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ENTROPY DISSIPATION ESTIMATES FOR THE LINEAR BOLTZMANN OPERATOR

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ABSTRACT. We prove a linear inequality between the entropy and entropy dissipation functionals for the linear Boltzmann operator (with a Maxwellian equilibrium background). This provides a positive answer to the analogue of Cercignani's conjecture for this linear collision operator. Our result covers the physically relevant case of hard-spheres interactions as well as Maxwellian kernels, both with and without a cut-off assumption. For Maxwellian kernels, the proof of the inequality is surprisingly simple and relies on a general estimate of the entropy of the gain operator due to Matthes and Toscani (2012); Villani (1998). For more general kernels, the proof relies on a comparison principle. Finally, we also show that in the grazing collision limit our results allow to recover known logarithmic Sobolev inequalities.

CONTENTS

1. INTRODUCTION

1.1. Setting of the problem and main result. The use of Lyapunov functionals is a well-known technique to study the asymptotic behavior of dynamical systems, and in the theory of the Boltzmann equation and related models it is now a classical tool. For the nonlinear, spatially homogeneous Boltzmann equation

$$\partial_t f = \mathcal{Q}(f,f), \qquad f(0,v) = f_0(v), \qquad v \in \mathbb{R}^d,\ t \geqslant 0, \tag{1.1}$$

posed for a function $f = f(t,v)$ depending on $t \geqslant 0$ and $v \in \mathbb{R}^d$, it is a well-known fact that $f(t,v)$ converges (as $t \to \infty$) towards the Maxwellian distribution M_f with same mass, momentum and energy as f_0,

$$M_f(v) = \frac{\varrho_f}{(2\pi\, E_f)^{d/2}} \exp\left(-\frac{|v-\mathbf{u}_f|^2}{2\, E_f}\right), \qquad v \in \mathbb{R}^d,$$

where

$$\left.\begin{aligned}\varrho_f &= \int_{\mathbb{R}^d} f(t,v)\,\mathrm{d}v = \int_{\mathbb{R}^d} f_0(v)\,\mathrm{d}v,\\ \varrho_f \mathbf{u}_f &= \int_{\mathbb{R}^d} f(t,v) v\,\mathrm{d}v = \int_{\mathbb{R}^d} f_0(v) v\,\mathrm{d}v,\\ d\,\varrho_f E_f &= \int_{\mathbb{R}^d} f(t,v)|v-\mathbf{u}_f|^2\,\mathrm{d}v = \int_{\mathbb{R}^d} f_0(v)\,|v-\mathbf{u}_f|^2\,\mathrm{d}v\end{aligned}\right\}\qquad \text{for all } t \geqslant 0.$$

Notice that eq. (1.1) conserves density, momentum and kinetic energy which explains why the above quantities ϱ_f, $\mathbf{u}_f$ and E_f are constant in time. The *Shannon-Boltzmann relative entropy* of f with respect to the Maxwellian distribution M_f

$$\mathcal{H}(f|M_f) := \int_{\mathbb{R}^d} f(v)\log\frac{f(v)}{M_f(v)}\,\mathrm{d}v \tag{1.2}$$

is a Lyapunov functional, that is, it is decreasing along solutions to (1.1): if $f = f(t,v)$ solves (1.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f|M_f) = -\mathcal{D}(f) \leqslant 0, \tag{1.3}$$

where the functional $\mathcal{D}$ is called the *entropy dissipation*. The question of whether one can find a functional inequality between $\mathcal{H}$ and $\mathcal{D}$ of the form

$$\mathcal{D}(f) \geqslant \lambda\,\mathbf{\Phi}(\mathcal{H}(f|M_f))$$

valid for some $\lambda > 0$, some nondecreasing continuous function $\mathbf{\Phi} : [0,+\infty) \to [0,+\infty)$ with $\mathbf{\Phi}(0) = 0$, and all functions f (with f possibly satisfying some additional suitable bounds), is generally known as *Cercignani's conjecture*. It has several variants and a long history (e.g. Carlen and Carvalho (1994); Toscani and Villani (1999); Villani (2003); see the recent review by Desvillettes et al. (2011) for further details). If true, this inequality gives a lot of information on the asymptotic behavior of (1.1), since then one obtains the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f(t)|M_f) \leqslant -\lambda\,\mathbf{\Phi}(\mathcal{H}(f(t)|M_f)),$$

from which one can deduce that $\mathcal{H}(f(t)|M_f)$ converges to 0 as $t \to +\infty$, with an explicit rate. Notice that due to the Csiszár-Kullback-Pinsker inequality the convergence of $\mathcal{H}(f(t)|M_f)$ towards 0 implies the convergence in $L^1(\mathbb{R}^d)$ of $f(t,v)$ towards M_f. Unfortunately, the available versions of Cercignani's conjecture do not yield an optimal rate of convergence of $f(t,v)$ towards M_f. However, the use of this Lyapunov functional approach combined with a careful spectral analysis of the linearized Boltzmann operator allow to recover an exponential convergence to equilibrium (Mouhot, 2006).

We are interested in studying the corresponding conjecture in the case of the *linear* Boltzmann equation which, though simpler, has not yet been settled. Let us describe the model in more detail before explaining our results. The homogeneous, linear Boltzmann equation is given by

$$\partial_t f = \mathcal{Q}(f,M) = \mathcal{L}f, \qquad f(0,v) = f_0(v), \qquad t \geqslant 0,\ v \in \mathbb{R}^d, \tag{1.4}$$

where $\mathcal{Q}$ is the bilinear Boltzmann operator,

$$\mathcal{Q}(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|q|,\xi)\Big(f(v')g(v'_*) - f(v)g(v_*)\Big)\,\mathrm{d}v_*\,\mathrm{d}n. \tag{1.5}$$

Here $q = v - v_*$ is the relative velocity, $\xi = |q\cdot n|/|q|$, and v' and v'_* are the pre-collisional velocities which result, respectively, in v and v_* after the elastic collision

$$v' = v - (q\cdot n)n, \qquad v'_* = v_* + (q\cdot n)n. \tag{1.6}$$

The particle distributions f and g are nonnegative functions of the velocity variable $v \in \mathbb{R}^d$ and $B(|q|,\xi)$ is a nonnegative function usually called the *collision kernel*. We will assume throughout this paper that the function M appearing in (1.4) is a given normalized Maxwellian distribution with unit mass:

$$M(v) = \Big(2\pi\theta\Big)^{-d/2}\exp\left(-\frac{|v-u_0|^2}{2\theta}\right), \qquad v \in \mathbb{R}^d, \tag{1.7}$$

where $u_0 \in \mathbb{R}^d$ is the *bulk velocity* and $\theta > 0$ is the *effective temperature*. We notice that the normalization of M is not a loss of generality since a time scaling of (1.4) easily translates into results for non-normalized

Maxwellians. Similarly, since (1.4) is linear, for simplicity we will assume throughout that the solution f also has mass 1:

$$\int_{\mathbb{R}^d} f(t,v)\,\mathrm{d}v = \int_{\mathbb{R}^d} f_0(v)\,\mathrm{d}v = 1, \qquad t \geqslant 0.$$

Galilean invariance and a scaling in v also easily show that one may study only the case $\theta = 1$, $u_0 = 0$. However, we will state all results for (1.7) in order to make clear how inequalities depend on them.

We shall investigate in this paper collision operators $\mathcal{L} = \mathcal{L}_B$ corresponding to various collision kernels $B = B(|q|,\xi)$ but shall most often deal with kernels that factor as

$$B(|q|,\xi) = \beta(|q|)\,b(\xi) \tag{1.8}$$

for some measurable nonnegative mappings $b : [0,1] \to [0,\infty)$ and $\beta(\cdot) : [0,\infty) \to [0,\infty)$. For the purposes of proofs we always work with the cut-off assumption that

$$\int_{\mathbb{S}^{d-1}} b(\tilde{q}\cdot n)\,\mathrm{d}n < +\infty, \tag{1.9}$$

(where $\tilde{q} = q/|q|$), though our results apply also to non-cutoff kernels (just because the entropy dissipation is larger in that case; see Remark 1.2). We always deal with *hard potential interactions*, that is, collision kernels with β nondecreasing.[1] In particular, we will deal with

$$B(|q|,\xi) = c_d|q|^\gamma \xi^{d-2}, \tag{1.10}$$

for $\gamma \geqslant 0$ (with c_d a normalization constant). In dimension $d = 3$, the case $\gamma = 1$ is the case of *hard-spheres interactions*, while the $\gamma = 0$ corresponds to the *Maxwell molecules interaction*, that is,

$$B(|q|,\xi) = B_{\mathrm{hs}}(|q|,\xi) = c_d|q\cdot n| = c_d|q|\xi \qquad \text{(Hard-spheres)}, \tag{1.11}$$

$$B(|q|,\xi) = B_{\max}(|q|,\xi) = c_d\frac{|q\cdot n|}{|q|} = c_d\xi \qquad \text{(Maxwell molecules)}. \tag{1.12}$$

We will also deal with general *Maxwellian collision kernels*, that is, kernels which depend only on ξ:

$$B(|q|,\xi) = b(\xi) \tag{1.13}$$

for some measurable function $b : [0,1] \to [0,+\infty)$. (The Maxwell molecules approximation (1.12) being a particular case.) We say a Maxwellian collision kernel is *normalized* when, for any $\tilde{q} \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} b(\tilde{q}\cdot n)\,\mathrm{d}n = |\mathbb{S}^{d-1}| \int_0^1 b(\xi)(1-\xi^2)^{\frac{d-3}{2}}\,\mathrm{d}\xi = 1, \tag{1.14}$$

where $|\mathbb{S}^{d-1}|$ represents the $(d-1)$-dimensional volume of $\mathbb{S}^{d-1}$.

