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Matrix representations of the real numbers

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Abstract

The main purpose of this paper is to determine all matrix representations of the real numbers. It is shown that every such representation is completely reducible, while all non-trivial irreducible representations must be of 2-dimensional and can be expressed in a unique form. It is found that those representations are essentially determined by the ways of embedding the real numbers into the complex numbers. This results in a one-to-one correspondence between the equivalent classes of irreducible representations and the equivalent classes of homomorphisms from the real number field to the complex number field. The matrix representations of the complex numbers are also determined.

MSC: 15A04, 12D99

Keywords: Matrix representation; real number field; complex number field, discontinuous homomorphisms.

1 Introduction and main theorem

The main purpose of this paper is to answer the question: for an arbitrary positive integer n, in how many different ways the real numbers can be represented by $n \times n$ real matrices such that the two basic operations, addition and multiplication, are preserved. In other words, we aim to determine all matrix representations of $\mathbb{R}$. Here a representation means a ring homomorphism between $\mathbb{R}$ and the ring of $n \times n$ matrices $\mathtt{M}_n(\mathbb{R})$ for an arbitrary $n \in \mathbb{N}$. For example, let $\rho : \mathbb{R} \to \mathtt{M}_2(\mathbb{R})$ be a matrix representation of $\mathbb{R}$. The restriction of ρ to $\mathbb{R}^+$, which is the multiplicative subgroup of $\mathbb{R}$ consisting of all positive real numbers, induces a group representation of $\mathbb{R}^+$. Note that $\mathbb{R}^+$ is a locally compact abelian group, of which all continuous representations are well known (cf. [5]). Hence if ρ is continuous, from the restricted group representation of $\mathbb{R}^+$ it is easy to deduce that up to conjugacy ρ must be of the form

$$\rho(a) = \begin{pmatrix} \tau(a) & 0 \\ 0 & \tau'(a) \end{pmatrix},\ \forall a \in \mathbb{R}$$

where τ and τ' are the zero map or the identity map of $\mathbb{R}$. Yet if discontinuity is granted, there exists a huge family of representations of form $\rho_\sigma : \mathbb{R} \to \mathtt{M}_2(\mathbb{R})$ defined by

$$\rho_\sigma(a) = \begin{pmatrix} \sigma(a)_r & -\sigma(a)_i \\ \sigma(a)_i & \sigma(a)_r \end{pmatrix}, \ \forall a \in \mathbb{R} \tag{1}$$

where σ is a discontinuous homomorphism from $\mathbb{R}$ to $\mathbb{C}$ while $\sigma(a)_r$ and $\sigma(a)_i$ are real and imaginary parts of $\sigma(a)$ respectively. The cardinality of this family of representations is $2^{\mathfrak{c}}$ where $\mathfrak{c}$ is the cardinality of $\mathbb{C}$ (cf. Corollary 3.1). We will see that up to equivalence the above representations actually provide a complete list for all two-dimensional representations of $\mathbb{R}$, which is a consequence of our main theorem.

In order to state our main results we recall some elementary notions. Let $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$ be a matrix representation of $\mathbb{R}$ for an arbitrary $n \in \mathbb{N}$. By identifying $\mathtt{M}_n(\mathbb{R})$ with the $\mathbb{R}$-linear endomorphisms $\mathtt{End}_{\mathbb{R}}(\mathbb{R}^n)$, the image $\rho(\mathbb{R})$ of $\mathbb{R}$ under ρ can be considered as a subset of linear transformations over the vector space $\mathbb{R}^n$. Then ρ is called irreducible if there is no non-trivial proper subspace of $\mathbb{R}^n$ stabilized by $\rho(\mathbb{R})$. A matrix representation of $\mathbb{R}$ is called completely reducible if it is a direct sum of some irreducible representations. Two n-dimensional representations ρ and ρ' are said equivalent if there exists a non-singular matrix $X \in \mathtt{M}_n(\mathbb{R})$ such that

$$\rho(a) = X\rho'(a)X^{-1}, \ \forall a \in \mathbb{R}.$$

We denote by ι_X the inner automorphism of $\mathtt{M}_n(\mathbb{R})$ via the conjugation by X.

