distribution $\boldsymbol{\beta}(\xi) := \xi\boldsymbol{\mu}(\mathrm{d}\xi)$ which, of course, is defined by the identity

$$\int_{\mathbb{R}} \boldsymbol{\beta}(\xi)\psi(\xi)\mathrm{d}\xi = \int_{\mathbb{R}} \xi\psi(\xi)\boldsymbol{\mu}(\mathrm{d}\xi) \qquad \text{for any } \psi \in \mathcal{C}^\infty_c(\mathbb{R})\,.$$

One sees from (1.13) that $\boldsymbol{\beta}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\boldsymbol{\beta}(\xi) = \mathcal{Q}(\boldsymbol{\mu},\boldsymbol{\mu}) \tag{4.1}$$

in the sense of distributions. Since $\boldsymbol{\mu} \in \mathcal{P}^0_{\max(\gamma,2)}(\mathbb{R})$, it follows that $\mathcal{Q}^\pm(\boldsymbol{\mu},\boldsymbol{\mu})$ belongs to $\mathcal{M}^+(\mathbb{R})$. Therefore, as a solution to (4.1), the measure $\boldsymbol{\beta}$ is a distribution whose derivative belongs to $\mathcal{M}(\mathbb{R})$. It follows from [20, Theorem 6.77] that $\boldsymbol{\beta} \in BV_{\mathrm{loc}}(\mathbb{R})$, where $BV(\mathbb{R})$ denotes the space of functions with bounded variations. This implies that the measure $\boldsymbol{\beta}$ is absolutely continuous. In particular, there exists $H \in L^1(\mathbb{R})$ such that $\boldsymbol{\beta}(\mathrm{d}\xi) = H(\xi)\mathrm{d}\xi$. This proves the result. ∎

PROPOSITION 4.2. *Let $\boldsymbol{\mu} \in \mathcal{P}^0_{\max(\gamma,2)}(\mathbb{R})$ be a steady solution to* (1.9). *Then, there exist $\kappa_0 \geqslant 0$ and $G \in L^1_{\max(\gamma,2)}(\mathbb{R})$ nonnegative such that*

$$\boldsymbol{\mu}(\mathrm{d}\xi) = G(\xi)\mathrm{d}\xi + \kappa_0\, \delta_0(\mathrm{d}\xi),$$

where δ_0 is the Dirac mass in 0.

Proof. Let us denote by $\mathcal{B}(\mathbb{R})$ the set of Borel subsets of $\mathbb{R}$. According to Lebesgue decomposition theorem [34, Theorem 8.1.3] there exists $G \in L^1(\mathbb{R})$ nonnegative and a measure $\boldsymbol{\mu}_s$ such that

$$\boldsymbol{\mu}(\mathrm{d}\xi) = G(\xi)\mathrm{d}\xi + \boldsymbol{\mu}_s(\mathrm{d}\xi),$$

where the measure $\boldsymbol{\mu}_s$ is singular to the Lebesgue measure over $\mathbb{R}$. More specifically, there is $\Gamma \in \mathcal{B}(\mathbb{R})$ with zero Lebesgue measure such that $\boldsymbol{\mu}_s(\mathbb{R}\backslash\Gamma) = 0$. The proof of the lemma consists then in proving that $\boldsymbol{\mu}_s$ is supported in $\{0\}$, i.e., $\Gamma = \{0\}$. This comes directly from Proposition 4.1. Indeed, by uniqueness of the Lebesgue decomposition, one has

$$\xi\boldsymbol{\mu}(\mathrm{d}\xi) = \xi G(\xi)\mathrm{d}\xi + \xi\boldsymbol{\mu}_s(\mathrm{d}\xi) = H(\xi)\mathrm{d}\xi\,,$$

with $H \in L^1(\mathbb{R})$, so that

$$\xi\boldsymbol{\mu}_s(\mathrm{d}\xi) = 0\,.$$

This implies that $\boldsymbol{\mu}_s(\mathbb{R}\backslash(-\delta;\delta)) = 0$ for any $\delta > 0$ and therefore that $\boldsymbol{\mu}_s$ is supported in $\{0\}$. Notice that since $\boldsymbol{\mu} \in \mathcal{P}^0_{\max(\gamma,2)}(\mathbb{R})$ one has $G \in L^1_{\max(\gamma,2)}(\mathbb{R})$. ∎

Proof of Theorem 1.5. With the notation of Proposition 4.2, our aim is to show that $\kappa_0 = 0$. Plugging the decomposition obtained in Proposition 4.2 in the weak formulation (1.13), we get

$$-\int_{\mathbb{R}} \xi\phi'(\xi)G(\xi)\mathrm{d}\xi = \int_{\mathbb{R}} \mathcal{Q}(G,G)(\xi)\phi(\xi)\mathrm{d}\xi + \kappa_0 \int_{\mathbb{R}} |\xi|^\gamma \Big(\phi\,(a\,\xi) + \phi\,(b\,\xi) - \phi(\xi) - \phi(0)\Big)G(\xi)\mathrm{d}\xi\,,$$

where we used that $\mathcal{Q}(\delta_0,\delta_0) = 0 = \int_{\mathbb{R}} \xi\phi'(\xi)\delta_0(\mathrm{d}\xi)$. Recall the hypothesis

$$\mathbf{m}_\gamma = \int_{\mathbb{R}} |\xi|^\gamma G(\xi)\mathrm{d}\xi > 0\,, \tag{4.2}$$

from which the above can be reformulated as

$$\kappa_0\,\mathbf{m}_\gamma\phi(0) = \int_{\mathbb{R}} \mathcal{Q}(G,G)(\xi)\phi(\xi)\mathrm{d}\xi + \int_{\mathbb{R}} \xi\phi'(\xi)G(\xi)\mathrm{d}\xi + \kappa_0 \int_{\mathbb{R}} |\xi|^\gamma \Big(\phi\,(a\,\xi) + \phi\,(b\,\xi) - \phi(\xi)\Big)G(\xi)\mathrm{d}\xi$$

for any $\phi \in \mathcal{C}^1_b(\mathbb{R})$. Notice that the above identity can be rewritten as

$$\kappa_0\,\phi(0)\mathbf{m}_\gamma = \int_{\mathbb{R}} \big(A(\xi)\phi(\xi) + B(\xi)\phi'(\xi)\big)\mathrm{d}\xi \tag{4.3}$$

for some L^1-functions

$$A(\xi) = \mathcal{Q}(G,G)(\xi) + \frac{\kappa_0}{a}\left|\frac{\xi}{a}\right|^\gamma G\left(\frac{\xi}{a}\right) + \frac{\kappa_0}{b}\left|\frac{\xi}{b}\right|^\gamma G\left(\frac{\xi}{b}\right) - \kappa_0\,|\xi|^\gamma\,G(\xi)$$

and $B(\xi) = \xi G(\xi)$. Let ϕ be a smooth function with support in $(-1,1)$ and satisfying $\phi(0) = 1$. For any $\varepsilon > 0$, $\phi\left(\frac{\cdot}{\varepsilon}\right)$ belongs to $\mathcal{C}^1_b(\mathbb{R})$, one can apply (4.3) to get

$$\kappa_0\;\mathbf{m}_\gamma = \int_{-\varepsilon}^{\varepsilon}\left(A(\xi)\phi\left(\frac{\xi}{\varepsilon}\right) + G(\xi)\frac{\xi}{\varepsilon}\,\phi'\left(\frac{\xi}{\varepsilon}\right)\right)\mathrm{d}\xi\,.$$

Hence,

$$0 \leqslant \kappa_0\;\mathbf{m}_\gamma \leqslant \|\phi\|_{L^\infty}\int_{-\varepsilon}^{\varepsilon}|A(\xi)|\mathrm{d}\xi + \sup_{\xi\in\mathbb{R}}|\xi\,\phi'(\xi)|\int_{-\varepsilon}^{\varepsilon}G(\xi)\mathrm{d}\xi\,.$$

Letting $\varepsilon \to 0$, one obtains $\kappa_0\,\mathbf{m}_\gamma = 0$, thus, using hypothesis (4.2) we must have $\kappa_0 = 0$. ∎

4.2. Proof of Theorem 1.6. We have all the previous machinery at hand to prove the existence of "physical" solutions to (1.9) in the sense of Definition 1.3 employing the dynamic fixed point Theorem 1.1. Let us distinguish here two cases:

(1) First, assume that $\gamma \in (0,1]$. Setting $\mathcal{Y}$ to be the space $\mathcal{M}(\mathbb{R})$ endowed with the weak-$\star$ topology, we introduce the nonempty closed convex set

$$\mathcal{Z} := \left\{\mu\in\mathcal{P}^0(\mathbb{R}) \text{ such that} \int_{\mathbb{R}} |\xi|^\gamma\mu(\mathrm{d}\xi) \geqslant \frac{\gamma}{\mathcal{K}_\gamma}\,, \text{ and} \int_{\mathbb{R}} |\xi|^p\,\mu(\mathrm{d}\xi) \leqslant \mathbf{M}_p \quad \forall\, p\geqslant 2\right\},$$

where $\mathbf{M}_p$ was defined in Proposition 3.9 and $\mathcal{K}_\gamma > 0$ is the positive constant given in Theorem 3.8. This set is a compact subset of $\mathcal{Y}$ thanks to the uniform moment estimates (recall that $\mathcal{Y}$ is endowed with the weak-$\star$ topology): indeed, for any compact $K \subset \mathbb{R}$ with $K \subset (-A,A)$ $(A > 0)$

$$\sup_{\mu\in\mathcal{Z}} \mu(\mathbb{R}\setminus K) \leqslant A^{-p}\mathbf{M}_p < \infty.$$