Equation (1.4) is sometimes known also as the *scattering* equation, and can be interpreted as giving the time evolution of the velocity distribution of a cloud of particles, homogeneously distributed in space. These particles do not interact among themselves, but only with background particles whose distribution is given by M, considered as a thermal bath in the sense that it remains unchanged even after interaction with the cloud of particles (this is reasonable if, for example, the total mass of the cloud of particles is much smaller than that of the background). Equation (1.4) conserves density (i.e., $\int_{\mathbb{R}^d} f(t,v)\,\mathrm{d}v = \int_{\mathbb{R}^d} f(0,v)\,\mathrm{d}v$ for all t), but in contrast with the nonlinear Boltzmann equation, momentum and kinetic energy are not conserved due to the interaction with the background. Except in the special case of *Maxwellian molecules*, no explicitly solvable differential equations can be derived for the evolution of the momentum and the kinetic energy. (For the explicit time evolution of momentum, energy and temperature in the case of Maxwell molecules see for example Spiga and Toscani (2004).)

It will be sometimes convenient to express the collision operator $\mathcal{L}$ in the following weak form:

$$\int_{\mathbb{R}^d} \psi(v)\mathcal{L}f(v)\,\mathrm{d}v = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|q|,\xi)f(v)M(v_*)\big(\psi(v') - \psi(v)\big)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}n \tag{1.15}$$

for any sufficiently regular ψ. On the other hand, $\mathcal{L}$ can also be written in the form

$$\mathcal{L}f(v) = \int_{\mathbb{R}^d} k_B(w,v)f(w)\,\mathrm{d}w - \sigma_B(v)f(v), \qquad v \in \mathbb{R}^d \tag{1.16}$$

[1] Notice that, for collision kernel of the above shape, if $\beta(\cdot)$ is such that $\liminf_{r\to\infty}\beta(r) = 0$, the convergence towards equilibrium is not expected to be exponential (for instance, the spectrum of $\mathcal{L}$ in the L^2 space with weight M^{-1} does not have a spectral gap)

for a kernel $k_B(v,w) \geqslant 0$ which depends of course on the collision kernel B (see for instance Carleman (1957)), and with

$$\sigma_B(v) = \int_{\mathbb{R}^d} k_B(v,w)\,\mathrm{d}w, \qquad v \in \mathbb{R}^d.$$

The kernel k_B can be written explicitly in some cases; see e.g. Arlotti and Lods (2007). For a general expression of k_B see the discussion leading to equation (3.9). One sees then that eq. (1.4) is the Kolmogorov forward equation for a Markov process on $\mathbb{R}^d$ with invariant measure, or equilibrium, M (notice that $\mathcal{L}(M) = \mathcal{Q}(M,M) = 0$ regardless of the collision kernel B), and it is well known that the relative entropy (1.2) with respect to the equilibrium is a Lyapunov functional for any equation of this type (see for example Chafaï (2004) or Michel et al. (2005).) In addition, $\mathcal{L}$ satisfies the *detailed balance* condition, that is,

$$M(v)k_B(v,w) = M(w)k_B(w,v), \qquad v,w \in \mathbb{R}^d, \tag{1.17}$$

which translates to the fact that $\mathcal{L}$ is symmetric in $L^2(\mathbb{R}^d, M(v)^{-1}\,\mathrm{d}v)$. Using this, we can explicitly write the time derivative of $\mathcal{H}(f|M)$ along solutions to (1.4):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f(t)|M) = \int_{\mathbb{R}^d} \mathcal{L}f(t,v)\log\left(\frac{f(t,v)}{M(v)}\right)\mathrm{d}v = -\mathcal{D}(f(t)) \tag{1.18}$$

where the *entropy dissipation* $\mathcal{D}(f)$ is

$$\mathcal{D}(f) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|q|,\xi)M(v)M(v_*)\Psi\left(\frac{f(v)}{M(v)},\frac{f(v')}{M(v')}\right)\mathrm{d}n\,\mathrm{d}v_*\,\mathrm{d}v, \tag{1.19}$$

with $\Psi(x,y) := (x-y)(\log x - \log y) \geqslant 0$. Alternatively, we can also write

$$\mathcal{D}(f) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} M(v)k_B(v,v')\Psi\left(\frac{f(v)}{M(v)},\frac{f(v')}{M(v')}\right)\mathrm{d}v\,\mathrm{d}v', \tag{1.20}$$

where k_B is the kernel appearing in (1.16).

It is interesting then to look for inequalities of the form $\mathcal{D}(f) \geqslant \lambda\mathcal{H}(f|M)$, for some $\lambda > 0$, since clearly this implies that any solution f to (1.4) with mass 1 satisfies

$$\mathcal{H}(f(t)|M) \leqslant \mathcal{H}(f_0|M)\exp(-\lambda t) \qquad \forall t \geqslant 0$$

yielding exponential convergence to the equilibrium M in the entropic sense (notice that our Maxwellian M was also normalized to have mass 1). The following is our main result regarding this:

Theorem 1.1. *Let $\mathcal{D}$ be the entropy dissipation functional* (1.19) *and consider either a hard-potential collision kernel B of the form* (1.10) *with $\gamma > 0$, or any normalized Maxwellian collision kernel (i.e., satisfying* (1.13) *and* (1.14)*). There exists a positive constant $\lambda = \lambda(B) > 0$ such that*

$$\mathcal{D}(f) \geqslant \lambda\,\mathcal{H}(f|M) \tag{1.21}$$

holds for any probability distribution $f \in L^1(\mathbb{R}^d)$.

If the collision kernel is Maxwellian then one may take

$$\lambda = \gamma_b := \int_{\mathbb{S}^{d-1}} (\tilde{q}\cdot n)^2 b(\tilde{q}\cdot n)\,\mathrm{d}n = |\mathbb{S}^{d-1}|\int_0^1 \xi^2 b(\xi)(1-\xi^2)^{\frac{d-3}{2}}\,\mathrm{d}\xi \in (0,1), \qquad \tilde{q} \in \mathbb{S}^{d-1}. \tag{1.22}$$

Notice that the value of γ_b does not depend on $\tilde{q}$ due to radial symmetry, and is a number strictly between 0 and 1 due to normalization.

Remark 1.2. *Notice that our results actually cover non cut-off kernels for which* (1.9) *is not satisfied, as long as they can be bounded below by a collision kernel to which the above theorem applies. Indeed, if $b : [0,1] \to \mathbb{R}^+$ is such that $\int_{\mathbb{S}^{d-1}} b(\tilde{q}\cdot n)\,\mathrm{d}n = +\infty$ (remember this integral does not depend on $\tilde{q} \in \mathbb{S}^{d-1}$) then removing the singularities, we can bound b from below by some measurable $b_0 : [0,1] \to \mathbb{R}^+$ satisfying*

$$\int_{\mathbb{S}^{d-1}} b_0(\tilde{q}\cdot n)\,\mathrm{d}n < +\infty, \qquad \tilde{q} \in \mathbb{S}^{d-1}.$$

Since, as one sees from (1.19)*, the* entropy dissipation *functional is monotone with respect to the collision kernel (while the relative entropy is obviously independent of the collision mechanism!), the result obtained for the cut-off kernel b_0 applies to the original kernel b. It is however likely that the obtained bound is far from being optimal.*

Also, for the hard spheres kernel (1.11) in dimension $d = 3$ we may take $\lambda = \sqrt{\theta}/4$ (see Example 3.6). In fact, we are able to give a general condition on B ensuring that inequality (1.21) holds; see Theorem 3.4. This inequality is part of a larger family of inequalities relating other Lyapunov functionals of (1.4) to their dissipations (see section 3.1), of which a prominent example is the *spectral gap inequality*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|q|,\xi)M(v)M(v_*)\left(\frac{f(v)}{M(v)}-\frac{f(v')}{M(v')}\right)^2 \mathrm{d}n\,\mathrm{d}v_*\,\mathrm{d}v \geqslant \frac{\lambda_2}{2}\int_{\mathbb{R}^d} M\left(\frac{f}{M}-1\right)^2 \mathrm{d}v. \qquad (1.23)$$

This inequality was already studied in Lods et al. (2008), and it implies exponential relaxation to equilibrium in the L^2 norm with weight M^{-1} for equation (1.4). However, it gives a different information from inequality (1.21), since convergence is given in a different distance: the spectral gap result gives convergence in a stronger topology, but also requires the initial condition to have stronger decay for large v. Of course, the best possible constants λ_2 and λ may be different as well (always with $\lambda \leqslant \lambda_2/2$, see Ané et al. (2000); Bakry et al. (2014)) implying different exponential relaxation speeds.

