N=4 enhancement of N=3 supersymmetry of Super Chern-Simons theories in D=3, Calabi HyperKähler metrics and M2-branes on the C(N[010]) conifold

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The $\mathscr{N}_3 = 3 \to \mathscr{N}_3 = 4$ enhancement of Super Chern-Simons theories in $D = 3$, Calabi HyperKähler metrics and M2-branes on the $\mathscr{C}(\mathrm{N}^{0,1,0})$ conifold

P. Fré [a,b,c*], A. Giambrone [a†], P. A. Grassi [b,c,d‡] and P. Vasko [a,b,c§]

[(a)] *Dipartimento di Fisica, Università di Torino, via P. Giuria 1, 10125 Torino, Italy.*

[(b)] *INFN, Sezione di Torino, via P. Giuria 1, 10125 Torino, Italy,*

[(c)] *Arnold-Regge Center, via P. Giuria 1, 10125 Torino, Italy,*

[(d)] *Dipartimento di Scienze e Innovazione Tecnologica, Università del Piemonte Orientale, viale T. Michel, 11, 15121 Alessandria, Italy.*

Abstract

Considering matter coupled supersymmetric Chern-Simons theories in three dimensions we extend the Gaiotto-Witten mechanism of supersymmetry enhancement $\mathscr{N}_3 = 3 \to \mathscr{N}_3 = 4$ from the case where the hypermultiplets span a flat HyperKähler manifold to that where they live on a curved one. We derive the precise conditions of this enhancement in terms of generalized Gaiotto-Witten identities to be satisfied by the tri-holomorphic moment maps. An infinite class of HyperKähler metrics compatible with the enhancement condition is provided by the Calabi metrics on $T^\star\mathbb{P}^n$. In this list we find, for $n = 2$ the resolution of the metric cone on $\mathrm{N}^{0,1,0}$ which is the unique homogeneous Sasaki Einstein 7-manifold leading to an $\mathscr{N}_4 = 3$ compactification of M-theory. This leads to challenging perspectives for the discovery of new relations between the enhancement mechanism in $D = 3$, the geometry of M2-brane solutions and also for the dual description of super Chern Simons theories on curved HyperKähler manifolds in terms of gauged fixed supergroup Chern Simons theories. The relevant supergroup is in this case $\mathrm{SU}(3|\mathrm{N})$ where $\mathrm{SU}(3)$ is the flavor group and $\mathrm{U}(\mathrm{N})$ is the color group.

*pfre@unito.it
†alfredo.giambrone@edu.unito.it
‡pietro.grassi@uniupo.it
§petr.vasko@to.infn.it

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1 Introduction

Matter coupled Chern-Simons gauge theories are of interest both as challenging paradigms in quantum field theory and as theoretical models for the description of certain condensed matter systems.

At the dawn of the new millennium Chern Simons matter coupled theories raised to prominence in association with the AdS_4/CFT_3 gauge-gravity correspondence. Indeed, after the discovery of the AdS_5/CFT_4 correspondence [1, 2, 3, 4, 5], the programme of the AdS_4/CFT_3 was an obvious development where all the results of Kaluza-Klein supergravity, accumulated at the beginning of the eighties could be recycled. In the years 1998-2000 a rush started to complete the derivation of the Kaluza-Klein spectra for all the compactifications of type $AdS_5 \times (G/H)_5$ and $AdS_4 \times (G/H)_7$. The idea was to compare such spectra with the towers of primary conformal fields in the dual gauge theory either in $D = 4$, for type IIB D3-branes, or in $D = 3$ for M2-branes. The case of the coset $T^{(1,1)} \equiv \frac{SU(2)_I \times SU(2)_{II}}{U(1)}$, where the denominator group is the diagonal of the standard $U_{I,II}(1) \subset SU_{I,II}(2)$ was studied and the results were published in the month of May 1999 [6]. The case of the Sasakian homogeneous seven manifolds listed in table 1 was actively studied and the results were published in [7, 8, 9, 10].

This was one side of the correspondence: that of supergravity. The other side, that of the gauge theory, required the determination of suitable candidates. The first mile-stone in this direction came in 1998 with the paper by Klebanov and Witten [11] where the geometrical description as a Kähler quotient of the metric cone $\mathscr{C}\left(T^{(1,1)}\right)$ on the coset $T^{(1,1)}$, denominated by them the *conifold*, was discussed. Indeed, the main point of [11] was the identification of the pivot role of the Kähler quotient in singling out the field content and the interactions of the dual gauge theory on the brane world-volume. In the case of $T^{(1,1)}$, the metric cone $\mathscr{C}\left(T^{(1,1)}\right)$ can be described as the Kähler quotient of $\mathbb{C}^2 \times \mathbb{C}^{2\star}$ with respect to a single $U(1)$. So Klebanov and Witten outlined a pattern that, about one year later and in presence of all the accomplished Kaluza Klein spectra for the relevant Sasakian manifolds, was generalized to the case AdS_4/CFT_3 in [12].

In all cases the (Hyper)-Kähler quotient description of the metric cone on a (tri)-Sasakian manifold is the starting point for the construction of the dual gauge theory on the brane world-volume. The coordinates of the linear space of which we perform the quotient are the (hyper/Wess-Zumino)-multiplets and the *color gauge group* is accordingly singled out by the quotient. Having singled out the principles for the second side of the correspondence, the explicit construction of the dual gauge theories became possible together with the definition of all the towers of conformal primaries to be compared with the Kaluza-Klein spectra. Both tasks were accomplished for the seven-dimensional Sasakians in the already quoted papers [7, 9, 13, 14]. In particular the case of the $\mathscr{N}_3 = 3, D = 3$ gauge theory, corresponding to the conifold of $N^{0,1,0}$, was derived in [13], leading to the mechanism of gaussian integration of the gauge multiplet degrees of freedom, leaving a quartic superpotential remnant, that anticipated of about nine years the scheme used in [15] to obtain the ABJM model. Indeed in [13] it was shown that, generalizing to non abelian gauge groups a mechanism already discovered in [16] for abelian ones, the addition of Chern-Simons interactions to an $\mathscr{N}_3 = 4, D = 3$ Yang-Mills gauge theory breaks supersymmetry to $\mathcal{N}_3 = 3$.

The $\mathcal{N}_3 = 4,3$ gauge theories can be identified as special subclasses of $\mathcal{N}_3 = 2, D = 3$ gauge theories, whose general form was described in [8] for linear representations, and was generalized to arbitrary Kähler and HyperKähler manifolds in [17], which introduced also a more compact and geometrical notation for the lagrangian. Utilizing the off-shell formulation of $\mathcal{N}_3 = 2, D = 3$ gauge theories of [8, 17], in [13] it was advocated that in the infrared strong coupling limit the gauge coupling constant g^2 goes to infinity while the dimensionless Chern-Simons coupling constant α states finite. In this way all kinetic terms of the fields belonging to the gauge multiplets are suppressed and the latter fields can be integrated out leaving an $\mathcal{N}_3 = 3$ matter coupled Chern-Simons gauge theory whose superpotential has the following very special form:

$$\mathscr{W} = -\frac{1}{8\alpha}\mathscr{P}^{+}_{\Lambda}\mathscr{P}^{+}_{\Sigma}\,\mathfrak{m}_{\Lambda\Sigma} \tag{1.1}$$

where $\mathscr{P}^{\Lambda}_{+}$ denote the holomorphic part of the moment-maps for the triholomorphic action of the gauge group generators T^{Λ} on the HyperKähler manifold HK_{2n} spanned by the hypermultiplets. The gauge group is generically denoted $\mathscr{G}$, its Lie algebra is denoted $\mathbb{G}$ and $\mathfrak{m}_{\Lambda\Sigma}$ is an invariant non-degenerated quadratic form on $\mathbb{G}$. As we stress later on, $\mathfrak{m}_{\Lambda\Sigma}$ is not necessarily the Cartan Killing form and it is not necessarily positive definite. The full scalar potential for these theories takes the form:

$$\begin{aligned} V_{scalar} &= \frac{1}{6}\left(\partial_i\mathscr{W}\,\partial_{j^\star}\overline{\mathscr{W}}\,g^{ij^\star} + \mathbf{m}^{\Lambda\Sigma}\,\mathscr{P}^{3}_{\Lambda}\mathscr{P}^{3}_{\Sigma}\right) \\ \mathbf{m}^{\Lambda\Sigma}(u,v) &\equiv \frac{1}{4\alpha^2}\,\mathfrak{m}^{\Lambda\Gamma}\mathfrak{m}^{\Sigma\Delta}k^{i}_{\Gamma}k^{j^\star}_{\Delta}\,g_{ij^\star} \end{aligned} \tag{1.2}$$

where $\mathscr{P}^{3}_{\Sigma}$ are the real components of the tri-holomorphic moment maps for the action of $\mathscr{G}$ on the HyperKähler manifold HK_{2n}, while $g_{ij^\star}$ denotes the components of its HyperKähler metric $\mathbf{g}$ and $k^{i}_{\Gamma}, k^{j^\star}_{\Gamma}$ are the components of the Killing vectors generating $\mathscr{G}$. Indeed the metric $\mathbf{g}$ must admit the gauge group $\mathscr{G}$ as isometry group.

In 2007 Bagger and Lambert presented their version of the $\mathcal{N}_3 = 8$ Chern-Simons theory [18, 19, 20]. Their work allowed us to understand how $\mathcal{N} > 3$ enhancements might arise starting from an $\mathcal{N}_3 = 3$ model. Few months after this discovery, all the formulations with $4 \leq \mathcal{N} \leq 8$ were constructed, utilizing the mechanism of gaussian integration of the physical fields of the vector multiplets, originally introduced for the case of the compactification on the tri-Sasakian $N^{0,1,0}$ manifold in [13]. Supersymmetric Chern-Simons theories were completely classified in the case when the scalar sector parameterizes a flat manifold. The key point was to understand how to specialize the $\mathcal{N}_3 = 3$ theory in order to enhance the R-symmetry.

An interesting construction is that presented by Gaiotto and Witten in [21]. Their starting point is an $\mathcal{N}_3 = 1$ theory with the field content of an $\mathcal{N}_3 = 3$ one. Adding a suitable superpotential the theory becomes $\mathcal{N}_3 = 3$ supersymmetric. By means of a restriction imposed on the superpotential one obtains an $\mathcal{N}_3 = 4$ supersymmetric theory. Further restrictions lead to higher $\mathcal{N}$-extended supersymmetric theories. An important feature is that these restrictions are equivalent to suitable choices of the gauge group and of the matter representation.

$\mathscr{N}$	Name	Coset	Holon. $\mathfrak{so}(8)$ bundle	Fibration
8	$\mathbb{S}^7$	$\frac{\mathrm{SO}(8)}{\mathrm{SO}(7)}$	1	$\begin{cases} \mathbb{S}^7 \stackrel{\pi}{\Longrightarrow} \mathbb{P}^3 \\ \forall p \in \mathbb{P}^3 ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	$\mathrm{M}^{1,1,1}$	$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)}$	SU(3)	$\begin{cases} M^{1,1,1} \stackrel{\pi}{\Longrightarrow} \mathbb{P}^2 \times \mathbb{P}^1 \\ \forall p \in \mathbb{P}^2 \times \mathbb{P}^1 ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	$\mathrm{Q}^{1,1,1}$	$\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)}$	SU(3)	$\begin{cases} Q^{1,1,1} \stackrel{\pi}{\Longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ \forall p \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	$V^{5,2}$	$\frac{\mathrm{SO}(5)}{\mathrm{SO}(2)}$	SU(3)	$\begin{cases} \mathrm{V}^{5,2} \stackrel{\pi}{\Longrightarrow} M_a \sim \text{quadric in } \mathbb{P}^4 \\ \forall p \in M_a ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
3	$\mathrm{N}^{0,1,0}$	$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$	SU(2)	$\begin{cases} \mathrm{N}^{0,1,0} \stackrel{\pi}{\Longrightarrow} \mathbb{P}^2 \\ \forall p \in \mathbb{P}^2 ; \pi^{-1}(p) \sim \mathbb{S}^3 \\ \hline \mathrm{N}^{0,1,0} \stackrel{\pi}{\Longrightarrow} \frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)} \\ \forall p \in \frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)} ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$

Table 1: The homogeneous 7-manifolds that admit at least 2 Killing spinors are all sasakian or tri-sasakian. This is evident from the fibration structure of the 7-manifold, which is either a fibration in circles $\mathbb{S}^1$ for the $\mathscr{N} = 2$ cases or a fibration in $\mathbb{S}^3$ for the unique $\mathscr{N} = 3$ case corresponding to the $\mathrm{N}^{0,1,0}$ manifold. Since this latter is also an $\mathscr{N} = 2$ manifold, there is in addition the $\mathbb{S}^1$ fibration.

The setup of [13] shows that for general groups and general couplings the Chern Simons interactions break R-symmetry from $\mathrm{SO}(4)$ to $\mathrm{SO}(3)$ and consequently also supersymmetry from $\mathscr{N}_3 = 4$ to $\mathscr{N}_3 = 3$, as we already explained.

Yet one can try to specialize the theory in order to recover $\mathrm{SO}(4)$ R-symmetry and this is the main issue of the present paper.

Another important discovery was made by Gaiotto and Witten, always dealing with the case when the scalar multiplets span a flat target manifold of Kähler, HyperKähler or even more restricted holonomy. They found that the enhancement to $\mathscr{N} \geq 4$ supersymmetry implies also the existence of a Lie super-algebra $\mathfrak{G}$ whose bosonic part is the Lie algebra $\mathbb{G}$ of the gauge group $\mathscr{G}$. This issue was thoroughly investigated in [22]. The authors of this paper worked directly with the formulation of the super Chern Simons matter coupled theories obtained after the elimination of the non-dynamical fields and with the final superpotential written in terms of dynamical fields. They showed that the crucial issues for the supersymmetry enhancement are the following:

1. suitable choices of the gauge group $\mathscr{G}$ with its related Lie algebra $\mathbb{G}$ which is not necessarily semisimple, rather it typically also involves abelian $\mathfrak{u}(1)$ factors,

2. suitable choices of complex or symplectic linear representations $\mathscr{D}(\mathbb{G})$ to which the scalar multiplets are assigned,

3. a suitable choice of a non-degenerate, yet not positive definite $\mathscr{G}$-invariant metric $\mathfrak{m}_{\Lambda\Sigma}$ on the Lie algebra $\mathbb{G}$.

In all instances classified in [22], the above enumerated choices correspond to the embedding $\mathbb{G} \hookrightarrow \mathfrak{G}$ of the bosonic Lie algebra into a super-Lie algebra, the representations of the scalar multiplets being the same of the fermionic generators of $\mathfrak{G}$. In certain cases the metric $\mathfrak{m}$ is the restriction to the bosonic generators of the super Cartan-Killing metric of $\mathfrak{G}$.

This provides a challenging Occam's razor in the classification of supersymmetric Chern-Simons theories. Indeed, this brings us to another interesting feature discovered by Kapustin and Saulina [23]. These authors showed that the same Lie super-algebra $\mathfrak{G}$ can be used to construct a Chern-Simons supergauge theory, namely a pure Chern-Simons theory whose gauge group is the supergroup $\mathfrak{G}$. Quantizing such a topological theory à la BRST and introducing the ghosts for the fermionic part of the supergauge symmetry, after topological twist, these latter can be identified with the matter multiplets of the standard supersymmetric Chern Simons theory of the bosonic subalgebra $\mathbb{G} \subset \mathfrak{G}$ [1] . This relation between the $\mathscr{N}_3 = 4$ supersymmetric Chern-Simons theory and the supergroup Chern Simons one, described by Kapustin and Saulina, is somehow reminiscent of the relation between the Neveu-Schwarz and the Green-Schwarz formulations of superstrings, where one trades world volume supersymmetry for supersymmetry in the target space. Kapustin and Saulina advocated that the supergroup formulation is helpful to build supersymmetric Wilson-loops [25].

The supergroup Chern Simons formulation is well established in the flat scalar manifold case. Instead, what might be the relevant supergroup $\mathfrak{G}$ and what might be its role in $\mathscr{N}_3 = 4$ enhanced super Chern Simons theories on curved HyperKähler manifolds is not clear yet. This issue will be addressed in future publications [26].

Indeed, as already noticed, supersymmetric Chern-Simons theories were mostly constructed assuming that the scalar sector parameterizes a flat Kähler manifold which, in the $\mathscr{N} \geq 3$ has to be HyperKähler. More general cases with curved HyperKähler manifolds were only sketched. In the formulation of [17] the scalar fields parameterize a generic Kähler or HyperKähler manifold and the gauge group is the isometry group of such a manifold. In addition, one has suitable superpotential functions.

The goal of the present paper is to show that, within the more general setup of [17], where the hypermultiplets span generic HyperKähler manifolds HK_{2n}, Chern-Simons $\mathscr{N}_3 = 3$ gauge theories are enhanced to $\mathscr{N}_3 = 4$ supersymmetry, if and only if the tri-holomorphic moment-maps $\mathscr{P}^{\pm,3}_{\Lambda}$ of the HK_{2n} isometry group $\mathscr{G}$ (the gauge group)[2], satisfy the following differential-algebraic constraints:

$$\begin{aligned} \partial_i \left(\mathscr{P}^+ \cdot \mathscr{P}^+\right) &= \partial_{\bar{\ell}} \left(\mathscr{P}^- \cdot \mathscr{P}^-\right) = 0 \\ \partial_i \left(\mathscr{P}^+ \cdot \mathscr{P}^3\right) &= \partial_{\bar{\ell}} \left(\mathscr{P}^- \cdot \mathscr{P}^3\right) = 0 \end{aligned}$$

[1] In the work of [24] the reversed path, from supergroup theory, in the case Achucarro-Tonwnsed supergravity to supersymmetric Chern-Simons theory has been used

[2] Here we refer only to those isometries acting tri-holomorphically on the HyperKähler space.

$$\partial_{\bar{\ell}}\left(\mathscr{P}^{+}\cdot\mathscr{P}^{3}\right)=\partial_{i}\left(\mathscr{P}^{-}\cdot\mathscr{P}^{3}\right)=0 \tag{1.3}$$

together with

$$\partial_{i}\partial_{\bar{\ell}}\left(2\mathscr{P}^{3}\cdot\mathscr{P}^{3}-\mathscr{P}^{+}\cdot\mathscr{P}^{-}\right)=0. \tag{1.4}$$

In the above formulae the scalar product is taken with respect to the previously mentioned non-degenerate invariant metric $\mathfrak{m}_{\Lambda\Sigma}$, whose signature is not necessarily positive (or negative) definite.

Those above are a weaker formulation of the constraints introduced by Gaiotto and Witten that have the same appearance without derivatives.

Once the constraints (1.3-1.4) have been established the obvious question is *which examples do we know of non-trivial HyperKähler manifolds endowed with continuous isometries whose moment maps satisfy these constraints?*. The first example was noted by Kapustin and Saulina and it is provided by the time honored Eguchi Hanson space EH. This HyperKähler manifold is $T^{\star}\mathbb{P}^{1}$, namely the total space of the cotangent bundle to the one-dimensional complex projective space: $\mathbb{P}^{1}\sim\mathbb{S}^{2}$. The isometry group acting tri-holomorphically on the corresponding Ricci flat HyperKähler metric is $\mathrm{SU}(2)$ and in appendix A we review the appropriate calculation of its moment maps, showing that they satisfy the necessary constraints for enhancement.

Actually the Eguchi-Hanson manifold is the first in an infinite series of HyperKähler manifolds, i.e. the $T^{\star}\mathbb{P}^{n}$ manifolds, endowed with the Calabi HyperKähler metrics that were explicitly constructed in [27], using a Maurer Cartan differential form approach. Such a construction is reviewed and applied to the case of interest to us in section 4. Indeed we make the conjecture that the enhancement constraints (1.3-1.4) hold true for the $\mathrm{SU}(\mathrm{n}+1)$ isometry of the Calabi HyperKähler metric on $T^{\star}\mathbb{P}^{n}$ for all values of $n\in\mathbb{N}$ and in appendix C.1 we explicitly prove our conjecture for the case $n=2$.

The Calabi metric on $T^{\star}\mathbb{P}^{2}$ is not a randomly chosen case rather it has a profound physical relevance. Indeed it corresponds to the resolution of the conic singularity at the tip of the metric cone $\mathscr{C}\left(\mathrm{N}^{0,1,0}\right)$. As displayed in table 1, the coset manifold $\mathrm{N}^{0,1,0}$ is the unique tri-holomorphic, homogeneous Sasaki-Einstein manifold that exists in 7-dimensions. Somehow, as we already remarked above, $\mathrm{N}^{0,1,0}$ is the 7-dimensional analogue of the Sasaki-Einstein homogeneous manifold $\mathrm{T}^{1,1}$ in 5-dimensions. It leads to a compactification of M-theory on

$$\mathscr{M}_{11} = \mathrm{AdS}_{4}\times\mathrm{N}^{0,1,0} \tag{1.5}$$

which preserves $\mathscr{N}_{4}=3$ supersymmetry and whose Kaluza Klein spectrum was explicitly calculated and organized into $\mathrm{Osp}(3|4)\times\mathrm{SU}(3)$ supermultiplets in [9, 13]. In particular in [13] the Kaluza Klein spectrum was compared with the spectrum of conformal operators of a dual conformal field theory whose structure follows from the description of the metric cone $\mathscr{C}\left(\mathrm{N}^{0,1,0}\right)$ as a HyperKähler quotient of $\mathbb{C}^{3}\times\mathbb{C}^{3^{\star}}$ with respect to the tri-holomorphic action of a $\mathrm{U}(1)$ group. All this is just synoptic with the Klebanov–Witten construction of the conformal gauge theory dual to the $\mathrm{AdS}_{5}\times\mathrm{T}^{1,1}$ compactification of type IIB supergravity [11]. There the metric cone $\mathscr{C}\left(\mathrm{T}^{1,1}\right)$ is described as the Kähler quotient of $\mathbb{C}^{2}\times\mathbb{C}^{2^{\star}}$ with respect to the holomorphic action of a $\mathrm{U}(1)$ group. In this synopsis the smooth Calabi HyperKähler metric on $T^\star\mathbb{P}^2$ is the analogue of the Ricci flat Kähler metric on the conifold resolution constructed and discussed in [28, 29, 30].

In the HyperKähler quotient the level κ of the moment map plays the role of resolution parameter. For $\kappa = 0$ we have the singular metric cone, while for $\kappa \neq 0$ we obtain the Calabi metric on the smooth manifold $T^\star\mathbb{P}^2$. From the M2-brane gauge-theory viewpoint the $\mathrm{U}(1)$ gauge group (which becomes U(N) for N M2-branes) is the *color group*, while $\mathrm{SU}(3)$ is the global *flavor group*.

It is interesting to remark that if we do not gauge the flavor symmetry $\mathrm{SU}(3)$, the Chern Simons conformal gauge theory of the color group U(N) on the boundary of AdS_4 is, as discussed in [13], an $\mathcal{N}_3 = 3$ superconformal theory with Osp(3|4) symmetry. On the other hand if we gauge also the flavor group $\mathrm{SU}(3)$, we obtain an $\mathcal{N}_3 = 4$ superconformal Chern Simons theory with Osp(4|4) symmetry. Indeed, as we prove in appendix C.2, with a suitable choice of the metric $m^{\Lambda\Sigma}$, that we specify there, the case of the Lie algebra:

$$\mathbb{G} = \mathfrak{su}(3) \oplus \mathfrak{su}(N) \oplus \mathfrak{u}(1) \quad N \neq 3 \tag{1.6}$$

falls into the classification of [22] and the corresponding moment maps satisfy the enhancement constraints (1.3-1.4). Such a flavor-color conformal theory is of the flat HyperKähler type and hence, according to Kapustin and Saulina, it is equivalent to a gauged–fixed supergroup Chern-Simons theory[3], the supergroup being:

$$\mathfrak{G} = \mathfrak{su}(3|N) \tag{1.7}$$

Integrating out the color degrees of freedom in the *supersymmetric bosonic Chern Simons formulation* one obtains an $\mathcal{N}_3 = 4$ theory with gauge group the $\mathrm{SU}(3)$ flavor group and target space the Calabi HyperKähler manifold $T^\star\mathbb{P}^2$. What happens after an analogous integration in the equivalent *supergroup Chern-Simons formulation* is what we plan to explore in [26].

Our paper is organized as follows. Section 2 summarizes the general structure of $\mathcal{N}_3 = 3$ Chern-Simons gauge theory on curved scalar manifolds as geometrically formulated in [17]. Section 3 is the main core of the present article. Utilizing the appropriate quaternionic vielbein formalism for HyperKähler and Quaternionic Kähler manifolds introduced in [31] and systematically reviewed in [32] we show that we can rewrite the $\mathcal{N}_3 = 3$ Chern Simons theory in a manifestly $\mathcal{N}_3 = 4$ form à la Gaiotto–Witten whenever the weak constraints (1.3-1.4) are satisfied. Section 4 deals with the case of the HyperKähler Calabi metric on $T^\star\mathbb{P}^2$ and its relation with the $N^{0,1,0}$-compactification of M-theory. In subsections 4.1,4.2 we recall the HyperKähler quotient construction of the metric cone $\mathscr{C}\left(\mathrm{N}^{0,1,0}\right)$ and how it was used in [13] to determine the structure of the superconformal theory dual to the (1.5) compactification. In subsection 4.3 we discuss the resolution of the conifold singularity which we do in two different but equivalent ways: in subsection 4.4 we resolve the singularity uplifting the moment map to a non vanishing level κ in the HyperKähler quotient, while in subsection 4.5, following the approach of [27], we perform the direct construction of the Calabi HyperKähler metric utilizing the Maurer Cartan forms

[3]See also the recent paper on Supergravity Chern-Simons theory [24]

of SU(3) on the coset $N^{0,1,0}$. In particular in eq.s (4.48) and (4.51) we present the intrinsic components of the Riemann tensor and of the Usp(4) curvature 2-form $\mathbb{R}^{\alpha\beta}$ that, up to our knowledge, were not yet explicitly available in the literature. Section 5 presents in the utilized notations the explicit form of the *square integrable self-dual closed* $(2,2)$*-form* existing on $T^\star\mathbb{P}^2$ equipped with the Calabi metric [27]. This item is very important in order to construct M2-brane solutions of D=11 supergravity with an internal self-dual 4-form flux on the transverse space, which preserves half of the supersymmetries preserved by the fluxless solution [33]. Finally section 6 contains our conclusions.

The several appendices contain the details of lengthy calculations, in particular those of the moment maps on curved and flat spaces.

2 $\mathscr{N}_3 = 3$ supersymmetric Chern-Simons theories

$\mathscr{N}_3 = 3, D = 3$ Chern Simons gauge theories are just a particular subclass of $\mathscr{N}_3 = 2, D = 3$ Chern Simons field theory. Hence we start from the general form of the latter that was systematized in [17].

