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Abstract

In this paper we provide a semantic reconstruction of rational closure. We first consider rational closure as defined by Lehman and Magidor [**?**] for propositional logic, and we provide a semantic characterization based on a minimal models mechanism on rational models. Then we extend the whole formalism and semantics to Description Logics, by focusing our attention to the standard $\mathcal{ALC}$: we first naturally adapt to Description Logics Lehman and Magidor's propositional rational closure, starting from an extension of $\mathcal{ALC}$ with a typicality operator $\mathbf{T}$ that selects the most typical instances of a concept C (hence $\mathbf{T}(C)$ stands for typical C). Then, for the Description Logics, we define a minimal model semantics for the logic $\mathcal{ALC}$ and we show that it provides a semantic characterization for the rational closure of a Knowledge base. We consider both the rational closure of the TBox and the rational closure of the ABox.

Keywords: Description Logics, Nonmonotonic Reasoning, Knowledge Representation, Rational Closure

1. Introduction

In [**?**] Kraus Lehmann and Magidor (henceforth KLM) proposed an axiomatic approach to nonmonotonic reasoning based on the notion of plausible inference. Plausible inferences are represented by conditionals of the form $A \mathrel{|\!\sim} B$, to be read as "typically or normally A entails B". For instance, the conditional assertion $monday \mathrel{|\!\sim} go\_work$ can be used in order to represent that "normally if it is Monday I go to work". Conditional entailment is nonmonotonic since from $A \mathrel{|\!\sim} B$ one cannot derive $A \wedge C \mathrel{|\!\sim} B$, in our example from $monday \mathrel{|\!\sim} go\_work$ one cannot monotonically derive $monday \wedge ill \mathrel{|\!\sim} go\_work$ ("normally if it is Monday, even if I am ill I go to work").

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KLM presented a hierarchy of axiomatic systems for plausible inference, each system specifies a set of postulates characterising plausible inference. The systems are, from the weakest to the strongest: cumulative logic C, loop-cumulative logic CL, and most important preferential logic P. In subsequent work [**?**] Preferential logic was strengthened to rational logic R and the latter was proposed as the most adequate system to represent (nonmontonic) plausible inference.

Although it is arguable whether, KLM systems, and in particular R, represent adequately all types of nonmonotonic inferences[1], we think that KLM systems and the strongest R in particular, are still a significant proposal for nonmonotonic reasoning for two reasons: on a theoretical level, they define a set of inferential properties which are useful (even if not necessarily wanted) to classify and analyze concrete nonmonotonic inference, (b) they provide a simple and *direct* language to express plausible inferences and to reason about them.

In this work we take KLM logic R as the basis of our approach to nonmonotonic reasoning. Even if R formalizes some properties of nonmonotonic inference it is too weak in itself to perform useful nonmonotonic inferences.

We have just seen that by the nonmonotonicity of $\mid\!\sim$, $A \mid\!\sim B$ does not entail $A \wedge C \mid\!\sim B$ ($monday \mid\!\sim go\_work$ does not entail $monday \wedge ill \mid\!\sim go\_work$), and this is a wanted property of $\mid\!\sim$: it is what allows to express sets of conditionals that in classical logic would lead to contradictory or absurd conclusions (for instance $\{monday \rightarrow go\_work, monday \wedge ill \rightarrow \neg go\_work\}$ gives $\neg(monday \wedge ill)$ in classical logic, that is that it is impossible to be ill on Monday). However, there are cases in which, in the absence of information to the contrary, we would like to be able to tentatively infer that also $A \wedge C \mid\!\sim B$, with the possibility of withdrawing the inference in case we discovered that it is inconsistent. For instance, we might want to infer that $A \wedge C \mid\!\sim B$ when C is irrelevant with respect to the property B: in the example, we might want to tentatively infer from $monday \mid\!\sim go\_work$ ("normally if it is Monday, I go to work") that $monday \wedge shines \mid\!\sim go\_work$ ("normally if it is Monday, even if the sun shines I go to work"), with the possibility of withdrawing the conclusion if we discovered that indeed the sun shining prevents from going to work. R cannot handle irrelevant information in conditionals, and the inferences just exemplified are not supported.

Partially motivated by this weakness, Lehmann and Magidor have proposed a true nonmonotonic mechanism on the top of R. *Rational closure* [**?**] on the one hand preserves the properties of R, on the other hand it allows to perform some truthful nonmonotonic inferences, like the one just mentioned ($monday \wedge shines \mid\!\sim go\_work$). In [**?**] the authors give a syntactic procedure to calculate the set of conditionals entailed by the rational closure as well as a quite complex semantic construction. It is worth noticing that a strongly related construction has been proposed by Pearl [**?**] with his notion of 1-entailment, originating from a probabilistic interpretation of conditionals within the well-established System Z.

[1] It has been shown that existing nonmonotonic systems do not satisfy in general all the properties of KLM systems: in particular circumscription (for well-founded theories) satisfies all postulates of preferential logic, but it does not satisfy rational monotony of R, whereas default logic fails to satisfy even the cumulativity postulate of the weakest logic C. Of course, a nonmonotonic mechanism may give rise to different inference relations (skeptical, credulous, etc) with different properties.

In this paper we provide a semantic reconstruction of rational closure for propositional logic as well as for Description Logics (DLs for short) with a specific attention to the standard $\mathcal{ALC}$. We first consider rational closure as defined by Lehman and Magidor [**?**] for propositional logic, and we provide a semantic characterization based on a minimal models mechanism on rational models. Then we extend the whole formalism and semantics to Description Logics: we first naturally adapt to DLs Lehman and Magidor's propositional rational closure, starting from an extension of $\mathcal{ALC}$ with a typicality operator $\mathbf{T}$ that selects the most typical instances of a concept C (the extension is called $\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}$). For $\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}$, we provide both a syntactic and a semantical notion of rational closure, along the same lines used for the propositional case: we first define rational closure over the TBox, and subsequently rational closure for the ABox.

The first problem we tackle in this work is that of giving a purely semantical characterization of the syntactic notion of rational closure. Our semantic characterization has as its main ingredient the modal semantics of logic R, over which we build a minimal models' mechanism, based on the minimization of the rank of worlds. Intuitively, we prefer the models that minimize the rank of domain elements: the lower the rank of a world, the more normal (or less exceptional) is the world and our minimization corresponds intuitively to the idea of minimizing less-normal or less-plausible worlds (or maximizing most plausible ones). We show that a semantic reconstruction of rational closure can be obtained as a specific instance of a general semantic framework for nonmonotonic reasoning. Within this general framework we give two characterizations of rational closure: one based on a fixed interpretations semantics and the other with a variable interpretations semantics.

The theoretical question we address in this first part of the paper is the following:

A) Given the fact that logic R is characterized by a specific class of Kripke models, what are the Kripke models that characterize the rational closure of a set of positive conditionals?

We notice in passim that our semantic characterization of rational closure in terms of minimal models is different from the one given by Lehmann and Magidor's in [**?**] which is based on a different notion of minimal models. Moreover we consider our semantic characterization as a specific case of a general minimal models' mechanism for nonmonotonic reasoning, and in this paper we show under what conditions we capture rational closure. The generality of our semantical characterization is well-suited to study variants of rational closure. Finally, the semantic characterization does also easily extend to other logics, as Description Logics ($\mathcal{ALC}$), that we discuss next. In the second part of the paper we consider Description Logics. If propositional KLM systems deal with propositions ("I go to work") and relations among propositions ("usually, if it is Monday, then I go to work"), Description Logics deal with concepts, relations among concepts, as well as with individuals. In Description Logics one can use concept inclusion in order to express that all the members of a class have a given property (thus $\mathit{Cats} \sqsubseteq \mathit{Mammal}$ expresses the general property that "cats are mammals", and $\mathit{Pet} \sqsubseteq \exists \mathit{HasOwner}.\top$ that "all pets have an owner"). One can also use assertions in order to represent the fact that an individual has a given property, e.g. $\mathit{Cat}(\mathit{tom})$ ("Tom is a cat") or $\exists \mathit{HasOwner}.\top(\mathit{tom})$ ("Tom has an owner") or

HasOwner(*tom*, *nadeem*) ("Nadeem is Tom's owner"). A distinguishing quality of Description Logics is their controlled complexity: the trade-off between expressivity of the languages and good computational complexities is one of the main reasons justifying the success of DLs.

Many works in the literature have considered how to extend the basic formalism of Description Logics with nonmonotonic reasoning features [**? ? ? ? ? ? ? ? ? ? ? ?**]; the purpose of these extensions is to allow to reason about prototypical properties of individuals or classes of individuals. In these extensions one can represent, for instance, knowledge expressing the fact that the heart is *usually* positioned in the left-hand side of the chest, with the exception of people with *situs inversus*, that have the heart positioned in the right-hand side. Also, one can infer that an individual enjoys all the *typical* properties of the classes it belongs to. So, for instance, in the absence of information that someone has situs inversus, one would assume that it has the heart positioned in the left-hand side.

In spite of the number of work in this direction, the problem of extending DLs for reasoning about prototypical properties seems far from being solved. The most well-known semantics for nonmonotonic reasoning have been used to the purpose, from default logic [**?**], to circumscription [**?**], from Lifschitz's nonmonotonic logic MKNF [**? ?**] to KLM logics. In particular, concerning KLM logics, in [**?**] a preferential extension of $\mathcal{ALC}$ (called $\mathcal{ALC} + \mathbf{T}$) is defined, based on the KLM logic P, and in [**?**] a defeasible description logic based on the KLM logic R is introduced. In [**?**] a minimal model semantics for the logic $\mathcal{ALC} + \mathbf{T}$ is presented.

An approach to the definition of rational closure for DLs has been proposed by Casini and Straccia in [**?**], where a notion of rational closure is defined for $\mathcal{ALC}$ through an algorithmic construction similar to the one introduced by Freund in [**?**] for the propositional calculus. For propositional logic, this construction can be proved to be equivalent to the notion of rational closure proposed by Lehmann and Magidor in [**?**]. [**?**] explores the axiomatic properties of this notion of rational closure for $\mathcal{ALC}$, and shows that the notion of *default assumption consequence* is a rational consequence relation validating the knowledge base. On the other hand, [**?**] does not consider a semantics for rational closure.

In this paper, we take our moves from the notion of propositional rational closure given by Lehmann and Magidor, and we show that it can be naturally extended to the description logic $\mathcal{ALC}$. Furthermore, we investigate its semantics, by extending to $\mathcal{ALC}$ the minimal model semantics introduced at the propositional level in order to address question *A*. The questions we address in the second part of the paper are therefore the following:

B) What is the natural extension of the well-established notion of rational closure in [**?**] to Description Logics?

C) What is the corresponding semantics?

D) How can this mechanism deal with the ABox?

As we will see, for concept inclusions (TBox) the extension of both the syntactic and the semantical characterization of rational closure from propositional logic to DLs is relatively direct, although the presence of typicality assertions in the ABox makes things not straightforward. Furthermore, the algorithmic construction we propose for ABox reasoning is novel and it entirely relies on the semantical characterization: only once we have extended the semantics for rational closure to take into account ABox individuals, we can provide the corresponding mechanism to compute rational closure of the ABox.

As matter of fact, we do not consider our adaption of Lehmann and Magidor's rational closure to DLs as the conclusive solution to the issue of nonmonotonic extensions of Description Logics. Rational closure has some known weaknesses that come together with its recognised advantages (among which, its computational lightness, which is crucial in Description Logics). Both advantages and weaknesses are inherited by its extension to Description Logics. Nevertheless, since rational closure is one of the most established formalisms for nonmonotonic reasoning and it has good computational properties, we think that its application to Description Logics significantly contributes to the quest of nonmonotonic extensions of Description Logics. Furthermore, this work can be regarded as a first step towards the exploration of semantics for more refined versions of rational closure, that overcome some of the known weaknesses of this mechanism (see for instance [**? ?**] which combines rational closure with inheritance networks).

To summarize the resulting approach: our starting point is the standard Description Logic $\mathcal{ALC}$, more precisely $\mathcal{ALC}$ extended with a typicality operator $\mathbf{T}$. The operator $\mathbf{T}$, first introduced in [**?**], allows to directly express typical properties such as $\mathbf{T}(\mathit{HeartPosition}) \sqsubseteq \mathit{Left}$, $\mathbf{T}(\mathit{Bird}) \sqsubseteq \mathit{Fly}$, and $\mathbf{T}(\mathit{Penguin}) \sqsubseteq \neg \mathit{Fly}$, whose intuitive meaning is that, normally, the heart is positioned in the left-hand side of the chest, that typical birds fly, whereas penguins do not. In this paper, the $\mathbf{T}$ operator is intended to enjoy the well-established properties of rational logic R . Even if $\mathbf{T}$ is a nonmonotonic operator (so that for instance $\mathbf{T}(\mathit{HeartPosition}) \sqsubseteq \mathit{Left}$ does not entail that $\mathbf{T}(\mathit{HeartPosition} \sqcap \mathit{SitusInversus}) \sqsubseteq \mathit{Left}$), the logic itself is monotonic. Indeed, in this logic it is not possible to monotonically infer from $\mathbf{T}(\mathit{Bird}) \sqsubseteq \mathit{Fly}$, in the absence of information to the contrary, that also $\mathbf{T}(\mathit{Bird} \sqcap \mathit{Black}) \sqsubseteq \mathit{Fly}$. Nor can it be nonmonotonically inferred from $\mathit{Bird}(\mathit{tweety})$, in the absence of information to the contrary, that $\mathbf{T}(\mathit{Bird})(\mathit{tweety})$. Nonmonotonicity is achieved first by adapting to $\mathcal{ALC}$ with $\mathbf{T}$ the propositional construction of rational closure. This nonmonotonic extension allows to infer typical subsumptions from the TBox (TBox reasoning). Intuitively and similarly to the propositional case, the rational closure construction amounts to assigning a *rank* (a level of exceptionality) to every concept; this rank is used to evaluate typical inclusions of the form $\mathbf{T}(C) \sqsubseteq D$: the inclusion is supported by the rational closure whenever the rank of C is strictly smaller than the rank of $C \sqcap \neg D$. From a semantic point of view, nonmonotonicity is achieved by defining, on the top of $\mathcal{ALC}$ with typicality, a minimal model semantics which is similar to the one in [**?**]. Differently from [**?**], the notion of minimality used here is based on the minimization of the ranks of the domain elements, rather than on the minimization of the extension of specific concepts. This semantics provides a characterization to the rational closure construction for $\mathcal{ALC}$.

Last, we tackle the problem of extending rational closure to ABox reasoning: in order to ascribe typical properties to individuals, we maximize the typicality of an individual. This is done by minimizing its rank (that is, its level of exceptionality).

As we will see, because of the interaction between individuals (due to roles) it is not possible to separately assign a unique minimal rank to each individual and alternative minimal ranks must be considered. We end up with a kind of *skeptical* inference with respect to the ABox.

The rational closure construction we propose for $\mathcal{ALC}$ has not just a theoretical interest and a simple minimal model semantics. We show that it retains the same complexity of the underlying description logic. For $\mathcal{ALC}$, the problem of deciding whether a typical inclusion belongs to the rational closure of the TBox is in EXPTIME as well as the problem of deciding whether an assertion $C(a)$ belongs to the rational closure of the knowledge base over the ABox. In this respect, the proposed approach is less complex than other approaches to nonmonotonic reasoning in DLs such as [**? ?**] and comparable in complexity with the approaches in [**? ? ?**], and thus a good candidate to define effective nonmonotonic extensions of DLs. The results on the rational closure in $\mathcal{ALC}$ (as an extension of Lehmann and Magidor's rational closure [**?**]) extensively rely on the finite model property, which holds for $\mathcal{ALC}$. However, the construction of rational closure can be extended to more expressive description logics that do not enjoy the finite model property. Some preliminary results on the rational closure for $\mathcal{SHIQ}$ [**?**] can be found in [**?**].

2. Propositional rational closure: a semantic characterization

2.1. KLM rational system R

The language of logic R consists just of conditional assertions $A \mid\sim B$. We here consider a richer language which also allows boolean combinations of assertions. Our language $\mathcal{L}$ is defined from a set of propositional variables *ATM*, the boolean connectives and the conditional operator $\mid\sim$. From propositional variables, propositional formulas are defined as usual in the propositional logic. We use $A, B, C, \ldots$ to denote propositional formulas (that do not contain conditional formulas), whereas $F, G, \ldots$ are used to denote all formulas (including conditionals). The formulas of $\mathcal{L}$ are defined as follows: if A is a propositional formula, $A \in \mathcal{L}$; if A and B are propositional formulas, $A \mid\sim B \in \mathcal{L}$; if F is a boolean combination of formulas of $\mathcal{L}$, then $F \in \mathcal{L}$. A *knowledge base* K is a set of conditional assertions $A \mid\sim B$. In this work, we restrict our attention to *finite knowledge bases*.

Before presenting the axiomatization of R, let us clarify one point: in its original presentation [**?**], a conditional $A \mid\sim B$ is considered as a consequence relation between a pair of propositional formulas A and B, so that their systems provide a set of "postulates" (or closure conditions) that the intended consequence relation must satisfy. Alternatively, these postulates may be seen as *rules* to derive new conditionals from given ones. We take a slightly different viewpoint, shared, among others, by Halpern and Friedman [**?**] (see Section 8) and Boutilier [**?**], who proposed a modal interpretation of R: in our understanding, this system is an ordinary logical system in which a conditional $A \mid\sim B$ is a formula belonging to the object language. Whenever we restrict our consideration, as done by Lehmann and Magidor in [**?**], to the entailment of a conditional from a set of conditionals, the two viewpoints *coincide*, and a conditional is a logical consequence of a set of conditionals in logic R if and only if it belongs to all rational consequence relations extending that set of conditionals, or (in semantic terms), it is valid in all rational models (as defined by [**?**]) of that set.

Here is the axiomatization of logic R. In our presentation Lehmann and Magidor's postulates/rules are just *axioms*. We use $\vdash_{PC}$ (resp. $\models_{PC}$) to denote provability (resp. validity) in the propositional calculus .

All axioms and rules of propositional logic	(PC)
$A \mid\!\sim A$	(REF)
if $\vdash_{PC} A \leftrightarrow B$ then $(A \mid\!\sim C) \rightarrow (B \mid\!\sim C)$	(LLE)
if $\vdash_{PC} A \rightarrow B$ then $(C \mid\!\sim A) \rightarrow (C \mid\!\sim B)$	(RW)
$((A \mid\!\sim B) \wedge (A \mid\!\sim C)) \rightarrow (A \wedge B \mid\!\sim C)$	(CM)
$((A \mid\!\sim B) \wedge (A \mid\!\sim C)) \rightarrow (A \mid\!\sim B \wedge C)$	(AND)
$((A \mid\!\sim C) \wedge (B \mid\!\sim C)) \rightarrow (A \vee B \mid\!\sim C)$	(OR)
$((A \mid\!\sim B) \wedge \neg(A \mid\!\sim \neg C)) \rightarrow (A \wedge C \mid\!\sim B)$	(RM)

The axiom (CM) is called cumulative monotony and it is characteristic of all KLM logics, axiom (RM) is called rational monotony and it characterizes the logic of rational entailment R (it is what distinguishes rational from the weaker preferential entailment). In [**?**], Friedman and Halpern have shown that the axiom system of R is complete with respect to a wide spectrum of different semantics (e.g. possibilistic structures and k-rankings), proposed in order to formalize some forms of nonmonotonic reasoning. This can be explained by the fact that all these models are examples of *plausibility structures*, and the truth in them is captured by the axioms of R.

The logic R enjoys a very simple modal semantics, actually it turns out that it corresponds to the flat fragment of the well-known conditional logic **VC** [**?**]. The modal semantics is defined by considering a set of worlds $\mathcal{W}$ equipped by an accessibility (or preference) relation $<$. Intuitively the meaning of $x < y$ is that x is more typical/more normal/less exceptional than y. We say that a conditional $A \mid\!\sim B$ is true in a model if B holds in all most normal worlds where A is true, i.e. in all $<$-minimal worlds satisfying A.

Definition 1. A *rational* model is a triple

$$\mathcal{M} = \langle \mathcal{W}, <, V \rangle$$

where:

- $\mathcal{W}$ is a non-empty set of worlds;
- $<$ is an irreflexive, transitive relation on $\mathcal{W}$ satisfying modularity: for all x, y, z, if $x < y$ then either $x < z$ or $z < y$. $<$ further satisfies the Smoothness condition defined below;
- V is a function $V : \mathcal{W} \longmapsto 2^{ATM}$, which assigns to every world w the set of atoms holding in that world. If F is a boolean combination of formulas, its truth conditions ($\mathcal{M}$, $w \models F$) are defined as for propositional logic. Let A be a

propositional formula; we define $Min_<^{\mathcal{M}}(A) = \{w \in \mathcal{W} \mid \mathcal{M}, w \models A$ and $\forall w'$, $w' < w$ implies $\mathcal{M}, w' \not\models A\}$. Moreover:

$$\mathcal{M}, w \models A \mathrel{|\!\sim} B$$

if for all w', if $w' \in Min_<^{\mathcal{M}}(A)$ then $\mathcal{M}, w' \models B$.

At this point we can define the *Smoothness condition*: if $\mathcal{M}, w \models A$, then either $w \in Min_<^{\mathcal{M}}(A)$ or there is $w' \in Min_<^{\mathcal{M}}(A)$ such that $w' < w$.

Validity and satisfiability of a formula are defined as usual. We say that a formula F is *satisfiable* if there is a rational model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ and a world $w \in \mathcal{W}$ such that $\mathcal{M}, w \models F$. We say that a formula F is valid in a rational model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, and we write $\mathcal{M} \models F$, if, for all $w \in \mathcal{W}$, it holds that $\mathcal{M}, w \models F$. We say that a formula F is *valid* if it is valid in all rational models, i.e. if, for all rational models $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, it holds that $\mathcal{M} \models F$.

Given a set of formulas K of $\mathcal{L}$ and a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, we say that $\mathcal{M}$ is a model of K, written $\mathcal{M} \models K$, if for every $F \in K$ and every $w \in \mathcal{W}$, we have that $\mathcal{M}, w \models F$. K *rationally entails* a formula F, written $K \models F$ if F is valid in all rational models of K.

It is easy to see from Definition 1 that the truth condition of $A \mathrel{|\!\sim} B$ is "global" in a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$: given a world w, we have that $\mathcal{M}, w \models A \mathrel{|\!\sim} B$ if, for all w', if $w' \in Min_<^{\mathcal{M}}(A)$ then $\mathcal{M}, w' \models B$. It immediately follows that $A \mathrel{|\!\sim} B$ holds in w if and only if $A \mathrel{|\!\sim} B$ is valid in a model, i.e. it holds that $\mathcal{M}, w' \models A \mathrel{|\!\sim} B$ for all w' in $\mathcal{W}$; for this reason we will often write $\mathcal{M} \models A \mathrel{|\!\sim} B$. Moreover, when the reference to the model $\mathcal{M}$ is unambiguous, we will simply write $Min_<(A)$ instead of $Min_<^{\mathcal{M}}(A)$.

Theorems 6.8 and 6.9 in [**?**] provide a constructive proof of the following finite model property of R.

Fact 1. *Given a set of formulas K, if it is satisfiable, then it is satisfiable in a* finite *model. Furthermore, if a given F is satisfiable in a model of K (for $K \not\models \neg F$), then F is satisfiable in a finite model of K.*

From now on, we will restrict our consideration to rational models with a finite set of worlds.

Given a rational model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, let us now define the rank $k_{\mathcal{M}}(w)$ of a world w and the rank $k_{\mathcal{M}}(F)$ of a formula F.

Definition 2 (Rank $k_{\mathcal{M}}(w)$ of a world in $\mathcal{M}$). Given a (finite) rational model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, the rank $k_{\mathcal{M}}$ of a world $w \in \mathcal{W}$, written $k_{\mathcal{M}}(w)$, is the length of the longest chain $w_0 < \ldots < w$ from w to a minimal w_0 (i.e. there is no w' such that $w' < w_0$).

This definition makes sense even if the relation $<$ is not modular. Observe that, for a modular relation on a finite set, all maximal chains [2] from an element w to a minimal w_0 have the same length.

[2] A chain $w_0 < w_1 < \ldots < w_n$ is maximal if there is no element w' such that for some $i = 0, \ldots, n-1$ it holds $w_i < w' < w_{i+1}$.

The previous definition defines from $<$ a rank function $k_{\mathcal{M}} : \mathcal{W} \longmapsto \mathbb{N}$. The opposite is also possible and in general in rational models the rank function $k_{\mathcal{M}}$ and $<$ can be defined from each other by letting $x < y$ if and only if $k_{\mathcal{M}}(x) < k_{\mathcal{M}}(y)$ (this is similarly stated by [**?**] where a rank function k over a possibly infinite set is used, since there is no restriction to finite models) Hence, modular preferential models are called *ranked models*.

Definition 3 (Rank $k_{\mathcal{M}}(F)$ of a formula in a model). The rank $k_{\mathcal{M}}(F)$ of a formula F in a model $\mathcal{M}$ is $i = min\{k_{\mathcal{M}}(w) : \mathcal{M}, w \models F\}$. If there is no w such that $\mathcal{M}, w \models F$, then we say F has no rank in $\mathcal{M}$.

It is easy to observe that:

Proposition 1. *For any $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, we have $\mathcal{M} \models A \mid\sim B$ if and only if $k_{\mathcal{M}}(A \wedge B) < k_{\mathcal{M}}(A \wedge \neg B)$ or A has no rank in $\mathcal{M}$.*

2.2. Lehmann and Magidor's definition of rational closure

Although the operator $\mid\sim$ is *nonmonotonic*, the notion of rational entailment (defined in Definition 1) in itself is *monotonic*: if $K \models F$ and $K \subseteq K^*$ then also $K^* \models F$.

In order to strengthen R, Lehmann and Magidor in [**?**] propose the well-known mechanism of rational closure. As already mentioned, the main motivation of Lehmann and Magidor leading to the definition of rational closure was *technical*: it turns out that the intersection of all rational consequence relations satisfying a set of conditionals coincides with the weaker *preferential* consequence relation satisfying that set (that is weaker in that it does not satisfy (RM)), so that (i) the axiom/rule (RM) does not add anything and (ii) such relation in itself *fails* to satisfy (RM). Lehmann and Magidor's notion of rational closure provides a solution to both problems and can be seen as the "minimal" (in some sense) rational consequence completing a set of conditionals.

Since in rational closure no boolean combination of conditionals is allowed, in the following, the knowledge base K is just a finite set of positive conditional assertions of the form $A \mid\sim B$. In such a case, rational entailment is equivalent to preferential entailment.