1.2. **Link to Logarithmic Sobolev inequalities.** In addition to being fundamental in the study of the asymptotic behavior of (1.4), entropy dissipation inequalities of the form of (1.21) have interesting links to results in the theory of Markov processes and have been the subject of several recent studies in discrete settings. In the framework of discrete, time-continuous Markov processes the study of inequalities such as (1.21) is relatively recent. They are often referred to as a type of "*modified logarithmic Sobolev inequalities*" in this context; see Bobkov and Tetali (2006); Bakry et al. (2014) for recent results and a summary of related literature. Comparatively, entropy dissipation inequalities for continuous-space processes have been little studied, so it is interesting to see whether more general techniques can be developed for them. The idea of studying the convergence to equilibrium of a Markov process in terms of the relative entropy to the invariant measure is in fact much older, but it has usually been done by means of logarithmic Sobolev inequalities instead of (1.21). For our linear operator $\mathcal{L}$, this would be an inequality of the form

$$\mathscr{E}\left(\sqrt{M}\sqrt{f}\right) \geqslant \lambda_0 H(f|M) \qquad (1.24)$$

for some $\lambda_0 > 0$ and all probability distributions f in $\mathbb{R}^d$, where $\mathscr{E}$ is the Dirichlet form associated to $\mathcal{L}$:

$$\mathscr{E}(g) := -\int_{\mathbb{R}^d} g(v)\mathcal{L}g(v)\,M^{-1}(v)\,\mathrm{d}v.$$

This approach is followed, for example, in Diaconis and Saloff-Coste (1996). Though it is written there for discrete models, one can easily follow the same arguments here in order to see that (1.24) would imply (1.21) with $\lambda = \lambda_0$. The interesting problem with this approach is that for our continuous model the logarithmic Sobolev inequality (1.24) *cannot hold*. The reason for this is that, as is well-known (Gross, 1975, 1993), the log-Sobolev inequality (1.24) is equivalent to an $L^q - L^p$ regularizing property of solutions of equation (1.4), known as *hypercontractivity* which does not hold for solutions to (1.4) (see a quick proof of this fact in Appendix A). Hence we have that

Theorem 1.3. *Under the cut-off assumption* (1.9)*, there is no constant $\lambda_0 > 0$ such that inequality* (1.24) *holds for all probability distributions f in $\mathbb{R}^d$.*

Hence, the linear Boltzmann operator is an interesting case in which the entropy dissipation (or modified log-Sobolev) inequality (1.21) holds, but the log-Sobolev inequality (1.24) does not!

The links between our modified log-Sobolev inequality (1.21) and true log-Sobolev inequalities turn out to be tighter than expected. Recall that the well-known Gaussian log-Sobolev inequality (also known as Stam-Gross inequality) asserts that

$$I(f|M) \geqslant \frac{2}{\theta} H(f|M) \qquad (1.25)$$

for any $f \in L^1(\mathbb{R}^d)$ with unit mass. Here $I(f|M)$ is the relative Fisher information

$$I(f|M) = \int_{\mathbb{R}^d} f(v)\left|\nabla \log\left(\frac{f(v)}{M(v)}\right)\right|^2 \mathrm{d}v.$$

The above functional inequality is, as well-known, the entropy-entropy dissipation estimate for the Fokker-Planck equation

$$\partial_t \varrho(t,v) = \nabla\cdot\left(\nabla\varrho(t,v) - \frac{\nabla M(v)}{M(v)}\varrho(t,v)\right) \qquad (1.26)$$

since the time derivative of $H(\varrho|M)$ along solutions to (1.26) exactly yields

$$\frac{\mathrm{d}}{\mathrm{d}t}H(\varrho(t)|M) = -I(\varrho(t)|M) \qquad \forall t \geqslant 0.$$

In Proposition 5.2 below we are able to show that inequality (1.21) also holds for the following family of collision kernels (depending on $\epsilon \in (0,1]$):

$$B_\epsilon(|q|,\xi) = |q| b_\epsilon(\xi), \qquad b_\epsilon(\xi) = \xi \mathbb{1}_{[0,\epsilon]}(\xi), \tag{1.27}$$

where $\mathbb{1}_{[0,\epsilon]}$ denotes the characteristic of the interval $[0,\epsilon]$. As $\epsilon \to 0$, a suitable scaling of equation (1.4) with this collision kernel approaches a Fokker-Planck equation (with a diffusion matrix different from the identity; see Lods and Toscani (2004).) The dependence of λ on ϵ actually enables us to recover in the limit $\epsilon \to 0$ a version of (1.25) for that diffusion matrix (notice that (1.25) corresponds to the case of an identity diffusion matrix); details of this are given in Section 5. This procedure can be understood as a microscopic validation of well-known logarithmic Sobolev inequalities.

There are interesting similarities between this result and one derived in Bobkov and Tetali (2006): it is shown there that one may obtain (1.25) as the limit of certain discrete modified log-Sobolev inequalities. We show a similar result here, but through a completely different limiting process.

1.3. **Method of proof.** Our proof of (1.21) consists in first proving the result for the dissipation $D_{\max}$ of the linear Boltzmann operator $\mathcal{L}_{\max}$ associated with a Maxwellian collision kernel and then deducing the result for other collision kernels by a comparison argument. Namely, one of the main steps in our proof is the following comparison result whose proof closely follows the lines of a similar result proved for the study of the spectral gap of $\mathcal{L}_{\rm hs}$ (Lods et al., 2008, Proposition 3.3):

Proposition 1.4. *Take $\gamma \geqslant 0$ and let $\mathcal{D}_\gamma$ denote the entropy dissipation functional of the linear Boltzmann operator associated to the collision potential* (1.10) *(so that $\gamma = 0$ corresponds to Maxwellian molecules interactions). There is some positive explicit constant $C > 0$, depending only on γ, such that*

$$\mathcal{D}_\gamma(f) \geqslant C\theta^{\gamma/2}\mathcal{D}_0(f)$$

for any probability distribution f.

For the proof of this (in a more general statement that allows for comparing dissipations of other Lyapunov functionals) see Proposition 3.5.

Then one sees that in order to prove Theorem 1.1 it is enough to prove it for a normalized Maxwellian collision kernel. This lends itself to significant simplification since, as is well-known, Maxwellian collision kernels generally allow for explicit computations. Here is the heart of the argument, which we give for simplicity in the Maxwellian molecules case (i.e., for B given by (1.12)). In this case, the linear Boltzmann operator $\mathcal{L}_{\max}$ can be written as

$$\mathcal{L}_{\max}(f) = \mathcal{L}^+_{\max}(f) - f.$$

Now, the operator $\mathcal{L}^+_{\max}(f) = \mathcal{Q}^+_{\max}(f,M)$ satisfies the following analog of the Shannon-Stam inequality (Villani, 1998, Corollary 4.3): for any probability densities f and g it holds that

$$H(\mathcal{Q}^+_{\max}(f,g)) \leqslant \frac{1}{2}H(f) + \frac{1}{2}H(g), \tag{1.28}$$

where H is the *Shannon-Boltzmann entropy*

$$H(f) := \int_{\mathbb{R}^d} f(v)\log f(v)\,\mathrm{d}v, \tag{1.29}$$

defined for any nonnegative $f \in L^1(\mathbb{R}^d)$ with finite energy. We show in Lemma 2.4 that this translates to a contraction property of $\mathcal{L}^+_{\max}$, measured in entropy:

$$\mathcal{H}(\mathcal{L}^+_{\max}f|M) \leqslant \frac{1}{2}\mathcal{H}(f|M).$$

This allows us to write

$$\begin{aligned}\mathcal{D}_{\max}(f) = \int_{\mathbb{R}^d} f\log\left(\frac{f}{M}\right)\mathrm{d}v - \int_{\mathbb{R}^d}\mathcal{L}^+_{\max}(f)\log\left(\frac{f}{M}\right)\mathrm{d}v \\ = \mathcal{H}(f|M) - \mathcal{H}(\mathcal{L}^+_{\max}f|M) + \mathcal{H}(\mathcal{L}^+_{\max}f|f) \geqslant \frac{1}{2}\mathcal{H}(f|M).\end{aligned}$$

Notice that we estimated $\mathcal{H}(\mathcal{L}^+_{\max}f|f) \geqslant 0$ since $\mathcal{L}^+_{\max}(f)$ and f have the same mass. This shows the inequality.

This provides an interesting link between the entropy dissipation inequality (1.21) and the convexity property (1.28) of the gain part $\mathcal{Q}_{\max}^{+}$ of the bilinear Boltzmann operator. The proof of (1.28) was based on a similar contraction property of $\mathcal{Q}_{\max}^{+}(f,g)$ with respect to the Fisher information, along with a representation of the Fisher information as the time-derivative of the entropy along the adjoint Ornstein-Uhlenbeck semigroup (see equation (2.7) in Section 2); we refer to Villani (1998) for a detailed proof. Estimates for other Maxwellian kernels (i.e., depending only on the ξ variable) may be obtained by using extensions of (1.28) which were essentially proved in Matthes and Toscani (2012). We refer to Section 2 for details on this.

