Towards a Rational Closure for expressive description logics: the case of SHIQ.

Towards a Rational Closure for expressive description logics: the case of $\mathcal{SHIQ}$

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Abstract. We explore the extension of the notion of rational closure to logics lacking the finite model property, considering the logic $\mathcal{SHIQ}$. We provide a semantic characterization of rational closure in $\mathcal{SHIQ}$ in terms of a preferential semantics, based on a finite rank characterization of minimal models. We show that the rational closure of a KB can be computed in EXPTIME based on a polynomial encoding of the rational extension of $\mathcal{SHIQ}$ into entailment in $\mathcal{SHIQ}$. We discuss the extension of rational closure to more expressive description logics.

1 Introduction

A lot of work has been done in order to extend the basic formalism of Description Logics (DLs) with nonmonotonic reasoning features [38, 1, 10, 12, 15, 18, 30, 4, 16, 3, 6, 37, 33, 31, 5, 2]. The purpose of these extensions is to allow reasoning about *prototypical properties* of individuals or classes of individuals, as well as combining DLs with nonmonotonic rule-based languages, such as Datalog under the answer set semantics. The most well known semantics for nonmonotonic reasoning have been used to the purpose: default logic [1], circumscription [3], Lifschitz's nonmonotonic logic MKNF [10, 37, 31], ASP [12, 11], Datalog +/- [26], preferential reasoning [15, 4, 5, 18], rational closure [6, 9, 5, 21].

The interest of rational closure for DLs is that it provides a significant and reasonable skeptical nonmonotonic inference mechanism, while keeping the same complexity as the underlying logic. In this work, we focus on rational closure for the description logic $\mathcal{SHIQ}$ [29]. In particular, we define the rational closure for $\mathcal{SHIQ}$, building on the notion of rational closure proposed by Lehmann and Magidor [35], as we have done for $\mathcal{ALC}$ [17, 21]. Our construction differs from the one introduced by Casini and Straccia [6] for $\mathcal{ALC}$, which is more similar to the construction by Freund [13] for propositional logic, as well as from the one introduced in [5]. Both [6] and [5] exploit (in different ways) the materialization of the knowledge base, while our notion of exceptionality directly exploits preferential entailment.

We provide a semantic characterization of rational closure for $\mathcal{SHIQ}$ in terms of a preferential semantics, by generalizing to $\mathcal{SHIQ}$ the results for rational closure for $\mathcal{ALC}$ presented in [17]. The generalization is not trivial since, differently from $\mathcal{ALC}$, $\mathcal{SHIQ}$ lacks the finite model property [29]. Our construction exploits an extension of $\mathcal{SHIQ}$ with a typicality operator $\mathbf{T}$, that selects the most typical instances of a concept

C, giving rise to the new concept $\mathbf{T}(C)$. We define a *minimal model semantics* and a notion of minimal entailment for the resulting logic, $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$, and we show that the inclusions belonging to the rational closure of a TBox are those minimally entailed by the TBox, when restricting to *canonical* models. This result exploits a characterization of minimal models, showing that we can restrict to (possibly infinite) models with finite ranks. We also show that the rational closure of a TBox can be computed by exploiting a linear encoding of $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ into $\mathcal{SHIQ}$, and that the problem of deciding if an inclusion is in the rational closure of a TBox is in EXPTIME.

The linear encoding of $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ into $\mathcal{SHIQ}$ is obtained by proving that rational entailment is equivalent to preferential entailment for arbitrary queries (provided the ABox does not contain typicality assertions). The same result also holds for all the description logics from $\mathcal{ALC}$ to $\mathcal{SROIQ}$. This provides an upper bound on the complexity of rational entailment for these logics as well as a way to compute subsumption and instance checking under the rational semantics and to construct the rational closure also for logics more expressive than $\mathcal{SHIQ}$. However, as we will see, the meaning of the rational closure for expressive logics including nominals can be sometimes problematic and we discuss the issue in Section 7.

This paper is an extended version of the work presented in [20, 19].

2 A nonmonotonic extension of $\mathcal{SHIQ}$

In this section, following the approach in [16, 18], we introduce an extension of $\mathcal{SHIQ}$ [29] with a typicality operator $\mathbf{T}$ in order to express typical inclusions, obtaining the logic $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$. Following [16, 18], we introduce a typicality operator $\mathbf{T}$ to express typicality inclusions. The idea is to allow concepts of the form $\mathbf{T}(C)$, whose intuitive meaning is that $\mathbf{T}(C)$ selects the *typical* instances of a concept C. We can therefore distinguish between the properties that hold for all instances of C ($C \sqsubseteq D$), and those that only hold for the typical instances of C ($\mathbf{T}(C) \sqsubseteq D$). The semantic of the typicality operator will be defined in terms of rational models [35]. We consider an alphabet of concept names $\mathcal{C}$, role names $\mathcal{R}$, transitive roles $\mathcal{R}^+ \subseteq \mathcal{R}$, and individual constants $\mathcal{O}$. Given $A \in \mathcal{C}$, $R \in \mathcal{R}$, and $n \in \mathbb{N}$ we define:

$$C_R := A \mid \top \mid \bot \mid \neg C_R \mid C_R \sqcap C_R \mid C_R \sqcup C_R \mid \forall S.C_R \mid \exists S.C_R \mid (\geq nS.C_R) \mid (\leq nS.C_R)$$
$$C_L := C_R \mid \mathbf{T}(C_R)$$
$$S := R \mid R^-$$

As usual, we assume that transitive roles cannot be used in number restrictions [29]. A *knowledge base* (KB) is a pair $K = (\mathcal{T}, \mathcal{A})$, where the TBox $\mathcal{T}$ contains a finite set of concept inclusions $C_L \sqsubseteq C_R$ and a finite set of role inclusions $R \sqsubseteq S$ and the ABox $\mathcal{A}$ contains a finite set of assertions of the form $C_R(a)$ and $S(a, b)$, with $a, b \in \mathcal{O}$. In the following we will call *non-extended* the concepts C_R in which the $\mathbf{T}$ operator does not occur. Differently from [21], here we assume that ABox does not contain typicality assertions $\mathbf{T}(C)(a)$. The reason for this limitation is explained after Proposition 3. A similar motivation holds for limiting the occurrences of $\mathbf{T}$ to the left hand side of concept inclusions, in agreement with all the other definitions of rational

closure for DLs which deal with defeasible inclusions [6, 9, 5, 21] stating that "normally the C's are D's".

Following the preferential approaches in [16, 4, 21], a semantics for the extended language is defined, adding to interpretations in $\mathcal{SHIQ}$ [29] a *preference relation* $<$ on the domain to evaluate defeasible inclusions. $<$ is intended to compare the "typicality" of domain elements, that is to say, $x < y$ means that x is more typical than y. The typical instances of a concept C (the instances of $\mathbf{T}(C)$) are the instances x of C that are minimal with respect to the preference relation $<$ (so that there is no other instance of C preferred to x). As here we consider a rational extension of $\mathcal{SHIQ}$, we assume the preference relation $<$ to be modular as in [4, 21].

In the following definition we will use the notions of modular and well-founded relations. An irreflexive and transitive relation $<$ is: *modular* if, for all $x, y, z \in \Delta$, if $x < y$ then $x < z$ or $z < y$; it is *well-founded* if, for all $S \subseteq \Delta$, for all $x \in S$, either $x \in min_<(S)$ or $\exists y \in min_<(S)$ such that $y < x$[4].

Definition 1 (Interpretations in $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$). *A $\mathcal{SHIQ}^{R}\mathbf{T}$ interpretation $\mathcal{M}$ is any structure $\langle \Delta, <, I \rangle$ where: Δ is a domain; $<$ is an irreflexive, transitive, well-founded, and modular relation over Δ; I is a function that maps: each concept $A \in \mathcal{C}$ to a set $A^I \subseteq \Delta$; each individual name $a \in \mathcal{O}$ to an element $a^I \in \Delta$; and each role $R \in \mathcal{R}$ to a relation $R^I \subseteq \Delta \times \Delta$ such that, for all $P \in \mathcal{R}$ and for all $R \in \mathcal{R}^+$,*

$$(x, y) \in P^I \textit{ iff } (y, x) \in (P^-)^I$$
$$\textit{if } (x, y) \in R^I \textit{ and } (y, z) \in R^I \textit{ then } (x, z) \in R^I$$

The interpretation function $\cdot^I$ is extended to complex concepts as usual:

$\top^I = \Delta; \quad \bot^I = \emptyset;$
$(C \sqcap D)^I = C^I \cap D^I;$
$(C \sqcup D)^I = C^I \cup D^I;$
$(\neg C)^I = \Delta - C^I;$
$(\exists R.C)^I = \{x \in \Delta \mid \textit{ there is a } y \in \Delta \textit{ with } (x, y) \in R^I \textit{ and } y \in C^I\};$
$(\forall R.C)^I = \{x \in \Delta \mid \textit{ for all } y \in \Delta, \textit{ if } (x, y) \in R^I, \textit{ then } y \in C^I\};$
$(\geq nR.C)^I = \{x \in \Delta \mid \sharp\{y \in \Delta \textit{ s.t. } (x, y) \in R^I \textit{ and } y \in C^I\} \geq n\};$
$(\leq nR.C)^I = \{x \in \Delta \mid \sharp\{y \in \Delta \textit{ s.t. } (x, y) \in R^I \textit{ and } y \in C^I\} \leq n\}$

For the $\mathbf{T}$ operator, we let:

$$(\mathbf{T}(C))^I = min_<(C^I), \textit{ where } min_<(S) = \{u : u \in S \textit{ and } \nexists z \in S \textit{ s.t. } z < u\}.$$

It can be proved that an irreflexive and transitive relation $<$ on Δ is well-founded if and only if there are no infinite descending chains $\ldots x_{i+1} < x_i < \ldots < x_0$ of elements of Δ.

The logic $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$, as well as the underlying $\mathcal{SHIQ}$, does not enjoy the finite model property [29]. As for rational models in [35] (see Proposition 3.7), $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$

[4] As observed in [21], this condition is stronger than the smoothness condition introduced by Kraus, Lehmann and Magidor [32]. Indeed, the condition above considers all subsets S of Δ and does not only apply to the interpretations C^I of the concepts C of the language. It is easy to prove that such a condition is equivalent to requiring that $(\Delta, <)$ is well-founded, i.e. there is no infinite descending chain of individuals.

interpretations can be equivalently defined by postulating the existence of a function $k_{\mathcal{M}} : \Delta \longmapsto Ord$ assigning an ordinal to each domain element, and then letting $x < y$ if and only if $k_{\mathcal{M}}(x) < k_{\mathcal{M}}(y)$. We call $k_{\mathcal{M}}(x)$ *the rank of element* x in $\mathcal{M}$. When finite, $k_{\mathcal{M}}(x)$ can be understood as the length of a chain $x_0 < \cdots < x$ from x to a minimal x_0 (an x_0 s.t. for no x', $x' < x_0$).

Notice that the meaning of $\mathbf{T}$ can be split into two parts [16]: for any x of the domain Δ, $x \in (\mathbf{T}(C))^I$ just in case (i) $x \in C^I$, and (ii) there is no $y \in C^I$ such that $y < x$. In order to isolate the second part of the meaning of $\mathbf{T}$, we introduce a new modality $\Box$, whose accessibility relation $R_\Box$ is such that $(x, y) \in R_\Box$ iff $y < x$. The well-foundedness of $<$ ensures that typical elements of C^I exist whenever $C^I \neq \emptyset$, by avoiding infinitely descending chains of elements. The interpretation of $\Box$ in $\mathcal{M}$ is as follows:

Definition 2. *Given an interpretation $\mathcal{M}$, we extend the definition of I with the following clause:*

$$(\Box C)^I = \{x \in \Delta \mid \textit{for every } y \in \Delta, \textit{ if } y < x \textit{ then } y \in C^I\}$$

It is easy to observe that, as for preferential interpretations in [16], also in ranked interpretations x is a typical instance of C if and only if it is an instance of C and $\Box\neg C$, that is to say:

Proposition 1. *Given an interpretation $\mathcal{M}$, given a concept C and an element $x \in \Delta$, we have that*

$$x \in (\mathbf{T}(C))^I \textit{ iff } x \in (C \sqcap \Box\neg C)^I$$

Since we only use $\Box$ to capture the meaning of $\mathbf{T}$, in the following we will always use the modality $\Box$ followed by a negated concept, as in $\Box\neg C$.

The notion of satisfiability of a KB in an interpretation is defined as usual:

Definition 3 (Satisfiability and entailment). *Given a $\mathcal{SHIQ}^R\mathbf{T}$ interpretation $\mathcal{M}=\langle\Delta, <, I\rangle$, we say that:*

- $\mathcal{M}$ satisfies an inclusion $C \sqsubseteq D$ if $C^I \subseteq D^I$, and similarly for role inclusions;

- $\mathcal{M}$ satisfies an assertion $C(a)$ if $a^I \in C^I$;

- $\mathcal{M}$ satisfies an assertion $R(a, b)$ if $(a^I, b^I) \in R^I$.