2.1 The Lagrangian of the $\mathscr{N}_3 = 2$ Chern Simons gauge theory

The lagrangian of $\mathscr{N}_3 = 2$ Chern-Simons Gauge Theory, as systematized in [17], takes the following form:

$$
\begin{aligned}
\mathscr{L}_{CSoff} \;=\; & -\alpha\,\mathrm{Tr}\left(\mathfrak{F}\wedge\mathscr{A}+\frac{2}{3}\mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A}\right)+\left(\frac{1}{2}g_{ij^\star}\,\Pi^{m|i}\,\nabla\overline{z}^{j^\star}+\overline{\Pi}^{m|j^\star}\,\nabla z^i\right)\wedge e^n\wedge e^p\,\varepsilon_{mnp} \\
& -\frac{1}{6}\,g_{ij^\star}\,\Pi^{m|i}\,\overline{\Pi}^{m|j^\star}\,e^r\wedge e^s\wedge e^t\,\varepsilon_{rst} \\
& +\mathrm{i}\frac{1}{2}\,g_{ij^\star}\left(\overline{\chi}^{j^\star}\,\gamma^m\,\nabla\chi^i+\overline{\chi}^i_c\,\gamma^m\,\nabla\chi^{i^\star}_c\right)\wedge e^n\wedge e^p\,\varepsilon_{mnp} \\
& +\Big(-\frac{1}{3}M^\Lambda\left(\partial_i k^j_\Lambda\,g_{j\ell^\star}\,\overline{\chi}^{\ell^\star}\,\chi^i-\partial_{i^\star}k^{j^\star}_\Lambda\,g_{j\ell^\star}\overline{\chi}^\ell_c\,\chi^{i^\star}_c\right)+\frac{\alpha}{3}\left(\overline{\lambda}^\Lambda\,\lambda^\Sigma+\overline{\lambda}^\Lambda_c\,\lambda^\Sigma_c\right)\mathfrak{m}_{\Lambda\Sigma} \\
& +\mathrm{i}\,\frac{1}{3}\left(\overline{\chi}^{j^\star}_c\,\lambda^\Lambda\,k^i_\Lambda-\overline{\chi}^i_c\,\lambda^\Lambda\,k^{j^\star}_\Lambda\right)g_{ij^\star}+\frac{1}{6}\left(\partial_i\partial_j\mathscr{W}\,\overline{\chi}^i_c\,\chi^j+\partial_{i^\star}\partial_{j^\star}\overline{\mathscr{W}}\,\overline{\chi}^{i^\star}\,\chi^{j^\star}_c\right) \\
& -V(M,D,\mathscr{H},z,\overline{z})+\mathfrak{a}\mathscr{R}_{ij^\star kl^\star}\overline{\chi}^{j^\star}\,\chi^i\overline{\chi}^k_c\chi^{j^\star}_c\Big)\,\varepsilon_{mnp}\,e^m\wedge e^n\wedge e^p
\end{aligned}
\tag{2.1}
$$

where:

1. The complex scalar fields z^i span a Kähler manifold $\mathscr{M}_K$, $g_{ij^\star}$ denoting its Kähler metric, $\mathscr{R}_{ij^\star kl^\star}$ denoting its curvature 2-form. The coefficient $\mathfrak{a}$ is fixed by supersymmetry.
2. $\Pi^{m|i}$ are auxiliary fields that are identified with the world volume derivatives of the scalar z^i by their own equation of motion.
3. The one–forms e^m denote the dreibein of the world volume.
4. $\mathscr{A}^\Lambda$ is the gauge-one form of the gauge group $\mathscr{G}$.

5. λ^Λ are the gauginos, namely the spin $\frac{1}{2}$ partners of the gauge bosons $\mathscr{A}^\Lambda$

6. χ^i are the chiralinos, namely the spin $\frac{1}{2}$ partners of the Wess-Zumino scalars z^i.

7. M^Λ are the real scalar fields in the adjoint of the gauge group that complete the gauge multiplet together with the gauginos and the gauge bosons.

8. $\mathscr{W}(z)$ is the superpotential.

9. k^i_Λ are the Killing vectors of the Kähler metric of $\mathscr{M}_K$, associated with the generators of the gauge group.

10. $\mathfrak{m}^{\Lambda\Sigma} = \mathfrak{m}^{\Sigma\Lambda}$ denotes a non degenerate, $\mathscr{G}$-invariant metric on the Lie Algebra $\mathbb{G}$, which is not necessarily semisimple. The metric $\mathfrak{m}^{\Lambda\Sigma}$ is not necessarily positive-definite and as a consequence the scalar potential is not necessarily positive definite.

11. The coeffcient $\mathfrak{a}$, which we do not calculate since we do not need it, is fixed by supersymmetry invariance of $\mathscr{L}_{CSoff}$.

The scalar potential in terms of physical and auxiliary fields is the following one:

$$\begin{aligned} V(M,D,\mathscr{H},z,\bar{z}) &= \left(\frac{\alpha}{3}M^\Lambda \mathfrak{m}_{\Lambda\Sigma} - \frac{1}{6}\mathscr{P}_\Sigma(z,\bar{z}) + \frac{1}{6}\zeta_I\,\mathfrak{C}^I_\Sigma\right) D^\Sigma + \frac{1}{6}M^\Lambda M^\Sigma k^i_\Lambda k^{j^\star}_\Sigma\, g_{ij^\star} \\ &\quad + \frac{1}{6}\left(\mathscr{H}^i\,\partial_i\mathscr{W} + \mathscr{H}^{\ell^\star}\,\partial_{\ell^\star}\overline{\mathscr{W}}\right) - \frac{1}{6}g_{i\ell^\star}\,\mathscr{H}^i\,\mathscr{H}^{\ell^\star} \end{aligned} \tag{2.2}$$

where $\mathscr{P}_\Sigma(z,\bar{z})$ are the moment maps associated with each generator of the gauge-group, ζ_I are the Fayet-Iliopoulos parameters associated with each generator of the center of the gauge Lie algebra, $\mathscr{H}^i$ are the complex auxiliary fields of the Wess-Zumino multiplets and D^Λ are the auxiliary scalars of the vector multiplets. By $\mathfrak{C}^I_\Sigma$ we denote the projector onto a basis of generators of the Lie Algebra center $\mathfrak{z}[\mathbb{G}]$.

In these theories the gauge multiplet does not propagate and it is essentially made of lagrangian multipliers for certain constraints. Indeed, the auxiliary fields, the gauginos and the vector multiplet scalars have algebraic field equations so that they can be eliminated by solving such equations of motion. The vector multiplet auxiliary scalars D^Λ appear only as lagrangian multipliers of the constraint:

$$M^\Lambda = \frac{1}{2\alpha}\mathfrak{m}^{\Lambda\Sigma}\left(\mathscr{P}_\Sigma - \zeta_I\,\mathfrak{C}^I_\Sigma\right) \tag{2.3}$$

while the variation of the auxiliary fields $\mathscr{H}^{j^\star}$ of the Wess Zumino multiplets yields:

$$\mathscr{H}^i = g^{ij^\star}\,\partial_{j^\star}\overline{\mathscr{W}} \quad ; \quad \overline{\mathscr{H}}^{j^\star} = g^{ij^\star}\,\partial_i\mathscr{W} \tag{2.4}$$

On the other hand, the equation of motion of the field M^Λ implies:

$$D^\Lambda = -\frac{1}{\alpha}\mathfrak{m}^{\Lambda\Gamma}g_{ij^\star}\,k^i_\Gamma k^{j^\star}_\Sigma\,M^\Sigma = -\frac{1}{2\alpha^2}\,g_{ij^\star}\,\mathfrak{m}^{\Lambda\Gamma}\,k^i_\Gamma k^{j^\star}_\Sigma\,\mathfrak{m}^{\Sigma\Delta}\left(\mathscr{P}_\Delta - \zeta_I\,\mathfrak{C}^I_\Delta\right) \tag{2.5}$$

which finally resolves all the auxiliary fields in terms of functions of the physical scalars.

Upon use of both constraints (2.3) and (2.4) the scalar potential takes the following positive definite form:

$$
\begin{aligned}
V(z,\bar{z}) &= \frac{1}{6}\left(\partial_i \mathscr{W}\, \partial_{j^\star}\overline{\mathscr{W}}\, g^{ij^\star} + \mathbf{m}^{\Lambda\Sigma}\left(\mathscr{P}_\Lambda - \zeta_I\, \mathfrak{C}^I_\Lambda\right)\left(\mathscr{P}_\Sigma - \zeta_J\, \mathfrak{C}^J_\Sigma\right)\right) \\
\mathbf{m}^{\Lambda\Sigma}(z,\bar{z}) &\equiv \frac{1}{4\alpha^2}\, \mathfrak{m}^{\Lambda\Gamma}\, \mathfrak{m}^{\Sigma\Delta}\, k^i_\Gamma\, k^{j^\star}_\Delta\, g_{ij^\star}
\end{aligned}
\qquad (2.6)
$$

In a similar way the gauginos can be resolved in terms of the chiralinos:

$$
\lambda^\Lambda = -\frac{1}{2\alpha}\, \mathfrak{m}^{\Lambda\Sigma}\, g_{ij^\star}\, \chi^i\, k^{j^\star}_\Sigma \quad ; \quad \lambda^\Lambda_c = -\frac{1}{2\alpha}\, \mathfrak{m}^{\Lambda\Sigma}\, g_{ij^\star}\, \chi^{j^\star}\, k^i_\Sigma \qquad (2.7)
$$

In this way if we were able to eliminate also the gauge one form $\mathscr{A}$, the Chern-Simons gauge theory would reduce to a theory of Wess-Zumino multiplets with additional interactions. The elimination of $\mathscr{A}$, however, is not possible in the nonabelian case and it is possible in the abelian case only through duality nonlocal transformations. This is the corner where interesting nonperturbative dynamics is hidden.

2.2 The structure of $\mathscr{N}_3 = 3$ Chern Simons gauge theories

The $\mathscr{N}_3 = 3$ case is just a particular case in the class of theories described in the previous section since a theory with $\mathscr{N}_3 = 3$ SUSY, must *a fortiori* be an $\mathscr{N}_3 = 2$ theory. In [8] , the case of $\mathscr{N}_3 = 4$ theories was also considered, within the $\mathscr{N}_3 = 2$ class. These latter are obtained through dimensional reduction of an $\mathscr{N}_4 = 2$ theory in four–dimensions. The main issue in such a dimensional reduction is the enhancement of the $D = 4$ R-symmetry, which is USp(2) to SO(4) in $D = 3$. Indeed, since each $D = 4$ Majorana spinor splits, under dimensional reduction on a circle $\mathbb{S}^1$, into two $D = 3$ Majorana spinors, the number of three–dimensional supercharges is just twice the number of $D = 4$ supercharges:

$$
\mathscr{N}_3 = 2 \times \mathscr{N}_4 \qquad (2.8)
$$

The mechanism of such enhancement of R-symmetry is analyzed in detail in Appendix D. Such analysis is quite relevant to the main issue of the present paper which is the retrieval of the $\mathfrak{so}(4)$ R-symmetry algebra naturally produced by the dimensional reduction when special conditions are satisfied by the hypermultiplet interactions. In the absence of such conditions the $D = 3$ R-symmetry being instead reduced to $\mathfrak{so}(3)$ by Chern Simons interaction.

Indeed the $\mathscr{N}_3 = 3$ case corresponds to an intermediate situation. It is an $\mathscr{N}_3 = 2$ theory with the field content of an $\mathscr{N}_3 = 4$ one, but with additional $\mathscr{N}_3 = 2$ interactions that respect three out of the four supercharges obtained through dimensional reduction. Using an $\mathscr{N}_3 = 2$ superfield formalism and the notion of twisted chiral multiplets it was shown in [34] that for abelian gauge theories these additional $\mathscr{N}_3 = 3$ interactions are

1. A Chern Simons term, with coefficient α

2. A mass-term with coefficient $\mu = \alpha$ for the chiral field Y^Λ in the adjoint of the color gauge group. By this latter we denote the complex field belonging, in four dimensions, to the $\mathcal{N}_4 = 2$ gauge vector multiplet.

In [13] the authors retrieved for non-abelian gauge theories the same result as that found by the authors of [34] for abelian theories. In [13] the construction was presented in the component formalism which is better suited to discus the relation between the world–volume gauge theory and the geometry of the transverse cone $\mathcal{C}(\mathcal{M}_7)$. Let us also remark that the arguments used in [15] are the same which were spelled out ten years earlier in [13]. In this section we summarize in the more general notations of [17], based on HyperKähler metrics and the tri-holomorphic moment maps, the general form of a non abelian $\mathcal{N}_3 = 3$ Chern Simons gauge theory in three dimensions as it was obtained in [13].

2.2.1 The field content and the interactions

The strategy of [13] was that of writing the $\mathcal{N}_3 = 3$ gauge theory as a special case of an $\mathcal{N}_3 = 2$ theory, whose general form was discussed in the previous section. For this latter the field content is given by:

multipl. type / SO(1,2) spin	1	$\frac{1}{2}$	0
vector multipl.	$\underbrace{A^\Lambda_\mu}_{\text{gauge field}}$	$\underbrace{\left(\lambda^{+\Lambda}, \lambda^{-\Lambda}\right)}_{\text{gauginos}}$	$\underbrace{M^\Lambda}_{\text{real scalar}}$
chiral multip.		$\underbrace{\left(\chi^{+i}, \chi^{-i^*}\right)}_{\text{chiralinos}}$	$\underbrace{z^i,\ \overline{z}^{i^*}}_{\text{complex scalars}}$

(2.9)

and the complete Lagrangian was given in the previous sections. In particular the complete Chern Simons Lagrangian before the elimination of the auxiliary fields was displayed in eq.(2.1).

The Chern-Simons $\mathcal{N}_3 = 3$ case is obtained when the following conditions are fulfilled:

- The spectrum of chiral multiplets is made of $\dim\mathcal{G} + 2n$ complex fields arranged in the following way

$$z^i = \begin{cases} Y^\Lambda &= \text{ complex fields} \quad \text{in the } \textbf{adjoint rep. of the color group} \\ q^j &= \begin{pmatrix} u^a \\ v_b \end{pmatrix} \quad \begin{cases} 2n \text{ complex fields spanning a } \textbf{HyperKähler manifold } HK_{2n} \\ \text{which is invariant under a} \\ \textbf{triholomorphic action} \text{ of the gauge group } \mathcal{G} \end{cases} \end{cases} . \tag{2.10}$$

- the Kähler potential has the following form:

$$\mathcal{K}(Y,u,v) = \widehat{\mathcal{K}}(u,v) \tag{2.11}$$

where $\widehat{\mathscr{K}}(u,v)$ is the Kähler potential of the Ricci-flat HyperKähler metric of the HyperKähler manifold HK_{2n}. The assumption that $\mathscr{K}(Y,u,v)$ does not depend on Y^Λ implies that the kinetic term of these scalars vanishes turning them into auxiliary fields that can be integrated away.

- The superpotential $\mathscr{W}(z)$ has the following form:

$$\mathscr{W}(Y,u,v) = \mathfrak{m}^{\Lambda\Sigma}\left(Y_\Lambda\,\mathscr{P}^+_\Sigma(u,v) + 2\,\alpha\,Y_\Lambda Y_\Sigma\right) \tag{2.12}$$

  where $\mathscr{P}^+_\Sigma(u,v)$ denotes the holomorphic part of the triholomorphic moment map induced by the triholomorphic action of the color group on HK_{2n}.

The reason why these two choices make the theory $\mathscr{N}_3 = 3$ invariant is simple: the first choice corresponds to assuming the field content of an $\mathscr{N}_3 = 4$ theory which is necessary since $\mathscr{N}_3 = 3$ and $\mathscr{N}_3 = 4$ supermultiplets are identical. The second choice takes into account that the metric of the hypermultiplets must be HyperKähler and that the gauge coupling constant was sent to infinity. The third choice introduces an interaction that preserves $\mathscr{N}_3 = 3$ supersymmetry but breaks (when $\alpha \neq 0$) $\mathscr{N}_3 = 4$ supersymmetry.

Going back to the off-shell Chern Simons lagrangian given in eq.(2.1) one can perform the elimination of the auxiliary fields that now include $Y^\Lambda, D^\Lambda, M^\Lambda, \mathscr{H}^i$ at the bosonic level and the gauginos $\lambda^\Lambda, \lambda^\Lambda_c, \chi^\Lambda, \chi^\Lambda_c$ at the fermionic level (note that there are two more non propagating gauginos coming from the chiral multiplet in the adjoint representation of the gauge group). We do not enter the details of the integration over the non propagating fermions and we just consider the bosonic lagrangian emerging from the integration over the auxiliary bosonic fields. The first integration to perform is that over the auxiliary field $\mathscr{H}^\Lambda$. This is simply the lagrangian multiplier of the constraint:

$$\partial_\Lambda \mathscr{W} = 0 \quad \Rightarrow \quad Y_\Lambda = \frac{1}{4\alpha}\,\mathscr{P}^+_\Lambda(u,v) \tag{2.13}$$

Substituting this back into the lagrangian yields a potential with the same structure as that in eq.(2.6) but with a modified superpotential which becomes quadratic in the holomorphic momentum maps:

$$\begin{aligned} V(u,v) &= \frac{1}{6}\left(\partial_i \mathfrak{W}\,\partial_{j^\star}\overline{\mathfrak{W}}\,g^{ij^\star} + \mathbf{m}^{\Lambda\Sigma}\,\mathscr{P}^3_\Lambda\,\mathscr{P}^3_\Sigma\right) \\ \mathbf{m}^{\Lambda\Sigma}(u,v) &\equiv \frac{1}{4\alpha^2}\,\mathfrak{m}^{\Lambda\Gamma}\,\mathfrak{m}^{\Sigma\Delta}\,k^i_\Gamma\,k^{j^\star}_\Delta\,g_{ij^\star} \qquad (2.14) \\ \mathfrak{W} &= -\frac{1}{8\alpha}\,\mathscr{P}^+_\Lambda\,\mathscr{P}^+_\Sigma\,\mathfrak{m}^{\Lambda\Sigma} \qquad (2.15) \end{aligned}$$

here by $\mathfrak{W}$ we mean the on-shell superpotential. Altogether the supersymmetric lagrangian of the $\mathscr{N}_3 = 3$ Chern Simons theory, after gaussian integration of the non propagating fields, takes the following form[4]:

$$\mathscr{L}_{CSon} = -\alpha\,\mathrm{Tr}\left(\mathfrak{F}\wedge\mathscr{A} + \frac{1}{3}\mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A}\right) + \frac{1}{6}g_{ij^\star}\,\nabla^m q^i\,\nabla_m \overline{q}^{j^\star}\,e^r\wedge e^s\wedge e^t\,\varepsilon_{rst}$$

[4]Here and in the sequel, we do not take care of the quartic fermionic interaction which is not relevant in our discussion.

$$
\begin{aligned}
&+\mathrm{i}\frac{1}{2}g_{ij^\star}\left(\overline{\chi}^{j^\star}\gamma^m\nabla\chi^i+\overline{\chi}^i_c\gamma^m\nabla\chi^{j^\star}_c\right)\wedge e^n\wedge e^p\,\varepsilon_{mnp}\\
&+\left(\frac{\mathrm{i}}{12\alpha}\left(\partial_\ell k^\Lambda_{j^\star}\overline{\chi}^{j^\star}\chi^\ell-\partial_{\ell^\star}k^\Lambda_j\overline{\chi}^j_c\chi^{\ell^\star}_c\right)\mathscr{P}^3_\Lambda-\frac{1}{6\alpha}\mathfrak{m}_{\Lambda\Sigma}\left(\overline{\chi}^{\ell^\star}k^\Lambda_{\ell^\star}\chi^j k^\Sigma_j+\overline{\chi}^j_c k^\Lambda_j\chi^{\ell^\star}_c k^\Sigma_{\ell^\star}\right)\right.\\
&+\frac{1}{6}\left(\partial_i\partial_j\mathfrak{W}\overline{\chi}^i_c\chi^j+\partial_{i^\star}\partial_{j^\star}\overline{\mathfrak{W}}\overline{\chi}^{i^\star}\chi^{j^\star}_c\right)+\frac{1}{6}\partial_i\mathfrak{W}\partial_{j^\star}\overline{\mathfrak{W}}g^{ij^\star}\\
&\left.+\frac{1}{24\alpha^2}\mathfrak{m}^{\Lambda\Gamma}\mathfrak{m}^{\Sigma\Delta}k^i_\Gamma k^{j^\star}_\Delta g_{ij^\star}\mathscr{P}^3_\Lambda\mathscr{P}^3_\Sigma\right)\wedge e^m\wedge e^n\wedge e^p\,\varepsilon_{mnp} \qquad (2.16)
\end{aligned}
$$

3 HyperKähler manifolds in the hypermultiplet sector and the supersymmetry enhancement

Given the above result we take the following two steps:

a) Still maintaining full generality we try to rearrange the items contained in the lagrangian (2.16) in such a way as to bring into evidence the HyperKähler structure of the scalar manifold and its holonomy group.

b) Next we introduce the constraints (1.3-1.4) on the moment maps and we show that when they hold true the lagrangian (2.16) can be further elaborated in such a way as to become structurally similar to the Lagrangian of the Gaiotto-Witten theory [21]. In this way the R-symmetry and the supersymmetry enhancements are revealed for general HyperKähler manifolds (curved ones included), whose tri-holomorphic isometries satisfy the constraint (1.3-1.4) at the level of their moment maps. Note also, that, as we already stressed, equations (1.3-1.4) encode weaker constraints with respect to those so far discussed in the literature.

Let us start with our programme.

3.1 Quaternionic vielbein for HyperKähler manifolds and moment maps

Following the notations of [31] we recall that a HyperKähler manifold HK_{2n} is a $4n$-dimensional real manifold endowed with a metric h:

$$
ds^2=h_{uv}(q)dq^u\otimes dq^v \quad ; \quad u,v=1,\dots,4m \qquad (3.1)
$$

and three complex structures

$$
(\mathbf{J}^x):\ T(HK_{2n})\longrightarrow T(HK_{2n}) \qquad (x=1,2,3) \qquad (3.2)
$$

that satisfy the quaternionic algebra

$$
\mathbf{J}^x\mathbf{J}^y=-\delta^{xy}\mathbf{1}+\varepsilon^{xyz}\mathbf{J}^z \qquad (3.3)
$$

and respect to which the metric is hermitian:

$$
\forall\mathbf{X},\mathbf{Y}\in THK_{2n}:\quad h\left(\mathbf{J}^x\mathbf{X},\mathbf{J}^x\mathbf{Y}\right)=h\left(\mathbf{X},\mathbf{Y}\right) \qquad (x=1,2,3) \qquad (3.4)
$$

From eq.(3.4) it follows that one can introduce a triplet of 2-forms

$$\mathbf{K}^x = \mathbf{K}^x_{uv} dq^u \wedge dq^v \quad ; \quad \mathbf{K}^x_{uv} = h_{uw} (\mathbf{J}^x)^w_v \tag{3.5}$$

that provides the generalization of the concept of Kähler form occurring in the complex case. The triplet $\mathbf{K}^x$ is named the *HyperKähler* form. It is an $\mathrm{SU}(2)$ Lie–algebra valued 2–form in the same way as the Kähler form is a $\mathrm{U}(1)$ Lie–algebra valued 2–form. The space is HyperKähler if the 2-forms in this triplet are all closed:

$$\mathrm{d}\mathbf{K}^x = 0 \tag{3.6}$$

As a consequence of the above structure the manifold HK_{2n} has a holonomy group of the following type:

$$\begin{aligned} \mathrm{Hol}(HK_{2n}) &= \mathbf{1} \otimes \mathscr{H} \quad \text{(HyperKähler)} \\ \mathscr{H} &\subset \mathrm{Usp(2n)} \end{aligned} \tag{3.7}$$

Hence introducing flat indices $\{A, B, C = 1, 2\}, \{\alpha, \beta, \gamma = 1, .., 2n\}$ that run, respectively, in the fundamental representations of SU(2) and USp(2n), we can find a vielbein 1-form

$$\mathscr{U}^{A\alpha} = \mathscr{U}^{A\alpha}_u(q) dq^u \tag{3.8}$$

such that

$$h_{uv} = \mathscr{U}^{A\alpha}_u \mathscr{U}^{B\beta}_v \mathbb{C}_{\alpha\beta} \varepsilon_{AB} \tag{3.9}$$

where $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ and $\varepsilon_{AB} = -\varepsilon_{BA}$ are, respectively, the flat USp(2n) and $\mathrm{USp}(2) \sim \mathrm{SU}(2)$ invariant metrics. The vielbein $\mathscr{U}^{A\alpha}$ is covariantly closed with respect to a flat SU(2)-connection ω^z:

$$d\omega^x + \tfrac{1}{2} \varepsilon^{xyz} \omega^y \wedge \omega^z = 0 \tag{3.10}$$

and to some USp(2n)-Lie Algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$:

$$\begin{aligned} \nabla \mathscr{U}^{A\alpha} &\equiv d\mathscr{U}^{A\alpha} + \frac{i}{2} \omega^x (\varepsilon \sigma_x \varepsilon^{-1})^A{}_B \wedge \mathscr{U}^{B\alpha} \\ &+ \Delta^{\alpha\beta} \wedge \mathscr{U}^{A\gamma} \mathbb{C}_{\beta\gamma} = 0 \end{aligned} \tag{3.11}$$

where $(\sigma^x)_A{}^B$ are the standard Pauli matrices. For them we utilize the conventions shown in formula (D.7) and we set $\varepsilon_{AB} = i\sigma_2$. Furthermore $\mathscr{U}^{A\alpha}$ satisfies the reality condition:

$$\mathscr{U}_{A\alpha} \equiv (\mathscr{U}^{A\alpha})^* = \varepsilon_{AB} \mathbb{C}_{\alpha\beta} \mathscr{U}^{B\beta} \tag{3.12}$$

Eq.(3.12) defines the rule to lower the symplectic indices by means of the flat symplectic metrics ε_{AB} and $\mathbb{C}_{\alpha\beta}$. More specifically we can write a stronger version of eq.(3.9):

$$
\begin{aligned}
(\mathscr{U}_u^{A\alpha}\mathscr{U}_v^{B\beta} + \mathscr{U}_v^{A\alpha}\mathscr{U}_u^{B\beta})\mathbb{C}_{\alpha\beta} &= h_{uv}\varepsilon^{AB} \\
(\mathscr{U}_u^{A\alpha}\mathscr{U}_v^{B\beta} + \mathscr{U}_v^{A\alpha}\mathscr{U}_u^{B\beta})\varepsilon_{AB} &= h_{uv}\frac{1}{n}\mathbb{C}^{\alpha\beta}
\end{aligned}
\tag{3.13}
$$

We have also the inverse vielbein $\mathscr{U}^u_{A\alpha}$ defined by the equation

$$
\mathscr{U}^u_{A\alpha}\mathscr{U}_v^{A\alpha} = \delta^u_v \tag{3.14}
$$

Flattening a pair of indices of the Riemann tensor $\mathscr{R}^{uv}{}_{ts}$ we obtain

$$
\mathscr{R}^{uv}{}_{ts}\mathscr{U}_u^{\alpha A}\mathscr{U}_v^{\beta B} = \mathbb{R}^{\alpha\beta}_{ts}\varepsilon^{AB} \tag{3.15}
$$

where $\mathbb{R}^{\alpha\beta}_{ts}$ is the field strength of the USp(2n) connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$:

$$
d\Delta^{\alpha\beta} + \Delta^{\alpha\gamma}\wedge\Delta^{\delta\beta}\mathbb{C}_{\gamma\delta} \equiv \mathbb{R}^{\alpha\beta} = \mathbb{R}^{\alpha\beta}_{ts}dq^t\wedge dq^s \tag{3.16}
$$

Eq. (3.15) is the explicit statement that the Levi Civita connection associated with the metric h has a holonomy group contained in $\mathbf{1}\otimes$ USp(2n). Consider now eq.s (3.3,3.5). We easily deduce the following relation:

$$
h^{st}\mathbf{K}^x_{us}\mathbf{K}^y_{tw} = -\delta^{xy}h_{uw} + \varepsilon^{xyz}\mathbf{K}^z_{uw} \tag{3.17}
$$

Eq.(3.17) implies that the intrinsic components of the HyperKähler 2-forms $\mathbf{K}^x$ yield a representation of the quaternion algebra. Hence we can write:

$$
\mathbf{K}^x = \frac{\mathrm{i}}{2}\mathbb{C}_{\alpha\beta}(\sigma^x)_{AB}\mathscr{U}^{\alpha A}\wedge\mathscr{U}^{\beta B} \tag{3.18}
$$

where the second index of the Pauli matrix has been lowered with ε_{BC}.