1.4. **Structure of the paper.** The plan of the paper is as follows. In Section 2 we prove our results for Maxwellian kernels (including the proof of Theorem 1.1 for Maxwellian kernels.) In Section 3 we prove a more general version of the comparison result in Proposition 1.4 in order to deduce Theorem 1.1 for hard potential interactions, thus completing the proof of Theorem 1.1. In Section 4 we show how the entropy dissipation inequality may be used to give an exponential rate of convergence to equilibrium for eq. (1.4) (which is straightforward) and for a nonlinear Boltzmann equation with particles bath. Finally, we describe in Section 5 the link between our inequality (1.21) and logarithmic Sobolev inequalities. In particular, we recall the Fokker-Planck limit of grazing collisions and some well-known features of log-Sobolev inequalities, and then show how some of them can be recovered from (1.21).

2. Inequalities for Maxwellian collision kernels

We begin in this section with a proof of the following entropy dissipation inequality for the linear Boltzmann operator with a Maxwellian collision kernel:

Theorem 2.1. *Let $B(|q|,\xi)=b(\xi)$ be a normalized Maxwellian collision kernel. Let $\mathcal{D}_{\max}$ denote the associated entropy dissipation functional. For any probability distribution $f=f(v)$ one has*

$$\mathcal{D}_{\max}(f)\geqslant\gamma_b\,\mathcal{H}(f|M) \tag{2.1}$$

with γ_b defined in (1.22).

In order to prove this we need several previous results; the proof of Theorem 2.1 is given at the end of this section. We first prove the following contraction property of the entropy which is essentially contained in Matthes and Toscani (2012):

Lemma 2.2. *Let $B(|q|,\xi)=b(\xi)$ be a normalized Maxwellian collision kernel. For any probability distributions f,g one has*

$$H(\mathcal{Q}_+(f,g))\leqslant(1-\gamma_b)H(f)+\gamma_b H(g) \tag{2.2}$$

where we recall that $H(\cdot)$ denotes the Shannon-Boltzmann entropy defined in (1.29).

Proof. In Matthes and Toscani (2012, eq. (3)) it is proved that for the Fisher information

$$I(f)=\int_{\mathbb{R}^d}\frac{|\nabla f(v)|^2}{f(v)}\,\mathrm{d}v \tag{2.3}$$

an analogous inequality holds

$$I(\mathcal{Q}_+(f,g))\leqslant(1-\gamma_b)I(f)+\gamma_b I(g). \tag{2.4}$$

(Notice that the estimates in Matthes and Toscani (2012) are written in terms of the σ-representation for Boltzmann's operator; here we have written the corresponding expression in the n-representation by a change of variables.) To deduce (2.2) from (2.4), we use a well-known strategy already used in Villani (1998), based on the nice property that the Boltzmann operator commutes with the adjoint Ornstein-Uhlenbeck semigroup. Namely, given a probability measure f, let $\mathcal{S}_t f(v)=\varrho(t,v)$ denote the unique solution (at time $t\geqslant 0$) to the Fokker-Planck equation (1.26) with initial datum $\varrho(0)=f$ (i.e. $(\mathcal{S}_t)_{t\geqslant 0}$ is the adjoint Ornstein-Uhlenbeck semigroup). Whenever $B(|q|,\xi)=b(\xi)$ is a normalized Maxwellian collision kernel we have (Bobylev, 1988)

$$\mathcal{Q}_+(\mathcal{S}_t f,\mathcal{S}_t f)=\mathcal{S}_t\mathcal{Q}_+(f,f)\qquad\forall t\geqslant 0, \tag{2.5}$$

for any $f\in L^1(\mathbb{R}^d)$ with finite energy. Moreover, a well-known property of Fisher information is that

$$H(f)-H(M)=\int_0^\infty\left(I(\mathcal{S}_t f)-I(M)\right)\mathrm{d}t, \tag{2.6}$$

for any $f \in L^1(\mathbb{R}^d)$ with unit mass and finite energy. Combining this with the above commutation property (2.5), one gets the following representation formula:

$$H(\mathcal{Q}_+(f,g)) - H(M) = \int_0^\infty (I(\mathcal{Q}_+(\mathcal{S}_t f, \mathcal{S}_t g) - I(M))\,\mathrm{d}t. \tag{2.7}$$

Applying (2.4) in (2.7) and then (2.6) gives

$$\begin{aligned}
H(\mathcal{Q}_+(f,g)) - &H(M) \\
&\leqslant (1-\gamma_b)\int_0^\infty (I(\mathcal{S}_t f) - I(M))\,\mathrm{d}t + \gamma_b \int_0^\infty (I(\mathcal{S}_t g) - I(M))\,\mathrm{d}t \\
&= (1-\gamma_b)H(f) + \gamma_b H(g) - H(M)
\end{aligned}$$

which completes the proof. □

Remark 2.3. *The proof of* (2.3) *as derived in* Matthes and Toscani (2012) *is based on an explicit representation of $\mathcal{Q}$ in Fourier variables. It is for this reason that it is crucial for their techniques to deal with Maxwellian collision kernels.*

We define the gain part of the linear operator $\mathcal{L}$ by $\mathcal{L}_+(f) = \mathcal{Q}_+(f, M)$. Next we show that $\mathcal{L}_+$ takes a function closer to the equilibrium in the relative entropy sense.

Lemma 2.4. *Let $B(|q|,\xi)$ be a normalized Maxwellian collision kernel and let $\mathcal{L}$ be associated linear Boltzmann operator. Then,*

$$\mathcal{H}(\mathcal{L}_+ f|M) \leqslant (1-\gamma_b)\mathcal{H}(f|M), \tag{2.8}$$

where γ_b is defined by (1.22).

Proof. We have, using Lemma 2.2 with $g = M$

$$\mathcal{H}(\mathcal{L}_+ f|M) = H(\mathcal{L}_+ f) - \int_{\mathbb{R}^d} \mathcal{L}_+ f \log M\,\mathrm{d}v \leqslant (1-\gamma_b)H(f) + \gamma_b H(M) - \int_{\mathbb{R}^d} \mathcal{L}_+ f \log M\,\mathrm{d}v.$$

Since $B(q,\xi)$ is a *normalized* Maxwellian collision kernel, we have that $\mathcal{L}f = \mathcal{L}_+(f) - f$ so that

$$\mathcal{H}(\mathcal{L}_+ f|M) \leqslant (1-\gamma_b)\mathcal{H}(f|M) - \gamma_b \int_{\mathbb{R}^d} (f-M)\log M\,\mathrm{d}v - \int_{\mathbb{R}^d} \mathcal{L}f \log M\;\mathrm{d}v.$$

Thus, (2.8) reduces to showing that

$$-\gamma_b \int_{\mathbb{R}^d} (f-M)\log M\,\mathrm{d}v \leqslant \int_{\mathbb{R}^d} \mathcal{L}f \log M\,\mathrm{d}v,$$

or, in other words, that

$$\int_{\mathbb{R}^d} \mathcal{L}f |v-u_0|^2\,\mathrm{d}v \leqslant -\gamma_b \int_{\mathbb{R}^d} (f-M)|v-u_0|^2\,\mathrm{d}v, \tag{2.9}$$

where we have used that $\displaystyle\int_{\mathbb{R}^d} \mathcal{L}f\,\mathrm{d}v = \int_{\mathbb{R}^d} (f-M)\,\mathrm{d}v = 0$. Actually, (2.9) holds with equality, which can be checked by an explicit calculation which we give in the following Lemma 2.5 for the convenience of the reader. □

The estimate we need in (2.9) can be obtained from the fact that the evolution of the temperature in equation (1.4) is explicit in the Maxwellian case (which was already known; see for example Spiga and Toscani (2004)). We give here a short proof for completeness:

Lemma 2.5. *Let $B(q,\xi) = b(\xi)$ be a normalized Maxwellian collision kernel and $f \in L^1(\mathbb{R}^d; (1+|v|^2)\,\mathrm{d}v)$. Then*

$$\int_{\mathbb{R}^d} \mathcal{L}f(v)|v-u_0|^2\,\mathrm{d}v = -\gamma_b \int_{\mathbb{R}^d} (f(v) - M(v))|v-u_0|^2\,\mathrm{d}v. \tag{2.10}$$

Proof. By using the weak form (1.15) of $\mathcal{L}$ and the fact that $\mathcal{L}(M) = 0$, and writing $h := f - M$ and $\xi := (q \cdot n)/|q|$,

$$\int_{\mathbb{R}^d} \mathcal{L}h(v)|v-u_0|^2\,\mathrm{d}v = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d} h(v)M(v_*)b(\xi)(|v'-u_0|^2 - |v-u_0|^2)\,\mathrm{d}v_*\,\mathrm{d}n\,\mathrm{d}v. \tag{2.11}$$

Notice that

$$|v' - u_0|^2 - |v - u_0|^2 = -\xi^2|v - u_0|^2 - |v_* - u_0|^2\xi^2 + 2(v - u_0)\cdot(v_* - u_0)\xi^2 \\ - 2((v_* - u_0)\cdot n)((v - u_0)\cdot n) + 2((v_* - u_0)\cdot n)^2.$$

When substituted inside (2.11), several of these terms vanish after integration due to either $\int h \, \mathrm{d}v = 0$ or the symmetry of M about u_0. Hence we obtain, using also the normalization of M, that

$$\int_{\mathbb{R}^d} \mathcal{L}h(v)|v - u_0|^2 \, \mathrm{d}v = -\int_{\mathbb{R}^d} |v - u_0|^2 h(v) \int_{\mathbb{S}^{d-1}} b(\xi)\xi^2 \, \mathrm{d}n \, \mathrm{d}v = -\gamma_b \int_{\mathbb{R}^d} |v - u_0|^2 h(v) \, \mathrm{d}v$$

which is the desired result. □

Remark 2.6. *Notice that* (2.9) *can be rewritten as*

$$\int_{\mathbb{R}^d} \mathcal{Q}_+(f, M)|v - u_0|^2 \, \mathrm{d}v \leqslant (1 - \gamma_b) \int_{\mathbb{R}^d} f|v - u_0|^2 \, \mathrm{d}v + \gamma_b \int_{\mathbb{R}^d} M|v - u_0|^2 \, \mathrm{d}v\,,$$

which strongly resembles (2.2) *for the temperature functional instead of the relative entropy. Equation* (2.10)*, written as*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |v - u_0|^2 (f - M) \, \mathrm{d}v = -\gamma_b \int_{\mathbb{R}^d} |v - u_0|^2 (f - M) \, \mathrm{d}v$$

is also analogous to (1.21)*, which can be written as*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}(f|M) \leqslant -\gamma_b \mathcal{H}(f|M)$$

for a Maxwellian kernel.