Choosing $A > 0$ large enough, $\sup_{\mu\in\mathcal{Z}} \mu(\mathbb{R}\setminus K)$ can be made arbitrarily small and this provides the tightness of $\mathcal{Z}$. We conclude thanks to Prokhorov's compactness Theorem (see [25, Theorem 1.7.6, p. 41]). Moreover, according to Proposition 3.9, there exists $\alpha > 0$ and $C(\alpha) > 0$ such that, for any $\mu \in \mathcal{Z}$,

$$\int_{\mathbb{R}} \exp(\alpha|\xi|^{\gamma})\,\mu(\mathrm{d}\xi) \leqslant C(\alpha) < \infty\,.$$

Thus, $\mu \in \mathcal{P}_{\exp,\gamma}(\mathbb{R})$. Therefore, using Theorem 1.4, $(\mathcal{S}_t(\mu))_{t\geqslant 0}$ and, consequently, $(\mathcal{F}_s(\mu))_{s\geqslant 0}$ are well defined. Setting $\boldsymbol{\nu}_s = \mathcal{F}_s(\mu)$, it follows from Theorem 3.8 that

$$\int_{\mathbb{R}} |\xi|^p \boldsymbol{\nu}_s(\mathrm{d}\xi) \leqslant \mathbf{M}_p \qquad \forall\, p \geqslant 2,\ s \geqslant 0\,.$$

Using the lower bound in (3.21a), we deduce that

$$\int_{\mathbb{R}} |\xi|^{\gamma}\,\boldsymbol{\nu}_s(\mathrm{d}\xi) \geqslant \frac{\gamma}{\mathcal{K}_\gamma} \qquad \forall s \geqslant 0\,.$$

This shows that $\boldsymbol{\nu}_s \in \mathcal{Z}$ for any $s \geqslant 0$, i.e., $\mathcal{F}_s(\mathcal{Z}) \subset \mathcal{Z}$ for all $s \geqslant 0$. Moreover, one deduces directly from Proposition 2.11 that $(\mathcal{F}_s)_s$ is continuous over $\mathcal{Z}$. As a consequence, it is possible to apply Theorem 1.1 to deduce the existence of a measure $\boldsymbol{\mu} \in \mathcal{Z}$ such that $\mathcal{F}_s(\boldsymbol{\mu}) = \boldsymbol{\mu}$, a steady measure solution to (1.9) in the sense of Definition 1.3. Finally, since $\boldsymbol{\mu} \in \mathcal{Z}$, its moment of order γ is bounded away from zero, and by Theorem 1.5, $\boldsymbol{\mu}$ is absolutely continuous with respect to the Lebesgue measure. This proves the result in the case $\gamma \in (0,1]$.

(2) Assume now $\gamma > 1$ and let $\gamma_* = \max(\gamma, 2)$. Then, consider $\mathcal{Y}$ to be the space $\mathcal{M}(\mathbb{R})$ endowed with the weak-$\star$ topology and we introduce the nonempty closed convex set

$$\mathcal{Z} := \left\{\mu \in \mathcal{P}^0(\mathbb{R}) \ \text{ such that } \int_{\mathbb{R}} |\xi|^p\,\mu(\mathrm{d}\xi) \leqslant \mathbf{M}_p \ \ \forall\, p \geqslant \gamma_*, \ \text{ and } \int_{\mathbb{R}} |\xi|\mu(\mathrm{d}\xi) \geqslant \ell\right\}$$

for some positive constant ℓ to be determined. In fact, we prove that there exists $\ell = \ell(\gamma)$ sufficiently small such that $\mathcal{Z}$ is invariant under the semiflow $(\mathcal{F}_s)_{s\geqslant 0}$. Indeed, according to (3.22a), for any $\ell > 0$ if μ_0 is such that $M_1(\mu_0) \geqslant \ell$, then

$$M_1\big(\mathcal{F}_s(\mu_0)\big) \geqslant \min\left\{\ell\,;\,\left(\frac{1}{C_{\gamma,1}\,A_1^{1-\frac{1}{\gamma}}}\right)^{\frac{1}{\gamma}}\right\} \qquad \forall\, s \geqslant 0\,,$$

where $C_{\gamma,1} > 0$ is some positive universal constant. And, according to (3.9), A_1 is *any* real number larger than $\max\{\tilde{C}_{\gamma,1}\,;\,\frac{M_{1+\gamma}(\mu_0)}{M_1(\mu_0)^{1+\gamma}}\}$, where $\tilde{C}_{\gamma,1}$ is another universal positive constant. In particular, choosing ℓ small enough such that

$$\frac{\mathbf{M}_{1+\gamma}}{\ell^{1+\gamma}} \geqslant \tilde{C}_{\gamma,1}\,,$$

it is possible to pick $A_1 := \frac{\mathbf{M}_{1+\gamma}}{\ell^{1+\gamma}}$, where we recall that, since $\mu_0 \in \mathcal{Z}$ and $1+\gamma \geqslant \gamma_*$, one has $M_{1+\gamma}(\mu_0) \leqslant \mathbf{M}_{1+\gamma}$. In such a case, one gets

$$\min\left\{\ell\,;\,\left(\frac{1}{C_{\gamma,1}\,A_1^{1-\frac{1}{\gamma}}}\right)^{\frac{1}{\gamma}}\right\}=\min\left\{\ell\,;\,\frac{\ell^{1-\frac{1}{\gamma^2}}}{C_{\gamma,1}^{\frac{1}{\gamma}}\,\mathbf{M}_{1+\gamma}^{\frac{\gamma-1}{\gamma^2}}}\right\}.$$

We set $\ell \leqslant C_{\gamma,1}^{-\gamma}\,\mathbf{M}_{1+\gamma}^{1-\gamma}$ in order to get

$$\min\left\{\ell\,;\,\left(\frac{1}{C_{\gamma,1}\,A_1^{1-\frac{1}{\gamma}}}\right)^{\frac{1}{\gamma}}\right\}=\ell\,,$$

and $M_1(\mathcal{F}_s(\mu_0)) \geqslant \ell$ for any $s \geqslant 0$. Arguing as in the case $\gamma \in (0,1]$, this shows that $\mathcal{F}_s(\mathcal{Z}) \subset \mathcal{Z}$ for any $s \geqslant 0$, and there exists a steady measure $\boldsymbol{\mu}$ which is absolutely continuous with respect to the Lebesgue measure.

5. Numerical simulations. This section contains numerical simulations for the rescaled equation

$$\partial_s g(s,\xi) - \tfrac{1}{2}\big(a^2+b^2-1\big)\partial_\xi\big(\xi\, g(s,\xi)\big) = \mathcal{Q}(g,g)(s,\xi), \qquad s\geqslant 0\,,\ \ \xi\in\mathbb{R}, \tag{5.1}$$

where $\mathcal{Q}(g,g)$ has been previously defined in (1.2) and (1.3). We recall that such a model has been studied in [18, 32] in the case of $\gamma = 0$ and it admits a *unique* steady state

$$\mathcal{M}_1(\xi) = \frac{2}{\pi}\left(\frac{1}{1+\xi^2}\right)^2 \tag{5.2}$$

such that

$$\int_{\mathbb{R}} \mathcal{M}_1(\xi)\begin{pmatrix}1\\ \xi\\ \xi^2\end{pmatrix}\mathrm{d}\xi = \begin{pmatrix}1\\ 0\\ 1\end{pmatrix}.$$

We will use the numerical solutions of (5.1) to verify the properties of our models for general values of γ. We shall consider here initial datum $g_0(\xi) = g(0,\xi)$ which shares the same first moments of $\mathcal{M}_1$, i.e.,

$$\int_{\mathbb{R}} g_0(\xi)\begin{pmatrix}1\\ \xi\\ \xi^2\end{pmatrix}\mathrm{d}\xi = \begin{pmatrix}1\\ 0\\ 1\end{pmatrix}. \tag{5.3}$$

The coefficient $c = -\frac{1}{2}\big(a^2+b^2-1\big) = ab > 0$ in (5.1) is the only one that gives a stationary self-similar profile with finite energy in the case $\gamma = 0$; see [18]. In contrast, as already noticed, the case $\gamma > 0$ accepts any arbitrary positive coefficient in the equation; thus, we will perform all numerical simulations with such a coefficient for comparison purposes. Recall that in our previous analysis we choose this coefficient to be 1.

5.1. Numerical scheme. To compute the solution, we have to make a technical assumption which is the truncation of $\mathbb{R}$ into a finite domain $\Omega = [-L, L]$. When L is chosen large enough so that g is machine zero at $\xi = \pm L$, this will not affect the quality of the solutions. Then, we use a discrete mesh consisting of N cells as follows:

$$-L = \xi_{\frac{1}{2}} < \xi_{\frac{3}{2}} < \cdots < \xi_{N+\frac{1}{2}} = L\,.$$

We denote cell $I_j = \big(\xi_{j-\frac{1}{2}}, \xi_{j+\frac{1}{2}}\big)$, with cell center $\xi_j = \frac{1}{2}\big(\xi_{j-\frac{1}{2}}+\xi_{j+\frac{1}{2}}\big)$ and length $\Delta\xi_j = \xi_{j+\frac{1}{2}} - \xi_{j-\frac{1}{2}}$. The scheme we use is the DG method [19], which has excellent conservation properties. The DG schemes employ the approximation space defined by

$$V_h^k = \{v_h : v_h|_{I_j} \in P^k(I_j),\ 1 \leqslant j \leqslant N\},$$

where $P^k(I_j)$ denotes all polynomials of degree at most k on I_j, and look for the numerical solution $g_h \in V_h^k$ such that

$$\begin{aligned}(5.4)\quad &\int_{I_j} \partial_s g_h(s,\xi)\, v_h(\xi)\,\mathrm{d}\xi + \tfrac{1}{2}\big(a^2+b^2-1\big)\int_{I_j} \xi\, g_h(s,\xi)\,\partial_\xi v_h(\xi)\,\mathrm{d}\xi \\ &\quad + \tfrac{1}{2}\big(a^2+b^2-1\big)\Big((\widehat{\xi g_h}\, v_h^+)_{j-\frac{1}{2}} - (\widehat{\xi g_h}\, v_h^-)_{j+\frac{1}{2}}\Big) \\ &\quad\quad = \int_{I_j} \mathcal{Q}(g_h,g_h)(s,\xi)\, v_h(\xi)\,\mathrm{d}\xi\,, \qquad j=1,\dots,N,\end{aligned}$$

holds true for any $v_h \in V_h^k$. In (5.4), $\widehat{\xi g_h}$ is the upwind numerical flux

$$\widehat{\xi g_h} = \begin{cases} \xi g_h^-(s,\xi) & \text{if } \xi \ge 0, \\ \xi g_h^+(s,\xi) & \text{if } \xi < 0, \end{cases}$$

where g_h^-, g_h^+ denote the left and right limits of g_h at the cell interface. Equation (5.4) is in fact an ordinary differential equation for the coefficients of $g_h(s,\xi)$. The system can then be solved by a standard ODE integrator, and in this paper we use the third-order TVD-Runge–Kutta methods [33] to evolve this method-of-lines ODE. Notice that the implementation of the collision term in (5.4) is done by recalling (1.3), and we only need to calculate it for all the basis functions in V_h^k. This is done before the time evolution starts to save computational cost.