Definition 4 (Exceptionality of propositional formulas and conditional formulas). Let K be a knowledge base (i.e. a finite set of positive conditional assertions) and A a propositional formula. A is said to be *exceptional* for K if and only if $K \models \top \mid\sim \neg A$. A conditional formula $A \mid\sim B$ is exceptional for K if its antecedent A is exceptional for K. The set of conditional formulas of K which are exceptional for K will be denoted as $E(K)$.

It is possible to define a non increasing sequence of subsets of K, $C_0 \supseteq C_1, C_1 \supseteq C_2, \ldots$ by letting $C_0 = K$ and, for $i > 0$, C_i the set of conditionals of C_{i-1} exceptional for C_{i-1}, i.e. $C_i = E(C_{i-1})$. Observe that, being K finite, there is an $n \geq 0$ such that $C_n = \emptyset$ or for all $m > n, C_m = C_n$. The sets C_i are used to define the rank of a formula, as in the next definition. Notice that if there is an m such that $C_m = C_{m+1}$, then for all $k > m$, it will hold that $C_m = C_k$ (indeed $E(C_m) = E(C_{m+1}) = \ldots = E(C_k)$).

Definition 5 (Rank of a formula). A propositional formula A has *rank* i (for K), written $rank(A) = i$, if and only if i is the least natural number for which A is not exceptional for C_i. If A is exceptional for all C_i then A has no rank.

As mentioned above, we can restrict our consideration to sequences $C_0, \dots, C_n$ where C_n is the first set in the sequence such that either $C_n = \emptyset$ or $C_n = C_{n+1}$: in both cases for all $t > n$, $C_t = C_n$, therefore the formulas exceptional for C_t and C_n coincide. For this reason, if a formula A has a rank, then $rank(A) \leq n$.

The notion of rank of a formula allows to define the rational closure of a knowledge base K.

Definition 6 (Rational closure $\overline{K}$ of K). Let K be a conditional knowledge base. The rational closure $\overline{K}$ of K is the set of all $A \mathrel{|\!\sim} B$ such that either (1) the rank of A is strictly less than the rank of $A \wedge \neg B$ (this includes the case A has a rank and $A \wedge \neg B$ has none), or (2) A has no rank.

This mechanism, which is now well-established, allows to overcome some weaknesses of R. First of all, it is closed under rational monotonicity (RM): if $(A \mathrel{|\!\sim} B) \in \overline{K}$ and $(A \mathrel{|\!\sim} \neg C) \notin \overline{K}$ then $(A \wedge C) \mathrel{|\!\sim} B \in \overline{K}$. Furthermore, rational closure supports some of the wanted inferences that R does not support. For instance rational closure allows to deal with irrelevance: from $monday \mathrel{|\!\sim} go\_work$, it does support the nonmonotonic conclusion that $monday \wedge shines \mathrel{|\!\sim} go\_work$. In order to see that $monday \wedge shines \mathrel{|\!\sim} go\_work$ belongs to the rational closure of $K = \{monday \mathrel{|\!\sim} go\_work\}$, observe that $K \not\models \top \mathrel{|\!\sim} \neg(monday \wedge shines)$, therefore $rank(monday \wedge shines) = 0$. On the other hand, $K \models \top \mathrel{|\!\sim} \neg(monday \wedge shines \wedge \neg go\_work)$, therefore $rank(monday \wedge shines \wedge \neg go\_work) > 0$, from which we derive our nonmonotonic conclusion.

2.3. A semantic characterization of rational closure

Can we capture rational closure semantically?

We aim to provide a semantic reconstruction of rational closure in terms of a minimal models' mechanism, thus providing an instantiation of the following general recipe for nonmonotonic reasoning:

(i) fix an underlying modal semantics for conditionals (here we concentrate on R but another possible choice could have been the weaker P, as done for instance in [**? ? ?**]),

(ii) obtain nonmonotonic inference by restricting semantic consequence to a class of "minimal" models. These minimal models should be chosen on the basis of semantic considerations, independent from the *language* and from the *set of conditionals* (knowledge base) whose nonmonotonic consequences we want to determine.

In some respects, this approach is similar in spirit to "minimal models" approaches to nonmonotonic reasoning, such as circumscription [**?**]. However, as a difference with circumscription, the models (i) have a modal semantics, and (ii) the preference relation among models is independent from the language. This second aspect is also what differentiates this general recipe from other previous proposals such as [**?**], in which the idea is that preferred models are those ones that minimize the truth of specific formulas of the form $\neg\Box\neg A$.

The minimal model mechanism is based on comparing different models in order to see which one is preferred. As for circumscription, there are mainly two ways of comparing models with the same domain:

- by keeping the valuation function fixed (only comparing $\mathcal{M}$ and $\mathcal{M}'$ if V and V' in the two models coincide);

or

- by comparing $\mathcal{M}$ and $\mathcal{M}'$ also in case $V \neq V'$.

We consider the two possible semantics resulting from these alternatives.

As already mentioned, in this paper we limit our attention to knowledge bases K that are finite and that contain only positive conditionals. We begin by proving a property that links the rank $k_{\mathcal{M}}$ of a formula in any rational model $\mathcal{M}$ of a given knowledge base K and the rank of that formula as calculated in the definition of rational closure (Definition 5). The proof is similar to that of Lemma 5.18 in [**?**].

In the next proposition we shall use the notion of $\mathcal{M}_i$ defined as follows. Let $\mathcal{M} = \langle \mathcal{W}, <, V\rangle$ be any rational model of K. Let $\mathcal{M}_0 = \mathcal{M}$ and, for all i, let $\mathcal{M}_i = \langle \mathcal{W}_i, <_i, V_i\rangle$ be the rational model obtained from $\mathcal{M}$ by removing all the worlds w with $k_{\mathcal{M}}(w) < i$, i.e., $\mathcal{W}_i = \{w \in \mathcal{W} \mid k_{\mathcal{M}}(w) \geq i\}$. The C_i sets are those ones used to define the rank of a formula in Definition 5.

Proposition 2. *Let $\mathcal{M}$= $\langle \mathcal{W}, <, V\rangle$ be any rational model of K. For any propositional formula A, if rank(A) $\geq i$, then 1) $k_{\mathcal{M}}(A) \geq i$, and 2) if $A \mathrel{|\!\sim} B$ is rationally entailed by C_i, then $\mathcal{M}_i$ satisfies $A \mathrel{|\!\sim} B$.*

Proof. By induction on i. For $i = 0$, statement 1) holds, since it always holds that $k_{\mathcal{M}}(A) \geq 0$. Statement 2) also holds trivially.

For $i > 0$, 1) holds: if rank(A)$\geq i$, then by Definition 5 for all $j < i$, $C_j \models \top \mathrel{|\!\sim} \neg A$. By inductive hypothesis on 2), for all $j < i$ we have $\mathcal{M}_j \models \top \mathrel{|\!\sim} \neg A$. Hence, for all w with $k_{\mathcal{M}}(w) < i$, $\mathcal{M}, w \models \neg A$, and $k_{\mathcal{M}}(A) \geq i$. To prove 2), we reason as follows. Since $C_i \subseteq C_0$, $\mathcal{M} \models C_i$. Furthermore by definition of rank , for all $A \mathrel{|\!\sim} B \in C_i$, rank($A$) $\geq i$, hence by 1) just proved $k_{\mathcal{M}}(A) \geq i$. Hence $Min^{\mathcal{M}}_{<}(A) \subseteq \mathcal{W}_i$, and (given that $\mathcal{M} \models A \mathrel{|\!\sim} B$) also $\mathcal{M}_i \models A \mathrel{|\!\sim} B$. Therefore $\mathcal{M}_i \models C_i$. $\square$

A consequence of the previous proposition is the following.

Proposition 3. *Let $\mathcal{M}$ =$\langle \mathcal{W}, <, V\rangle$ be any rational model of K. For all w such that $k_{\mathcal{M}}(w) = i$, it holds that $\mathcal{M}, w \models \{A \rightarrow B \mid A \mathrel{|\!\sim} B \in C_i\}$.*

Proof. Let $\mathcal{M}$ =$\langle \mathcal{W}, <, V\rangle$ be any rational model of K. If $i = 0$, then for a contradiction suppose for some w with $k_{\mathcal{M}}(w) = 0$, and for some $A \rightarrow B : A \mathrel{|\!\sim} B \in C_0$, $\mathcal{M}, w \models A \wedge \neg B$. In this case obviously $w \in Min^{\mathcal{M}}_{<}(A)$, which contradicts that $Min^{\mathcal{M}}_{<}(A) \subseteq \{w \in \mathcal{W} \mid \mathcal{M}, w \models B\}$ (being $\mathcal{M}$ a model of K and $A \mathrel{|\!\sim} B \in K$). Therefore the proposition must hold. If $i > 0$ we repeat the same reasoning just done by considering $\mathcal{M}_i$ instead of $\mathcal{M}$: by Proposition 2, $\mathcal{M}_i$ satisfies C_i. By reasoning as for $i = 0$ we conclude that for all w with $k_{\mathcal{M}_i}(w) = 0$, $\mathcal{M}_i, w \models \{A \rightarrow B : A \mathrel{|\!\sim} B \in C_i\}$. By definition of $\mathcal{M}_i$ it follows that, for all w, it holds $k_{\mathcal{M}}(w) = i$, then $\mathcal{M}$,$w \models \{A \rightarrow B : A \mathrel{|\!\sim} B \in C_i\}$. □

Before we conclude the section we introduce one last proposition that we will use in the following.

Proposition 4. *For all K and A, if $K \models A \mathrel{|\!\sim} \bot$, then for all C_i, $C_i \models A \mathrel{|\!\sim} \bot$, and $C_i \models \top \mathrel{|\!\sim} \neg A$, i.e. A has no rank.*

Proof. Suppose for a contradiction that $K \models A \mathrel{|\!\sim} \bot$, but for some i, $C_i \not\models A \mathrel{|\!\sim} \bot$. In particular, let us consider the least i such that $C_i \not\models A \mathrel{|\!\sim} \bot$. By definition of C_i we can assume that $C_0 \supset \ldots \supset C_{i-1} \supset C_i$. Consider a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ of C_i in which it does not hold that $A \mathrel{|\!\sim} \bot$, i.e. in which $\{w \in \mathcal{W} \mid \mathcal{M} \models A\} \neq \emptyset$. By definition of C_i, for all conditionals $A_1 \mathrel{|\!\sim} B_1 \ldots A_n \mathrel{|\!\sim} B_n$ in $C_{i-1} - C_i$, it holds that $C_{i-1} \not\models \top \mathrel{|\!\sim} \neg A_1, \ldots, C_{i-1} \not\models \top \mathrel{|\!\sim} \neg A_n$, i.e. there are rational models $\mathcal{M}_1 = \langle \mathcal{W}_1, <_1, V_1 \rangle, \ldots, \mathcal{M}_n = \langle \mathcal{W}_n, <_n, V_n \rangle$ of C_{i-1} in which $\top \mathrel{|\!\sim} \neg A_1, \ldots, \top \mathrel{|\!\sim} \neg A_n$ does not hold, respectively, i.e., in which there are worlds $x_1, \ldots, x_n$ (respectively) such that $k_{\mathcal{M}_1}(x_1) = 0, \ldots, k_{\mathcal{M}_n}(x_n) = 0$, and $\mathcal{M}_1, x_1 \models A_1, \ldots, \mathcal{M}_n, x_n \models A_n$. Consider now the model $\mathcal{M}' = \langle \mathcal{W}', <', V' \rangle$ obtained from $\mathcal{M}$ by letting $\mathcal{W}' = \mathcal{W} \cup \{x_1, \ldots, x_n\}$, $V' = V$ for all worlds of $\mathcal{W}$, whereas $V = V_1, \ldots, V_n$ for the worlds $x_1, \ldots, x_n$ respectively. Let $k_{\mathcal{M}'}(x_1) = 0, \ldots, k_{\mathcal{M}'}(x_n) = 0$, whereas for all $w \in \mathcal{W}$, let $k_{\mathcal{M}'}(w) = k_{\mathcal{M}}(w) + 1$. Define $<'$ accordingly. We can prove that $\mathcal{M}'$ satisfies C_{i-1}: for conditionals $A_i \mathrel{|\!\sim} B_i$ in C_i this follows since for sure the minimal A_i-worlds will be worlds already in the starting $\mathcal{M}$ (since $C_{i-1} \models \top \mathrel{|\!\sim} \neg A_i$ hence none of the $x_1, \ldots, x_n$ is an A_i-world), and keep satisfying $A_i \mathrel{|\!\sim} B_i$ as they did it in $\mathcal{M}$. For $A_i \mathrel{|\!\sim} B_i \in C_{i-1} - C_i$, by construction of $\mathcal{M}'$, the minimal A_i-worlds will be one of the $x_1, \ldots, x_n$ just introduced, and they satisfy the conditional since they did so in the original models. Furthermore in $\mathcal{M}'$ there is an A-world (for the A of the proposition), which shows that $C_{i-1} \not\models A \mathrel{|\!\sim} \bot$. This contradicts the assumption that i is the least natural number such that $C_i \not\models A \mathrel{|\!\sim} \bot$. □

2.3.1. Fixed versus Variable Interpretations Minimal Models Semantics

The first semantics we consider is a *fixed interpretations minimal semantics*, for short *FIMS*.

Definition 7 (*FIMS*). Given models $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ and $\mathcal{M}' = \langle \mathcal{W}', <', V' \rangle$, we say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ with respect to the fixed interpretations minimal semantics, and we write $\mathcal{M} <_{FIMS} \mathcal{M}'$, if

- $\mathcal{W} = \mathcal{W}'$
- $V = V'$
- for all x, $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$ whereas there exists x' such that $k_{\mathcal{M}}(x') < k_{\mathcal{M}'}(x')$.

Given a knowledge base K, we say that $\mathcal{M}$ is a minimal model of K with respect to $<_{FIMS}$ if $\mathcal{M}$ is a model of K and there is no $\mathcal{M}'$ such that $\mathcal{M}'$ is a model of K and $\mathcal{M}' <_{FIMS} \mathcal{M}$. We say that K minimally entails a formula F with respect to *FIMS*, and we write $K \models_{FIMS} F$, if F is valid in all models of K that are minimal with respect to $<_{FIMS}$ (among all the possible models of K).

Proposition 5. *Given a finite model $\mathcal{M}$ of K, either $\mathcal{M}$ is a minimal FIMS model of K or there is a finite minimal FIMS model $\mathcal{M}'$ of K such that $\mathcal{M}' <_{FIMS} \mathcal{M}$.*

In our second semantics, we let the interpretations vary. The semantics is called variable interpretations minimal semantics, for short *VIMS*.

Definition 8 (*VIMS*). Given models $\mathcal{M} = \langle \mathcal{W}, <, V\rangle$ and $\mathcal{M}' = \langle \mathcal{W}', <', V'\rangle$ we say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ with respect to the variable interpretations minimal semantics, and write $\mathcal{M} <_{VIMS} \mathcal{M}'$, if

- $\mathcal{W} = \mathcal{W}'$
- for all x, $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$ whereas there exists x' such that $k_{\mathcal{M}}(x') < k_{\mathcal{M}'}(x')$.

Given a knowledge base K, we say that $\mathcal{M}$ is a minimal model of K with respect to $<_{VIMS}$ if $\mathcal{M}$ is a model of K and there is no $\mathcal{M}'$ such that $\mathcal{M}'$ is a model of K and $\mathcal{M}' <_{VIMS} \mathcal{M}$. K minimally entails a formula F with respect to *VIMS*, and we write $K \models_{VIMS} F$, if F is valid in all models of K that are minimal with respect to $<_{VIMS}$ (among all the possible models of K).

It is easy to realize that the two semantics, *FIMS* and *VIMS*, define different sets of minimal models. This is illustrated by the following example.

Example 1. Let $K = \{penguin \mathrel{|\!\sim} bird, penguin \mathrel{|\!\sim} \neg fly, bird \mathrel{|\!\sim} fly\}$. We derive that $K \not\models_{FIMS} penguin \wedge black \mathrel{|\!\sim} \neg fly$. Indeed in *FIMS* there can be a model $\mathcal{M}$ in which $\mathcal{W} = \{x, y, z\}$, $V(x) = \{penguin, bird, fly, black\}$, $V(y) = \{penguin, bird\}$, $V(z) = \{bird, fly\}$, and $z < y < x$. $\mathcal{M}$ is a model of K, and it is minimal with respect to *FIMS* (indeed once fixed $V(x), V(y), V(z)$ as above, it is not possible to lower the rank of x nor of y nor of z unless we falsify K). Furthermore, in $\mathcal{M}$ x is a typical world in which "it is a penguin" and "it is black" hold (since there is no other world satisfying the same propositions which is preferred to it) where "it flies" holds. Therefore, $K \not\models_{FIMS} penguin \wedge black \mathrel{|\!\sim} \neg fly$.

On the other hand, $\mathcal{M}$ is not minimal with respect to *VIMS*. Indeed, consider the model $\mathcal{M}' = \langle \mathcal{W}, <', V'\rangle$ obtained from $\mathcal{M}$ by letting $V'(x) = \{penguin, bird, black\}$, $V'(y) = V(y)$, $V'(z) = V(z)$ and by defining $<'$ as: $z <' y$ and $z <' x$. Clearly $\mathcal{M}' \models K$, and $\mathcal{M}' <_{VIMS} \mathcal{M}$, since $k_{\mathcal{M}'}(x) < k_{\mathcal{M}}(x)$, while $k_{\mathcal{M}'} = k_{\mathcal{M}}$ for all other worlds.

The example above shows that *FIMS* and *VIMS* lead to different sets of minimal models for a given K. Notice, however, that the model $\mathcal{M}'$ we have used to illustrate this fact is not a minimal model for K in *VIMS*. A minimal model in *VIMS* for K that can be defined on the set of worlds $\mathcal{W}$ is given by $V(x) = V(y) = V(z) = \{bird, fly\}$, and the empty relation $<$. This is quite a degenerate model of K in which "it is a penguin" is never true. This illustrates the strength of *VIMS*: in case of knowledge bases that only contain positive conditionals, logical entailment in *VIMS* collapses into classical logic entailment. This feature corresponds to a similar feature of the nonmonotonic logic P_{min} in [**?**] (see Section 2.4), can be proven in the same way, and leads to the following proposition.

Proposition 6. *Let K be a set of positive conditionals. Let us replace all formulas of the form $A \mathrel{|\!\sim} B$ in K with $A \rightarrow B$, and call K° the resulting set of formulas. We have that $K \models_{VIMS} A \mathrel{|\!\sim} B$ if and only if $K^\circ \models_{PC} A \rightarrow B$.*

As for P_{min} this strong feature of *VIMS* can be prevented by adding *existence assertions* to the knowledge base, in the example we could add, for instance, $\neg(penguin \mathrel{|\!\sim} \bot)$ to force us to consider non-trivial models where the proposition "it is a penguin" is satisfied. In the next section, we will apply *VIMS* in a similar way, by restricting our consideration to knowledge bases that include existence assertions (expressed by negated conditionals).

2.3.2. A semantic reconstruction of Rational Closure

Can we capture rational closure within one or the other of the semantics above? A first conjecture might be that the *FIMS* of Definition 7 could capture rational closure. However, we are soon forced to recognize that this is not the case. For instance, Example 1 above illustrates that $\{penguin \mathrel{|\!\sim} bird, penguin \mathrel{|\!\sim} \neg fly, bird \mathrel{|\!\sim} fly\} \not\models_{FIMS} penguin \land black \mathrel{|\!\sim} \neg fly$. On the contrary, it can be easily verified that $penguin \land black \mathrel{|\!\sim} \neg fly$ is in the rational closure of $\{penguin \mathrel{|\!\sim} bird, penguin \mathrel{|\!\sim} \neg fly, bird \mathrel{|\!\sim} fly\}$. Therefore, *FIMS* as it is does not allow us to define a semantics corresponding to rational closure. Things change if we consider *FIMS* applied to models that contain *all possible valuations compatible with a given knowledge base K*. We call these models *canonical models*.

Example 2. Consider Example 1 above. If we restrict our attention to models that also contain a world w with $V(w) = \{penguin, bird, black\}$ which satisfies "it is a penguin", "it is black" and "it does not fly" in which w is a typical world satisfying "it is a penguin", we are able to conclude that *typically it holds that if it is a penguin and it is black then it does not fly*, the same as in rational closure. Indeed, in all minimal *FIMS* models of K that also contain w with $V(w) = \{penguin, bird, black\}$, it holds that $penguin \land black \mathrel{|\!\sim} \neg fly$ (in particular, in Example 1 above, adding w to $\mathcal{M}$ would give $z < w$ and $w < x$).

We are led to the conjecture that *FIMS* restricted to canonical models could be the right semantics for rational closure. Canonical models are defined with respect to the language $\mathcal{L}$ restricted to the propositional variables occurring in the knowledge base and in the query. Given a knowledge base K and a query Q, let $ATM_{K,Q}$ be the set of all the propositional variables of *ATM* occurring in K or in the query Q, and let $\mathcal{L}_{K,Q}$ be the restriction of the language $\mathcal{L}$ to the propositional variables in $ATM_{K,Q}$.

A truth assignment $v : ATM_{K,Q} \longrightarrow \{true, false\}$ is *compatible* with K, if there is no propositional formula $A \in \mathcal{L}_{K,Q}$ such that $v(A) = true$ and $K \models A \mathrel{|\!\sim} \bot$ (where v is extended to arbitrary propositional formulas as usual).

Definition 9 (Canonical Model). A model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ satisfying a knowledge base K is said to be *canonical* if it contains (at least) a world associated to each truth assignment compatible with K, that is to say: if v is compatible with K, then there exists a world w in $\mathcal{W}$ such that, for all propositional formulas $B \in \mathcal{L}_{K,Q}$, $\mathcal{M}, w \models B$ if and only if $v(B) = true$.

It can be easily shown that, for any knowledge base, a minimal canonical *FIMS* model exists: this is any canonical model in which every possible world w has the rank associated to the conjunction of all atoms and negated atoms in $\mathcal{L}_{K,Q}$ that it satisfies. This is stated by the following theorem.

Theorem 1. *For any satisfiable K there exists a finite minimal canonical FIMS model* $\mathcal{M}$*.*

Proof. Since K is satisfiable consider a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ of K. Given the finite model property of R, we can assume, without loss of generality that $\mathcal{M}$ has a finite set of worlds. Let $v_1, \dots, v_r$ be any enumeration of the truth assignments $v_i : ATM_{K,Q} \longrightarrow \{true, false\}$ compatible with K. Observe that, since $ATM_{K,Q}$ is a finite set of propositional variables, the truth assignments v_i as defined above are finitely many.

We proceed starting from $\mathcal{M}$ and extending it by the addition of new worlds. Let $\mathcal{M}_0 = \mathcal{M}$. For each i, from 1 to r, we reason as follows. If there is no world in $\mathcal{M}_{i-1}$ associated to v_i, consider a model $\mathcal{M}' = \langle \mathcal{W}', <', V' \rangle$ of K in which there is at least a world associated to v_i. Such a model $\mathcal{M}'$ exists since v_i is compatible with K. By Fact 1, we can assume $\mathcal{M}'$ to be finite as well. We add to $\mathcal{M}_{i-1}$ all the worlds in $\mathcal{M}'$, to get $\mathcal{M}_i = \langle \mathcal{W}_i, <_i, V_i \rangle$, where: (1) $\mathcal{W}_i = \mathcal{W}_{i-1} \cup \mathcal{W}'$; (2) $<_i$ is defined as $<_{i-1}$ on the worlds in $\mathcal{W}_{i-1}$; it is defined as $<'$ on the worlds in $\mathcal{W}'$ and, for all $x \in \mathcal{W}_{i-1}$ and $y \in \mathcal{W}'$, $x <_i y$; (3) V_i is defined as V_{i-1} on the worlds in $\mathcal{W}_{i-1}$ and it is defined as V' on the worlds in $\mathcal{W}'$.

Observe, that the resulting model $\mathcal{M}_i$ is the juxtaposition of the two models $\mathcal{M}_{i-1}$ and $\mathcal{M}'$, where the rank of each world in $\mathcal{M}_{i-1}$ is lower than the rank of each world in $\mathcal{M}'$. It is finite, as both $\mathcal{M}_{i-1}$ and $\mathcal{M}'$ are finite.

It is easy to see that, if $\mathcal{M}_{i-1}$ satisfies K, then $\mathcal{M}_i$ satisfies K as well. Consider any conditional $C \mathrel{|\!\sim} B \in K$, and any world $w \in Min^{\mathcal{M}_i}_{<_i}(C)$. Then either $w \in \mathcal{W}_{i-1}$ or $w \in \mathcal{W}'$. If $w \in \mathcal{W}_{i-1}$, then $w \in Min^{\mathcal{M}_{i-1}}_{<_i}(C)$, by the definition of $<_i$. Since $\mathcal{M}_{i-1}$ is a model of K, $\mathcal{M}_{i-1} \models C \mathrel{|\!\sim} B$ and $\mathcal{M}_{i-1}, w \models B$. By construction, $V_i(w) = V_{i-1}(w)$, so that $\mathcal{M}_i, w \models B$, and $\mathcal{M}_i \models C \mathrel{|\!\sim} B$. If $w \in \mathcal{W}'$, then $w \in Min^{\mathcal{M}'}_{<_i}(C)$, by the definition of $<_i$. Since $\mathcal{M}'$ is a model of K, $\mathcal{M}' \models C \mathrel{|\!\sim} B$ and $\mathcal{M}', w \models B$. By construction, $V_i(w) = V'(w)$, so that $\mathcal{M}_i, w \models B$, and $\mathcal{M}_i \models C \mathrel{|\!\sim} B$.

Given that $\mathcal{M}_0 = \mathcal{M}$ is a model of K, we conclude that all the $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_r$ are models of K. After all the valuations $v_1, \dots, v_r$ have been considered, we obtain a model $\mathcal{M}_r$ of K which is canonical and is finite as well, as we have only considered finite models in the construction of $\mathcal{M}_r$. From $\mathcal{M}_r$, by Proposition 5, we can obtain a minimal canonical *FIMS* model. □

In the following, we show that the canonical models that are minimal with respect to *FIMS* are an adequate semantic counterpart of rational closure.

Proposition 7. *Let* $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ *be a canonical model of K, minimal with respect to* $<_{FIMS}$*. Given* $i \in \mathbb{N}$*, for all* $w \in \mathcal{W}$ *it holds that: if* $\mathcal{M}, w \models A \rightarrow B$ *for all* $A \mathrel{|\!\sim} B$ *in* C_i*, then* $k_{\mathcal{M}}(w) \leq i$*.*

Proof. The proof is by induction on i. If $i = 0$, suppose for a contradiction that there is a w such that $\mathcal{M}, w \models A \rightarrow B$ for all $A \mathrel{|\!\sim} B$ in C_0, but $k_{\mathcal{M}}(w) > 0$. Then it can be easily seen that the canonical model obtained from $\mathcal{M}$ by simply changing $k_{\mathcal{M}}(w)$ into 0 is still a model of $C_0 = K$ and it is preferred to $\mathcal{M}$, thus contradicting the minimality of $\mathcal{M}$.