Denote by $\mathtt{Hom}(\mathbb{R}, \mathbb{C})$ the set of ring homomorphisms from $\mathbb{R}$ to $\mathbb{C}$. The cardinality of this set is infinite (cf. [3] and [7]). Two homomorphisms $\sigma, \tau \in \mathtt{Hom}(\mathbb{R}, \mathbb{C})$ are said equivalent if $\sigma = \alpha\tau$, where α is either the identity map or the complex conjugation of $\mathbb{C}$. This is obviously an equivalent relation of $\mathtt{Hom}(\mathbb{R}, \mathbb{C})$. For convenience we call the the zero map and the identity map of $\mathbb{R}$ the trivial representations of $\mathbb{R}$.

Our main result is as follows.

Theorem 1. *Let $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$ be a matrix representation of $\mathbb{R}$ where $n \in \mathbb{N}$.*

1. *The representation ρ is completely reducible.*
2. *If ρ is irreducible and non-trivial, then $n = 2$ and there exist a homomorphism $\sigma \in \mathtt{Hom}(\mathbb{R}, \mathbb{C})$ and a non-singular matrix $X \in \mathtt{M}_2(\mathbb{R})$ such that*

$$\rho = \iota_X \cdot \rho_\sigma \tag{2}$$

   where ρ_σ is defined by the identity (1).

3. *The above σ is unique up to equivalence. More over, there exists a one-to-one correspondence between the equivalent classes of irreducible representations of $\mathbb{R}$ and the equivalent classes of $\mathtt{Hom}(\mathbb{R}, \mathbb{C})$*

The proof of this theorem is developed in the following sections. Mean while, from this theorem we determine also the matrix representations for the complex numbers (cf. Theorem 2).

2 Preliminaries

Throughout this paper for a representation we always mean a real matrix representation unless otherwise explained. Let $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$ be a representation. Then by identifying $\mathtt{M}_n(\mathbb{R})$ with $\mathtt{End}_{\mathbb{R}}(\mathbb{R}^n)$, $\mathbb{R}^n$ is a $\mathbb{R}$-$\rho(\mathbb{R})$ bimodule. Equivalently, if we denote by R_ρ the subring of $\mathtt{M}_n(\mathbb{R})$ generated by $\mathbb{R}I$ and $\rho(\mathbb{R})$ where I is the $n \times n$ identity matrix, then $\mathbb{R}^n$ is a R_ρ-module in natural way. The following property is obvious.

Proposition 2.1. *A representation $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$ is irreducible if and only if $\mathbb{R}^n$ is an irreducible R_ρ-module.*

Recall that the radical of a module M is by definition the intersection of all maximal submodules of M. Given a representation $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$, we denote by $\mathtt{Rad}_\rho(\mathbb{R}^n)$ the radical of the R_ρ-module $\mathbb{R}^n$.

Proposition 2.2. *Let $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$ be a representation. The following statements are equivalent.*

1. *ρ is completely reducible.*
2. *$\mathbb{R}^n$ is a semisimple R_ρ-module.*
3. $\mathtt{Rad}_\rho(\mathbb{R}^n) = \{0\}$.

Proof. The equivalence of the the first two statements are obvious. The equivalence of the last two statements comes from the fact that, since $\mathbb{R}^n$ is an artinian module, it is semisimple if and only if its radical is trivial (cf. [1, p. 129]). $\square$

We need some elementary properties from linear algebra. Recall that an element $x \in \mathtt{M}_n(\mathbb{R})$ is nilpotent if $x^m = 0$ for some $m \in \mathbb{N}$ and that x is unipotent if $x - I$ is nilpotent, where I is the identity matrix. An element $x \in \mathtt{M}_n(\mathbb{R})$ is semisimple if it is conjugated to a diagonal matrix in $\mathtt{M}_n(\mathbb{C})$.

Lemma 2.1. *(Jordan-Chevalley decomposition) Let x be a non-zero element of $\mathtt{M}_n(\mathbb{R})$.*

1. *There exist a unique semisimple element $x_s \in \mathtt{M}_n(\mathbb{R})$ and a unique nilpotent element $x_n \in \mathtt{M}_n(\mathbb{R})$ such that $x = x_s + x_n$ and $x_s x_n = x_n x_s$. More over, there exist polynomials without constant term $f(t), g(t) \in \mathbb{R}[t]$ such that $x_s = f(x)$ and $x_n = g(x)$.*
2. *If x is non-singular, there exists a unique unipotent element $x_u \in \mathtt{M}_n(\mathbb{R})$ such that $x = x_s x_u$ and $x_s x_u = x_u x_s$. More over, every subspace of $\mathbb{R}^n$ stabilized by x is also stabilized by x_u.*

The elements x_s, x_n and x_u are called semisimple, nilpotent and unipotent parts of x respectively. We refer to [2, p. 80] for more details of the Jordan-Chevalley decompositions.