The DG method described above when $k \geqslant 1$ (i.e., we use a scheme with at least piecewise linear polynomial space) will preserve mass and momentum up to discretization error from the boundary and numerical quadratures. This can be easily verified by using appropriate test functions v_h in (5.4). For example, if we take $v_h = 1$ for any j, and sum up on j, we obtain

$$\int_\Omega \partial_s g_h\,\mathrm{d}\xi = \frac{L}{2}\big(a^2+b^2-1\big)\big(g_h^-(\xi=L) + g_h^+(\xi=-L)\big)\,.$$

If L is taken large enough so that g_h achieves machine zero at $\pm L$, this implies mass conservation. Similarly, we can prove

$$\int_\Omega \partial_s g_h\,\xi\,\mathrm{d}\xi = -\tfrac{1}{2}\big(a^2+b^2-1\big)\int_\Omega g_h\,\xi\,\mathrm{d}\xi + \frac{L^2}{2}(a^2+b^2-1)\big(g_h^-(\xi=L) - g_h^+(\xi=-L)\big)\,.$$

Again, when L is large enough and the initial momentum is zero, this shows conservation of momentum for the numerical solution.

5.2. Discussion of numerical results. We use as the initial state the discontinuous initial profile

$$g(0,\xi) = \begin{cases} \frac{1}{2\sqrt{3}} & \text{if } |\xi| \le \sqrt{3}, \\ 0 & \text{otherwise.} \end{cases}$$

This profile clearly satisfies the moment conditions (5.3). We take the domain to be $[-40, 40]$ and use piecewise quadratic polynomials on a uniform mesh of size 2000. Four sets of numerical results have been computed, corresponding to $(\gamma, a) = (1, 0.1)$, $(1, 0.3)$, $(1, 0.5)$, $(2, 0.5)$, and $(3, 0.5)$, respectively. The computation is stopped when the residual

$$\sqrt{\int_\Omega \left(\frac{g_h(s^{n+1},\xi) - g_h(s^n,\xi)}{\Delta s}\right)^2 \mathrm{d}\xi}$$

reduces to a threshold below 10^{-4} indicating convergence to a steady state.

In Figure 1 we plot the objects of study in this document, that is, the equilibrium solutions for different values of γ. In this plot, the amplitude of the solutions has been normalized to one at the origin for comparison purposes. The numerical solutions are used for the cases $\gamma = 1, 2, 3$, while for $\gamma = 0$, we use the theoretical equilibrium $\mathcal{M}_1$ as defined in (5.2). In general terms, these smooth patterns are expected with exponential tails happening for any $\gamma > 0$. The behavior of the profiles at the origin is quite subtle and will depend nonlinearly on the potential, for instance, the case $\gamma = 2$ renders a wider profile relative to $\gamma = 0$ in contrast to $\gamma = 1$ or $\gamma = 3$. This is not to say that such behavior is discontinuous with respect to γ; it is simply the net result of the contributions of short- and long-range interactions of the particles in equilibrium.

In Figure 2, we fix $\gamma = 1$ and compare the stationary solution for different values of a. Recall that the parameter a measures the "inelasticity" degree of the system with $a = 0$ being elastic particles and with $a = 0.5$ being sticky particles. As expected, smaller values of a will render a wider distribution profile at the origin keeping the tails unchanged. Near the origin, the distribution of particles for less inelastic systems will be underpopulated relative to more inelastic systems which force particles to a more concentrated state. Tails, however, are more dependent on the growth of the potential and should remain relatively unchanged despite changes in inelasticity. Interestingly, the numerical simulation shows an unexpected effect: the maximum density of particles is not necessarily located at the origin.

In Figure 3, we plot the evolution of energy as a function of time in a system of sticky particles $a = 0.5$ using different values of γ. Changes in the relaxation times are expected since the potential growth γ impacts directly on the spectral gap of the linearized interaction operator. This numerical result seems to confirm, in one dimension, the natural idea that higher γ implies a higher spectral gap, hence faster relaxation to equilibrium. Refer to [30] for ample discussion in higher dimensions for the so-called quasi-elastic regime. Additionally, the results of Figure 3 are numerical confirmation of the optimal cooling rate given in our Theorems 3.5 and 3.8.

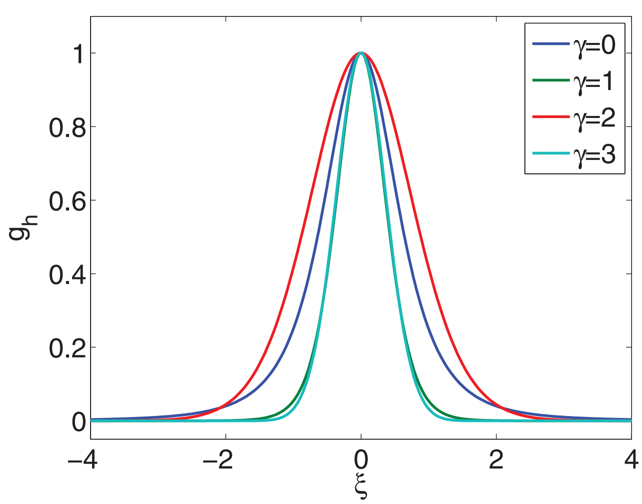

Fig. 1. *Rescaled equilibrium solutions for different values of* γ *with* $a = 0.5$ *(sticky particles). Curves corresponding to* $\gamma = 1, 2, 3$ *are computed numerically, while the curve for* $\gamma = 0$ *is obtained from the known steady state* $\mathcal{M}_1$ *in* (5.2).

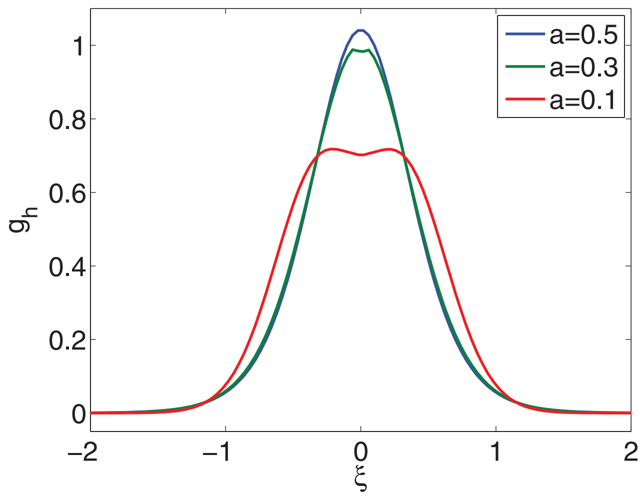

FIG. 2. *Equilibrium solutions for different values of inelasticity a when $\gamma = 1$.*

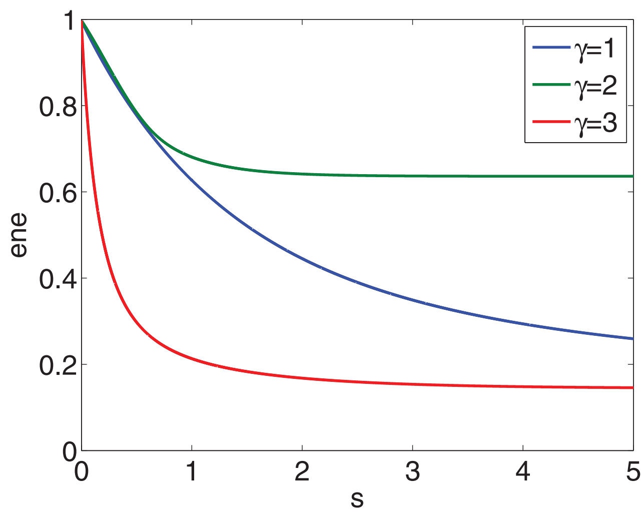

FIG. 3. *Evolution of energy for $\gamma = 1, 2, 3$ when $a = 0.5$.*

In Figure 4, we investigate the evolution of the distribution function and its discontinuities for the case $\gamma = 1$ and $a = 0.5$. The simulation shows, in our 1-D setting, a well-established phenomena happening in elastic and quasi-elastic Boltzmann equations in higher dimensions: discontinuities are damped at an exponential rate [30]. As a consequence, points of low regularity which are contributed by $\mathcal{Q}^+(g, g)$ due to such discontinuities will be smoothed out exponentially fast as well. This is the case for the point $\xi = 0$ in this particular simulation. A numerical simulation was also performed using an initial Gaussian profile (not included). Both numerical simulations showed an evolution toward the same equilibrium profile which reinforces the belief of the uniqueness of the self-similar profile.