For $i > 0$, we reason in a similar way: let us consider $w \in \mathcal{W}$ such that for all $A \mathrel{|\!\sim} B$ in C_i, $\mathcal{M}$,$w \models A \rightarrow B$ but $k_{\mathcal{M}}(w) > i$. Let $\mathcal{M}'$ be a model obtained from $\mathcal{M}$ by changing $<$ in order to have $k_{\mathcal{M}'}(w) = i$. $\mathcal{M}'$ is preferred to $\mathcal{M}$ and it is a model of K, as it satisfies all the conditionals in K. Let $A \mathrel{|\!\sim} B \in K$. It is clear that, for all the worlds $w' \in \mathcal{W}$ with $w' \neq w$, w' satisfies $A \mathrel{|\!\sim} B$ in $\mathcal{M}'$, as it satisfies it in $\mathcal{M}$. To show that w satisfies $A \mathrel{|\!\sim} B$, let $w \in Min_<^{\mathcal{M}'}(A)$. If $A \mathrel{|\!\sim} B$ in C_i, we know from the hypothesis that w satisfies $A \rightarrow B$, and hence, w satisfies B. If $A \mathrel{|\!\sim} B$ in $K - C_i$, there is a $j < i$ such that $A \mathrel{|\!\sim} B \in C_j$, $C_j \not\models \top \mathrel{|\!\sim} \neg A$ while $C_{j-1} \models \top \mathrel{|\!\sim} \neg A$. From $C_j \not\models \top \mathrel{|\!\sim} \neg A$, it follows that there is a model $\mathcal{M}_j$ of C_j with a w° such that $k_{\mathcal{M}_j}(w^\circ) = 0$ and w° satisfies A. By Proposition 3, we have that $\mathcal{M}_j, w^\circ$ satisfies $\{A \rightarrow B : A \mathrel{|\!\sim} B \in C_j\} \cup \{A\}$, hence $C_j \not\models A_1 \rightarrow B_1 \wedge \ldots \wedge A_m \rightarrow B_m \wedge A \mathrel{|\!\sim} \bot$ and, by Proposition 4, we have that $K \not\models A_1 \rightarrow B_1 \wedge \ldots \wedge A_m \rightarrow B_m \wedge A \mathrel{|\!\sim} \bot$. Since $\mathcal{M}'$ (as $\mathcal{M}$) is canonical, it follows that there is a world $w^* \in \mathcal{W}$ such that w^* satisfies all the implications $A' \rightarrow B'$ s.t. $A' \mathrel{|\!\sim} B'$ in C_j and w^* satisfies A. By inductive hypothesis, $k_{\mathcal{M}}(w^*) < i$, and therefore $k_{\mathcal{M}}(A) < i$. By construction of $\mathcal{M}'$, $k_{\mathcal{M}'}(w^*) < i$, and therefore $k_{\mathcal{M}'}(A) < i$ which contradicts the hypothesis that $w \in Min_<^{\mathcal{M}'}(A)$. Hence, $\mathcal{M}'$ satisfies all the conditionals in K. The fact that $k_{\mathcal{M}}(w) > i$ and $k_{\mathcal{M}'}(w) = i$ contradicts the minimality of $\mathcal{M}$. Hence, it must be $k_{\mathcal{M}}(w) \leq i$, and the proof is over. □

Proposition 8. *Let $\mathcal{M}$ be a canonical model of K minimal with respect to $<_{FIMS}$. Then, given $i \in \mathbb{N}$, $rank(A) = i$ if and only if $k_{\mathcal{M}}(A) = i$.*

Proof. *(Only if part)* Let us assume that $rank(A) = i$. By definition of rank, we know that $C_i \not\models \top \mathrel{|\!\sim} \neg A$. Then there is a rational model $\mathcal{M}'$ of C_i that does not satisfy $\top \mathrel{|\!\sim} \neg A$. In $\mathcal{M}'$ there must be a world w', with $k_{\mathcal{M}'}(w') = 0$ such that $\mathcal{M}', w' \models A$. For all propositional formulas $B \in \mathcal{L}$, such that $\mathcal{M}', w' \models B$, it must be the case that C_i that does not satisfy $\top \mathrel{|\!\sim} \neg B$ in $\mathcal{M}'$. Hence, for all propositional formulas $B \in \mathcal{L}$, such that $\mathcal{M}', w' \models B$, $C_i \not\models \top \mathrel{|\!\sim} \neg B$. Let B' be the conjunction of all these Bs. Clearly, A is one of the conjuncts of B'. Furthermore, $C_i \not\models \top \mathrel{|\!\sim} \neg B'$. By Proposition 4, from $C_i \not\models \top \mathrel{|\!\sim} \neg B'$, it follows that $K \not\models B' \mathrel{|\!\sim} \bot$. Let v be the truth assignment associated with the world w' of $\mathcal{M}'$. Then v is compatible with K. Since $\mathcal{M}$ is a canonical model, there must be a world $w \in W$ of $\mathcal{M}$ such that for all propositional formulas $B \in \mathcal{L}$, $\mathcal{M}, w \models B$ if and only if $v(B) = true$. In particular, we have that $\mathcal{M}, w \models A$. We show that, for all $D \mathrel{|\!\sim} B \in C_i$, $\mathcal{M}, w \models D \rightarrow B$. Observe that D and B are propositional formulas and that their valuation is the same in w and in w'. Hence it is sufficient to show that $\mathcal{M}', w' \models D \rightarrow B$, for all $D \mathrel{|\!\sim} B \in C_i$. This follows from the fact that $\mathcal{M}', w' \models D \mathrel{|\!\sim} B$ holds for all $D \mathrel{|\!\sim} B \in C_i$. Indeed, if $\mathcal{M}', w' \not\models D$, it trivially holds that $\mathcal{M}', w' \models D \rightarrow B$. If $\mathcal{M}', w' \models D$, then (since $k_{\mathcal{M}'}(w') = 0$), $w' \in Min_{<'}^{\mathcal{M}'}(D)$, and hence $\mathcal{M}', w' \models B$. Thus, $\mathcal{M}', w' \models D \rightarrow B$.

Now, there is a world $w \in W$ such that, for all $D \mathrel{|\!\sim} B \in C_i$, w satisfies $D \rightarrow B$. By Proposition 7, $k_{\mathcal{M}}(w) \leq i$. Since w satisfies A, $k_{\mathcal{M}}(A) \leq i$. As by Proposition 2 we know that $k_{\mathcal{M}}(A) \geq i$, we can conclude that $k_{\mathcal{M}}(A) = i$.

(If part) This direction is obvious, given the *only if* part: if $k_{\mathcal{M}}(A) = i$, then

$rank(A) = i$. Indeed, by absurd, if $rank(A) = j \neq i$, then $k_{\mathcal{M}}(A) = j \neq i$, against the hypothesis. □

A direct consequence of Proposition 8 together with the observation that if a formula has a rank then its maximal value is n where n is the last element of $C_0 \supset \ldots \supset C_n$ such that $C_n = \emptyset$ or such that for all $m > n$, $C_m = C_n$ is stated in the following proposition.

Proposition 9. *Let n be the last element of $C_0 \supset \ldots \supset C_n$ such that $C_n = \emptyset$ or such that for all $m > n$ $C_m = C_n$, then in all minimal canonical models $\mathcal{M}$, for all worlds w, $k_{\mathcal{M}}(w) \leq n$.*

We can now prove the following theorem:

Theorem 2. *Let K be a knowledge base and $\mathcal{M}$ be a canonical model of K minimal with respect to $<_{FIMS}$. We show that, for all conditionals $A \mathrel{|\!\sim} B \in \mathcal{L}$:*

$$\mathcal{M} \models A \mathrel{|\!\sim} B \text{ if and only if } A \mathrel{|\!\sim} B \in \overline{K},$$

where $\overline{K}$ is the rational closure of K.

Proof. *(Only if part)* Let us assume that $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ satisfies $A \mathrel{|\!\sim} B$. Then, for each world $w \in Min_<(A)$, w satisfies B. If $Min_<(A) = \emptyset$, then there is no w s.t. $\mathcal{M}, w \models A$, hence A has no rank in $\mathcal{M}$ and, by Proposition 8, A has no rank. In this case, by Definition 6, $A \mathrel{|\!\sim} B \in \overline{K}$. Let us assume that $k_{\mathcal{M}}(A) = i$. As $k_{\mathcal{M}}(A \land B) < k_{\mathcal{M}}(A \land \neg B)$, then $k_{\mathcal{M}}(A \land \neg B) > i$. By Proposition 8, $rank(A) = i$ and $rank(A \land \neg B) > i$. Hence, by Definition 6, $A \mathrel{|\!\sim} B \in \overline{K}$.

(If part) If $A \mathrel{|\!\sim} B$ belongs to $\overline{K}$, then, by Definition 6, either (a) $rank(A) < rank(A \land \neg B)$ (or A has a rank and $A \land \neg B$ has not), or (b) A has no rank. In the first case (a), by Proposition 8 we have that $k_{\mathcal{M}}(A) < k_{\mathcal{M}}(A \land \neg B)$, which entails $k_{\mathcal{M}}(A \land B) < k_{\mathcal{M}}(A \land \neg B)$. Hence $\mathcal{M}$ satisfies $A \mathrel{|\!\sim} B$. In case, A has a rank and $A \land \neg B$ has not, suppose $rank(A) = i$. By Proposition 8, $k_{\mathcal{M}}(A) = i$. It is easy to show that $k_{\mathcal{M}}(A \land \neg B) > i$. If, by absurdum, $k_{\mathcal{M}}(A \land \neg B) \leq i$, by Proposition 8, we would have $rank(A \land \neg B) \leq i$, against the hypothesis that $A \land \neg B$ has no rank.

In case (b), by Proposition 8, A has no rank in $\mathcal{M}$, hence $\mathcal{M}$ satisfies $A \mathrel{|\!\sim} B$. □

In Theorem 2 we have shown a correspondence between rational closure and minimal models with fixed interpretations, *on the proviso that* we restrict our attention to minimal *canonical* models. We can obtain the same effect by extending K into K' by adding negated conditionals:

Definition 10. Let K be a knowledge base. We define

$$K' = K \cup \{\neg(C \mathrel{|\!\sim} \bot) \mid C = (\neg)A_1 \land (\neg)A_2 \land \ldots \land (\neg)A_n,$$
$$\text{such that } A_i \in ATM_{K,Q}, \text{ with } i = 1, 2, \ldots, n, \text{ and } K \not\models (C \mathrel{|\!\sim} \bot)\}$$

(that is C is a conjunction of literals whose propositional variables occur in the knowledge base or in the query).

Indeed it can be easily verified that all models of K' are canonical, hence restricting *FIMS* to canonical models on the one hand and considering the extension of K as K' on the other hand amounts to the same effect. We can therefore restate Theorem 2 above as follows:

Theorem 3. *Let K be a knowledge base and let K' be defined as in Definition 10. It holds that*

$$K' \models_{FIMS} A \mathrel{|\!\sim} B \text{ if and only if } A \mathrel{|\!\sim} B \in \overline{K},$$

where $\overline{K}$ is the rational closure of K.

Notice that the size of K' is exponential in that of K.

Before we go any further, let us point out that this characterization of rational closure, in terms of minimal canonical *FIMS* models, is related to Lehmann and Magidor's semantical characterization in [**?**]: we use canonical models, as they do, and we show a correspondence between the rank of a formula (syntactically defined in terms of exceptionality) and the rank of the formula in minimal canonical *FIMS* models. However the definition of minimal canonical *FIMS* models that we use here, based on a specific preference relation between different canonical models, is different from the definition provided in [**?**] (see Section 5.3, Definition 20) where the involved preference relation is defined in terms of conditionals satisfied in the compared models.

We may wonder whether the restriction to canonical models can be lifted by adopting a semantics based on variable valuations. In general the answer is negative. We have already mentioned that, if we consider knowledge bases containing only positive conditionals, logical entailment in *VIMS* collapses into classical logic entailment. To avoid this collapse, we can require that, when we are checking for entailment of a conditional $A \mathrel{|\!\sim} B$ from a K, at least an $A \land B$-world and an $A \land \neg B$-world be present in the models of K. This can be obtained by adding to K the conditionals $\neg(A \land B \mathrel{|\!\sim} \bot)$ and $\neg(A \land \neg B \mathrel{|\!\sim} \bot)$. Also in this case, however, we cannot give a positive answer to the above question. Indeed, it is possible to build a model of K, minimal with respect to *VIMS*, which falsifies a conditional $A \mathrel{|\!\sim} B$ which, on the contrary, is satisfied in all the canonical minimal models of K under *FIMS*. This is shown by the following:

Example 3. Let K be as follows:

$\{\top \mathrel{|\!\sim} S,$
$S \mathrel{|\!\sim} \neg D,$
$L \mathrel{|\!\sim} P,$
$R \mathrel{|\!\sim} Q,$
$E \mathrel{|\!\sim} F,$
$H \mathrel{|\!\sim} G,$
$D \mathrel{|\!\sim} \neg P \land \neg Q \land \neg F \land \neg G,$
$S \mathrel{|\!\sim} \neg(L \land R),$
$S \mathrel{|\!\sim} \neg(L \land E),$
$S \mathrel{|\!\sim} \neg(L \land H),$
$S \mathrel{|\!\sim} \neg(R \land E),$
$S \mathrel{|\!\sim} \neg(R \land H),$
$S \mathrel{|\!\sim} \neg(E \land H)\}.$

Let

$A = D \land S \land R \land L \land E \land H,$
$B = \neg Q \land \neg P \land \neg F \land \neg G$

and let

$$K' = K \cup \{\neg(A \wedge B \mathrel{|\!\sim} \bot),\ \neg(A \wedge \neg B \mathrel{|\!\sim} \bot)\}.$$

We define a model $\mathcal{M} = (\mathcal{W}, <, V)$ of K', which is minimal with respect to *VIMS*, as follows: $\mathcal{W} = \{x, w, y_1.y_2, y_3\}$, where:

$V(y_1) = \{S, \neg D, \neg R, \neg L, \neg E, \neg H, P, Q, F, G\}$
$V(y_2) = \{\neg S, \neg D, R, L, E, H, P, Q, F, G\}$
$V(y_3) = \{\neg S, D, \neg P, \neg Q, \neg F, \neg G, \neg R, \neg L, \neg E, \neg H\}$
$V(x) = \{D, S, R, L, E, H, \neg Q, \neg P, \neg F, \neg G\}$
$V(w) = \{D, S, R, L, E, H, Q, \neg P, \neg F, \neg G\}$

with $k_{\mathcal{M}}(y_1) = 0$, $k_{\mathcal{M}}(y_2) = 1$, $k_{\mathcal{M}}(y_3) = 1$, $k_{\mathcal{M}}(x) = 2$ and $k_{\mathcal{M}}(w) = 2$. Observe that: x is an $A \wedge B$-minimal world; w is an $A \wedge \neg B$-minimal world; y_1 is an S-minimal world; y_2 is a minimal world for R, L, E and H; and y_3 is a D-minimal world.

$\mathcal{M}$ is a model of K which is minimal with respect to *VIMS*. Also, $A \mathrel{|\!\sim} B$ is falsified in $\mathcal{M}$, while, on the contrary, $A \mathrel{|\!\sim} B$ holds in all the canonical models minimal with respect to *FIMS*. Indeed, in all such models the rank of $k(A \wedge B) = 1$ while $k(A \wedge \neg B) = 2$. However, it is not possible to construct a model $\mathcal{M}'$ with 5 worlds so that $\mathcal{M}' <_{VIMS} \mathcal{M}$. In particular, lowering the rank of w is never possible, since w is a non-typical D-world, and typical $D-$worlds are non typical $\top$-worlds, hence w will always have rank at least 2. For x we reason in a different way: although in principle it could have rank 1, assigning to x rank 1 entails that there are at least 4 distinct R, L, E and H-worlds with rank 0. But this is impossible given that we have only 5 worlds in the model. In order to satisfy all these formulas by a single world, we have to introduce a world at level 1 (which can be a non S and therefore satisfy pairs of these formulas). This is world y_2, whose rank cannot therefore be lowered. y_2 cannot be a $D-$world, we therefore need y_3 which is a minimal $D-$world that can have rank at least 1 and whose rank cannot therefore be lowered.

As suggested by this example, in order to characterize rational closure in terms of *VIMS*, we should restrict our consideration to models which contain "enough" worlds. In the following, as in Theorem 3, we enrich K with negated conditionals but, as a difference with K' of Theorem 3, we only need to add to K a polynomial number of negated conditionals (instead of an exponential number). The purpose of the addition is that of restricting our attention to models that are minimal with respect to $<_{VIMS}$ and that have a set of worlds "large" enough to have, in principle, a distinct most-preferred world for each antecedent of conditionals in K. Intuitively, this condition discards the models, as the one illustrated by the example above, in which a formula (e.g. $A \wedge B$) has a rank higher than the rank it could have just because there are not enough worlds (and lowering the rank of a formula would lead to the falsification of some conditionals in K).

For this reason, we expand K into K'' by adding, for each antecedent C of a conditional formulas in K, a new corresponding atom ϕ_C, and by requiring that all these new atoms are mutually disjoint. This will guarantee that all models of K'' will have a distinct world satisfying each newly introduced atom ϕ_C and its corresponding formula C. Furthermore, if the problem to be addressed is that of knowing whether $A \mathrel{|\!\sim} B$

is logically entailed by K, we also introduce $\phi_{A\wedge B}$ and $\phi_{A\wedge\neg B}$ in order to also have a distinct world associated to $A \wedge B$ and $A \wedge \neg B$. This is stated in a formal way in the following definition.

Definition 11. Given a knowledge base K, we define:

- $A_{K,A\mid\sim B} = \{C \mid$ either, for some D, $C \mid\sim D \in K$ or $C = A \wedge B$ or $C = A \wedge \neg B$, and $K \not\models C \mid\sim\bot\}$;
- $K'' = K \cup \{\neg(C \wedge \phi_C \mid\sim\bot) : C \in A_{K,A\mid\sim B}\} \cup \{(\phi_{C_i} \wedge \phi_{C_j} \mid\sim\bot) : C_i, C_j \in A_{K,A\mid\sim B}\}$.

We can now establish a correspondence between *FIMS* and *VIMS*. By virtue of Theorem 2, this allows us to establish a correspondence between rational closure and *VIMS*, as stated by Theorem 1.

Theorem 4. *Let $\mathcal{M}$ be a canonical model of K, minimal with respect to FIMS, and let K'' be the extension of K defined as in Definition 11. We have that:*

$$\mathcal{M} \models A \mid\sim B \textit{ if and only if } K'' \models_{VIMS} A \mid\sim B.$$

Proof. We show the contrapositive of the two directions.

First, suppose $K'' \not\models_{VIMS} A \mid\sim B$. Let $\mathcal{M}' = \langle \mathcal{W}', <', V' \rangle$ be a model of K'' minimal with respect to $<_{VIMS}$ that does not satisfy $A \mid\sim B$, i.e., such that $k_{\mathcal{M}'}(A \wedge \neg B) \leq k_{\mathcal{M}'}(A \wedge B)$. We want to show that also $\mathcal{M} \not\models A \mid\sim B$, i.e., $k_{\mathcal{M}}(A \wedge \neg B) \leq k_{\mathcal{M}}(A \wedge B)$. For a contradiction, suppose in the canonical $\mathcal{M}$, $k_{\mathcal{M}}(A \wedge \neg B) = j > k_{\mathcal{M}}(A \wedge B) = i$. By Propositions 2 and 8, $k_{\mathcal{M}'}(A \wedge \neg B) \geq j$ and $k_{\mathcal{M}'}(A \wedge B) \geq i$, and since by hypothesis $k_{\mathcal{M}'}(A \wedge \neg B) \leq k_{\mathcal{M}'}(A \wedge B)$, it follows that $k_{\mathcal{M}'}(A \wedge B) \geq j > i$. We show that this goes against the minimality of $\mathcal{M}'$.

From $\mathcal{M}$ and $\mathcal{M}'$ we build a model $\mathcal{M}^* = \langle \mathcal{W}^*, <^*, V^* \rangle$ such that $\mathcal{M}^*$ is a model of K'' and $\mathcal{M}^* <_{VIMS} \mathcal{M}'$. In particular, for each formula in $A_{K,A\mid\sim B}$, we include in $\mathcal{W}^*$ a minimal world from $\mathcal{M}$ satisfying that formula. More precisely, we introduce in $\mathcal{W}^*$ the following worlds from $\mathcal{M}$: $x \in Min^{\mathcal{M}}_<(A \wedge B)$, $x' \in Min^{\mathcal{M}}_<(A \wedge \neg B)$ and a world $y \in Min^{\mathcal{M}}_<(C)$, for each C antecedent of a conditional in K s.t. $K \not\models C \mid\sim\bot$. For these worlds, we define $V^* = V$ and $k_{\mathcal{M}^*} = k_{\mathcal{M}}$. If the same element y is associated to two different formulas it must be duplicated into y and y' (and $V^*(y') = V^*(y)$ and $k_{\mathcal{M}^*}(y') = k_{\mathcal{M}^*}(y)$). Furthermore, for each world y introduced as a representative of $Min^{\mathcal{M}}_<(C)$, $V^*(y)$ is extended in order to include ϕ_C. $<^*$ is straightly defined from $k_{\mathcal{M}^*}$ in the obvious way. The construction is almost finished. Notice that up to this point we have introduced in $\mathcal{W}^*$ no more elements than those in $\mathcal{W}'$. To conclude we have to rename the elements of $\mathcal{W}^*$ with the names as the elements of $\mathcal{W}'$ that satisfy the same ϕ_C, and we have to add to $\mathcal{W}^*$ the elements of $\mathcal{W}'$ that are eventually missing (we let for these cases $V^* = V'$ and $k_{\mathcal{M}^*} = k_{\mathcal{M}'}$).

It can be shown that $\mathcal{M}^*$ is a model of K'', and $\mathcal{M}^* <_{VIMS} \mathcal{M}'$, against the minimality of $\mathcal{M}'$. First of all, we show that $\mathcal{M}^*$ is a model of K''. Indeed, by construction we have introduced a new element y of $\mathcal{M}$ for each C antecedent of a conditional in K or equal to $A \wedge B$ or $A \wedge \neg B$, and this element is still in $Min^{\mathcal{M}^*}_<(C)$ (otherwise, $k_{\mathcal{M}^*}(C) < k_{\mathcal{M}^*}(y) = k_{\mathcal{M}}(y) = k_{\mathcal{M}}(C)$, against Propositions 2 and 8). Furthermore, $V^*(y)$ includes ϕ_C. Hence, $\mathcal{M}^*$ satisfies all conditionals introduced in K'' with form

$\neg(C \wedge \phi_C) \mid\!\sim \perp$. Consider now the positive conditionals $C \mid\!\sim D$ in K'', that were already in K. Hence, consider any y inserted in $\mathcal{M}^*$ from $\mathcal{M}$. Let $y \in Min_<^{\mathcal{M}^*}(C)$. Then also $y \in Min_<^{\mathcal{M}}(C)$ (otherwise there would be another $y' \in Min_<^{\mathcal{M}}(C)$ with $\mathcal{M}, y' \models C$ and $k_{\mathcal{M}}(y') < k_{\mathcal{M}}(y)$ that would have been taken in the construction; and by construction in $\mathcal{M}^*$ it would hold that $\mathcal{M}^*, y' \models C$ and $k_{\mathcal{M}^*}(y') < k_{\mathcal{M}^*}(y)$, against $y \in Min_<^{\mathcal{M}^*}(C)$). Since $\mathcal{M}$ is a model of K, and $C \mid\!\sim D \in K$, $\mathcal{M}, y \models D$, hence also $\mathcal{M}^*, y \models D$. Consider now y introduced in $\mathcal{M}^*$ from $\mathcal{M}'$. If $y \in Min_<^{\mathcal{M}^*}(C)$, then we reason as follows to show that $y \in Min_<^{\mathcal{M}'}(C)$. First of all, we know that $k_{\mathcal{M}^*}(y) = k_{\mathcal{M}}(C)$. Indeed in $\mathcal{M}^*$ we have inserted a y' that was in $Min_<^{\mathcal{M}}(C)$. As shown above, $y' \in Min_<^{\mathcal{M}^*}(C)$. Hence $k_{\mathcal{M}^*}(y) = k_{\mathcal{M}^*}(y')$ and $k_{\mathcal{M}^*}(y) = k_{\mathcal{M}}(C)$. But by construction $k_{\mathcal{M}^*}(y) = k_{\mathcal{M}'}(y)$ and if $y \notin Min_<^{\mathcal{M}'}(C)$, there would be a y' s.t. $\mathcal{M}', y' \models C$ and $k_{\mathcal{M}'}(y') < k_{\mathcal{M}'}(y)$, hence $k_{\mathcal{M}'}(C) < k_{\mathcal{M}}(C)$, against Propositions 2 and 8. Hence, since $C \mid\!\sim D$ holds in $\mathcal{M}'$, $\mathcal{M}', y \models D$ and by construction also $\mathcal{M}, y \models D$.

For the conditionals with form $\phi_{C_i} \wedge \phi_{C_j} \mid\!\sim \perp$: they hold in $\mathcal{M}^*$ since we have suitably extended V^* in order to include at most one ϕ_C at a time.

Last, it obviously holds that $\mathcal{M}^* <_{VIMS} \mathcal{M}'$. Indeed the set of worlds of the two models coincide, and for all y taken from $\mathcal{M}'$, $k_{\mathcal{M}^*}(y) = k_{\mathcal{M}'}(y)$, and for all y taken from $\mathcal{M}$, they were introduced as representatives of a given C antecedent of a conditional or equal to $A \wedge B$, $A \wedge \neg B$. For all these formulas by Proposition 2 and 8, it holds that $k_{\mathcal{M}^*}(C) = k_{\mathcal{M}}(C) \leq k_{\mathcal{M}'}(C)$, hence $k_{\mathcal{M}^*}(y) \leq k_{\mathcal{M}'}(C)$. Furthermore, for $A \wedge B$ we have shown above that $k_{\mathcal{M}^*}(A \wedge B) = k_{\mathcal{M}}(A \wedge B) = i < k_{\mathcal{M}'}(A \wedge B)$, hence $\mathcal{M}^* <_{VIMS} \mathcal{M}'$, which contradicts the minimality of $\mathcal{M}'$. We conclude that if $K'' \not\models_{VIMS} A \mid\!\sim B$, then also $K \not\models_{FIMS} A \mid\!\sim B$.