Recalling now that a HyperKähler manifold is also a complex Kähler manifold we can introduce complex coordinates and vielbein with respect to a reference complex structure that we choose to be that associated with $\mathbf{K}^z$. Than $\mathbf{K}^z$ is the Kähler 2-form of HK_{2n} and we have:

$$
\begin{aligned}
\mathbf{K}^z &= i\,g_{ij^\star}\,dz^i\wedge d\bar{z}^{j^\star} = i\,\mathbf{e}^\alpha\wedge\bar{\mathbf{e}}_\alpha \\
&= i\,\mathscr{U}^{1\alpha}\wedge\mathscr{U}^{2\beta}\mathbb{C}_{\alpha\beta}
\end{aligned}
\tag{3.19}
$$

where $\mathbf{e}^\alpha$ is a set of complex vielbein one-forms such that:

$$
ds^2 = g_{ij^\star}\,dz^i\otimes d\bar{z}^{j^\star} = \sum_{\alpha=1}^{n}\mathbf{e}^\alpha\otimes\bar{\mathbf{e}}_\alpha \quad ; \quad \bar{\mathbf{e}}_\alpha \equiv \left(\mathbf{e}^\alpha\right)^\star \tag{3.20}
$$

Utilizing our basis of Pauli matrices we also find:

$$\mathbf{K}^{\pm} = \mathbf{K}^{x} \pm i\mathbf{K}^{y} \quad ; \quad \mathbf{K}^{+} = -i\mathscr{U}^{1\alpha}\wedge\mathscr{U}^{1\beta}\mathbb{C}_{\alpha\beta} \quad ; \quad \mathbf{K}^{-} = i\mathscr{U}^{2\alpha}\wedge\mathscr{U}^{2\beta}\mathbb{C}_{\alpha\beta} \tag{3.21}$$

Once the complex vielbein $\mathbf{e}^{\alpha}$ are found there is a universal way of writing the quaternionic vielbein $\mathscr{U}^{A\alpha}$ so that eq.s(3.19-3.21) are satisfied, namely:

$$\mathscr{U}^{1\alpha} = \mathbf{e}^{\alpha} \quad ; \quad \mathscr{U}^{2\beta} = \mathbb{C}^{\alpha\beta}\,\overline{\mathbf{e}}^{\beta} \tag{3.22}$$

In this way we get:

$$\mathbf{K}^{+} = -i\,\mathbf{e}^{\alpha}\wedge\mathbf{e}^{\beta}\,\mathbb{C}_{\alpha\beta} \quad ; \quad \mathbf{K}^{-} = i\,\overline{\mathbf{e}}_{\alpha}\wedge\overline{\mathbf{e}}_{\beta}\,\mathbb{C}^{\alpha\beta} \tag{3.23}$$

The above structure is very useful for the calculation of the relation between Killing vectors of an isometry group $\mathscr{G}$ of the HyperKähler metric and their associated moment maps. Let us denote $\mathbf{k}_{\Lambda}$ such Killing vectors closing the Lie algebra $\mathbb{G}$, whose structure constants we denote $f^{\Lambda}{}_{\Gamma\Delta}$ as usual:

$$\begin{aligned} \mathbf{k}_{\Lambda} &= k^{i}_{\Lambda}\,\partial_{i} + k^{i^\star}_{\Lambda}\,\partial_{i^\star} \\ [\mathbf{k}_{\Gamma}\,,\mathbf{k}_{\Delta}] &= f^{\Lambda}{}_{\Gamma\Delta}\,\mathbf{k}_{\Lambda} \end{aligned} \tag{3.24}$$

Utilizing the complex vielbein:

$$\mathbf{e}^{\alpha} = e^{\alpha}_{i}\,dz^{i} + e^{\alpha}_{i^\star}\,d\overline{z}^{i^\star} \quad ; \quad \overline{\mathbf{e}}_{\alpha} = \overline{e}_{\alpha\,i}\,dz^{i} + \overline{e}_{\alpha\,i^\star}\,dz^{i^\star} \tag{3.25}$$

it is convenient to introduce the flat components of the Killing vectors:

$$k^{\alpha}_{\Lambda} = e^{\alpha}_{i}\,k^{i}_{\Lambda} + e^{\alpha}_{i^\star}\,k^{i^\star}_{\Lambda} \quad ; \quad k_{\alpha,\Lambda} = \overline{e}_{\alpha\,i}k^{i}_{\Lambda} + \overline{e}_{\alpha\,i^\star}\,k^{i^\star}_{\Lambda} \tag{3.26}$$

and from the definition of the tri-holomorphic moment maps:

$$\mathbf{i}_{\Lambda}\mathbf{K}^{x} = -\,\mathrm{d}\mathscr{P}^{x}_{\Lambda} \tag{3.27}$$

we obtain[5]:

$$\begin{aligned} k^{\alpha}_{\Lambda} &= i\partial^{\alpha}\,\mathscr{P}^{3}_{\Lambda} = -\frac{i}{2}\,\mathbb{C}^{\alpha\beta}\,\partial_{\beta}\,\mathscr{P}^{+}_{\Lambda} \\ k_{\Lambda\alpha} &= -i\partial_{\alpha}\,\mathscr{P}^{3}_{\Lambda} = \frac{i}{2}\,\mathbb{C}_{\alpha\beta}\,\partial^{\beta}\,\mathscr{P}^{-}_{\Lambda} \end{aligned} \tag{3.28}$$

[5]As usual we denote anholonomic derivatives $\partial_{\alpha} = \mathbf{e}^{i}_{\alpha}\partial_{i} + \mathbf{e}^{j^\star}_{\alpha}\,\partial_{j^\star}$ where $\mathbf{e}^{i}_{\alpha}$ is the inverse vielbein. ∂^{α} is the complex conjugate of ∂_{α}.

3.2 Making the R-symmetry exiplicit and the enhancement

In this new frame we can make R-symmetry explicit and we can conveniently study its enhancement. The scalar kinetic term will involves the gauged quaternionic vielbein

$$\frac{1}{24}\mathbb{C}_{\alpha\beta}\varepsilon_{AB}\langle\hat{\mathscr{U}}^{A\alpha},\hat{\mathscr{U}}^{B\beta}\rangle e^r\wedge e^s\wedge e^t\varepsilon_{rst} \tag{3.29}$$

where

$$\begin{aligned}
\langle\hat{\mathscr{U}}^{A\alpha},\hat{\mathscr{U}}^{B\beta}\rangle &= \eta^{mn}\hat{\mathscr{U}}^{A\alpha}_m\hat{\mathscr{U}}^{A\alpha}_n &(3.30)\\
\hat{\mathscr{U}}^{A\alpha} &= \mathscr{U}^{A\alpha}_i\nabla q^i+\mathscr{U}^{A\alpha}_{j^\star}\nabla\overline{q}^{j^\star} &(3.31)\\
\nabla q^i &= dq^i+k^i_\Lambda\mathscr{A}^\Lambda=\nabla_m q^i dx^m &(3.32)
\end{aligned}$$

The fermionic kinetic term can be rewritten in an $\mathrm{SU}(2)$ invariant form:

$$\frac{\mathrm{i}}{2}\chi_{\dot{A}\alpha}\gamma^m\nabla\chi^{\dot{A}\alpha}\wedge e^n\wedge e^p\varepsilon_{mnp} \tag{3.33}$$

where

$$g_{ij^\star}=\mathbb{C}_{\alpha\beta}\mathscr{U}^{1\alpha}_i\mathscr{U}^{2\beta}_{j^\star}=\mathbb{C}_{\alpha\beta}\mathscr{U}^{1\alpha}_{j^\star}\mathscr{U}^{2\beta}_i \tag{3.34}$$

$$\begin{aligned}
\left\{\chi^{\dot{A}\alpha}\right\} &= \left\{\chi^i\mathscr{U}^{1\alpha}_i,\chi^{j^\star}_c\mathscr{U}^{2\alpha}_{j^\star}\right\}\\
\left\{\chi_{\dot{A}\alpha}\right\} &= \mathbb{C}_{\alpha\beta}\left\{\overline{\chi}^{j^\star}\mathscr{U}^{2\beta}_{j^\star},-\overline{\chi}^i_c\mathscr{U}^{1\beta}_i\right\} &(3.35)
\end{aligned}$$

We can also think of this latter as the reduction from four to three dimensions of the kinetic term for a Majorana spinor, $\chi^{\dot{1}}$ and $\chi^{\dot{2}}$ being its opposite chirality projections. In three dimensions the $\dot{A}$ index plays the role of the $\mathrm{SU}(2)_\mathrm{L}$ R-symmetry while the $\mathrm{SU}(2)$ rotating the complex structures plays the role of the $\mathrm{SU}(2)_\mathrm{R}$ R-symmetry. To recover $\mathscr{N}_3=4$ supersymmetry we should be able to write the interactions in an $\mathrm{SU}(2)_\mathrm{L}\times\mathrm{SU}(2)_\mathrm{R}$ invariant form. We can do this when the constraints (1.3-1.4) hold true. They can be rewritten with anholonomic derivatives as follows

$$\begin{aligned}
\partial_\alpha\left(\mathscr{P}^+\cdot\mathscr{P}^+\right) &= 0 &(3.36)\\
\partial^\alpha\left(\mathscr{P}^-\cdot\mathscr{P}^-\right) &= 0 &(3.37)\\
\partial_\alpha\left(\mathscr{P}^3\cdot\mathscr{P}^+\right) &= 0 &(3.38)\\
\partial^\alpha\left(\mathscr{P}^3\cdot\mathscr{P}^-\right) &= 0 &(3.39)\\
\partial^\beta\partial_\alpha\left(2\mathscr{P}^3\cdot\mathscr{P}^3-\mathscr{P}^+\cdot\mathscr{P}^-\right) &= 0 &(3.40)
\end{aligned}$$

Considering the scalar potential

$$V_{pot} = \frac{1}{24\alpha^2}\mathfrak{m}^{\Lambda\Gamma}\mathfrak{m}^{\Sigma\Delta}k^i_\Gamma k^{j^\star}_\Delta g_{ij^\star}\mathscr{P}^3_\Lambda\mathscr{P}^3_\Sigma = \frac{1}{48\alpha^2}\mathfrak{m}^{\Lambda\Gamma}\mathfrak{m}^{\Sigma\Delta}k^\alpha_\Gamma k_{\alpha\Delta}\mathscr{P}^3_\Lambda\mathscr{P}^3_\Sigma \tag{3.41}$$

we can use the definition of the tri-holomorphic moment maps and eq.(3.39) to rewrite it as

$$V_{pot} = \frac{1}{192\alpha^2}\mathfrak{m}^{\Lambda\Gamma}\mathfrak{m}^{\Sigma\Delta}\partial^\alpha\mathscr{P}^-_\Gamma\partial_\alpha\mathscr{P}^+_\Delta\mathscr{P}^3_\Lambda\mathscr{P}^3_\Sigma = -\frac{1}{192\alpha^2}\mathfrak{m}^{\Lambda\Gamma}\mathfrak{m}^{\Sigma\Delta}\mathscr{P}^-_\Gamma\partial_\alpha\mathscr{P}^+_\Delta\partial^\alpha\mathscr{P}^3_\Lambda\mathscr{P}^3_\Sigma \tag{3.42}$$

thanks to the moment map equivariance we obtain

$$V_{pot} = \frac{\mathrm{i}}{192\alpha^2}f^{\Gamma\Sigma\Pi}\mathscr{P}^-_\Gamma\mathscr{P}^3_\Sigma\mathscr{P}^+_\Pi \tag{3.43}$$

where we define

$$f^{\Lambda\Sigma\Pi} \equiv \mathfrak{m}^{\Lambda\Gamma}\mathfrak{m}^{\Sigma\Delta}f^\Pi_{\Gamma\Delta} \tag{3.44}$$

In this definition $\mathfrak{m}^{\Lambda\Sigma}$ is a non-degenerate invariant quadratic form on the Lie algebra $\mathbb{G}$, a priori different from the Cartan Killing metric, which might be degenerate if the Lie algebra is not semisimple. So $f^{\Lambda\Sigma\Pi}$ is not necessarily completely antisymmetric. $f^{\Lambda\Sigma\Pi}$ is completely antisymmetric if $\mathfrak{m}^{\Lambda\Sigma} = \kappa^{\Lambda\Sigma}$ is the Cartan-Killing metric of a simple Lie algebra. In the case of a Lie algebra which is the direct sum of some finite number of simple Lie algebras and abelian ones, $\mathfrak{m}^{\Lambda\Sigma}$ can be chosen to be block-diagonal. Each block corresponding to a simple part is proportional to the respective Cartan-Killing metric. Each block corresponding to an abelian addend is a generic non-degenerate invariant quadratic form on it. Also in this case $f^{\Lambda\Sigma\Pi}$ is totally antisymmetric. It turns out that this freedom in the definition of $\mathfrak{m}^{\Lambda\Sigma}$ is essential in order to satisfy the moment map constraints in the case of free hypermultiplets. In any case, assuming that $f^{\Lambda\Sigma\Pi}$ is completely antisymmetric we obtain

$$V_{pot} = \frac{\mathrm{i}}{192\alpha^2}f^{\Gamma\Sigma\Pi}\mathscr{P}^-_\Gamma\mathscr{P}^3_\Sigma\mathscr{P}^+_\Pi = \frac{1}{576\alpha^2}f^{\Gamma\Sigma\Pi}\mathscr{P}^x_\Gamma\mathscr{P}^y_\Sigma\mathscr{P}^z_\Pi\varepsilon_{xyz} \tag{3.45}$$

Thanks to eq.(3.37) the other contributions to the scalar potential vanish. For the same reason the only surviving interactions from the Yukawa coupling are

$$Yuk = -\frac{1}{12\alpha}\mathfrak{m}^{\Lambda\Sigma}\mathscr{P}^3_\Sigma\partial_\alpha\partial^\gamma\mathscr{P}^3_\Lambda\left(\chi_{\dot{1}\gamma}\chi^{\dot{1}\alpha} - \mathbb{C}_{\beta\gamma}\mathbb{C}^{\alpha\rho}\chi_{\dot{2}\rho}\chi^{\dot{2}\beta}\right) - \frac{1}{6\alpha}\mathfrak{m}^{\Lambda\Sigma}\left(j_{2\dot{1}\Lambda}j^{2\dot{1}}_\Sigma + j_{1\dot{2}\Lambda}j^{1\dot{2}}_\Sigma\right) \tag{3.46}$$

where

$$\begin{aligned} j^{A\dot{B}}_\Lambda &= k^{A\alpha}_\Lambda\chi^{\dot{B}\beta}\mathbb{C}_{\alpha\beta} \\ j_{A\dot{B}\Lambda} &= k_{A\alpha\Lambda}\chi^{\dot{B}\beta}\mathbb{C}^{\alpha\beta} \qquad (3.47) \\ k^{A\alpha}_\Lambda &= \mathscr{U}^{A\alpha}(k_\Lambda) \\ k_{A\alpha\Lambda} &= \varepsilon_{AB}\mathbb{C}_{\alpha\beta}\mathscr{U}^{B\beta}(k_\Lambda) \qquad (3.48) \end{aligned}$$

Now we use eq.(3.40) to obtain

$$
\begin{aligned}
Yuk &= -\frac{1}{12\alpha}\mathfrak{m}^{\Lambda\Sigma}\left(\frac{1}{4}\partial_\alpha\mathscr{P}^+_\Sigma\partial^\gamma\mathscr{P}^-_\Lambda - \partial_\alpha\mathscr{P}^3_\Sigma\partial^\gamma\mathscr{P}^3_\Lambda\right)\left(\chi_{\dot{1}\gamma}\chi^{\dot{1}\alpha} - \mathbb{C}_{\beta\gamma}\mathbb{C}^{\alpha\rho}\chi_{\dot{2}\rho}\chi^{\dot{2}\beta}\right)\\
&\quad -\frac{1}{6\alpha}\mathfrak{m}^{\Lambda\Sigma}\left(j_{2\dot{1}\Lambda}j^{2\dot{1}}_\Sigma + j_{1\dot{2}\Lambda}j^{1\dot{2}}_\Sigma\right) \qquad (3.49)
\end{aligned}
$$

We can express derivatives of the moment maps in terms of Killing vectors thanks to eq.(3.28). We obtain an $\mathrm{SU}(2)_\mathrm{L}\otimes\mathrm{SU}(2)_\mathrm{R}$ invariant interaction

$$
-\frac{1}{12\alpha}\mathfrak{m}^{\Lambda\Sigma}\left(j_{2\dot{1}\Lambda}j^{2\dot{1}}_\Sigma + j_{1\dot{2}\Lambda}j^{1\dot{2}}_\Sigma + j_{1\dot{1}\Lambda}j^{1\dot{1}}_\Sigma + j_{2\dot{2}\Lambda}j^{2\dot{2}}_\Sigma\right) = -\frac{1}{12\alpha}j_{A\dot{B}}\cdot j^{A\dot{B}} \qquad (3.50)
$$

This result was obtained utilizing the following relations

$$
\mathbb{C}_{\alpha\beta}\mathscr{U}^{1\alpha}_i\mathscr{U}^{2\beta}_j = \mathbb{C}_{\alpha\beta}\mathscr{U}^{1\alpha}_{i^\star}\mathscr{U}^{2\beta}_{j^\star} = \mathbb{C}_{\alpha\beta}\mathscr{U}^{2\alpha}_i\mathscr{U}^{2\beta}_j = \mathbb{C}_{\alpha\beta}\mathscr{U}^{1\alpha}_{i^\star}\mathscr{U}^{1\beta}_{j^\star} = \mathbb{C}_{\alpha\beta}\mathscr{U}^{2\alpha}_i\mathscr{U}^{2\beta}_{j^\star} = \mathbb{C}_{\alpha\beta}\mathscr{U}^{1\alpha}_i\mathscr{U}^{1\beta}_{j^\star} = 0
$$

Summarizing, we have shown that when the moment map constraints (1.3-1.4) are satisfied the $\mathscr{N}_3 = 3$ supersymmetric Chern-Simons theory takes the following $\mathscr{N}_3 = 4$ form:

$$
\begin{aligned}
\mathscr{L}^{\mathscr{N}=4}_{CS} &= -\alpha\,\mathrm{Tr}\left(\mathfrak{F}\wedge\mathscr{A} + \frac{1}{3}\mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A}\right)\\
&\quad + \left(\frac{1}{4}\mathbb{C}_{\alpha\beta}\varepsilon_{AB}\langle\hat{\mathscr{U}}^{A\alpha}, \hat{\mathscr{U}}^{B\beta}\rangle + \mathrm{i}\,\chi_{\dot{A}\alpha}\gamma^\mathrm{m}\nabla_\mathrm{m}\chi^{\dot{A}\alpha}\right.\\
&\quad \left. -\frac{1}{2\alpha}j_{A\dot{B}}\cdot j^{A\dot{B}} + \frac{1}{96\alpha^2}f^{\Gamma\Sigma\Pi}\mathscr{P}^x_\Gamma\mathscr{P}^y_\Sigma\mathscr{P}^z_\Pi\varepsilon_{xyz}\right)\wedge\mathrm{Vol}(\mathscr{M}_3) \qquad (3.51)
\end{aligned}
$$

4 The Calabi HyperKähler manifold $T^\star\mathbb{P}^2$ and the resolution of the conifold $\mathscr{C}(\mathrm{N}^{0,1,0})$

The manifolds $\mathrm{N}^{\mathrm{p,q,r}}$:

$$
\mathrm{N}^{\mathrm{p,q,r}} = \frac{\mathrm{SU}(3)\times\mathrm{U}_\mathrm{Y}(1)}{\mathrm{U}_\mathrm{I}(1)\times\mathrm{U}_\mathrm{II}(1)} \qquad (4.1)
$$

were introduced by Castellani and Romans in 1984 [35] as 7-dimensional Einstein manifolds with Killing spinors, useful in the programme of Kaluza-Klein supergravity, namely for Freund-Rubin compactifications of D=11 supergravity of the type:

$$
\mathscr{M}_{11} = \mathrm{AdS}_4\times\left(\frac{\mathrm{G}}{\mathrm{H}}\right)_7 \qquad (4.2)
$$

The manifolds $\mathrm{N}^{\mathrm{p,q,r}}$ are defined as follows (see [36], 2nd vol., sect. V.6.2).

Let λ_Σ ($\Sigma = 1, \dots, 8$) be the standard Gell–Mann matrices[6]

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad ; \quad \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}
\end{aligned} \tag{4.3}$$

The eight generators of the $\mathrm{SU}(3)$ group in the fundamental defining representation can be chosen as the following eight anti-hermitian matrices:

$$\mathbf{t}_\Sigma \equiv \tfrac{i}{2}\lambda_\Sigma \quad ; \quad [\mathbf{t}_\Lambda, \mathbf{t}_\Sigma] = \mathfrak{f}_{\Lambda\Sigma}{}^{\Delta}\mathbf{t}_\Delta \tag{4.4}$$

The tensor $\mathfrak{f}_{\Lambda\Sigma}{}^{\Delta}$ encodes the $\mathrm{SU}(3)$ structure constants. Let moreover $i\mathbf{Y}$ denote the generator of the extra group $\mathrm{U_Y}(1)$. The coset manifold (4.1) is completely determined by specifying the two generators of the $\mathrm{U_I}(1)$ and $\mathrm{U_{II}}(1)$ factors of the subgroup $\mathrm{H} \subset \mathrm{G} = \mathrm{SU}(3) \times \mathrm{U}(1)$. One sets:

$$\begin{aligned}
\mathbf{h}_{\mathrm{I}} &= -\frac{2}{\sqrt{3p^2+q^2+2r^3}\,\sqrt{3p^2+q^2}}\left(\sqrt{3}rp\,\mathbf{t}_8 + rq\,\mathbf{t}_3 - \tfrac{i}{2}\left(3p^2+q^2\right)\mathbf{Y}\right) \\
\mathbf{h}_{\mathrm{II}} &= -\frac{1}{\sqrt{3p^2+q^2}}\left(-q\,\mathbf{t}_8 + \sqrt{3}\,p\,\mathbf{t}_3\right)
\end{aligned} \tag{4.5}$$

where p,q,r are coprime integers. As shown in the original paper and in [37], the local geometry of the manifolds (4.1) depends only on the ratio $3p/q$ while the integer r is related with their fundamental group. When we set $p=0, r=0$ the generator $\mathbf{h}_{\mathrm{I}}$ just becomes $i\mathbf{Y}$, while the generator $\mathbf{h}_{\mathrm{II}}$ becomes $\mathbf{t}_8$. Hence we find:

$$\mathrm{N}^{0,1,0} \sim \frac{\mathrm{SU}(3)}{\mathrm{U}_8(1)} \tag{4.6}$$

where $\mathrm{U}_8(1)$ is generated by $\mathbf{t}_8$.

[6]We recall the explicit expression of the Gell-Mann matrices since we need them in the sequel. In this way we fix normalizations.

The manifold $N^{0,1,0}$ figures in the very short list of homogeneous Sasaki-Einstein 7 manifolds which is recalled in table 1. Since the subgroup $U_8(1)$ has an $SU(2)$ normalizer in $SU(3)$, it follows that we can also write:

$$N^{0,1,0} \sim \frac{SU(3)}{U_8(1)} \sim \frac{SU(3)\times SU(2)}{SU(2)\times U(1)} \tag{4.7}$$

showing that $N^{0,1,0}$ is not only sasakian, rather it is also tri-sasakian, admitting two kind of fibrations.

In the first fibration $N^{0,1,0}$ is seen as a circle-bundle over the flag manifold:

$$\mathfrak{m}_6^F \equiv \frac{SU(3)}{U(1)\times U(1)} \tag{4.8}$$

the group $U(1)\times U(1)$ being the maximal torus. Namely we have:

$$\frac{SU(3)}{U_8(1)} \sim N^{0,1,0} \stackrel{\pi}{\longrightarrow} \mathfrak{m}_6^F \quad ; \quad \forall p \in \mathfrak{m}_6^F : \pi^{-1}(p) \sim U(1) \tag{4.9}$$

In the second fibration $N^{0,1,0}$ is seen as an $\mathbb{S}^3$-fibration over $\mathbb{P}^2$

$$N^{0,1,0} \sim \frac{SU(3)\times SU(2)}{SU(2)\times U_8(1)} \stackrel{\pi}{\longrightarrow} \mathbb{P}^2 \quad ; \quad \forall p \in \mathbb{P}^2 : \pi^{-1}(p) \sim SU(2) \tag{4.10}$$

The peculiarity of M-theory compactification on

$$\mathscr{M}_{11} = AdS_4 \times N^{0,1,0} \tag{4.11}$$

is that it leads to $\mathscr{N}_4 = 3$ rather than $\mathscr{N}_4 = 4$ supersymmetry in $D = 4$, as one might expect from the SU(2)-holonomy of the internal seven-manifold. Indeed, notwithstanding such holonomy, the differential equation for the Killing spinors can be integrated only for three, rather than for four of them [35].

Correspondingly in [38],[9] the complete Kaluza Klein spectrum of M-theory on the background (4.11) was derived and organized into supermultiplets of the supergroup $Osp(3|4)$ each supermultiplet being assigned to a tower of irreducible representations of the isometry group $SU(3)$ that determines its mass and charges.

4.1 The $N^{0,1,0}$ manifold from the $D = 3$ gauge theory viewpoint

As discussed in general terms in [17] and summarized in [39], whenever we have an $AdS_4 \times \mathscr{M}_7$ solution of $D = 11$ supergravity we can construct its associated $M2$-brane solution that interpolates between a locally Minkowskian $Mink_{1,10}$ flat manifold at infinity and the $AdS_4 \times \mathscr{M}_7$ manifold at the brane-horizon $r \to 0$. The general structure of the $\mathscr{M}_{11}$ metric in the M2-brane solution is of the form:

$$ds^2_{M2-brane} = H^{-2/3}(y)\, ds^2_{Mink_{1,2}} + H^{1/3}(y)\, ds^2_{\mathscr{C}(\mathscr{M}_7)} \tag{4.12}$$

where $ds^2_{Mink_{1,2}}$ is the flat Minkowski metric on the three-dimensional brane world volume and

$$ds^2_{\mathscr{C}(\mathscr{M}_7)} = dr^2 + r^2\, ds^2_{\mathscr{M}_7} \tag{4.13}$$

is the metric of the metric cone $\mathscr{C}(\mathscr{M}_7)$ over the Einstein 7-manifold $\mathscr{M}_7$.

Whenever $\mathscr{M}_7$ is sasakian, namely it admits at least two Killing spinors, the metric cone $\mathscr{C}(\mathscr{M}_7)$ is, according to an equivalent definition of sasakian manifolds, a Ricci-flat Kähler manifold K_4. This does not exclude that K_4 might be singular. Indeed, for all sasakian homogeneous 7-manifolds different from the round 7-sphere, K_4 has a singularity at the tip of the cone and therefore is a *conifold*.

One is therefore interested in *crepant resolutions* of this conifold singularity and we shall address this problem from the point of view of the gauge theory living on the brane world-volume.