Given a knowledge base $K = (\mathcal{T}, \mathcal{A})$, we say that: an interpretation $\mathcal{M}$ satisfies $\mathcal{T}$ (resp. $\mathcal{A}$) if $\mathcal{M}$ satisfies all inclusions in $\mathcal{T}$ (resp. assertions in $\mathcal{A}$); $\mathcal{M}$ is a model of K if $\mathcal{M}$ satisfies $\mathcal{T}$ and $\mathcal{A}$.

Let a query *F be an inclusion $C_L \sqsubseteq C_R$ or an assertion $C(a)$, $\mathbf{T}(C)(a)$ or $R(a, b)$. We say that F* is entailed by *K, written $K \models_{\mathcal{SHIQ}^R\mathbf{T}} F$, if for all models $\mathcal{M} =\langle\Delta, <, I\rangle$ of K, $\mathcal{M}$ satisfies F.*

Let us now introduce the notion of rank of a $\mathcal{SHIQ}$ concept.

Definition 4 (Rank of a concept $k_{\mathcal{M}}(C_R)$). *Given an interpretation $\mathcal{M} =\langle\Delta, <, I\rangle$, we define the* rank *$k_{\mathcal{M}}(C_R)$ of a concept C_R in the interpretation $\mathcal{M}$ as $k_{\mathcal{M}}(C_R) = min\{k_{\mathcal{M}}(x) \mid x \in {C_R}^I\}$. If ${C_R}^I = \emptyset$, then C_R has no rank in $\mathcal{M}$ and we write $k_{\mathcal{M}}(C_R) = \infty$.*

It is immediate to verify that:

Proposition 2. *For any interpretation $\mathcal{M} = \langle \Delta, <, I \rangle$, $\mathcal{M}$ satisfies $\mathbf{T}(C) \sqsubseteq D$ if and only if $k_{\mathcal{M}}(C \sqcap D) < k_{\mathcal{M}}(C \sqcap \neg D)$ or $k_{\mathcal{M}}(C) = \infty$.*

The following theorem states that, for the knowledge bases considered in this paper (that neither contain typicality assertions in ABox nor allow the typicality operator on the r.h.s.of inclusions), reasoning in $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ has the same complexity as reasoning in $\mathcal{SHIQ}$, i.e. it is in EXPTIME.

Theorem 1. *Given a $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ knowledge base $K = (\mathcal{T}, \mathcal{A})$ and a query F, the entailment $K \models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} F$ can be decided in* EXPTIME.

We prove the theorem above by providing a linear encoding of entailment in $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ into entailment in $\mathcal{SHIQ}$. First of all, let us remember from [35] that rational entailment is equivalent to preferential entailment for knowledge bases only containing positive conditionals $A \mid\!\sim B$. We show that a similar result also holds for the rational extension of $\mathcal{SHIQ}$ with typicality. Let $\mathcal{SHIQ}^{\mathsf{P}}\mathbf{T}$ be the logic that we obtain removing the requirement of modularity in the definition of interpretation in $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ (Definition 1). In this logic the typicality operator has a preferential semantics [32], based on the preferential models rather than on the ranked models [35]. An extension of $\mathcal{ALC}$ with typicality based on preferential models has been studied in [16], while an extension of $\mathcal{ALC}$ with defeasible inclusions based on ranked models has been developed in [4]. As the TBox of a KB in $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ is a set of strict inclusions and defeasible inclusions (i.e., positive conditionals) and ABox is a set of assertions that do not contain the operator $\mathbf{T}$, it can be proved that:

Proposition 3. *Given a knowledge base $K = (\mathcal{T}, \mathcal{A})$ and a query F (an inclusion or an assertion),*

$$K \models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} F \textit{ iff } K \models_{\mathcal{SHIQ}^{\mathsf{P}}\mathbf{T}} F$$

Proof. The (if) direction is trivial, thus we consider the (only if) one. Suppose that $K \not\models_{\mathcal{SHIQ}^{\mathsf{P}}\mathbf{T}} F$, let $\mathcal{M} = \langle \Delta, <, I \rangle$ be a preferential model of K, where $<$ is transitive, irreflexive, and well-founded, which falsifies F. Then, there must be an $x \in \Delta$ such that: if $F = E \sqsubseteq D$, $x \in E^I$ and $x \notin D^I$; if $F = C(a)$ (resp., $F = \mathbf{T}(C)(a)$), $a^I = x$ and $x \notin C^I$ (resp., $x \notin (\mathbf{T}(C))^I$). Define first a model $\mathcal{M}_1 = \langle \Delta, <_1, I_1 \rangle$, where $I_1 = I$ and the relation $<_1$ is defined as follows:

$$<_1 = < \cup \{(u, v) \mid (u = x \vee u < x) \wedge v \neq x \wedge v \not< x\}$$

It can be proved that:

1. $<_1$ is transitive and irreflexive
2. $<_1$ is well-founded
3. if $u < v$ then $u <_1 v$, for all $u, v \in \Delta$
4. if $u <_1 x$ then $u < x$, for all $u \in \Delta$.

We can show that $\mathcal{M}_1$ is a model of K. This is obvious for the assertions in ABox (as they do not contain $\mathbf{T}$), for the role inclusions in TBox and for the concept inclusions in TBox that do not involve $\mathbf{T}$, as the interpretation I is the same in $\mathcal{M}$ and in $\mathcal{M}_1$. Given an inclusion $\mathbf{T}(G) \sqsubseteq H \in K$, if it holds in $\mathcal{M}$ then it holds also in $\mathcal{M}_1$ as $min_{<_1}^{\mathcal{M}_1}(G) \subseteq min_{<}^{\mathcal{M}}(G)$. We show that $\mathcal{M}_1$ falsifies F by cases on the form of F. If F is $E \sqsubseteq D$, then $x \in E$ and $x \notin D$. The only interesting case is when $E = \mathbf{T}(C)$. To this regard, we know that $x \notin D^{\mathcal{M}_1}$, as the interpretation I is the same in $\mathcal{M}$ and $\mathcal{M}_1$. Suppose by absurd that $x \notin (\mathbf{T}(C))^{I_1}$. Since $x \in (\mathbf{T}(C))^I$, we have that $x \in C^I = C^{I_1}$, thus there must be a $y <_1 x$ with $y \in C^{I_1} = C^I$. But then, by 4, $y < x$ and we get a contradiction. Thus $x \in (\mathbf{T}(C))^{I_1}$ and $x \notin D^{I_1}$, that is, x falsifies $E \sqsubseteq D$ in $\mathcal{M}_1$. If the query F is $C(a)$, then $a^I = x$ and $x \notin C^I$. Hence, $a^{I_1} = x$ and $x \notin C^{I_1}$, so that $C(a)$ is falsified in $\mathcal{M}_1$. If the query F is $\mathbf{T}(C)(a)$, then $a^I = x$ and $x \notin (\mathbf{T}(C))^I$. Hence, $a^{I_1} = x$. From $x \notin (\mathbf{T}(C))^I$, there is $y \in \Delta$ such that $y < x$ and $y \in C^I$. By 3, $y <_1 x$ and, from $y \in C^{I_1}$, we conclude $x \notin (\mathbf{T}(C))^{I_1}$. Therefore, $\mathbf{T}(C)(a)$ is falsified in $\mathcal{M}_1$.

Observe that $<_1$ in model $\mathcal{M}_1$ satisfies:

$$(*)\ \forall z \neq x\ (z <_1 x \vee x <_1 z)$$

As a next step we define a ***modular*** model $\mathcal{M}_2 = \langle \Delta, <_2, I_2 \rangle$, where $I_2 = I_1 = I$ and the relation $<_2$ is defined as follows. Considering $\mathcal{M}_1$ where $<_1$ is well-founded, we can define by recursion the following function k from Δ to ordinals :

- $k(u) = 0$ if u is minimal in $\mathcal{M}_1$ (whence the set $\{y \mid y <_1 u\}$ is empty)
- $k(u) = max\{k(y) \mid y <_1 u\} + 1$
  if the set $\{k(y) \mid y <_1 u\}$ is non-empty and has a maximal element.
- $k(u) = sup\{k(y) \mid y <_1 u\}$
  otherwise, that is the set $\{k(y) \mid y <_1 u\}$ is non-empty, but there is no a maximal $k(y)$ for $y <_1 u$.

Observe that if $u <_1 v$ then $k(u) < k(v)$. We now define:

$$u <_2 v \text{ iff } k(u) < k(v)$$

Notice that $<_2$ is clearly transitive, modular, and well-founded; moreover $u <_1 v$ implies $u <_2 v$. We can prove as before that $\mathcal{M}_2$ is a model of K and that it falsifies the query F by x. For the latter, let F be $E \sqsubseteq D$. We know that $\mathcal{M}_1$ falsifies F by x, i.e., $x \in E^{I_1}$ and $x \notin D^{I_1}$ We consider again the only interesting case when $E = \mathbf{T}(C)$, so that $x \in (\mathbf{T}(C))^{I_1}$. Suppose by absurd that $x \notin (\mathbf{T}(C))^{I_2}$. Since $x \in (\mathbf{T}(C))^{I_1}$, we have that $x \in C^{I_2} = C^{I_1}$, thus there must be a $y <_2 x$ with $y \in C^{I_2} = C^{I_1}$. But $y <_2 x$ means that $k(y) < k(x)$. We can conclude that it must be also $y <_1 x$, otherwise by (*) we would have $x <_1 y$ which entails $k(x) < k(y)$, a contradiction. We have shown that $y <_1 x$ and $y \in C^{I_1}$, thus $x \notin (\mathbf{T}(C))^{I_1}$ a contradiction. Therefore $x \in (\mathbf{T}(C))^{I_2}$ and $x \notin D^{I_2}$, that is x falsifies $E \sqsubseteq D$ in $\mathcal{M}_2$. We have shown that $K \not\models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} F$, when F is $E \sqsubseteq D$. The case when $F = C(a)$ is trivial. When F is $\mathbf{T}(C)(a)$, as $\mathcal{M}_1$ falsifies F, then $a^{I_1} = x$ and $x \notin (\mathbf{T}(C))^{I_1}$. Hence, $a^{I_2} = x$. Also, from $x \notin (\mathbf{T}(C))^{I_1}$, there is $y \in \Delta$ such that $y <_1 x$ and $y \in C^{I_1} = C^{I_2}$. As $y <_1 x$ then $k(y) < k(x)$, hence $y <_2 x$. From $y \in C^{I_2}$, we conclude $x \notin (\mathbf{T}(C))^{I_2}$. Therefore, $\mathbf{T}(C)(a)$ is falsified in $\mathcal{M}_2$.

The proof above does not rely on specific properties of the logic $\mathcal{SHIQ}$ and Proposition 3 also holds for other description logics, provided the typicality operator may only occur on the left hand side of concept inclusions (as we have assumed in this paper). In particular, Proposition 3 holds for the rational extension of $\mathcal{SROIQ}$ introduced in [14]. However, Proposition 3 would not hold if typicality assertions on individuals were contained in the ABox, as typicality assertions entail negative conditionals (in fact, although $\neg(\mathbf{T}(C) \sqsubseteq A)$ is not in the language, in any interpretation satisfying the ABox $\mathcal{A} = \{\mathbf{T}(C)(a), \neg A(c)\}$, the inclusion $\mathbf{T}(C) \sqsubseteq A$ is false, and $\neg(\mathbf{T}(C) \sqsubseteq A)$ follows from the KB.

Given the result in Proposition 3, to prove Theorem 2.7 it is enough to show that reasoning in $\mathcal{SHIQ}^{\mathsf{P}}\mathbf{T}$ can be polynomially reduced to reasoning in $\mathcal{SHIQ}$. To this purpose, we show that for all queries F in $\mathcal{SHIQ}^{\mathsf{P}}\mathbf{T}$:

$$K \models_{\mathcal{SHIQ}^{\mathsf{P}}\mathbf{T}} F \text{ iff } K' \models_{\mathcal{SHIQ}} F' \tag{1}$$

for some polynomial encoding K' and F' of K and F in $\mathcal{SHIQ}$. The existence of such a polynomial encoding has been proved in [14] for a preferential extension of $\mathcal{SROIQ}$ [27], but it holds for all the logics containing the constructs of $\mathcal{ALC}$. Here we provide a simplified encoding for $\mathcal{SHIQ}$.

The idea of the encoding exploits the definition of the typicality operator $\mathbf{T}$ in terms of the modality $\Box$ recalled in Section 2: $\mathbf{T}(C)$ is defined as $C \sqcap \Box\neg C$, where the accessibility relation of the modality $\Box$ is the inverse of the preference relation $<$ in preferential models. The encoding introduces a new concept name $\Box_{\neg C}$, for each typicality concept $\mathbf{T}(C)$ occurring in K (or in the query) and replaces each occurrence of a concept $\mathbf{T}(C)$ with the concept $C \sqcap \Box_{\neg C}$.