Lemma 2.2. *Let $\{x_1, x_2, ..., x_m\}$ be a commutative subset of $\mathtt{M}_n(\mathbb{R})$ and let $x_{i,s}$ be the semisimple part of x_i for $1 \leq i \leq m$. Then an element $\sum_{i=1}^{m} a_i x_i \in \mathtt{M}_n(\mathbb{R})$, where $a_i \in \mathbb{R}$ for $1 \leq i \leq m$, is nilpotent if and only if*

$$\sum_{i=1}^{m} a_i x_{i,s} = 0$$

Proof. It follows from the polynomial property of Jordan decomposition that $\{x_{1,s}, x_{2,s}, ..., x_{m,s}\}$ is a commutative set since so is $\{x_1, x_2, ..., x_m\}$, therefore $\{a_1 x_{1,s}, a_2 x_{2,s}, ..., a_m x_{m,s}\}$ is also a commutative set. Note that the sum and the multiplication of a finite number of commutative semisimple elements are still semisimple. Then $a_i x_{i,s}$ is semisimple for all $1 \leq i \leq m$ and therefore $\sum_{i=1}^{m} a_i x_{i,s}$ is also semisimple. Let $x_{i,n}$ be the nilpotent part of x_i for all $1 \leq i \leq m$. Then $\{x_{1,n}, x_{2,n}, ..., x_{m,n}\}$ is a commutative set since so is $\{x_1, x_2, ..., x_m\}$. Since $a_i x_{i,n}$ is nilpotent for all $1 \leq i \leq m$, $\sum_{i=1}^{m} a_i x_{i,n}$ is also nilpotent. We have

$$\sum_{i=1}^{m} a_i x_i = \sum_{i=1}^{m} a_i (x_{i,s} + x_{i,n}) = \sum_{i=1}^{m} a_i x_{i,s} + \sum_{i=1}^{m} a_i x_{i,n},$$

meanwhile obviously $\sum_{i=1}^{m} a_i x_{i,s}$ and $\sum_{i=1}^{m} a_i x_{i,n}$ commute with each other. Then the uniqueness of Jordan decomposition implies that

$$(\sum_{i=1}^{m} a_i x_i)_s = \sum_{i=1}^{m} a_i x_{i,s}.$$

Consequently the lemma follows from the fact that an element of $\mathtt{M}_n(\mathbb{R})$ is nilpotent if and only if the semisimple part of the element vanishes. □

Lemma 2.3. *let G be a commutative subgroup of $\mathtt{GL}_n(\mathbb{R})$ and G_u be the set of the unipotent parts of all $x \in G$. If V is a subspace of $\mathbb{R}^n$ stabilized by G, then V is also stabilized by G_u. More over, there exists a non-zero vector $v \in V$ which is fixed by every element of G_u.*

Proof. It is a consequence of lemma 2.1 that each subspace $V \subseteq \mathbb{R}^n$ stabilized by G must be also stabilized by G_u. More over, since G is commutative, it is easy to check that G_u is a unipotent subgroup of $\mathtt{GL}_n(\mathbb{R})$. Then the existence of a non-trivial vector of V fixed by G_u is a consequence of Lie-Kolchin Theorem (cf. [2, p. 87]). □

Lemma 2.4. *Given a representation $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$ and a non-zero element $v \in \mathbb{R}^n$, let*

$$R_\rho v = \{xv \in \mathbb{R}^n | \forall x \in R_\rho\}$$

which is a R_ρ-module. Then the radical $\mathtt{Rad}(R_\rho v)$ of $R_\rho v$ is trivial.

Proof. Note that the set of non-zero elements of $\rho(\mathbb{R})$, denoted by $\rho(\mathbb{R})^*$, is a commutative multiplicative subgroup of $\mathtt{GL}_n(\mathbb{R})$, which obviously stabilizes the subspace $R_\rho v$ of $\mathbb{R}^n$. Let $\rho(\mathbb{R})_u$ be the group consisting of the unipotent parts of all elements of $\rho(\mathbb{R})^*$. Then by Lemma 2.3, $R_\rho v$ is also stabilized by $\rho(\mathbb{R})_u$. Again it follows from Lemma 2.3 that there exists a non-zero element $v_0 \in R_\rho v$ such that