The lagrangian of $\mathscr{N}_3 = 3$ Chern-Simons Gauge Theory, as systematized in [17] within the family of $\mathscr{N}_3 = 2$ Chern Simons gauge theories, takes the form discussed in section 2 and presented in eq.s (2.1,2.11,2.12,2.15).

4.2 The $\mathscr{N}_3 = 3$ gauge theory corresponding to the $\mathrm{N}^{0,1,0}$ compactification

Having clarified the structure of a generic $\mathscr{N}_3 = 3$ gauge theory let us consider, as an illustration, the specific one associated with the $\mathrm{N}^{0,1,0}$ seven–manifold following the presentation of [13]. As explained above the manifold $\mathrm{N}^{0,1,0}$ is the circle bundle inside $\mathscr{O}(1,1)$ over the flag manifold $\mathfrak{m}_6^F$ (see eq.s(4.8-4.9). Furthermore as also explained in [12] (see eq.(B.2)), the base manifold $\mathfrak{m}_6^F$ can be algebraically described as the following quadric

$$\sum_{i=1}^{3} u^i\, v_i = 0 \tag{4.14}$$

in $\mathbb{P}^2 \times \mathbb{P}^{2*}$, where u^i and v_i $(i = 1,2,3)$ are the homogeneous coordinates of $\mathbb{P}^2$ and $\mathbb{P}^{2*}$, respectively.

Hence a complete description of the metric cone $\mathscr{C}\left(\mathrm{N}^{0,1,0}\right)$ can be given by writing the following equations in $\mathbb{C}^3 \times \mathbb{C}^{3*}$:

$$\mathscr{C}\left(\mathrm{N}^{0,1,0}\right) = \begin{cases} |u^i|^2 - |v_i|^2 &=& 0 & \text{fixes equal the radii of } \mathbb{P}^2 \text{ and } \mathbb{P}^{2*} \\ 2\,u^i\, v_i &=& 0 & \text{cuts out the quadric locus} \\ \left(u^i\, e^{i\theta}, v_i\, e^{-i\theta}\right) &\simeq& \left(u^i, v_i\right) & \text{identifies points of } \mathrm{U}(1) \text{ orbits} \end{cases} \tag{4.15}$$

Eq.s (4.26) can be easily interpreted as the statement that the cone $K_4 = \mathscr{C}\left(\mathrm{N}^{0,1,0}\right)$ is the HyperKähler quotient of a flat three-dimensional quaternionic space with respect to the triholomorphic action of a $\mathrm{U}(1)$ group. Indeed the first two equations in (4.26) can be rewritten as the vanishing of the triholomorphic moment map of a $\mathrm{U}(1)$ group. It suffices to identify:

$$\begin{aligned} \mathscr{P}_3 &= -\left(|u^i|^2 - |v_i|^2\right) \\ \mathscr{P}_- &= 2 v_i u^i \end{aligned} \tag{4.16}$$

Comparing with eq.s (2.15) we see that the cone $\mathscr{C}(\mathrm{N}^{0,1,0})$ can be correctly interpreted as the space of classical vacua in an abelian $\mathcal{N}_3 = 3$ gauge theory with 3 hypermultiplets in the fundamental representation of a flavor group $\mathrm{SU}(3)$.

If the color group is $\mathrm{U}(1)$ there is only one value for the index Λ. The potential is a positive definite quadratic form in the moment maps with minimum at zero which is attained when the moment map vanishes.

Relying on this geometrical picture of the transverse space to an $M2$–brane living on $\mathrm{AdS}_4 \times \mathrm{N}^{0,1,0}$, in [13] was conjectured that the $\mathcal{N}_3 = 3$ non–abelian gauge theory whose infrared conformal point is dual to $D = 11$ supergravity compactified on $\mathrm{AdS}_4 \times \mathrm{N}^{0,1,0}$ should have the following structure:

$$
\begin{array}{lccc}
\text{gauge group} & \mathscr{G}_{gauge} & = & \mathrm{SU(N)}_1 \times \mathrm{SU(N)}_2 \\
 & & & \\
\text{flavor group} & \mathscr{G}_{flavor} & = & \mathrm{SU}(3) \\
 & & & \\
\text{color representations of the hypermultiplets} & \begin{bmatrix} u \\ v \end{bmatrix} & \Rightarrow & \begin{bmatrix} \left(\mathbf{N}_1, \overline{\mathbf{N}}_2\right) \\ \left(\overline{\mathbf{N}}_1, \mathbf{N}_2\right) \end{bmatrix} \\
 & & & \\
\text{flavor representations of the hypermultiplets} & \begin{bmatrix} u \\ v \end{bmatrix} & \Rightarrow & \begin{bmatrix} \mathbf{3} \\ \overline{\mathbf{3}} \end{bmatrix}
\end{array}
\tag{4.17}
$$

More explicitly and using an $\mathcal{N}_3 = 2$ notation we can say that the field content of the theory proposed in [13] is given by the following chiral fields, that are all written as $N \times N$ matrices:

$$
\begin{array}{rclc}
Y_1 & = & (Y_1)^{\Lambda_1}{}_{\Sigma_1} & \text{adjoint of } \mathrm{SU(N)}_1 \\
Y_2 & = & (Y_2)^{\Lambda_2}{}_{\Sigma_2} & \text{adjoint of } \mathrm{SU(N)}_2 \\
u^i & = & \left(u^i\right)^{\Lambda_1}{}_{\Sigma_2} & \text{in the } (\mathbf{3}, \mathbf{N}_1, \overline{\mathbf{N}}_2) \\
v_i & = & (v_i)_{\Sigma_1}{}^{\Lambda_2} & \text{in the } (\mathbf{3}, \overline{\mathbf{N}}_1, \mathbf{N}_2)
\end{array}
\tag{4.18}
$$

and the superpotential before integration on the auxiliary fields Y can be written as follows:

$$
\mathscr{W} = 2\left[\mathrm{Tr}\left(Y_1\, u^i\, v_i\right) + \mathrm{Tr}\left(Y_2\, v_i\, u_i\right) + \alpha_1 \mathrm{Tr}\left(Y_1 Y_1\right) + \alpha_2 \mathrm{Tr}\left(Y_2 Y_2\right)\right] \tag{4.19}
$$

where $\alpha_{1,2}$ are the Chern Simons coefficients associated with the $\mathrm{SU(N)}_{1,2}$ simple gauge groups, respectively. Setting:

$$
\alpha_1 \quad = \quad \pm\alpha_2 = \alpha \tag{4.20}
$$

and integrating out the two fields $Y_{1,2}$ that have received a mass by the Chern Simons mechanism, in [13] it was obtained the following effective quartic superpotential:

$$\mathfrak{W}^{eff} = -\frac{1}{2}\frac{1}{\alpha}\left[\mathrm{Tr}\left(v_i u^i v_j u^j\right) \pm \mathrm{Tr}\left(u^i v_i u^j u_j\right)\right] \tag{4.21}$$

The vanishing relations one can derive from the above superpotential are the following ones:

$$u^i v_j u^j = \pm u^j v_j u^i \quad ; \quad v_i u^j v_j = \pm v_j u^j v_i \tag{4.22}$$

Consider now the chiral conformal superfields one can write in this theory:

$$\Phi^{i_1\, i_2 \ldots i_k}_{j_1\, j_2 \ldots j_k} \equiv \mathrm{Tr}\left(u^{(i_1} v_{(j_1} u^{i_2} v_{j_2} \ldots u^{i)_k} v_{j_k)}\right) \tag{4.23}$$

where the round brackets denote symmetrization on the indices. The above operators have k indices in the fundamental representation of $\mathrm{SU}(3)$ and k indices in the antifundamental one, but they are not yet assigned to the irreducible representation:

$$M_1 = M_2 = k \tag{4.24}$$

as it is predicted both by general geometric arguments and by the explicit evaluation of the Kaluza Klein spectrum of hypermultiplets [9]. To be irreducible the operators (4.23) have to be traceless. This is what is implied by the vanishing relation (4.22) if we choose the minus sign in eq.(4.20).

The field content and the structure of this $\mathcal{N}_3 = 3$ Chern Simons gauge theory is encoded in the quiver diagram displayed in fig.1. The similar quiver diagram associated with the Eguchi Hanson space is pictured in fig.2

In [13] it was noticed that for $\mathrm{N}^{0,1,0}$ the form of the superpotential, which is dictated by the Chern-Simons term, is strongly reminiscent of the superpotential considered in [11]. Indeed, the CFT theory associated with $\mathrm{N}^{0,1,0}$ has many analogies with the simpler cousin $\mathrm{T}^{1,1}$ [6]. However it was stressed in [13] that there is also a crucial difference, pertaining to a general phenomenon that was discussed for the case of compactifications on $\mathrm{M}^{1,1,1}$ and $\mathrm{Q}^{1,1,1}$ in [7] and [12]. The moduli space of vacua of the abelian theory is isomorphic to the cone $\mathscr{C}\left(\mathrm{N}^{0,1,0}\right)$. When the theory is promoted to a non-abelian one, there are naively conformal operators whose existence is in contradiction with geometric expectations and with the KK spectrum, in this case the hypermultiplets that do not satisfy relation (4.24). Differently from what happens for $\mathrm{T}^{1,1}$ [11], the superpotential in eq. (4.21) is not sufficient for eliminating these redundant non-abelian operators.

Ten years later in a paper by Gaiotto et al [40], it was advocated that, maintaining the same flavor-group assignments and the same color group, the color representation assignments of the hypermultiplets that lead to the correct dual CFT are slightly different from those shown in eq. (4.18) since in addition to the bi-fundamental representation one needs also the two fundamental ones.

In any case it is appropriate to stress that, on the basis of the general form of the $\mathcal{N}_3 = 3$ gauge theory discussed above as a particular case of the general $\mathcal{N}_3 = 2$ theory, it was just in [13] that the structure of an

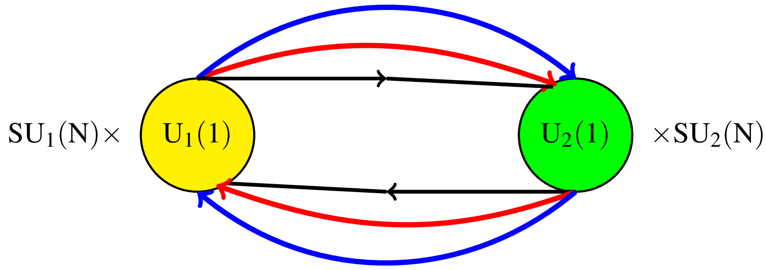

Figure 1: The quiver diagram describing the D=3 gauge theory corresponding to the a stack of M2-branes with transverse 8-dimensional space provided by the metric cone on the coset manifold N^{010}

$\mathcal{N}_3 = 3, D = 3$ Chern-Simons gauge theory, corner stone of the famous ABJM model[15], was for the first time derived in the literature. Indeed in [13] it was just conjectured that the gauge coupling constant g flows to infinity at the infrared conformal point, so that the effective lagrangian is obtained from the general one by letting $\mathbf{e} = \frac{1}{g^2} \to 0$. It was in [13] that the conversion of the Y^Λ field into a lagrangian multiplier was for the first time observed, leading to the generation of an effective superpotential of type (4.21).

4.3 Resolution of the conifold singularity for $\mathscr{C}(\mathrm{N}^{0,1,0})$

The shaking news of paper [27] is that the resolution of the singularity for the conifold $\mathscr{C}(\mathrm{N}^{0,1,0})$ is provided by a HyperKähler 8-dimensional manifold which is the total space of the cotangent bundle of $\mathbb{P}^2$, namely:

$$\mathscr{M}_8 = HK^{(2)}_{Calabi} \sim T^\star\mathbb{P}^2 \tag{4.25}$$

The HyperKähler metric on $T^\star\mathbb{P}^2$ is the Calabi metric which admits, as the authors show, a general representation for all $T^\star\mathbb{P}^{1+n}$ and this justifies the name $HK^{(n)}_{Calabi}$ for these HyperKähler manifolds of real dimensions $4n+4$.

In the present section, following the guide-lines of [27] we explicitly derive the HyperKähler metric of $HK^{(2)}_{Calabi}$ as the resolution of the singular conifold $\mathscr{C}(\mathrm{N}^{0,1,0})$ and we advocate that this resolution just corresponds, in eq. (4.26), to lifting the real component of the moment map to a non vanishing level:

$$HK^{(1)}_{Calabi} = \left\{ \begin{array}{rcccl} \mathscr{P}_3 & \equiv & |u^i|^2 - |v_i|^2 & = & \kappa \neq 0 \\ \mathscr{P}_+ & \equiv & 2\,u^i\,v_i & = & 0 \\ \left(u^i\,e^{i\theta}, v_i\,e^{-i\theta}\right) & \simeq & \left(u^i, v_i\right) & & \end{array} \right. \tag{4.26}$$

The Calabi metric is a generalization of the Eguchi Hanson metric and it is indeed HyperKähler. What happens

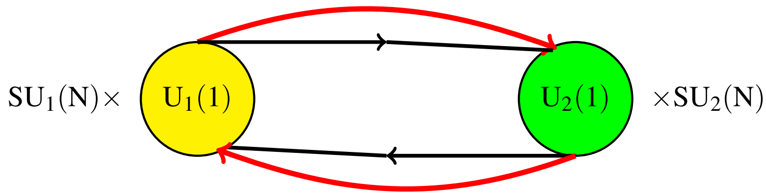

Figure 2: The quiver diagram associated with the $\mathbb{C}^2/\mathbb{Z}_2$ Kleinian singularity, whose resolution is the Eguchi Hanson HyperKähler manifold. In the two nodes, which correspond to the two irreducible one-dimensional representations of $\mathbb{Z}_2$ we place the two gauge groups $\mathrm{SU}_{1,2}(\mathrm{N}) \times \mathrm{U}_{1,2}(1)$. The scalar multiplets correspond to the two directed lines going from one to the other node and are in the bi-fundamental representation of the mentioned node groups. The HyperKähler quotient is done with respect to the relative $\mathrm{U}(1)$ group, the overall $\mathrm{U}(1)$ being the irrelevant barycentric group.

is that there are two routes one can follow to generalize the Eguchi Hanson case:

$$EH = \begin{cases} T^\star\mathbb{P}^1 & \overset{\text{generalization}}{\longrightarrow} & T^\star\mathbb{P}^{n+1} & \dim_{\mathbb{R}} = 4n+4 & \text{HyperKähler} \\ & & & & \\ \mathcal{O}_{\mathbb{P}^1}(-2) & \overset{\text{generalization}}{\longrightarrow} & \mathcal{O}_{\mathbb{P}^{1+n}}(-2-n) & \dim_{\mathbb{R}} = 2n+4 & \text{Kähler} \end{cases} \tag{4.27}$$

The route in the first line of eq.(4.27) is that followed by the authors of [27] who indeed constructed HyperKähler Calabi metrics for all values of n utilizing the tri-Sasaki Einstein manifold $\frac{\mathrm{SU(n+2)}}{\mathrm{U(n)}}$ as a starting point. The route in the second line is that followed in the resolution of $\mathbb{C}^{n+2}/\mathbb{Z}_{n+2}$ singularities. For $n =$ odd the second line exists and it is always Kähler but it can have no comparison with the first line. Instead for $n =$ even we might conjecture some relation as in the Eguchi-Hanson case. This issue will be addressed elsewhere.

Next we derive the mentioned resolution step by step.

4.4 The resolution via HyperKähler quotient

Here we perform the HyperKähler quotient $\mathbb{C}^3 \times \mathbb{C}^{\star 3} / /_{HK} U(1)$. We introduce three complex coordinates $\{u^i\}$ $i : 1,2,3$ of $\mathbb{C}^3$ and the dual coordinates $\{v_i\}$ of $\mathbb{C}^{\star 3}$. We introduce the flat Kähler potential

$$\mathcal{K}_{\mathbb{C}^3\times\mathbb{C}^{\star 3}} = \overline{u}_i u^i + \overline{v}^i v_i \tag{4.28}$$

We define the tri-holomorphic $\mathrm{U}(1)$ action, $\{u^i\,,v_i\} \to \{e^{\mathrm{i}\phi}u^i\,, e^{-\mathrm{i}\phi}v_i\}$. In order to perform the HyperKähler quotient we identify points of $\mathrm{U}(1)$ orbits and we set the $\mathrm{U}(1)$ tri-holomorphic moment map levels:

$$\begin{aligned} \mathscr{P}^3(u,v,\overline{u},\overline{v}) &= \overline{u}_i u^i - \overline{v}^i v_i = \kappa \\ \mathscr{P}^+(u,v) &= u^i v_i = 0 \end{aligned}$$

$$\{u^i\,,v_i\} \quad\simeq\quad \{e^{\mathrm{i}\phi}u^i\,,e^{-\mathrm{i}\phi}v_i\} \tag{4.29}$$

This defines the HyperKähler manifold $\widetilde{\mathscr{C}(N^{0,1,0})}$ which is the resolution of the conifold $\mathscr{C}(N^{0,1,0})$, κ being related to the resolution parameter.

4.4.1 Solving the algebraic constraints

First, we consider the complexification of U(1), $e^{i\phi} \to e^{-\Phi}$, and we set the following gauge

$$\begin{aligned} w^i &= \frac{u^i}{u^3} = e^{\Phi}u^i \\ z_j &= u^3 v_j = e^{-\Phi}v_j \end{aligned} \tag{4.30}$$

This implies $w^3 = 1$ while from $\mathscr{P}^+ = 0$ we obtain $z^3 = -w^a z_a$ where $a = 1,2$. We can identify w^a with the inhomogenous coordinates of $\mathbb{P}^2$ and z_a with the fibre coordinates of $T^\star\mathbb{P}^2$. Now, we find the element Φ that lifts the real moment map from 0 to κ

$$\mathscr{P}^3 = \kappa \Leftrightarrow \Phi_\pm = -\frac{1}{2}\log\left(\frac{\kappa \pm \sqrt{\kappa^2+4R}}{2(1+\overline{w}_a w^a)}\right) \tag{4.31}$$

where $R = (1+\overline{w}_a w^a)(\overline{z}^a z_a + \overline{(w^a z_a)}(w^a z_a))$.
This solution defines the immersion $i : T^\star\mathbb{P}^2 \simeq \widetilde{\mathbb{C}(N^{0,1,0})} \longrightarrow \mathbb{C}^3\times\mathbb{C}^{\star 3}$.

4.4.2 The Kähler potential and the metric

The Kähler potential of the HyperKähler manifold described by (4.29) has the following form:

$$\begin{aligned} \mathscr{K}_{T^\star\mathbb{P}^2} &= i^\star\mathscr{K}_{\mathbb{C}^3\times\mathbb{C}^{\star 3}} - \alpha\,\kappa\,\Phi_+ \\ &= \sqrt{4R+\kappa^2} + \frac{\alpha}{2}\,\kappa\,\log\left(\kappa+\sqrt{\kappa^2+4R}\right) - \frac{\alpha}{2}\,\kappa\,\log\left(1+\overline{w}_a w^a\right) \\ &\equiv F(R) - \frac{\alpha}{2}\,\kappa\,\log\left(1+\overline{w}_a w^a\right) \end{aligned} \tag{4.32}$$

from which we obtain the Kähler metric $g_{T^\star\mathbb{P}^2} = g_{IJ^\star}dq^I\otimes d\overline{q}^{J^\star} = \partial_I\partial_{J^\star}\mathscr{K}_{T^\star\mathbb{P}^2}\,dq^I\otimes d\overline{q}^{J^\star}$, $q^I = (w^a, z_a)$.
The coefficient α has to be fixed. Indeed, the above metric must be Ricci-flat. First, we compute the determinant of the metric. Then, we solve $det(g_{IJ^\star}) = \delta \in \mathbb{R}$ in terms of α.

We obtain:

$$det(g_{IJ^\star}) = \frac{1}{4}F'(R)\left(RF''(R)+F'(R)\right)\Big(\alpha^2\kappa^2+8R^2F'(R)^2-6\alpha\kappa RF'(R)\Big) = \delta \tag{4.33}$$

From this latter we find

$$F'(R) = \frac{\alpha\kappa+\sqrt{\alpha^2\kappa^2+16R\sqrt{\delta}}}{4R} \tag{4.34}$$

while from (4.32)

$$F'(R) = \frac{\alpha\kappa^3-\alpha\kappa^2\sqrt{\kappa^2+4R}+4\alpha\kappa R+8R\sqrt{\kappa^2+4R}}{16R^2+4\kappa^2R} \tag{4.35}$$

So we find the unique solution $\alpha=-2$ and $\delta=1$.

4.5 The resolution via Maurer Cartan equations and the Calabi HyperKähler manifold

Let $\mathscr{E}^\Lambda$ be a set of left-invariant one forms associated with the generators of $\mathrm{SU}(3)$ normalized as in eq.(4.4). Eventually they will be obtained as in equation (B.11) from the coset representative $\mathbb{L}_{\mathrm{N}^{010}}$ of the 7-manifold of interest to us. Yet this is not relevant for the explicit construction of the Calabi HyperKähler manifolds. What is relevant is that they satisfy the Maurer Cartan equations of $\mathrm{SU}(3)$ explicitly written below

$$\begin{aligned}
0 &= \mathrm{d}\mathscr{E}^1-\mathscr{E}^2\wedge\mathscr{E}^3-\frac{1}{2}\mathscr{E}^4\wedge\mathscr{E}^7+\frac{1}{2}\mathscr{E}^5\wedge\mathscr{E}^6\\
0 &= \mathrm{d}\mathscr{E}^2+\mathscr{E}^1\wedge\mathscr{E}^3-\frac{1}{2}\mathscr{E}^4\wedge\mathscr{E}^6-\frac{1}{2}\mathscr{E}^5\wedge\mathscr{E}^7\\
0 &= \mathrm{d}\mathscr{E}^3-\mathscr{E}^1\wedge\mathscr{E}^2-\frac{1}{2}\mathscr{E}^4\wedge\mathscr{E}^5+\frac{1}{2}\mathscr{E}^6\wedge\mathscr{E}^7\\
0 &= \mathrm{d}\mathscr{E}^4+\frac{1}{2}\mathscr{E}^1\wedge\mathscr{E}^7+\frac{1}{2}\mathscr{E}^2\wedge\mathscr{E}^6+\frac{1}{2}\mathscr{E}^3\wedge\mathscr{E}^5-\frac{1}{2}\sqrt{3}\mathscr{E}^5\wedge\mathscr{E}^8\\
0 &= \mathrm{d}\mathscr{E}^5-\frac{1}{2}\mathscr{E}^1\wedge\mathscr{E}^6+\frac{1}{2}\mathscr{E}^2\wedge\mathscr{E}^7-\frac{1}{2}\mathscr{E}^3\wedge\mathscr{E}^4+\frac{1}{2}\sqrt{3}\mathscr{E}^4\wedge\mathscr{E}^8\\
0 &= \mathrm{d}\mathscr{E}^6+\frac{1}{2}\mathscr{E}^1\wedge\mathscr{E}^5-\frac{1}{2}\mathscr{E}^2\wedge\mathscr{E}^4-\frac{1}{2}\mathscr{E}^3\wedge\mathscr{E}^7-\frac{1}{2}\sqrt{3}\mathscr{E}^7\wedge\mathscr{E}^8\\
0 &= \mathrm{d}\mathscr{E}^7-\frac{1}{2}\mathscr{E}^1\wedge\mathscr{E}^4-\frac{1}{2}\mathscr{E}^2\wedge\mathscr{E}^5+\frac{1}{2}\mathscr{E}^3\wedge\mathscr{E}^6+\frac{1}{2}\sqrt{3}\mathscr{E}^6\wedge\mathscr{E}^8\\
0 &= \mathrm{d}\mathscr{E}^8-\frac{1}{2}\sqrt{3}\mathscr{E}^4\wedge\mathscr{E}^5-\frac{1}{2}\sqrt{3}\mathscr{E}^6\wedge\mathscr{E}^7
\end{aligned} \tag{4.36}$$

To construct in one stroke both the metric cone and its resolution, namely the Calabi HyperKähler metric $HK^{(2)}_{Calabi}$, we introduce an additional coordinate that we name τ and we introduce the following vielbein for an 8-dimensional manifold:

$$\begin{aligned}
V^1 &= A(\tau)\mathscr{E}^1\\
V^2 &= A(\tau)\mathscr{E}^2\\
V^3 &= B(\tau)\mathscr{E}^4
\end{aligned}$$

$$
\begin{aligned}
V^4 &= B(\tau)\,\mathscr{E}^5 \\
V^5 &= C(\tau)\,\mathscr{E}^6 \\
V^6 &= C(\tau)\,\mathscr{E}^7 \\
V^7 &= F(\tau)\,\mathscr{E}^3 \\
V^8 &= \frac{\mathrm{d}\tau}{\sqrt{1-\frac{\ell^4}{\tau^4}}}
\end{aligned}
\tag{4.37}
$$

where $A(\tau), B(\tau), C(\tau), F(\tau)$ are functions of the variable τ to be determined and ℓ is a real parameter.

The choice (4.37) needs to be properly explained. We have introduced a different scaling factor $A(\tau)$, $B(\tau)$, $C(\tau)$, for each of the three doublets of Maurer Cartan forms that are rotated one into the other by the generator $\mathbf{t}_3$, as it is evident from the Maurer Cartan equations (4.36). This choice respects the $\mathrm{U}(1)$ fibration of $\mathrm{N}^{0,1,0}$ and it is mandatory. The fourth function $F(\tau)$ multiplies the one-form $\mathscr{E}^3$ which is a $\mathrm{U}(1)$ singlet. Hence it is a priori independent. The choice of the function of τ appearing in V^8 is not any limitation of the ansatz, since any other function would amount to a redefinition of the τ coordinate. It is just an educated guess that simplifies the subsequent differential equations.

Given the ansatz (4.37) we could start constructing the spin connection Ω^{IJ} and the curvature 2-form $\mathfrak{R}^{IJ}$ for generic functions, yet, as the authors of [27] do, imposing that the final manifold should be HyperKähler is much more restrictive and determines all the undetermined functions.