To capture the properties of the $\Box$ modality, a new transitive role name $P_<$ is introduced to represent the relation $<$ in preferential models, and the following concept inclusion axioms are introduced in K' (for all concepts C such that $\mathbf{T}(C)$ occurs in K):

$$\Box_{\neg C} \sqsubseteq \forall P_< .\neg C \qquad\qquad \neg\Box_{\neg C} \sqsubseteq \exists P_< .(C \sqcap \Box_{\neg C})$$

The first inclusion encodes the semantics of the modality. Since $P_<$ is a transitive role, the inclusion is a simplification of the corresponding inclusion $\Box_{\neg C} \sqsubseteq \forall P_<.(\neg C \sqcap \Box_{\neg C})$ in [14], as transitive roles are not available in $\mathcal{SROIQ}$. The second inclusion accounts for the well-foundedness: if a domain element is not a typical C element then there must be a typical C element preferred to it. The encoding is linear.

Observe that the same role name $P_<$ is used in the inclusions for any concept C, and $P_<$ is not required to satisfy irreflexivity and modularity. It is proven in [14] that this is enough to establish equivalence (1). We refer to [14] for the proof, which is essentially the same for $\mathcal{SHIQ}^{\mathsf{P}}\mathbf{T}$ as for $\mathcal{SROIQ}^{\mathsf{P}}\mathbf{T}$ apart from a slight simplification thanks to transitivity of $P_<$[5].

From the encoding of $\mathcal{SHIQ}^{\mathsf{P}}\mathbf{T}$ into $\mathcal{SHIQ}$, given Proposition 3, the following proposition can be obtained, which proves Theorem 1:

[5] In particular, for the simplified encoding, one can simplify the (only if) part of the proof of Proposition 2 in [15], by defining $<$ starting directly from $P_<^{I'}$ rather than from its transitive closure $(P_<^{I'})^+$.

Proposition 4. *Let $K = (\mathcal{T}, \mathcal{A})$ be a knowledge base in $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ and F a query. $K \models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} F$ iff $K' \models_{\mathcal{SHIQ}} F'$, where K' and F' are polynomial encodings in $\mathcal{SHIQ}$ of K and F, respectively.*

As a consequence, rational entailment in $\mathcal{SROIQ}^{\mathsf{R}}\mathbf{T}$ can be computed by optimized DL reasoners over linear encodings of the KB and query. Observe that Proposition 4 holds as well for $\mathcal{ALC}$ (based on the encoding in [14]) and for all the description logics ranging from $\mathcal{ALC}$ to $\mathcal{SROIQ}$ (or from $\mathcal{ALC}$ to $\mathcal{SHOIQ}$), provided the typicality operator only occurs on the left hand side of concept inclusions in the KB. Let L be such a logic (that contains at least the constructs of $\mathcal{ALC}$) and let $L^{\mathsf{R}}\mathbf{T}$ be the rational extension of L.

Corollary 1. *Entailment in the rational extension $L^{\mathsf{R}}\mathbf{T}$ of L is in the same complexity class as entailment in L.*

As a consequence of the corollary above, given a knowledge base K in the rational extension $L^{\mathsf{R}}\mathbf{T}$ of any logic L from $\mathcal{ALC}$ to $\mathcal{SROIQ}$ [27], the *instance checking* problem (i.e. checking whether $C(a)$, $\mathbf{T}(C)(a)$ or $R(a, b)$ is entailed by K) and the *subsumption* problem (i.e., checking whether $C \sqsubseteq D$ is entailed by K, where C can contain the typicality operator $\mathbf{T}$) have the same complexity as the instance checking problem and the subsumption problem in L (respectively), provided that in K the typicality operator is only allowed to occur on the left hand side of concept inclusions, as we have assumed in this paper. Such problems can be solved using the linear encoding in [14], or the simplified one described above if transitive roles are available, and solving the corresponding instance checking (resp., subsumption) problem in L. This is stated by the following theorem (where L and $L^{\mathsf{R}}\mathbf{T}$ are defined as above):

Theorem 2. *The instance checking and subsumption problems in the rational extension $L^{\mathsf{R}}\mathbf{T}$ of L are in the same complexity class as the instance checking and subsumption problems in L (respectively).*

A Protégé plug-in for reasoning in DLs under the rational closure of TBox has been presented in [22].

3 Minimal Model Semantics

It is easy to see that the typicality operator $\mathbf{T}$ itself is nonmonotonic, i.e., $\mathbf{T}(C) \sqsubseteq D$ does not imply $\mathbf{T}(C \sqcap E) \sqsubseteq D$. This nonmonotonicity of $\mathbf{T}$ allows to express the properties that hold for the typical instances of a class but, possibly, not for all the members of that class. However, the logic $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ is monotonic: what is inferred from a KB can still be inferred from any KB' with KB $\subseteq$ KB'. This is a clear limitation in DLs. As a consequence of the monotonicity of $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$, one cannot deal with irrelevance. For instance, a KB

$VIP \sqsubseteq Person$
$\mathbf{T}(Person) \sqsubseteq \leq 1\ HasMarried.Person$
$\mathbf{T}(VIP) \sqsubseteq \geq 2\ HasMarried.Person$

does not entail $\mathbf{T}(VIP \sqcap Tall) \sqsubseteq \geq 2\ HasMarried.Person$, even if the property of being tall is irrelevant with respect to the number of marriages. Observe that we do not want to draw this conclusion in a monotonic way from $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$, since otherwise we would not be able to retract it when knowing, for instance, that typical tall VIPs have just one marriage (see also Example 1). Rather, we would like to obtain this conclusion in a nonmonotonic way. In order to obtain this nonmonotonic behavior, we strengthen the semantics of $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ by defining a minimal models mechanism which is similar, in spirit, to circumscription. Given a KB, the idea is to: 1. define a *preference relation* among $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ models, giving preference to the model in which domain elements have a lower rank; 2. restrict entailment to *minimal* $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ models (w.r.t. the above preference relation) of the KB.

Definition 5 (Minimal models). *Given $\mathcal{M} = \langle \Delta, <, I\rangle$ and $\mathcal{M}' = \langle \Delta', <', I'\rangle$ we say that $\mathcal{M}$ is preferred to $\mathcal{M}'$ ($\mathcal{M} \prec \mathcal{M}'$) if (i) $\Delta = \Delta'$, (ii) $C^I = C^{I'}$ for all (non-extended) concepts C, and (iii) for all $x \in \Delta$, $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$ whereas there exists $y \in \Delta$ such that $k_{\mathcal{M}}(y) < k_{\mathcal{M}'}(y)$. Given a knowledge base K, we say that $\mathcal{M}$ is a minimal model of K with respect to $\prec$ if it is a model satisfying K and there is no $\mathcal{M}'$ model satisfying K such that $\mathcal{M}' \prec \mathcal{M}$.*

To prove the existence of minimal models of a consistent KB, let us define $K = K_F \cup K_D$ where:

$$K_F = \{C \sqsubseteq D \in \mathcal{T} : \mathbf{T} \text{ does not occur in } C\} \cup \{R \sqsubseteq S \in \mathcal{T}\} \cup \mathcal{A}$$
$$K_D = \{\mathbf{T}(C) \sqsubseteq D \in \mathcal{T}\},$$

Proposition 5 (Existence of minimal models). *If K is a satisfiable knowledge base, then it has a minimal model.*

Proof. Let $\mathcal{M} = \langle \Delta, <, I\rangle$ be a model of K, where we assume that $k_{\mathcal{M}} : \Delta \longrightarrow Ord$ determines $<$ and Ord is the class of ordinals. We show that a minimal model $\mathcal{M}_{min} = \langle \Delta_{min}, <^{min}, I^{min}\rangle$ of K can be constructed, with $\Delta_{min} = \Delta$ and $I_{min} = I$. Define the relation

$\mathcal{M} \approx \mathcal{M}'$ if $\mathcal{M}' = \langle \Delta', <', I'\rangle$, $\Delta' = \Delta$ and $C^I = C^{I'}$ for all (non-extended) concepts C

where $<'$ is also determined by some rank function $k_{\mathcal{M}'}$ on ordinals[6]. Define further $Mod_K(\mathcal{M}) = \{\mathcal{M}' \mid \mathcal{M}' \models K \text{ and } \mathcal{M}' \approx \mathcal{M}\}$. Clearly, $Mod_K(\mathcal{M})$ is non-empty as $\mathcal{M} \in Mod_K(\mathcal{M})$. We define $\mathcal{M}_{min} = \langle \Delta_{min}, <^{min}, I^{min}\rangle$, with $\Delta_{min} = \Delta$, $I^{min} = I$ and $<^{min}$ determined by the rank function $k_{\mathcal{M}_{min}}$ defined as follows:

$$k_{\mathcal{M}_{min}}(x) = min\{k_{\mathcal{M}'}(x) \mid \mathcal{M}' \in Mod_K(\mathcal{M})\}, \text{ for all } x \in \Delta$$

Observe that $k_{\mathcal{M}_{min}}(x)$ is well-defined for any element $x \in \Delta$ and $k_{\mathcal{M}_{min}}(C) = min\{k_{\mathcal{M}_{min}}(x) \mid x \in C^{I^{min}}\}$ is well-defined for any concept C (a set of ordinals has always a least element). We now show that $\mathcal{M}_{min} \models K$. Since $I_{min} = I$ and $\mathcal{M} \models K_F$, it follows immediately that $\mathcal{M}_{min} \models K_F$.

[6] Notice that there is no requirement that in $\mathcal{M}'$ the interpretation of individual and role names must be the same as in $\mathcal{M}$.

We prove that $\mathcal{M}_{min} \models K_D$. Let $\mathbf{T}(C) \sqsubseteq E \in K_D$. Suppose by absurdity that $\mathcal{M}_{min} \not\models \mathbf{T}(C) \sqsubseteq E$, this means that $k_{\mathcal{M}_{min}}(C \sqcap \neg E) \leq k_{\mathcal{M}_{min}}(C \sqcap E)$. Let $\mathcal{M}_1 \in Mod_K(\mathcal{M})$, such that $k_{\mathcal{M}_{min}}(C \sqcap \neg E) = k_{\mathcal{M}_1}(C \sqcap \neg E)$. $\mathcal{M}_1$ exists. Similarly, let $\mathcal{M}_2 \in Mod_K(\mathcal{M})$, such that $k_{\mathcal{M}_{min}}(C \sqcap E) = k_{\mathcal{M}_2}(C \sqcap E)$. We then have $k_{\mathcal{M}_1}(C \sqcap \neg E) = k_{\mathcal{M}_{min}}(C \sqcap \neg E) \leq k_{\mathcal{M}_{min}}(C \sqcap E) = k_{\mathcal{M}_2}(C \sqcap E) \leq k_{\mathcal{M}_1}(C \sqcap E)$, as $k_{\mathcal{M}_2}(C \sqcap E)$ is minimal. Thus we get that $k_{\mathcal{M}_1}(C \sqcap \neg E) \leq k_{\mathcal{M}_1}(C \sqcap E)$ against the fact that $\mathcal{M}_1$ is a model of K.

The minimal model semantics introduced above is the same used in [21] for defining a semantic characterization of the rational closure in $\mathcal{ALC}$. Although it has strong similarities with the minimal model semantics for $\mathcal{ALC}$ presented in [18], it is worth noticing that the notion of minimality here (and in [21]) is based on the minimization of the ranks of domain elements, while in [18] it is based on the minimization of the instances of the concepts $\neg\Box\neg C$. Both kinds of minimization, roughly speaking, are intended to maximize the typicality of the individuals belonging to a concept. The choice of the kind of minimization, however, makes a big difference from the point of view of the complexity of the resulting minimal entailment. In Section 4, we show that there is a correspondence between the minimal models of a KB (according to the semantics in Definition 5) and the rational closure of a KB.

3.1 Infinite Minimal Models with finite ranks

In the following we provide a characterization of minimal models of a KB in terms of their rank: intuitively minimal models are exactly those where each domain element has rank 0 if it satisfies all defeasible inclusions, and otherwise has the smallest rank greater than the rank of any concept C occurring in a defeasible inclusion $\mathbf{T}(C) \sqsubseteq D$ of the KB falsified by the element. Exploiting this intuitive characterization of minimal models, we are able to show that, for a finite KB, minimal models have always a *finite* ranking function, no matter whether they have a finite domain or not. This result allows us to provide a semantic characterization of rational closure for logics, like $\mathcal{SHIQ}$, that do not have the finite model property. Let $K = K_F \cup K_D$, as defined in Section 3, where K_D is the set of all defeasible inclusions in K and K_F is the set of all the strict concept inclusions, role inclusions and assertions in K.

Given a model $\mathcal{M} = \langle \Delta, <, I \rangle$, let us define the set $S_x^{\mathcal{M}}$ of defeasible inclusions falsified by a domain element $x \in \Delta$, as $S_x^{\mathcal{M}} = \{\mathbf{T}(C) \sqsubseteq D \in K_D \mid x \in (C \sqcap \neg D)^I\}$.