For the other direction, suppose $\mathcal{M} \not\models A \mid\!\sim B$, i.e., $k_{\mathcal{M}}(A \wedge \neg B) \leq k_{\mathcal{M}}(A \wedge B)$. Let $k_{\mathcal{M}}(A \wedge \neg B) = i$ and $k_{\mathcal{M}}(A \wedge B) = j$. Consider the model $\mathcal{M}^*$ built as in the first part of the construction used above. More precisely $\mathcal{M}^* = \langle \mathcal{W}^*, <^*, V^* \rangle$ is built from $\mathcal{M}$ by cutting out its portion containing: x in $Min_<^{\mathcal{M}}(A \wedge B)$, $x' \in Min_<^{\mathcal{M}}(A \wedge \neg B)$ and an element $y \in Min_<^{\mathcal{M}}(C)$ for each antecedent C of a conditional in K. $V^* = V$ and $k_{\mathcal{M}^*} = k_{\mathcal{M}}$. If the same element y is associated to two different formulas, it must be duplicated into y and y' (and $V^*(y') = V^*(y)$ and $k_{\mathcal{M}^*}(y') = k_{\mathcal{M}^*}(y)$). Furthermore, for each world y associated to a formula C, $V^*(y)$ is extended in order to include ϕ_C. Last, $<^*$ is defined from $k_{\mathcal{M}^*}$ in the obvious way. By reasoning similarly to what we have done above, we can show that $\mathcal{M}^*$ is a model of K''. Furthermore, there cannot be a $\mathcal{M}^{*'} <_{VIMS} \mathcal{M}^*$. Indeed, any model of K'' must have a distinct element x satisfying $C \wedge \phi_C$ for each C in $A_{K, A \mid\sim B}$. Now suppose there was a model $\mathcal{M}^{*'}$ of K'' with $\mathcal{M}^{*'} <_{VIMS} \mathcal{M}^*$. If $\mathcal{M}^{*'} <_{VIMS} \mathcal{M}^*$, then for some x, $k_{\mathcal{M}^{*'}}(x) < k_{\mathcal{M}^*}(x)$. Suppose in $\mathcal{M}^*, x \models C \wedge \phi_C$ (and hence also $\mathcal{M}^{*'}, x \models C \wedge \phi_C$). By construction of $\mathcal{M}^*$, $k_{\mathcal{M}^*}(x) = k_{\mathcal{M}}(C)$. If $k_{\mathcal{M}^{*'}}(x) < k_{\mathcal{M}^*}(x)$, then $k_{\mathcal{M}^{*'}}(C) < k_{\mathcal{M}^*}(C)$, against Propositions 2 and 8. We conclude that it cannot hold $\mathcal{M}^{*'} <_{VIMS} \mathcal{M}^*$, hence $\mathcal{M}^*$ is a minimal *VIMS* model of K''. Furthermore by construction $k_{\mathcal{M}^*}(A \wedge \neg B) \leq k_{\mathcal{M}^*}(A \wedge B)$. We conclude that $K'' \not\models_{VIMS} A \mid\!\sim B$. □

From Theorem 2 and Theorem 4 just shown, it follows that:

Corollary 1. *Let K be a knowledge base. Given K'' defined as in Definition 11, it holds that*

$$A \mid\!\sim B \in \overline{K} \textit{ if and only if } K'' \models_{VIMS} A \mid\!\sim B,$$

where $\overline{K}$ is the rational closure of K.

We conclude the section with a comparison with the related works on rational closure.

2.4. Relation with P_{min} and Pearl's System Z

In [?] an alternative nonmonotonic extension of preferential logic P called P_{min} is proposed. Similarly to the semantics presented in this work, P_{min} is based on a minimal modal semantics. However the preference relation among models is defined in a different way. Intuitively, in P_{min} the fact that a world x is a minimal A-world is expressed by the fact that x satisfies $A \land \Box\neg A$, where $\Box$ is defined with respect to the inverse of the preference relation (i.e. with respect to the accessibility relation given by Ruv if and only if $v < u$). The idea is that preferred models are those that minimize the set of worlds where $\neg\Box\neg A$ holds, that is A-worlds which are not minimal. As a difference from the approach presented in this work, the semantics of P_{min} is defined starting from preferential models, in which the relation $<$ is irreflexive and transitive (thus, no longer modular).

P_{min} is a nonmonotonic logic considering only P models that, intuitively, minimize the non-typical worlds. More precisely, given a set of formulas K, a model $\mathcal{M} = \langle \mathcal{W}_\mathcal{M}, <_\mathcal{M}, V_\mathcal{M}\rangle$ of K and a model $\mathcal{N} = \langle \mathcal{W}_\mathcal{N}, <_\mathcal{N}, V_\mathcal{N}\rangle$ of K, we say that $\mathcal{M}$ is preferred to $\mathcal{N}$ if $\mathcal{W}_\mathcal{M} = \mathcal{W}_\mathcal{N}$, and the set of pairs $(w, \neg\Box\neg A)$ such that $\mathcal{M}, w \models \neg\Box\neg A$ is strictly included in the corresponding set for $\mathcal{N}$. A model $\mathcal{M}$ is a *minimal model* for K if it is a model of K and there is not a model $\mathcal{M}'$ of K which is preferred to $\mathcal{M}$. Entailment in P_{min} is restricted to minimal models of a given set of formulas K. In Section 3 of [?] it is observed that the logic P_{min} turns out to be quite strong. In general, if we only consider knowledge bases containing only positive conditionals, we get the same trivialization result (part of Proposition 1 in [?]) as the one contained in Proposition 6 for *VIMS*. This does not hold for rational closure. This is the reason why we have introduced the additional assumptions in order to obtain an equivalence with rational closure. Similarly, in order to tackle this trivialization in P_{min}, Section 3 in [?] is focused on the so called $well-behaved\ knowledge\ bases$, that explicitly include that A is possible ($\neg(A \mathrel{|\!\sim} \bot)$) for all conditional assertions $A \mathrel{|\!\sim} B$ in the knowledge base.

We may now wonder whether P_{min} is equivalent to *VIMS*, which is seemingly the closer semantics.

Or whether *VIMS* is equivalent to a stronger version of P_{min} obtained by replacing P with R as the underlying logic. We call R_{min} this stronger version of P_{min}.

Example 4. Let $K = \{PhD \mathrel{|\!\sim} \neg worker, PhD \mathrel{|\!\sim} adult, adult \mathrel{|\!\sim} worker, italian \mathrel{|\!\sim} house\_owner, PhD \mathrel{|\!\sim} \neg house\_owner\}$. What do we derive in P_{min} and R_{min}, and what in *VIMS*? By what said above, since K only contains positive conditionals, both in P_{min} and R_{min}, on the one side, and in *VIMS*, on the other side, we derive that $italian \land PhD \mathrel{|\!\sim} \bot$. So let us add to K the constraint that people who are italian and have a PhD do exist by introducing in K a conditional $\neg(italian \land PhD \mathrel{|\!\sim} \bot)$, thus obtaining: $K' = \{PhD \mathrel{|\!\sim} \neg worker, PhD \mathrel{|\!\sim} adult, adult \mathrel{|\!\sim} worker, italian \mathrel{|\!\sim} house\_owner, PhD \mathrel{|\!\sim} \neg house\_owner, \neg(italian \land PhD \mathrel{|\!\sim} \bot)\}$.

Notice that, since $\neg(italian \land PhD \mathrel{|\!\sim} \bot)$ entails both that $\neg(italian \mathrel{|\!\sim} \bot)$ and that $\neg(PhD \mathrel{|\!\sim} \bot)$, and that this in turn entails $\neg(adult \mathrel{|\!\sim} \bot)$, K' is also well-behaved.

It can be easily verified that the logical consequences of K' in P_{min}, R_{min} and *VIMS* differ. In both P_{min} and R_{min}, for instance, we derive neither that $italian \land PhD \mid\!\sim house\_owner$ nor that $italian \land PhD \mid\!\sim \neg house\_owner$: the two alternatives are equivalent. On the other hand, in *VIMS* we derive that $italian \land PhD \mid\!\sim \neg house\_owner$.

The previous example shows that in some cases *VIMS* is stronger than both P_{min} and R_{min}. The following one shows that the two approaches are incomparable, since there are also logical consequences that hold for both P_{min} and R_{min} but not for *VIMS*.

Example 5. Let $K = \{PhD \mid\!\sim adult, adult \mid\!\sim work, PhD \mid\!\sim \neg work, italian \mid\!\sim house\_owner\}$. What do we derive about typical $italian \land PhD \land work$, for instance? Do they inherit the property of typical Italians of being *house_owner*?

Again, in order to prevent the entailment of $italian \land PhD \land work \mid\!\sim \bot$ from K both in *VIMS* and in P_{min} and R_{min}, we add to K the constraint that italians with a PhD who work exist, henceforth they also have typical instances. Therefore we expand K into:

$$K' = \{PhD \mid\!\sim adult, adult \mid\!\sim work, PhD \mid\!\sim \neg work,$$
$$italian \mid\!\sim house\_owner, \neg(italian \land PhD \land work \mid\!\sim \bot)\}.$$

By reasoning as in Example 4 we can show that K' is a well-behaved knowledge base.

Now it can be easily shown that the conditional assertion

$$italian \land PhD \land work \mid\!\sim house\_owner$$

is entailed in P_{min} and R_{min}, whereas nothing is entailed in *VIMS*. This difference can be explained intuitively as follows. The set of properties for which an individual is atypical matters in P_{min} and R_{min}, where one has to minimize the set of distinct $\neg\Box\neg C$: even if an $italian \land PhD \land work$ is an atypical PhD, P_{min} and R_{min} still maximize its typicality as an italian, and therefore entail that it is a house_owner, as all typical italians. As a difference, in *VIMS*, what matters is *the set of individuals which are more typical* than a given x, rather than *the set of properties* by which they are more typical. As a consequence, since an x which is $italian \land PhD \land work$ is an atypical PhD, there is no need to maximize its typicality as an italian, as long as this does not increase the set of individuals more typical than x.

In [**?**] Pearl has introduced two notions of 0-entailment and 1-entailment to perform nonmonotonic reasoning. We recall here the semantic definition of both and then we remark upon their relation with our semantics and rational closure. A model $\mathcal{M}$ for a finite knowledge base K has the form $\mathcal{M}$= $(\{true, false\}^{ATM}, k_{\mathcal{M}})$ where $\{true, false\}^{ATM}$ is the set of propositional interpretations for, say, a fixed finite propositional language, and $k_{\mathcal{M}}$ is our height function mapping propositional interpretations to $\mathbb{N}$, the definition of height $k_{\mathcal{M}}(A)$ of a formula is the same as in our semantic. A conditional $A \mid\!\sim B$ is true in a model $\mathcal{M}$ if $k_{\mathcal{M}}(A \land B) < k_{\mathcal{M}}(A \land \neg B)$. Then the two entailment relations are defined as follows:

$K \models_{0-ent} A \mid\!\sim B$ if $A \mid\!\sim B$ is true in all models of K
$K \models_{1-ent} A \mid\!\sim B$ if $A \mid\!\sim B$ is true in the (unique) model $\mathcal{M}$ of K which is *minimal* with respect to $k_{\mathcal{M}}$,

where minimal with respect to $k_{\mathcal{M}}$ means that no other model $\mathcal{M}'$ assigns a lower value $k_{\mathcal{M}'}$ to any propositional interpretation. First, observe that Pearl's semantics (both 0 and 1 entailment) cannot cope with conditionals having an inconsistent antecedent. This limitation is deliberate and is motivated by a probabilistic interpretation of conditionals: in asserting $A \mid\sim B$, A must not be impossible, no matter how it is unlikely. For this reason, a knowledge base such as $K = \{A \mid\sim P, A \mid\sim \neg P, B \mid\sim Q\}$ is out of the scope of Pearl's semantics, and nothing can be said about its consequences. As a difference with respect to Pearl's approach we are able to consider such K, we just derive that A is impossible, without concluding that K is inconsistent or trivial, in the sense that everything follows from it. Moreover both 0-entailment and 1-entailment fail to validate:

$$\emptyset \models_{0-ent/1-ent} A \mid\sim \bot \text{ whenever } \vdash_{PC} \neg A$$

which is valid in any KLM logic, whence in rational closure (as well as in our semantics). However, two definitions should make apparent the relations with our semantics and rational closure. If we consider a K such that $\forall A \mid\sim B \in K, K \not\models_R A \mid\sim \bot$, we get an obvious correspondence between our *canonical* models (which will contain worlds for very possible propositional interpretation) and models of Pearl's semantics. The correspondence preserves *FIMS* minimality, so that we immediately get:

Proposition 10. *$K \models_{1-ent} A \mid\sim B$ if and only if $\mathcal{M} \models A \mid\sim B$ for all canonical models $\mathcal{M}$ of K that are minimal with respect to FIMS.*

By Theorem 2, we therefore obtain $K \models_{1-ent} A \mid\sim B$ if and only if $A \mid\sim B \in \bar{K}$. This is not a surprise, the correspondence between 1-entailment and rational closure was already observed by Pearl in [**? ?**]. However, it only works for knowledge bases with the strong consistency assumption as above.

3. Rational closure in Description Logics

As recalled in the Introduction, nonmonotonic reasoning in Description Logic has attracted an increasing interest in the last years [**? ? ? ? ? ? ? ? ? ? ?**]. Our purpose is to investigate whether rational closure can be extended in order to support nonmonotonic reasoning to Description Logics.

In this section, we extend to $\mathcal{ALC}$ the notion of rational closure proposed by Lehmann and Magidor [**?**], recalled in Section 2.2, and we define a semantic characterization of this notion of rational closure by introducing a minimal model semantics for $\mathcal{ALC}$ with typical inclusions. This semantics is a direct generalization of the minimal (canonical) model semantics introduced in Section 2.3

To express typical inclusions, $\mathcal{ALC}$ is extended with a typicality operator **T**, following the approach in [**? ?**]. Differently from [**?**], here we consider special kinds of preferential models, namely, *rational* models, to define the semantics of the **T** operator, and we use a different notion of preference between models, namely, the preference relation $<_{FIMS}$, introduced in Section 2.3. Given the typicality operator, the typical assertion $\mathbf{T}(C) \sqsubseteq D$ (all the typical C's are D's) plays the role of the conditional assertion $C \mid\sim D$ in R. We show that the correspondence result established by Theorem 2 can be lifted from the propositional calculus to $\mathcal{ALC}$.

3.1. The logic $\mathcal{ALC} + \mathbf{T}_R$

In order to apply rational closure to DLs we proceed in two steps. First, similarly to [**?**], we extend the standard $\mathcal{ALC}$ by a typicality operator $\mathbf{T}$ that allows to single out the typical instances of a concept. Since we are dealing here with rational closure (that builds over R), we attribute to $\mathbf{T}$ properties related to R. The resulting logic is called $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$. As a second step, we build over $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$ a rational closure mechanism.

Our starting point is therefore the extension of logic $\mathcal{ALC}$ with a typicality operator $\mathbf{T}$: we allow concepts of the form $\mathbf{T}(C)$, whose intuitive meaning is that $\mathbf{T}(C)$ selects the *typical* instances of a concept C. We can therefore distinguish between the properties that hold for all instances of concept C ($C \sqsubseteq D$), and those that only hold for the typical such instances ($\mathbf{T}(C) \sqsubseteq D$).

Definition 12. We consider an alphabet of concept names $\mathcal{C}$, of role names $\mathcal{R}$, and of individual constants $\mathcal{O}$. Given $A \in \mathcal{C}$ and $R \in \mathcal{R}$, we define:

$$C_R := A \mid \top \mid \bot \mid \neg C_R \mid C_R \sqcap C_R \mid C_R \sqcup C_R \mid \forall R.C_R \mid \exists R.C_R$$
$$C_L := C_R \mid \mathbf{T}(C_R)$$

A knowledge base is a pair (TBox, ABox). TBox contains a finite set of concept inclusions $C_L \sqsubseteq C_R$. ABox contains assertions of the form $C_L(a)$ and $R(a, b)$, where $a, b \in \mathcal{O}$.

The semantics of $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$ can be formulated in terms of rational models: ordinary models of $\mathcal{ALC}$ are equipped with a *preference relation* $<$ on the domain, whose intuitive meaning is to compare the "typicality" of domain elements, that is to say $x < y$ means that x is more typical than y. Typical members of a concept C, that is members of $\mathbf{T}(C)$, are the members x of C that are minimal with respect to this preference relation (s.t. there is no other member of C more typical than x).

Definition 13 (Semantics of $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$). A model $\mathcal{M}$ of $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$ is any structure $\langle \Delta, <, I \rangle$ where:

- Δ is the domain;
- $<$ is an irreflexive, transitive and modular (if $x < y$ then either $x < z$ or $z < y$) relation over Δ;
- I is the extension function that maps each concept C to $C^I \subseteq \Delta$, and each role R to $R^I \subseteq \Delta \times \Delta$. For concepts of $\mathcal{ALC}$, C^I is defined in the usual way. For the $\mathbf{T}$ operator, we have
$$(\mathbf{T}(C))^I = Min_<(C^I),$$
where $Min_<(S) = \{u : u \in S \text{ and } \nexists z \in S \text{ s.t. } z < u\}$.

Furthermore, $<$ satisfies the *Well − Foundedness Condition*, i.e., for all $S \subseteq \Delta$, for all $x \in S$, either $x \in Min_<(S)$ or $\exists y \in Min_<(S)$ such that $y < x$. [3]

[3] Observe that, although in [**? ?**] we have called the above condition Smoothness condition, this condition

The semantics with one single preference relation $<$ is the one that, as we will show, corresponds to rational closure. One may think of considering a sharper semantics with several preference relations, we briefly discuss this variant in the last section.

An alternative equivalent semantics of the $\mathbf{T}$ operator by means of a set of postulates that are essentially a reformulation of axioms and rules of nonmonotonic entailment in rational logic R can be found in the Appendix, together with the proof of the equivalence.

Definition 14 (Model satisfying a knowledge base). Given an $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ model $\mathcal{M}$= $\langle \Delta, <, I \rangle$, we assume that I is extended to assign a domain element a^I of Δ to each individual constant a of O. We say that:

- a model $\mathcal{M}$ satisfies an inclusion $C \sqsubseteq D$ (written $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} C \sqsubseteq D$) if it holds $C^I \subseteq D^I$;
- $\mathcal{M}$ satisfies an assertion $C(a)$ (written $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} C(a)$) if $a^I \in C^I$ and $\mathcal{M}$ satisfies an assertion $R(a, b)$ (written $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} R(a, b)$) if $(a^I, b^I) \in R^I$.

Given a knowledge base K=(TBox,ABox), we say that:

- $\mathcal{M}$ satisfies TBox if $\mathcal{M}$ satisfies all inclusions in TBox (written $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}}$ TBox);
- $\mathcal{M}$ satisfies ABox if $\mathcal{M}$ satisfies all assertions in ABox (written $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}}$ ABox);
- $\mathcal{M}$ satisfies K if it satisfies both its TBox and its ABox (written $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} K$);
- a concept C is satisfiable with respect to K, if there is a model $\mathcal{M} = \langle \Delta, <, I \rangle$ satisfying K and such that $C^I \neq \emptyset$.

It is worth noticing that, as a difference with our previous approach in [**?**], here we do not assume the *unique name assumption*, that is to say we do not assume that, in a model $\mathcal{M}$, I is extended to assign a distinct element a^I of Δ to each individual constant a of O. In [**?**], UNA is needed since the properties of the preference relation $<$ are built from preferential logic P: in that case, the unique name assumption avoids that models in which two names are mapped into the same individual of the domain are preferred to those in which they are mapped into distinct ones. This is needed in order to perform useful reasoning about two different individuals named in the ABox. As we will see in Definition 23 below, we restrict our concern to the only case of an FIMS semantics based on the minimization of ranks, therefore the unique name assumption is no longer needed.

is stronger than the smoothness condition introduced in the propositional case (Definition 1). Indeed, the condition above considers all subsets S of Δ and does not only apply to the interpretations C^I of the concepts C of the language. It is easy to prove that such a condition is equivalent to require that $(\Delta, <)$ is well-founded, i.e. there is no infinite descending chain of individuals. In the following, we keep the same condition as in previous work, but we call it *well-foundedness* condition.

By a construction similar to that used in Theorem 2.3 of [**?**] for the weaker logic $\mathcal{ALC} + \mathbf{T}$, we can prove the following theorem. The proofs and further details are provided in the technical report [**?**].

Theorem 5 (Complexity of $\mathcal{ALC}$+$\mathbf{T}_{\mathsf{R}}$)**.** *Given an $\mathcal{ALC}$+$\mathbf{T}_{R}$ knowledge base K=(TBox, ABox), the problem of deciding satisfiability of K is* ExpTime*-complete.*

The finite model property of $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ follows as an easy consequence of the terminating tableau construction in Section 4.1 of [**?**].

Theorem 6 (Finite model property for $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$)**.** *Given a knowledge base K, if it is satisfiable in $\mathcal{ALC} + \mathbf{T}_{R}$ then there exists a finite $\mathcal{ALC} + \mathbf{T}_{R}$ model satisfying K, i.e. $\mathcal{ALC} + \mathbf{T}_{R}$ has the finite model property.*

Let us define the derivability of an inclusion and of an assertion in $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$:

Definition 15. Given a knowledge base K, an inclusion $C_L \sqsubseteq C_R$ and an assertion $C_L(a)$, with $a \in O$, we say that:

- the inclusion $C_L \sqsubseteq C_R$ is *entailed* from K, written $K \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} C_L \sqsubseteq C_R$, if ${C_L}^I \subseteq {C_R}^I$ holds in all models $\mathcal{M} = \langle \Delta, <, I \rangle$ satisfying K;
- the assertion $C_L(a)$ is *entailed* from K, written $K \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} C_L(a)$, if $a^I \in {C_L}^I$ holds in all models $\mathcal{M} = \langle \Delta, <, I \rangle$ satisfying K.

As usual, when, for a given knowledge base K and a concept C, it holds that $K \not\models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} C \sqsubseteq \perp$ we say that C is *satisfiable* with respect to K.

As an easy consequence of Theorem 6, we prove the following corollary:

Corollary 2. *Given a knowledge base K and a concept C satisfiable with respect to K, then there exists a finite $\mathcal{ALC} + \mathbf{T}_{R}$ model $\mathcal{M} = \langle \Delta, <, I \rangle$ satisfying K, such that $C^I \neq \emptyset$.*

Proof. Let K=(TBox,ABox) and let us assume that C is satisfiable with respect to K. Then there is a model $\mathcal{M} = \langle \Delta, <, I \rangle$ satisfying K such that $C^I \neq \emptyset$. Let $x \in C^I$, let d be a new individual name not occurring in K and let K' =(TBox,ABox'), where ABox'= ABox $\cup \{C(d)\}$. Clearly, K' is satisfiable, as the model obtained from $\mathcal{M}$ by letting $d^I = x$ satisfies K'. By the finite model property (Theorem 6) there exists a finite model satisfying K'. Let $\mathcal{M}'$ be such a model. $\mathcal{M}'$ is a finite model of K such that $C^{I'} \neq \emptyset$. □

As for propositional rational models, finite $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ models (to which we can restrict attention by Theorem 6) can be equivalently defined by postulating the existence of a function $k_{\mathcal{M}} : \Delta \longmapsto \mathbb{N}$, where $k_{\mathcal{M}}$ assigns a finite rank to each world, and is defined as follows.

Definition 16 (Rank of a domain element $k_{\mathcal{M}}(x)$). Given a model $\mathcal{M} = \langle \Delta, <, I \rangle$, the rank $k_{\mathcal{M}}$ of a domain element $x \in \Delta$, is the length of the longest chain $x_0 < \ldots < x$ from x to a minimal x_0 (i.e. such that there is no x' such that $x' < x_0$).

As for the propositional case, the rank function $k_{\mathcal{M}}$ and $<$ can be defined from each other by letting $x < y$ if and only if $k_{\mathcal{M}}(x) < k_{\mathcal{M}}(y)$.

Definition 17 (Rank of a concept $k_{\mathcal{M}}(C_R)$ in a model). Given a model $\mathcal{M} = \langle \Delta, <, I \rangle$, the *rank $k_{\mathcal{M}}(C_R)$ of a concept C_R in the model $\mathcal{M}$* is defined as

$$k_{\mathcal{M}}(C_R) = min\{k_{\mathcal{M}}(x) \mid x \in {C_R}^I\}.$$

If ${C_R}^I = \emptyset$, then C_R has no rank and we write $k_{\mathcal{M}}(C_R) = \infty$.

It is immediate to verify that:

Proposition 11. *For any $\mathcal{M} = \langle \Delta, <, I \rangle$, we have that $\mathcal{M}$ satisfies $\mathbf{T}(C) \sqsubseteq D$ if and only if $k_{\mathcal{M}}(C \sqcap D) < k_{\mathcal{M}}(C \sqcap \neg D)$.*

As already mentioned, although the typicality operator $\mathbf{T}$ itself is nonmonotonic (i.e. $\mathbf{T}(C) \sqsubseteq D$ does not imply $\mathbf{T}(C \sqcap E) \sqsubseteq D$), the logic $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ is monotonic: what is inferred from K can still be inferred from any K' with $K \subseteq K'$. This is a clear limitation in DLs. As a consequence of the monotonicity of $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$, one cannot deal with irrelevance, for instance. So one cannot derive from $K = \{Penguin \sqsubseteq Bird, \mathbf{T}(Bird) \sqsubseteq Fly, \mathbf{T}(Penguin) \sqsubseteq \neg Fly\}$ that $K \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} \mathbf{T}(Penguin \sqcap Black) \sqsubseteq \neg Fly$, even if the property of being black is irrelevant with respect to flying. In the same way, if we add to K the information that Jim is a bird ($Bird(jim)$), in $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ one cannot tentatively derive, in the absence of information to the contrary, that it is a typical bird and therefore it flies ($\mathbf{T}(Bird)(jim)$ and $Fly(jim)$).

In the following section we investigate the possibility of overcoming this weakness by extending to $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ the notion of rational closure. As we will see, this extension allows to deal with irrelevance and allows to attribute typical properties to individuals.

3.2. Rational Closure of the TBox in $\mathcal{ALC} + \mathbf{T}_R$

In this section, we extend to $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ the definition of rational closure introduced by Lehmann and Magidor for the propositional case.

We first consider the rational closure with respect to TBox, in which essentially we only consider which concept inclusions belong to the rational closure of K. Next we will consider rational closure with respect to ABox, in which we consider the individuals explicitly named in the ABox, and derive their properties.

Let us first define the notion of *query*: a query is either an inclusion relation or an assertion of the ABox; we want to check whether it is entailed from a given knowledge base.