We are finally able to complete the proof of Theorem 2.1:

Proof of Theorem 2.1. Since in the Maxwellian case we have $\mathcal{L}f = \mathcal{L}_+ f - f$, using the expression of $D_{\max}$ in (1.18) one gets that

$$\mathcal{D}_{\max}(f) = \int_{\mathbb{R}^d} f \log\left(\frac{f}{M}\right) \mathrm{d}v - \int_{\mathbb{R}^d} \mathcal{L}_+(f) \log\left(\frac{f}{M}\right) \mathrm{d}v \\ = \mathcal{H}(f|M) - \mathcal{H}(\mathcal{L}_+ f|M) + \mathcal{H}(\mathcal{L}_+ f|f) \geqslant \mathcal{H}(f|M) - \mathcal{H}(\mathcal{L}_+ f|M),$$

since $\int_{\mathbb{R}^d} \mathcal{L}_+ f \, \mathrm{d}v = 1 = \int_{\mathbb{R}^d} f \, \mathrm{d}v$ and

$$\mathcal{H}(g|f) = \int_{\mathbb{R}^d} g \log \frac{f}{g} \, \mathrm{d}v \geqslant 0$$

whenever f and g share the same mass. Finally, using Lemma 2.4 to estimate $\mathcal{H}(\mathcal{L}_+ f|M)$ gives

$$\mathcal{D}(f) \geqslant \gamma_b \mathcal{H}(f|M)$$

which is the desired result. □

3. Inequalities for non-Maxwellian collision kernels

3.1. **Comparison of dissipations for general kernels.** As explained in the Introduction, the rest of entropy dissipation inequalities which we derive are based on Theorem 2.1, valid for Maxwellian collision kernels. We then obtain similar inequalities by comparing the dissipation for a given kernel B with a Maxwellian dissipation. This strategy was already used in Lods, Mouhot, and Toscani (2008) in order to estimate the spectral gap for the operator $\mathcal{L}$ and comes from Baranger and Mouhot (2005) where it was used to estimate the spectral gap of the linearized operator $\mathcal{Q}(f, M) + \mathcal{Q}(M, f)$. For the linear Boltzmann operator $\mathcal{L}$, we give here an improved version which enables us, for example, to estimate the entropy dissipation functional for the physical case of hard-spheres interactions (we will also use this comparison principle for grazing collisions kernels in Section 5).

Since the linear equation (1.4) is, as remarked before, the Kolmogorov forward equation of a Markov process with equilibrium M, it is well known (see for example Chafaï (2004)) that all functionals of the form

$$\mathcal{H}_\Phi(f|M) = \int_{\mathbb{R}^d} M(v)\Phi\left(\frac{f(v)}{M(v)}\right) \mathrm{d}v, \tag{3.1}$$

for $\Phi : [0,\infty) \to [0,\infty)$ convex, are decreasing along solutions to (1.4). In fact, using the detailed balance property (1.17) one sees formally that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\Phi(f(t)|M) = \int_{\mathbb{R}^d} \mathcal{L}f(t,v)\,\Phi'\left(\frac{f(t,v)}{M(v)}\right)\mathrm{d}v = -\mathcal{D}_\Phi(f(t)),$$

for any solution $f(t,v)$ to (1.4), where

$$\mathcal{D}_\Phi(f) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|q|,\xi)\,M(v)M(v_*)\Psi\left(\frac{f(v)}{M(v)},\frac{f(v')}{M(v')}\right)\mathrm{d}n\,\mathrm{d}v\,\mathrm{d}v_* \tag{3.2}$$

$$\Psi(x,y) := (x-y)(\Phi'(x)-\Phi'(y)) \geqslant 0, \qquad x,y\in[0,+\infty). \tag{3.3}$$

Alternatively, we can write

$$\mathcal{D}_\Phi(f) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} M(v)k_B(v,v')\Psi\left(\frac{f(v)}{M(v)},\frac{f(v')}{M(v')}\right)\mathrm{d}v\,\mathrm{d}v', \tag{3.4}$$

with k_B the kernel of the linear operator $\mathcal{L}$ (see (1.16)). (Note, however, that $\mathcal{H}_\Phi(f|M)$ is decreasing along solutions also for equations without detailed balance, though the expression of the dissipation is different in that case.)

We call $\mathcal{H}_\Phi(f|M)$ the *relative Φ-entropy* of f with respect to M. Particular examples of it are given by $\Phi(x) = x\log x - x + 1$, which gives the usual relative entropy (1.2) when f has the same mass as M; and $\Phi(x) = (x-1)^2$, which gives the distance of f to the equilibrium M in the L^2 norm with weight M^{-1}. Since $\mathcal{L}$ depends on the collision kernel B, it will be sometimes convenient to rather write $\mathcal{D}^B_\Phi(f)$ to emphasize the collision kernel B. Since our arguments apply to general relative Φ-entropies with no modification, we state our results for them as well.

Proposition 3.1 (Comparison of dissipations)**.** *Let B, $\tilde{B}$ be two collision kernels defined by*

$$B(|q|,\xi) = \beta(|q|)b(\xi), \qquad \tilde{B}(|q|,\xi) = b(\xi) \tag{3.5}$$

where $\beta : [0,\infty) \to [0,\infty)$ is a nondecreasing mapping and $b(\cdot)$ satisfies the normalization condition (1.14). Call M_0 the normalized Maxwellian with mean velocity 0 and temperature $\theta > 0$ (that is, $M_0(v) = M(v+u_0)$). Assume that there exists $\varrho_0 > 0$ such that

$$\tilde{C}_\theta := \inf_{\substack{\bar{v}\in\mathbb{R}^{d-1}\\ s\in[0,\varrho_0]}} \frac{\displaystyle\int_{\mathbb{R}^{d-1}} \beta\Big(\big(|\bar{v}-\bar{v}_*|^2+s^2\big)^{1/2}\Big)b\left(\frac{s}{\big(|\bar{v}-\bar{v}_*|^2+s^2\big)^{1/2}}\right)M_0(\bar{v}_*)\,\mathrm{d}\bar{v}_*}{\displaystyle\int_{\mathbb{R}^{d-1}} b\left(\frac{s}{\big(|\bar{v}-\bar{v}_*|^2+s^2\big)^{1/2}}\right)M_0(\bar{v}_*)\,\mathrm{d}\bar{v}_*} > 0. \tag{3.6}$$

(Where, for $w\in\mathbb{R}^{d-1}$, $M_0(w)$ is understood as $M_0(w,0)$.) Then

$$\mathcal{D}^B_\Phi(f) \geqslant C_\theta \mathcal{D}^{\tilde{B}}_\Phi(f) \tag{3.7}$$

for any probability distribution $f \in L^1(\mathbb{R}^d)$, with $C_\theta := \min\{\beta(\varrho_0), \tilde{C}_\theta\}$.

Remark 3.2. *At first sight, the comparison of convolution integrals (3.6) may seem difficult to check. However, we shall see further on that it holds true for hard potential interactions (see Prop. 3.5) and for the kernels used in the grazing collision limit (see Proposition 5.2).*

In order to give the proof of Proposition 3.1 we follow the ideas in (Lods et al., 2008, Proposition 3.3), but we rephrase the argument in a simplified way. Particularly, we show that Proposition 3.1 can actually be deduced from a comparison of the kernels k_B, $k_{\tilde{B}}$ of $\mathcal{L}$ corresponding to B and $\tilde{B}$.