Proposition 6. *Let $\mathcal{M} = \langle \Delta, <, I \rangle$ be a model of K and $x \in \Delta$, then: (a) if $k_{\mathcal{M}}(x) = 0$ then $S_x^{\mathcal{M}} = \emptyset$; (b) if $S_x^{\mathcal{M}} \neq \emptyset$ then $k_{\mathcal{M}}(x) > k_{\mathcal{M}}(C)$ for every C such that, for some D, $\mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}$.*

Proof. Observe that (a) follows from (b), since if $k_{\mathcal{M}}(x) = 0$ then it cannot be $k_{\mathcal{M}}(x) > k_{\mathcal{M}}(C)$, for any C, whence by (b) it must be $S_x^{\mathcal{M}} = \emptyset$. Let us prove (b). Suppose for a contradiction that (b) is false, so that $S_x^{\mathcal{M}} \neq \emptyset$ and for some C such that, for some D, $\mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}$, we have $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}}(C)$. We have also that $x \in (C \sqcap \neg D)^I$ as $\mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}$. But $\mathcal{M} \models K$, in particular $\mathcal{M} \models \mathbf{T}(C) \sqsubseteq D$, thus it must be $x \notin (\mathbf{T}(C))^I$; but since $x \in C^I$, we have that $x \notin Min(C^I)$, that is there is $y \in C^I$, with $k_{\mathcal{M}}(y) < k_{\mathcal{M}}(x)$, which means $k_{\mathcal{M}}(x) > k_{\mathcal{M}}(C)$, and we get a contradiction.

Proposition 7. *Let $K = K_F \cup K_D$ and $\mathcal{M} = \langle \Delta, <, I \rangle$ be a model of K_F; suppose that for any $x \in \Delta$ it holds:*

(a) if $k_{\mathcal{M}}(x) = 0$ then $S_x^{\mathcal{M}} = \emptyset$

(b) if $S_x^{\mathcal{M}} \neq \emptyset$ then $k_{\mathcal{M}}(x) > k_{\mathcal{M}}(C)$ for every C such that, for some D, $\mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}$.

Then $\mathcal{M} \models K$.

Proof. Let $\mathbf{T}(C) \sqsubseteq D \in K_D$, suppose that for some $x \in C^I$, it holds $x \in (\mathbf{T}(C))^I - D^I$, then $\mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}$. By hypothesis, we have $k_{\mathcal{M}}(x) > k_{\mathcal{M}}(C)$, against the fact that $x \in (\mathbf{T}(C))^I$.

Proposition 8. *Let $K = K_F \cup K_D$ and $\mathcal{M} = \langle \Delta, <, I \rangle$ a* minimal model *of K, for every $x \in \Delta$, it holds:*

(a) if $S_x^{\mathcal{M}} = \emptyset$ then $k_{\mathcal{M}}(x) = 0$

(b) if $S_x^{\mathcal{M}} \neq \emptyset$ then $k_{\mathcal{M}}(x) = 1 + max\{k_{\mathcal{M}}(C)$ such that $\mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}\}$.

Proof. Let $\mathcal{M} = \langle \Delta, <, I \rangle$ be a minimal model of K. Define another model $\mathcal{M}' = \langle \Delta, <', I \rangle$, where $<'$ is determined by a ranking function $k_{\mathcal{M}'}$ as follows:

- $k_{\mathcal{M}'}(x) = 0$ if $S_x^{\mathcal{M}} = \emptyset$,
- $k_{\mathcal{M}'}(x) = 1 + max\{k_{\mathcal{M}}(C) \mid \mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}\}$ if $S_x^{\mathcal{M}} \neq \emptyset$.

It is easy to see that (i) for every x $k_{\mathcal{M}'}(x) \leq k_{\mathcal{M}}(x)$. Indeed, if $S_x^{\mathcal{M}} = \emptyset$ then it is obvious; if $S_x^{\mathcal{M}} \neq \emptyset$, then $k_{\mathcal{M}'}(x) = 1 + max\{k_{\mathcal{M}}(C) \mid \mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}\} \leq k_{\mathcal{M}}(x)$ by Proposition 6. It equally follows that (ii) for every concept C, $k_{\mathcal{M}'}(C) \leq k_{\mathcal{M}}(C)$. To see this: let $z \in C^I$ such that $k_{\mathcal{M}}(z) = k_{\mathcal{M}}(C)$, either $k_{\mathcal{M}'}(C) = k_{\mathcal{M}'}(z) \leq k_{\mathcal{M}}(z)$ and we are done, or there exists $y \in C^I$, such that $k_{\mathcal{M}'}(C) = k_{\mathcal{M}'}(y) < k_{\mathcal{M}'}(z) \leq k_{\mathcal{M}}(z)$.

Observe that $S_x^{\mathcal{M}} = S_x^{\mathcal{M}'}$, since the evaluation function I is the same in the two models. By definition of $\mathcal{M}'$, we have $\mathcal{M}' \models K_F$; moreover by (i) and (ii) it follows that:

(iii) if $k_{\mathcal{M}'}(x) = 0$ then $S_x^{\mathcal{M}'} = \emptyset$.

(iv) if $S_x^{\mathcal{M}'} \neq \emptyset$: $k_{\mathcal{M}'}(x) = 1 + max\{k_{\mathcal{M}}(C) \mid \mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}\} \geq 1 + max\{k_{\mathcal{M}'}(C) \mid \mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}'}\}$, that is $k_{\mathcal{M}'}(x) > k_{\mathcal{M}'}(C)$ for every C such that for some D, $\mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}'}$.

By Proposition 7 we obtain that $\mathcal{M}' \models K$; but by (i) $k_{\mathcal{M}'}(x) \leq k_{\mathcal{M}}(x)$ and by hypothesis $\mathcal{M}$ is minimal. Thus it must be that for every $x \in \Delta$, $k_{\mathcal{M}'}(x) = k_{\mathcal{M}}(x)$ (whence $k_{\mathcal{M}'}(C) = k_{\mathcal{M}}(C)$) which entails that $\mathcal{M}$ satisfies (a) and (b) in the statement of the theorem.

Also the opposite direction holds:

Proposition 9. *Let $K = K_F \cup K_D$, let $\mathcal{M} = \langle \Delta, <, I \rangle$ be a model of K_F, suppose that for every $x \in \Delta$, it holds:*

(a) $S_x^{\mathcal{M}} = \emptyset$ iff $k_{\mathcal{M}}(x) = 0$

(b) if $S_x^{\mathcal{M}} \neq \emptyset$ then $k_{\mathcal{M}}(x) = 1 + max\{k_{\mathcal{M}}(C) \mid \mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}\}$.

then $\mathcal{M}$ is a minimal model of K.

Proof. In light of previous Propositions 6 and 7, it is sufficient to show that $\mathcal{M}$ is minimal. To this aim, let $\mathcal{M}' = \langle \Delta, <', I \rangle$, with associated ranking function $k_{\mathcal{M}'}$, be another model of K, we show that for every $x \in \Delta$, it holds $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$. We proceed by induction on $k_{\mathcal{M}'}(x)$. If $S_x^{\mathcal{M}} = S_x^{\mathcal{M}'} = \emptyset$, we have that $k_{\mathcal{M}}(x) = 0 \leq k_{\mathcal{M}'}(x)$ (no need of induction). If $S_x^{\mathcal{M}} = S_x^{\mathcal{M}'} \neq \emptyset$, then since $\mathcal{M}' \models K$, by Proposition 6: $k_{\mathcal{M}'}(x) \geq 1 + max\{k_{\mathcal{M}'}(C) \mid \mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}'}\}$. Let $S_x^{\mathcal{M}'} = S_x^{\mathcal{M}} = \{\mathbf{T}(C_1) \sqsubseteq D_1, \ldots, \mathbf{T}(C_u) \sqsubseteq D_u\}$. For $i = 1, \ldots, u$ let $k_{\mathcal{M}'}(C_i) = k_{\mathcal{M}'}(y_i)$ for some $y_i \in \Delta$. Observe that $k_{\mathcal{M}'}(y_i) < k_{\mathcal{M}'}(x)$, thus by induction hypothesis $k_{\mathcal{M}}(y_i) \leq k_{\mathcal{M}'}(y_i)$, for $i = 1, \ldots, u$. But then $k_{\mathcal{M}}(C_i) \leq k_{\mathcal{M}}(y_i)$, so that we finally get:

$$\begin{aligned}
k_{\mathcal{M}'}(x) &\geq 1 + max\{k_{\mathcal{M}}(C) \mid \mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}'}\} \\
&= 1 + max\{k_{\mathcal{M}'}(C_1), \ldots, k_{\mathcal{M}'}(C_u)\} \\
&= 1 + max\{k_{\mathcal{M}'}(y_1), \ldots, k_{\mathcal{M}'}(y_u)\} \\
&\geq 1 + max\{k_{\mathcal{M}}(y_1), \ldots, k_{\mathcal{M}}(y_u)\} \\
&\geq 1 + max\{k_{\mathcal{M}}(C_1), \ldots, k_{\mathcal{M}}(C_u)\} \\
&= 1 + max\{k_{\mathcal{M}}(C) \mid \mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}\} \\
&= k_{\mathcal{M}}(x)
\end{aligned}$$

Putting Propositions 8 and 9 together, we obtain the following theorem which provides a characterization of minimal models.

Theorem 3. *Let $K = K_F \cup K_D$, and let $\mathcal{M} = \langle \Delta, <, I \rangle$ be a model of K_F. The following are equivalent:*

- *$\mathcal{M}$ is a minimal model of K*
- *For every $x \in \Delta$ it holds: (a) $S_x^{\mathcal{M}} = \emptyset$ iff $k_{\mathcal{M}}(x) = 0$ (b) if $S_x^{\mathcal{M}} \neq \emptyset$ then $k_{\mathcal{M}}(x) = 1 + max\{k_{\mathcal{M}}(C) \mid \mathbf{T}(C) \sqsubseteq D \in S_x^{\mathcal{M}}\}$.*

The following proposition shows that in any minimal model the *rank* of each domain element is finite.

Proposition 10. *Let $K = K_F \cup K_D$ and $\mathcal{M} = \langle \Delta, <, I \rangle$ a minimal model of K, for every $x \in \Delta$, $k_{\mathcal{M}}(x)$ is a finite ordinal ($k_{\mathcal{M}}(x) < \omega$).*

The previous proposition is essential for establishing a correspondence between the minimal model semantics of a KB and its rational closure. From now on, we can assume that the ranking function assigns to each domain element in Δ a natural number, i.e. that $k_{\mathcal{M}} : \Delta \longrightarrow \mathbb{N}$. From the statement of Proposition 9 we can also conclude that the rank of a domain element x in any minimal model of the KB cannot be higher than the number of typicality inclusions $\mathbf{T}(C) \sqsubseteq D$ in the KB. Both the finite rank result (Proposition 10) and the existence of minimal models result (Proposition 5) hold as well for more expressive logics such as $\mathcal{SHOIQ}$ and $\mathcal{SROIQ}$, as their proof does not depend on the underlying description logic.

In the next section we will extend to $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ the notion of rational closure: this extension allows to deal with irrelevance and allows to attribute typical properties to concepts. Based on the finite rank proposition (Proposition 10), we can now prove that the rational closure of $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ is semantically characterized by (a specific class of) the minimal models introduced in Defintion 5.

4 Rational Closure for $\mathcal{SHIQ}$

In this section, we extend to $\mathcal{SHIQ}$ the notion of rational closure introduced by Lehmann and Magidor [35]. Given the typicality operator, the typicality inclusions $\mathbf{T}(C) \sqsubseteq D$ (all the typical C's are D's) play the role of conditional assertions $C \mid\!\sim D$ in [35]. Here, we adopt for $\mathcal{SHIQ}$ the rational closure construction introduced for $\mathcal{ALC}$ in [21]. However, as we restrict our consideration to the case when the ABox does not contain typicality assertions $\mathbf{T}(C)(a)$, the construction is simpler as it does not require ABox to be modified when computing the closure of the TBox. Nevertheless ABox has to be taken into account to make the construction general enough to work for expressive logics which allow the TBox to be internalized into the ABox. This construction is similar, but not equivalent, to the rational closure constructions for $\mathcal{ALC}$ in [6, 5], which exploit materialization. As a difference, we use rational entailment in the construction, and we show that the rational closure w.r.t. TBox can be computed by exploiting a polynomial encoding of $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ into $\mathcal{SHIQ}$, so that that the problem of deciding whether a (defeasible) inclusion belongs to the rational closure of a TBox is in EXPTIME. In Section 6 we shortly discuss the rational closure over the ABox.