Finally, we verify the performance of our scheme by plotting the distribution's mass, momentum, and energy. Only the plot for $\gamma = 1, a = 0.5$, is shown since

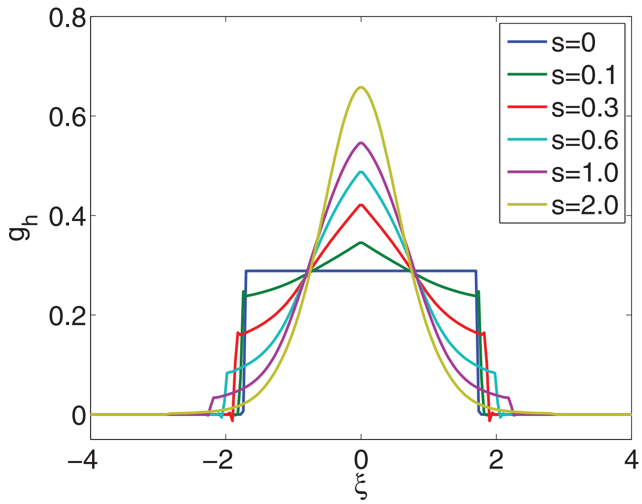

Fig. 4. *Evolution of* $g(s, \xi)$ *when* $\gamma = 1, a = 0.5$.

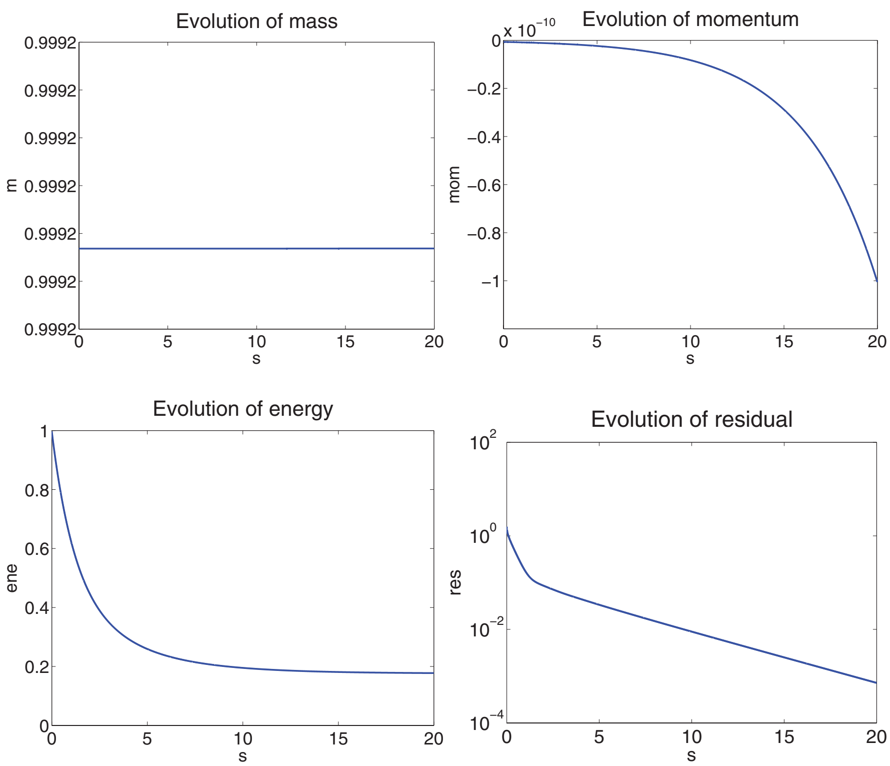

Fig. 5. $\gamma = 1$, $a = 0.5$, *evolution of mass, momentum, energy, and residual.*

the other cases display similar accuracy. In Figure 5 we plot the evolution of mass, momentum, energy, and residual (in the log scale). The decay of residual shows convergence to steady state, while mass and momentum are preserved up to 10 digits of accuracy verifying the performance of the DG method.

6. Conclusion and perspectives. In the present paper we studied the large-time behavior of the solution to the dissipative Boltzmann equation in one dimension. The main achievement of the document is threefold: (1) give a proof for the well-posedness of such a problem in the measure setting, (2) provide a careful study of the moments, including the optimal rate of convergence of solutions toward the Dirac mass at 0 in the Wasserstein metric, and (3) prove the existence of "physical" steady solutions in the self-similar variables, that is, steady measure solutions that are in fact absolutely continuous with respect to the Lebesgue measure. Let us make a few comments about the perspectives and related open problems.

6.1. Regularity propagation for inelastic Boltzmann in one dimension. The numerical simulations performed in section 5 seem to confirm that many of the known results given for inelastic Boltzmann in higher dimensions should extend to inelastic Boltzmann in one dimension, at least under suitable conditions. More specifically, rigorous results about propagation of Lebesgue and Sobolev norms and exponential attenuation of discontinuities for the time evolution problem should hold. Similarly, the study of optimal regularity for the stationary problem is an interesting aspect of the equation which is unknown.

6.2. Alternative approach à la Fournier–Laurençot. Exploiting the analogy between (1.8) and the self-similar Smoluchowski equation, one may wonder if the approach performed by Fournier and Laurençot in [22] can be adapted to (1.9). We recall that the approach in [22] consists in finding a suitable discrete approximation of the steady problem for which a discrete steady solution can be constructed. If such a discrete solution exhibits all the desired properties (positivity, uniform upper bounds, and suitable lower bounds) uniformly with respect to the discretization parameter, then one can pass to the limit to obtain the desired steady solution to (1.9). Such an approach fully exploits the 1-D feature of the problem. Besides, it does not resort to the evolution equation (1.8), a fact that makes it very elegant. The main contrast with respect to [22] lies in the fact that no estimates for moments of *negative* order seem available for our problem. Moreover, Smoluchowski's equation is such that the collision-like operator sends mass to infinity while the drift term brings it back to zero. The model (1.9) has the opposite behavior: the collision tends to concentrate mass in zero while the drift term sends it to infinity.

6.3. Uniqueness and stability of the self-similar profile. Now that the existence of a steady solution to (1.9) has been settled, the next challenge is to prove that such a self-similar profile is unique and that it attracts solutions to (1.8) as $s \to \infty$ or, at least, to find conditions for this to hold. This is certainly the case in the simulations performed in section 5, which show, in addition, an exponential rate of attraction. For the 3-D inelastic Boltzmann equation, such a result has been proven in [30] in the so-called weakly inelastic regime (a perturbation of the elastic problem). Since the 1-D Boltzmann equation is meaningless for elastic interactions a perturbative approach seems inadequate. Once a uniqueness theory is at hand, it would be desirable to obtain rate of convergence; see, for instance, [4]. This would render a more complete picture of the large-time behavior of the dissipative Boltzmann equation on the line.

6.4. The rod alignment problem by Aranson and Tsimring. Aranson and Tsimring in [5] have introduced the following model for rod alignment (the rods have distinguishable beginnings and ends):

$$\partial_t P(t,\theta) = \int_{-\pi}^{\pi} P\Big(t,\theta - \frac{\theta_*}{2}\Big)\, P\Big(t,\theta + \frac{\theta_*}{2}\Big)\, \big|\theta_*\big|^{\gamma}\, \mathrm{d}\theta_* - P(t,\theta) \int_{-\pi}^{\pi} P(t,\theta+\theta_*)\, \big|\theta_*\big|^{\gamma}\, \mathrm{d}\theta_* \tag{6.1}$$

having initial condition $P(0) = P_0$ and angle $\theta \in (-\pi,\pi)$. The authors introduced the model for Maxwellian interactions $\gamma = 0$, yet the model is sound for any $\gamma \geqslant 0$. We refer to [9, 17] for other variations of such a model. Here $P(t,\theta)$ is the time distribution of rods having orientation $\theta \in [-\pi,\pi)$. Equation (6.1) models a system of many discrete rods aligning by the pairwise irreversible law

$$\big(\theta - \tfrac{\theta_*}{2}, \theta + \tfrac{\theta_*}{2}\big) \to \big(\theta,\theta\big). \tag{6.2}$$

Let us explain the interaction law (6.2). We start by fixing a horizontal frame and picking two interacting rods with orientation $\theta_1, \theta_2 \in [-\pi,\pi)$. Define $\theta_* \in [-\pi,\pi)$ as the angle between the ends of the rods. Bisect the rods and define $\theta \in [-\pi,\pi)$ as the angle between the horizontal frame and the bisecting line. Thus, we can express the rods' orientation, up to modulo 2π, by the relation $\theta_1 = \theta - \frac{\theta_*}{2}$ and $\theta_2 = \theta + \frac{\theta_*}{2}$ with respect to the horizontal frame. After interaction, both rods will align with the bisection angle θ. This law produces the alignment of rods; we refer to [5, 9] for an interesting discussion and simulations. The law (6.2) can be written in terms of the rod orientations θ_1, $\theta_2 \in [-\pi,\pi)$ as

$$\big(\theta_1,\theta_2\big) \to \begin{cases} \big(\frac{\theta_1+\theta_2}{2}, \frac{\theta_1+\theta_2}{2}\big), & \big|\theta_1-\theta_2\big| \leqslant \pi, \\ \big(\frac{\theta_1+\theta_2}{2}+\pi, \frac{\theta_1+\theta_2}{2}+\pi\big), & \big|\theta_1-\theta_2\big| > \pi. \end{cases} \tag{6.3}$$

Note that in the case $\big|\theta_1 - \theta_2\big| > \pi$ the addition of π is needed since we choose the alignment to occur in the direction of the bisecting angle associated to the ends of the rods (as opposed to the beginnings of the rods). The interaction law (6.3) is discontinuous; thus intuitively we understand that model (6.1) will not have conservation of momentum because there is a choice of alignment direction. Let us fix this by considering an initial datum P_0 with compact support in $(-\pi/2,\pi/2)$:

$$\mathrm{Supp}\, P_0 \subset (-\pi/2,\pi/2)\,.$$

Such a property is conserved by the dynamic of (6.1) and it corresponds to a system of rods where a *rod's beginning and end are indistinguishable*; thus we can always assign an angle $\theta \in (-\pi/2,\pi/2)$ to each rod. For such a model, the weak formulation is very similar to that of (1.1) except for the fact that all integrals are considered now over the finite interval $(-\pi/2,\pi/2)$. For this reason, the decay of the moments of the solution $P(t)$ to (6.1) is identical to that of (1.1). Consequently, this translates into the convergence of $P(t)$ toward a Dirac mass centered at 0 as $t \to \infty$ in the Wasserstein metric. The question is to understand the model after self-similar rescaling where the support of solutions is no longer fixed and given by $\big(-V(t)\pi/2, V(t)\pi/2\big) \to (-\infty,\infty)$ as $t \to \infty$. Thus, it is natural to expect that the self-similar solution to (6.1) will converge toward the steady solution to (1.9).