Notice that the kernel k_B of $\mathcal{L}$ (see expression (1.16)) can be calculated by the use of Carleman's representation (originally described by Carleman (1957); see also Villani (2002, section 1.4.6)):

$$\mathcal{L}_+f(v) = \mathcal{Q}_+(f,M)(v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|q|,\xi)f(v')M(v'_*)\,\mathrm{d}v_*\,\mathrm{d}n$$
$$= 2\int_{\mathbb{R}^d}\frac{f(v')}{|v-v'|^{d-1}}\int_{E_{v,v'}} B(|q|,\xi)M(v'_*)\,\mathrm{d}v'_*\,\mathrm{d}v',$$

where $E_{v,v'}$ is the hyperplane $\{v'_*\in\mathbb{R}^d \mid (v'_*-v)\cdot(v'-v)=0\}$, and it is understood that the $\mathrm{d}v'_*$ integral above is with respect to the $(d-1)$-dimensional Lebesgue measure on this hyperplane. Since $|q|$ and ξ

must now be written in terms of v, v'_* and v', note that

$$|q| = |2v - v' - v'_*|, \qquad |q \cdot n| = |v - v'|, \qquad \xi = \frac{|v - v'|}{|2v - v' - v'_*|}. \tag{3.8}$$

Hence we have, for any $B = B(|q|, \xi)$:

$$k_B(v', v) = \frac{1}{|v - v'|^{d-1}} \int_{E_{v,v'}} B(|q|, \xi) M(v'_*) \, \mathrm{d}v'_*, \qquad v', v \in \mathbb{R}^d. \tag{3.9}$$

We now prove the following which clearly implies Proposition 3.1 by virtue of (3.4):

Proposition 3.3 (Comparison of kernels). *Assume that the collision kernels B and $\tilde{B}$ satisfy* (3.6). *Then, for the same constant C_θ as in Proposition 3.1,*

$$k_B(v', v) \geqslant C_\theta \, k_{\tilde{B}}(v', v) \quad \text{for all } v, v' \in \mathbb{R}^d. \tag{3.10}$$

Proof. By translation invariance of (1.4) (i.e., $\mathcal{Q}(f, M)(v + u) = \mathcal{Q}(f(\cdot + u), M(\cdot + u))(v)$) one sees that it is enough to show the result when the mean velocity M, namely u_0, is equal to 0 (in fact, the kernels k_B corresponding to different mean velocities are just translations of one another). Hence we assume $u_0 = 0$ throughout the proof, so $M = M_0$ (this will make calculations easier).

Using (3.8) and (3.9), (3.10) is equivalent to

$$\int_{E_{v,v'}} \beta(|2v - v' - v'_*|) \, b\left(\frac{|v - v'|}{|2v - v' - v'_*|}\right) M_0(v'_*) \, \mathrm{d}v'_* \geqslant C_\theta \int_{E_{v,v'}} b\left(\frac{|v - v'|}{|2v - v' - v'_*|}\right) M_0(v'_*) \, \mathrm{d}v'_*$$
$$\text{for any } v', v \in \mathbb{R}^d. \tag{3.11}$$

Take n to be the unit vector along the direction of $v - v'$. We now write

$$v'_* = rn + \bar{v}'_*$$

for (uniquely determined) $r \in \mathbb{R}$ and $\bar{v}'_*$ orthogonal to n. Write also

$$v = rn + \bar{v}$$

for some v orthogonal to n (note that r must have the same value as before, since $v'_* - v$ is orthogonal to n in $E_{v,v'}$) and

$$v' = (r + s)n + \bar{v}$$

for some $s \in \mathbb{R}$ (and the same $\bar{v}$ as before, since $v - v'$ is parallel to n.) With this and the expressions in (3.8) we have

$$|2v - v' - v'_*|^2 = |v - v'|^2 + |v - v'_*|^2 = |\bar{v} - \bar{v}'_*|^2 + s^2, \qquad \frac{|v - v'|}{|2v - v' - v'_*|} = \frac{s}{\sqrt{|\bar{v} - \bar{v}'_*|^2 + s^2}}. \tag{3.12}$$

Changing variables to $\bar{v}'_*$, we obtain that (3.11) reads

$$\int_{n^\perp} \beta\left(\sqrt{|\bar{v} - \bar{v}'_*|^2 + s^2}\right) b\left(\frac{s}{\sqrt{|\bar{v} - \bar{v}'_*|^2 + s^2}}\right) M_0(\bar{v}'_*) \, \mathrm{d}\bar{v}'_* \geqslant C_\theta \int_{n^\perp} b\left(\frac{s}{\sqrt{|\bar{v} - \bar{v}'_*|^2 + s^2}}\right) M_0(\bar{v}'_*) \, \mathrm{d}\bar{v}'_*.$$

Notice that we have used here that $M(v'_*) = (2\pi\theta)^{-d/2} M(\bar{v}'_*) M(rn)$, with $M(rn)$ independent of the integration variable and hence cancelling from both sides of the inequality.

By rotational symmetry we may also take $v - v'$ parallel to $(0, \cdots, 0, 1) \in \mathbb{R}^d$, so that $n^\perp = \mathbb{R}^{d-1}$, identified as the set of points in $\mathbb{R}^d$ with zero last coordinate (so the variables with a bar just represent the first $d - 1$ coordinates of the variables without a bar). Then, (3.11) is equivalent to

$$\int_{\mathbb{R}^{d-1}} \beta\left(\sqrt{|\bar{v} - \bar{v}'_*|^2 + s^2}\right) b\left(\frac{s}{\sqrt{|\bar{v} - \bar{v}'_*|^2 + s^2}}\right) M_0(\bar{v}'_*) \, \mathrm{d}\bar{v}'_* \geqslant C_\theta \int_{\mathbb{R}^{d-1}} b\left(\frac{s}{\sqrt{|\bar{v} - \bar{v}'_*|^2 + s^2}}\right) M_0(\bar{v}'_*) \, \mathrm{d}\bar{v}'_*$$
$$\text{for any } \bar{v} \in \mathbb{R}^{d-1} \text{ and } s \geqslant 0. \tag{3.13}$$

Now, given $\varrho_0 > 0$, since $\beta(\cdot)$ is nondecreasing, it is clear that, for any $\bar{v} \in \mathbb{R}^{d-1}$ and any $s \geqslant \varrho_0$ it holds

$$\int_{\mathbb{R}^{d-1}} \beta\left(\sqrt{|\bar{v} - \bar{v}'_*|^2 + s^2}\right) b\left(\frac{s}{\sqrt{|\bar{v} - \bar{v}'_*|^2 + s^2}}\right) M_0(\bar{v}'_*) \, \mathrm{d}\bar{v}'_*$$
$$\geqslant \beta(\varrho_0) \int_{\mathbb{R}^{d-1}} b\left(\frac{s}{\sqrt{|\bar{v} - \bar{v}'_*|^2 + s^2}}\right) M_0(\bar{v}'_*) \, \mathrm{d}\bar{v}'_*.$$

It is then enough to show that (3.13) holds for some constant $\tilde{C}_\theta$, uniformly for $\bar{v} \in \mathbb{R}^{d-1}$ and $s \in [0, \varrho_0)$. This is exactly assumption (3.6). This achieves the proof and, in particular, shows that (3.10) holds with $C_\theta = \min(\beta(\varrho_0), \tilde{C}_\theta)$. □

By using Theorem 2.1, Proposition 3.1 directly implies the following inequality for non-Maxwellian collision kernels:

Theorem 3.4. *Assume that the collision kernel B is given by (3.5) where $b(\xi)$ is a normalized Maxwellian collision kernel and $\beta(\cdot)$ satisfies (3.6). Then for all nonnegative probability distributions f we have*

$$\mathcal{D}_B(f) \geqslant C_\theta \gamma_b \, \mathcal{H}(f|M), \tag{3.14}$$

where $C_\theta > 0$ is the constant in Proposition 3.1 and γ_b was defined in (1.22).

Remark 1.2 also applies here: the above result is valid for any collision kernels which can be bounded below by a collision kernel satisfying the hypotheses of the theorem, and in particular it applies to non-cut-off hard collision kernels.

3.2. **Application to hard-potential interactions.** We show here how the above Proposition applies to the fundamental model of hard-potential interactions (including the hard-spheres case) for which

$$B(|q|, \xi) = c_d \, |q|^\gamma \, \xi^{d-2} \tag{3.15}$$

where $c_d > 0$ is a normalization constant given by

$$c_d := \left(|\mathbb{S}^{d-1}| \int_0^1 \xi^{d-2} \, \left(1 - \xi^2\right)^{\frac{d-3}{2}} \, \mathrm{d}\xi \right)^{-1}.$$

Introducing then $\tilde{B}(|q|, \xi) = b(\xi) = c_d \xi^{d-2}$, we see that $\tilde{B}$ is a normalized Maxwellian collision kernel. As a consequence of Proposition 3.1 we obtain the following, which completes the proof of Theorem 1.1:

Proposition 3.5. *Let B be a hard-potential collision kernel of the form (3.15) with $\gamma \geqslant 0$, in dimension $d \geqslant 2$. There exists some explicit $C > 0$ such that*

$$\mathcal{D}^B_\Phi(f) \geqslant C \theta^{\gamma/2} \mathcal{D}^{\tilde{B}}_\Phi(f). \tag{3.16}$$

Proof. The proof consists simply in checking that Assumption (3.6) is met by the kernels $\beta(|q|) = |q|^\gamma$ and $b(\xi) = c_d \xi^{d-2}$. Actually, the dependence on θ is easily obtained: call, for $\mu > 0$,

$$M_\mu(v) := \mu^d M(\mu v), \quad f_\mu(v) := \mu^d f(\mu v).$$

Note that the temperature of M_μ is μ^{-2} times that of M. Then we have the scaling

$$\mathcal{D}^B_{\Phi, M_\mu}(f_\mu) = \mu^{-\gamma} \mathcal{D}^B_{\Phi, M}(f), \quad \mathcal{D}^{\tilde{B}}_{\Phi, M_\mu}(f_\mu) = \mathcal{D}^B_{\Phi, M}(f),$$

where we have denoted the dependence on M as an additional subscript. One sees then that it is enough to show (3.16) when the temperature θ of M is equal to 1, so we assume this in the rest of the proof.