Definition 6 (Exceptionality of concepts and inclusions). *Let $K = (\mathcal{T}, \mathcal{A})$ be a KB and C a concept. C is* exceptional *for K if and only if $K \models_{\mathcal{SHIQ}^{R}\mathbf{T}} \mathbf{T}(\top) \sqsubseteq \neg C$. A $\mathbf{T}$-inclusion $\mathbf{T}(C) \sqsubseteq D$ is exceptional for K if C is exceptional for K. The set of $\mathbf{T}$-inclusions of K which are exceptional in K will be denoted by $\mathcal{E}(K)$.*

Given a DL knowledge base $K = (\mathcal{T}, \mathcal{A})$, it is possible to define a sequence of non increasing subsets of TBoxes $\mathcal{T} = \mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \mathcal{T}_2, \ldots$ by letting,

$\mathcal{T}_0 = \mathcal{T}$;
$K_0 = (\mathcal{T}_0, \mathcal{A})$;
$\mathcal{T}_i = \mathcal{E}(K_{i-1}) \cup \{C \sqsubseteq D \in \mathcal{T}$ s.t. operator $\mathbf{T}$ does not occur in $C\}$, for $i > 0$;
$K_i = (\mathcal{T}_i, \mathcal{A})$.

Observe that, being K finite, there is an $n \geq 0$ such that, for all $m > n, K_m = K_n$ or $K_m = \emptyset$. Observe also that the definition of the K_i's is similar to the definition of the C_i's in Lehmann and Magidor's rational closure [32] but, at each step, "strict" inclusions $C \sqsubseteq D$ are also added in $\mathcal{T}_i$.

Definition 7 (Rank of a concept). *A concept C has* rank *i (denoted by $rank(C) = i$) for $K = (\mathcal{T}, \mathcal{A})$, iff i is the least natural number for which C is not exceptional for K_i. If C is exceptional for all K_i then $rank(C) = \infty$, and we say that C has no rank.*

The notion of rank of a formula allows to define the rational closure of the TBox of a knowledge base.

Definition 8 (Rational closure of a TBox). *Let $K = (\mathcal{T}, \mathcal{A})$ be a DL knowledge base. We define, $\overline{\mathcal{T}}$, the* rational closure *of $\mathcal{T}$, as*

$$\overline{\mathcal{T}} = \{\mathbf{T}(C) \sqsubseteq D \mid \textit{either } rank(C) < rank(C \sqcap \neg D) \textit{ or } rank(C) = \infty\} \\ \cup \{C \sqsubseteq D \mid K \models_{\mathcal{SHIQ}^{R}\mathbf{T}} C \sqsubseteq D\}$$

where C and D are arbitrary $\mathcal{SHIQ}$ concepts.

As for the rational closure by Lehmann and Magidor [35], the rational closure construction above allows to strengthen inference in $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ and, for instance, it allows to deal with irrelevance:

Example 1. Let $K = (\mathcal{T}, \mathcal{A})$, with $\mathcal{T} = \{\mathbf{T}(Actor) \sqsubseteq Charming\}$ and $\mathcal{A} = \emptyset$. It can be verified that $\mathbf{T}(Actor \sqcap Comic) \sqsubseteq Charming \in \overline{\mathcal{T}}$. This is a nonmonotonic inference that does no longer follow if we discover that indeed comic actors are not charming (and, in this respect, they are atypical actors): indeed given a TBox $\mathcal{T}' = \mathcal{T} \cup \{\mathbf{T}(Actor \sqcap Comic) \sqsubseteq \neg Charming\}$, we have that $\mathbf{T}(Actor \sqcap Comic) \sqsubseteq Charming \notin \overline{\mathcal{T}'}$. Indeed, as for the propositional case, rational closure is closed under rational monotonicity [32]: from $\mathbf{T}(Actor) \sqsubseteq Charming \in \overline{\mathcal{T}}$ and $\mathbf{T}(Actor) \sqsubseteq Bold \notin \overline{\mathcal{T}}$ it follows that $\mathbf{T}(Actor \sqcap \neg Bold) \sqsubseteq Charming \in \overline{\mathcal{T}}$.

Although the rational closure $\overline{\mathcal{T}}$ is an infinite set, its definition is based on the construction of a finite sequence $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n$ of subsets of $\mathcal{T}$, and the problem of verifying that an inclusion $\mathbf{T}(C) \sqsubseteq D \in \overline{\mathcal{T}}$ can be shown to be in EXPTIME. Note that, in the sequence $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n$, n is $O(|K_D|)$ and hence $O(|K|)$, where $|K|$ is the size of K. Computing $\mathcal{E}(K_{i-1})$, for each $i = 1, \ldots, n$, requires to check, for all concepts A occurring on the left hand side of a $\mathbf{T}$-inclusion in $\mathcal{T}_{i-1}$ whether $K_{i-1} \models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} \mathbf{T}(\top) \sqsubseteq \neg A$. Using the encoding in $\mathcal{SHIQ}$ (Proposition 1) it is enough to check that $K'_{i-1} \models_{\mathcal{SHIQ}} \top \sqcap \Box_{\neg\top} \sqsubseteq \neg A$, which requires exponential time in the size of K'_{i-1} (and hence in the size of K, since the encoding is linear). If not already checked, the exceptionality of C and of $C \sqcap \neg D$ have to be checked for each K_i, to determine the ranks of C and of $C \sqcap \neg D$ (which can be computed using the encoding in $\mathcal{SHIQ}$ as well). Hence, computing the ranks of all the concepts C, such that $\mathbf{T}(C)$ occurs in the TBox or in the query, requires a quadratic number of calls (in the number of typicality inclusions and, hence, in the size of K) of an EXPTIME procedure (in the size of K) which checks entailment in $\mathcal{SHIQ}$.

Theorem 4 (Complexity of rational closure over TBox). *Given a knowledge base $K = (\mathcal{T}, \mathcal{A})$, the problem of deciding whether $\mathbf{T}(C) \sqsubseteq D \in \overline{\mathcal{T}}$ is in* EXPTIME.

The argument above shows that the rational closure of a TBox can be computed simply using the entailment in $\mathcal{SHIQ}$, through the encoding of $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ into $\mathcal{SHIQ}$. Observe that the rational closure construction above can be used as well to define the rational closure of a TBox in any standard description logic L containing at least the constructs of $\mathcal{ALC}$, by replacing entailment in $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ in Definitions 6 and 8 with entailment in the rational extension $L^{\mathsf{R}}\ \mathbf{T}$ of L. In particular, the construction of rational closure above can be adopted for expressive DLs such as $\mathcal{SHOIQ}$ and $\mathcal{SROIQ}$ [27] (the description logic at the bases of OWL2 DL), exploiting the linear encodings defined in Section 2 (see Theorem 2) or the one in [14]. As we will see, the meaning of the rational closure for more expressive logics including nominals can sometimes be problematic and we defer the discussion of this issue to Section 7.

5 A Minimal Model Semantics for Rational Closure in $\mathcal{SHIQ}$

In previous sections we have extended to $\mathcal{SHIQ}$ the notion of rational closure introduced in [35] for propositional logic. To provide a semantic characterization of rational

closure, we define a special class of minimal models, exploiting the fact that, by Proposition 10, in all minimal $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ models the *rank* of each domain element is always finite. First of all, we can observe that the minimal model semantics in Definition 5, as it is, cannot capture the rational closure of a TBox.

Consider $K = (\mathcal{T}, \emptyset)$, where $\mathcal{T}$ contains: $VIP \sqsubseteq Person$, $\mathbf{T}(Person) \sqsubseteq \leq 1\ HasMarried.Person$, $\mathbf{T}(VIP) \sqsubseteq \geq 2\ HasMarried.\ Person$. We observe that $\mathbf{T}(VIP \sqcap Tall) \sqsubseteq \geq 2\ HasMarried.Person$ does not hold in all minimal $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ models of K w.r.t. Definition 5. Indeed there is a minimal model $\mathcal{M} = \langle \Delta, <, I \rangle$ of K in which $\Delta = \{x, y, z\}$, $VIP^I = \{x, y\}$, $Person^I = \{x, y, z\}$, $(\leq 1\ HasMarried.\ Person)^I = \{x, z\}$, $(\geq 2\ HasMarried.Person)^I = \{y\}$, $Tall^I = \{x\}$, and $z < y < x$. Also, x is a typical tall VIP in $\mathcal{M}$ (since there is no other tall VIP preferred to him) and has no more than one spouse, therefore $\mathbf{T}(VIP \sqcap Tall) \sqsubseteq \geq 2\ HasMarried.Person$ does not hold in $\mathcal{M}$. On the contrary, it can be verified that $\mathbf{T}(VIP \sqcap Tall) \sqsubseteq \geq 2\ HasMarried.Person \in \overline{\mathcal{T}}$.

Things change if we consider the minimal models semantics applied to models that contain a domain element for *each combination of concepts consistent with* K. We call these models *canonical models*. Therefore, in order to semantically characterize the rational closure of a $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ knowledge base K, we restrict our attention to *minimal canonical models*. First, we define $\mathcal{S}_K$ as the set of all the (non-extended) concepts (and subconcepts) occurring in K or in the query F together with their complements.

In order to define canonical models, we consider all the sets of (non-extended) concepts $\{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{S}_K$ that are *consistent with* K, i.e., s.t. $K \not\models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n \sqsubseteq \bot$.

Definition 9 (Canonical model with respect to $\mathcal{S}_K$). *Given $K = (\mathcal{T}, \mathcal{A})$ and a query F, a model $\mathcal{M} = \langle \Delta, <, I \rangle$ of K is* canonical with respect to $\mathcal{S}_K$ *if it contains at least a domain element $x \in \Delta$ s.t. $x \in (C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n)^I$, for each set of concepts $\{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{S}_K$ that is consistent with K.*

Next we define the notion of minimal canonical model.

Definition 10 (Minimal canonical models (w.r.t. $\mathcal{S}_K$)). *$\mathcal{M}$ is a minimal canonical model of K if it is a canonical model of K and it is minimal with respect to $\prec$ (see Definition 5) among all the canonical models of K.*

Proposition 11 (Existence of minimal canonical models). *Let K be a finite knowledge base, if K is satisfiable then it has a minimal canonical model.*

Proof. Let $\mathcal{M} = \langle \Delta, <, I \rangle$ be a minimal model of K (which exists by Proposition 5), and let $\{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{S}_K$ any subset of $\mathcal{S}_K$ consistent with K.

We show that we can expand $\mathcal{M}$ in order to obtain a model of K that contains an instance of $C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n$. By repeating the same construction for all maximal subsets $\{C_1, C_2, \ldots, C_n\}$ of $\mathcal{S}_K$, we eventually obtain a canonical model of K.

For each $\{C_1, C_2, \ldots, C_n\}$ consistent with K, it holds that $K \not\models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} C_1 \sqcap C_2 \sqcap \cdots \sqcap C_n \sqsubseteq \bot$, i.e. there is a model $\mathcal{M}' = \langle \Delta', <', I' \rangle$ of K that contains an instance of $\{C_1, C_2, \ldots, C_n\}$.

Let $\mathcal{M}'^*$ be the model obtained by combining $\mathcal{M}$ and $\mathcal{M}'$ as follows. We let $\mathcal{M}'^* = \langle \Delta'^*, <'^*, I'^* \rangle$, where $\Delta'^* = \Delta \cup \Delta^*$. As far as individuals $a \in \mathcal{O}$ named in the ABox,

$a^{I'^*} = a^I$, whereas for the concepts and roles, $I'^* = I$ on Δ and $I'^* = I'$ on Δ'. Also, $k_{\mathcal{M}'^*} = k_{\mathcal{M}}$ for the elements in Δ, and $k_{\mathcal{M}'^*} = k_{\mathcal{M}'}$ for the elements in Δ'. We let $x <'^* y$ iff $k_{\mathcal{M}'^*}(x) = k_{\mathcal{M}'^*}(y)$, for all $x, y \in \Delta'^*$.

The model $\mathcal{M}'^*$ is still a model of K. For the set K_F in the previous definition this is obviously true. For K_D, for each $\mathbf{T}(C) \sqsubseteq D$ in K_D, if $x \in min_{<'^*}(C^{I'^*})$ in $\mathcal{M}'^*$, then $x \in min_{<}(C^I)$ in $\mathcal{M}$ or $x \in min_{<'}(C^{I'})$ in $\mathcal{M}'$. In both cases x is an instance of D (since both $\mathcal{M}$ and $\mathcal{M}'$ satisfy K_D), therefore $x \in D^{I'^*}$, and $\mathcal{M}'^*$ satisfies K_D.

By repeating the same construction for all maximal subsets $\{C_1, C_2, \ldots, C_n\}$ of $\mathcal{S}_K$, we obtain a canonical model of K, call it $\mathcal{M}^*$. We do not know whether the model is minimal. However, by applying the construction used in the proof of Proposition 5, we can construct a minimal model of K, ${\mathcal{M}^*}_{min}$, with the same domain and interpretation function as $\mathcal{M}^*$. ${\mathcal{M}^*}_{min}$ is therefore a canonical model of K, and furthermore it is minimal. Therefore K has a minimal canonical model.