6.5. Extension to other collision-like problems. It seems that the present approach is robust enough to be applied to various contexts. In particular, the argument may be helpful to tackle notoriously difficult questions, such as the existence of a stationary self-similar solution to the Smoluchowski equation with *ballistic kernel interactions*. It may be possible, also, to give a more natural treatment of the stationary inelastic Boltzmann equation in the framework of probability measures. The difficulty will be to find a dynamical stable set in order to apply the dynamical fixed point and a suitable regularization theory for the stationary equation of the particular problem.

Appendix A. Cauchy theory in both the L^1-context and the measure setting. In this appendix, we give a detailed proof of the existence and stability estimates of section 2.2 yielding to Theorem 1.4. We fix here $\gamma > 0$ and set $\gamma_* = \max(\gamma, 2)$. We begin with an existence and uniqueness result for the Cauchy problem (1.1) in the special case in which the initial datum μ_0 is absolutely continuous with respect to the Lebesgue measure, i.e.,

$$\mu_0(\mathrm{d}x) = f_0(x)\mathrm{d}x\,.$$

THEOREM A.1. *Fix $\delta > 0$. Let a nonnegative $f_0 \in L^1_{\gamma_*+\gamma+\delta}(\mathbb{R})$ be given with $f_0 \neq 0$. Setting $\mu_0(\mathrm{d}x) = f_0(x)\mathrm{d}x$, there exists a* unique *family $(f_t)_{t\geqslant 0} \subset L^1_{\gamma_*+\gamma+\delta}(\mathbb{R})$ such that $\mu_t(\mathrm{d}x) = f_t(x)\mathrm{d}x$ is a weak measure solution to* (1.1) *associated to μ_0. Moreover,*

$$\begin{aligned}&\int_{\mathbb{R}} f_t(x)\mathrm{d}x = \int_{\mathbb{R}} f_0(x)\mathrm{d}x, \qquad \int_{\mathbb{R}} x f_t(x)\mathrm{d}x = \int_{\mathbb{R}} x f_0(x)\mathrm{d}x,\\ &\textit{and} \quad \int_{\mathbb{R}} |x|^2 f_t(x)\mathrm{d}x \leqslant \int_{\mathbb{R}} |x|^2 f_0(x)\mathrm{d}x \qquad \forall\, t\geqslant 0\,.\end{aligned} \tag{A.1}$$

In addition to this, if one assumes that

$$f_0 \in \bigcap_{k\geqslant 0} L^1_k(\mathbb{R})\,,$$

then $(f_t)_{t\geqslant 0} \subset \bigcap_{k\geqslant 0} L^1_k(\mathbb{R})$.

Proof. We follow the approach of some unpublished notes by Bressan [16]. For $K > 0$ and $\delta > 0$, we introduce

$$\Omega_{K,\delta} = \left\{0 \leqslant f \in L^1(\mathbb{R}), \quad \int_{\mathbb{R}} f(x)\,\mathrm{d}x = \int_{\mathbb{R}} f_0(x)\,\mathrm{d}x, \quad \int_{\mathbb{R}} f(x)|x|^{\gamma_*+\gamma+\delta}\mathrm{d}x \leqslant K\right\},$$

where we recall that $\gamma_* := \max(\gamma, 2)$. We consider $\Omega_{K,\delta}$ as a subset of $L^1_{\gamma_*}(\mathbb{R})$ recalling the notation $\langle x\rangle = \sqrt{1+x^2}$ with $x \in \mathbb{R}$. For $f \in \Omega_{K,\delta}$, a change of variables in the collision operator leads to

$$\begin{aligned}\|\mathcal{Q}(f,f)\|_{L^1_{\gamma_*}} \leqslant &\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(y)|x-y|^\gamma \langle bx+ay\rangle^{\gamma_*}\mathrm{d}x\,\mathrm{d}y\\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(y)|x-y|^\gamma\langle x\rangle^{\gamma_*}\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Now, for any x, $y \in \mathbb{R}$,

$$|x-y|^\gamma \leqslant C_\gamma\langle x\rangle^\gamma\langle y\rangle^\gamma, \qquad \langle ay+bx\rangle \leqslant \sqrt{2}\langle x\rangle\langle y\rangle\,,$$

with $C_\gamma = 2^{\gamma/2}$ so that

$$\|\mathcal{Q}(f,f)\|_{L^1_{\gamma_*}} \leqslant C_{\gamma+\gamma_*}\big(m_{\gamma+\gamma_*}(f)\big)^2 + C_\gamma\, m_{\gamma+\gamma_*}(f)\, m_\gamma(f),$$

where $m_k(f) := \int_{\mathbb{R}} f(x)\langle x\rangle^k\mathrm{d}x$ for any $k \geqslant 0$. Now, for any $0 < k < \gamma_*+\gamma+\delta$ one has

(A.2)
$$m_k(f) \leqslant m_{\gamma_*+\gamma+\delta}(f) \leqslant 2^{\frac{\gamma_*+\gamma+\delta}{2}-1} \int_{\mathbb{R}} f(x)(1+|x|^{\gamma_*+\gamma+\delta})\mathrm{d}x \leqslant 2^{\frac{\gamma_*+\gamma+\delta}{2}-1}(\|f_0\|_{L^1}+K)$$

from which we deduce that there exists $C>0$ such that

$$\|\mathcal{Q}(f,f)\|_{L^1_{\gamma_*}} \leqslant C\left(\|f_0\|_{L^1}+K\right)^2 .$$

Consequently, $\mathcal{Q}(f,f) \in L^1_{\gamma_*}(\mathbb{R})$. Let us now prove that the restriction to $\Omega_{K,\delta}$ of the mapping $f \mapsto \mathcal{Q}(f,f) \in L^1_{\gamma_*}(\mathbb{R})$ is Hölder continuous. For $f,g \in \Omega_{K,\delta}$,

$$\begin{aligned}\|\mathcal{Q}(f,f)-\mathcal{Q}(g,g)\|_{L^1_{\gamma_*}} &\leqslant \int_{\mathbb{R}}\int_{\mathbb{R}} |(f-g)(x)|f(y)|x-y|^{\gamma}\langle bx+ay\rangle^{\gamma_*}\mathrm{d}x\,\mathrm{d}y\\ &\quad+\int_{\mathbb{R}}\int_{\mathbb{R}} g(x)|(f-g)(y)||x-y|^{\gamma}\langle bx+ay\rangle^{\gamma_*}\mathrm{d}x\,\mathrm{d}y\\ +\int_{\mathbb{R}}\int_{\mathbb{R}} |(f-g)(x)|f(y)|x-y|^{\gamma}\langle x\rangle^{\gamma_*}\mathrm{d}x\,\mathrm{d}y &+\int_{\mathbb{R}}\int_{\mathbb{R}} g(x)|(f-g)(y)||x-y|^{\gamma}\langle x\rangle^{\gamma_*}\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Proceeding as previously, one notices that

$$\begin{aligned}\|\mathcal{Q}(f,f)-\mathcal{Q}(g,g)\|_{L^1_{\gamma_*}} &\leqslant C_{\gamma+\gamma_*}\|g\|_{L^1_{\gamma+\gamma_*}}\|f-g\|_{L^1_\gamma}\\ &+C_{\gamma+\gamma_*}\left(\|f\|_{L^1_{\gamma+\gamma_*}}+\|g\|_{L^1_{\gamma+\gamma_*}}+\|f\|_{L^1_\gamma}\right)\int_{\mathbb{R}}|(f-g)(x)|\,\langle x\rangle^{\gamma_*+\gamma}\mathrm{d}x;\end{aligned}$$

thus, by (A.2), there exists $C>0$ such that

$$\|\mathcal{Q}(f,f)-\mathcal{Q}(g,g)\|_{L^1_{\gamma_*}} \leqslant C\left(\|f_0\|_{L^1}+K\right)\left(\|f-g\|_{L^1_\gamma}+\int_{\mathbb{R}}|(f-g)(x)|\,\langle x\rangle^{\gamma_*+\gamma}\mathrm{d}x\right).$$

Thanks to the Hölder inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}}|(f-g)(x)|\,\langle x\rangle^{\gamma_*+\gamma}\mathrm{d}x\\ &\leqslant\left(\int_{\mathbb{R}}|(f-g)(x)|\,\langle x\rangle^{\gamma_*+\gamma+\delta}\mathrm{d}x\right)^{\frac{\gamma}{\gamma+\delta}}\left(\int_{\mathbb{R}}|(f-g)(x)|\,\langle x\rangle^{\gamma_*}\mathrm{d}x\right)^{\frac{\delta}{\gamma+\delta}}\\ &\leqslant\left(2^{\frac{\gamma_*+\gamma+\delta}{2}}(\|f_0\|_{L^1}+K)\right)^{\frac{\gamma}{\gamma+\delta}}\|f-g\|_{L^1_{\gamma_*}}^{\frac{\delta}{\gamma+\delta}}.\end{aligned}$$

Combining the previous two inequalities, we deduce that the mapping $f \mapsto \mathcal{Q}(f,f)$ is uniformly Hölder continuous on $L^1_{\gamma_*}(\mathbb{R})$ when restricted to $\Omega_{K,\delta}$. Let us look for a one-sided Lipschitz condition. For $f,g \in L^1_{\gamma_*}(\mathbb{R})$, we introduce

$$\left[f,g\right]_- = \lim_{s\to 0^-}\frac{\|f+sg\|_{L^1_{\gamma_*}}-\|f\|_{L^1_{\gamma_*}}}{s} .$$