$$x_u v_0 = v_0,\ \forall x_u \in \rho(\mathbb{R})_u. \tag{3}$$

Let $R_\rho v_0$ be the submodule of $R_\rho v$ generated by v_0. We claim that

$$R_\rho v_0 = R_\rho v. \tag{4}$$

In fact, consider $R_\rho v$ as a $\mathbb{R}$-space and let α be a non-singular $\mathbb{R}$-linear transformation of $R_\rho v$ such that $\alpha(v) = v_0$. Since v_0 belongs to $R_\rho v$, we have $v_0 = yv$ for an element $y \in R_\rho$. Let $\lambda_\alpha : R_\rho v \to R_\rho v$ be a map defined by

$$\lambda_\alpha(xv) = yxv,\ \forall x \in R_\rho.$$

Clearly λ_α is an endomorphism of R_ρ-module since R_ρ is a commutative ring. More over, since

$$yxv = xyv = xv_0,\ \forall x \in R_\rho$$

we have $\lambda_\alpha(R_\rho v) = R_\rho v_0$. Since α^{-1} is also a $\mathbb{R}$-linear transformation of $R_\rho v$, in a similar way we define an endomorphism $\lambda_{\alpha^{-1}} \in \mathtt{End}_{R_\rho}(R_\rho v)$. It is easy to check that $\lambda_{\alpha^{-1}}$ is the inverse of λ_α. Hence λ_α is an automorphism of $R_\rho v$. Then the identity (4) holds because

$$R_\rho v = \lambda_\alpha(R_\rho v) = R_\rho v_0.$$

Now we show that the radical $\mathtt{Rad}(R_\rho v_0)$ of the module $R_\rho v_0$ is trivial. Let J be the Jacobson radical of the ring R_ρ. Since $R_\rho v$ is a finitely generated module, we have (cf. [1, p. 172])

$$\mathtt{Rad}(R_\rho v_0) = J \cdot R_\rho v_0 = R_\rho J v_0 = J v_0. \tag{5}$$

For an arbitrary $x \in J$, we can write

$$x = \sum_{i=1}^{m} a_i x_i,\ \mathtt{for}\ a_i \in \mathbb{R},\ x_i \in \rho(\mathbb{R}),\ 1 \leq i \leq m.$$

Let $x_{i,s}$ and $x_{i,u}$ be the semisimple and the unipotent parts of x_i respectively. Then $x_{i,u} \in \rho(\mathbb{R})_u$ for all $1 \leq i \leq m$. We have by Lemma 2.1 and the identity (3) that

$$xv_0 = \sum_{i=1}^{m} a_i x_{i,s} x_{i,u} v_0 = \sum_{i=1}^{m} a_i x_{i,s} v_0$$

Note that all elements of J are nilpotent since R_ρ is finitely generated. It follows from Lemma 2.2 and the nilpotency of $x \in J$ that

$$\sum_{i=1}^{m} a_i x_{i,s} = 0.$$

Hence

$$xv_0 = 0, \ \forall x \in J.$$

This implies by the identities (4) and (5) that

$$\mathtt{Rad}(R_\rho v) = \mathtt{Rad}(R_\rho v_0) = \{0\}.$$

□

Lemma 2.5. *Let R be a $\mathbb{R}$-subalgebra of $\mathtt{M}_n(\mathbb{R})$. If R is an integral domain, then it is a division ring.*

Proof. We show that every non-zero element $s \in R$ has an inverse in R. Denote by $\mathbb{R}[s]$ the subalgebra of R generated by $\mathbb{R}I$ and s. Note that s is algebraic over $\mathbb{R}$ because

$$\mathtt{dim}_{\mathbb{R}}\mathbb{R}[s] \leq \mathtt{dim}_{\mathbb{R}}\mathtt{M}_n(\mathbb{R}) < \infty.$$

Let $f(x) \in \mathbb{R}[x]$ be minimal polynomial of s over $\mathbb{R}$ and denote by $(f(x))$ the ideal of $\mathbb{R}[x]$ generated by $f(x)$. Then we have an isomorphism

$$\mathbb{R}[s] \cong \mathbb{R}[x]/(f(x)).$$

We claim that $f(x)$ is irreducible, which implies by above isomorphism that $\mathbb{R}[s]$ is a field and consequently $s^{-1} \in \mathbb{R}[s] \subseteq R$. In fact if otherwise there exist non-constant polynomials $g(x), h(x) \in \mathbb{R}[x]$ such that $f(x) = g(x)h(x)$. Then

$$g(s)h(s) = 0.$$

However, since both $g(x)$ and $h(x)$ have degree strictly less than that of $f(x)$,

$$g(s) \neq 0, \ h(s) \neq 0.$$

This is contrary to the assumption that R is an integral domain. Thus we obtain that R is a division ring. □