We observe that this proof would not go through for $\mathcal{SHOIQ}$, and we will see some counterexample in Section 7. To prove the correspondence between minimal canonical models and the rational closure of a TBox, we need to introduce some propositions. We recall that by $rank(C)$ we denote the rank of C in the rational closure construction (Def. 7), while by $k_{\mathcal{M}}(C)$ we denote the rank of concept C in the model $\mathcal{M}$ (Def. 4). Next proposition concerns all $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ models. Given a $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ model $\mathcal{M} = \langle \Delta, <, I \rangle$, we define a sequence $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \ldots$ of models as follows: we let $\mathcal{M}_0 = \mathcal{M}$ and, for all i, we let $\mathcal{M}_i = \langle \Delta, <_i, I \rangle$ be the $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ model obtained from $\mathcal{M}$ by assigning a rank 0 to all the domain elements x with $k_{\mathcal{M}}(x) \leq i$. More precisely, we let $k_{\mathcal{M}_i}(x) = k_{\mathcal{M}}(x) - i$ if $k_{\mathcal{M}}(x) > i$, and $k_{\mathcal{M}_i}(x) = 0$ otherwise. We can prove the following:

Proposition 12. *Let $K = (\mathcal{T}, \mathcal{A})$ and let $\mathcal{M} = \langle \Delta, <, I \rangle$ be any $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ model of K. For any concept C, if $rank(C) \geq i$, then: 1) $k_{\mathcal{M}}(C) \geq i$, and 2) if $\mathbf{T}(C) \sqsubseteq D$ is entailed by K_i, then $\mathcal{M}_i$ satisfies $\mathbf{T}(C) \sqsubseteq D$.*

Proof. By induction on i. For $i = 0$: 1) holds (since it always holds that $k_{\mathcal{M}}(\mathrm{C}) \geq 0$). 2) holds trivially as $\mathcal{M}_0 = \mathcal{M}$.

For $i > 0$: Let us prove 1). If $rank(C) \geq i$, then, by Definition 7, for all $j < i$, we have that $K_j \models \mathbf{T}(\top) \sqsubseteq \neg C$. By inductive hypothesis on 2), for all $j < i$ $\mathcal{M}_j \models \mathbf{T}(\top) \sqsubseteq \neg C$. Hence, for all x with $k_{\mathcal{M}}(x) < i$, $x \notin C^I$, and $k_{\mathcal{M}}(C) \geq i$.

To prove 2), we reason as follows. Since $K_i \subseteq K_0$, $\mathcal{M} \models K_i$. All strict inclusions in K_i are satisfied in $\mathcal{M}_i$. Furthermore by construction, for all $\mathbf{T}(C) \sqsubseteq D \in K_i$, $rank(C) \geq i$ and hence, by 1) just proved, $k_{\mathcal{M}}(C) \geq i$. Thus, in $\mathcal{M}$, $min_{<}(C^I) \geq i$, and also $\mathcal{M}_i \models \mathbf{T}(C) \sqsubseteq D$. Therefore $\mathcal{M}_i \models K_i$.

Let us now consider minimal canonical models and prove a correspondence between the rank of a formula in the rational closure (Definition 7) and the rank of a formula in a model (Definition 4).

Proposition 13. *Given $K = (\mathcal{T}, \mathcal{A})$, for all $C \in \mathcal{S}_K$, if $rank(C) = i$, then: 1) there is a $\{C_1 \ldots C_n\} \subseteq \mathcal{S}_K$ maximal and consistent with K such that $C \in \{C_1 \ldots C_n\}$ and $rank(C_1 \sqcap \cdots \sqcap C_n) = i$; 2) for any $\mathcal{M}$ minimal canonical model of K, $k_{\mathcal{M}}(C) = i$.*

Proof. By induction on i. Let us first consider the base case in which $i = 0$. We have that $K \not\models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} \mathbf{T}(\top) \sqsubseteq \neg C$. Then there is a model $\mathcal{M}_1$ of K with a domain element x such that $k_{\mathcal{M}_1}(x) = 0$ and x satisfies C. By Proposition 5 we can assume without loss of generality that $\mathcal{M}_1$ is minimal. For 1): consider the maximal set of concepts $\{C_1 \dots C_n\}$ in $\mathcal{S}_K$ of which x is an instance in $\mathcal{M}_1$. By construction, $\{C_1 \dots C_n\}$ is consistent with K and contains C. Furthermore, $rank(C_1 \sqcap \dots \sqcap C_n) = 0$ since clearly $K \not\models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} \mathbf{T}(\top) \sqsubseteq \neg(C_1 \sqcap \dots \sqcap C_n)$. For 2): by definition of canonical model, in any canonical model $\mathcal{M}$ of K, $\{C_1 \dots C_n\}$ is satisfiable by an element y. Furthermore, in any minimal canonical $\mathcal{M}$, $k_{\mathcal{M}}(y) = 0$, since otherwise we could build $\mathcal{M}'$ identical to $\mathcal{M}$ except from the fact that $k_{\mathcal{M}'}(y) = 0$. It can be easily proven that $\mathcal{M}'$ would still be a model of K (indeed $\{C_1 \dots C_n\}$ was already satisfiable in $\mathcal{M}_1$ by an element with rank 0) and $\mathcal{M}' \prec \mathcal{M}$, against the minimality of $\mathcal{M}$. Therefore, in any minimal canonical model $\mathcal{M}$ of K, it holds $k_{\mathcal{M}}(C) = 0$.

For the inductive step, consider the case in which $i > 0$. We have that $K_i \not\models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}}$ $\mathbf{T}(\top) \sqsubseteq \neg C$, then there must be a model $\mathcal{M}_1 = \langle \Delta_1, <_1, I_1 \rangle$ of K_i, and a domain element x such that $k_{\mathcal{M}_1}(x) = 0$ and x satisfies C. Consider the maximal set of concepts $\{C_1, \dots Cn\} \subseteq \mathcal{S}_K$ of which x is an instance in $\mathcal{M}_1$. Clearly, $\{C_1 \dots C_n\}$ is consistent with K_i and $C \in \{C_1, \dots Cn\}$. Furthermore, $rank(C_1 \sqcap \dots \sqcap C_n) = i$. Indeed $K_{i-1} \models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} \mathbf{T}(\top) \sqsubseteq \neg(C_1 \sqcap \dots \sqcap C_n)$ (since $K_{i-1} \models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} \mathbf{T}(\top) \sqsubseteq \neg C$ and $C \in \{C_1, \dots Cn\}$), whereas clearly by the existence of x, $K_i \not\models_{\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}} \mathbf{T}(\top) \sqsubseteq \neg(C_1 \sqcap \dots \sqcap C_n)$. In order to prove 1) we are left to prove that the set $\{C_1, \dots Cn\}$ (that we will call Γ in the following) is consistent with K.

To prove this, take any minimal canonical model $\mathcal{M} = \langle \Delta, <, I \rangle$ of K. First observe that, by inductive hypothesis we know that for all concepts C' such that $rank(C') < i$, there is a maximal consistent set of concepts $\{C'_1, \dots C'_n\}$ with $C' \in \{C'_1, \dots C'_n\}$ and $rank(C'_1 \sqcap \dots \sqcap C'_n) = j < i$. Furthermore, we know that $k_{\mathcal{M}}(C') = j < i$.

For a contradiction, if $\mathcal{M}$ did not contain any element satisfying Γ we could expand it by adding to $\mathcal{M}$ a portion of the model $\mathcal{M}_1$ including $x \in \Delta_1$. More precisely, we add to $\mathcal{M}$ a new set of domain elements $\Delta_x \subseteq \Delta_1$, containing the domain element x of $\mathcal{M}_1$ and all the domain elements of Δ_1 which are reachable from x in $\mathcal{M}_1$ through a sequence of relations $R_i^{I_1}$s or $(R_i^-)^{I_1}$s, more precisely: $\Delta_x = \{z \in \Delta_1 : (x, z) \in (\bigcup_i (R_i^{I_1} \cup (R_i^-)^{I_1}))^*\}$. Let $\mathcal{M}'$ be the resulting model. We define I' on the elements of Δ as in $\mathcal{M}$, while we define I' on the elements of Δ_x as in $\mathcal{M}_1$. Finally, we let, for all $w \in \Delta$, $k_{\mathcal{M}'}(w) = k_{\mathcal{M}}(w)$ and, for all $y \in \Delta_x$, $k_{\mathcal{M}'}(y) = i + k_{\mathcal{M}_1}(y)$. In particular, $k_{\mathcal{M}'}(x) = i$. The resulting model $\mathcal{M}'$ would still be a model of K. Indeed, the ABox would still be satisfied by the resulting model (being the $\mathcal{M}$ part unchanged). For the TBox $\mathcal{T}$: all domain elements already in $\mathcal{M}$ still satisfy all the inclusions. For all $y \in \Delta_x$ (including x): for all inclusions in K_i, y satisfies them (since it does in $\mathcal{M}_1$); for all typicality inclusions $\mathbf{T}(D) \sqsubseteq G \in K - K_i$, $rank(D) < i$, hence by inductive hypothesis $k_{\mathcal{M}}(D) < i$, hence $k_{\mathcal{M}'}(D) < i$, and y is not a typical instance of D and trivially satisfies the inclusion. It is easy to see that $\mathcal{M}'$ also satisfies role inclusions $R \sqsubseteq S$ and that, for each transitive roles R, $R^{I'}$ is transitive. Observe that, indeed, any role R may relate elements in Δ, or it may relate elements in Δ_x, but it may not relate an element in Δ with an element in Δ_x or vice-versa.

We have then built a model of K satisfying Γ. Therefore Γ is consistent with K. As we have proven that Γ is maximal and consistent with K, it contains C and has rank i, we conclude that point 1) holds.

In order to prove point 2) we need to prove that any minimal canonical model $\mathcal{M}$ of K not only satisfies Γ but satisfies it with rank i, i.e. $k_{\mathcal{M}}(C_1 \sqcap \cdots \sqcap C_n) = i$, which entails $k_{\mathcal{M}}(C) = i$ (since $C \in \{C_1, \ldots C_n\}$) By Proposition 12 we know that $k_{\mathcal{M}}(C_1 \sqcap \cdots \sqcap C_n) \geq i$. We need to show that also $k_{\mathcal{M}}(C_1 \sqcap \cdots \sqcap C_n) \leq i$. We reason as above: for a contradiction suppose $k_{\mathcal{M}}(C_1 \sqcap \cdots \sqcap C_n) > i$, i.e., for all the minimal domain elements y instances of $C_1 \sqcap \cdots \sqcap C_n$, $k_{\mathcal{M}}(y) > i$. We show that this contradicts the minimality of $\mathcal{M}$. Indeed consider $\mathcal{M}'$ obtained from $\mathcal{M}$ by letting $k_{\mathcal{M}'}(y) = i$, for some minimal domain element $y \in (C_1 \sqcap \cdots \sqcap C_n)^I$, and leaving all the rest unchanged. $\mathcal{M}'$ would still be a model of K: the only thing that changes with respect to $\mathcal{M}$ is that y might have become in $\mathcal{M}'$ a minimal instance of a concept of which it was only a non-typical instance in $\mathcal{M}$. This might compromise the satisfaction in $\mathcal{M}$ of a typical inclusion as $\mathbf{T}(E) \sqsubseteq G$. However: if $rank(E) < i$, we know by inductive hypothesis that $k_{\mathcal{M}}(E) < i$ hence also $k_{\mathcal{M}'}(E) < i$ and y is not a minimal instance of E in $\mathcal{M}'$. If $rank(E) \geq i$, then $\mathbf{T}(E) \sqsubseteq G \in K_i$. As $y \in (C_1 \sqcap \cdots \sqcap C_n)^I$ (where $\{C_1, \ldots C_n\} \subseteq \mathcal{S}_K$ is maximal and consistent with K), we have that: $y \in F^I$ iff $x \in F^{I_1}$, for all (non-extended) concepts F. If $y \in E^I$, then $E \in \{C_1, \ldots C_n\}$. Hence, in $\mathcal{M}_1$, $x \in E^{I_1}$. But $\mathcal{M}_1$ is a model of K_i, and satisfies all the inclusions in K_i. Therefore $x \in G^{I_1}$ and, thus, $y \in G^I$.

It follows that $\mathcal{M}'$ would be a model of K, and $\mathcal{M}' \prec \mathcal{M}$, against the minimality of $\mathcal{M}$. We are therefore forced to conclude that $k_{\mathcal{M}}(C_1 \sqcap \cdots \sqcap C_n) = i$, and hence also $k_{\mathcal{M}}(C) = i$, and 2) holds.

The following theorem follows from the propositions above:

Theorem 5. *Let $K = (\mathcal{T}, \mathcal{A})$ be a knowledge base and $C \sqsubseteq D$ a query. We have that $C \sqsubseteq D \in \overline{\mathcal{T}}$ if and only if $C \sqsubseteq D$ holds in all minimal canonical models of K with respect to $\mathcal{S}_K$.*

Proof. ($\Leftarrow$) Assume that $C \sqsubseteq D$ holds in all minimal canonical models of K with respect to $\mathcal{S}_K$, and let $\mathcal{M} = \langle \Delta, <, I \rangle$ be a minimal canonical model of K satisfying $C \sqsubseteq D$. Observe that C and D (and their complements) belong to $\mathcal{S}_K$. The proof considers two cases: (1) the left end side of the inclusion C does not contain the typicality operator, and (2) the left end side of the inclusion is $\mathbf{T}(C)$.