The dominated convergence theorem implies that

$$\left[f,g\right]_- \leqslant \int_{\mathbb{R}}\mathrm{sign}(f(x))g(x)\langle x\rangle^{\gamma_*}\,\mathrm{d}x\,.$$

Our aim is to show that there exists a constant $L>0$ such that for any $f,g\in\Omega_{K,\delta}$,

$$\left[f-g,\mathcal{Q}(f,f)-\mathcal{Q}(g,g)\right]_- \leqslant L\|f-g\|_{L^1_{\gamma_*}} .$$

But,

$$
\begin{aligned}
&\int_{\mathbb{R}} \operatorname{sign}\big((f-g)(x)\big)\big(\mathcal{Q}(f,f)-\mathcal{Q}(g,g)\big)(x)\langle x\rangle^{\gamma_*}\,\mathrm{d}x \\
&\qquad\leqslant \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |(f-g)(x-ay)|\,(f+g)(x+by)|y|^{\gamma}\langle x\rangle^{\gamma_*}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad+\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} (f+g)(x-ay)\,|(f-g)(x+by)|\,|y|^{\gamma}\,\langle x\rangle^{\gamma_*}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad+\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |(f-g)(x+y)|\,(f+g)(x)|y|^{\gamma}\langle x\rangle^{\gamma_*}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad-\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |(f-g)(x)|(f+g)(x+y)|y|^{\gamma}\langle x\rangle^{\gamma_*}\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}
$$

Thus, changing variables leads to

(A.3)

$$
\begin{aligned}
&\big[f-g,\,\mathcal{Q}(f,f)-\mathcal{Q}(g,g)\big]_- \\
&\quad\leqslant \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |(f-g)(x)|(f+g)(y)|x-y|^{\gamma}\big(\langle ax+by\rangle^{\gamma_*}+\langle bx+ay\rangle^{\gamma_*}+\langle y\rangle^{\gamma_*}\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad-\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} |(f-g)(x)|(f+g)(y)|x-y|^{\gamma}\,\langle x\rangle^{\gamma_*}\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}
$$

Now, since $\gamma_*\geqslant 2$ the mapping $x\mapsto\langle x\rangle^{\gamma_*}$ is convex over $\mathbb{R}$; thus, for any $x,\,y\in\mathbb{R}$ (recall that $a+b=1$)

$$
\begin{aligned}
\langle ax+by\rangle^{\gamma_*}+\langle bx+ay\rangle^{\gamma_*}-\langle x\rangle^{\gamma_*}-\langle y\rangle^{\gamma_*} &= \langle ax+by\rangle^{\gamma_*}-a\langle x\rangle^{\gamma_*}-b\langle y\rangle^{\gamma_*} \\
&\quad+\langle bx+ay\rangle^{\gamma_*}-b\langle x\rangle^{\gamma_*}-a\langle y\rangle^{\gamma_*}\leqslant 0\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\big[f-g,\mathcal{Q}(f,f)-\mathcal{Q}(g,g)\big]_- &\leqslant \int_{\mathbb{R}}\int_{\mathbb{R}} |(f-g)(x)|(f+g)(y)|x-y|^{\gamma}\langle y\rangle^{\gamma_*}\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant L\|f-g\|_{L^1_{\gamma}}\leqslant L\|f-g\|_{L^1_{\gamma_*}}\,,\ \text{with } L=2^{\frac{\gamma_*+\gamma+\delta}{2}}C_{\gamma}(\|f_0\|_{L^1}+K)\,.
\end{aligned}
$$

Next, let us look for a subtangent condition. Given $f\in\Omega_{K,\delta}$ and $h\geqslant 0$, one notices that

$$
f(x)+h\mathcal{Q}(f,f)(x)=h\int_{\mathbb{R}} f(x-ay)f(x+by)|y|^{\gamma}\mathrm{d}y+f(x)\left(1-h\int_{\mathbb{R}} f(x+y)|y|^{\gamma}\mathrm{d}y\right)\,.
$$

In particular, what prevents $f+h\mathcal{Q}(f,f)$ from being a.e. nonnegative is the influence of large x in the last convolution integral. To overcome this difficulty, for any $R>0$, we introduce the truncation $f_R(x)=f(x)\chi_{\{|x|<R\}}$. Then, since $f\geqslant f_R$ one deduces from the above identity that

$$
\begin{aligned}
f(x)+h\mathcal{Q}(f_R,f_R)(x) &\geqslant h\int_{\mathbb{R}} f_R(x-ay)f_R(x+by)|y|^{\gamma}\mathrm{d}y \\
&\quad+f_R(x)\left(1-h\int_{\mathbb{R}} f_R(x+y)|y|^{\gamma}\mathrm{d}y\right)\qquad \text{a.e. } x\in\mathbb{R}\,.
\end{aligned}
\tag{A.4}
$$

Now,

$$
\int_{\mathbb{R}} f_R(x+y)|y|^{\gamma}\mathrm{d}y=\int_{\mathbb{R}} f_R(y)|x-y|^{\gamma}\mathrm{d}y\leqslant \max(2^{\gamma-1},1)\int_{\mathbb{R}} f(y)\,(|x|^{\gamma}+|y|^{\gamma})\,\mathrm{d}y
$$

and using Young's inequality, one sees that there exists some positive constant $C_0 > 0$ depending only on K, δ, γ, and $\|f_0\|_{L^1}$, but not on R, such that

$$\int_{\mathbb{R}} f_R(x+y)|y|^{\gamma}\mathrm{d}y \leqslant C_0(1+|x|^{\gamma}) \qquad \text{for any } x \in \mathbb{R}\,,\ R>0 \text{ and } f \in \Omega_{K,\delta}\,.$$

Therefore, recalling that f_R is supported on $\{|x| < R\}$, one deduces from (A.4) that

$$f(x) + h\mathcal{Q}(f_R, f_R)(x) \geqslant 0 \qquad \text{for a.e. } x \in \mathbb{R} \quad \forall\, 0 < h < h_R := \frac{1}{C_0(1+R^{\gamma})}\,.$$

Moreover, since $\mathcal{Q}$ preserves the mass,

$$\int_{\mathbb{R}} \big(f(x) + h\mathcal{Q}(f_R, f_R)(x)\big)\mathrm{d}x = \int_{\mathbb{R}} f(x)\mathrm{d}x = \int_{\mathbb{R}} f_0(x)\mathrm{d}x\,.$$

Finally, using (2.7) it follows that, for any $R > 0$,

$$\int_{\mathbb{R}} \mathcal{Q}(f_R, f_R)(x)|x|^{\gamma+\gamma_*+\delta}\mathrm{d}x \\ \leqslant -\frac{1}{2}\left(1 - a^{\gamma+\gamma_*+\delta} - b^{\gamma+\gamma_*+\delta}\right)\int_{\mathbb{R}}\int_{\mathbb{R}} f_R(x)f_R(y)|x-y|^{2\gamma+\gamma_*+\delta}\mathrm{d}x\mathrm{d}y \leqslant 0\,.$$

Consequently,

$$\int_{\mathbb{R}} \big(f(x) + h\mathcal{Q}(f_R, f_R)(x)\big)|x|^{\gamma+\gamma_*+\delta}\mathrm{d}x \leqslant \int_{\mathbb{R}} f(x)|x|^{\gamma+\gamma_*+\delta}\mathrm{d}x \leqslant K \qquad \forall R > 0\,.$$

We have thus shown that, for any $R > 0$ and any $0 < h < h_R$, one has $f + h\mathcal{Q}(f_R, f_R) \in \Omega_{K,\delta}$. In particular, for any $R > 0$ and any $0 < h < h_R$ one has

$$\begin{aligned}\mathrm{dist}(f + h\mathcal{Q}(f,f), \Omega_{K,\delta}) &\leqslant \|f + h\mathcal{Q}(f,f) - (f + h\mathcal{Q}(f_R, f_R)\|_{L^1_{\gamma_*}} \\ &= h\,\|\mathcal{Q}(f,f) - \mathcal{Q}(f_R, f_R)\|_{L^1_{\gamma_*}}\,.\end{aligned}$$

Now, for $f \in \Omega_{K,\delta}$, one can make $\|\mathcal{Q}(f,f) - \mathcal{Q}(f_R, f_R)\|_{L^1_{\gamma_*}}$ arbitrarily small provided $R > 0$ is large enough; thus, the subtangent condition

$$\liminf_{h \to 0^+} h^{-1}\mathrm{dist}\,(f + h\mathcal{Q}(f,f), \Omega_{K,\delta}) = 0$$

holds true.