Lemma 2.6. *Let $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$ be an irreducible representation. Then R_ρ is a field. In particular, if ρ is non-trivial, there exists an isomorphism $\beta : \mathbb{C} \to R_\rho$ such that $\beta(a) = aI$ for all $a \in \mathbb{R}$.*

Proof. Since $\mathbb{R}^n$ is an irreducible R_ρ-module, it comes from Proposition 2.1 and Schur's lemma that $\mathtt{End}_{R_\rho}(R^n)$ is a division ring. It is obvious that both $\mathbb{R}I$ and $\rho(\mathbb{R})$ are contained in $\mathtt{End}_{R_\rho}(R^n)$. Hence we have

$$R_\rho \subseteq \mathtt{End}_{R_\rho}(R^n).$$

This means that R_ρ is an integral domain. Then it follows from Lemma 2.6 that R_ρ is a field since it is commutative. In particular, if ρ is non-trivial, then $\rho(\mathbb{R})$ is not contained in $\mathbb{R}I$. Hence R_ρ is a non-trivial field extension of finite degree over $\mathbb{R}I$. This implies that R_ρ has to be isomorphic to $\mathbb{C}$ which is the unique non-trivial field extension of finite degree over $\mathbb{R}$. The existence of an isomorphism as β is obvious. □

Denote by $C_{\mathtt{M}_n(\mathbb{R})}R_\rho$ the centralizer of R_ρ in $\mathtt{M}_n(\mathbb{R})$.

Lemma 2.7. *Let $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$ be a non-trivial and irreducible representation. Then*

$$C_{\mathtt{M}_n(\mathbb{R})}R_\rho = R_\rho. \tag{6}$$

Proof. By identifying $\mathtt{M}_n(\mathbb{R})$ with $\mathtt{End}_{\mathbb{R}}(\mathbb{R}^n)$ we have

$$R_\rho \subseteq C_{\mathtt{M}_n(\mathbb{R})}R_\rho = \mathtt{End}_{R_\rho}(\mathbb{R}^n).$$

On the other hand, for an arbitrary $s \in C_{\mathtt{M}_n(\mathbb{R})}R_\rho$, the subring $R_\rho(s)$ of $C_{\mathtt{M}_n(\mathbb{R})}R_\rho$ generated by R_ρ and s is a commutative $\mathbb{R}$-subalgebra. Note that $C_{\mathtt{M}_n(\mathbb{R})}R_\rho$ is a division ring since so is $\mathtt{End}_{R_\rho}(\mathbb{R}^n)$ by Schur's lemma. It follows from Lemma 2.5 that $R_\rho(s)$ is a field. More over, since s must be algebraic over $\mathbb{R}I$, $R_\rho(s)$ is an algebraic extension of the field R_ρ. This implies that $R_\rho(s) = R_\rho$ since R_ρ is isomorphic to $\mathbb{C}$ by Lemma 2.6. Then s belongs to R_ρ. Hence we have

$$C_{\mathtt{M}_n(\mathbb{R})}R_\rho \subseteq R_\rho$$

from which follows the identity (6). □

Lemma 2.8. *If a representation $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$ is non-trivial and irreducible, then $n = 2$.*

Proof. It follows from the double center theorem of simple algebras (cf. [4, p. 232]) that

$$\mathtt{dim}_{\mathbb{R}}R_\rho \cdot \mathtt{dim}_{\mathbb{R}}C_{\mathtt{M}_n(\mathbb{R})}R_\rho = \mathtt{dim}_{\mathbb{R}}\mathtt{M}_n(\mathbb{R}) = n^2.$$

Note that by Lemma 2.6 and Lemma 2.7 we have

$$\mathtt{dim}_{\mathbb{R}}C_{\mathtt{M}_n(\mathbb{R})}R_\rho = \mathtt{dim}_{\mathbb{R}}R_\rho = 2.$$