In case (1), the minimal canonical model $\mathcal{M}$ of K satisfies $C \sqsubseteq D$, i.e., $C^I \subseteq D^I$. For a contradiction, let us assume that $C \sqsubseteq D \notin \overline{\mathcal{T}}$. Then, by definition of $\overline{\mathcal{T}}$, it must be: $K \not\models_{\mathcal{SHIQ}} C \sqsubseteq D$. Hence, $K \not\models_{\mathcal{SHIQ}} C \sqcap \neg D \sqsubseteq \bot$, and the set of concepts $\{C, \neg D\}$ is consistent with K. As $\mathcal{M}$ is a canonical model of K, there must be a element $x \in \Delta$ such that $x \in (C \sqcap \neg D)^I$. This contradicts the fact that $C^I \subseteq D^I$.

In case (2), assume $\mathcal{M}$ satisfies $\mathbf{T}(C) \sqsubseteq D$. Then, $(\mathbf{T}(C))^I \subseteq D^I$, i.e., for each $x \in min_<(C^I)$, $x \in D^I$. If $min_<(C^I) = \emptyset$, then there is no $x \in C^I$ (by the well-foundedness condition), hence C has no rank in $\mathcal{M}$ and, by Proposition 13, C has no rank ($rank(C) = \infty$). In this case, by Definition 8, $\mathbf{T}(C) \sqsubseteq D \in \overline{\mathcal{T}}$. Otherwise, let us assume that $k_{\mathcal{M}}(C) = i$. As $k_{\mathcal{M}}(C \sqcap D) < k_{\mathcal{M}}(C \sqcap \neg D)$, then $k_{\mathcal{M}}(C \sqcap \neg D) > i$.

By Proposition 13, $rank(C) = i$ and $rank(C \sqcap \neg D) > i$. Hence, by Definition 8, $\mathbf{T}(C) \sqsubseteq D \in \overline{\mathcal{T}}$.

($\Rightarrow$) If $C \sqsubseteq D \in \overline{\mathcal{T}}$ (for C non-extended concept), then, by definition of $\overline{\mathcal{T}}$, $K \models_{\mathcal{SHIQ}} C \sqsubseteq D$. Therefore, each minimal canonical model $\mathcal{M}$ of K satisfies $C \sqsubseteq D$. If $\mathbf{T}(C) \sqsubseteq D \in \overline{\mathcal{T}}$, then by Definition 8, either (a) $rank(C) < rank(C \sqcap \neg D)$, or (b) $rank(C) = \infty$. Let $\mathcal{M}$ be any minimal canonical model of K. In the case (a), by Proposition 13, $k_{\mathcal{M}}(C) < k_{\mathcal{M}}(C \sqcap \neg D)$, which entails $k_{\mathcal{M}}(C \sqcap D) < k_{\mathcal{M}}(C \sqcap \neg D)$. Hence $\mathcal{M}$ satisfies $\mathbf{T}(C) \sqsubseteq D$. In case (b), we can show that C has no rank in $\mathcal{M}$ (i.e., $k_{\mathcal{M}}(C) = \infty$) and hence $\mathcal{M}$ satisfies $\mathbf{T}(C) \sqsubseteq D$. For a contradiction, let $k_{\mathcal{M}}(C) = j$ for some finite j, and let $i = j + 1$. As $k_{\mathcal{M}}(C) < i$, by Proposition 12 it cannot be the case that $rank(C) \geq i$. Hence, $rank(C) < i$, which contradicts the fact that $rank(C) = \infty$.

6 Rational Closure over the ABox

The construction of rational closure in Section 4 only accounts for TBox minimization. However, the minimal canonical model semantics can be easily adapted to maximize the typicality of individual names: as for any domain element, we would like to attribute to each individual constant named in the ABox the lowest possible rank. Therefore we further refine Definition 10 of minimal canonical models with respect to TBox by taking into account the interpretation of individual constants of the ABox.

Definition 11 (Minimal canonical model w.r.t. ABox). *Given $K = (\mathcal{T}, \mathcal{A})$, let $\mathcal{M} = \langle \Delta, <, I \rangle$ and $\mathcal{M}' = \langle \Delta', <', I' \rangle$ be two models of K. We say that $\mathcal{M}$* is preferred to $\mathcal{M}'$ w.r.t. ABox *($\mathcal{M} <_{ABox} \mathcal{M}'$) if, for all individual constants a occurring in $\mathcal{A}$, $k_{\mathcal{M}}(a^I) \leq k_{\mathcal{M}'}(a^{I'})$ and there is at least one individual constant b occurring in $\mathcal{A}$ such that $k_{\mathcal{M}}(b^I) < k_{\mathcal{M}'}(b^{I'})$. $\mathcal{M}$ is a* minimal canonical model of K w.r.t. ABox *if $\mathcal{M}$ is a minimal canonical model of K (w.r.t. Definition 10) and there is no minimal canonical model $\mathcal{M}'$ of K such that $\mathcal{M}' <_{ABox} \mathcal{M}$.*

We can prove that:

Theorem 6. *For any consistent $K = (\mathcal{T}, \mathcal{A})$ there exists a minimal canonical model of K w.r.t. ABox.*

Indeed, by Proposition 11 the set of minimal canonical models of a consistent K is non-empty and (by Proposition 10) each domain element of a model in this set has a finite rank. Hence, it cannot be the case that there is an infinite descending chain $\ldots <_{ABox} \mathcal{M}_1 <_{ABox} \mathcal{M}_0$ of models in the set, and a minimal model w.r.t. $<_{ABox}$ must exist.

In order to see the strength of the above semantics, consider our example about marriages and VIPs.

Example 2. Let $K = (\mathcal{T}, \mathcal{A})$ be a knowledge base where: $\mathcal{T}$={$\mathbf{T}(Person) \sqsubseteq \leq 1\ HasMarried.\ Person$, $\mathbf{T}(VIP) \sqsubseteq \geq 2\ HasMarried.Person$, $VIP \sqsubseteq Person$}, and $\mathcal{A}$ = { $VIP(demi)$, $Person(marco)$}. Knowing that Marco is a person and Demi is a VIP, we would like to be able to assume, in the absence of other information, that Marco

is a typical person, whereas Demi is a typical VIP, and therefore Marco has at most one spouse, whereas Demi has at least two. Consider any minimal canonical model $\mathcal{M}$ of KB. Being canonical, $\mathcal{M}$ will contain, among other elements, the following:

$x \in (Person)^I$, $x \in (\leq 1\ HasMarried.Person)^I$, $x \in (\neg VIP)^I$, $k_{\mathcal{M}}(x) = 0$;
$y \in (Person)^I$, $y \in (\geq 2\ HasMarried.Person)^I$, $y \in (\neg VIP)^I$, $k_{\mathcal{M}}(y) = 1$;
$z \in (VIP)^I$, $z \in (Person)^I$, $z \in (\geq 2\ HasMarried.Person)^I$, $k_{\mathcal{M}}(z) = 1$;
$w \in (VIP)^I$, $w \in (Person)^I$, $w \in (\leq 1\ HasMarried.Person)^I$, $k_{\mathcal{M}}(w) = 2$.

so that x is a typical person and z is a typical VIP. According to Definition 11, there is a unique minimal canonical model w.r.t. ABox in which $(marco)^I = x$ and $(demi)^I = z$.

An algorithmic construction for computing the rational closure over the ABox has been presented in [21] for $\mathcal{ALC}$. It is possible to see that the same construction can be used for $\mathcal{SHIQ}$ as well. The construction is based on the idea of considering all the possible minimal consistent assignments of ranks to the individuals explicitly named in the ABox, and adopts a skeptical view considering only those conclusions $C(a)$ which hold for all assignments. Rather than reporting the details of the construction and of the related complexity results for $\mathcal{SHIQ}$, for which we refer to [20], let us recall Example 14 from [21] which shows that alternative minimal canonical models can be obtained from the minimal model semantics in Definition 11, with different ranks for individual names. We will use a variant of the same example in the next section to comment on the rational closure for the more expressive logics which allows for *nominals*.

Example 3. Normally computer science courses (CS) are taught only by academic members (A), whereas business courses (B) are taught only by consultants (C), consultants and academics are disjoint, this gives the following TBox $\mathcal{T}$: $\mathbf{T}(CS) \sqsubseteq \forall taught.A$, $\mathbf{T}(B) \sqsubseteq \forall taught.C$, $C \sqsubseteq \neg A$. Suppose the ABox $\mathcal{A}$ contains: $CS(c1)$, $B(c2)$, $taught(c1, joe)$, $taught(c2, joe)$ and let $K = (\mathcal{T}, \mathcal{A})$. From the rational closure $\overline{\mathcal{T}}$, we get that all atomic concepts have rank 0. Observe, however, that there is no minimal model (w.r.t. ABox) in which both $c1^I$ and $c2^I$ have rank 0 (otherwise, joe would both be an A and a C, which is inconsistent). In the minimal models of K either $c1^I$ has rank 0 and $c2^I$ has rank 1, or vice-versa. $(A \sqcup C)(joe)$ holds in all the minimal models of K (according to Def. 2).

7 Extending the correspondence to more expressive logics

As we have seen, the definition of the rational closure can be extended to expressive logics such as $\mathcal{SHOIQ}$ [28] and $\mathcal{SROIQ}$ [27], provided ABox is considered in the construction of the rational closure of TBox (which makes the definition in Section 4 different from [19, 20]). A natural question is whether the correspondence between the rational closure construction and the minimal canonical model semantics of the previous section can be extended to stronger DLs. In the general case, we can see that this is not possible already for $\mathcal{SHOIQ}$. As we have seen, the results on the rational extension of $\mathcal{SHIQ}$ in Section 2 and on the minimal model semantics in Section 3 extend to more expressive logics. In particular, by Proposition 2, subsumption in $\mathcal{SHOIQ}^{\mathsf{R}}\mathbf{T}$ (resp. $\mathcal{SROIQ}^{\mathsf{R}}\mathbf{T}$) can be polynomially encoded in $\mathcal{SHOIQ}$ (resp. $\mathcal{SROIQ}$) and used to compute the rational closure. However, Propositions 11 (existence of a minimal

canonical model) and 13 do not hold for more expressive logics. Due to the presence of nominals and their interaction with number restrictions, a consistent $\mathcal{SHOIQ}$ knowledge base may have no canonical model (whence no *minimal* canonical ones). Consider, for instance, the following example:

Example 4. Consider the knowledge base $K = (\mathcal{T}, \mathcal{A})$, where $\mathcal{A} = \emptyset$ and
$\mathcal{T} = \{\{mary\} \sqsubseteq \leq 2\ has\_friend^{-}.Person,\ \ \neg\{mary\} \sqsubseteq\ \exists has\_friend.\{mary\},$
$\mathbf{T}(Student) \sqsubseteq\ Young\}$
According to TBox, Mary is friend of at most 2 persons. Furthermore, all elements in the model which do not correspond to Mary are in the relation $hasFriend$ with Mary. Clearly, the models of K cannot contain more than three elements in their domain (including the interpretation of $mary$). Observe that the knowledge base K is consistent and the concepts $Student \sqcap Young$, $Student \sqcap \neg Young$, $\neg Student \sqcap Young$ and $\neg Student \sqcap \neg Young$ are all satisfiable in K. However, there is no model for K which might satisfy all of the above four concept, as any model of K contains at most three domain elements. Hence, *there is no canonical model* for the knowledge base K above. Indeed, the property of existence of a (minimal) canonical model fails to hold for $\mathcal{SHOIQ}$ knowledge bases.

The notion of canonical model as defined in this paper is, therefore, too strong to characterize rational closure for logics as expressive as $\mathcal{SHOIQ}$. However, if we consider the rational closure construction of K above, the inclusion $\mathbf{T}(Student \sqcap Tall) \sqsubseteq Young$ would belong to the rational closure, as $rank(Student \sqcap Tall \sqcap Young) = 0 < rank$ $(Student \sqcap Tall \sqcap \neg Young) = 1$, which is perfectly reasonable, as we want to conclude that typical tall students are young. The fact that there is no canonical model for K does not impair, in this case, the significance of the rational closure of K. In other cases, however, the rational closure construction may be not adequate to deal with a $\mathcal{SHOIQ}$ knowledge base.