To prove (3.6) it suffices clearly to show that there exists $C > 0$ such that

$$\int_{\mathbb{R}^{d-1}} M_0(\bar{v}_*) (\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2})^{2-d+\gamma} \, \mathrm{d}\bar{v}_* \geqslant C \int_{\mathbb{R}^{d-1}} M_0(\bar{v}_*) (\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2})^{2-d} \, \mathrm{d}\bar{v}_* \tag{3.17}$$

for any $\bar{v} \in \mathbb{R}^{d-1}$ and any $0 < s \leqslant 1$. Choose $\delta > 0$. In the region where $|\bar{v} - \bar{v}_*| \geqslant \delta$ we have $\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2} \geqslant \delta$ and hence

$$\int_{|\bar{v} - \bar{v}_*| \geqslant \delta} M_0(\bar{v}_*) (\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2})^{2-d+\gamma} \, \mathrm{d}\bar{v}_* \geqslant \delta^\gamma \int_{|\bar{v} - \bar{v}_*| \geqslant \delta} M_0(\bar{v}_*) (\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2})^{2-d} \, \mathrm{d}\bar{v}_*.$$

So it is enough to show that

$$\int_{|\bar{v} - \bar{v}_*| < \delta} M_0(\bar{v}_*) (\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2})^{2-d} \, \mathrm{d}\bar{v}_* \leqslant K \int_{\mathbb{R}^{d-1}} M_0(\bar{v}_*) (\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2})^{2-d+\gamma} \, \mathrm{d}\bar{v}_* \tag{3.18}$$

for some $K > 0$, all $\bar{v} \in \mathbb{R}^d$ and all $0 < s \leqslant 1$, which would imply (3.17) to hold with $C = \min\{\delta^\gamma, 1/K\}$. Let us bound the left-hand-side of (3.18) first. On the integration region we have $|\bar{v}_*| \geqslant (|\bar{v}| - \delta)_+ =: \max\{|\bar{v}| - \delta, 0\}$. Hence

$$M_0(\bar{v}_*) \leqslant K_1 \exp(-|\bar{v}_*|^2/2) \leqslant K_1 \exp\left(-\frac{(|\bar{v}| - \delta)_+^2}{2} \right)$$

for some $K_1 > 0$. Using this, the left hand side of (3.18) is bounded above by

$$\begin{aligned}\int_{|\bar v-\bar v_*|<\delta} M_0(\bar v_*)(\sqrt{|\bar v-\bar v_*|^2+s^2})^{2-d}\,\mathrm{d}\bar v_* &\leqslant \int_{|\bar v-\bar v_*|<\delta} M_0(\bar v_*)|\bar v-\bar v_*|^{2-d}\,\mathrm{d}\bar v_* \\ &\leqslant K_2\exp\left(-\frac{(|\bar v|-\delta)_+^2}{2}\right),\end{aligned} \tag{3.19}$$

for some $K_2 > 0$. On the other hand, using that for $|\bar v_*| < 1$ we have

$$\sqrt{|\bar v-\bar v_*|^2+1} \leqslant \sqrt{(|\bar v|+1)^2+1} \leqslant K_4(|\bar v|+1)$$

for some $K_4 > 1$, we see that the right hand side of (3.18) is bounded below by

$$\begin{aligned}\int_{\mathbb{R}^{d-1}} M_0(\bar v_*)(\sqrt{|\bar v-\bar v_*|^2+s^2})^{2-d+\gamma}\,\mathrm{d}\bar v_* &\geqslant \int_{|\bar v_*|<1} M_0(\bar v_*)(\sqrt{|\bar v-\bar v_*|^2+1})^{2-d+\gamma}\,\mathrm{d}\bar v_* \\ &\geqslant K_5\,(1+|\bar v|)^{2-d+\gamma}\end{aligned} \tag{3.20}$$

when $d \geqslant 2+\gamma$, or simply by

$$\int_{\mathbb{R}^{d-1}} M_0(\bar v_*)(\sqrt{|\bar v-\bar v_*|^2+s^2})^{2-d+\gamma}\,\mathrm{d}\bar v_* \geqslant \int_{\mathbb{R}^{d-1}} M_0(\bar v_*)|\bar v-\bar v_*|^{2-d+\gamma}\,\mathrm{d}\bar v_* \geqslant K_6, \tag{3.21}$$

for some $K_6 > 0$, when $d < 2+\gamma$. The bounds (3.19), (3.20) and (3.21) clearly show (3.18), finishing the proof of the lemma. □

Example 3.6 (Hard-spheres case in dimension 3)**.** Let us estimate the constant C above in general dimension $d \geqslant 2$ whenever $\gamma = d-2$, which happens to be slightly easier and covers in particular the physically relevant case of hard-spheres in dimension $d = 3$ for which $\gamma = 1$. Let us then assume that $d \geqslant 2$ and let

$$\beta(|q|) = |q|^{d-2} \qquad \text{and} \qquad b(\xi) = c_d\,\xi^{d-2}.$$

Then, for any $s \geqslant 0$, with the notations of Proposition 3.1,

$$\beta\Big(\big(|\bar v-\bar v_*|^2+s^2\big)^{1/2}\Big)\, b\left(\frac{s}{\big(|\bar v-\bar v_*|^2+s^2\big)^{1/2}}\right) = c_d\, s^{d-2}$$

so that, to check (3.6), it is enough to show the inequality

$$\int_{\mathbb{R}^{d-1}} M_0(\bar v_*)\,\mathrm{d}\bar v_* \geqslant \tilde C_\theta \int_{\mathbb{R}^{d-1}} \frac{M_0(\bar v_*)}{|\bar v-\bar v_*|^{d-2}}\,\mathrm{d}\bar v_* \tag{3.22}$$

for any $\bar v \in \mathbb{R}^{d-1}$. Now, the left hand side is a given number, namely

$$\int_{\mathbb{R}^{d-1}} M_0(\bar v_*)\,\mathrm{d}\bar v_* = \|M_0\|_{L^1(\mathbb{R}^{d-1})} = \frac{1}{\sqrt{2\pi\theta}}$$

while the right hand side is bounded for $\bar v \in \mathbb{R}^{d-1}$: one can write, for any $\bar v \in \mathbb{R}^{d-1}$ and any $r > 0$,

$$\begin{aligned}\int_{\mathbb{R}^{d-1}} \frac{M_0(\bar v_*)}{|\bar v-\bar v_*|^{d-2}}\,\mathrm{d}\bar v_* &\leqslant \|M_0\|_{L^\infty(\mathbb{R}^{d-1})} \int_{\{|\bar v-\bar v_*|<r\}} \frac{\mathrm{d}\bar v_*}{|\bar v-\bar v_*|^{d-2}} + r^{-(d-2)}\|M_0\|_{L^1(\mathbb{R}^{d-2})} \\ &= r\,\|M_0\|_{L^\infty(\mathbb{R}^{d-1})}\,|\mathbb{S}^{d-2}| + r^{-(d-2)}\|M_0\|_{L^1(\mathbb{R}^{d-1})}\end{aligned}$$

and, optimizing the parameter $r > 0$, one finds that the constant $\tilde C_\theta > 0$ in (3.22) can be chosen as

$$C_0 := \frac{1}{d-1}\left(\frac{(d-2)\|M_0\|_{L^1(\mathbb{R}^{d-1})}}{\|M_0\|_{L^\infty(\mathbb{R}^{d-1})}\,|\mathbb{S}^{d-2}|}\right)^{\frac{d-2}{d-1}}$$

In particular, in dimension $d = 3$, one can choose $\tilde C_\theta = \frac{\sqrt{\theta}}{2}$. For the special case of the *Shannon-Boltzmann relative entropy*, i.e. for $\Phi(x) = x\log x - x + 1$, one simply denotes by $\mathcal{D}_{\max}$ the dissipation associated to $\tilde B(\xi) = c_d\xi$ and, bearing in mind that c_d is a normalization constant for collision kernel, deduces from Theorem 2.1 that:

$$\mathcal{D}_{\max}(f) \geqslant \frac{1}{2}\mathcal{H}(f|M).$$

Therefore, if $\mathcal{D}_{\text{hs}}$ denotes the entropy dissipation associated to hard-spheres interactions in dimension $d = 3$ we immediately deduce from Proposition 3.5 that

$$\mathcal{D}_{\text{hs}}(f) \geqslant \frac{\sqrt{\theta}}{4}\mathcal{H}(f|M).$$

4. Some applications

4.1. **Speed of convergence to equilibrium for the linear Boltzmann equation.** Once we have Theorem 1.1 and the relation (1.18) it is straightforward to deduce the following result:

Theorem 4.1. *Let $B = B(|q|, \xi)$ denote a hard-potential collision kernel given by (1.10) or any normalized Maxwellian collision kernel $B = b(\xi)$ satisfying (1.14). Let $f_0 \in L^1(\mathbb{R}^d, (1 + |v|^2)\,\mathrm{d}v)$ be a given probability density with finite entropy and let $f(t) = f(t, \cdot)$ be the associated solution to the linear Boltzmann equation (1.4). Then*

$$\mathcal{H}(f(t)|M) \leqslant \exp(-\lambda\, t)\mathcal{H}(f_0|M) \quad \text{for} \quad t \geqslant 0, \tag{4.1}$$

with λ given in Theorem 1.1. In particular, the Csiszár-Kullback-Pinsker inequality yields

$$\|f(t) - M\|_{L^1(\mathbb{R}^d)} \leqslant \sqrt{2} \exp\left(-\frac{\lambda}{2}\, t\right) \mathcal{H}(f_0|M) \qquad \forall t \geqslant 0.$$

Again, this result applies also to any collision kernels that can be bounded below by a kernel satisfying the assumptions; see Remark 1.2.