This implies that $n = 2$. □

3 Proof of Theorem 1 and miscellaneous results

The proof of Theorem 1.1. Given a representation $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$, we have the $\mathbb{R}$-space $\mathbb{R}^n$ becoming a R_ρ-module where R_ρ is the subalgebra of $\mathtt{M}_n(\mathbb{R})$ defined at the beginning of Section 2. Let $\{v_1, v_2, \ldots, v_n\}$ be a $\mathbb{R}$-basis of $\mathbb{R}^n$. Then we have

$$\mathbb{R}^n = \bigoplus_{i=1}^{n} \mathbb{R}v_i = \sum_{i=1}^{n} R_\rho v_i.$$

Note that since $R_\rho v_i$ is a finitely generated module with trivial radical by Lemma 2.4, it has to be semisimple (cf. [1, p. 129]) for all $1 \leq i \leq n$. Therefore as a sum of semisimple submodules, $\mathbb{R}^n$ is also semisimple. This is equivalent to the complete reducibility of ρ by Proposition 2.2. □

The poof of Theorem 1.2. Suppose that $\rho : \mathbb{R} \to \mathtt{M}_n(\mathbb{R})$ is irreducible and non-trivial. Then it follows from Lemma 2.8 that $n = 2$. More over, it comes from Lemma 2.6 that there exists an isomorphism $\beta : \mathbb{C} \to R_\rho$ such that $\beta(a) = aI$ for all $a \in \mathbb{R}$. Let $\mu : \mathbb{C} \to \mathtt{M}_2(\mathbb{R})$ be the canonical ring homomorphism defined by

$$\mu(z) = \begin{pmatrix} z_r & -z_i \\ z_i & z_r \end{pmatrix}, \ \forall z \in \mathbb{C} \tag{7}$$

where z_r and z_i are real and imaginary parts of z respectively. The composition $\mu\beta^{-1}$ is a $\mathbb{R}$-algebra homomorphism from R_ρ to $\mathtt{M}_2(\mathbb{R})$. Note that R_ρ is a simple $\mathbb{R}$-subalgebra of $\mathtt{M}_2(\mathbb{R})$ by Lemma 2.6. Then by Noether-Skolem Theorem (cf. [4, p. 230]) there exists a non-singular matrix $Y \in \mathtt{M}_2(\mathbb{R})$ such that $\mu\beta^{-1}$ is just the restriction of the inner automorphism ι_Y of $\mathtt{M}_2(\mathbb{R})$ which is the conjugation via Y. Denote by σ the composition $\beta^{-1}\rho$. It is easy to check that the following diagram is commutative.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\rho} & \mathtt{M}_2(\mathbb{R}) \\ \sigma\big\downarrow & & \big\downarrow\iota_Y \\ \mathbb{C} & \xrightarrow{\mu} & \mathtt{M}_2(\mathbb{R}) \end{array}$$

Then we have, for all $a \in \mathbb{R}$,

$$\rho(a) = \iota_Y^{-1}\mu\sigma(a) = Y^{-1} \begin{pmatrix} \sigma(a)_r & -\sigma(a)_i \\ \sigma(a)_i & \sigma(a)_r \end{pmatrix} Y.$$

By setting $X = Y^{-1}$, we obtain the identity (2) of Theorem 1. □

In order to prove Theorem 1.3 we need some properties of endomorphisms as well as representations of complex numbers. Note that the properties and the discussion used for proving Theorem 1.1 and 1.2 are extendable to the representations of complex numbers by simply substituting $\mathbb{C}$ for $\mathbb{R}$ in the argument. The substitution leads to a description of the matrix representations of complex numbers, which is similar to that of real numbers.

Theorem 2. *Every matrix representation $\rho : \mathbb{C} \to \mathtt{M}_n(\mathbb{R})$ is completely reducible. More over, if ρ is irreducible, then $n = 2$ and there exists a $\sigma \in \mathtt{End}(\mathbb{C})$ such that*

$$\rho = \iota_X \mu\sigma$$

where ι_X is the inner automorphism of $\mathtt{M}_2(\mathbb{R})$ via the conjugation by a non-singular matrix X and μ is the homomorphism defined by the identity (7).

Proof. This is analogous to the proofs of Theorem 1.1 and 1.2. □

We consider $\mathtt{Hom}(\mathbb{R}, \mathbb{C})$. It is a trivial fact that each homomorphism $\sigma \in \mathtt{Hom}(\mathbb{R}, \mathbb{C})$ can be extend to an endomorphism of $\mathbb{C}$.