Definition 18 (Query). A *query F* is either an assertion $C_L(a)$ or an inclusion relation $C_L \sqsubseteq C_R$. Given a model $\mathcal{M} = \langle \Delta, <, I \rangle$, a query F holds in $\mathcal{M}$ if $\mathcal{M}$ satisfies F, i.e. if $a^I \in (C_L(a))^I$ or $C_L^I \subseteq C_R^I$, respectively. [4]

[4]The notion of query we have just defined does not consider the case of querying about role instances, that is to say of the form $R(a, b)$, where R is a role name and a, b are individual names occurring in the ABox. The reason is that in $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$, like in the basic $\mathcal{ALC}$, for any knwoledge base $K = TBox \cup ABox$, and any role instance $R(a, b)$ as above, it holds that if K is satisfiable, then $K \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} R(a, b)$ if and only if $R(a, b) \in ABox$ (if K is not satisfiable everything follows), thus neither the logic $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$, nor the rational closure construction add any inferential power. This of course would not necessarily be true in extensions of $\mathcal{ALC}$ containing for instance role constructors.

Definition 19 (Exceptionality of concepts and inclusions). Let K=(TBox,ABox) be a knowledge base. A concept C is said to be *exceptional* for K if and only if $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}}$ $\mathbf{T}(\top) \sqsubseteq \neg C$. A $\mathbf{T}$-inclusion $\mathbf{T}(C) \sqsubseteq D$ is exceptional for K if C is exceptional for K. The set of $\mathbf{T}$-inclusions of K which are exceptional in K will be denoted as $\mathcal{E}(K)$.

Note that, differently from Lehmann and Magidor's notion of exceptionality in Section 2.2, the exceptionality of a concept is defined also taking into account the *ABox*. This is needed when the ABox contains typicality assertions of the form $\mathbf{T}(C)(a)$. Indeed, as we will see later with an example, the construction of the rational closure of the TBox of a knowledge base K is affected by the presence of typicality assertions in the ABox: if the assertions $\mathbf{T}(C)(a)$ and $\neg D(a)$ are in the ABox, it is not the case that all the typical C's are D's, so that the defeasible inclusion $\mathbf{T}(C) \sqsubseteq D$ does not hold.

Similarly to the propositional case, in the following we introduce a sequence of knowledge bases, starting from the initial one, K, in order to iteratively use exceptionality in the construction of the rational closure. At each step, in order to reason about the following exceptional subset of K, we remove the inclusions $\mathbf{T}(C) \sqsubseteq D$ of K that are not exceptional for K. Before we do this, if there is an assertion $\mathbf{T}(C)(a)$ in ABox, we add to a all the typical properties of C that we are removing. Because we want to reason in the same way for equivalent concepts, this leads us to the slightly more complicated formulation of ABox$_i$ below.

Definition 20. Given a DL knowledge base K=(TBox,ABox), it is possible to define a sequence of knowledge bases $E_0, \dots, E_i, \dots, E_n$ by letting E_0 = (TBox$_0$, ABox$_0$) where TBox$_0$ = TBox and ABox$_0$ = ABox and, for $i > 0$, E_i = (TBox$_i$, ABox$_i$) where

- TBox$_i$ = $\mathcal{E}(E_{i-1}) \cup \{C \sqsubseteq D \in$ TBox $\mid \mathbf{T}$ does not occur in $C\}$
- ABox$_i$ = ABox$_{i-1}$ $\cup$ $\{(\neg C \sqcup D)(a) \mid \mathbf{T}(C) \sqsubseteq D$ in $(E_{i-1} - E_i)$ and there is a $\mathbf{T}(B)(a) \in$ ABox such that $E_{i-1} \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg B$ and $E_j \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg B$ for all $j < i-1\}$

(as a consequence of the next Definition 21, these are the Bs such that $rank(B) = i-1$).

Clearly TBox$_0 \supseteq$ TBox$_1 \supseteq$ TBox$_2, \dots$, while ABox$_0 \subseteq$ ABox$_1 \subseteq$ ABox$_2, \dots$

Observe that, being K finite, there is a least $n \geq 0$ such that, for all $m > n$, TBox$_m$ = TBox$_n$ or TBox$_m = \emptyset$. We take (TBox$_n$, ABox$_n$) as the last element of the sequence of knowledge bases starting from K. Observe also that the definition of the TBox$_i$'s is the same as the definition of the C_i's in Lehmann and Magidor's definition of rational closure in Section 2.2, except for the fact that here, at each step, we also add all the "strict" inclusions $C \sqsubseteq D$ (where $\mathbf{T}$ does not occur in C).

Informally, for the definition of ABox$_i$, if $\mathbf{T}(B)(a) \in$ABox (i.e., a is a typical B-element), and B has rank $i-1$, then, for all the inclusions $\mathbf{T}(C) \sqsubseteq D$ in $(E_{i-1} - E_i)$, since C has also rank $i-1$ we have that: if a is a C-element, then it is a typical C-element and the assertion $(\neg C \sqcup D)(a)$ must hold.

Note that, when the ABox does not contain typicality assertions of the form $\mathbf{T}(C)(a)$, we have that, for all i, ABox$_i$ = ABox. In this case, ABox$_i$ is irrelevant to determine the exceptionality of concepts as $E_i \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}}$ $\mathbf{T}(\top) \sqsubseteq \neg C$ if and only if

$\text{TBox}_i \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg C$, for all concepts C. As in this case the definition of exceptionality of concepts does not depend on the ABox, the construction above can be simplified, by taking E_0 = TBox and $E_i = \text{TBox}_i$, and evaluating exceptionality only with respect to the TBox. Hence, we can avoid the computation of the ABox_is and the construction becomes quite similar to the one of Lehmann and Magidor recalled in Section 2.2. This simplified construction can be found in [**?**].

Definition 21 (Rank of a concept). A concept C has *rank* i (denoted by $rank(C) = i$) for K=(TBox,ABox), if and only if i is the least natural number for which C is not exceptional for E_i. If C is exceptional for all E_i then $rank(C) = \infty$, and we say that C has no rank.

Consider the least $n \geq 0$ such that, for all $m > n, \text{TBox}_m = \text{TBox}_n$ or $\text{TBox}_m = \emptyset$. Then from the above definition it follows that if a concept C has a rank, its highest possible value is n. As for propositional logic, the notion of rank of a formula allows to define the rational closure of a knowledge base K with respect to TBox .

Definition 22 (Rational closure of TBox). Let K=(TBox,ABox) be a DL knowledge base. We define $\overline{TBox}$, the *rational closure* of TBox, as

$$\begin{aligned}\overline{TBox} = \{\mathbf{T}(C) \sqsubseteq D \mid \text{ either } rank(C) < rank(C \sqcap \neg D)\\ \text{or } rank(C) = \infty\} \cup \{C \sqsubseteq D \mid K \models_{\mathcal{ALC}+\mathbf{T}_R} C \sqsubseteq D\}\end{aligned}$$

It can be easily seen that the rational closure of TBox is a nonmonotonic strengthening of $\mathcal{ALC} + \mathbf{T}_R$. For instance, it allows to deal with irrelevance, as the following example shows.

Example 6. Let K = (TBox, ABox) where ABox = $\emptyset$ and TBox = $\{Penguin \sqsubseteq Bird$, $\mathbf{T}(Bird) \sqsubseteq Fly$, $\mathbf{T}(Penguin) \sqsubseteq \neg Fly\}$. It can be verified that $\mathbf{T}(Bird \sqcap Black) \sqsubseteq Fly \in \overline{TBox}$. This is a nonmonotonic inference that does no longer follow if we know that typical black birds do not fly: given $TBox'$= TBox $\cup$ $\{\mathbf{T}(Bird \sqcap Black) \sqsubseteq \neg Fly\}$, we have that $\mathbf{T}(Bird \sqcap Black) \sqsubseteq Fly \notin \overline{TBox'}$. Similarly, as for the propositional case, rational closure is closed under rational monotonicity: from $\mathbf{T}(Bird) \sqsubseteq Fly \in \overline{TBox}$ and $\mathbf{T}(Bird) \sqsubseteq \neg LivesEurope \notin \overline{TBox}$ it follows that $\mathbf{T}(Bird \sqcap LivesEurope) \sqsubseteq Fly \in \overline{TBox}$.

We can show that the presence of typicality assertions in the ABox has an impact on the construction of the rational closure.

Example 7. Let K = (TBox, ABox), where TBox is as in Example 6 and ABox= $\{\mathbf{T}(Bird \sqcap Black)(opus), \neg Fly(opus)\}$. As *opus* is a typical black bird and it does not fly, it is clear the we are no longer ready to accept that typical black birds fly, otherwise we get an inconsistency with the ABox. Indeed, using the construction of rational closure given above, we have that $K \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg(Bird \sqcap Black)$, so that $rank(Bird \sqcap Black) \neq 0$. In particular, $rank(Bird \sqcap Black) = 1$ and $rank(Bird \sqcap Black \sqcap \neg Fly) = 1$ as well. Hence, $\mathbf{T}(Bird \sqcap Black) \sqsubseteq Fly \notin \overline{TBox}$.

The next example shows that a sequence of ABoxes in the construction of the rational closure is actually needed.

Example 8. Let K = (TBox, ABox) where TBox = $\{Penguin \sqsubseteq Bird$, $\mathbf{T}(Bird) \sqsubseteq Fly$, $\mathbf{T}(Penguin) \sqsubseteq \neg Fly$, $\mathbf{T}(Bird) \sqsubseteq \forall HasFriend.Fly\}$ and ABox= $\{\mathbf{T}(Bird)(opus)$, $HasFriend(opus, pio), \mathbf{T}(Penguin \sqcap Violet)(pio)\}$.

From the construction in Definition 20, we have:

- TBox$_0$ =$\{Penguin \sqsubseteq Bird$, $\mathbf{T}(Bird) \sqsubseteq Fly$, $\mathbf{T}(Penguin) \sqsubseteq \neg Fly$, $\mathbf{T}(Bird) \sqsubseteq \forall HasFriend.Fly\}$,
  ABox$_0$ = $\{\mathbf{T}(Bird)(opus), HasFriend(opus, pio), \mathbf{T}(Penguin \sqcap Violet)(pio)\}$.
- TBox$_1$ =$\{Penguin \sqsubseteq Bird$, $\mathbf{T}(Penguin) \sqsubseteq \neg Fly\}$,
  ABox$_1$ = $\{\mathbf{T}(Bird)(opus), HasFriend(opus, pio), \mathbf{T}(Penguin \sqcap Violet)(pio), (\neg Bird \sqcup Fly)(opus), (\neg Bird \sqcup \forall HasFriend.Fly)(opus)\}$.
- TBox$_2$ =$\{Penguin \sqsubseteq Bird\}$,
  ABox$_2$ = $\{\mathbf{T}(Bird)(opus), HasFriend(opus, pio), \mathbf{T}(Penguin \sqcap Violet)(pio), (\neg Bird \sqcup Fly)(opus), (\neg Bird \sqcup \forall HasFriend.Fly)(opus)\}$.

Observe that the last two assertions in ABox$_1$ have been introduced as $\mathbf{T}(Bird)(opus) \in$ ABox, and *Bird* is not exceptional in E_0. Observe also that $E_1 \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg(Penguin \sqcap Violet)$ and the assertion $(\neg Bird \sqcup \forall HasFriend.Fly)(opus)$ in ABox$_1$ is needed to infer that *pio* flies and hence, although it is a typical violet penguin, *pio* cannot be a typical penguin.

We get $rank(Penguin \sqcap Violet) = 2$, while $rank(Penguin) = 1$, and $rank(Bird) = 0$. Hence, we can conclude that typical penguins are not violet, $\mathbf{T}(Penguin) \sqsubseteq \neg Violet \in \overline{TBox}$, as $rank(Penguin) < rank(Penguin \sqcap Violet)$.

So far we have extended to $\mathcal{ALC} + \mathbf{T}_R$ the syntactic notion of rational closure. We wonder whether we provide a semantic characterization of this notion by extending the semantic characterization given at the propositional level.

As for the propositional case (in the case of *FIMS*), in order to semantically characterize the rational closure, we first restrict our attention to minimal rational models that minimize *the rank of domain elements*. Informally, given two models of K, one in which a given domain element x has rank 2 (because for instance $z < y < x$) , and another in which it has rank 1 (because only $y < x$), we prefer the latter, as in this model the element x is assumed to be "more typical" than in the former.

Definition 23 (Minimal models). Given $\mathcal{M} =\langle \Delta, <, I\rangle$ and $\mathcal{M}' = \langle \Delta', <', I'\rangle$ we say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ ($\mathcal{M} <_{FIMS} \mathcal{M}'$) if:

- $\Delta = \Delta'$
- $C^I = C^{I'}$ for all concepts C
- for all $x \in \Delta$, it holds that $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$ whereas there exists $y \in \Delta$ such that $k_{\mathcal{M}}(y) < k_{\mathcal{M}'}(y)$.

Given a knowledge base K, we say that $\mathcal{M}$ is a minimal model of K with respect to $<_{FIMS}$ if it is a model satisfying K and there is no $\mathcal{M}'$ model satisfying K such that $\mathcal{M}' <_{FIMS} \mathcal{M}$.

It is worth noticing that roles are not considered in Definition 23, in other words, they are allowed to *vary* in the proposed preferential semantics. Subsequently, as for the propositional case, we restrict our attention to minimal canonical models. We define $\mathcal{S}$ as the set of all the concepts (and subconcepts) occurring in K or in the query F together with their complements (observe that $\mathcal{S}$ is finite).

In order to define canonical models, we consider all the sets of concepts $\{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{S}$ that are *consistent with* K, i.e., s.t. $K \not\models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n \sqsubseteq \bot$.

Definition 24 (Canonical model with respect to $\mathcal{S}$). Given K=(TBox,ABox) and a query F, a model $\mathcal{M}$ =$\langle\Delta, <, I\rangle$ satisfying K is *canonical with respect to* $\mathcal{S}$ if, for each set of concepts $\{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{S}$ consistent with K, there exists (at least) a domain element $x \in \Delta$ such that $x \in (C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n)^I$.

The intuition is that a canonical model contains all the individuals that enjoy properties that are consistent with the knowledge base. This is needed when reasoning about the (relative) rank of the concepts: it is important to have them all represented. As we will see in Theorem 7, in $\mathcal{ALC}$ the existence of a canonical model is guaranteed for any consistent knowledge base. However, this may be not true for more expressive logics and, in particular, this is not true for $\mathcal{SHOIQ}$ [**?**] (see example 4 in [**?**]).

Next we define the notion of minimal canonical model.

Definition 25 (Minimal canonical models (with respect to TBox)). $\mathcal{M}$ is a minimal canonical model of K if it satisfies K, it is minimal (with respect to Definition 23) and it is canonical (according to Definition 24).

We can now prove the following:

Theorem 7. *For any consistent knowledge base K, there exists a finite, minimal canonical model of K with respect to TBox.*

Proof. Let $\mathcal{M} = \langle\Delta, <, I\rangle$ be a finite model of K (which exists by the finite model property, since K is consistent), and let $\{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{S}$ be any subset of $\mathcal{S}$ consistent with K. We show that we can expand $\mathcal{M}$ in order to obtain a finite model of K that contains an instance of $C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n$. By repeating the same construction for all maximal consistent subsets $\{C_1, C_2, \ldots, C_n\}$ of $\mathcal{S}$, we eventually obtain a finite canonical model of K.

Indeed, for each $\{C_1, C_2, \ldots, C_n\}$ consistent with K, it holds that $K \not\models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n \sqsubseteq \bot$, i.e. concept $C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n$ is satisfiable with respect to K. By Corollary 2 there exists a finite $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ model $\mathcal{M}' = \langle\Delta', <', I'\rangle$ satisfying K, such that $(C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n)^{I'} \neq \emptyset$.

Let $\mathcal{M}^{'*}$ be the union of $\mathcal{M}$ and $\mathcal{M}'$, i.e. $\mathcal{M}^{'*} = \langle\Delta^{'*}, <^{'*}, I^{'*}\rangle$, where $\Delta^{'*} = \Delta \cup \Delta^{'}$. As far as individuals named in the ABox are concerned, we define $I^{'*}$ as I, that is to say $a^{I^{'*}} = a^I$ for all $a \in \mathcal{O}$ occurring in ABox. For concepts and roles, $I^{'*}$ is defined as I for elements in Δ and as I' on elements in Δ', that is to say, for all atomic concepts $C \in \mathcal{C}$ and all roles $R \in \mathcal{R}$:

- $x \in C^{I^{'*}}$ for all $x \in \Delta$, if $x \in C^I$;
- $x \in C^{I^{'*}}$ for all $x \in \Delta'$, if $x \in C^{I'}$;

- $(x, y) \in R^{I'^*}$ for all $x, y \in \Delta$, if $(x, y) \in R^I$;
- $(x, y) \in R^{I'^*}$ for all $x, y \in \Delta'$, if $(x, y) \in R^{I'}$.

Also, $k_{\mathcal{M}'^*}(x) = k_{\mathcal{M}}(x)$ for the elements in $x \in \Delta$, and $k_{\mathcal{M}'^*}(x) = n + k_{\mathcal{M}'}(x)$ for all the elements $x \in \Delta'$, where n is the maximum value of $k_{\mathcal{M}}$ in $\mathcal{M}$ (n is finite, as each element of $\mathcal{M}$ has a finite rank). $<'^*$ is straightforwardly defined from $k_{\mathcal{M}'^*}$ by letting $x <'^* y$ if and only if $k_{\mathcal{M}'^*}(x) < k_{\mathcal{M}'^*}(y)$. It can be verified that $\mathcal{M}'^*$ is a finite model of K which contains an instance of $C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n$. For the inclusions and assertions of K that do not contain $\mathbf{T}$ this is obviously true. For the inclusions containing $\mathbf{T}$, for each $\mathbf{T}(C) \sqsubseteq D$, if $x \in Min_{<'^*}(C)$ in $\mathcal{M}'^*$, also $x \in Min_{<}(C)$ in $\mathcal{M}$ or $x \in Min_{<'}(C)$ in $\mathcal{M}'$. In both cases x is an instance of D (since both $\mathcal{M}$ and $\mathcal{M}'$ satisfy K), therefore $x \in D^{I'^*}$, and $\mathcal{M}'^*$ satisfies K.

By repeating the same construction for all the (finitely many) maximal consistent subsets $\{C_1, C_2, \ldots, C_n\}$ of $\mathcal{S}$, we obtain a finite canonical model of K, call it $\mathcal{M}^+$. We do not know whether $\mathcal{M}^+$ is minimal. Observe that, as the domain Δ^+ of $\mathcal{M}^+$ is finite, the rank of each element in Δ^+ is finite. If $\mathcal{M}^+$ is not minimal, then there is a model $\mathcal{M}_1$ (over the same domain Δ^+) preferred to $\mathcal{M}^+$, such that, for all $x \in \Delta^+$ $k_{\mathcal{M}_1}(x) \leq k_{\mathcal{M}^+}(x)$ and for some $y \in \Delta^+$ $k_{\mathcal{M}_1}(y) < k_{\mathcal{M}^+}(y)$. Again, if $\mathcal{M}_1$ is not minimal there must be another $\mathcal{M}_2$ preferred to $\mathcal{M}_1$. And so on, lowering the ranks. As the domain Δ^+ is finite, this descending chain of models cannot be infinite and, eventually, we reach a minimal canonical model of K. □

To prove the correspondence between minimal canonical models and the rational closure of a TBox, we need to introduce some propositions. Given an $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ model $\mathcal{M} = \langle \Delta, <, I \rangle$, we define a sequence $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \ldots$ of models as follows: We let $\mathcal{M}_0 = \mathcal{M}$ and, for all i, we let $\mathcal{M}_i = \langle \Delta, <_i, I \rangle$ be the $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ model obtained from $\mathcal{M}$ by assigning a rank 0 to all the domain elements x with $k_{\mathcal{M}}(x) \leq i$, i.e., $k_{\mathcal{M}_i}(x) = k_{\mathcal{M}}(x) - i$ if $k_{\mathcal{M}}(x) > i$, and $k_{\mathcal{M}_i}(x) = 0$ otherwise.

Proposition 12. *Let $K = \langle TBox, ABox \rangle$ and let $\mathcal{M} = \langle \Delta, <, I \rangle$ be a minimal canonical $\mathcal{ALC} + \mathbf{T}_R$ model satisfying K. For any concept C, if rank(C) $\geq i$, then*

1) $k_{\mathcal{M}}(C) \geq i$, and
2) $\mathcal{M}_i$ satisfies E_i.

Proof. By induction on i. For $i = 0$, 1) holds (since it always holds that $k_{\mathcal{M}}(\mathrm{C}) \geq 0$). 2) holds trivially as $\mathcal{M}_0 = \mathcal{M}$.

For $i > 0$, 1) holds: if rank(C)$\geq i$, then, by Definition 21, for all $j < i$, we have that $E_j \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} \mathbf{T}(\top) \sqsubseteq \neg C$. By inductive hypothesis on 2), for all $j < i$ $\mathcal{M}_j \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} \mathbf{T}(\top) \sqsubseteq \neg C$. Hence, for all x with $k_{\mathcal{M}}(x) < i$, $x \notin C^I$, and $k_{\mathcal{M}}(C) \geq i$.

To prove 2), we reason as follows. Since $\mathrm{TBox}_i \subseteq \mathrm{TBox}_0$, $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} \mathrm{TBox}_i$. Furthermore by definition of rank, for all $\mathbf{T}(B) \sqsubseteq D \in \mathrm{TBox}_i$, $rank(B) \geq i$, hence by 1) just proved $k_{\mathcal{M}}(B) \geq i$. Hence, in $\mathcal{M}$, the rank of all elements in $Min_{<}(B^I)$ is $\geq i$, and also $\mathcal{M}_i \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} \mathbf{T}(B) \sqsubseteq D$.

To prove that $\mathcal{M}_i \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} E_i$, we also need to show that $\mathcal{M}_i \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} \mathrm{ABox}_i$. By construction, for all the assertions $C(a) \in \mathrm{ABox}$, $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} C(a)$ and there is an element $x \in \Delta$ such that $x \in C^I$ and $a^I = x$. As $\mathcal{M}_i$ only differs from $\mathcal{M}$ for the ranks, if $C \neq \mathbf{T}(B)$, $\mathcal{M}_i \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C(a)$. If $C = \mathbf{T}(B)$, as in $\mathcal{M}$ it holds that $x \in (\mathbf{T}(B))^I$, x is a B-minimal element in $\mathcal{M}$, and it can be proven that it remains a B-minimal element in $\mathcal{M}_i$. Thus, $\mathcal{M}_i$ satisfies $\mathbf{T}(B)(a)$.

For each assertion $(\neg C \sqcup D)(a) \in$ ABox$_i$ such that $(\neg C \sqcup D)(a) \notin$ABox, we distinguish two cases: either $(\neg C \sqcup D)(a) \in$ ABox$_{i-1}$ or $(\neg C \sqcup D)(a) \notin$ ABox$_{i-1}$. In the first case, by inductive hypothesis, $\mathcal{M}_{i-1} \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} E_{i-1}$, and hence $\mathcal{M}_{i-1} \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} (\neg C \sqcup D)(a)$ and also $\mathcal{M}_i \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} (\neg C \sqcup D)(a)$ (since $\mathbf{T}$ does not occur in C). In the second case, the assertion $(\neg C \sqcup D)(a)$ has been added to ABox$_i$ and was not in ABox$_{i-1}$. Hence, there is an inclusion $\mathbf{T}(C) \sqsubseteq D$ in $(E_{i-1} - E_i)$ and there is a $\mathbf{T}(B)(a) \in$ABox (and hence in ABox$_i$) such that $E_{i-1} \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg B$. As $\mathbf{T}(B)(a) \in$ABox, $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(B)(a)$ and for some $x \in \Delta$, $x \in Min_<(B^I)$ and $x = a^I$. We want to show that $x \in (\neg C \sqcup D)^I$, for all $\mathbf{T}(C) \sqsubseteq D$ in E_{i-1}, so that $(\neg C \sqcup D)(a)$ is satisfied in $\mathcal{M}_{i-1}$ and hence in $\mathcal{M}_i$.

By construction, $rank(B) = i-1$, and by inductive hypothesis, part 1), $k_{\mathcal{M}}(B) \geq i-1$. We show that $k_{\mathcal{M}}(B) = i-1$ and $k_{\mathcal{M}_{i-1}}(B) = 0$.

From $E_{i-1} \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg B$, we know there is a model, $\mathcal{M}''$ satisfying E_{i-1} and such that, for some domain element y, $k_{\mathcal{M}''}(y) = 0$ and $y \in B^{I''}$. Clearly, for all $\mathbf{T}(C) \sqsubseteq D \in$ TBox$_{i-1}$, $y \in (\neg C \sqcup D)^{I''}$. Let $\{C_1, \dots, C_r\}$ be the maximal consistent set of concepts of which y is an instance. We can show that $\{C_1, \dots, C_r\}$ is consistent with K. Indeed, we can define a new model of K by adding to $\mathcal{M}$ all the domain elements in $\mathcal{M}''$, including y, by keeping the interpretation of concepts and relations on such elements as in $\mathcal{M}''$ and by letting the rank $k_{\mathcal{M}}(y) = i-1$ and $k_{\mathcal{M}}(z) = n+1$ (where n is the highest rank in $\mathcal{M}$), for all $z \in \Delta''$ such that $z \neq y$. The obtained model is clearly a model of K satisfying $\{C_1, \dots, C_r\}$, which proves the consistency of this set w.r.t. K.

As $\mathcal{M}$ is a canonical model, and $\{C_1, \dots, C_r\}$ is consistent with K, there must be a y' in Δ such that y' is an instance of $C_1 \sqcap \dots \sqcap C_r$. Furthermore, for all $\mathbf{T}(C) \sqsubseteq D \in TBox_{i-1}$, $y' \in (\neg C \sqcup D)^I$, and y' must have rank $k_{\mathcal{M}}(y') = i-1$ (as $\mathcal{M}$ is a minimal model of K). Hence, $k_{\mathcal{M}_{i-1}}(B) = 0$, and, since $\mathcal{M}_{i-1}$ satisfies $\mathbf{T}(B)(a)$, it must be $k_{\mathcal{M}_{i-1}}(x) = 0$ for $x = a^I$. Thus, in $\mathcal{M}_{i-1}$, if $x \in C^I$, then $x \in (T(C))^I$, and from the fact that $\mathcal{M}_{i-1}$ satisfies $\mathbf{T}(C) \sqsubseteq D$, we can conclude that $x \in D^I$. Hence, $x \in (\neg C \sqcup D)^I$ for $x = a^I$, so that $(\neg C \sqcup D)(a)$ is satisfied in $\mathcal{M}_{i-1}$. It is easy to see that $(\neg C \sqcup D)(a)$ is satisfied in $\mathcal{M}_i$ as well. Therefore $\mathcal{M}_i \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} E_i$. □

The next proposition is still concerned with minimal canonical models, to prove the correspondence between the rank of a concept (as in Definition 21) and the rank of a concept in a minimal canonical model (as in Definition 17).