4.5.1 The three complex structures and the three HyperKähler forms

The advantage of working in the intrinsic vielbein basis is that in this frame the three complex structures $\mathbf{J}^x$ that must satisfy the algebra of quaternion imaginary units:

$$
\mathbf{J}^x \cdot \mathbf{J}^y = -\delta^{xy}\,\mathbf{Id}_{8\times 8} + \varepsilon^{xyz}\,\mathbf{J}^y \quad ; \quad x,y,z = 1,2,3 \tag{4.38}
$$

are constant antisymmetric matrices. An explicit representation representation of the algebra (4.38) that up to SO(8) rotations is unique is provided by the following matrices:

$$
\mathbf{J}^1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad ; \quad
\mathbf{J}^2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\tag{4.39}
$$

$$\mathbf{J}^3 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.40}$$

Correspondingly, by setting:

$$\mathbf{K}^x = \mathbf{J}^x_{IJ} V^I \wedge V^J \tag{4.41}$$

we find the following three candidate HyperKähler forms:

$$\begin{aligned} \mathbf{K}^1 &= 2(V^1 \wedge V^8 - V^2 \wedge V^7 - V^3 \wedge V^6 + V^4 \wedge V^5) \\ \mathbf{K}^2 &= 2(V^1 \wedge V^7 + V^2 \wedge V^8 - V^3 \wedge V^5 - V^4 \wedge V^6) \\ \mathbf{K}^3 &= -2(V^1 \wedge V^2 + V^3 \wedge V^4 - V^5 \wedge V^6 - V^7 \wedge V^8) \end{aligned} \tag{4.42}$$

In order to define a bona-fide HyperKähler structure the above triplet of 2-forms must be closed. Imposing

$$\mathrm{d}\mathbf{K}^1 = \mathrm{d}\mathbf{K}^2 = \mathrm{d}\mathbf{K}^3 = 0 \tag{4.43}$$

inserting the ansatz (4.37) and utilizing the Maurer-Cartan equations (4.36) one obtains a collection of first order differential and algebraic constraints on the four functions $A(\tau)$, $B(\tau)$, $C(\tau)$, $F(\tau)$ which has a unique, easily retrievable solution:

$$A(\tau) = \frac{\tau}{2}, \quad B(\tau) = \frac{\sqrt{\tau^2 + \ell^2}}{2\sqrt{2}}, \quad C(\tau) = \frac{\sqrt{\tau^2 - \ell^2}}{2\sqrt{2}}, \quad F(\tau) = \frac{\sqrt{\tau^4 - \ell^4}}{2\tau} \tag{4.44}$$

As one sees in the case $\ell = 0$ all the functions degenerate in a coefficient times τ. This means that the corresponding metric line element is the metric cone over $\mathrm{N}^{0,1,0}$. As we know such a manifold is singular. For $\ell \neq 0$ we have instead the Calabi HyperKähler metric which is a smooth HyperKähler manifold and corresponds to the resolution of the conifold singularity. Obviously these are the same manifold and the same metric as the manifold and the metric obtained in section 4.4 by means of Kähler quotient. A precise correspondence requires an identification between the coordinates w^a, z_a utilized there and the 7 coordinates used for the $\mathrm{N}^{0,1,0}$ coset manifold plus the radial like coordinate τ. We did not dwell, at this level, on this cumbersome and boring exercise. The precise identification of coordinates will also provide the precise relation between the level parameter κ and the resolution parameter ℓ appearing in the present discussion.

4.5.2 Spin connection and curvature

Having fixed all the functions we can calculate the spin connection and the curvature of the Calabi HyperKähler manifold. From the torsion equation:

$$\mathrm{d}V^I + \Omega^{IJ} \wedge V^J = 0 \tag{4.45}$$

we obtain a unique solution, as it is always the case, encoded in the following one-form valued 8×8 matrix:

$$\Omega^{IJ} =$$

$$\begin{pmatrix}
0 & \frac{(\ell^4+\tau^4)V_7}{\tau^3\sqrt{\tau^4-\ell^4}} & \frac{\sqrt{\frac{\ell^2+\tau^2}{\tau^2-\ell^2}}V_6}{\tau} & -\frac{\sqrt{\frac{\ell^2+\tau^2}{\tau^2-\ell^2}}V_5}{\tau} & \frac{\sqrt{1-\frac{2\ell^2}{\ell^2+\tau^2}}V_4}{\tau} & -\frac{\sqrt{1-\frac{2\ell^2}{\ell^2+\tau^2}}V_3}{\tau} & -\frac{\sqrt{\tau^4-\ell^4}V_2}{\tau^3} & \frac{\sqrt{\tau^4-\ell^4}V_1}{\tau^3} \\
-\frac{(\ell^4+\tau^4)V_7}{\tau^3\sqrt{\tau^4-\ell^4}} & 0 & \frac{\sqrt{\frac{\ell^2+\tau^2}{\tau^2-\ell^2}}V_5}{\tau} & \frac{\sqrt{\frac{\ell^2+\tau^2}{\tau^2-\ell^2}}V_6}{\tau} & -\frac{\sqrt{1-\frac{2\ell^2}{\ell^2+\tau^2}}V_3}{\tau} & -\frac{\sqrt{1-\frac{2\ell^2}{\ell^2+\tau^2}}V_4}{\tau} & \frac{\sqrt{\tau^4-\ell^4}V_1}{\tau^3} & \frac{\sqrt{\tau^4-\ell^4}V_2}{\tau^3} \\
-\frac{\sqrt{\frac{\ell^2+\tau^2}{\tau^2-\ell^2}}V_6}{\tau} & -\frac{\sqrt{\frac{\ell^2+\tau^2}{\tau^2-\ell^2}}V_5}{\tau} & 0 & \frac{V_7\ell^2}{\tau\sqrt{\tau^4-\ell^4}} + \frac{\sqrt{3}\mathscr{E}_8}{2} & 0 & 0 & -\frac{\sqrt{\tau^4-\ell^4}V_4}{\tau^3+\ell^2\tau} & \frac{\sqrt{\tau^4-\ell^4}V_3}{\tau^3+\ell^2\tau} \\
\frac{\sqrt{\frac{\ell^2+\tau^2}{\tau^2-\ell^2}}V_5}{\tau} & -\frac{\sqrt{\frac{\ell^2+\tau^2}{\tau^2-\ell^2}}V_6}{\tau} & -\frac{V_7\ell^2}{\tau\sqrt{\tau^4-\ell^4}} - \frac{\sqrt{3}\mathscr{E}_8}{2} & 0 & 0 & 0 & \frac{\sqrt{\tau^4-\ell^4}V_3}{\tau^3+\ell^2\tau} & \frac{\sqrt{\tau^4-\ell^4}V_4}{\tau^3+\ell^2\tau} \\
-\frac{\sqrt{1-\frac{2\ell^2}{\ell^2+\tau^2}}V_4}{\tau} & \frac{\sqrt{1-\frac{2\ell^2}{\ell^2+\tau^2}}V_3}{\tau} & 0 & 0 & 0 & \frac{V_7\ell^2}{\tau\sqrt{\tau^4-\ell^4}} + \frac{\sqrt{3}\mathscr{E}_8}{2} & \frac{(\ell^2+\tau^2)V_6}{\tau\sqrt{\tau^4-\ell^4}} & \frac{\sqrt{1-\frac{\ell^4}{\tau^4}}\,\tau V_5}{\tau^2-\ell^2} \\
\frac{\sqrt{1-\frac{2\ell^2}{\ell^2+\tau^2}}V_3}{\tau} & \frac{\sqrt{1-\frac{2\ell^2}{\ell^2+\tau^2}}V_4}{\tau} & 0 & 0 & -\frac{V_7\ell^2}{\tau\sqrt{\tau^4-\ell^4}} - \frac{\sqrt{3}\mathscr{E}_8}{2} & 0 & -\frac{(\ell^2+\tau^2)V_5}{\tau\sqrt{\tau^4-\ell^4}} & \frac{\sqrt{1-\frac{\ell^4}{\tau^4}}\,\tau V_6}{\tau^2-\ell^2} \\
\frac{\sqrt{\tau^4-\ell^4}V_2}{\tau^3} & -\frac{\sqrt{\tau^4-\ell^4}V_1}{\tau^3} & \frac{\sqrt{\tau^4-\ell^4}V_4}{\tau^3+\ell^2\tau} & -\frac{\sqrt{\tau^4-\ell^4}V_3}{\tau^3+\ell^2\tau} & -\frac{(\ell^2+\tau^2)V_6}{\tau\sqrt{\tau^4-\ell^4}} & \frac{(\ell^2+\tau^2)V_5}{\tau\sqrt{\tau^4-\ell^4}} & 0 & \frac{(\ell^4+\tau^4)V_7}{\tau^3\sqrt{\tau^4-\ell^4}} \\
-\frac{\sqrt{\tau^4-\ell^4}V_1}{\tau^3} & -\frac{\sqrt{\tau^4-\ell^4}V_2}{\tau^3} & -\frac{\sqrt{\tau^4-\ell^4}V_3}{\tau^3+\ell^2\tau} & -\frac{\sqrt{\tau^4-\ell^4}V_4}{\tau^3+\ell^2\tau} & -\frac{(\ell^2+\tau^2)V_5}{\tau\sqrt{\tau^4-\ell^4}} & -\frac{(\ell^2+\tau^2)V_6}{\tau\sqrt{\tau^4-\ell^4}} & -\frac{(\ell^4+\tau^4)V_7}{\tau^3\sqrt{\tau^4-\ell^4}} & 0
\end{pmatrix} \tag{4.46}$$

Next we calculate the curvature 2-form:

$$\mathfrak{R}^{IJ} = d\Omega^{IJ} + \Omega^{IK} \wedge \Omega^{KJ} \tag{4.47}$$

and for it we get the following explicit rather simple form:

$$
\begin{aligned}
\mathfrak{R}^{1,2} &= \frac{2\ell^2\left(\tau^2\left(V^3\wedge V^4+V^5\wedge V^6\right)+2\left(V^1\wedge V^2+V^7\wedge V^8\right)\ell^2\right)}{\tau^6}\\
\mathfrak{R}^{1,3} &= \frac{\left(V^1\wedge V^3+V^2\wedge V^4+V^5\wedge V^7+V^6\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{1,4} &= \frac{\left(-V^2\wedge V^3+V^1\wedge V^4+V^6\wedge V^7-V^5\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{1,5} &= \frac{\left(-V^1\wedge V^5+V^2\wedge V^6+V^3\wedge V^7-V^4\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{1,6} &= \frac{\left(-V^2\wedge V^5-V^1\wedge V^6+V^4\wedge V^7+V^3\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{1,7} &= \frac{2\left(V^2\wedge V^8-V^1\wedge V^7\right)\ell^4}{\tau^6}\\
\mathfrak{R}^{1,8} &= -\frac{2\left(V^2\wedge V^7+V^1\wedge V^8\right)\ell^4}{\tau^6}\\
\mathfrak{R}^{2,3} &= \frac{\left(V^2\wedge V^3-V^1\wedge V^4-V^6\wedge V^7+V^5\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{2,4} &= \frac{\left(V^1\wedge V^3+V^2\wedge V^4+V^5\wedge V^7+V^6\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{2,5} &= \frac{\left(-V^2\wedge V^5-V^1\wedge V^6+V^4\wedge V^7+V^3\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{2,6} &= \frac{\left(V^1\wedge V^5-V^2\wedge V^6-V^3\wedge V^7+V^4\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{2,7} &= -\frac{2\left(V^2\wedge V^7+V^1\wedge V^8\right)\ell^4}{\tau^6}\\
\mathfrak{R}^{2,8} &= \frac{2\left(V^1\wedge V^7-V^2\wedge V^8\right)\ell^4}{\tau^6}\\
\mathfrak{R}^{3,4} &= \frac{2\left(2\tau^2\left(V^3\wedge V^4+V^5\wedge V^6\right)+\left(V^1\wedge V^2+V^7\wedge V^8\right)\ell^2\right)}{\tau^4}\\
\mathfrak{R}^{3,5} &= -\frac{2\left(V^3\wedge V^5-V^4\wedge V^6\right)}{\tau^2}\\
\mathfrak{R}^{3,6} &= -\frac{2\left(V^4\wedge V^5+V^3\wedge V^6\right)}{\tau^2}\\
\mathfrak{R}^{3,7} &= \frac{\left(V^1\wedge V^5-V^2\wedge V^6-V^3\wedge V^7+V^4\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{3,8} &= \frac{\left(V^2\wedge V^5+V^1\wedge V^6-V^4\wedge V^7-V^3\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{4,5} &= -\frac{2\left(V^4\wedge V^5+V^3\wedge V^6\right)}{\tau^2}\\
\mathfrak{R}^{4,6} &= \frac{2\left(V^3\wedge V^5-V^4\wedge V^6\right)}{\tau^2}\\
\mathfrak{R}^{4,7} &= \frac{\left(V^2\wedge V^5+V^1\wedge V^6-V^4\wedge V^7-V^3\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{4,8} &= \frac{\left(-V^1\wedge V^5+V^2\wedge V^6+V^3\wedge V^7-V^4\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{5,6} &= \frac{2\left(2\tau^2\left(V^3\wedge V^4+V^5\wedge V^6\right)+\left(V^1\wedge V^2+V^7\wedge V^8\right)\ell^2\right)}{\tau^4}\\
\mathfrak{R}^{5,7} &= \frac{\left(V^1\wedge V^3+V^2\wedge V^4+V^5\wedge V^7+V^6\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{5,8} &= \frac{\left(V^2\wedge V^3-V^1\wedge V^4-V^6\wedge V^7+V^5\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{6,7} &= \frac{\left(-V^2\wedge V^3+V^1\wedge V^4+V^6\wedge V^7-V^5\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{6,8} &= \frac{\left(V^1\wedge V^3+V^2\wedge V^4+V^5\wedge V^7+V^6\wedge V^8\right)\ell^2}{\tau^4}\\
\mathfrak{R}^{7,8} &= \frac{2\ell^2\left(\tau^2\left(V^3\wedge V^4+V^5\wedge V^6\right)+2\left(V^1\wedge V^2+V^7\wedge V^8\right)\ell^2\right)}{\tau^6}
\end{aligned}
\tag{4.48}
$$

From eq.(4.48) we easily extract the components of the Riemann tensor:

$$\mathfrak{R}^{IJ} = \mathrm{Rie}^{IJ}{}_{KL}\,V^K\wedge V^L \tag{4.49}$$

and calculating the Ricci tensor we find that it duely vanishes:

$$\text{Ricci}^{I}_{K} \equiv \text{Rie}^{IL}{}_{KL} = 0 \tag{4.50}$$

The above result inserted into eq. (3.15), together with eq. (3.22) yields the explicit form of the USp(4) curvature two-form which is the following one:

$$\begin{aligned}
\mathbb{R}^{1,1} &= \frac{2\ell^4 \mathbf{e}^1 \wedge \bar{\mathbf{e}}_2}{\tau^6} \\
\mathbb{R}^{1,2} &= \frac{\ell^2\left(\tau^2(\mathbf{e}^4 \wedge \bar{\mathbf{e}}_4 - \mathbf{e}^3 \wedge \bar{\mathbf{e}}_3) + 2\ell^2 \mathbf{e}^1 \wedge \bar{\mathbf{e}}_1 - 2\ell^2 \mathbf{e}^2 \wedge \bar{\mathbf{e}}_2\right)}{\tau^6} \\
\mathbb{R}^{1,3} &= -\frac{\ell^2(\mathbf{e}^1 \wedge \bar{\mathbf{e}}_4 + \mathbf{e}^3 \wedge \bar{\mathbf{e}}_2)}{\tau^4} \\
\mathbb{R}^{1,4} &= \frac{\ell^2(\mathbf{e}^4 \wedge \bar{\mathbf{e}}_2 - \mathbf{e}^1 \wedge \bar{\mathbf{e}}_3)}{\tau^4} \\
\mathbb{R}^{2,2} &= -\frac{2\ell^4 \mathbf{e}^2 \wedge \bar{\mathbf{e}}_1}{\tau^6} \\
\mathbb{R}^{2,3} &= \frac{\ell^2(\mathbf{e}^2 \wedge \bar{\mathbf{e}}_4 - \mathbf{e}^3 \wedge \bar{\mathbf{e}}_1)}{\tau^4} \\
\mathbb{R}^{2,4} &= \frac{\ell^2(\mathbf{e}^2 \wedge \bar{\mathbf{e}}_3 + \mathbf{e}^4 \wedge \bar{\mathbf{e}}_1)}{\tau^4} \\
\mathbb{R}^{3,3} &= \frac{2\mathbf{e}^3 \wedge \bar{\mathbf{e}}_4}{\tau^2} \\
\mathbb{R}^{3,4} &= \frac{2\tau^2(\mathbf{e}^3 \wedge \bar{\mathbf{e}}_3 - \mathbf{e}^4 \wedge \bar{\mathbf{e}}_4) + \ell^2(-(\mathbf{e}^1 \wedge \bar{\mathbf{e}}_1)) + \ell^2 \mathbf{e}^2 \wedge \bar{\mathbf{e}}_2}{\tau^4} \\
\mathbb{R}^{4,4} &= -\frac{2\mathbf{e}^4 \wedge \bar{\mathbf{e}}_3}{\tau^2}
\end{aligned} \tag{4.51}$$

5 The self-dual closed $\Omega^{2,2}$-form on the Calabi HyperKähler manifold $T^{\star}\mathbb{P}^2$ and the associated deformed M2-Brane solution

For the reasons specified in the introduction we want to find $\Omega^{2,2} \in \bigwedge^{2,2}\left(T^{\star}HK^{(2)}_{Calabi}\right)$ such that

$$\star\Omega^{2,2} = \Omega^{2,2} \tag{5.1}$$

$$d\Omega^{2,2} = 0 \tag{5.2}$$

It is convenient to work in the coset frame where we can decompose $\Omega^{2,2}$ along the real vielbein, $\left\{V^I\right\} I: 1,...,8$. Using the complex structure $\mathbf{J}^3$ one can always go back to the complex vielbein, $\left\{\mathbf{e}^{\alpha}, \bar{\mathbf{e}}_{\beta}\right\}$, $\alpha, \beta = 1, \ldots, 4$, which are its eigenstates with eigenvalues $\{\mathrm{i}, -\mathrm{i}\}$, respectively. We find 21 $(2,2)$-self-dual independent basis elements $\{\mathfrak{S}^{\alpha}\}$. A generic solution of (5.1) is: $\Omega^{2,2} = \gamma_p \mathfrak{S}^p$. We choose $\gamma_p = \gamma_p(\tau)$. Now we solve [7] the differential equation (5.2). We get a 4 parameter solution. We can use this solution to deform the M2-Brane $D = 11$ Supergravity background

$$\begin{aligned}
ds^2_{11} &= H(\tau)^{-\frac{2}{3}} ds^2_{Mink_{1,2}} + H(\tau)^{\frac{1}{3}} ds^2_{HK^{(2)}_{Calabi}} \\
\mathbf{A}^{[3]} &= H(\tau)^{-1} \text{Vol}_{Mink_{1,2}}
\end{aligned}$$

[7] This step involves the torsionless equation $dV^I = -\Omega^I_J \wedge V^J$. One gets some equations along $\mathscr{E}^8$ which is outside the coset $\frac{\mathrm{SU(3)}}{\mathrm{U(1)}}$. These latter are 14 independent algebraic equations for $\{\gamma_\alpha\}$.

$$\begin{aligned} \mathbf{F}^{[4]} &= d\mathbf{A}^{[3]} + \Omega^{2,2} \\ \Box H(\tau) &= \star(\Omega^{2,2} \wedge \Omega^{2,2}) \end{aligned} \tag{5.3}$$

To make this deformation consistent we impose $L^2(\ell, +\infty)$-integrability[8] and reality for the source $\star(\Omega^{2,2} \wedge \Omega^{2,2})$. Up to an overall constant $\mathfrak{c}$ we find the following unique solution:

$$\begin{aligned} \Omega^{2,2} &= -\frac{\mathfrak{c}}{2\ell^2\tau^4\left(\ell^2+\tau^2\right)}\Big(\mathbf{e}^1\wedge\mathbf{e}^2\wedge\overline{\mathbf{e}}_1\wedge\overline{\mathbf{e}}_2+\mathbf{e}^3\wedge\mathbf{e}^4\wedge\overline{\mathbf{e}}_3\wedge\overline{\mathbf{e}}_4 \\ &\quad -\mathbf{e}^2\wedge\mathbf{e}^4\wedge\overline{\mathbf{e}}_2\wedge\overline{\mathbf{e}}_4-\mathbf{e}^1\wedge\mathbf{e}^3\wedge\overline{\mathbf{e}}_1\wedge\overline{\mathbf{e}}_3\Big) \\ &\quad +\frac{\mathfrak{c}}{\tau^2\left(\ell^2+\tau^2\right)^3}\Big(\mathbf{e}^1\wedge\mathbf{e}^4\wedge\overline{\mathbf{e}}_1\wedge\overline{\mathbf{e}}_4+\mathbf{e}^1\wedge\mathbf{e}^4\wedge\overline{\mathbf{e}}_2\wedge\overline{\mathbf{e}}_3 \\ &\quad +\mathbf{e}^2\wedge\mathbf{e}^3\wedge\overline{\mathbf{e}}_1\wedge\overline{\mathbf{e}}_4+\mathbf{e}^2\wedge\mathbf{e}^3\wedge\overline{\mathbf{e}}_2\wedge\overline{\mathbf{e}}_3\Big) \\ \star(\Omega^{2,2}\wedge\Omega^{2,2}) &= \frac{\mathfrak{c}^2\left(\ell^8+6\ell^6\tau^2+16\ell^4\tau^4+6\ell^2\tau^6+\tau^8\right)}{\ell^4\tau^8\left(\ell^2+\tau^2\right)^6} \end{aligned} \tag{5.4}$$

Plugging (5.4) in (5.3) we obtain a solution involving only a new integration constant, namely the value of the inhomogeneous harmonic function at infinity H_∞:

$$H(\tau) = \frac{\mathfrak{c}^2\left(5\tau^6+40\ell^6+48\tau^2\ell^4+25\tau^4\ell^2\right)}{320\tau^2\ell^8\left(\tau^2+\ell^2\right)^5} + H_\infty \tag{5.5}$$

In this way, as already done in [27], we have shown that there exists an exact M2-brane solution of $D = 11$ supergravity with the Calabi HyperKähler manfiold $HK^{(2)}_{Calabi}$ as transverse space and a self dual flux of the 4-form. For large values of τ with respect to the resolution parameter ℓ the transverse space metric reduces to the metric cone on the tri-sasakian manifold $\mathrm{N}^{0,1,0}$.

6 Conclusions

In this paper, as we explained in the introduction, we have generalized in a systematic way to curved HyperKähler manifolds the Gaiotto-Witten type of lagrangian for $\mathcal{N}_3 \geq 4$ Chern-Simons gauge theories in $D = 3$. The enhancement conditions are fully geometrical and are encoded in the weaker constraints (1.3-1.4) to be satisfied by the tri-holomorphic moment maps of the gauged isometries.

In the perspective of the gauge/gravity correspondence, the supersymmetric Chern Simons gauge theory is supposed to live on the boundary of an asymptotic AdS_4 manifold and a challenging opportunity emerges since an infinite series of HyperKähler metrics satisfying the enhancement constraints are the HK^n_{Calabi} constructed on the total space of the cotangent bundles $T^\star\mathbb{P}^n$ where the cases $n = 1$ and $n = 2$ respectively correspond to the

[8]In this section we are searching for a self-dual $(2,2)$-form. One could search for an antiself-dual $(2,2)$-form $\mathfrak{w}$, the L^2-integrability condition will imply $\mathfrak{w} = 0$.

Eguchi-Hanson gravitational instanton and to the smooth resolution of the metric cone $\mathscr{C}(\mathrm{N}^{0,1,0})$. The second example is the most interesting one and provides the inspiration for further inquiries and developments that we presently list:

a) The compactification of $D = 11$ supergravity on $\mathrm{AdS}_4 \times \mathrm{N}^{0,1,0}$ is the unique one on a Sasakian homogeneous 7-manifold that yields $\mathscr{N}_4 = 3$ supersymmetry in $D = 4$. In view of the discovered enhancement to $\mathscr{N}_3 = 4$ of the dual Chern-Simons theory on the AdS_4-boundary when the flavor group $\mathrm{SU}(3)$ is gauged, we would like to study the $\mathrm{AdS}_4 \times \mathrm{N}^{0,1,0}$ vacuum in terms of an appropriate gauging of $\mathscr{N}_4 = 3, D = 4$ supergravity.

b) Utilizing the supergravity potential provided by the above mentioned gauging it would be interesting to find its moduli and deformations, looking for other extrema of the potential.

c) As shown in this paper the Chern Simons theory on the AdS_4 boundary where we gauge both the color group $\mathrm{U(N)}$ and the flavor group $\mathrm{SU}(3)$ is enhanced to $\mathscr{N}_3 = 4$ supersymmetry and admits a dual description in terms of a gauged-fixed supergroup Chern Simons theory with supergroup $\mathrm{SU(3|N)}$. Integrating out the color degrees of freedom we obtain a flavor Chern Simons gauge theory with gauge group $\mathrm{SU}(3)$ and target manifold HK^2_{Calabi} which still preserves $\mathscr{N}_3 = 4$. It would be interesting to describe the same theory in the dual supergroup formulation.

d) Since HK^2_{Calabi} admits a self-dual (2,2) harmonic form we can consider M2-brane solutions with self-dual internal fluxes. It would be interesting, in the framework of the gauge/gravity correspondence to retrieve the role of this flux in the Chern-Simons gauge theory on the boundary and to explore all the relations between the supergroup formulation, the $D = 4$ supergravity approach and the $D = 11$ supergravity M2-brane solution.

We plan to investigate such multi-faceted questions in new research projects based on collaborations with Mario Trigiante, Daniele Ruggeri and Laura Andrianapoli.

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A The example of the Eguchi-Hanson space

As mentioned in the introduction, the simplest example of curved HyperKähler manifold whose moment maps satisfy the constraints for supersymmetry enhancement is the time honored Eguchi Hanson space [41].
The EH space can be obtained as the HyperKähel quotient $\mathbb{C}^2 \times \mathbb{C}^{\star 2}//_{HK}U(1)$. We do not perform it explicitly since we would repeat the steps in 4.4. Here, we briefly describe the EH geometry and we give the expression for the tri-holomorphic moment maps associated with the SU(2) action. The geometry of $T^\star\mathbb{P}^1$ (i. e. Eguchi-Hanson) is encoded in the following Kähler potential:

$$\mathscr{K} = \sqrt{4R+\kappa^2} - \kappa log\left(\frac{\sqrt{4R+\kappa^2}+\kappa}{\sqrt{R}}\right) \tag{A.1}$$

$$R = |v|^2(1+|u|^2)^2 \tag{A.2}$$

where u is the coordinate on $\mathbb{P}^1$ and v is the fibre coordinate. The metric is obtained as $g_{ij^\star} = \frac{\partial}{\partial z^i}\frac{\partial}{\partial z^{j^\star}}\mathscr{K}$, $z^i = (u,v)$. The Eguchi-Hanson space is an HyperKähler space. The HyperKähler form is the following:

$$\begin{aligned}
\mathbf{K}^3 &= \mathrm{i} g_{ij^\star} dz^i \wedge d\bar{z}^{j^\star} \\
\mathbf{K}^+ &= 2(du \wedge dv) \\
\mathbf{K}^- &= 2(d\bar{u} \wedge d\bar{v})
\end{aligned} \tag{A.3}$$

An element of the isometry group $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2)$ acts in the following way

$$u \longrightarrow \frac{au+b}{cu+d}\ ,\ v \longrightarrow v(cu+d)^2 \tag{A.4}$$

This isometry is generated by the following holomorphic Killing vectors[9]

$$k = k_\Lambda T^\Lambda\ ,\ T^\Lambda = \tfrac{\mathrm{i}}{2}\sigma^\Lambda$$

$$k_1 = \frac{\mathrm{i}}{2}(u^2-1)\frac{\partial}{\partial u} + \mathrm{i}uv\frac{\partial}{\partial v}\ , \quad k_2 = \tfrac{1}{2}(u^2+1)\tfrac{\partial}{\partial u} - uv\tfrac{\partial}{\partial v} \quad , \ k_3 = \mathrm{i}u\frac{\partial}{\partial u} - \mathrm{i}v\frac{\partial}{\partial v} \tag{A.5}$$

Thanks to the HyperKähler structure we can define the tri-holomorphic moment maps

$$\begin{aligned}
d\mathscr{P}^3_\Lambda &= i_{k_\Lambda}\mathbf{K}^3 \\
d\mathscr{P}^+_\Lambda &= i_{k_\Lambda}\mathbf{K}^+ \\
d\mathscr{P}^-_\Lambda &= i_{k_\Lambda}\mathbf{K}^-
\end{aligned}$$

[9] σ^Λ , $\Lambda = 1,2,3$ are the standard Pauli matrices

These latter imply that $\mathscr{P}^3_\Lambda$ is defined modulo the real part of an holomorphic function while $\mathscr{P}^+_\Lambda$ is defined modulo a constant. Thanks to this freedom we can make the tri-holomorphic moment map equivariant, namely:

$$\{\mathscr{P}^x_\Lambda, \mathscr{P}^x_\Sigma\} \equiv i_{k_\Lambda} i_{k_\Sigma} \mathbf{K}^x = f^\Gamma_{\Lambda\Sigma} \mathscr{P}^x_\Gamma \tag{A.6}$$

The equivariant real moment maps are

$$\begin{aligned}
\mathscr{P}^3_1 &= -\frac{(u+\overline{u})\sqrt{4R+\kappa^2}}{2(1+|u|^2)} \\
\mathscr{P}^3_2 &= -\mathrm{i}\frac{(u-\overline{u})\sqrt{4R+\kappa^2}}{2(1+|u|^2)} \\
\mathscr{P}^3_3 &= -\frac{(1-|u|^2)\sqrt{4R+\kappa^2}}{2(1+|u|^2)}
\end{aligned} \tag{A.7}$$

The equivariant holomorphic moment maps are

$$\begin{aligned}
\mathscr{P}^+_1 &= \mathrm{i}(1-u^2)v \\
\mathscr{P}^+_2 &= (1+u^2)v \\
\mathscr{P}^+_3 &= 2\mathrm{i}uv
\end{aligned}$$

Now we choose $\mathfrak{m}_{\Lambda\Sigma} = \kappa_{\Lambda\Sigma}$ to be the Cartan-Killing metric of $\mathrm{SU}(2)$. We find

$$\mathscr{P}^+ \cdot \mathscr{P}^+ = 0 \ , \ \mathscr{P}^+ \cdot \mathscr{P}^3 = 0 \tag{A.8}$$

and

$$\mathscr{P}^3 \cdot \mathscr{P}^3 = \frac{\kappa^2}{4} + R \ , \ \mathscr{P}^+ \cdot \mathscr{P}^- = 2R \tag{A.9}$$

so that

$$2\mathscr{P}^3 \cdot \mathscr{P}^3 - \mathscr{P}^+ \cdot \mathscr{P}^- = \frac{\kappa^2}{2} \tag{A.10}$$

From these identities we see that the constraints (1.3-1.4) hold true.