Consider again Example 3 above, but replacing the assertions $CS(c1)$, $B(c2)$ in ABox with the inclusions: $\{c1\} \sqsubseteq CS$ and $\{c2\} \sqsubseteq B$. The rational closure construction, in this example, would assign rank 0 to all atomic concepts as well as to concepts $\{c1\}$ and $\{c2\}$, although there is no model of K at all in which both $c1^I$ and to $c2^I$ have rank 0. In this example, the rational closure of K is inconsistent, as it contains both $\mathbf{T}(\{c1\}) \sqsubseteq \forall taught.A$ and $\mathbf{T}(\{c2\}) \sqsubseteq \forall taught.C$ (and, clearly, $\mathbf{T}(\{c\}) \equiv \{c\}$).

We must observe, however, that to determine if a query F belongs to the rational closure of K, the rational closure construction does not need to check the exceptionality of all the concepts in the KB, but only of the concepts C occurring in F or in $\mathbf{T}(C)$ on the left hand side of some defeasible inclusion in K. In the example above, rational closure does not compute the rank of $\{c1\}$, unless $\{c1\}$ occurs in F. This observation motivates the definition of an alternative semantics for rational closure, the $\mathbf{T}$-*minimal model semantics* introduced in [23] to deal with the inadequacy of the canonical model semantics in a language with nominals. Although such a semantics was introduced for the rational extension of $\mathcal{SROEL}(\sqcap, \times)$ [34] (at the basis of the OWL EL ontology language), it could be adopted for expressive DLs as well. It weakens the requirement on canonical models as "only for the concepts C such that $\mathbf{T}(C)$ occurs in the KB K (or in the query), an instance of C is required to exist in the model, when C is satisfiable

in K" [23]. In Examples 4 and 3 above, the $\mathbf{T}$-minimal models of K exist and give the same ranks to concepts as the rational closure, when excluding nominals.

The $\mathbf{T}$-minimal models allows to capture the meaning of the rational closure in cases when minimal canonical models do not, nevertheless there are KBs with multiple $\mathbf{T}$-minimal models or no $\mathbf{T}$-minimal models (we refer to [23] for examples). The syntactic condition that nominals should not occur in the scope of $\mathbf{T}$ in the KB and in the query (and the same holds for all concepts equivalent to some nominal), is not sufficient to fill the mismatch between the $\mathbf{T}$-minimal model semantics and the rational closure as the rational closure construction checks the exceptionality of each concept 'per se', and it does not take into consideration the general constraints in the KB. This mismatch is already evident in $\mathcal{SROEL}(\sqcap, \times)^{\mathsf{R}}\mathbf{T}$ from the fact that computing $\mathbf{T}$-minimal entailment is a Π_2^P-hard problem [23] while computing the rational closure is in P as rational entailment $\models_{sroelrt}$ can be polynomially encoded into Datalog [24].

Consider the following KB $K' = (\mathcal{T}, \emptyset)$ in $\mathcal{SROIQ}^{\mathsf{R}}\mathbf{T}$, where $\mathcal{T} = \{\mathbf{T}(A) \sqsubseteq C$, $\mathbf{T}(B) \sqsubseteq D$, $\exists U.C \sqcap \exists U.D \sqsubseteq \bot\}$ and U is the universal role. The rational closure of K' assigns rank 0 to both A and B, but there is *no model* of K' containing both an A element and a B element. In the minimal models of K', either A has rank 0 and B has no rank, or vice versa. In particular, K' has no $\mathbf{T}$-minimal models.

Given the examples above, we cannot expect that for any KB the rational closure might always provide reliable consequences. A natural way of identifying those KBs which are suitable for defining the rational closure is to check for the existence of a model $\mathcal{M}$ of the KB whose rank assignment is coherent with the rank assignment of the rational closure, in the following sense: (1) $k_{\mathcal{M}}(C) < k_{\mathcal{M}}(D)$, for all concepts C, D such that $rank(C) < rank(D)$ and (2) $k_{\mathcal{M}}(C) = k_{\mathcal{M}}(D)$, for all concepts C, D such that $rank(C) = rank(D)$. This consistency check would exclude, for instance the knowledge base K' above, as there is no model of K' in which both A and B have rank 0, while it would accept the KBs in Examples 4 and 3 as suitable for defining the rational closure.

Verifying the existence of a model of a knowledge base K coherent with the rank assignment of the rational closure is not immediate and can be done by checking the satisfiability of a KB K_L obtained form K by adding the following inclusions (where $L_0, \ldots, L_n$ are new concept names, for n the maximum rank $< \infty$ in the rational closure construction): $L_0 \equiv \top,\ L_i \sqsubseteq L_{i-1},\ L_i \sqsubseteq \neg\mathbf{T}(L_{i-1})$ and

$A \sqsubseteq \mathbf{T}(\top)$, for all concepts A such that $rank(A) = 0$;

$A \sqsubseteq \mathbf{T}(L_i)$, for all concepts A such that $rank(A) = i$.

These inclusions enforce conditions (1) and (2) above. Nevertheless, a correspondence between the rational closure and a minimal model semantics for the KBs which satisfy the consistency check above is still to be developed and requires further investigation.

Due to the considerations above, we can regard the correspondence result for $\mathcal{SHIQ}$ only as a first step in the definition of a semantic characterization of rational closure for expressive description logics. The correspondence between $\mathbf{T}$-minimal models and the rational closure has to be investigated both for low complexity and for expressive DLs, possibly identifying more restricted fragments of the language for which the rational closure construction provides reliable information. Further refinements of the seman-

tics and of the rational closure construction might be needed to deal with knowledge bases as K' above. We leave this investigation for future work.

8 Related Works

As mentioned in the Introduction, in the literature there are many proposals of non-monotonic extensions of description logics [38, 1, 10, 12, 15, 18, 30, 4, 16, 3, 6, 37, 33, 31, 5, 2]. In the following we mostly restrict our discussion to approaches explicitly dealing with defeasible inclusions and we refer to [18, 21] for further comparisons.

In [16, 18], nonmonotonic extensions of DLs based on the **T** operator have been proposed. In these extensions, focused on the basic DL $\mathcal{ALC}$, the semantics of **T** is based on preferential models [32] without requiring modularity. Nonmonotonic inference is obtained by restricting entailment to *minimal models*, where minimal models are those minimizing the interpretation of $\neg\Box\neg C$ concepts. This notion of minimal model semantics has strong relation with the semantics of circumscriptive KBs [3] and the complexity of minimal entailment is already CO-NEXP^{NP} for $\mathcal{ALC}$ [18]. In this work, we have presented an alternative notion of minimal model semantics, in which the notion of minimality is independent from the language and is only determined by the relational structure of models. Under this notion of minimality, the complexity of minimal entailment drops to EXPTIME for both $\mathcal{ALC}^{\mathsf{R}}\mathbf{T}$ and $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$.

The first notion of rational closure for DLs was defined by Casini and Straccia in [6], based on the construction proposed by Freund [13] for propositional logic. In [5] a semantic characterization of a variant of the rational closure in [6] has been presented. The major difference between our construction and those is [6, 5] is in the notion of exceptionality: our definition exploits preferential entailment, while [6, 5] directly use entailment in $\mathcal{ALC}$ over a materialization of the KB. In this paper we have shown that, under the condition that the **T** operator only occurs on the left hand side of defeasible inclusions in a $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ knowledge base, the rational closure of TBox can be computed using entailment in $\mathcal{SHIQ}$ and the rational closure construction requires a quadratic number of calls (in the number of typicality assertions in the KB) to a $\mathcal{SHIQ}$ reasoner.

It is well known that rational closure has some weaknesses that accompany its well-known qualities. Among the weaknesses is the fact that one cannot separately reason property by property, so that, if a subclass of C is exceptional for a given aspect, it is exceptional tout court and does not inherit any of the typical properties of C. Among the strengths of rational closure is its computational lightness, which is crucial in Description Logics. To overcome the limitations of rational closure, in [7, 9] an approach is introduced based on the combination of rational closure and *Defeasible Inheritance Networks*, in [8] the lexicographic closure introduced by Lehmann [36] is extended to $\mathcal{ALC}$, and in [25] a refinement of the semantics of rational closure has been developed, where models are equipped with several preference relations. The approach in [38], based on default inheritance reasoning and on the use of specificity to resolve conflicts among defaults, also avoids the "all or nothing" problem of rational closure. However, computing an extension (or deciding if there is an extension) is shown to be NP-hard already for tractable DL fragments.

A new non monotonic description logics, which also deals with the above mentioned problem, is $\mathcal{DL}^N$ [2], which supports normality concepts and enjoys good computational properties. In particular, $\mathcal{DL}^N$ preserves the tractability of low complexity

DLs, including $\mathcal{EL}^{\bot^{++}}$ and $DL\text{-}lite$. The logic incorporates a notion of overriding, namely the idea that more specific inclusions override less specific ones. A difference with rational closure is that, in case there are unresolved conflicts among defeasible inclusions with the same preference, in $\mathcal{DL}^N$ inheritance is not blocked and the conflict is made explicit through the inconsistency of some normality concept. This happens, for instance, in the Nixon diamond (see Example 9 in [2]), where the normality concept $NRepQuaker$ (representing the prototypical republican quakers) can be inferred to be inconsistent, showing an unresolved conflict between the two defeasible inclusions $Quaker \sqsubseteq_n Pacifist$ and $Republican \sqsubseteq_n \neg Pacifist$. In this example, in the rational closure construction the concept $\mathbf{T}(Quaker \sqcap Republican)$ is not inconsistent but it does neither inherits the properties of typical Quakers nor those of typical Republicans (as rational closure assigns rank 1 to $Quaker \sqcap Republican$, while rank 0 to $Quaker$ and to $Republican$). We can observe, however that rational closure is not always able to accommodate conflicts by blocking inheritance, and in Section 7 we have seen examples in which the rational closure is inconsistent as well.

Recent works discuss the combination of open and closed world reasoning in DLs. In particular, some combinations of DLs with LP languages have been proposed, for instance under the answer set semantics and the well-founded semantics in [12, 11], under the MKNF semantics [37], as well as in Datalog +/- [26]. In [31] a general DL language is introduced, which extends $\mathcal{SROIQ}$ with nominal schemas and epistemic operators as defined in [37], and encompasses some of the most prominent nonmonotonic rule languages (including Datalog under the answer set semantics). A grounded circumscription approach for DLs with local closed world capabilities has been defined in [33]. In [23] the $\mathbf{T}$-minimal model semantics is introduced to strengthen rational entailment for $\mathcal{SROEL}(\sqcap, \times)^{\mathsf{R}}\mathbf{T}$ KBs, and Answer Set Preferences are used for reasoning under minimal entailment. Rational closure and its relations with the $\mathbf{T}$-minimal model semantics are not studied in [23].

9 Conclusions

In this paper we have studied an extension of the rational closure defined by Lehmann and Magidor to the Description Logic $\mathcal{SHIQ}$, for knowledge bases containing typicality inclusions of the form $\mathbf{T}(C) \sqsubseteq D$. Extending the semantic characterization of rational closure to a logic like $\mathcal{SHIQ}$, which does not enjoy the finite model property [29], rises new problems with respect to the case of $\mathcal{ALC}$ studied in [21], for which the finite model property holds. We have shown that in all minimal models of a finite KB in $\mathcal{SHIQ}$ the rank of domain elements is always finite, although the domain might be infinite, and we have exploited this result to establish the correspondence between minimal models under the canonical model semantics and the rational closure construction for $\mathcal{SHIQ}$. We have proved an EXPTIME upper bound for reasoning with the rational closure and shown that the rational closure of a TBox in $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ can be computed using entailment in $\mathcal{SHIQ}$, showing that, for the knowledge bases in which the typicality operator can only occur on the left hand side of typicality inclusions, entailment in $\mathcal{SHIQ}^{\mathsf{R}}\mathbf{T}$ can be reduced to entailment in $\mathcal{SHIQ}^{\mathsf{P}}\mathbf{T}$ which, in turn, has a linear encoding into entailment in $\mathcal{SHIQ}$.

A more general rational closure construction for dealing with KBs allowing for arbitrary occurrences of the typicality operator has not been studied so far. As already

observed (see Section 7), for arbitrary KBs Proposition 2.8 would not work but, to exploit the linear encoding to $\mathcal{SHIQ}$, a definition of rational closure in this case could be given using preferential entailment rather then rational entailment. However, the correspondence with the preferential semantics (if any) might then become less direct.

The rational closure construction can be applied in principle to more expressive description logics. However, we have seen in Section 7 that the semantic characterization developed for $\mathcal{ALC}$ and $\mathcal{SHIQ}$ cannot be directly extended to stronger logics, such as $\mathcal{SHOIQ}$ or $\mathcal{SROIQ}$, as the notion of canonical model is too strong to deal with nominals and with the universal role. Even if the **T**-minimal model semantics allows to capture a larger set of KBs, and we can check for the existence of a model of the KB which is coherent with the rank assignment in the rational closure, further investigation is needed to provide a semantic characterization of rational closure for more expressive description logics.

To address the weakness of rational closure mentioned in Section 8 and to avoid the "all or nothing" problem, in [25] a finer grained semantics where models are equipped with several preference relations is considered, which is shown to correspond to a refinement of the rational closure semantics. The extension of the rational closure construction to accommodate this refinement is left for future work.

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