Using the Hölder continuity, the subtangent condition, and [28, Theorem VI.2.2] we have conditions (C1)–(C3) in [28, p. 229]. Adding the one-sided Lipschitz condition, we can apply [28, Theorem VI.4.3] and deduce the existence and the uniqueness of a global solution f to (1.1) such that $f(t) \in \Omega_{K,\delta}$ for every $t \geqslant 0$. Moreover, (A.1) holds; thus, it follows from (2.7) that for every $k \geqslant \gamma_*$ and every $t \geqslant 0$,

$$\int_{\mathbb{R}} f(t,x)\,|x|^k\mathrm{d}x \leqslant \int_{\mathbb{R}} f_0(x)\,|x|^k\mathrm{d}x\,.$$

This implies, together with the conservation of the mass, that $f(t) \in L^1_k(\mathbb{R})$ for any $t \geqslant 0$ and any $k \in \mathbb{R}$. Finally, it is easily checked that the family $(\mu_t)_{t\geqslant 0}$ defined by $\mu_t(\mathrm{d}x) = f(t,x)\mathrm{d}x$ for any $t \geqslant 0$ is a weak measure solution to (1.1). ∎

Proof of Proposition 2.10. Let $T > 0$ be fixed. For any $\phi \in \mathrm{Lip}_1(\mathbb{R})$ and $t \in [0, T]$ define

$$W_\phi(t) := \int_{\mathbb{R}} \phi(x)\mu_t(\mathrm{d}x) - \int_{\mathbb{R}} \phi(x)\nu_t(\mathrm{d}x)\,.$$

We will also use in the proof the notation

$$W(t) = d_{\mathrm{KR}}(\mu_t, \nu_t) = \sup_{\phi \in \mathrm{Lip}_1(\mathbb{R})} W_\phi(t)$$

and recall that

$$W(t) = \int_{\mathbb{R}^2} |x - y|\pi_t(\mathrm{d}x, \mathrm{d}y) \;\text{ for some } \pi_t \in \Pi(\mu_t, \nu_t)\,.$$

Let now $\phi \in \mathrm{Lip}_1(\mathbb{R})$ be fixed. One has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}W_\phi(t) &= \frac{1}{2}\int_{\mathbb{R}^2} |x - y|^\gamma \Delta\phi(x, y)\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)\\ &\qquad - \frac{1}{2}\int_{\mathbb{R}^2} |v - w|^\gamma \Delta\phi(v, w)\nu_t(\mathrm{d}v)\nu_t(\mathrm{d}w)\\ &= \int_{\mathbb{R}^2} \pi_t(\mathrm{d}x, \mathrm{d}v) \int_{\mathbb{R}^2} \Big[|x - y|^\gamma \Delta_0\phi(x, y) - |v - w|^\gamma \Delta_0\phi(v, w)\Big]\pi_t(\mathrm{d}y, \mathrm{d}w),\end{aligned} \tag{A.5}$$

where $\Delta_0\phi(x, y) = \phi(ax + by) - \phi(x)$ for any $(x, y) \in \mathbb{R}^2$. Now,

$$\begin{aligned}&|x - y|^\gamma \Delta_0\phi(x, y) - |v - w|^\gamma \Delta_0\phi(v, w)\\ &\qquad = \Big(|x - y|^\gamma - |v - w|^\gamma\Big)\Delta_0\phi(x, y) + |v - w|^\gamma \Big(\Delta_0\phi(x, y) - \Delta_0\phi(v, w)\Big)\,,\end{aligned} \tag{A.6}$$

and, recalling that the Lipschitz constant of ϕ is at most one, the second term readily yields

$$\begin{aligned}\Big|\Delta_0\phi(x, y) - \Delta_0\phi(v, w)\Big| &= \Big|\phi(ax + by) - \phi(av + bw) - \phi(x) + \phi(v)\Big|\\ &\leqslant (1 + a)|x - v| + b|y - w|\,.\end{aligned} \tag{A.7}$$

For the first term in (A.6), we use the identity $A = \min\{A, B\} + (A - B)_+$, valid for any $A, B \geqslant 0$, to obtain the estimate

$$\begin{aligned}\big|A^\gamma - B^\gamma|A &= \big|A^\gamma - B^\gamma\big|\big(\min\{A, B\} + (A - B)_+\big)\\ &\leqslant \big|A^\gamma - B^\gamma\big| \min\{A, B\} + \big|A^\gamma - B^\gamma\big|\,\big|A - B\big|\\ &\leqslant (1 + \gamma)\max\{A, B\}^\gamma\big|A - B\big|\,.\end{aligned}$$

The last inequality follows noticing that $\big|A^\gamma - B^\gamma\big| \min\{A, B\} \leqslant \gamma\max\{A, B\}^\gamma\big|A - B\big|$. Since $\big|\Delta_0\phi(x, y)\big| \leqslant b|x - y|$ we can choose $A = |x - y|$ and $B = |v - w|$ to conclude that

$$\begin{aligned}\Big(|x - y|^\gamma - |v - w|^\gamma\Big)\Delta_0\phi(x, y) &\leqslant b(1 + \gamma)\max\{|x - y|, |v - w|\}^\gamma\big||x - y| - |v - w|\big|\\ &\leqslant b(1 + \gamma)\max\{|x - y|, |v - w|\}^\gamma\big(|x - v| + |y - w|\big)\,.\end{aligned} \tag{A.8}$$

Gathering the estimates (A.6), (A.7), and (A.8) in (A.5) and using symmetry of the expression, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}W_\phi(t) \leqslant 3(1+\gamma)H(t)$, where we introduced

$$H(t) := \int_{\mathbb{R}^2} \pi_t(\mathrm{d}x,\mathrm{d}v) \int_{\mathbb{R}^2} \big(|x|^\gamma + |y|^\gamma + |v|^\gamma + |w|^\gamma\big)|x-v|\pi_t(\mathrm{d}y,\mathrm{d}w)\,. \tag{A.9}$$

Expand $H(t) = H_1(t) + H_2(t)$, where

$$H_1(t) := \int_{\mathbb{R}^2} \pi_t(\mathrm{d}x,\mathrm{d}v) \int_{\mathbb{R}^2} \big(|y|^\gamma + |w|^\gamma\big)|x-v|\pi_t(\mathrm{d}y,\mathrm{d}w)\,,$$
$$H_2(t) := \int_{\mathbb{R}^2} \big(|x|^\gamma + |v|^\gamma\big)|x-v|\pi_t(\mathrm{d}x,\mathrm{d}v)\,.$$

Notice that

$$\begin{aligned} H_1(t) &= \int_{\mathbb{R}^2} \big(|y|^\gamma + |w|^\gamma\big)\pi_t(\mathrm{d}y,\mathrm{d}w) \int_{\mathbb{R}^2} |x-v|\pi_t(\mathrm{d}x,\mathrm{d}v) \\ &= W(t)\int_{\mathbb{R}} |x|^\gamma(\mu_t+\nu_t)(\mathrm{d}x) \leqslant C\,W(t)\,. \end{aligned} \tag{A.10}$$

The last inequality follows because the weak measure solutions μ_t and ν_t have the γ-moment uniformly bounded in $t \in [0,T]$, and additionally, $\pi_t \in \Pi(\mu_t,\nu_t)$ achieves the Kantorovich–Rubinstein distance. We estimate now $H_2(t)$ as in [23, Corollary 2.3]. Namely, for any $t\in[0,T]$ and any $r>0$, one has

$$\begin{aligned} \int_{\mathbb{R}^2} \Big(|x|^\gamma + |v|^\gamma\Big)\,|x-v|\,\pi_t(\mathrm{d}x,\mathrm{d}v) &\leqslant 2r^\gamma \int_{\mathbb{R}^2} |x-v|\,\pi_t(\mathrm{d}x,\mathrm{d}v) \\ &\quad + \int_{\min(|x|,|v|)\geqslant r} \Big(|x|^\gamma + |v|^\gamma\Big)\,|x-v|\,\pi_t(\mathrm{d}x,\mathrm{d}v) \\ &= 2r^\gamma W(t) + \int_{\min(|x|,|v|)\geqslant r} \Big(|x|^\gamma + |v|^\gamma\Big)\,|x-v|\,\pi_t(\mathrm{d}x,\mathrm{d}v) \end{aligned}$$

since $\pi_t \in \Pi(\mu_t,\nu_t)$ achieves the Kantorovich–Rubinstein distance. Setting now R_ε such that

$$\begin{aligned} &\Big(|x|^\gamma + |v|^\gamma\Big)\,|x-v|\left(\exp\left(\frac{\varepsilon|x|^\gamma}{2}\right) + \exp\left(\frac{\varepsilon|v|^\gamma}{2}\right)\right) \\ &\qquad \leqslant R_\varepsilon\left(\exp(\varepsilon|x|^\gamma) + \exp(\varepsilon|v|^\gamma)\right) \qquad \forall\,(x,v)\in\mathbb{R}^2\,, \end{aligned}$$

it follows that

$$H_2(t) \leqslant 2r^\gamma W(t) + R_\varepsilon\,C_T(\varepsilon)\exp\left(-\frac{\varepsilon r^\gamma}{2}\right)\,.$$

Choosing

$$r^\gamma = |2\log W(t)/\varepsilon|$$

we obtain

$$H_2(t) \leqslant \frac{4}{\varepsilon}W(t)\,|\log W(t)| + R_\varepsilon\,C_T(\varepsilon)\,W(t)\,. \tag{A.11}$$

Estimates (A.10) and (A.11) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}W_\phi(t) \leqslant K_\varepsilon C_T(\varepsilon)\,W(t)\big(1+|\log W(t)|\big) \tag{A.12}$$

with a constant $K_\varepsilon > 0$ depending only on γ and $\varepsilon > 0$. Integrating (A.12) and taking the supremum over $\phi \in \mathrm{Lip}_1(\mathbb{R})$ we get the conclusion. ∎

Appendix B. Slowly increasing entropy bounds. We show in this appendix that the entropy of the solution to (1.2) is increasing logarithmically in time, which we believe is an interesting a priori estimate. For simplicity, we restrict ourselves to the case $a = b = 1/2$.

Let $f_0 \in L^1(\mathbb{R})$ be a nonnegative initial datum such that

$$\int_{\mathbb{R}} f_0(x)\mathrm{d}x = 1, \qquad \int_{\mathbb{R}} f_0(x)x\mathrm{d}x = 0,$$

and

$$\int_{\mathbb{R}} \exp(\varepsilon|x|^\gamma) f_0(x)\mathrm{d}x < \infty$$

for some $\varepsilon > 0$, in such a way that there exists a unique solution $(f_t)_{t\geqslant 0} \subset L^1(\mathbb{R})$ to (1.2) with

$$f(t,x) \geqslant 0 \qquad \text{for a. e. } x \in \mathbb{R}, \qquad \int_{\mathbb{R}} f(t,x)\mathrm{d}x = 1 \qquad \forall t \geqslant 0.$$

Set

$$\mathcal{H}(f(t)) = \int_{\mathbb{R}} f(t,x)\log f(t,x)\mathrm{d}x \qquad \forall t \geqslant 0$$

and assume that $\mathcal{H}(f_0) < \infty$. Recall that $M_k(t) = \int_{\mathbb{R}} f(t,x)|x|^k\mathrm{d}x$ for any $k \geqslant 0$. We have first the following proposition.