Proposition 13. *Given a consistent K and $\mathcal{S}$, for all $C \in \mathcal{S}$, if $rank(C) = i$, then:*

(1) there is a $\{C_1, C_2, \dots, C_r\} \subseteq \mathcal{S}$ maximal and consistent with K such that $C \in \{C_1, C_2, \dots, C_r\}$ and $rank(C_1 \sqcap C_2 \sqcap \dots \sqcap C_r) = i$

(2) for any $\mathcal{M}$ minimal canonical model of K, it holds that $k_{\mathcal{M}}(C) = i$

Proof. We prove (1). If $i = 0$, we have that $K \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg C$. Then there is a model $\mathcal{M}_0$ of K with a domain element x such that $k_{\mathcal{M}_0}(x) = 0$ and x is an instance of C. Consider the maximal consistent set of concepts in $\mathcal{S}$ of which x is an instance in

$\mathcal{M}_0$. This is a maximal consistent set $\{C_1, C_2, \dots, C_r\} \subseteq \mathcal{S}$ containing C. Furthermore, $rank(C_1 \sqcap C_2 \sqcap \dots \sqcap C_r) = 0$ since clearly $K \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg(C_1 \sqcap C_2 \sqcap \dots \sqcap C_r)$ (given that $k_{\mathcal{M}_0}(x) = 0$).

For all $i > 0$ we proceed as follows. We have that $E_i \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg C$, then there must be a model $\mathcal{M}_i = \langle \Delta_i, <_i, I_i \rangle$ of E_i, and a domain element x such that $k_{\mathcal{M}_i}(x) = 0$ and x is an instance of C. Consider the maximal consistent set of concepts $\{C_1, \dots, C_r\} \subseteq \mathcal{S}$ of which x is an instance in $\mathcal{M}_i$. Clearly, $C \in \{C_1, \dots, C_r\}$. Furthermore, $rank(C_1 \sqcap C_2 \sqcap \dots \sqcap C_r) = i$. Indeed $E_{i-1} \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg(C_1 \sqcap C_2 \sqcap \dots \sqcap C_r)$ (since $E_{i-1} \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg C$ and $C \in \{C_1, \dots, C_r\}$), whereas clearly by the existence of x, $E_i \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg(C_1 \sqcap C_2 \sqcap \dots \sqcap C_r)$.

We have to prove that the set $\{C_1, \dots, C_r\}$ is consistent with K. The proof is the same for $i = 0$ and for $i > 0$. Let $\mathcal{M}_i = \langle \Delta_i, <_i, I_i \rangle$ be the model, considered few lines above in this proof, such that $x \in \Delta_i$ is an instance of C. Starting from a finite model $\mathcal{M} = \langle \Delta, <, I \rangle$ of K ($\mathcal{M}$ exists by the finite model property, Theorem 6), we add to $\mathcal{M}$ all the domain elements of $\mathcal{M}_i$.

We define the resulting model $\mathcal{M}' = \langle \Delta', <', I' \rangle$ as follows: $\Delta' = \Delta \cup \Delta_i$; I' is defined on the elements of Δ as I in $\mathcal{M}$, and on the elements of Δ_i as I_i in $\mathcal{M}_i$. For the interpretation of concepts: for $x \in \Delta$, $x \in C^{I'}$ if and only if $x \in C^I$; for $x \in \Delta_i$, $x \in C^{I'}$ if and only if $x \in C^{I_i}$. For the interpretation of roles: for $x, y \in \Delta$, $(x, y) \in R^{I'}$ if and only if $(x, y) \in R^I$; for $x, y \in \Delta_i$, $(x, y) \in R^{I'}$ if and only if $(x, y) \in R^{I_i}$; and, for any two elements $x \in \Delta$ and $y \in \Delta_i$, $(x, y) \notin R^{I'}$ and $(y, x) \notin R^{I'}$. For all individual constants $a \in \mathcal{O}$, we let $a^{I'} = a^I$. Finally, for all $w \in \Delta$, we let $k_{\mathcal{M}'}(w) = k_{\mathcal{M}}(w)$ and, for all $y \in \Delta_i$, we let $k_{\mathcal{M}'}(y) = n + 1 + k_{\mathcal{M}_i}(y)$, where n is the highest value of $k_{\mathcal{M}}$ in $\mathcal{M}$ (n is finite as each element in $\mathcal{M}$ has a finite rank).

We can show that by construction the resulting model satisfies K. Let $C \sqsubseteq D$ be an inclusion in TBox. We distinguish two cases: C does not contain the typicality operator and $C = T(B)$ for some B. In the first case, $C \subseteq D$ is a strict inclusion. Let $x \in C^{I'}$. There are two cases: either $x \in \Delta$ or $x \in \Delta_i$. In the first case, $x \in C^I$ in $\mathcal{M}$. As $\mathcal{M}$ satisfies K, $x \in D^I$ and, by definition of $\mathcal{M}'$, $x \in D^{I'}$. In the second case, $x \in C^{I_i}$. As $\mathcal{M}_i$ satisfies all the strict inclusions in K (which belong to E_i), $x \in D^{I_i}$ and, by definition of $\mathcal{M}'$, $x \in D^{I'}$.

In case $C = T(B)$ for some B, observe that if $x \in (\mathbf{T}(B))^{I'}$, then either $x \in \Delta$ or $x \in \Delta_i$. In the first case, x is B-minimal in $\mathcal{M}$ and $x \in D^I$. Hence, by definition of $\mathcal{M}'$, $x \in D^{I'}$. In the second case, x is B-minimal in $\mathcal{M}_i$ and $x \in D^{I_i}$. Hence, by definition of $\mathcal{M}'$, $x \in D^{I'}$.

Observe that all the assertions in the ABox are satisfied in $\mathcal{M}$ and we have interpreted individual constants over the elements of Δ as in $\mathcal{M}$: $a^{I'} = a^I$, for all $a \in \mathcal{O}$. By construction, for $x \in \Delta$, $x \in C^{I'}$ iff $x \in C^I$. Hence, if $B(a) \in$ ABox is satisfied in $\mathcal{M}$, then it is satisfied in $\mathcal{M}'$ as well.

From this, we can conclude that $\mathcal{M}'$ is a model satisfying K and $(C_1 \sqcap C_2 \sqcap \dots \sqcap C_r)^{I'} \neq \emptyset$. From this, point (1) follows.

Let us prove point (2). By point (1), if $rank(C) = i$ there is a $\{C_1, C_2, \dots, C_r\} \subseteq \mathcal{S}$ maximal and consistent with K containing C and such that $rank(C_1 \sqcap C_2 \sqcap \dots \sqcap C_r) = i$. By Definition 24, we know that in all canonical models there is at least an instance of $(C_1 \sqcap C_2 \sqcap \dots \sqcap C_r)$. To prove point (2) we show that in all minimal canonical models $\mathcal{M}$ of K, $k_{\mathcal{M}}(C_1 \sqcap C_2 \sqcap \dots \sqcap C_r) = i$, which entails $k_{\mathcal{M}}(C) = i$ (since $C \in \{C_1, C_2, \dots, C_r\}$).

By Proposition 12 we know that $k_{\mathcal{M}}(C_1 \sqcap C_2 \sqcap \ldots \sqcap C_r) \geq i$. We need to show that also $k_{\mathcal{M}}(C_1 \sqcap C_2 \sqcap \ldots \sqcap C_r) \leq i$. For a contradiction suppose $k_{\mathcal{M}}(C_1 \sqcap C_2 \sqcap \ldots \sqcap C_r) > i$, i.e., for all the domain elements x instances of $C_1 \sqcap C_2 \sqcap \ldots \sqcap C_r$, $k_{\mathcal{M}}(x) > i$. We show that this contradicts the minimality of $\mathcal{M}$. From $\mathcal{M}$ we build another model $\mathcal{M}' = \langle \Delta', <', I' \rangle$ of K by lowering the ranks of some elements in $\mathcal{M}$ and leaving all the rest unchanged. We let $\Delta' = \Delta$ and $I' = I$. For each element $y \in \Delta$, let $\{C_1, C_2, \ldots, C_r\} \subseteq \mathcal{S}$ be the maximal set of concepts consistent with K of which y is an instance. If $rank(C_1 \sqcap C_2 \sqcap \ldots \sqcap C_r) = i < k_{\mathcal{M}}(y)$, we let $k_{\mathcal{M}'}(y) = i$. Otherwise, we let $k_{\mathcal{M}'}(y) = k_{\mathcal{M}}(y)$. Observe that we can obtain $\mathcal{M}'$ from the model $\mathcal{M}$ by repeatedly lowering the rank of the elements in Δ rank by rank, starting from rank $i = 0$.

$\mathcal{M}'$ would still be a model of $TBox$: at each step, when the rank of an element y is lowered to i (together with all the other elements whose rank is lowered to i), the only thing that changes with respect to $\mathcal{M}$ is that y might have become in $\mathcal{M}'$ a minimal instance of a concept of which it was only a non-typical instance in $\mathcal{M}$. This might compromise the satisfaction in $\mathcal{M}'$ of a typicality inclusion as $\mathbf{T}(E) \sqsubseteq G$. We show that this cannot happen by reasoning by induction on i to prove that, after lowering the rank of an element y in Δ, the modified model still satisfies all the inclusions in K. Let $rank(C_1 \sqcap C_2 \sqcap \ldots \sqcap C_r) = i < k_{\mathcal{M}}(y)$, consider a step in which we let $k_{\mathcal{M}'}(y) = i$.

For $i = 0$, let $\mathbf{T}(E) \sqsubseteq G \in K$. It can be easily proven that being rank$(C_1 \sqcap C_2 \sqcap \ldots \sqcap C_r) = 0$, then if $E \in \{C_1, C_2, \ldots, C_r\}$ also $G \in \{C_1, C_2, \ldots, C_r\}$ (indeed if on the contrary $\neg G \in \{C_1, C_2, \ldots, C_r\}$, then clearly $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg(C_1 \sqcap C_2 \sqcap \ldots \sqcap C_r)$, against the hypothesis that rank$(C_1 \sqcap C_2 \sqcap \ldots \sqcap C_r) = 0$). Therefore if $y \in E^{I'}$, also $y \in G^{I'}$, and $\mathbf{T}(E) \sqsubseteq G$ holds in $\mathcal{M}'$.

For $i > 0$, let $\mathbf{T}(E) \sqsubseteq G \in K$. We consider two cases: $rank(E) \geq i$ and $rank(E) < i$. If $rank(E) \geq i$ we reason as above (with E_i instead of K and i instead of 0) to conclude that if $E \in \{C_1, C_2, \ldots, C_r\}$ also $G \in \{C_1, C_2, \ldots, C_r\}$, hence if $y \in E^{I'}$, also $y \in G^{I'}$, and $\mathbf{T}(E) \sqsubseteq G$ holds in $\mathcal{M}'$. If $rank(E) < i$, then $rank(E) \leq i - 1$, and we know by construction that $k_{\mathcal{M}'}(E) < i$ and y is not a minimal instance of E in $\mathcal{M}'$. Hence lowering the rank of y does not compromise the satisfaction of $\mathbf{T}(E) \sqsubseteq G \in E_i$.

The resulting $\mathcal{M}'$ is such that for all maximal set of concepts consistent with K, $\{C_1, \ldots, C_r\}$, $k_{\mathcal{M}'}(C_1 \sqcap \ldots \sqcap C_r) = rank(C_1 \sqcap \ldots \sqcap C_r)$. Furthermore, by the above reasoning, $\mathcal{M}'$ satisfies TBox. We show that $\mathcal{M}'$ also satisfies ABox, and in particular it is not the case that a $\mathbf{T}(B)(a) \in ABox$ might turn false in $\mathcal{M}'$.

For all assertions $\mathbf{T}(B)(a)$ $\in$ABox, from the hypothesis we know that $\mathcal{M}$ satisfies $\mathbf{T}(B)(a)$. Hence, there is a $z \in \Delta$ such that $a^I = z$ and $z \in (T(B))^I$. We show that it must be the case that $z \in (T(B))^{I'}$ and, therefore, $\mathbf{T}(B)(a)$ is satisfied in $\mathcal{M}'$ as well. Let $\{C_1, \ldots, C_r\}$ be the maximal consistent set of concepts of which z is an instance in $\mathcal{M}$. We prove that, rank$(C_1 \sqcap \ldots \sqcap C_r) =$ rank(B).

Clearly rank$(C_1 \sqcap \ldots C_r) \geq$ rank(B) (since $B \in \{C_1, \ldots, C_r\}$). Suppose for a contradiction that rank$(C_1 \sqcap \ldots \sqcap C_r) >$ rank(B) , i.e. there is an E_i s.t. $E_i \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg B$ but $E_i \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg(C_1 \sqcap \ldots \sqcap C_k)$. Take the minimal i for which this happens, we show a contradiction. As $\mathbf{T}(B)(a) \in ABox$, for all $\mathbf{T}(C) \sqsubseteq D \in E_i$, $(\neg C \sqcup D)(a)$ has been added to ABox$_i$. We know by Proposition 12 that $\mathcal{M}_i$ satisfies E_i and, in particular, it satisfies ABox$_i$. Thus $\mathcal{M}_i \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} (\neg C \sqcup D)(a)$, and $z \in (\neg C \sqcup D)^I$, for all $\mathbf{T}(C) \sqsubseteq D \in E_i$. As $\mathcal{M}$ is a minimal model, it must be the case that $k_{\mathcal{M}}(z) = i$ (otherwise we can define a canonical model $\mathcal{M}''$ such that $\mathcal{M}'' <_{FIMS} \mathcal{M}$). Therefore, $k_{\mathcal{M}_i}(z) = 0$

and, as $\mathcal{M}_i$ satisfies E_i, $\mathcal{M}_i$ is a model of E_i such that $k_{\mathcal{M}_i}(C_1 \sqcap \ldots \sqcap C_k) = 0$, thus contradicting the fact that $E_i \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg(C_1 \sqcap \ldots \sqcap C_k)$. Hence, rank$(C_1 \sqcap \ldots \sqcap C_r)$ = rank(B).

As by the construction of $\mathcal{M}'$, it must be that $k_{\mathcal{M}'}(a^{I'})$=rank$(C_1 \sqcap \ldots \sqcap C_r)$. To conclude that $z \in (\mathbf{T}(B))^{I'}$, observe that it is not possible that there is an element $y \in B^{I'}$ such that $k_{\mathcal{M}'}(y) < k_{\mathcal{M}'}(a^{I'})$. In fact, otherwise it would be: $k_{\mathcal{M}'}(y) <$ rank(B), which contradicts Proposition 12, point (a). This concludes the proof that $\mathcal{M}'$ satisfies ABox.

It follows that $\mathcal{M}'$ would be a model of K, and $\mathcal{M}' <_{FIMS} \mathcal{M}$, against the minimality of $\mathcal{M}$. We are therefore forced to conclude that $k_{\mathcal{M}}(C_1 \sqcap C_2 \sqcap \ldots \sqcap C_r) = i$, hence also $k_{\mathcal{M}}(C) = i$, and 2) holds. □

As a consequence of Proposition 13 and by what we know about the highest rank of a concept (in case it has a rank) we state the following proposition.

Proposition 14. *Let us consider the least $n \geq 0$ such that, for all $m > n, TBox_m = TBox_n$ or $TBox_m = \emptyset$. Then, in all minimal canonical models $\mathcal{M}$, for all domain elements x, $k_{\mathcal{M}}(x) \leq n$.*

We can now prove the following theorem:

Theorem 8. *Let K=(TBox,ABox) be a knowledge base and $C \sqsubseteq D$ a query. We have that $C \sqsubseteq D \in \overline{TBox}$ if and only if $C \sqsubseteq D$ holds in all minimal canonical models of K with respect to TBox.*

Proof. (If part) Assume that $C \sqsubseteq D$ holds in all minimal canonical models of K with respect to TBox, and let $\mathcal{M} = \langle \Delta, <, I \rangle$ be a minimal canonical model of K satisfying $C \sqsubseteq D$. Observe that C and D (and their complements) belong to $\mathcal{S}$. We consider two cases: (1) the left hand side of the inclusion C does not contain the typicality operator, and (2) the left hand side of the inclusion is $\mathbf{T}(C)$.

In case (1), the minimal canonical model $\mathcal{M}$ of K satisfies $C \sqsubseteq D$. Then, $C^I \subseteq D^I$. For a contradiction, let us assume that $C \sqsubseteq D \notin \overline{TBox}$. Then, by definition of $\overline{TBox}$, it must be: $K \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C \sqsubseteq D$. Hence, $K \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C \sqcap \neg D \sqsubseteq \bot$, and the set of concepts $\{C, \neg D\}$ is consistent with K. As $\mathcal{M}$ is a canonical model of K, there must be an element $x \in \Delta$ such that $x \in (C \sqcap \neg D)^I$. This contradicts the fact that $C^I \subseteq D^I$.

In case (2), assume $\mathcal{M}$ satisfies $\mathbf{T}(C) \sqsubseteq D$. Then, $\mathbf{T}(C)^I \subseteq D^I$, i.e., for each $x \in Min_<(C^I)$, $x \in D^I$. If $Min_<(C^I) = \emptyset$, then there is no $x \in C^I$ (by the smoothness condition), hence C has no rank $k_{\mathcal{M}}$ in $\mathcal{M}$ and, by Proposition 13, C has no rank ($rank(C) = \infty$). In this case, by Definition 22, $\mathbf{T}(C) \sqsubseteq D \in \overline{TBox}$. Otherwise, let us assume that $k_{\mathcal{M}}(C) = i$. Since $\mathcal{M}$ satisfies $\mathbf{T}(C) \sqsubseteq D$, $k_{\mathcal{M}}(C \sqcap D) < k_{\mathcal{M}}(C \sqcap \neg D)$, then $k_{\mathcal{M}}(C \sqcap \neg D) > i$. By Proposition 13, $rank(C) = i$ and $rank(C \sqcap \neg D) > i$. Hence, by Definition 22, $\mathbf{T}(C) \sqsubseteq D \in \overline{TBox}$.

(Only if part) If $C \sqsubseteq D \in \overline{TBox}$, then, by definition of $\overline{TBox}$, $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C \sqsubseteq D$. Therefore, each minimal canonical model $\mathcal{M}$ of K satisfies $C \sqsubseteq D$.

If $\mathbf{T}(C) \sqsubseteq D \in \overline{TBox}$, then by Definition 22, either (a) $rank(C) < rank(C \sqcap \neg D)$, or (b) C has no rank. Let $\mathcal{M}$ be any minimal canonical model of K. In the case (a), by Proposition 13, $k_{\mathcal{M}}(C) < k_{\mathcal{M}}(C \sqcap \neg D)$, which entails $k_{\mathcal{M}}(C \sqcap D) < k_{\mathcal{M}}(C \sqcap \neg D)$. Hence $\mathcal{M}$ satisfies $\mathbf{T}(C) \sqsubseteq D$. In case (b), by Proposition 13, C has no rank in $\mathcal{M}$, hence $\mathcal{M}$ satisfies $\mathbf{T}(C) \sqsubseteq D$. □

For a strict inclusion $C \sqsubseteq D$ the problem of deciding whether $C \sqsubseteq D \in \overline{TBox}$ is clearly in EXPTIME as, by definition of $\overline{TBox}$ (Definition 22), it amounts to check whether $\underline{K} \models_{\mathcal{ALC}+\mathbf{T}_R} C \sqsubseteq D$ (Theorem 5). The problem of deciding whether $\mathbf{T}(C) \sqsubseteq D \in \overline{TBox}$ is in EXPTIME as well.

Theorem 9 (Complexity of rational closure over the TBox)**.** *Given a knowledge base $K = (TBox, ABox)$, the problem of deciding whether $\mathbf{T}(C) \sqsubseteq D \in \overline{TBox}$ is in* EXPTIME.

Proof. Checking if $\mathbf{T}(C) \sqsubseteq D \in \overline{TBox}$ can be done by computing the finite sequence $\text{TBox}_0, \text{TBox}_1, \ldots, \text{TBox}_n$ of non increasing subsets of TBox inclusions and the sequence $\text{ABox}_0, \text{ABox}_1, \ldots, \text{ABox}_n$ of non decreasing supersets of ABox in the construction of the rational closure. Note that the number n of the TBox_i (and ABox_i) is $O(|K|)$, where $|K|$ is the size of the knowledge base K.

Computing each $\text{TBox}_i = \mathcal{E}(\text{TBox}_{i-1})$, requires to check, for all concepts C' occurring on the left hand side of a $\mathbf{T}$-inclusion in the TBox, whether $\text{TBox}_{i-1} \models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg C'$, which requires an exponential time [**?**] in the size of TBox_{i-1} (and hence in the size of K). The number of the concepts C' to be considered is $O(|K|)$.

Computing each ABox_i requires to to check whether $E_{i-1} \not\models_{\mathcal{ALC}+\mathbf{T}_R} \mathbf{T}(\top) \sqsubseteq \neg B$ which requires an exponential time in the size of E_{i-1} (and hence in the size of K).

If not already checked, the exceptionality of C and of $C \sqcap \neg D$ have to be checked for each TBox_i, to determine the ranks of C and of $C \sqcap \neg D$ (which also requires an exponential time in the size of K). Hence, verifying if $\mathbf{T}(C) \sqsubseteq D \in \overline{TBox}$ is in EXPTIME. □

The above result provides an EXPTIME upper bound for deciding whether $\mathbf{T}(C) \sqsubseteq D \in \overline{TBox}$ (the EXPTIME lower bound comes from the fact that subsumption in $\mathcal{ALC}$ is EXPTIME-hard). It requires a quadratic (in the size of K) number of calls to an EXPTIME algorithm for checking subsumption in $\mathcal{ALC} + \mathbf{T}_R$. In the case the ABox does not contain typicality assertions, it is possible to see that subsumption in $\mathcal{ALC}+\mathbf{T}_R$ can be polynomialy reduced to subsumption in $\mathcal{ALC}$ so that optimized $\mathcal{ALC}$ prover can be used to this purpose. The encoding is the same as the one introduced in [**?**] for reducing subsumption in $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ to subsumption in $\mathcal{SHIQ}$ (see [**?**] Proposition 3).

To conclude the session, we want to observe that our definition of exceptionality (Definition 19), which exploits preferential entailment, cannot be equivalently replaced with a notion of exceptionality which directly exploits entailment in $\mathcal{ALC}$ over the materialization of the KB, in the spirit of the other proposals of rational closure in [**?** **?**]. In particular, consider a knowledge base K =(TBox,ABox) and let $K_S = \{A \sqsubseteq B \mid A \sqsubseteq B \in \text{TBox}\}$ be the set of strict inclusions in K and $\tilde{K_D} = \bigsqcap \{\neg A \sqcup B \mid \mathbf{T}(A) \sqsubseteq B \in \text{TBox}\}$ be the materialization of the defeasible inclusions in K. One can wonder whether the following notion of exceptionality: *"B is exceptional with respect to K if and only if* $(K_S, ABox) \models_{\mathcal{ALC}} \tilde{K_D} \sqsubseteq \neg B$" is equivalent to the notion of exceptionality introduced in Definition 19 . The next example shows that this is not the case at least in the context of our rational closure construction (Definition 20).

Example 9. Let K = (TBox, ABox) where TBox = $\{Faun \sqsubseteq \exists HasFriend.WingedHorse$, $\mathbf{T}(WingedHorse) \sqsubseteq Fly$, $\mathbf{T}(WingedHorse) \sqsubseteq \neg Fly\}$ and ABox= $\emptyset$.

From the construction in Definition 20, we have that ABox_m = ABox and TBox_m = TBox, for all m, as *WingedHorse* is exceptional for K, that is, $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg WingedHorse$. Furthermore, *Faun* is exceptional for K (that is, $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(\top) \sqsubseteq \neg Faun$) and is exceptional for all the $E_i = (\text{ABox}_i, \text{TBox}_i)$ in the construction. Hence $rank(Faun) = \infty$. Observe that, in $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$, any model $\mathcal{M}$ satisfying K contains neither a *WingedHorse* nor a *Faun*-element, i.e., $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} WingedHorse \sqsubseteq \bot$, $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} Faun \sqsubseteq \bot$ and, of course, also $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mathbf{T}(Faun) \sqsubseteq \bot$. Therefore, $\mathbf{T}(Faun) \sqsubseteq \bot$ holds in all the minimal canonical models of K and this is in accordance with the fact that, being $rank(Faun) = \infty$, $\mathbf{T}(Faun) \sqsubseteq \bot$ is in the rational closure of TBox.

If we adopt the definition of exceptionality introduced just above, we get a different result. We have: $K_S = \{Faun \sqsubseteq \exists HasFriend.WingedHorse\}$ and $\tilde{K_D} = ((\neg WingedHorse \sqcup Fly) \sqcap (\neg WingedHorse \sqcup \neg Fly))$, therefore

$K_S \models_{\mathcal{ALC}} \tilde{K_D} \sqsubseteq \neg WingedHorse$ but
$K_S \not\models_{\mathcal{ALC}} \tilde{K_D} \sqsubseteq \neg Faun$

For the second statement, observe that there is an $\mathcal{ALC}$ model satisfying K_S containing a *Faun*-element x, which is an instance of $\tilde{K_D}$ and is not a Winged Horse, but is in the relation *HasFriend* with a *WingedHorse*-element y. Also, y is not required to be an instance of $\tilde{K_D}$. Hence, *Faun* is not exceptional with respect to K while *WingedHorse* is exceptional, and we get $rank(Faun) = 0$ and $rank(WingedHorse) = \infty$. Therefore, with this notion of exceptionality, $\mathbf{T}(Faun) \sqsubseteq \bot$ would not be in the rational closure of TBox, as $rank(Faun) \not< rank(Faun \sqcap \neg\bot)$, since , clearly, $rank(Faun \sqcap \neg\bot) = rank(Faun)$.

The same example knowledge base K above can be used to show the difference between our notion of exceptionality in Definition 19 and the notion of exceptionality in [**?**], which exploits the materialization of both the strict and the defeasible part in the TBox. For simplicity, let us consider the case when ABox is empty and is not considered in the construction of the rational closure of TBox. Following [**?**], we could define exceptionality as follows: *"B is exceptional with respect to K if and only if* $\models_{\mathcal{ALC}} \tilde{K_S} \sqcap \tilde{K_D} \sqsubseteq \neg B$*"*, where $\tilde{K_S} = \sqcap\{\neg A \sqcup B \mid A \sqsubseteq B \in K_S\}$ is the materialization of the strict inclusions in K and $\tilde{K_D}$ is the materialization of the defeasible inclusions in K (as defined above). Consider the following example:

Example 10. Let K be the knowledge base in Example 9. We have $K_S = (\neg Faun \sqcup \exists HasFriend.WingedHorse)$ and $\tilde{K_D} = ((\neg WingedHorse \sqcup Fly) \sqcap (\neg WingedHorse \sqcup \neg Fly))$. Therefore, $\not\models_{\mathcal{ALC}} \tilde{K_S} \sqcap \tilde{K_D} \sqsubseteq \neg Faun$, i.e., *Faun* is not exceptional for K if we adopt the notion of exceptionality from [**?**] and hence $rank(Faun) = 0$. Again, with this notion of exceptionality, $\mathbf{T}(Faun) \sqsubseteq \bot$ would not be in the rational closure of TBox, as $rank(Faun) \not< rank(Faun \sqcap \neg\bot)$), while, as we have seen in Example 9, with the notion of exceptionality in Definition 19 we get $rank(Faun) = \infty$ and $\mathbf{T}(Faun) \sqsubseteq \bot \in \overline{\text{TBox}}$.