The Eguchi-Hanson space is the first element in the infinite series of HyperKähler manifolds $T^\star\mathbb{P}^{1+n}$. Now we present the next case which is physically more interesting.

B Parameterizing the $\mathrm{N}^{0,1,0}$ coset representative

In this appendix we study a suitable parameterization of the $\mathrm{N}^{0,1,0}$ coset which reflects its double fibration structure.

B.1 The double fibration and the coset representative of the flag manifold

The first step in our construction consists of establishing a good parameterization of the coset $N^{0,1,0}$ as described in equation (4.9). To this effect we use a double fibration, namely we regard the flag manifold $\mathfrak{m}_6^F$ as a $\mathbb{P}^1$ fibration over $\mathbb{P}^2$:

$$\mathfrak{m}_6^F \xrightarrow{\pi} \mathbb{P}^2 \quad ; \quad \forall p \in \mathbb{P}^2 \quad \pi^{-1}(p) \sim \mathbb{P}^1 \tag{B.1}$$

Regarding $\mathbb{P}^2$ as the standard coset manifold $\mathrm{SU}(3)/\mathrm{SU}(2) \times \mathrm{U}(1)$, the usual complex coordinates $u_{1,2}$ in which the $\mathrm{SU}(3)$ invariant Kähler metric on $\mathbb{P}^2$ takes the familiar Fubini-Study form are encoded in the following coset representative[10]:

$$\mathbb{L}_{\mathbb{P}^2} = \begin{pmatrix} \frac{\frac{u_1\overline{u}_1}{\sqrt{|\mathbf{u}|^2+1}}+u_2\overline{u}_2}{|\mathbf{u}|^2} & \frac{u_1\left(\frac{1}{\sqrt{|\mathbf{u}|^2+1}}-1\right)\overline{u}_2}{|\mathbf{u}|^2} & \frac{u_1}{\sqrt{|\mathbf{u}|^2+1}} \\ \frac{u_2\left(\frac{1}{\sqrt{|\mathbf{u}|^2+1}}-1\right)\overline{u}_1}{|\mathbf{u}|^2} & \frac{\frac{u_2\overline{u}_2}{\sqrt{|\mathbf{u}|^2+1}}+u_1\overline{u}_1}{|\mathbf{u}|^2} & \frac{u_2}{\sqrt{|\mathbf{u}|^2+1}} \\ -\frac{\overline{u}_1}{\sqrt{|\mathbf{u}|^2+1}} & -\frac{\overline{u}_2}{\sqrt{|\mathbf{u}|^2+1}} & \frac{1}{\sqrt{|\mathbf{u}|^2+1}} \end{pmatrix} \in \mathrm{SU}(3) \quad ; \quad |\mathbf{u}|^2 \equiv |u_1|^2 + |u_2|^2 \tag{B.2}$$

Indeed, calculating the left-invariant 1-form:

$$\Lambda_{\mathbb{P}^2} = \mathbb{L}^\dagger_{\mathbb{P}^2}\, \mathrm{d}\mathbb{L}_{\mathbb{P}^2} \tag{B.3}$$

and defining the vierbein of the manifold $\mathbb{P}^2$ as[11] :

$$\left\{E^1_{\mathbb{P}^2}, E^2_{\mathbb{P}^2}, E^3_{\mathbb{P}^2}, E^4_{\mathbb{P}^2}\right\} = -2\left\{\mathrm{Tr}\left(\mathbf{t}_4\,\Lambda_{\mathbb{P}^2}\right), \mathrm{Tr}\left(\mathbf{t}_5\,\Lambda_{\mathbb{P}^2}\right), \mathrm{Tr}\left(\mathbf{t}_6\,\Lambda_{\mathbb{P}^2}\right), \mathrm{Tr}\left(\mathbf{t}_7\,\Lambda_{\mathbb{P}^2}\right)\right\} \tag{B.4}$$

we obtain the standard Fubini-Study line element:

$$\begin{aligned} ds^2_{\mathbb{P}^2} &= \sum_{I=1}^{4}\left(E^I_{\mathbb{P}^2}\right)^2 \\ &= \frac{\mathrm{d}u_1\,\mathrm{d}\overline{u}_1(1+u_2\,\overline{u}_2)+\mathrm{d}u_2\,\mathrm{d}\overline{u}_2(1+u_1\,\overline{u}_1)-u_1\,\overline{u}_2\mathrm{d}u_2\mathrm{d}\overline{u}_1-u_2\,\overline{u}_1\mathrm{d}u_1\mathrm{d}\overline{u}_2}{(|\mathbf{u}|^2+1)^2} \end{aligned} \tag{B.5}$$

Next we introduce the coset representative of $\mathrm{SU}(2)/\mathrm{U}(1)$ immersed in $\mathrm{SU}(3)$. We set:

$$\Lambda_{\mathbb{P}^1} = \begin{pmatrix} \sqrt{\frac{1}{|v|^2+1}} & \frac{v}{\sqrt{|v|^2+1}} & 0 \\ -\frac{\overline{v}}{\sqrt{|v|^2+1}} & \sqrt{\frac{1}{|v|^2+1}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SU}(3) \tag{B.6}$$

[10] In this definition, u and v must not be confused with the flat $\mathbb{C}^3 \oplus \mathbb{C}^{\star 3}$ coordinates related to the HyperKähler quotient construction.

[11] The formula below is justified because the four generators of the subalgebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ are $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_8$, so that the coset generators are $\mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6, \mathbf{t}_7$.

which, with the same logic as before, produces the standard Fubini-Study metric on $\mathbb{P}^1$. In this case the zweibein of $\mathbb{P}^1$ is obtained by tracing the left invariant one form:

$$\Lambda_{\mathbb{P}^1} = \mathbb{L}^\dagger_{\mathbb{P}^1} \, \mathrm{d}\mathbb{L}_{\mathbb{P}^1} \tag{B.7}$$

with $\mathbf{t}_1$ and $\mathbf{t}_2$ that are the coset generators inside the SU(2) subalgebra of SU(3) spanned by the generators $\mathbf{t}_{1,2,3}$.

In this way a convenient dense chart for the flag manifold $\mathfrak{m}_6^F$ is provided by the three complex coordinates u_1, u_2, v. Correspondingly we can define the coset representative for $\mathfrak{m}_6^F$ as follows:

$$\mathbb{L}_{flag} = \mathbb{L}_{\mathbb{P}^2} \, \mathbb{L}_{\mathbb{P}^1} \tag{B.8}$$

So doing the left invariant 1-form of $\mathfrak{m}_6^F$ takes the form:

$$\Lambda_{flag} \equiv \mathbb{L}^\dagger_{flag} \, \mathrm{d}\mathbb{L}_{flag} = \Lambda_{\mathbb{P}^1} + \mathbb{L}^\dagger_{\mathbb{P}^1} \, \Lambda_{\mathbb{P}^2} \mathbb{L}_{\mathbb{P}^1} \tag{B.9}$$

which exposes the fibred structure (B.1) of the flag manifold. The sechsbein of $\mathfrak{m}_6^F$ is provided by[12]:

$$\begin{aligned} &\left\{E^1_{flag}, E^2_{flag}, E^3_{flag}, E^4_{flag}, E^5_{flag}, E^6_{flag},\right\} = \\ &-2\left\{\mathrm{Tr}\left(\mathbf{t}_1 \, \Lambda_{flag}\right), \mathrm{Tr}\left(\mathbf{t}_2 \, \Lambda_{flag}\right), \mathrm{Tr}\left(\mathbf{t}_4 \, \Lambda_{flag}\right), \mathrm{Tr}\left(\mathbf{t}_5 \, \Lambda_{flag}\right), \mathrm{Tr}\left(\mathbf{t}_6 \, \Lambda_{flag}\right), \mathrm{Tr}\left(\mathbf{t}_7 \, \Lambda_{flag}\right)\right\} \end{aligned} \tag{B.10}$$

In the explicit calculation, if needed, of the coordinate dependence of the vielbein E^I_{flag}, the structure of (B.9) of the left-invariant one-form is very useful. Indeed naming:

$$\mathscr{E}^\Sigma_{\mathbf{n}} = -2 \, \mathrm{Tr}\left(\mathbf{t}_\Sigma \, \Lambda_{\mathbf{n}}\right) \quad ; \quad \Lambda_{\mathbf{n}} \equiv \mathbb{L}^\dagger_{\mathbf{n}} \mathrm{d}\mathbb{L}_{\mathbf{n}} \tag{B.11}$$

the components along the generators (4.4) of any SU(3) left-invariant form we see that under the subgroup $\mathrm{SU}(2) \subset \mathrm{SU}(3)$ the generators and hence the corresponding 1-forms are organized in the following representations:

$$\underbrace{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3}_{\text{triplet} = \text{adjoint}} \quad \oplus \quad \underbrace{\mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6, \mathbf{t}_7}_{\text{complex doublet} = \text{4-dim irrep}} \quad \oplus \quad \underbrace{\mathbf{t}_8}_{\text{singlet}} \tag{B.12}$$

We name $x, y, \cdots = 1, 2, 3$ the indices of the triplet, $\alpha, \beta, \cdots = 4, 5, 6, 7$ the indices of the real quadruplet and we keep 8 for the singlet. According to this we conclude that:

$$\begin{aligned} \mathbb{L}_{\mathbb{P}^1} \, \mathbf{t}_x \mathbb{L}^\dagger_{\mathbb{P}^1} &= \mathscr{S}_{xy}(v, \overline{v}) \, \mathbf{t}_y \\ \mathbb{L}_{\mathbb{P}^1} \, \mathbf{t}_\alpha \mathbb{L}^\dagger_{\mathbb{P}^1} &= \mathscr{T}_{\alpha\beta}(v, \overline{v}) \, \mathbf{t}_\beta \\ \mathbb{L}_{\mathbb{P}^1} \, \mathbf{t}_8 \mathbb{L}^\dagger_{\mathbb{P}^1} &= \mathbf{t}_8 \end{aligned} \tag{B.13}$$

[12] The generators of the coset manifold in this case are $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6, \mathbf{t}_7$

where the 3×3 matrix $\mathscr{S}_{xy}(v,\overline{v})$ and the 4×4 one $\mathscr{T}_{\alpha\beta}(v,\overline{v})$ depend only on the fibre coordinates $v,\overline{v}$. Looking now at the traces appearing in equation (B.10), we see that the final form of the sechsbein is as follows:

$$\begin{array}{rcl} E^x_{flag} & = & \mathscr{E}^x_{\mathbb{P}^1} + \mathscr{S}^{xy}(v,\overline{v})\,\mathscr{E}^y_{\mathbb{P}^2} \quad ; \quad x=1,2, \quad ; \quad y=1,2,3 \\ E^\alpha_{flag} & = & \mathscr{T}^{\alpha\beta}(v,\overline{v})\,\mathscr{E}^\beta_{\mathbb{P}^2} \quad ; \quad \alpha,\beta=4,5,6,7 \end{array} \tag{B.14}$$

By construction the 1-forms $\mathscr{E}^x_{\mathbb{P}^1}$ depend only on the coordinates $v,\overline{v}$, while $\mathscr{E}^x_{\mathbb{P}^2}$ and $\mathscr{E}^\alpha_{\mathbb{P}^2}$ depend only on the coordinates $u_1,u_2,\overline{u}_1,\overline{u}_2$ of the $\mathbb{P}^2$ base manifold.

B.2 The coset representative of $\mathrm{N}^{0,1,0}$

The next step of the construction consists of building the coset representative of the 7-dimensional coset $\mathrm{N}^{0,1,0}$ regarded as a $\mathrm{U}(1)$ fibration over the flag manifold that we studied in section B.1. The strategy is identical to that used in the construction of the flag manifold vielbein. We introduce the $\mathrm{U}(1)$ group element obtained by exponentiating the generator $\mathbf{t}_3$:

$$\mathbb{L}_{\mathrm{U}(1)} = \begin{pmatrix} e^{\frac{i\psi}{2}} & 0 & 0 \\ 0 & e^{-\frac{i\psi}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SU}(3) \tag{B.15}$$

and we write the complete coset representative as follows:

$$\mathbb{L}_{\mathrm{N}^{010}} = \mathbb{L}_{flag}\,\mathbb{L}_{\mathrm{U}(1)} = \mathbb{L}_{\mathbb{P}^2}\,\mathbb{L}_{\mathbb{P}^1}\,\mathbb{L}_{\mathrm{U}(1)} \tag{B.16}$$

Introducing the complete left-invariant one form:

$$\Lambda_{\mathrm{N}^{010}} = \mathbb{L}^\dagger_{\mathrm{N}^{010}}\,\mathrm{d}\mathbb{L}_{\mathrm{N}^{010}} \tag{B.17}$$

The vielbein of the 7-manifold are given by:

$$\begin{array}{l} \left\{E^1_{\mathrm{N}^{010}}, E^2_{\mathrm{N}^{010}}, E^3_{\mathrm{N}^{010}}, E^4_{\mathrm{N}^{010}}, E^5_{\mathrm{N}^{010}}, E^6_{\mathrm{N}^{010}}, E^7_{\mathrm{N}^{010}}\right\} = \\ -2\left\{\mathrm{Tr}\left(\mathbf{t}_1\,\Lambda_{\mathrm{N}^{010}}\right), \mathrm{Tr}\left(\mathbf{t}_2\,\Lambda_{\mathrm{N}^{010}}\right), \mathrm{Tr}\left(\mathbf{t}_4\,\Lambda_{\mathrm{N}^{010}}\right), \mathrm{Tr}\left(\mathbf{t}_5\,\Lambda_{\mathrm{N}^{010}}\right), \mathrm{Tr}\left(\mathbf{t}_6\,\Lambda_{\mathrm{N}^{010}}\right), \mathrm{Tr}\left(\mathbf{t}_7\,\Lambda_{\mathrm{N}^{010}}\right), \mathrm{Tr}\left(\mathbf{t}_3\Lambda_{\mathrm{N}^{010}}\right)\right\} \end{array} \tag{B.18}$$

and the doubled fibred-structure displayed in eq.(B.16) can be utilized to work out the explicit dependence of the 7-vielbein on the seven well-adapted coordinates ψ,v,u_1,u_2 (one real and three complex) if this is needed. It suffices to specialize to $\mathrm{U}_3(1)\subset\mathrm{SU}(2)$ the analysis performed in eq.s (B.13) for the full subgroup $\mathrm{SU}(2)\subset\mathrm{SU}(3)$. The nested fibred structure is also useful to work out the explicit transformations of the coordinates $\mathbf{u},v,\psi$ of the manifold $\mathrm{N}^{0,1,0}$ under the isometry group $\mathrm{SU}(3)$. Indeed the compensator subgroup H of each of the three factors in eq.(B.16) is the G group of the next factor. Hence we expect the following.

Let:

$$\mathfrak{g} = \left(\begin{array}{c|c} A_{2\times 2} & B_{2\times 1} \\ \hline C_{1\times 2} & D_{1\times 1} \end{array}\right) \in \mathrm{SU}(3) \tag{B.19}$$

be a group element of $\mathrm{SU}(3)$. The transformation induced on the $\mathbb{P}^2$ coordinates will be holomorphic and projective linear fractional:

$$\mathbf{u}' = (A\,\mathbf{u} + B)\cdot(C\mathbf{u} + D)^{-1} \tag{B.20}$$

that induced on the fibre coordinate will also be fractional linear, but $\mathbf{u}$-dependent:

$$v' = \big(a(\mathfrak{g},\mathbf{u})\,v + b(\mathfrak{g},d\mathbf{u})\big)\cdot\big(c(\mathfrak{g},\mathbf{u})v + d(\mathfrak{g},\mathbf{u})\big)^{-1} \tag{B.21}$$

where the coefficients a,b,c,d appearing in the above formula are those displayed by the compensator matrix in the subgroup $\mathrm{SU}(2)\times \mathrm{U}_8(1)$:

$$\mathfrak{H}_{comp|\mathbb{P}^2}(\mathfrak{g},\mathbf{u}) \;=\; \left(\begin{array}{cc|c} a(\mathfrak{g},\mathbf{u}) & b(\mathfrak{g},\mathbf{u}) & 0 \\ c(\mathfrak{g},\mathbf{u}) & d(\mathfrak{g},\mathbf{u}) & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\cdot \underbrace{\left(\begin{array}{cc|c} \exp[i\mu(\mathfrak{g},\mathbf{u})] & 0 & 0 \\ 0 & \exp[i\mu(\mathfrak{g},\mathbf{u})] & 0 \\ \hline 0 & 0 & \exp[-2i\mu(\mathfrak{g},\mathbf{u})] \end{array}\right)}_{\mathfrak{H}_8(\mathfrak{g},\mathbf{u})} \tag{B.22}$$

As it happens in all coset manifolds and for any choice of the coset representative, the compensator depends both on the point $\mathbf{u}$ and on the choice of the group element $\mathfrak{g}$ acting as an isometry. At the next step we will have a compensator depending both on the coordinates $\mathbf{u}$ and on the coordinate v

$$\mathfrak{H}_{comp|\mathbb{P}^1}(\mathfrak{g},\mathbf{u},v) = \left(\begin{array}{cc|c} \exp[\frac{i}{2}\lambda(\mathfrak{g},\mathbf{u},v)] & 0 & 0 \\ 0 & \exp[-\frac{i}{2}\lambda(\mathfrak{g},\mathbf{u},v)] & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\cdot \mathfrak{H}_8(\mathfrak{g},\mathbf{u}) \tag{B.23}$$

and this determines the transformation of the coordinate ψ:

$$\psi' = \psi + \lambda(\mathfrak{g},\mathbf{u},v) \tag{B.24}$$

The procedure outlined above allows the construction of all Killing vectors for the manifold $\mathrm{N}^{0,1,0}$ and eventually for the resolution of its metric cone that is given by the Calabi HyperKähler manifold $HK^{(2)}_{Calabi}$. This is also the starting point for the calculation of the moment maps of the $\mathrm{SU}(3)$ isometries. For the actual construction of $HK^{(2)}_{Calabi}$ we do not need any explicit parameterization of the coset manifold. The Maurer-Cartan equations satisfied by the left-invariant one-forms are completely sufficient.

C Calculation of moment maps in two cases

In this section we calculate the moment maps in two cases relevant to our discussion:

a) For the $\mathrm{SU}(3)$ isometry of the curved Calabi HyperKähler manifold $HK^{(2)}_{Calabi}$

b) For the linearly realized $\mathrm{SU(m)} \times \mathrm{SU(n)} \times \mathrm{U}(1)$ isometry of a flat HyperKähler manifold with $2 \times m \times n$ complex coordinates.

C.1 The moment maps of the $\mathrm{SU}(3)$ isometries of the Calabi metric on $T^*\mathbb{P}^2$

In this section we calculate the moment maps of the relevant isometry group $\mathrm{SU}(3)$ in the case of the curved HyperKähler manifold $T^*\mathbb{P}^2$ endowed with the Calabi metric and we verify that they satisfy the weak constraint (1.3-1.4) necessary for the supersymmetry enhancement discussed in the main text.

C.1.1 Transformation of the $T^*\mathbb{P}^2$ complex coordinates under the isometry group $\mathrm{SU}(3)$

We denote the complex coordinates of $T^*\mathbb{P}^2$ as $q = (w^1, w^2 | z_1, z_2)$. The first pair provides a chart on the base $\mathbb{P}^2$, while the second pair on the fibre. Then let us write an element of the isometry group in a block form

$$\mathfrak{g} = \left(\begin{array}{c|c} A_{2\times 2} & B_{2\times 1} \\ \hline C_{1\times 2} & D_{1\times 1} \end{array}\right) \in \mathrm{SU}(3). \tag{C.1}$$

The transformation of the $\mathbb{P}^2$ coordinates will be holomorphic and projective linear fractional

$$\mathbf{w}' = (A\mathbf{w} + B)(C\mathbf{w} + D)^{-1}. \tag{C.2}$$

This induces a transformation of the fibre coordinates given by the inverse transformation of the differentials of the base coordinates. Defining

$$\delta = C\mathbf{w} + D \in \mathbb{C}, \qquad M = (A\mathbf{w} + B) \otimes C \in \mathrm{Mat}_{2\times 2} \tag{C.3}$$

we obtain the transformation of the differentials of $\mathbf{w}$

$$d\mathbf{w}' = \frac{\delta A - M}{\delta^2} \equiv F^{-1} d\mathbf{w}. \tag{C.4}$$

Thus the transformation of the fibre coordinates $\mathbf{z}$ takes the form

$$\mathbf{z}' = F\mathbf{z}. \tag{C.5}$$

Next, we verify that the above transformations implied by the structure of a cotangent bundle leave invariant the Kähler potential as derived by the HyperKähler quotient construction (see section 4.4). This check supports the claim that the manifold HK_8 emerging from the HyperKähler quotient construction can in fact be identified with $T^*\mathbb{P}^2$. The Kähler potential reads

$$\mathscr{K} = F(R) + \kappa \log(1 + \mathbf{w}^\dagger \mathbf{w}). \tag{C.6}$$

It is not hard to observe that the second term is invariant up to a real part of a holomorphic function, which can be absorbed by a Kähler transformation

$$\log(1 + \mathbf{w}'^\dagger \mathbf{w}') = \log(1 + \mathbf{w}^\dagger \mathbf{w}) - \underbrace{\left(\log\delta + \log\overline{\delta}\right)}_{\text{Kähler transformation}}. \tag{C.7}$$

Therefore it is enough to show that the function [13]

$$R(q,\overline{q}) = (1 + \mathbf{w}^\dagger \mathbf{w})\left[\mathbf{z}^\dagger(\mathbf{1}_{2\times 2} + \overline{\mathbf{w}}\mathbf{w}^T)\mathbf{z}\right] = (1 + \|\mathbf{w}\|^2)(\|\mathbf{z}\|^2 + |\mathbf{w}\cdot\mathbf{z}|^2) \tag{C.8}$$

is invariant. This can be achieved by computing finite transformations of the coordinates with respect to all one-parameter subgroups of $\mathrm{SU}(3)$

$$\mathfrak{g}_\Lambda(t) = e^{tT_\Lambda} \text{ with } T_\Lambda = \frac{i}{2}\lambda_\Lambda \in \mathfrak{su}(3), \;\; \Lambda = 1,\ldots,8, \tag{C.9}$$

where λ_Λ are the Gell-Mann matrices. We do not list here the finite transformations of coordinates, rather write down their infinitesimal form in terms of Killing vectors in the next section. In any case, it can be explicitly verified that the function R is invariant under all finite transformations associated with individual generators of the isometry group. This concludes the proof that the Kähler potential of the HyperKähler quotient is invariant under the isometry group of $T^*\mathbb{P}^2$.

C.1.2 Killing vectors

The action of the isometry group $\mathrm{SU}(3)$ on the coordinates of $T^*\mathbb{P}^2$ yields at the infinitesimal level the (holomorphic) Killing vectors

$$k^i_\Lambda = \frac{d}{dt}\left(\mathfrak{g}_\Lambda(t) q^i\right)\Big|_{t=0}. \tag{C.10}$$

[13] We are thinking of $\mathbf{w}$ and $\mathbf{z}$ as column vectors.

They are displayed in matrix form and the Killing vector corresponding to the generator T_Λ is stored in the Λ-th row [14]

$$k^i_\Lambda = \begin{pmatrix} \frac{iw_2}{2} & \frac{iw_1}{2} & -\frac{1}{2}(iz_2) & -\frac{1}{2}(iz_1) \\ \frac{w_2}{2} & -\frac{w_1}{2} & \frac{z_2}{2} & -\frac{z_1}{2} \\ \frac{iw_1}{2} & -\frac{1}{2}(iw_2) & -\frac{1}{2}(iz_1) & \frac{iz_2}{2} \\ -\frac{1}{2}i\left(w_1^2-1\right) & -\frac{1}{2}iw_1w_2 & \frac{1}{2}i(2w_1z_1+w_2z_2) & \frac{1}{2}iw_1z_2 \\ \frac{1}{2}\left(w_1^2+1\right) & \frac{w_1w_2}{2} & -w_1z_1-\frac{w_2z_2}{2} & -\frac{1}{2}w_1z_2 \\ -\frac{1}{2}iw_1w_2 & -\frac{1}{2}i\left(w_2^2-1\right) & \frac{1}{2}iw_2z_1 & \frac{1}{2}i(w_1z_1+2w_2z_2) \\ \frac{w_1w_2}{2} & \frac{1}{2}\left(w_2^2+1\right) & -\frac{1}{2}w_2z_1 & -\frac{1}{2}w_1z_1-w_2z_2 \\ \frac{1}{2}i\sqrt{3}w_1 & \frac{1}{2}i\sqrt{3}w_2 & -\frac{1}{2}i\sqrt{3}z_1 & -\frac{1}{2}i\sqrt{3}z_2 \end{pmatrix}. \tag{C.11}$$

These Killing vectors satisfy the $\mathfrak{su}(3)$ Lie algebra

$$[k_\Lambda, k_\Sigma] = -f^\Gamma_{\Lambda\Sigma} k_\Gamma \tag{C.12}$$

with the structure constants defined as

$$f^\Gamma_{\Lambda\Sigma} = -2\mathrm{Tr}\left([T_\Lambda, T_\Sigma]\, T^\Gamma\right). \tag{C.13}$$

C.1.3 The $\mathrm{SU}(3)$ moment maps

The knowledge of Killing vectors allows us to look for their potentials, i.e. the associated moment maps $\mathscr{P}^x_\Lambda$. They are defined by the formula

$$d\mathscr{P}^x_\Lambda = i_{k_\Lambda}\mathbf{K}^x, \;\; x = 1,2,3. \tag{C.14}$$

Here $\mathbf{K}^x$ is the triplet of Kähler forms. We pick $\mathbf{K}^3$ and associate it with the real moment map $\mathscr{P}^3$. The remaining two Kähler forms form (anti)-holomorphic combinations $\mathbf{K}^\pm = \mathbf{K}^1 \pm i\mathbf{K}^2$, which correspond to (anti)-holomorphic moment maps $\mathscr{P}^\pm = \mathscr{P}^1 \pm i\mathscr{P}^2$

$$d\mathscr{P}^3_\Lambda = i_{k_\Lambda}\mathbf{K}^3 \tag{C.15}$$

$$d\mathscr{P}^\pm_\Lambda = i_{k_\Lambda}\mathbf{K}^\pm. \tag{C.16}$$

The Kähler form $\mathbf{K}^3$ is associated with the metric and therefore also with the Kähler potential as $\mathbf{K}^3 = i\partial\overline{\partial}\mathscr{K}$. Consequently, a general solution can be constructed for the real moment map in (C.15). Indeed, projecting it to

[14] For practical reasons we have shifted the indices of $\mathbf{w}$ coordinates from top to the bottom.