PROPOSITION B.1. *Assume that $\gamma \geqslant 1$. Then, there exists $C_\gamma > 0$ (depending only on γ and not on f_0) such that*

$$\mathcal{H}(f(t)) \leqslant \mathcal{H}(f_0) + \frac{2^{\gamma+1}}{C_\gamma e}\log\left(1 + C_\gamma\sqrt{M_{2\gamma}(0)t}\right) \qquad \forall t \geqslant 0.$$

Proof. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f(t)) = \int_{\mathbb{R}} \mathcal{Q}(f(t,x), f(t,x))\log f(t,x)\mathrm{d}x \qquad \forall t \geqslant 0$$

the proof consists simply in estimating this last integral. We forget about the dependence with respect to t to simplify notation and set

$$I := \int_{\mathbb{R}} \mathcal{Q}(f,f)\log f(x)\mathrm{d}x.$$

Applying (1.3) to $\psi(x) = \log f(x)$ we get

$$I = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(y)\big|x-y\big|^\gamma \log\left(\frac{f\left(\frac{x+y}{2}\right)}{\sqrt{f(x)f(y)}}\right)\mathrm{d}x\mathrm{d}y.$$

Set

$$Z_\gamma = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(y)\big|x-y\big|^\gamma\mathrm{d}x\mathrm{d}y$$

so that $\mu(\mathrm{d}x,\mathrm{d}y) = \frac{1}{Z_\gamma}f(x)f(y)\big|x-y\big|^\gamma\mathrm{d}x\mathrm{d}y$ is a probability measure over $\mathbb{R}^2$. From Jensen's inequality we have

$$I \leqslant Z_\gamma \log\left(\int_{\mathbb{R}^2} \frac{f\left(\frac{x+y}{2}\right)}{\sqrt{f(x)f(y)}}\mu(\mathrm{d}x,\mathrm{d}y)\right).$$

Setting now

$$\frac{1}{J} := \int_{\mathbb{R}^2} \frac{f\left(\frac{x+y}{2}\right)}{\sqrt{f(x)f(y)}}\mu(\mathrm{d}x,\mathrm{d}y) = \frac{1}{Z_\gamma}\int_{\mathbb{R}^2} f\left(\frac{x+y}{2}\right)\sqrt{f(x)f(y)}|x-y|^\gamma \mathrm{d}x\mathrm{d}y$$

we get easily that

$$I \leqslant -J\log J \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f\left(\frac{x+y}{2}\right)\sqrt{f(x)f(y)}|x-y|^\gamma \mathrm{d}x\mathrm{d}y.$$

Since $-s\log s \leqslant \frac{1}{e}$ for any $s>0$, we get

$$I \leqslant \frac{1}{e}\int_{\mathbb{R}^2} f\left(\frac{x+y}{2}\right)\sqrt{f(x)f(y)}|x-y|^\gamma \mathrm{d}x\mathrm{d}y,$$

and estimating $|x-y|^\gamma \leqslant 2^{\gamma-1}\left(|x|^\gamma+|y|^\gamma\right)$ and using symmetry we find

$$I \leqslant \frac{2^\gamma}{e}\int_{\mathbb{R}} \sqrt{f(x)}\,|x|^\gamma \mathrm{d}x \int_{\mathbb{R}} f\left(\frac{x+y}{2}\right)\sqrt{f(y)}\mathrm{d}y.$$

Setting $g(z)=f(z/2)$ and $h(x)=|x|^\gamma\sqrt{f(x)}$ we see that

$$I \leqslant \frac{2^\gamma}{e}\int_{\mathbb{R}} h(x)\left(g*\sqrt{f}\right)(x)\mathrm{d}x,$$

where $*$ denotes the convolution product. A simple use of Young's convolution inequality yields

$$I \leqslant \frac{2^\gamma}{e}\|h\|_{L^2(\mathbb{R})}\,\|g*\sqrt{f}\|_{L^2(\mathbb{R})} \leqslant \frac{2^\gamma}{e}\|h\|_{L^2(\mathbb{R})}\,\|g\|_{L^1(\mathbb{R})}\,\left\|\sqrt{f}\right\|_{L^2(\mathbb{R})}.$$

Since

$$\|h\|_{L^2(\mathbb{R})} = \sqrt{\left(\int_{\mathbb{R}} f(x)|x|^{2\gamma}\mathrm{d}x\right)} = \sqrt{M_{2\gamma}(f)}, \quad \|g\|_{L^1(\mathbb{R})} = 2\|f\|_{L^1(\mathbb{R})} = 2$$

and $\|\sqrt{f}\|_{L^2} = \|f\|_{L^1}^{1/2} = 1$ we finally obtain that

$$I \leqslant \frac{2^{\gamma+1}}{e}\sqrt{M_{2\gamma}(f)}.$$

In other words,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f(t)) \leqslant \frac{2^{\gamma+1}}{e}\sqrt{M_{2\gamma}(t)} \qquad \forall t\geqslant 0. \tag{B.1}$$

Using Theorem 3.5, (3.19b) (remember that $\gamma>1$), we have

$$M_{2\gamma}(t) \leqslant \frac{M_{2\gamma}(0)}{(1+C_\gamma\sqrt{M_{2\gamma}(0)}t)^2}, \tag{B.2}$$

where $C_\gamma = \frac{1}{4}\left(1-2^{1-2\gamma}\right)$. Plugging this into (B.1) one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f(t)) \leqslant \frac{2^{\gamma+1}}{e}\frac{\sqrt{M_{2\gamma}(0)}}{1+C_\gamma\sqrt{M_{2\gamma}(0)}t} \qquad \forall t\geqslant 0,$$

which yields the result after integration. ∎

We can actually prove the optimality of the above upper bound using a general comparison result between entropy and moments which can be tracked back to Nash [31]. We state here the result in its general form in dimension $n \geqslant 1$ and give a complete proof for completeness.

PROPOSITION B.2. *Let $n \geqslant 1$ and $f \in L^1(\mathbb{R}^n)$ nonnegative to be given with $\int_{\mathbb{R}^n} f(x)\mathrm{d}x = 1$. For any $k \geqslant 0$, set*

$$M_k(f) = \int_{\mathbb{R}^n} f(x)|x|^k \mathrm{d}x \qquad \text{and} \qquad \mathcal{H}(f) = \int_{\mathbb{R}^n} f(x)\log f(x)\mathrm{d}x.$$

Then,

$$M_k(f) \geqslant C(k,n)\exp\left(-\frac{k}{n}\mathcal{H}(f)\right) \qquad \forall k \geqslant 1,$$

where $C(k,n) = \exp\left(-k - \frac{k}{n}\log\gamma(k,n) + \log n\right)$, $\gamma(k,n) = \frac{|\mathbb{S}^{n-1}|}{k}\Gamma(\frac{n}{k})$.

Proof. For any $\lambda \in \mathbb{R}$, notice that

$$\min_{s>0}\,(s\log s + \lambda s) = -\exp(-\lambda - 1).$$

Applying this to $s = f(x)$, $\lambda = a|x|^k + b$ (with $a, b > 0$ to be fixed later on) and integrating over $\mathbb{R}^n$ one gets

$$\mathcal{H}(f) + aM_k(f) + b \geqslant -\exp(-b-1)\int_{\mathbb{R}^n} \exp(-a|x|^k)\mathrm{d}x.$$

One easily checks that

$$\int_{\mathbb{R}^n} \exp(-a|v|^k)\mathrm{d}v = a^{-\frac{n}{k}}\int_{\mathbb{R}^n} \exp(-|v|^k)\mathrm{d}v = a^{-\frac{n}{k}}\gamma(k,n)$$

with $\gamma(k,n) = \frac{|\mathbb{S}^{n-1}|}{k}\Gamma\left(\frac{n}{k}\right)$. Therefore, for any $k \geqslant 0$, it holds that

$$\mathcal{H}(f) + aM_k(f) + b \geqslant -\exp(-b-1)\gamma(k,n)\,a^{-\frac{n}{k}} \qquad \forall a, b \in \mathbb{R}.$$

One optimizes with respect to the parameters a, b choosing, for instance, $a = \frac{n}{M_k}$ and b in such a way that $\exp(-b-1)\gamma(k,n)\,a^{-\frac{n}{k}} = 1$. This leads to

$$\mathcal{H}(f) + n \geqslant -1 - b = \log\left(\frac{a^{\frac{n}{k}}}{\gamma(k,n)}\right) = \frac{n}{k}\log a - \log\gamma(k,n).$$

Since $a = \frac{n}{M_k(f)}$, we finally obtain

$$\frac{n}{k}\log M_k(f) \geqslant -\mathcal{H}(f) - n - \log\gamma(k,n) + \frac{n}{k}\log n, \tag{B.3}$$

which is the desired estimate. ∎

We deduce from the above proposition that the upper bound provided by Proposition B.1 is almost optimal.

PROPOSITION B.3. *For $\gamma > 1$, one has*

$$\mathcal{H}(f(t)) \geqslant \frac{1}{\gamma}\log\left(1 + C_\gamma\sqrt{M_{2\gamma}(0)t}\right) - 1 - \log 2 - \frac{1}{2\gamma}\log(M_{2\gamma}(0)) \qquad \forall t \geqslant 0.$$

Proof. We apply the estimate provided by Proposition B.2 to $n = k = 1$ and $f = f(t, x)$. We notice that $\gamma(1,1) = 2\Gamma(1) = 2$ and obtain from (B.3)

$$\log M_1(t) \geqslant -\mathcal{H}(f(t)) - 1 - \log 2 \qquad \forall t \geqslant 0.$$

Using now the fact that $M_1(t) \leqslant M_{2\gamma}(t)^{\frac{1}{2\gamma}}$ together with (B.2) we get that

$$\log M_1(t) \leqslant \frac{1}{2\gamma}\log(M_{2\gamma}(0)) - \frac{1}{\gamma}\log\left(1 + C_\gamma\sqrt{M_{2\gamma}(0)}t\right),$$

which, combined with the previous estimate, yields the result. ◻

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