Lemma 3.1. *Let C be a field which is isomorphic to $\mathbb{C}$. Then each homomorphism $\sigma \in \mathtt{Hom}(\mathbb{R}, C)$ has exactly two extensions $\bar\sigma, \bar\sigma' \in \mathtt{Hom}(\mathbb{C}, C)$ and, more over, $\bar\sigma = \bar\sigma'\epsilon$ where ϵ is the complex conjugation.*

Proof. This comes from the fact that, for the imaginary i, the extensions of σ have only two possible images $\{i, -i\}$ since the minimal polynomial of i is $x^2 + 1 \in \mathbb{Q}[x]$. $\square$

Two endomorphisms $\beta, \tau \in \mathtt{End}(\mathbb{C})$ are said equivalent if $\beta = \alpha\tau\alpha'$, where α and α' are the identity map or the complex conjugation. This is clearly an equivalent relation for $\mathtt{End}(\mathbb{C})$. The following property is obvious.

Proposition 3.1. *The extension of each element of* $\mathtt{Hom}(\mathbb{R}, \mathbb{C})$ *to an element of* $\mathtt{End}(\mathbb{C})$ *induces a one-to-one correspondence between the family of equivalent classes of* $\mathtt{Hom}(\mathbb{R}, \mathbb{C})$ *and that of* $\mathtt{End}(\mathbb{C})$.

Proposition 3.2. *Each irreducible representation* $\rho : \mathbb{R} \to \mathtt{M}_2(\mathbb{R})$ *has exactly two extensions* $\bar{\rho} : \mathbb{C} \to \mathtt{M}_2(\mathbb{R})$. *In precise, if* $\rho = \iota_X \rho_\sigma$ *for some* $X \in \mathtt{GL}_2(\mathbb{R})$ *and* $\sigma \in \mathtt{Hom}(\mathbb{R}, \mathbb{C})$, *then* $\bar{\rho}$ *is either* $\iota_X \mu\bar{\sigma}$ *or* $\iota_X \mu\bar{\sigma}\epsilon$, *where* $\bar{\sigma} \in \mathtt{End}(\mathbb{C})$ *is an extension of* σ *and* ϵ *is the complex conjugation.*

Proof. The existence of extensions for ρ is obvious. Note that every extension $\bar{\rho}$ of ρ is irreducible since so is ρ. Denote by $R_{\bar{\rho}}$ the subring of $\mathtt{M}_2(\mathbb{R})$ generated by $\mathbb{R}I$ and $\bar{\rho}(\mathbb{C})$. Then $\mathbb{R}^2$ is an irreducible $R_{\bar{\rho}}$-module. Since, by Schur's lemma, $\mathtt{End}_{R_{\bar{\rho}}}(\mathbb{R}^2)$ is a division ring which contains the $\mathbb{R}$-subalgebra $R_{\bar{\rho}}$ of $\mathtt{M}_2(\mathbb{R})$, it follows from Lemma 2.5 that $R_{\bar{\rho}}$ is a field extension of degree 2 over $\mathbb{R}I$. Note that by the same lemma R_ρ is also an extension of degree 2 over $\mathbb{R}I$ and that $R_{\bar{\rho}} \supseteq R_\rho$. We obtain that $R_{\bar{\rho}} = R_\rho$. Then by Lemma 3.1 ρ has only two possible extensions. In consequence, the second assertion of the lemma is obvious. $\square$

The proof of Theorem 1.3. Given a non-trivial and irreducible representation $\rho : \mathbb{R} \to \mathtt{M}_2(\mathbb{R})$, suppose that there exist $X, Y \in \mathtt{GL}_2(\mathbb{R})$ and $\sigma, \tau \in \mathtt{Hom}(\mathbb{R}, \mathbb{C})$ such that

$$\rho = \iota_X \rho_\sigma = \iota_Y \rho_\tau.$$

Set $T = Y^{-1}X$. Then the above identity implies that

$$\iota_T \rho_\sigma = \rho_\tau. \tag{8}$$

Note that each extension of an element of $\mathtt{Hom}(\mathbb{R}, \mathbb{C})$ either fix the imaginary i or send i to $-i$. For convenience we denote by $\bar{\sigma} \in \mathtt{End}(\mathbb{C})$ the extension of σ such that $\bar{\sigma}(i) = i$ and similarly by $\bar{\tau}$ the extension of τ which fixes the imaginary i. Since ρ_τ has two extensions $\mu\bar{\tau}$ and $\mu\bar{\tau}\epsilon$ by Proposition 3.2, the identity (8) implies that either