An alternative notion of exceptionality can be defined along the lines of [**?**]. Consider a knowledge base K =(TBox,ABox) (again we assume ABox is empty). We can define exceptionality as follows: *"B is exceptional with respect to K if and only if* $K_S \cup K_D \models_{\mathcal{ALC}} \top \sqsubseteq \neg B$*"*, where $K_S = \{A \sqsubseteq B \mid A \sqsubseteq B \in \text{TBox}\}$ is the set of strict inclusions in K and $K_D = \{A \sqsubseteq B \mid \mathbf{T}(A) \sqsubseteq B \in \text{TBox}\}$ is the set containing a strict inclusion for each defeasible inclusion in K. This notion of exceptionality is not equivalent to the one in Definition 19 when used in the context of our rational closure construction, as shown by the following example.

Example 11. Let $K = (\text{TBox}, \text{ABox})$ where TBox $= \{Penguin \sqsubseteq Bird, Bird \sqsubseteq \exists HasEnemy.Penguin, \mathbf{T}(Bird) \sqsubseteq Fly, \mathbf{T}(Penguin) \sqsubseteq \neg Fly\}$ and ABox$= \emptyset$.

We have $K_S = \{Penguin \sqsubseteq Bird, Bird \sqsubseteq \exists HasEnemy.Penguin\}$ and $K_D = \{Bird \sqsubseteq Fly, Penguin \sqsubseteq \neg Fly\}$.

It holds that: $K_S \cup K_D \models_{\mathcal{ALC}} \top \sqsubseteq \neg Penguin$ and $K_S \cup K_D \models_{\mathcal{ALC}} \top \sqsubseteq \neg Bird$. For the first entailment, if $\mathcal{M}$ were an $\mathcal{ALC}$ model satisfying the inclusions $K_S \cup K_D$ and x an instance of *Penguin* in $\mathcal{M}$, then x would also be an instance of *Bird* and, by the inclusions $Bird \sqsubseteq Fly$, $Penguin \sqsubseteq \neg Fly$ in K_D, x would be an instance of both *Fly* and $\neg Fly$. For the second entailment, as there is no model satisfying $K_S \cup K_D$ that contains an instance of *Penguin*, then, there is no model containing an instance of *Bird*, since any instance of *Bird* must be in the relation *HasEnemy* with an instance of *Penguin*.

Therefore, *Penguin* and *Bird* are both exceptional for K, so that $rank(Bird) = \infty$ and $rank(Penguin) = \infty$. Hence, with this notion of exceptionality, $\mathbf{T}(Bird) \sqsubseteq \bot$ and $\mathbf{T}(Bird) \sqsubseteq \neg Fly$ would be in the rational closure of TBox. Conversly, with our notion of exceptionality in Definition 19, we get that *Bird* is not exceptional for K, and that $rank(Bird) = 0$. Thus, $\mathbf{T}(Bird) \sqsubseteq \bot$ and $\mathbf{T}(Bird) \sqsubseteq \neg Fly$ are not in the rational closure of TBox (in agreement with the fact that these inclusions do not hold in all the minimal models of K).

In conclusion, if we replace, in our definition of rational closure (Definition 20), the notion of exceptionality in Definition 19 (based on the entailment in $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$) with a different notion of exceptionality which exploits the materialization of the KB and entailment in $\mathcal{ALC}$, inspired to the notions of exceptionality used in [**? ?**], the rational closure we obtain is different from the rational closure obtained based on exceptionality in Definition 19.

3.3. Rational Closure Over the ABox: Maximizing the Typicality of Named Individuals

In this section we extend the notion of rational closure defined in the previous one in order to take into account the individual constants in the ABox. Consider, for instance, a K with TBox=$\{\mathbf{T}(Bird) \sqsubseteq Fly\}$ and ABox=$\{Bird(tweety)\}$. We would like to be able to conclude that Tweety flies although the ABox does not contain the information that Tweety is a typical bird. The rational closure of the TBox, in the previous section, does not say anything about the individual constants in the ABox, although its construction exploits the information in the ABox for consistency. We therefore address the question: what does the rational closure of a knowledge base K allow to infer about a specific individual constant a occurring in the ABox of K?

The definition of rational closure of a knowledge base K considered so far only exploits the ABox (and, in particular, the typicality assertions $\mathbf{T}(C)(a)$ in the ABox) to determine the exceptionality of concepts and hence to build the sequence $\text{TBox}_0, \text{TBox}_1,$

$\ldots, \text{TBox}_n$ of subsets of TBox required to define $\overline{TBox}$, and to reason about concept inclusions. We address the question of the ABox by first considering the semantic aspect, in order to treat individuals explicitly mentioned in the ABox in a uniform way with respect to the other domain elements: as for all the domain elements we would like to attribute to each individual constant named in the ABox the lowest possible rank. So we further refine Definition 25 by taking into account the interpretation of individual constants of the ABox: given two minimal canonical models $\mathcal{M}$ and $\mathcal{M}'$, we will prefer $\mathcal{M}$ to $\mathcal{M}'$ if there is an individual constant b occurring in ABox such that $k_{\mathcal{M}}(b^I) < k_{\mathcal{M}}(b^{I'})$ (whereas $k_{\mathcal{M}}(a^I) \leq k_{\mathcal{M}}(a^{I'})$ for all other individual constants occurring in ABox).

Definition 26 (Minimal canonical model of K minimally satisfying ABox). Given K=(TBox,ABox), let $\mathcal{M} = \langle \Delta, <, I \rangle$ and $\mathcal{M}' = \langle \Delta', <', I' \rangle$ be two canonical models of K which are minimal with respect to Definition 25. We say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ with respect to ABox, and we write $\mathcal{M} <_{ABox} \mathcal{M}'$, if, for all individual constants a occurring in ABox, it holds that $k_{\mathcal{M}}(a^I) \leq k_{\mathcal{M}'}(a^{I'})$ and there is at least one individual constant b occurring in ABox such that $k_{\mathcal{M}}(b^I) < k_{\mathcal{M}'}(b^{I'})$.

As a consequence of Theorem 7 we prove the following:

Theorem 10. *For any $K = (TBox, ABox)$ there exists a finite minimal canonical model of K minimally satisfying ABox.*

Proof. Observe that, as a consequence of Theorem 7, a finite minimal canonical model $\mathcal{M}$ of K (with respect to TBox) exists. In this model the rank of each element is finite (hence for each individual constant a, $k_{\mathcal{M}}(a^I)$ is finite). If $\mathcal{M}$ is not minimally satisfying ABox, then there must be a canonical model $\mathcal{M}_1$ such that $\mathcal{M}_1 <_{ABox} \mathcal{M}$, i.e., such that: $k_{\mathcal{M}_1}(a^{I_1}) \leq k_{\mathcal{M}}(a^I)$ for all individual constants a of ABox, and for some individual constant b_1 occurring in ABox $k_{\mathcal{M}_1}(b_1^{I_1}) < k_{\mathcal{M}}(b_1^I)$. In turn, if $\mathcal{M}_1$ is not minimally satisfying ABox, there must be a canonical model $\mathcal{M}_2$, such that $\mathcal{M}_2 <_{ABox} \mathcal{M}_1$, i.e., such that: $k_{\mathcal{M}_2}(a^{I_2}) \leq k_{\mathcal{M}_1}(a^{I_1})$ for all individual constants a of ABox, and for some individual constant b_2 occurring in ABox $k_{\mathcal{M}_2}(b_2^{I_2}) < k_{\mathcal{M}_1}(b_2^{I_1})$. And so on. Observe that the number of individual constants of $ABox$ is finite, as well as the rank associated to each constant in each model in the chain. Hence, any descending chain of models in the relation $<_{ABox}$ must be finite, and a minimal canonical model minimally satisfying ABox exists. □

In order to see the power of the above semantic notion, consider the standard birds and penguins example.

Example 12. Suppose we have a knowledge base K where TBox = $\{\mathbf{T}(Bird) \sqsubseteq Fly, \mathbf{T}(Penguin) \sqsubseteq \neg Fly, Penguin \sqsubseteq Bird\}$, and ABox = $\{Penguin(pio), Bird(tweety)\}$. Knowing that tweety is a bird and pio is a penguin, we would like to be able to assume, in the absence of other information, that tweety is a typical bird, whereas pio is a typical penguin, and therefore tweety flies whereas pio does not. Consider any minimal canonical model $\mathcal{M}$ of K. Being canonical, $\mathcal{M}$ will contain, among other elements, the following:

- $x \in (Bird)^I$, $x \in (Fly)^I$, $x \in (\neg Penguin)^I$, $k_{\mathcal{M}}(x) = 0$;
- $y \in (Bird)^I$, $y \in (\neg Fly)^I$, $y \in (\neg Penguin)^I$, $k_{\mathcal{M}}(y) = 1$;
- $z \in (Penguin)^I$, $z \in (Bird)^I$, $z \in (\neg Fly)^I$, $k_{\mathcal{M}}(z) = 1$;
- $w \in (Penguin)^I$, $w \in (Bird)^I$, $w \in (Fly)^I$, $k_{\mathcal{M}}(w) = 2$;

Notice that, in the definition of minimal canonical model, there is no constraint on the interpretation of the ABox constants tweety and pio. As far as Definition 25 is concerned, for instance, tweety can be mapped onto x, that is to say $tweety^I = x$, or onto y, i.e. $tweety^I = y$: the minimality of $\mathcal{M}$ with respect to Definition 25 is not affected by this choice. However in the first case it would hold that tweety is a typical bird, in the second tweety is not a typical bird. We want to prefer the first case, and this is what derives from Definition 26: if in $\mathcal{M}$ $tweety^I = x$ whereas in $\mathcal{M}_1$ (which for the rest is identical to $\mathcal{M}$) it holds that $tweety^I = y$, then $\mathcal{M}$ is preferred to $\mathcal{M}_1$. Similarly for pio. As a result, in all models of K, minimal with respect to both TBox and ABox (Definition 26), it holds what we wanted: that tweety is a typical bird, i.e. $\mathbf{T}(Bird)(tweety)$, and therefore it flies, whereas pio is a typical penguin, i.e. $\mathbf{T}(Penguin)(pio)$, and therefore it does not fly.

Our purpose is to give an algorithmic construction that we call rational closure of the ABox, which captures entailment determined by minimal canonical models of the ABox. The idea is that of considering all the possible minimal consistent assignments of ranks to the individuals explicitly named in the ABox. Each assignment adds some properties to named individuals which can be used to infer new conclusions. We adopt a skeptical view of considering only those conclusions which hold for all assignments. The equivalence with the semantics shows that the minimal entailment captures a skeptical approach when reasoning about the ABox.

More formally, in order to calculate the rational closure of ABox, written $\overline{ABox}$, for all individual constants of the ABox we find which is the lowest possible rank they can have in minimal canonical models with respect to Definition 25: the idea is that an individual constant a_i can have a given rank $k_j(a_i)$ just in case it is compatible with all the inclusions of the TBox that do not contain the $\mathbf{T}$ operator or that have a $\mathbf{T}(C)$ on the left side with C's rank $\geq k_j(a_i)$ (the inclusions whose antecedent C's rank is $< k_j(a_i)$ do not matter since, in the minimal canonical model, there will be an instance of C with rank $< k_j(a_i)$ and therefore a_i will not be a typical instance of C). The minimal possible rank assignment k_j for all a_i is computed in the algorithm below: μ_i^j computes all the concepts that a_i would need to satisfy in case it had the rank $k_j(a_i)$. The algorithm verifies whether μ_i^j is compatible with ($\overline{TBox}$, ABox) and whether it is minimal. Notice that, in this phase, all constants are considered simultaneously (indeed, the possible ranks of different individual constants depend on each other, as Example 14 below shows). For this reason, μ^j (which is the union of all μ_i^j for all a_i) takes into account the ranks attributed to all individual constants. Examples 13 and 14 below illustrate the use of the algorithm.

Definition 27 ($\overline{ABox}$: rational closure of ABox). Let $a_1, \ldots, a_m$ be the individuals explicitly named in the ABox. Let $k_1, k_2, \ldots, k_h$ be all the possible rank assignments (ranging from 1 to n, for n in Proposition 14) to the individuals occurring in ABox.

- Given a rank assignment k_j we define:
  - for each a_i: $\mu_i^j = \{(\neg C \sqcup D)(a_i)$ s.t. $C, D \in \mathcal{S}$, $\mathbf{T}(C) \sqsubseteq D$ in $\overline{TBox}$, and $k_j(a_i) = rank(C)\} \cup \{(\neg C \sqcup D)(a_i)$ s.t. $C \sqsubseteq D$ in TBox$\}$;
  - let $\mu^j = \mu_1^j \cup \ldots \cup \mu_m^j$ for all $\mu_1^j \ldots \mu_m^j$ just calculated for all $a_1, \ldots, a_m$ in the ABox
- We say that k_j is *consistent* with ($\overline{TBox}$, ABox) if:
  - if $\mathbf{T}(C)(a_i) \in ABox$, then $k_j(a_i) = \text{rank}(C)$;
  - TBox $\cup$ ABox $\cup\, \mu^j$ is consistent in $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$;
- We say that k_j is *minimal and consistent* with ($\overline{TBox}$, ABox) if k_j is consistent with ($\overline{TBox}$, ABox) and there is no k_i consistent with ($\overline{TBox}$, ABox) s.t. for all a_i, $k_i(a_i) \leq k_j(a_i)$ and for some b, $k_i(b) < k_j(b)$.
- The rational closure of ABox ($\overline{ABox}$) is the set of all assertions derivable in $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ from TBox $\cup$ ABox $\cup\, \mu^j$ for all minimal consistent rank assignments k_j, i.e:

$$\overline{ABox} = \bigcap_{k_j \text{minimal consistent}} \{C(a) : \text{ TBox} \cup \text{ABox} \cup \mu^j \models_{\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}} C(a)\}$$

Before we provide soundness and completeness of the algorithm, let us illustrate its use by the two following examples. The first example is the syntactic counterpart of the semantic Example 12 above.

Example 13. Consider the standard penguin example. Let K = (TBox, ABox), where TBox = $\{\mathbf{T}(Bird) \sqsubseteq Fly, \mathbf{T}(Penguin) \sqsubseteq \neg Fly, Penguin \sqsubseteq Bird\}$, and ABox = $\{Penguin(pio), Bird(tweety)\}$.

Computing the ranking of concepts we get that $rank(Bird) = 0$, $rank(Penguin) = 1$, $rank(Bird \sqcap \neg Fly) = 1$, $rank(Penguin \sqcap Fly) = 2$. It is easy to see that a rank assignment k_0 with $k_0(pio) = 0$ is inconsistent with K as μ^0 would contain $(\neg Penguin \sqcup Bird)(pio)$, $(\neg Bird \sqcup Fly)(pio)$, $(\neg Penguin \sqcup \neg Fly)(pio)$ and $Penguin(pio)$. Thus we are left with only two ranks k_1 and k_2 with respectively $k_1(pio) = 1, k_1(tweety) = 0$ and $k_2(pio) = k_2(tweety) = 1$.

The set μ^1 contains, among the others, $(\neg Penguin \sqcup \neg Fly)(pio)$, $(\neg Bird \sqcup Fly)(tweety)$. It is tedious but easy to check that $K \cup \mu^1$ is consistent and that k_1 is the only minimal consistent assignment (being k_1 preferred to k_2), thus both $\neg Fly(pio)$ and $Fly(tweety)$ belong to $\overline{ABox}$.

Example 14. This example shows the need of considering multiple ranks of individual constants: normally computer science courses (CS) are taught only by academic members (A), whereas business courses (B) are taught only by consultants (C), consultants and academics are disjoint, this gives the following TBox: $\mathbf{T}(CS) \sqsubseteq \forall taught.A$, $\mathbf{T}(B) \sqsubseteq \forall taught.C$, $C \sqsubseteq \neg A$. Suppose the ABox contains: $CS(c1)$, $B(c2)$, $taught(c1, joe)$,

$taught(c2, joe)$ and let K= (TBox, ABox). Computing the rational closure of TBox, we get that all atomic concepts have rank 0. Any rank assignment k_i, with $k_i(c1) = k_i(c2) = 0$, is inconsistent with K since the respective μ^i will contain both $(\neg CS \sqcup \forall taught.A)(c1)$ and $(\neg B \sqcup \forall taught.C)(c2)$, from which both $C(joe)$ and $A(joe)$ follow, which gives an inconsistency.

There are two minimal consistent ranks: k_1, such that $k_1(joe) = 0, k_1(c1) = 0, k_1(c2) = 1$, and k_2, such that $k_2(joe) = 0, k_2(c1) = 1, k_2(c2) = 0$. We have that ABox $\cup\, \mu 1 \models A(joe)$ and ABox $\cup\, \mu 2 \models C(joe)$. According to the skeptical definition of $\overline{ABox}$, neither $A(joe)$, nor $C(joe)$ belongs to $\overline{ABox}$, however $(A \sqcup C)(joe)$ belongs to $\overline{ABox}$.

We are now ready to show the completeness and soundness of the algorithm with respect to the semantic definition of rational closure of ABox.

Theorem 11 (Completeness of $\overline{ABox}$)**.** *Given K=(TBox, ABox), for all individual constants a in ABox, we have that if $C(a)$ holds in all minimal canonical models of K minimally satisfying ABox, then $C(a) \in \overline{ABox}$.*

Proof. We show the contrapositive. Suppose $C(a) \notin \overline{ABox}$, i.e. there is a minimal k_j consistent with ($\overline{TBox}$, ABox) s.t. TBox $\cup$ ABox $\cup\, \mu^j \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C(a)$. This means that there is an $\mathcal{M}' = \langle \Delta', <', I' \rangle$ such that for all $a_i \in ABox$, $k_{\mathcal{M}'}(a_i) = k_j(a_i)$, $\mathcal{M}' \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}}$ TBox $\cup$ ABox $\cup\, \mu^j$ and $\mathcal{M}' \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C(a)$. We build a minimal canonical model $\mathcal{M} = \langle \Delta, <, I \rangle$ of K, minimally satisfying ABox and such that $C(a)$ does not hold in $\mathcal{M}$ as follows. Since we do not know whether $\mathcal{M}'$ is minimal or canonical, we cannot use it directly; rather, we only use it as a support to the construction of $\mathcal{M}$. In particular we use it for the following Δ_1 component of $\mathcal{M}$ concerning the individuals explicitly named in ABox. Let $\Delta = \Delta_1 \cup \Delta_2$ where $\Delta_1 = \{a_i : a_i$ in ABox $\}$ and $\Delta_2 = \{\{C_1, \ldots, C_k\} \subseteq \mathcal{S}$: $\{C_1, \ldots, C_k\}$ is maximal consistent with K and $\mathbf{T}$ does not occur in $\{C_1, \ldots, C_k\}\}$. Notice that Δ_2 is necessary to make the model canonical. We define the rank $k_{\mathcal{M}}$ of each domain element as follows: for Δ_1, $k_{\mathcal{M}}(a_i) = k_j(a_i)$, and for Δ_2, $k_{\mathcal{M}}(\{C_1, \ldots, C_k\}) = rank(C_1 \sqcap \ldots \sqcap C_k)$. We then define $<$ in the obvious way: $x < y$ if and only if $k_{\mathcal{M}}(x) < k_{\mathcal{M}}(y)$.

We then define I as follows. First, for all a_i in ABox we let $a_i^I = a_i$. For the interpretation of concepts we reason in two different ways for Δ_1 and Δ_2. For Δ_1, we use $\mathcal{M}'$: for all atomic concepts C', we let $a_i \in C'^I$ in $\mathcal{M}$ if $(a_i)^{I'} \in C'^{I'}$ in $\mathcal{M}'$. For Δ_2, for all atomic concepts C', we let $\{C_1, \ldots, C_k\} \in C'^I$ if and only if $C' \in \{C_1, \ldots, C_k\}$. I then extends to boolean combinations of concepts in the usual way.

In order to conclude the model's construction, for each role R, we define R^I as follows. For $a_i, a_j \in \Delta_1$, $(a_i, a_j) \in R^I$ if and only if $((a_i)^{I'}, (a_j)^{I'}) \in R^{I'}$ in $\mathcal{M}'$. For $X, Y \in \Delta_2$, $(X, Y) \in R^I$ if and only if $\{C' \colon \forall R.C' \in X\} \subseteq Y$.

For $a_i \in \Delta_1$, $X \in \Delta_2$, $(a_i, X) \in R^I$ if and only if there is an $x \in \Delta'$ of $\mathcal{M}'$ such that $(a_i^{I'}, x) \in R^{I'}$ in $\mathcal{M}'$ and, for all concepts C', we have $x \in C'^{I'}$ if and only if $X \in C'^I$.

I is extended to quantified concepts in the usual way.

By definition of R^I and of I, it follows that for all $X \in \Delta_2$, $X \in \forall R.C^I$ iff $\forall R.C \in X$. Also, by maximality and consistency of X, for all $X \in \Delta_2$, $X \in \exists R.C^I$ iff $\exists R.C \in X$, as can be easily verified. If $X \in \exists R.C^I$, then by what just stated, $\forall R.\neg C \notin X$, and by maximality of X, $\exists R.C \in X$. For the other direction, if $\exists R.C \in X$ then by consistency of X $\forall R.\neg C \notin X$, hence by what just stated, $X \notin \forall R.\neg C^I$, and therefore $X \in \exists R.C^I$. For $a_i \in \Delta_1$, it obviously holds that $a_i \in \forall R.C^I$ iff $a_i \in \forall R.C^{I'}$ in $\mathcal{M}'$.

We first consider the TBox. $\mathcal{M}$ satisfies TBox: for elements $a_i \in \Delta_1$, for the inclusion $C_l \sqsubseteq C_j \in$ TBox, if $\mathbf{T}$ does not occur in C_l this obviously follows from definition of I since it holds in $\mathcal{M}'$. For $\mathbf{T}(C_l) \sqsubseteq C_j$, for all a_i we reason as follows. First of all, if $k_j(a_i) > \text{rank}(C_l)$ then $a_i \notin Min_<(C_l{}^I)$ and the inclusion trivially holds. On the other side, if $k_j(a_i) = \text{rank}(C_l)$, $(\neg C_l \sqcup C_j)(a_i) \in \mu^j$, and therefore $(a_i)^{I'} \in (\neg C_l \sqcup C_j)^{I'}$ in $\mathcal{M}'$, hence $(a_i)^I \in (\neg C_l \sqcup C_j)^I$ in $\mathcal{M}$. Last, if $k_j(a_i) < \text{rank}(C_l)$, by Proposition 12 (for $\mathcal{M}'$) then $a_i \notin (C_l)^I$, and we are done.

For the elements $X \in \Delta_2$: let $C_l \sqsubseteq C_j \in$ TBox. If $X \notin (C_l)^I$ the property trivially holds. Let $X \in (C_l)^I$, i.e. $C_l \in X$. We show that $X \in (C_j)^I$. We consider two cases: either C_l is different from $\mathbf{T}(C')$ or C_l is $\mathbf{T}(C')$. Let us consider the first case. Suppose, for a contradiction, that $X \notin (C_j)^I$ and, hence, $C_j \notin X$. As $X = \{C_1, \dots, C_k\}$ is consistent with K, $K \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C_1 \sqcap \dots \sqcap C_n \sqsubseteq \bot$. As $C_j \notin X$ and X is maximal among the consistent sets of concepts in $\mathcal{S}$, $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C_1 \sqcap \dots \sqcap C_n \sqcap C_j \sqsubseteq \bot$. Therefore, $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C_1 \sqcap \dots \sqcap C_n \sqsubseteq \neg C_j$. But, from the fact that $C_l \sqsubseteq C_j \in$ TBox and $C_l \in X$, we get $K \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C_1 \sqcap \dots \sqcap C_n \sqsubseteq C_j$. A contradiction. Let us consider the case that C_l is $\mathbf{T}(C')$. Since $X \in (\mathbf{T}(C'))^I$ also $X \in C'^I$ and by inductive hypothesis $C' \in X$. We reason by contradiction: suppose $C_j \notin X$, hence $\neg C_j \in X$. Since $\mathbf{T}(C') \sqsubseteq C_j \in$ TBox, it can be easily verified that $rank(C' \sqcap \neg C_j) > rank(C')$. Consider an $Y \in \Delta_2$ s.t. $C' \in Y$ and $rank(Y) = rank(C')$ (by Proposition 13 this Y exists). Hence by definition of $k_\mathcal{M}$, $k_\mathcal{M}(X) > k_\mathcal{M}(Y) = k_\mathcal{M}(C)$, which contradicts the possibility that $X \in Min(C')^I$, and hence that $X \in (\mathbf{T}(C'))^I$. Also in this case we can conclude that $C_j \in X$. Notice that by what said above about quantified concepts, this also holds in case C_i or C_j are quantified.

Furthermore, $\mathcal{M}$ is a minimal canonical model: it is canonical by construction. It is minimal with respect to Definition 23: for all $X \in \Delta_2$, we have that $k_\mathcal{M}(X)$ is the lowest possible rank it can have in any model (by Proposition 13).

We now consider the ABox. $\mathcal{M}$ satisfies ABox by definition of I and since $\mathcal{M}'$ satisfies it. This is obvious for ABox assertions that do not contain the $\mathbf{T}$ operator. If $\mathbf{T}(C)(a_i) \in ABox$, then by the algorithm $k_j(a_i) = k_\mathcal{M}(a_i) = \text{rank}(C)$. By Proposition 13, and since $\mathcal{M}$ is minimal and canonical, we know that rank (C)= $k_\mathcal{M}(C)$, therefore $(a_i)^I \in Min_<(C^I)$ and $\mathcal{M}$ satisfies $\mathbf{T}(C)(a_i)$.