$(1,0)$ and $(0,1)$ components leads to a system of two equations

$$\partial_i \mathscr{P}^3_\Lambda = -i g_{ij^\star} \overline{k}^{j^\star}_\Lambda = -i\overline{k}^{j^\star}_\Lambda \partial_i \overline{\partial}_{j^\star} \mathscr{K} = \partial_i \left(-i\overline{k}^{j^\star}_\Lambda \overline{\partial}_{j^\star} \mathscr{K} \right) \tag{C.17}$$

$$\overline{\partial}_{j^\star} \mathscr{P}^3_\Lambda = i g_{ij^\star} k^i_\Lambda = i k^i_\Lambda \partial_i \overline{\partial}_{j^\star} \mathscr{K} = \overline{\partial}_{j^\star} \left(i k^i_\Lambda \partial_i \mathscr{K} \right). \tag{C.18}$$

From (C.17) follows

$$\mathscr{P}^3_\Lambda = -i\overline{k}^{j^\star}_\Lambda \overline{\partial}_{j^\star} \mathscr{K} + \overline{f}^1_\Lambda(\overline{q}), \tag{C.19}$$

while from (C.18) one gets

$$\mathscr{P}^3_\Lambda = i k^j_\Lambda \partial_j \mathscr{K} + f^2_\Lambda(q). \tag{C.20}$$

Reality of P^3 implies $f^1_\Lambda(q) = f^2_\Lambda(q) \equiv h_\Lambda(q)$ and one needs to take the symmetric combination to make it manifestly real

$$\mathscr{P}^3_\Lambda = \frac{i}{2} \left(k^j_\Lambda \partial_j \mathscr{K} - \overline{k}^{j^\star}_\Lambda \overline{\partial}_{j^\star} \mathscr{K} \right) + \frac{1}{2} \left(h_\Lambda(q) + \overline{h}_\Lambda(\overline{q}) \right). \tag{C.21}$$

Thus we just showed that there is freedom in shifting the real moment map by a real part of a holomorphic function. It is important as we will exploit this fact to impose equivariance on the moment maps (with respect to the SU(3) action). Ultimately, only the equivariant moment maps are supposed to fulfill constraints, which allow for supersymmetry enhancement from $\mathscr{N}_3 = 3$ to $\mathscr{N}_3 = 4$. On the other hand, the difference of the above equations fixes the imaginary part of $h_\Lambda(q)$

$$h_\Lambda(q) - \overline{h}_\Lambda(\overline{q}) = -i \left(k^j_\Lambda \partial_j \mathscr{K} + \overline{k}^{j^\star}_\Lambda \overline{\partial}_{j^\star} \mathscr{K} \right). \tag{C.22}$$

In (C.21) reality of $\mathscr{P}^3$ is manifest, however it obscures the canonical relation for the moment maps as potentials for the Killing vectors

$$k^i_\Lambda = -i g^{j^\star i} \overline{\partial}_{j^\star} \mathscr{P}^3_\Lambda. \tag{C.23}$$

The apparent problem gets resolved once we apply the equation that fixes $\mathrm{Im}(h_\Lambda)$. In fact substituting either for $h_\Lambda(q)$ or $\overline{h}_\Lambda(\overline{q})$ brings us back to (C.19) or (C.20), respectively. This operation of course does not break reality, just makes it less manifest. In other words, the expression for $\mathrm{Im}(h_\Lambda)$ allows us to transfer between two equivalent forms for $\mathscr{P}^3$ – one that is manifestly real and the other one that clearly displays the canonical relation (C.23) between the Killing vector and the associated moment map.

In order to get an explicit expression for $\mathscr{P}^3$, we plug in the Killing vectors (C.11) and the Kähler potential given in (C.6) into (C.21). It is best to keep the function $F(R)$ implicit during the computation. Even so, the final formulae for the real moment map are not as neat as for the rest of the objects. We list the result component-wise for each generator of $\mathfrak{su}(3)$

$$\mathscr{P}^3_1 = -\frac{\kappa \left(w_2 \overline{w}_1 + w_1 \overline{w}_2 \right)}{2\left(\|\mathbf{w}\|^2 + 1 \right)} - \frac{1}{2} \Bigg[\overline{z}_1 \Bigg(z_1 \left(|w_1|^2 + 1 \right) \left(w_2 \overline{w}_1 + w_1 \overline{w}_2 \right)$$

$$
\begin{aligned}
&+z_2\left(w_2^2\overline{w}_1^2+|w_1|^2|w_2|^2-(1+\|\mathbf{w}\|^2)\right)\Bigg)+\overline{z}_2\Bigg(z_2\left(w_2\overline{w}_1+w_1\overline{w}_2\right)\left(|w_2|^2+1\right)\\
&+z_1\left(w_1^2\overline{w}_2^2+|w_1|^2|w_2|^2-(1+\|\mathbf{w}\|^2)\right)\Bigg)\Bigg]F'(R)\\
\mathscr{P}_2^3&=\frac{i\kappa\left(w_2\overline{w}_1-w_1\overline{w}_2\right)}{2(\|\mathbf{w}\|^2+1)}+\frac{i}{2}\Bigg[\overline{z}_1\Bigg(z_1(|w_1|^2+1)\left(w_2\overline{w}_1-w_1\overline{w}_2\right)\\
&+z_2\left(w_2^2\overline{w}_1^2-|w_1|^2|w_2|^2+(1+\|\mathbf{w}\|^2)\right)\Bigg)+\overline{z}_2\Bigg(z_2\left(w_2\overline{w}_1-w_1\overline{w}_2\right)\left(|w_2|^2+1\right)\\
&-z_1\left(w_1^2\overline{w}_2^2-|w_1|^2|w_2|^2+(1+\|\mathbf{w}\|^2)\right)\Bigg)\Bigg]F'(R)\\
\mathscr{P}_3^3&=-\frac{\kappa(|w_1|^2-|w_2|^2)}{2(\|\mathbf{w}\|^2+1)}+\frac{1}{2}\Bigg[z_1\Bigg(\overline{z}_2w_1\overline{w}_2(|w_2|^2-|w_1|^2)+\overline{z}_1\Bigg(|w_1|^2\left(|w_2|^2-|w_1|^2\right)\\
&+2|w_2|^2+1\Bigg)+z_2\Bigg(\overline{z}_1\overline{w}_1w_2\left(|w_1|^2-|w_2|^2\right)\\
&+\overline{z}_2\Bigg(|w_2|^2\left(|w_2|^2-|w_1|^2\right)-2|w_1|^2-1\Bigg)\Bigg)\Bigg]F'(R)\\
\mathscr{P}_4^3&=\frac{\kappa(\|\mathbf{w}\|^2-1)\left(\overline{w}_1+w_1\right)}{4(\|\mathbf{w}\|^2+1)}-\frac{1}{2}\Bigg[\overline{z}_1\Bigg(z_1\left(\overline{w}_1+w_1\right)\left(1+|w_1|^2+1+\|\mathbf{w}\|^2\right)\\
&-w_2z_2\left(\overline{w}_1(w_1+\overline{w}_1)+\|\mathbf{w}\|^2+1\right)\Bigg)+\overline{z}_2\Bigg(z_1\overline{w}_2\left(w_1(w_1+\overline{w}_1)+\|\mathbf{w}\|^2+1\right)\\
&+z_2\left(\overline{w}_1+w_1\right)(1+|w_2|^2)\Bigg)\Bigg]F'(R)
\end{aligned}
\tag{C.24}
$$

$$
\begin{aligned}
\mathscr{P}_5^3&=\frac{i\kappa(\|\mathbf{w}\|^2-1)\left(w_1-\overline{w}_1\right)}{4(\|\mathbf{w}\|^2+1)}+\frac{i}{2}\Bigg[\overline{z}_2\Bigg(z_1\overline{w}_2\Big(w_1(\overline{w}_1-w_1)+\|\mathbf{w}\|^2+1)\Big)\\
&-z_2\left(w_1-\overline{w}_1\right)(1+|w_2|^2)\Bigg)-\overline{z}_1\Bigg(z_1\left(w_1-\overline{w}_1\right)(1+|w_1|^2+1+\|\mathbf{w}\|^2)\\
&+w_2z_2\left(\overline{w}_1(w_1-\overline{w}_1)+1+\|\mathbf{w}\|^2\right)\Bigg)\Bigg]F'(R)
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{P}_6^3 &= \frac{\kappa(\|\mathbf{w}\|^2-1)\,(\overline{w}_2+w_2)}{4(\|\mathbf{w}\|^2+1)} - \frac{1}{2}\left[\overline{z}_1\left(z_2\overline{w}_1\left(w_2(w_2+\overline{w}_2)+1+\|\mathbf{w}\|^2\right)\right.\right.\\
&\quad \left. + z_1\,(\overline{w}_2+w_2)\,(1+|w_1|^2)\right) + \overline{z}_2\left(z_2\,(\overline{w}_2+w_2)\,(1+|w_2|^2+1+\|\mathbf{w}\|^2)\right.\\
&\quad \left.\left. + w_1z_1\left(\overline{w}_2(w_2+\overline{w}_2)+1+\|\mathbf{w}\|^2\right)\right)\right]F'(R)\\
\mathscr{P}_7^3 &= \frac{i\kappa(\|\mathbf{w}\|^2-1)\,(w_2-\overline{w}_2)}{4(\|\mathbf{w}\|^2+1)} - \frac{i}{2}\left[z_2\left(\overline{w}_1\overline{z}_1\left(w_2(w_2-\overline{w}_2)-\left(1+\|\mathbf{w}\|^2\right)\right)\right.\right.\\
&\quad \left. + \overline{z}_2\,(w_2-\overline{w}_2)\,(1+|w_2|^2+1+\|\mathbf{w}\|^2)\right) + z_1\left(w_1\overline{z}_2\left(\overline{w}_2(w_2-\overline{w}_2)+1+\|\mathbf{w}\|^2\right)\right.\\
&\quad \left.\left. + \overline{z}_1\,(w_2-\overline{w}_2)\,(1+|w_1|^2)\right)\right]F'(R)\\
\mathscr{P}_8^3 &= -\frac{\sqrt{3}\kappa\|\mathbf{w}\|^2}{2(\|\mathbf{w}\|^2+1)} - \frac{\sqrt{3}}{2}\left[\|\mathbf{w}\|^2|\mathbf{w}\cdot\mathbf{z}|^2-\|\mathbf{z}\|^2\right]F'(R).
\end{aligned} \tag{C.25}
$$

In the expressions above we still keep the freedom of adding $\mathrm{Re}(h_\Lambda)$.

In the next step we wish to solve for the holomorphic moment map. At this point we have to make an ansatz for the holomorphic Kähler form. Let us propose that it has the canonical form (based on our experience with $T^*\mathbb{P}^1$, i.e. the Eguchi–Hanson space)

$$
\mathbf{K}^+ = 2\left(dw^1\wedge dz_1 + dw^2\wedge dz_2\right). \tag{C.26}
$$

In the following we verify that this assumption is indeed correct. To do so, we recall the relation between the triplet of Kähler forms, complex structures and the metric

$$
\mathbf{J}^x = \mathbf{K}^x g^{-1}. \tag{C.27}
$$

With the definitions $\mathbf{J}^\pm = \mathbf{J}^1 \pm i\mathbf{J}^2$, the quaternionic algebra $\mathbf{J}^x\mathbf{J}^y = -\delta^{xy}\mathbf{1} + \varepsilon^{xy}{}_z\mathbf{J}^z$ translates to

$$
\left[\mathbf{J}^+,\mathbf{J}^3\right] = 2i\mathbf{J}^+ \tag{C.28}
$$

$$
\left[\mathbf{J}^-,\mathbf{J}^3\right] = -2i\mathbf{J}^- \tag{C.29}
$$

$$
\left[\mathbf{J}^+,\mathbf{J}^-\right] = -4i\mathbf{J}^3 \tag{C.30}
$$

Writing (C.27) in matrix notation leads to

$$\left(\begin{array}{c|c} i\mathbf{1} & \mathbf{J}^{+} \\ \hline \mathbf{J}^{-} & -i\mathbf{1} \end{array}\right) \equiv \left(\begin{array}{c|c} \mathbf{K}^{+} & ig \\ \hline -i\overline{g} & \mathbf{K}^{-} \end{array}\right) \left(\begin{array}{c|c} 0 & \overline{g}^{-1} \\ \hline g^{-1} & 0 \end{array}\right) = \left(\begin{array}{c|c} i\mathbf{1} & \mathbf{K}^{+}\overline{g}^{-1} \\ \hline \mathbf{K}^{-}g^{-1} & -i\mathbf{1} \end{array}\right). \tag{C.31}$$

It is immediate to check that relations (C.28), (C.29) are trivially satisfied. A true restriction is provided by equation (C.30). Utilizing the matrix notation (C.31), it turns it into the following constraints on $\mathbf{K}^{\pm}$

$$\mathbf{K}^{+}\overline{g}^{-1}\mathbf{K}^{-}g^{-1} = 4\mathbf{1}_{4\times 4} \tag{C.32}$$

$$\mathbf{K}^{-}g^{-1}\mathbf{K}^{+}\overline{g}^{-1} = 4\mathbf{1}_{4\times 4}. \tag{C.33}$$

We showed that these relations are satisfied (it is eaiser to check the inverse of them to avoid inverting the metric), which proves that our ansatz for the holomorphic Kähler form in (C.26) is consistent with the quaternionic algebra of the triplet of complex structures and standard formulae of HyperKähler geometry.

Once we are sure that we have the correct holomorphic Kähler form, we can compute its associated holomorphic moment map $\mathscr{P}^{+}$. To get it, one has to solve a very simple (in our case) system of first order partial differential equations given in (C.16). The solution is straightforward and takes the form

$$\mathscr{P}^{+} = \begin{pmatrix} i(w^1 z_2 + w^2 z_1) \\ (-w^1 z_2 + w^2 z_1) \\ i(w^1 z_1 - w^2 z_2) \\ i(-w^1(w^1 z_1 + w^2 z_2) + z_1) \\ (w^1(w^1 z_1 + w^2 z_2) + z_1) \\ i(-w^2(w^1 z_1 + w^2 z_2) + z_2) \\ (w^2(w^1 z_1 + w^2 z_2) + z_2) \\ i\sqrt{3}(w^1 z_1 + w^2 z_2). \end{pmatrix} \tag{C.34}$$

At this stage we have to impose equivariance on the moment maps

$$\{\mathscr{P}^{x}_{\Lambda}, \mathscr{P}^{x}_{\Sigma}\} \equiv i_{k_\Lambda} i_{k_\Sigma} \mathbf{K}^{x} = f^{\Gamma}_{\Lambda\Sigma}\mathscr{P}^{x}_{\Gamma}, \tag{C.35}$$

which fixes the freedom of shifts by $\mathrm{Re}(h_\Lambda)$. Equivariance requires

$$h_\Lambda(q) = -\frac{\kappa}{2}\left(0,0,0,w_1,iw_1,w_2,iw_2,-\frac{4}{\sqrt{3}}\right). \tag{C.36}$$

C.1.4 Verification of the supersymmetry enhancing conditions on moment maps

Having settled equivariance of the moment maps, we are finally in a position for checking the supersymmetry enhancing constraints on the moment maps. These constraints were spelled out in eq.s(1.3,1.4). In this particular case we choose $\mathfrak{m}_{\Lambda\Sigma} = \kappa_{\Lambda\Sigma}$ to be the Cartan-Killing metric of SU(3).

Substituting our explicit expressions for the moment maps summarized in (C.24) and (C.34) to the above equations gives

$$\left(\mathscr{P}^+ \cdot \mathscr{P}^+\right) = \left(\mathscr{P}^- \cdot \mathscr{P}^-\right) = \left(\mathscr{P}^+ \cdot \mathscr{P}^3\right) = \left(\mathscr{P}^- \cdot \mathscr{P}^3\right) = 0, \tag{C.37}$$

while the argument of the most interesting constraint (1.4) reduces to

$$2\mathscr{P}^3 \cdot \mathscr{P}^3 - \mathscr{P}^+ \cdot \mathscr{P}^- = \frac{2\kappa^2}{3}. \tag{C.38}$$

It is actually interesting to show the individual pieces from which the constraint is built. They depend in a very simple way on the function R defined in (C.8)

$$\mathscr{P}^3 \cdot \mathscr{P}^3 = \frac{\kappa^2}{3} + R \tag{C.39}$$

$$\mathscr{P}^+ \cdot \mathscr{P}^- = 2R. \tag{C.40}$$

It is plausible that such structure for the various scalar products of moment maps holds true for the whole series of HyperKähler Calabi manifolds $T^*\mathbb{P}^n$ (the function R generalizes in a straightforward way for the whole series).

Since the argument of the constraint (1.4) depends only on the resolution parameter κ (related to the scale of the metric ℓ), all constraints on the moment maps are satisfied, which implies supersymmetry enhancement from $\mathscr{N}_3 = 3$ to $\mathscr{N}_3 = 4$. Thus the conclusion of this analysis is that super Chern–Simons theory with target space $T^*\mathbb{P}^2$ and gauge group SU(3) acting non-linearly on the target space has actually $\mathscr{N}_3 = 4$ supersymmetry.

C.2 The moment maps of $\mathfrak{su}(n) \oplus \mathfrak{su}(m) \oplus \mathfrak{u}(1)$ acting on a flat HyperKähler manifold

Let us start by specifying the gauge Lie algebra $\mathfrak{g} = \mathfrak{su}(n) \oplus \mathfrak{su}(m) \oplus \mathfrak{u}(1)$[15] and the representations of the hypermultiplet scalars that provide coordinates of the flat (Hyper Kähler) target space $\mathbb{C}^{nm} \oplus \mathbb{C}^{nm}$

$$\mathbf{u} \in (n, \overline{m}), \quad \mathbf{v} \in (m, \overline{n}). \tag{C.41}$$

To be completely explicit we write $\mathbf{u}$, $\mathbf{v}$ in matrix form

$$\mathbf{u} \equiv u^i{}_{\widehat{k}} \in \mathrm{Mat(n{\times}m)}, \quad \mathbf{v} \equiv v^{\widehat{k}}{}_i \in \mathrm{Mat(m{\times}n)}, \qquad i = 1, \dots, n, \quad \widehat{k} = 1, \dots, m \tag{C.42}$$

[15] In this discussion we are assuming $m \neq n$.

and treat them as independent, i.e. not hermitian conjugate. In these coordinates the triplet of canonical HyperKähler forms reads

$$\mathbf{K}^{+} = \mathrm{Tr}(d\mathbf{u}\wedge d\mathbf{v}) \tag{C.43}$$

$$\mathbf{K}^{-} = -\mathrm{Tr}(d\mathbf{v}^{\dagger}\wedge d\mathbf{u}^{\dagger}) \tag{C.44}$$

$$\mathbf{K}^{3} = \frac{i}{2}\mathrm{Tr}(d\mathbf{u}\wedge d\mathbf{u}^{\dagger} - d\mathbf{v}^{\dagger}\wedge d\mathbf{v}). \tag{C.45}$$

The action of the gauge group on $\mathbf{u}$ is by $A \in \mathrm{SU(n)}$ on the left and the dual of $B \in \mathrm{SU(m)}$ on the right and similarly for $\mathbf{v}$

$$\mathbf{u} \mapsto \mathbf{A}\mathbf{u}\left(\mathbf{B}^{-1}\right)^{T} = \mathbf{A}\mathbf{u}\overline{\mathbf{B}}, \quad \mathbf{v} \mapsto \mathbf{B}\mathbf{v}\left(\mathbf{A}^{-1}\right)^{T} = \mathbf{B}\mathbf{v}\overline{\mathbf{A}}, \tag{C.46}$$

while the $\mathfrak{u}(1)$ acts as

$$\mathbf{u} \mapsto e^{i\phi}\mathbf{u}, \quad \mathbf{v} \mapsto e^{-i\phi}\mathbf{v}. \tag{C.47}$$

Writing $\mathbf{A}$ and $\mathbf{B}$ in infinitesimal form[16]

$$\mathbf{A}^{(a)} = \exp^{it\mathbf{T}^{a}}, \;\; \mathbf{T}^{a}: \text{ generator of } \mathfrak{su}(n) \tag{C.48}$$

$$\mathbf{B}^{(\widehat{a})} = \exp^{it\mathbf{T}^{\widehat{a}}}, \;\; \mathbf{T}^{\widehat{a}}: \text{ generator of } \mathfrak{su}(m) \tag{C.49}$$

and defining the Killing vectors as generators of the gauge group action in (C.46)

$$k^{a}_{\mathfrak{su}(n)} = \mathrm{Tr}\left[\left(\frac{d}{dt}\,\mathbf{u}^{(a)}_{\text{new}}\Big|_{t=0}\right)\frac{\partial}{\partial\mathbf{u}} + \left(\frac{d}{dt}\,\mathbf{v}^{(a)}_{\text{new}}\Big|_{t=0}\right)\frac{\partial}{\partial\mathbf{v}} + c.c.\right], \tag{C.50}$$

leads to

$$k^{a}_{\mathfrak{su}(n)} = i\mathrm{Tr}\left(\frac{\partial}{\partial\mathbf{u}}\mathbf{T}^{a}\mathbf{u} - \mathbf{u}^{\dagger}\overline{\mathbf{T}}^{a}\frac{\partial}{\partial\mathbf{u}^{\dagger}} - \mathbf{v}\overline{\mathbf{T}}^{a}\frac{\partial}{\partial\mathbf{v}} + \frac{\partial}{\partial\mathbf{v}^{\dagger}}\mathbf{T}^{a}\mathbf{v}^{\dagger}\right). \tag{C.51}$$

A word by word derivation holds true also for $\mathfrak{su}(m)$ and $\mathfrak{u}(1)$. The final formulae for the corresponding Killing vectors are

$$k^{\widehat{a}}_{\mathfrak{su}(m)} = i\mathrm{Tr}\left(-\mathbf{u}\overline{\mathbf{T}}^{\widehat{a}}\frac{\partial}{\partial\mathbf{u}} + \frac{\partial}{\partial\mathbf{u}^{\dagger}}\mathbf{T}^{\widehat{a}}\mathbf{u}^{\dagger} + \frac{\partial}{\partial\mathbf{v}}\mathbf{T}^{\widehat{a}}\mathbf{v} - \mathbf{v}^{\dagger}\overline{\mathbf{T}}^{\widehat{a}}\frac{\partial}{\partial\mathbf{v}^{\dagger}}\right) \tag{C.52}$$

$$k_{\mathfrak{u}(1)} = i\mathrm{Tr}\left(\mathbf{u}\frac{\partial}{\partial\mathbf{u}} - \mathbf{u}^{\dagger}\frac{\partial}{\partial\mathbf{u}^{\dagger}} - \mathbf{v}\frac{\partial}{\partial\mathbf{v}} + \mathbf{v}^{\dagger}\frac{\partial}{\partial\mathbf{v}^{\dagger}}\right). \tag{C.53}$$

In the next step one computes the moment maps defined as

$$d\mathscr{P}^{+a}_{\mathfrak{su}(n)} = i_{k^{a}}\mathbf{K}^{+} \tag{C.54}$$

[16] Here the adjoint index Λ splits in $(a,\widehat{a},\bullet)$ for $(\mathfrak{su}(n),\mathfrak{su}(m),\, \mathfrak{u}(1))$ respectively.

$$d\mathscr{P}^{-a}_{\mathfrak{su}(n)} = i_{k^a}\mathbf{K}^- \tag{C.55}$$

$$d\mathscr{P}^{3a}_{\mathfrak{su}(n)} = i_{k^a}\mathbf{K}^3 \tag{C.56}$$

for $\mathfrak{su}(n)$ and equivalently also for $\mathfrak{su}(m)$ and $\mathfrak{u}(1)$. The resulting expressions are respectively

$$\mathscr{P}^{+a}_{\mathfrak{su}(n)} = i\mathrm{Tr}\left(\mathbf{v}\mathbf{T}^a\mathbf{u}\right) \tag{C.57}$$

$$\mathscr{P}^{-a}_{\mathfrak{su}(n)} = -i\mathrm{Tr}\left(\mathbf{u}^\dagger\mathbf{T}^a\mathbf{v}^\dagger\right) \tag{C.58}$$

$$\mathscr{P}^{3a}_{\mathfrak{su}(n)} = -\frac{1}{2}\mathrm{Tr}\left(\mathbf{u}^\dagger\mathbf{T}^a\mathbf{u} - \mathbf{v}\mathbf{T}^a\mathbf{v}^\dagger\right) \tag{C.59}$$

for $\mathfrak{su}(n)$,

$$\mathscr{P}^{+\widehat{a}}_{\mathfrak{su}(m)} = -i\mathrm{Tr}\left(\mathbf{u}\mathbf{T}^{\widehat{a}}\mathbf{v}\right) \tag{C.60}$$

$$\mathscr{P}^{-\widehat{a}}_{\mathfrak{su}(m)} = i\mathrm{Tr}\left(\mathbf{v}^\dagger\mathbf{T}^{\widehat{a}}\mathbf{u}^\dagger\right) \tag{C.61}$$

$$\mathscr{P}^{3\widehat{a}}_{\mathfrak{su}(m)} = -\frac{1}{2}\mathrm{Tr}\left(-\mathbf{u}\mathbf{T}^{\widehat{a}}\mathbf{u}^\dagger + \mathbf{v}^\dagger\mathbf{T}^{\widehat{a}}\mathbf{v}\right) \tag{C.62}$$

$$\tag{C.63}$$

for $\mathfrak{su}(m)$ and finally

$$\mathscr{P}^{+}_{\mathfrak{u}(1)} = i\mathrm{Tr}\left(\mathbf{u}\mathbf{v}\right) \tag{C.64}$$

$$\mathscr{P}^{-}_{\mathfrak{u}(1)} = -i\mathrm{Tr}\left(\mathbf{v}^\dagger\mathbf{u}^\dagger\right) \tag{C.65}$$

$$\mathscr{P}^{3}_{\mathfrak{u}(1)} = -\frac{1}{2}\mathrm{Tr}\left(\mathbf{u}\mathbf{u}^\dagger - \mathbf{v}^\dagger\mathbf{v}\right) \tag{C.66}$$

for $\mathfrak{u}(1)$.