$$\iota_T \bar{\sigma} = \bar{\tau} \tag{9}$$

or

$$\iota_T \bar{\sigma} = \bar{\tau}\epsilon. \tag{10}$$

Suppose that the first identity (9) holds. By applying to the imaginary i from both sides of the identity we have an equation

$$T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This implies that

$$T = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

for some $a, b \in \mathbb{R}$. In particular, since all matrices of $\rho_\sigma(\mathbb{R})$ have same form as that of T, it is easy to check that

$$\iota_T \rho_\sigma(a) = \rho_\sigma(a), \ \forall a \in \mathbb{R}$$

which means that $\rho_\sigma = \rho_\tau$ by the identity (8). We obtain hence $\sigma = \tau$ in this case.

If the second identity (10) holds, by applying to i from both sides we have an equation

$$T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This implies that

$$T = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} b & -a \\ a & b \end{pmatrix}$$

for some $a, b \in \mathbb{R}$. If we denote by S the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then

$$\iota_T \rho_\sigma(a) = \iota_S \rho_\sigma(a) = \rho_{\epsilon\sigma}(a), \ \forall a \in \mathbb{R}.$$

We obtain again by the identity (8) that $\epsilon\sigma = \tau$, which means that both σ and τ belong to the same equivalent class. Thus we complete the proof of Theorem 1. $\square$

We observe from Theorem 1.3 that the cardinality of the equivalent classes of the irreducible representations of $\mathbb{R}$ is the same as that of $\mathtt{Hom}(\mathbb{R}, \mathbb{C})$, which is equal to the cardinality of $\mathtt{End}(\mathbb{C})$ by Proposition 3.1. Note that the set of functions from $\mathbb{C}$ to itself has cardinality $\mathfrak{c}^\mathfrak{c}$ since $|\mathbb{C}| = \mathfrak{c}$ and that the cardinality of automorphisms $\mathtt{Aut}(\mathbb{C})$ is $2^\mathfrak{c}$. We have

$$2^\mathfrak{c} = |\mathtt{Aut}(\mathbb{C})| \leq |\mathtt{End}(\mathbb{C})| \leq \mathfrak{c}^\mathfrak{c} = 2^{\aleph_0 \mathfrak{c}} = 2^\mathfrak{c}$$

which means that the cardinality of $\mathtt{End}(\mathbb{C})$ is $2^\mathfrak{c}$. Hence the irreducible representations of $\mathbb{R}$ can be quantified as follows.

Corollary 3.1. *Let $\mathfrak{c}$ represents $2^{\aleph_0}$. The cardinality of the family of equivalence classes of irreducible representations of $\mathbb{R}$ is $2^\mathfrak{c}$.*

Remark. Among various applications of the matrix representations of $\mathbb{R}$, we mention its connection with so-called "abstract" representations of Lie groups where the continuity is not an assumption imposed on the actions of Lie groups on real spaces (see for instance [6]). An example is that each discontinuous irreducible representation ρ of $\mathbb{R}$ induces a discontinuous irreducible representation of a Lie group $\bar{\rho} : \mathtt{SL}_2(\mathbb{R}) \to \mathtt{GL}_4(\mathbb{R})$ where $\bar{\rho}$ is obtained by applying ρ to the entries of each matrix in $\mathtt{SL}_2(\mathbb{R})$, which sheds light on determining all discontinuous actions of $\mathtt{SL}_2(\mathbb{R})$ on real spaces.

References

[1] F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, GTM 13, Springer-Verlag, New York, 1974.

[2] A. Borel, Linear algebraic groups, 2nd ed., GTM 126, Springer-Verlag, New York, 1991

[3] H. Kestelman, Automorphisms of the field of complex numbers, Proc. London Math. Soc., vol. 53 (1951), 1-12.

[4] R. S. Pierce, Associative Algebra, GTM 88, Springer-Verlag, New York, 1982.

[5] A. Robert, Introduction to the Representation Theory of Compact and Locally Compact Groups, London Math. Soc. Lecture Note Ser., vol. 80, Cambridge Univ. Press, 1983.

[6] J. Tits, Homomorphismes "abstrait" de groupes de Lie, in Symp. Math., Inst. Naz. Alta Mat., vol. 13, 479-499, Bologna, 1974.

[7] P. B. Yale, Automorphisms of the complex numbers, Math. Mag., vol. 39 (1966), 135-141.