Last, $\mathcal{M}$ minimally satisfies ABox. This follows by minimality of k_j. Suppose for a contradiction that there is another canonical model $\mathcal{M}' = \langle \Delta', <', I' \rangle$ of K such that $\mathcal{M}' <_{ABox} \mathcal{M}$, for all a_i $k_{\mathcal{M}'}(a_i) \leq k_\mathcal{M}(a_i)$, and for at least one b, $k_{\mathcal{M}'}(b) < k_\mathcal{M}(b)$. Consider $k_{j'}$, the rank assignment corresponding to $\mathcal{M}'$ (s.t. for all $a_i \in ABox$, $k_{j'}(a_i) = k_{\mathcal{M}'}(a_i)^{I'}$). Clearly $k_{j'}$ threatens the minimality of k_j. Furthermore $\mathcal{M}' \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}}$ TBox $\cup$ ABox $\cup$ $\mu^{j'}$: it satisfies TBox $\cup$ ABox because it is a model of K. It satisfies $\mu^{j'}$: for the inclusions without the $\mathbf{T}$ operator this is obvious. Let $a_i \in ABox$, and let $\mathbf{T}(C) \sqsubseteq D$ with rank $C \geq k_{j'}(a_i)$. It clearly holds that $(a_i)^{I'} \in (\neg C \sqcup D)^{I'}$ in $\mathcal{M}'$: indeed if rank(C) > $k_{j'}(a_i)$, then by Proposition 13 $(a_i)^{I'} \in (\neg C)^{I'}$. On the other hand, if rank(C) = $k_{j'}(a_i)$ always by Proposition 13, $a_i \in min(C)^{I'}$, and since by hypothesis $\mathcal{M}'$ satisfies TBox, also $a_i \in (D)^{I'}$. However, if all this holds, this contradicts the hypothesis that k_j is a minimal consistent assignment. Therefore, TBox $\cup$ ABox $\cup$ $\mu^{j'}$ is consistent in $\mathcal{ALC} + \mathbf{T}_\mathsf{R}$, which contradicts the minimality of k_j. It follows that such

$\mathcal{M}'$ cannot exist, and therefore $\mathcal{M}$ minimally satisfies *ABox*.

Last, $C(a)$ does not hold in $\mathcal{M}$, since it does not hold in $\mathcal{M}'$.

We have then built a minimal canonical model of K minimally satisfying ABox in which $C(a)$ does not hold. The theorem follows by contraposition. □

Theorem 12 (Soundness of $\overline{ABox}$)**.** *Given K=(TBox, ABox), for each individual constant a in ABox, we have that if* $C(a) \in \overline{ABox}$ *then* $C(a)$ *holds in all minimal canonical models of K minimally satisfying ABox.*

Proof. Let $C(a) \in \overline{ABox}$, and suppose for a contradiction that there is a minimal canonical model $\mathcal{M}$ of K minimally satisfying ABox s.t. $C(a)$ does not hold in $\mathcal{M}$. Consider now the rank assignment k_j corresponding to $\mathcal{M}$ (such that $k_j(a_i) = k_{\mathcal{M}}(a_i)$). if $\mathbf{T}(C)(a_i) \in ABox$, then $k_j(a_i) = k_{\mathcal{M}}(C) = \text{rank}(C)$ (by Proposition 13). k_j is clearly minimal. Suppose it was not so, and there was a $k_{j'}$ such that for all a_i $k_{j'}(a_i) \leq k_j(a_i)$, and for some a_l, $k_{j'}(a_l) < k_j(a_l)$. By repeating the same construction in the proof of Theorem 11, there is a minimal canonical model $\mathcal{M}'$ of K minimally satisfying ABox such that $k_{j'}(a_i) = k_{\mathcal{M}'}(a_i)$, therefore $\mathcal{M}' <_{ABox} \mathcal{M}$, against the hypothesis of minimality of $\mathcal{M}$. Clearly $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \mu^j$. Indeed, for all a_i let $(\neg C \sqcup D)(a_i) \in \mu^j_i$. We distinguish two cases. If $(\neg C \sqcup D)(a_i)$ has been introduced in μ^j_i because of a $C \sqsubseteq D$ in *TBox*, clearly $a_i{}^I \in (\neg C \sqcup D)^I$. If $(\neg C \sqcup D)(a_i)$ has been introduced in μ^j_i because of a $\mathbf{T}(C) \sqsubseteq D$ in $\overline{TBox}$: if $a_i{}^I \in (\neg C)^I$ clearly $(\neg C \sqcup D)(a_i)$ holds in $\mathcal{M}$. On the other hand, if $a_i{}^I \in (C)^I$: by hypothesis $rank(C) = k_j(a_i)$ hence by the correspondence between rank of a formula in the rational closure and in minimal canonical models (see Proposition 13) also $k_{\mathcal{M}}(C) = k_{\mathcal{M}}(a_i{}^I)$, but since $a_i{}^I \in (C)^I$, $k_{\mathcal{M}}(C) = k_{\mathcal{M}}(a_i{}^I)$, therefore $a_i{}^I \in (\mathbf{T}(C))^I$. By definition of μ_i, and since by Theorem 8, $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} \overline{TBox}$, $D(a_i)$ holds in $\mathcal{M}$ and therefore also $a_i{}^I \in (\neg C \sqcup D)^I$. Furthermore by hypothesis $\mathcal{M} \models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}}$ ABox.

Since by hypothesis $\mathcal{M} \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C(a)$, it follows that TBox $\cup$ ABox $\cup\, \mu^j \not\models_{\mathcal{ALC}+\mathbf{T}_\mathsf{R}} C(a)$, and by definition of $\overline{ABox}$, $C(a) \notin \overline{ABox}$, against the hypothesis.

The theorem follows by contraposition. □

Let us conclude this section by estimating the complexity of computing the rational closure of the ABox:

Theorem 13 (Complexity of rational closure over the ABox)**.** *Given a knowledge base K =(TBox,ABox), an individual constant a and a concept C, the problem of deciding whether* $C(a) \in \overline{ABox}$ *is* ExpTime*-complete.*

Proof. Let $|K|$ be the size of the knowledge base K and let the size of the query be $O(|K|)$. As the number of inclusions in the knowledge base is $O(|K|)$, then the number n of non-increasing subsets E_i in the construction of the rational closure is $O(|K|)$. Moreover, the number k of named individuals in the knowledge base is $O(|K|)$. Hence, the number k^n of different rank assignments to individuals is such that both k and n are $O(|K|)$. Observe that $k^n = 2^{Log\, k^n} = 2^{nLog\, k}$. Hence, k^n is $O(2^{nk})$, with n and k linear in $|K|$, i.e., the number of different rank assignments is exponential in $|K|$.

To evaluate the complexity of the algorithm for computing the rational closure of the ABox, observe that:

(i) For each j, the number of sets μ_i^j is k (which is linear in $|K|$). The number of inclusions in each μ_i^j is $O(|K|^2)$, as the size of $\mathcal{S}$ is $O(|K|)$ and the number of $\mathbf{T}$-inclusions $\mathbf{T}(C) \sqsubseteq D \in \overline{TBox}$, with $C, D \in \mathcal{S}$ is $O(|K|^2)$, while the number of $\mathbf{T}$-inclusions $C \sqsubseteq D \in TBox$ is $O(|K|)$. Hence, the size of set μ^j is $O(|K|^3)$.
(ii) For each k_j, the consistency with ($\overline{TBox}$, ABox) can be verified by checking the consistency of TBox $\cup$ ABox $\cup\, \mu^j$ in $\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}$, which requires exponential time in the size of the set of formulas TBox $\cup$ ABox $\cup\, \mu^j$ (which, as we have seen, is polynomial in the size of K). Hence, the consistency of each k_j can be verified in exponential time in the size of K.
(iii) The identification of the minimal assignments k_j among the consistent ones requires the comparison of each consistent assignment with each other (i.e. k^{2n} comparisons), where each comparison between k_j and $k_{j'}$ requires k steps. Hence, the identification of the minimal assignments requires $k \times k^{2n}$ steps, i.e. a number of steps exponential in $|K|$.
(iv) To define the rational closure $\overline{ABox}$ of ABox, for each concept C occurring in K or in the query (there are $O(|K|)$ many concepts), and for each named individual a_i, we have to check if $C(a_i)$ is derivable in $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ from TBox $\cup$ ABox $\cup\, \mu^j$ for all minimal consistent rank assignments k_j. As the number of different minimal consistent assignments k_j is exponential in $|K|$, this requires an exponential number of checks, each one requiring exponential time in the size of the knowledge base $|K|$. The cost of the overall algorithm is therefore exponential in the size of the knowledge base. Completeness comes from the complexity of the underlying $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$, as stated in Theorem 5. □

4. Conclusions and Related works

In the first part of the paper we have provided a semantic reconstruction of the well known notion of propositional rational closure. We have provided two minimal model semantics, based on the idea that preferred rational models are those in which the rank of the worlds is minimized. We have then shown that when adding suitable possibility assumptions to a knowledge base, these two minimal model semantics correspond to rational closure.

The correspondence between the proposed minimal model semantics and rational closure suggests the possibility of defining variants of rational closure by varying the three ingredients underlying our approach, namely: (i) the properties of the preference relation $<$: for instance just preorder, or multi-linear or weakly-connected; (ii) the comparison relation on models: based for instance on the rank of the worlds or on the inclusion between the relations $<$, or on a special kind of formulas satisfied by a world, as in the logic P_{min} [**?**]; (iii) the choice between fixed or variable interpretations. The systems obtained by various combinations of the three ingredients are largely unexplored and may give rise to useful nonmonotonic logics.

In the second part of the paper we have defined a rational closure construction for the Description Logic $\mathcal{ALC}$ extended with a typicality operator and provided a minimal model semantics for it based on the idea of minimizing the rank of objects in the domain, that is their level of "untypicality". This semantics corresponds to a natural extension to DLs of Lehmann and Magidor's notion of rational closure. We have also extended the notion of rational closure to the ABox, by providing an algorithm for computing it that is sound and complete with respect to the minimal model semantics. Last, we have shown an EXPTIME upper bound for the algorithm. The work presented in this paper is an extension of the work in [**?**] and in [**?**].

In another direction, we aim to develop a generalization of the notion of rational closure introduced in this paper and of its minimal model semantics to deal with more expressive DLs and, in particular, with DLs which do not enjoy the finite model property, such as $\mathcal{ALCOIQ}$ and $\mathcal{SHOIQ}$, for which the notion of canonical model as introduced in this paper appears to be too strong.

As far as rational closure is concerned, it is worth noticing that rational closure for Description Logics inherits both the virtues and the weakness of propositional rational closure. We have already said about the strengths, among which there are the good computational properties. For what concerns the weaknesses, rational closure does not allow to separately reason about the inheritance of different properties. For instance, in the classical birds and penguins example, rational closure does not allow to reason in this way: penguins inherit all typical properties of birds, except those for which we know they are an exception (as the property of flying). On the contrary, once penguins are recognized as non typical birds, no inheritance of typical properties is possible. In order to solve this problem, a strengthening of a rational closure-like algorithm with defeasible inheritance networks has been studied by [**?**].

In future work, we aim to explore possible strengthening of the notion of rational closure at the semantic level, to overcome the weaknesses mentioned above. One possible direction we briefly discuss here, could be to "relativize" the notion of typicality enforced by the semantics. In order to achieve this, we aim to refine the semantics by considering models equipped with multiple preference relations, whence with multiple "typicality" operators. In this variant, it should be possible to distinguish different aspects of typicality/exceptionality and consequently to avoid the "all or nothing" behavior of rational closure with respect to property inheritance. For the time being, we just notice that in order to make this variant interesting and meaningful, one should deal with issues like: what does differentiate one preference relation from another? What are the dependencies between different preference relations? Can different preference relations or (syntactically) different typicality operators be combined? All these issues require a suitable analysis/understanding which is preliminary to the technical development. Furthermore, one should also study an algorithmic counterpart of this semantics, that is to say, a suitable reformulation of the rational closure mechanism, with the hope of keeping a reasonable complexity.

In [**? ?**] nonmonotonic extensions of DLs based on the **T** operator have been proposed. In these extensions, the semantics of **T** is based on preferential logic P. Nonmonotonic inference is obtained by restricting entailment to *minimal models*, where minimal models are those that minimize the truth of formulas of a special kind. In this work, we have presented an alternative approach. First, the semantics underlying the **T** is R. Moreover and more importantly, we have adopted a minimal model semantics, where, as a difference with the previous approach, the notion of minimal model is completely independent from the language and is determined only by the relational structure of models.

Casini and Straccia in [?] develop a notion of rational closure for DLs. They propose a construction to compute the rational closure of an $\mathcal{ALC}$ knowledge base, which is not directly based on Lehmann and Magidor definition of rational closure [?], but is similar to the construction of rational closure proposed by Freund in [?] for the propositional calculus. [?] keeps the ABox into account, and defines closure operations over individuals. It introduces a consequence relation $\Vdash$ among a knowledge base K and assertions, under the requirement that the TBox is unfoldable and the ABox is closed under completion rules, such as, for instance, that if $a : \exists R.C \in$ ABox, then both aRb and $b : C$ (for some individual constant b) must belong to the ABox, too. Under such restrictions, a procedure is defined to compute the rational closure of the ABox, assuming that the individuals explicitly named are linearly ordered, and different orders determine different sets of consequences. The authors show that, for each order s, the consequence relation $\Vdash_s$ is rational and can be computed in PSPACE. In a subsequent work [? ?], the authors introduce an approach based on the combination of rational closure and *Defeasible Inheritance Networks* (INs). The authors first develop their approach at a propositional level, then they extend it to DLs, addressing both TBox and ABox reasoning. The resulting construction is a nonmonotonic mechanism enjoying the logical properties of rational entailment, but not suffering from the "all-or- nothing" behavior with respect to inheritance of defeasible properties. The nonmonotonic mechanism proposed by the authors corresponds to an algorithm to compute inferences, however, as a difference with our proposal, no declarative characterization of those inferences is provided. To overcome the limitations of rational closure, in [?] Casini and Straccia also define a notion of lexicographic closure for $\mathcal{ALC}$.

In [?] a semantic characterization of a variant of the notion of rational closure introduced in [?] has been presented, which is based on a generalization to $\mathcal{ALC}$ of our semantics in [?]. In [?], defeasible subsumption statements have the form $C \sqsubsim D$ and typicality assertions are not allowed in the ABox, which is defined as a standard $\mathcal{ALC}$ ABox. As we have seen, in this paper the presence of typicality assertions in the ABox may force some typicality inclusion not to hold, which is similar to allowing negated conditionals in KLM logics. While the minimal model semantics naturally deals with the presence of typicality assertions, the presence of typicality assertions in the ABox has to be taken into account, as we have done, in the definition of rational closure of the TBox and of the ABox.

A further difference of our construction with those in [? ?] is in the notion of exceptionality: our definition of exceptionality exploits preferential entailment, while [? ?] directly use entailment in $\mathcal{ALC}$ over a materialization of the knowledge base. We have seen in Section 3.2 that we cannot replace entailment in $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ by entailment in $\mathcal{ALC}$ over a materialization of the knowledge base. However, when typicality assertions are not allowed in the ABox, our notion of rational closure for TBox can be computed in $\mathcal{ALC}$ by defining a linear encoding of $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ entailment into $\mathcal{ALC}$ (the encoding is exactly the same as the one provided in [?] for encoding of $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ entailment into $\mathcal{SHIQ}$).

A related approach can be found in [?]. The basic idea of their semantics for the propositional case is similar to ours: to consider models of the K where the rank of each world is as small as possible. This idea has its roots in the work by Pearl [?]

and by Lehmann and Magidor [**?**]. The construction of [**?**] differs from ours as the very notion of model is different (although equivalent): a model is a sequence of sets of "atoms" (conjunctions of literals for every propositional variable). Each set of the sequence represents a set of worlds with the same ranking. A unique model of the rational closure is then defined by considering all models of the K and by taking for each level, starting from the bottom one, the union of the worlds (not already considered) at that level. This construction corresponds to building a model where each world has a minimal rank. In contrast, we proceed in a different way: our semantics is defined in terms of standard Kripke models where the rank is given by the preference (or accessibility) relation, and models of the rational closure are defined as the minimal ones with respect to a comparison relation on models. Our presentation is then more abstract and declarative than the one proposed in [**?**], whilst theirs is more "operational", as it relies on a specific representation of models and it provides a recipe to build a model of the rational closure, rather than a characterization of its properties.

The logic $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ we consider as our base language is equivalent to the logic for defeasible subsumptions in DLs proposed by [**?**]. At a syntactic level the two logics differ, so that in [**?**] one finds the defeasible inclusions $C \sqsubsim D$ instead of $\mathbf{T}(C) \sqsubseteq D$ of $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$, however it has be shown in [**?**] that the logic of defeasible subsumption can be translated into $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ by replacing $C \sqsubsim D$ with $\mathbf{T}(C) \sqsubseteq D$.

In [**?**] the semantics of the logic of defeasible subsumptions is strengthened by a preferential semantics. Intuitively, given a TBox, the authors first introduce a preference ordering $\ll$ on the class of all subsumption relations $\sqsubsim$ including TBox, then they define the rational closure of TBox as the most preferred relation $\sqsubsim$ with respect to $\ll$, i.e. such that there is no other relation $\sqsubsim'$ such that TBox $\subseteq \sqsubsim'$ and $\sqsubsim' \ll \sqsubsim$. Furthermore, the authors describe an EXPTIME algorithm in order to compute the rational closure of a given TBox. [**?**] does not address the problem of dealing with the ABox. In [**?**] a plug-in for the Protégé ontology editor implementing the mentioned algorithm for computing the rational closure for a TBox for OWL ontologies is described.

Recent works discuss the combination of open and closed world reasoning in DLs. In particular, formalisms have been defined for combining DLs with logic programming rules (see, for instance, [**?**] and [**?**]). A grounded circumscription approach for DLs with local closed world capabilities has been defined in [**?**].

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References

Appendix: an alternative semantics for $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$

An alternative semantic characterization of $\mathbf{T}$ can be given by means of a set of postulates that are essentially a reformulation of axioms and rules of nonmonotonic entailment in rational logic R: in this respect, the $\mathbf{T}$-assertion $\mathbf{T}(C) \sqsubseteq D$ is equivalent to the conditional assertion $C \mathrel{|\!\sim} D$ in R[5]. Given a domain Δ and a valuation function I, one can define the function $f_{\mathbf{T}}(S)$ for $S \subseteq \Delta$ that selects the *typical* instances of S, and in case $S = C^I$ for a concept C, it selects the typical instances of C. In this semantics, we define $(\mathbf{T}(C))^I = f_{\mathbf{T}}(C^I)$, and $f_{\mathbf{T}}$ has the intuitive properties for all subsets S of Δ of Definition 28 below:

Definition 28 (Semantics of T with selection function). A model is any structure

$$\langle \Delta, f_{\mathbf{T}}, I \rangle$$

where:

- Δ is the domain;
- $f_{\mathbf{T}} : Pow(\Delta) \longmapsto Pow(\Delta)$ is a function satisfying the following properties (given $S \subseteq \Delta$):

  $(f_{\mathbf{T}} - 1)$ $f_{\mathbf{T}}(S) \subseteq S$
  $(f_{\mathbf{T}} - 2)$ if $S \neq \emptyset$, then also $f_{\mathbf{T}}(S) \neq \emptyset$
  $(f_{\mathbf{T}} - 3)$ if $f_{\mathbf{T}}(S) \subseteq R$, then $f_{\mathbf{T}}(S) = f_{\mathbf{T}}(S \cap R)$
  $(f_{\mathbf{T}} - 4)$ $f_{\mathbf{T}}(\bigcup S_i) \subseteq \bigcup f_{\mathbf{T}}(S_i)$
  $(f_{\mathbf{T}} - 5)$ $\bigcap f_{\mathbf{T}}(S_i) \subseteq f_{\mathbf{T}}(\bigcup S_i)$
  $(f_{\mathbf{T}} - \mathsf{R})$ if $f_{\mathbf{T}}(S) \cap R \neq \emptyset$, then $f_{\mathbf{T}}(S \cap R) \subseteq f_{\mathbf{T}}(S)$

- I is the extension function that maps each extended concept C to $C^I \subseteq \Delta$, and each role R to $R^I \subseteq \Delta \times \Delta$ as follows:
  - I maps each role $R \in \mathcal{R}$ to its extension R^I;
  - I maps each atomic concept $A \in \mathcal{C}$ to its extension A^I;
  - I is extended to complex concepts in the usual way for constructors in $\mathcal{ALC}$, whereas for $(\mathbf{T}(C))$ is as follows:
    - $(\mathbf{T}(C))^I = f_{\mathbf{T}}(C^I)$

$(f_{\mathbf{T}} - 1)$ enforces that typical elements of S belong to S. $(f_{\mathbf{T}} - 2)$ enforces that if there are elements in S, then there are also *typical* such elements. $(f_{\mathbf{T}} - 3)$ expresses a weak form of monotonicity, namely *cautious monotonicity*. The next properties constraint the behavior of $f_{\mathbf{T}}$ with respect to $\cap$ and $\cup$ in such a way that they do not entail monotonicity. Last, $(f_{\mathbf{T}} - \mathsf{R})$ corresponds to rational monotonicity, and forces again a form

[5] This can be easily proven given Proposition 5.1 of [**?**] that shows the equivalence between the weaker logic $\mathcal{ALC} + \mathbf{T}$ in which $f_{\mathbf{T}}$ satisfies $(f_{\mathbf{T}} - 1) - (f_{\mathbf{T}} - 5)$ above but does not satisfy $(f_{\mathbf{T}} - \mathsf{R})$ and the KLM logic P which is weaker than R.

of monotonicity: if there is a typical S having the property R, then all typical S-and-Rs inherit the properties of typical Ss.

The following representation theorem shows that the above semantics for $\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}$ in Definition 28 is equivalent to the one in Definition 13.

First of all, we need to recall Lemma 2.1 in [**?**]:

Lemma 1 (Lemma 2.1 in [**?**], page 5)**.** *If $f_{\mathbf{T}}$ satisfies $(f_{\mathbf{T}}-1)-(f_{\mathbf{T}}-5)$, then $f_{\mathbf{T}}(S \cup R) \cap S \subseteq f_{\mathbf{T}}(S)$.*

Now we are able to prove the representation theorem:

Theorem 14. *A knowledge base is satisfiable in an $\mathcal{ALC}+\mathbf{T}_{\mathsf{R}}$ model described in Definition 13 if and only if it is satisfiable in a model $\mathcal{M} = \langle \Delta, f_{\mathbf{T}}, I \rangle$ where $f_{\mathbf{T}}$ satisfies $(f_{\mathbf{T}}-1)-(f_{\mathbf{T}}-5)$ and $(f_{\mathbf{T}}-\mathsf{R})$, and $(\mathbf{T}(C))^I = f_{\mathbf{T}}(C^I)$.*

Proof. Here we only consider the property $(f_{\mathbf{T}}-\mathsf{R})$. For the other properties, we refer to the proof of the Representation Theorem for $\mathcal{ALC}+\mathbf{T}$, as presented in [**?**], Theorem 2.1, page 5. The *only if* direction is trivial and left to the reader. For the *if* direction, as in [**?**], we define the $<$ relation as follows:

- for all $x, y \in \Delta$, we let $x < y$ if $\forall S \subseteq \Delta$, if $y \in f_{\mathbf{T}}(S)$, then (a) $x \notin S$ and (b) $\exists R \subseteq \Delta$ such that $S \subset R$ and $x \in f_{\mathbf{T}}(R)$.

Notice that given $(f_{\mathbf{T}}-\mathsf{R})$, this condition is equivalent to the simplified condition that only contains (a). Indeed, if (a) holds, it follows that also (b) holds. To be convinced, take any S such that $y \in f_{\mathbf{T}}(S)$, and $x \notin S$. We show that $x \in f_{\mathbf{T}}(S \cup \{x\})$, hence (b) holds. For a contradiction, suppose $x \notin f_{\mathbf{T}}(S \cup \{x\})$, then by $(f_{\mathbf{T}}-1)$ and $(f_{\mathbf{T}}-2)$, $f_{\mathbf{T}}(S \cup \{x\}) \cap S \neq \emptyset$, and by $(f_{\mathbf{T}}-\mathsf{R})$, $f_{\mathbf{T}}(S) = f_{\mathbf{T}}((S \cup \{x\}) \cap S) \subseteq f_{\mathbf{T}}(S \cup \{x\})$. Hence, $y \in f_{\mathbf{T}}(S \cup \{x\})$, which contradicts (a), given that $x \in S \cup \{x\}$. Therefore, we will consider the simplified definition of $<$:

- for all $x, y \in \Delta$, we let $x < y$ if $\forall S \subseteq \Delta$, if $y \in f_{\mathbf{T}}(S)$, then $x \notin S$.

We then show that if $f_{\mathbf{T}}$ satisfies $(f_{\mathbf{T}}-\mathsf{R})$, then $<$ is modular. Let $x < y$. Consider z and suppose $z \not< y$. This means that there is R such that $y \in f_{\mathbf{T}}(R)$, and $z \in R$ We reason as follows. First, notice that by Lemma 1, $y \in f_{\mathbf{T}}(\{y, z\})$ (given that $y, z \in R$, $y \in f_{\mathbf{T}}(R \cup \{y, z\}) \cap \{y, z\}$, hence $y \in f_{\mathbf{T}}(\{y, z\})$). In order to show that $<$ is modular, we want to show that $x < z$. For a contradiction, suppose that $x \not< z$. Then there is Z such that $z \in f_{\mathbf{T}}(Z)$ and $x \in Z$. Consider $Z \cup \{y, z\}$, by $(f_{\mathbf{T}}-1)$, $f_{\mathbf{T}}(Z \cup \{y, z\}) \subseteq Z \cup \{y, z\}$, and by $(f_{\mathbf{T}}-2)$, $f_{\mathbf{T}}(Z \cup \{y, z\}) \neq \emptyset$. Hence, either $f_{\mathbf{T}}(Z \cup \{y, z\}) \cap Z \neq \emptyset$ or $f_{\mathbf{T}}(Z \cup \{y, z\}) \cap Z = \emptyset$, and $f_{\mathbf{T}}(Z \cup \{y, z\}) \cap \{y, z\} \neq \emptyset$. In the last case, $y \in f_{\mathbf{T}}(Z \cup \{y, z\})$. In the first case, by $(f_{\mathbf{T}}-\mathsf{R})$, $f_{\mathbf{T}}(Z) = f_{\mathbf{T}}((Z \cup \{y, z\}) \cap Z) \subseteq f_{\mathbf{T}}(Z \cup \{y, z\})$, hence $z \in f_{\mathbf{T}}(Z \cup \{y, z\})$. From this, we derive that $f_{\mathbf{T}}(Z \cup \{y, z\}) \cap \{y, z\} \neq \emptyset$, hence, by $(f_{\mathbf{T}}-\mathsf{R})$, $f_{\mathbf{T}}(\{y, z\}) = f_{\mathbf{T}}((Z \cup \{y, z\}) \cap \{y, z\}) \subseteq f_{\mathbf{T}}(Z \cup \{y, z\})$, and $y \in f_{\mathbf{T}}(Z \cup \{y, z\})$. In both cases, we have that $y \in f_{\mathbf{T}}(Z \cup \{y, z\})$, however this is impossible, given that $x \in Z \cup \{y, z\}$ and $x < y$. We therefore conclude that if $z \not< y$, then $x < z$, hence modularity holds. $\square$