Next, we would like to verify that the moment map constraints

$$\begin{aligned}
\partial_i\left(\mathscr{P}^+\cdot\mathscr{P}^+\right) &= \partial_{\bar{j}}\left(\mathscr{P}^-\cdot\mathscr{P}^-\right) = 0\\
\partial_i\left(\mathscr{P}^+\cdot\mathscr{P}^3\right) &= \partial_{\bar{j}}\left(\mathscr{P}^-\cdot\mathscr{P}^3\right) = 0\\
\partial_{\bar{j}}\left(\mathscr{P}^+\cdot\mathscr{P}^3\right) &= \partial_i\left(\mathscr{P}^-\cdot\mathscr{P}^3\right) = 0
\end{aligned} \tag{C.67}$$

$$\partial_i\partial_{\bar{j}}\left(2\mathscr{P}^3\cdot\mathscr{P}^3 - \mathscr{P}^+\cdot\mathscr{P}^-\right) = 0, \tag{C.68}$$

which imply supersymmetry enhancement are satisfied. This requires in particular finding the correct quadratic form (denoted by a $\cdot$ in the formulae above) on the gauge Lie algebra $\mathfrak{su}(n)\oplus\mathfrak{su}(m)\oplus\mathfrak{u}(1)$. In fact, we will see that for a flat target space a stronger version of the constraints holds true, such that the products of the moment maps in parenthesis vanish by themselves. We will explicitly check the most involved constraint in the last line. The rest of the constraints can be easily verified to vanish as well. Parameterizing the quadratic form on

$\mathfrak{su}(n) \oplus \mathfrak{su}(m) \oplus \mathfrak{u}(1)$ by a relative sign between the Killing form on $\mathfrak{su}(n)$ and $\mathfrak{su}(m)$ and by a constant c for the $\mathfrak{u}(1)$ factor one gets for instance

$$\mathscr{P}^3 \cdot \mathscr{P}^3 = \kappa_{\mathfrak{su}(n)} \left(\mathscr{P}^3_{\mathfrak{su}(n)}, \mathscr{P}^3_{\mathfrak{su}(n)} \right) \pm \kappa_{\mathfrak{su}(m)} \left(\mathscr{P}^3_{\mathfrak{su}(m)}, \mathscr{P}^3_{\mathfrak{su}(m)} \right) + c \mathscr{P}^3_{\mathfrak{u}(1)} \mathscr{P}^3_{\mathfrak{u}}(1). \tag{C.69}$$

Employing the completeness relation for $\mathfrak{su}(n)$ (which implicitly fixes the normalization of the generators)

$$(\mathbf{T}^a)^j{}_i (\mathbf{T}_a)^p{}_q = \frac{1}{2} \left(\delta^j{}_q \delta^p{}_i - \frac{1}{n} \delta^j{}_i \delta^p{}_q \right) \tag{C.70}$$

we arrive at

$$\begin{aligned} 2\mathscr{P}^3 \cdot \mathscr{P}^3 = & \left[\frac{1}{4} \left(\mathrm{Tr}(\mathbf{u}^\dagger \mathbf{u} \mathbf{u}^\dagger \mathbf{u}) - \frac{1}{n} \left(\mathrm{Tr}(\mathbf{u}^\dagger \mathbf{u}) \right)^2 \right) - \frac{1}{2} \left(\mathrm{Tr}(\mathbf{u}^\dagger \mathbf{v}^\dagger \mathbf{v} \mathbf{u}) - \frac{1}{n} \mathrm{Tr}(\mathbf{u}^\dagger \mathbf{u}) \mathrm{Tr}(\mathbf{v}^\dagger \mathbf{v}) \right) \right. \\ & \left. + \frac{1}{4} \left(\mathrm{Tr}(\mathbf{v}\mathbf{v}^\dagger \mathbf{v}\mathbf{v}^\dagger) - \frac{1}{n} \left(\mathrm{Tr}(\mathbf{v}\mathbf{v}^\dagger) \right)^2 \right) \right] \\ & \pm \left[\frac{1}{4} \left(\mathrm{Tr}(\mathbf{u}\mathbf{u}^\dagger \mathbf{u}\mathbf{u}^\dagger) - \frac{1}{m} \left(\mathrm{Tr}(\mathbf{u}\mathbf{u}^\dagger) \right)^2 \right) - \frac{1}{2} \left(\mathrm{Tr}(\mathbf{u}\mathbf{v}\mathbf{v}^\dagger \mathbf{u}^\dagger) - \frac{1}{m} \mathrm{Tr}(\mathbf{u}\mathbf{u}^\dagger) \mathrm{Tr}(\mathbf{v}^\dagger \mathbf{v}) \right) \right. \\ & \left. + \frac{1}{4} \left(\mathrm{Tr}(\mathbf{v}^\dagger \mathbf{v}\mathbf{v}^\dagger \mathbf{v}) - \frac{1}{m} \left(\mathrm{Tr}(\mathbf{v}^\dagger \mathbf{v}) \right)^2 \right) \right] \\ & + c \left[\frac{1}{2} \left(\mathrm{Tr}(\mathbf{u}\mathbf{u}^\dagger) \right)^2 - \mathrm{Tr}(\mathbf{u}\mathbf{u}^\dagger) \mathrm{Tr}(\mathbf{v}\mathbf{v}^\dagger) + \frac{1}{2} \left(\mathrm{Tr}(\mathbf{v}\mathbf{v}^\dagger) \right)^2 \right] \end{aligned} \tag{C.71}$$

and

$$\begin{aligned} \mathscr{P}^+ \cdot \mathscr{P}^- = & \frac{1}{2} \left(\mathrm{Tr}(\mathbf{v}\mathbf{v}^\dagger \mathbf{u}^\dagger \mathbf{u}) - \frac{1}{n} \mathrm{Tr}(\mathbf{v}\mathbf{u}) \mathrm{Tr}(\mathbf{u}^\dagger \mathbf{v}^\dagger) \right) \\ & \pm \left(\mathrm{Tr}(\mathbf{u}\mathbf{u}^\dagger \mathbf{v}^\dagger \mathbf{v}) - \frac{1}{m} \mathrm{Tr}(\mathbf{u}\mathbf{v}) \mathrm{Tr}(\mathbf{v}^\dagger \mathbf{u}^\dagger) \right) + c \mathrm{Tr}(\mathbf{u}\mathbf{v}) \mathrm{Tr}(\mathbf{v}^\dagger \mathbf{u}^\dagger). \end{aligned} \tag{C.72}$$

Subtracting the two expressions above and imposing the result to vanish fixes the relative minus sign between the Killing forms for $\mathfrak{su}(n)$ and $\mathfrak{su}(m)$ and the constant c as

$$c = \frac{m-n}{2mn}. \tag{C.73}$$

Therefore we can conclude that with the choice of $\mathfrak{m}$ for $\mathfrak{su}(n) \oplus \mathfrak{su}(m) \oplus \mathfrak{u}(1)$[17]

$$\pm \left(\mathbf{1}_{\mathfrak{su}(n)}, -\mathbf{1}_{\mathfrak{su}(m)}, \frac{m-n}{2mn} \right) \tag{C.74}$$

the constraints (C.67) and (C.68) are satisfied and thus supersymmetry is enhanced from $\mathcal{N} = 3$ to $\mathcal{N} = 4$.

D Gamma matrix conventions and R-symmetry

In view of what we have discussed in the introduction we provide here a careful consideration of the R–symmetry enhancement that occurs when we dimensionally reduce an $\mathcal{N}_4 = 2$ gauge theory from $D = 4$ down to $D = 3$

D.1 The enhancement of R–symmetry

Prior to dimensional reduction $D = 4 \to D = 3$ the R–symmetry of an $\mathcal{N}_4$-extended gauge theory in four-dimensions is $\mathrm{U}(\mathcal{N}_4) = \mathrm{U}(1) \times \mathrm{SU}(\mathcal{N}_4)$, whose infinitesimal action on the $\mathcal{N}_4$ Majorana supercharges is the following:

$$\delta Q_A = [A_{AB} + iS_{AB}\gamma_5]\, Q_B \quad ; \quad A, B = 1, \ldots, \mathcal{N}_4 \tag{D.1}$$

where $A_{AB} = -A_{BA}$ is an antisymmetric matrix and $S_{AB} = S_{BA}$ is a symmetric one. Taking the chiral projection of the Majorana spinor:

$$\begin{aligned} \mathcal{Q}_A &= \tfrac{1}{2}(1+\gamma_5)\, Q^A \\ \mathcal{Q}^A &= \tfrac{1}{2}(1-\gamma_5)\, Q^A \end{aligned} \tag{D.2}$$

we obtain the standard complex action of $u(\mathcal{N}_4)$:

$$\delta \mathcal{Q}_A = [A_{AB} + iS_{AB}] = \mathcal{Q}_B \equiv U_A{}^B \mathcal{Q}_B \tag{D.3}$$

and the complex conjugate transformation for $\mathcal{Q}^B$. The same transformations apply to the other spinors (gauginos, hyperinos, etc) and bosons (the HyperKählerian vielbein $U^{\alpha A}$) with the same R–symmetry index.

After dimensional reduction the R–symmetry of the three–dimensional theory is enhanced from $\mathrm{U}(\mathcal{N}_4)$ to $\mathrm{SO}(2\mathcal{N}_4)$. This is essentially due to the splitting of each four–component Majorana spinor into a doublet of two–component Majorana spinors. It is important to follow the details of this enhancement mechanism since it is at the level of this symmetry that one finds the new dynamical patterns possible in $D = 3$ and not available in $D = 4$ in particular the breaking of $\mathcal{N}_3 = 4$ supersymmetry down to $\mathcal{N}_3 = 3$ via the introduction of a Chern Simons term. The re-enhancement mechanism from $\mathcal{N}_3 = 3 \to \mathcal{N}_3 = 4$, that constitutes the main issue of the

[17] We can also conclude that if $m = n$ the gauge Lie algebra would be $\mathfrak{su}(n) \oplus \mathfrak{su}(n)$. The choice of $\mathfrak{m}$ would be the same but the $\mathfrak{u}(1)$ piece would be excluded.

present paper, is nothing else but the restoration of the original $\mathrm{SO}(4)$ arising in dimensional reduction.

To begin with let us recall the embedding of $\mathrm{U}(\mathcal{N}_4)$ into $\mathrm{SO}(2\mathcal{N}_4)$ and the structure of the coset space $\mathrm{SO}(2\mathcal{N}_4)/\mathrm{U}(\mathcal{N}_4)$. Let us work at the Lie algebra level and set:

$$\begin{array}{rclcrcl} \mathbb{G} & \equiv & \mathfrak{so}(2\mathcal{N}_4) & ; & \mathbb{H} & \equiv & \mathfrak{u}(\mathcal{N}_4) \\ \mathbb{G} & = & \mathbb{H}\oplus\mathbb{K} & ; & [\mathbb{H},\mathbb{H}] & = & \mathbb{H} \quad , \quad [\mathbb{H},\mathbb{K}]=\mathbb{K} \quad , \quad [\mathbb{K},\mathbb{K}]=\mathbb{H} \end{array} \tag{D.4}$$

A generic antisymmetric $2\mathcal{N}_4\times 2\mathcal{N}_4$ matrix can be decomposed as follows:

$$\mathbf{M}\in\mathfrak{so}(2\mathcal{N}_4) \quad \Rightarrow \quad \mathbf{M} = \underbrace{\left(\begin{array}{c|c} A & -S \\ \hline S & A \end{array}\right)}_{h\in\mathbb{H}=\mathfrak{u}(\mathcal{N}_4)} \oplus \underbrace{\left(\begin{array}{c|c} B & C \\ \hline C & -B \end{array}\right)}_{k\in\mathbb{K}}$$

where:

$$A=-A^T \quad , \quad B=-B^T \quad , \quad C=-C^T \tag{D.5}$$

are antisymmetric $\mathcal{N}_4\times\mathcal{N}_4$ matrices and

$$S=S^T \tag{D.6}$$

is instead symmetric. The first matrix on the left-hand side of (D.5 belongs to $\mathfrak{u}(\mathcal{N}_4)$ subalgebra, while the second matrix belongs to the orthogonal subspace $\mathbb{K}$ whose dimension is $\mathcal{N}_4(\mathcal{N}_4-1)$. We will see how this decomposition is relevant to the enhancement of R–symmetry after dimensional reduction.

To grasp this phenomenon in a clean way we need to choose a well adapted gamma matrix basis.

D.2 The gamma matrix basis

In three dimensions we follow the conventions of [8] and we set:

$$\begin{array}{rclcrcl} \gamma^0 & = & \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & ; & \gamma^1 & = & -i\sigma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ \gamma^2 & = & -i\sigma^1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} & ; & C_{[3]} & = & -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array} \tag{D.7}$$

Then if we explicitly write the Majorana condition for a spinor θ we obtain:

$$\theta=\theta^c\equiv C_{[3]}\,\overline{\theta}=i\,\theta^* \tag{D.8}$$

so that we can write:

$$\theta = \exp[i\pi/4] \left(\begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right) \quad \text{with} \quad \theta^*_{1,2} = \theta_{1,2} \quad \text{real} \tag{D.9}$$

To make nice contact between four and three dimensions which is instrumental in order to derive the $D = 3$ supersymmetry transformations of hypermultiplet fields from their $D = 4$ susy transformations we choose the following basis of $D = 4$ gamma matrices:

$$\begin{array}{rcccl}
\gamma^0_{[4]} & = & \left(\begin{array}{c|c} \gamma^0 & 0 \\ \hline 0 & -\gamma^0 \end{array} \right) & = & \sigma^2 \otimes \sigma^3 \\
\gamma^1_{[4]} & = & \left(\begin{array}{c|c} \gamma^1 & 0 \\ \hline 0 & -\gamma^1 \end{array} \right) & = & -i\,\sigma^3 \otimes \sigma^3 \\
\gamma^2_{[4]} & = & \left(\begin{array}{c|c} \gamma^2 & 0 \\ \hline 0 & -\gamma^2 \end{array} \right) & = & -i\,\sigma^1 \otimes \sigma^3 \\
\gamma^3_{[4]} & = & \left(\begin{array}{c|c} 0 & -\mathbf{1} \\ \hline \mathbf{1} & 0 \end{array} \right) & = & -i\,\mathbf{1} \otimes \sigma^2 \\
\gamma^5_{[4]} & = & \left(\begin{array}{c|c} 0 & -\mathbf{1} \\ \hline -\mathbf{1} & 0 \end{array} \right) & = & -\mathbf{1} \otimes \sigma^1 \\
C_{[4]} & = & \left(\begin{array}{c|c} C_{[3]} & 0 \\ \hline 0 & C_{[3]} \end{array} \right) & = & -i\,\sigma^2 \otimes \mathbf{1}
\end{array} \tag{D.10}$$

We can now verify the decomposition of a four dimensional Majorana spinor under dimensional reduction. We set:

$$\psi = \left(\begin{array}{c} \vartheta_1 \\ \vartheta_2 \end{array} \right) \tag{D.11}$$

and we obtain:

$$\psi = \left(\begin{array}{c} \vartheta_1 \\ \vartheta_2 \end{array} \right) = C_{[4]}\, \overline{\psi}^T = \left(\begin{array}{c} i\,\vartheta^*_1 \\ -i\,\vartheta^*_2 \end{array} \right) \tag{D.12}$$

so that we can conclude:

$$\vartheta_1 = \xi_1 \quad ; \quad \vartheta_2 = -i\xi_2, \quad \text{where} \quad \xi_{1,2} = \text{Majorana spinors in } D = 3 \tag{D.13}$$

D.3 Dimensional reduction of the supersymmetry algebra

Let us now consider the dimensional reduction of the $\mathcal{N}_4$–extended supersymmetry algebra that we write in its dual form utilizing Maurer Cartan equations:

$$dV^{\widehat{a}} = \frac{i}{2}\overline{\Psi}_A \wedge \gamma^{\widehat{a}}_{[4]}\Psi_A \quad ; \quad \widehat{a},\widehat{b} = 0,1,2,3 \tag{D.14}$$

where $V^{\widehat{a}}$ is the vielbein 1–form of rigid superspace and Ψ_A ($A = 1,\dots,\mathcal{N}_4$) is the gravitino 1–form, namely an $\mathcal{N}_4$-tuplet of Majorana spinor fermionic one-forms. Using the above defined gamma matrix basis we immediately find:

$$dV^a = \frac{i}{2}\left[\overline{\xi}^1_A \wedge \gamma^a_{[3]}\xi^1_A + \overline{\xi}^2_A \wedge \gamma^a_{[3]}\xi^2_A\right] \quad ; \quad a = 0,1,2 \tag{D.15}$$

$$dV^3 = \frac{1}{2}\left[\overline{\xi}^1_A \wedge \xi^2_A + \overline{\xi}^2_A \wedge \xi^1_A\right] \tag{D.16}$$

The supersymmetry algebra (D.14) is invariant against the $\mathfrak{u}(\mathcal{N}_4)$ transformations (D.1) where Q_A is replaced by Ψ_A. The same is obviously true of eq.s (D.15,D.16) which are just a transcription of the same algebra. However if we delete eq.(D.16), then eq. (D.15) which is the $2\mathcal{N}_4$–extended supersymmetry algebra in $D = 3$ is invariant against $\mathfrak{so}(2\mathcal{N}_4)$ transformations: it suffices to consider $\left(\xi^1_A,\xi^2_A\right)$ as a column $2\mathcal{N}_4$ vector in the defining representation of $\mathfrak{so}(2\mathcal{N}_4)$. Disregarding eq.(D.16) has a clearcut physical meaning. Indeed V^3 is the 1–form dual to the translation generator in the 3-rd direction, namely P_3. Hence, in the dual language, disregarding eq.(D.16) means that we set $P_3 = 0$. This is just the very idea of dimensional reduction: we restrict our attention to field configurations that have zero momentum in the third direction, namely that are independent from x^3. On the $P_3 = 0$ slice we have an enhancement of R–symmetry which is promoted from $\mathfrak{u}(\mathcal{N}_4)$ to $\mathfrak{so}(2\mathcal{N}_4)$.

D.4 The relevant case $\mathcal{N}_4 = 2$

Let us now consider the chiral projections of the Majorana gravitino one-forms Ψ_A pertaining to the four-dimensional theory. We have:

$$\begin{aligned}\psi_A &= \frac{1}{2}\left(1+\gamma^5_{[4]}\right)\Psi_A = \begin{pmatrix}\chi_A\\ -\chi_A\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\left(\xi^1_A + i\,\xi^2_A\right)\\ -\frac{1}{2}\left(\xi^1_A + i\,\xi^2_A\right)\end{pmatrix}\\ \psi^A &= \frac{1}{2}\left(1-\gamma^5_{[4]}\right)\Psi_A = \begin{pmatrix}\chi^A\\ \chi^A\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\left(\xi^1_A - i\,\xi^2_A\right)\\ \frac{1}{2}\left(\xi^1_A - i\,\xi^2_A\right)\end{pmatrix}\end{aligned} \tag{D.17}$$

where χ_A is a generic 2–component spinor in $D = 3$ (no Majorana condition) and χ^A is just its conjugate:

$$\chi_A^c = C_{[3]}\overline{\chi}_A^T \equiv \chi^A \tag{D.18}$$

The R–symmetry transformations on the χ_A gravitino 1–forms
are easily derived by comparing with equations (D.5). We find:

$$\begin{aligned} \delta\chi_A &= U_A{}^B\chi_B + \mathscr{A}_{AB}\chi^B && \text{(D.19)} \\ \delta\chi^A &= U^A{}_B\chi^B + \overline{\mathscr{A}}^{AB}\chi_B && \text{(D.20)} \end{aligned}$$

where:

$$\begin{aligned} U_A{}^B &= A_{AB} + iS_{AB} \in \mathbb{H} = \mathfrak{u}(\mathscr{N}_4) && \text{(D.21)} \\ \mathscr{A}_{AB} &= B_{AB} + iC_{AB} \in \mathbb{K} = \mathfrak{so}(2\mathscr{N}_4)/\mathfrak{u}(\mathscr{N}_4) && \text{(D.22)} \end{aligned}$$

Eq.s (D.19,D.20) are the holomorphic transcription of eq.s (D.5), namely the decomposition of the adjoint of $\mathfrak{so}(2\mathscr{N}_4)$ with respect to $\mathfrak{u}(\mathscr{N}_4)$:

$$\mathrm{adj}\,\mathfrak{so}(2\mathscr{N}_4) = \mathrm{adj}\,\mathfrak{u}(\mathscr{N}_4) \oplus \wedge^2\text{fundamental} \oplus \wedge^2\text{anti-fundamental} \tag{D.23}$$

On the other hand recalling eq.(D.17) we see that the transformation under R-symmetry of the doublet of spinors $\{\xi_A^1, \xi_A^2\}$ is the following one:

$$\delta_R \left(\begin{array}{c} \xi_A^1 \\ \hline \xi_A^2 \end{array}\right) = \mathbf{M} \left(\begin{array}{c} \xi_A^1 \\ \hline \xi_A^2 \end{array}\right) = \left(\begin{array}{c|c} A+B & C-S \\ \hline C+S & A-B \end{array}\right) \left(\begin{array}{c} \xi_A^1 \\ \hline \xi_A^2 \end{array}\right) \tag{D.24}$$

where the 4×4 antisymmetric matrix $\mathbf{M}$ is that defined in eq. (D.5). It follows that the doublet of spinors $\{\xi_A^1, \xi_A^2\}$ transforms in the fundamental defining representation of $\mathfrak{so}(2\mathscr{N}_4)$. The subalgebra $\mathfrak{u}(\mathscr{N}_4)$ inherited from higher dimensions is that which does not mix the complex supercharges with their conjugates. The enhancement of R–symmmetry produced by the dimensional reduction is given by the antisymmetric representation $\wedge^2$fundamental that mixes conjugate supercharges.

There are two important observations:

1. When $\mathscr{N}_4 = 1$ there is no R–symmetry enhancement. This is the only case where the $\wedge^2$fundamental vanishes and we have $\mathrm{U}(1) \sim \mathrm{SO}(2)$ both in four and three dimensions.

2. When $\mathscr{N}_4 = 2$ the enhanced R-symmetry algebra is:

$$\mathfrak{so}(4) \sim \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \tag{D.25}$$

which, because of the accidental isomorphism splits into two simple subalgebras. The supercharges, which before reduction were in the fundamental 2 of $\mathfrak{u}(2)$ are, after reduction in the fundamental **4** of $\mathfrak{so}(4)$. With respect to the decomposition (D.25) one has $4 \sim (2,2)$. With respect to the diagonal subalgebra $\mathfrak{su}(2)_{diag} = \text{diag}\left(\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R\right)$ we have: $\mathbf{4} \to \mathbf{3} \oplus \mathbf{1}$, so that we can decompose the supercharges into a triplet plus a singlet and consider new terms in the lagrangian that violate the fourth supercharge preserving the other three. This is the way to construct $\mathcal{N}_3 = 3$ theories in three dimensions.

3. The enhancement of specially constructed Chern Simons theories from $\mathcal{N}_3 = 3$ to $\mathcal{N}_3 = 4$ is associated with a full reinstallment of the natural $\mathfrak{so}(4)$ produced by a hypothetical dimensional reduction from $D = 4$.

Let us now focus on $\mathcal{N}_4 = 2$ and reconsider the specific form of the decomposition (D.5) in this case. A complete basis for the antisymmetric 4×4 matrices, namely for the $\mathfrak{so}(4)$ Lie algebra is provided by the 't Hooft matrices. These are 4×4 real antisymmetric, (anti)self–dual matrices which satisfy the following relations:

$$\left.\begin{array}{rcl} \overset{\pm}{J}{}^{ab}_x & = & \pm\,\varepsilon^{abcd}\,\overset{\pm}{J}{}^{ab}_x \\ \overset{\pm}{J}_x \overset{\pm}{J}_y & = & -\delta_{xy} - \varepsilon^{xyz}\,\overset{\pm}{J}_z \\ \left[\overset{+}{J}_x\,,\overset{-}{J}_x\right] & = & 0 \end{array}\right\} \quad (x,y,.. = 1,2,3 \quad ; \quad a,b,c,.. = 1,2,3,4) \tag{D.26}$$

The explicit form for the 't Hooft matrices is the following:

$$\begin{array}{rclrcl} \overset{+}{J}_1 & = & \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \overset{-}{J}_1 & = & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ \overset{+}{J}_2 & = & \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \overset{-}{J}_2 & = & \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ \overset{+}{J}_2 & = & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & \overset{-}{J}_3 & = & \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array} \tag{D.27}$$

Comparing eq.(D.5) with eq.(D.27) one sees that the generators of the $\mathfrak{u}(2) \subset \mathfrak{so}(4)$ subalgebra already present in four dimensions are:

$$\mathfrak{u}(2) = \text{span}\left[\overset{+}{J}_1, \overset{+}{J}_2, \overset{+}{J}_3, \overset{-}{J}_3\right] \tag{D.28}$$

Indeed $\overset{-}{J}_2$ contains the trace part of the symmetric 2×2 matrix S_{AB}. On the other hand the $\wedge^2$fundamental that enhances R-symmetry in three dimensions is provided by:

$$\wedge^2 \text{ fundamental} = \text{span}\left[\overset{-}{J}_1, \overset{-}{J}_2\right] \tag{D.29}$$

If we consider the explicit form of the diagonal $\mathfrak{su}(2)_{diag}$ generators we find:

$$\begin{aligned}
J^1_{diag} &= \frac{1}{2}\left(\overset{+}{J}_1 + \overset{-}{J}_1\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
J^2_{diag} &= \frac{1}{2}\left(\overset{+}{J}_2 + \overset{-}{J}_2\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
J^3_{diag} &= \frac{1}{2}\left(\overset{+}{J}_3 + \overset{-}{J}_3\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{D.30}$$

As one sees these are the rotation generators on the three space spanned by the axes (2-3-4). The first direction being left invariant. Hence the triplet of unbroken supersymmetries of an $\mathcal{N}_3 = 3$ theory are given by:

$$\begin{aligned}
1st &= \xi^2_1 = -i\left(\chi_1 - \chi^1\right) \\
2nd &= \xi^1_2 = \left(\chi_2 + \chi^2\right) \\
3rd &= \xi^2_2 = -i\left(\chi_2 - \chi^2\right)
\end{aligned} \tag{D.31}$$

The above conclusion applies to the supersymmetry parameters in the same way. Utilizing the above described gamma matrix basis and naming κ_A, κ^A the chiral supersymmetry parameters in $D = 4$ we can write:

$$\eta_A = \begin{pmatrix} \frac{1}{2}(\varepsilon^1_A + i\,\varepsilon^2_A) \\ -\frac{1}{2}(\varepsilon^1_A + i\,\varepsilon^2_A) \end{pmatrix} \quad ; \quad \eta^A = \begin{pmatrix} \frac{1}{2}(\varepsilon^1_A - i\,\varepsilon^2_A) \\ \frac{1}{2}(\varepsilon^1_A - i\,\varepsilon^2_A) \end{pmatrix} \tag{D.32}$$

where ε_A^X $(X = 1,2,\ A = 1,2)$ are $2 \times 2 = 4$ anticommuting $D = 3$ Majorana spinors that constitute the supersymmetry parameters of an $\mathcal{N}_3 = 4$ supersymmetry algebra in the three-dimensional theory. The generic $\mathcal{N}_3 = 3$ Chern Simons theory admits only ε_A^2, ε_1^2. When there is enhancement, the missing parameter ε_1^1 is reinstalled.

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