For completeness we gather here some facts on the well-posedness of equation (1.4) and the rigorous derivation of the entropy relation (1.18). Assume for the rest of this paragraph that

$$B(|q|, \xi) = |q|^\gamma b(\xi)$$

for some $0 \leqslant \gamma \leqslant 2$ and some collision kernel $b(\cdot)$ satisfying (1.14). First, we notice that the operator $\mathcal{L}(f)$ is well defined for $f \in L^1(\mathbb{R}^d; (1 + |v|^\gamma)\,\mathrm{d}v)$. When considered as an operator on $L^2(\mathbb{R}^d; M^{-1})$ (a smaller space than $L^1(\mathbb{R}^d; (1+|v|^\gamma))$ then $\mathcal{L}$ is a self-adjoint operator with domain $L^2(\mathbb{R}^d; M(v)^{-1}|v|^\gamma\,\mathrm{d}v)$ (see Carleman (1957)). Regarding the evolution equation (1.4), $\mathcal{L}$ (with its natural domain) generates a C_0-semigroup in several spaces; for example, in $L^2(\mathbb{R}^d; M^{-1})$ and in $L^1(\mathbb{R}^d)$. By considering a suitable regularization $\Phi_\epsilon : [0, +\infty)$ of the function $\Phi(x) := x \log x - x + 1$, with Φ_ϵ differentiable on $[0, +\infty)$, one directly sees that, for any $\epsilon > 0$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\Phi_\epsilon}(f(t)) = -\mathcal{D}_{\Phi_\epsilon}(f(t))$$

for any solution $f(t)$ to (1.4) (in the semigroup sense) with initial condition in the domain of $\mathcal{L}$. One can then pass to the limit in $\epsilon \to 0$ in order to show that (1.18) holds rigorously for an initial condition f with finite energy and entropy.

4.2. **Trend to equilibrium for the nonlinear Boltzmann equation with particle bath.** We consider now the nonlinear (elastic) Boltzmann operator with particles bath

$$\partial_t f(t, v) = \alpha \mathcal{Q}(f, f)(t, v) + \mathcal{L}f, \qquad f(0, v) = f_0(v), \qquad t \geqslant 0,\ v \in \mathbb{R}^d, \tag{4.2}$$

where $\alpha \geqslant 0$ is a given constant while, as above, $\mathcal{L}f$ denotes the linear Boltzmann operator and $\mathcal{Q}(f, f)$ is the quadratic Boltzmann operator. Equation (4.2) models the evolution of particles (typically hard-spheres) according to the following rules: particles are suffering binary collision with themselves and also interact with the particles of a host medium at thermodynamical equilibrium. Notice that (4.2) has been recently considered in Bisi et al. (2011) (for inelastic interactions) and can also be seen as the spatially homogeneous version of the model recently investigated in Fröhlich and Gang (2012).

In the above, one assumes that $\mathcal{Q} = \mathcal{Q}_{B_1}$ is associated to a general collision kernel $B_1(|q|, \xi) \geqslant 0$ (including soft interactions, see Villani (2002)). Moreover, one assumes that $\mathcal{L}$ is associated to a collision kernel $B(|q|, \xi) = \beta(|q|)b(\xi)$ where $b(\cdot)$ satisfies the normalization condition (1.14) while $\beta(\cdot) : [0, \infty) \to [0, \infty)$ satisfies (3.6). Notice that we do not need here B_1 and B to be equal.

The well-posedness of the Cauchy problem associated to (4.2) can be handled with standard methods from spatially homogeneous kinetic theory and we do not address this question here, referring for instance to Villani (2002) for more details (see also Bisi et al. (2008) where a similar equation has been investigated for inelastic interactions). Moreover, it is also easy to prove that the unique steady state of the operator $\alpha\mathcal{Q}(f, f) + \mathcal{L}(f)$ is the host Maxwellian M, namely (see Bisi et al. (2011)):

Proposition 4.2. *For any $\alpha \geqslant 0$, the unique nonnegative solution $F \in L^1(\mathbb{R}^d; (1 + |v|)^2\,\mathrm{d}v)$ with unit mass to the stationary problem*

$$\alpha\mathcal{Q}(F, F) + \mathcal{L}(F) = 0$$

is given by $F(v) = M(v)$.

Concerning the long time behavior of solution, we prove in a simple way exponential trend towards equilibrium:

Theorem 4.3. *For any $\alpha \geqslant 0$, let $f_0 \in L^1(\mathbb{R}^d, (1+|v|)^3\,\mathrm{d}v)$ be such that $\mathcal{H}(f_0|M) < \infty$ and let $f(t,v)$ be the unique associated global solution to (4.2). Then, there exists $C > 0$ depending only B such that*

$$\mathcal{H}(f(t)|M) \leqslant \exp(-C\,\gamma_b t)\mathcal{H}(f_0|M) \qquad \forall t \geqslant 0 \tag{4.3}$$

where $\gamma_b > 0$ is the constant appearing in Theorem 2.1 while $C > 0$ is the constant appearing in Prop. 3.1.

Remark 4.4. *Notice that the long-time behavior of the solution $f(t,v)$ is completely driven by $\mathcal{L}$ and not by the quadratic operator $\mathcal{Q}$. In particular, the speed of convergence does not depend on α and is entirely determined by the collision kernel $B = B(|q|,\xi)$.*

Proof. The proof follows from standard arguments. Namely, direct computations yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f(t)|M) = \alpha \int_{\mathbb{R}^d} \mathcal{Q}(f,f)(t,v)\log\left(\frac{f(t,v)}{M(v)}\right)\mathrm{d}v - D(f)$$

Now, one sees that, since $\mathcal{Q}(f,f)$ conserved mass, momentum *and kinetic energy*,

$$\int_{\mathbb{R}^d} \mathcal{Q}(f,f)(t,v)\log\left(\frac{f(t,v)}{M(v)}\right)\mathrm{d}v = \int_{\mathbb{R}^d} \mathcal{Q}(f,f)(t,v)\log f(t,v)\,\mathrm{d}v$$

and, by well-known arguments that can be traced back to Boltzmann himself, this last quantity is nonnegative (this is exactly the classical Boltzmann's H-Theorem; see eq. (1.3)). Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f(t)|M) \leqslant -D(f(t))$$

and, one deduces from Theorem 3.4 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f(t)|M) \leqslant -C\gamma_b\mathcal{H}(f(t)|M) \qquad \forall t \geqslant 0$$

which achieves the proof. □

5. Grazing collisions limit and logarithmic Sobolev inequalities

In this section we show how general functional inequalities of the type (1.21) allow to recover, in a suitable limit, a well-known entropy-entropy dissipation estimate for a certain linear Fokker-Planck equation. The limit procedure is the so-called grazing collisions limit for which we assume that the collision kernel B is concentrated on small angle deviations. Before describing how this grazing collisions limit allows to recover a well-known logarithmic Sobolev inequality, we describe in more detail the asymptotic procedure.

5.1. The asymptotics of grazing collisions.

Whenever collisions concentrate around $|q\cdot n|/|q| \simeq 0$, it is well documented that $\mathcal{L}$ becomes close (in a sense to be made precise) to a certain linear Fokker-Planck operator (associated to a certain diffusion matrix $\mathbf{D}(v)$ that depends on M). We explain here the general mathematical framework following the lines of Lods and Toscani (2004).

We restrict ourselves to dimension $d = 3$ for simplicity. For any $\epsilon \in (0,1]$, we consider

$$B(|q|,\xi) = |q|^{\gamma}\, b_\epsilon(\xi) \tag{5.1}$$

for $\gamma = 0$ or $\gamma = 1$ and with $b_\epsilon(\cdot)$ given by

$$b_\epsilon(\xi) = \frac{\xi}{2\pi\epsilon}\,\mathbb{1}_{[0,\epsilon]}(\xi)$$

where we recall that $\xi = |q\cdot n|/|n|$. Notice that b_ϵ is a *normalized* Maxwellian collision kernel. Let $\mathcal{L}_\epsilon$ denote the associated linear Boltzmann operator (we do not distinguish here the two cases $\gamma = 0$ — corresponding to Maxwellian collision kernel — and $\gamma = 1$ corresponding to hard-spheres). Given $f_0 \in L^1(\mathbb{R}^3, (1+|v|^2)\,\mathrm{d}v)$, let $h = h_\epsilon(t,v)$ denote the unique solution to

$$\partial_t h = \mathcal{L}_\epsilon h, \qquad h(0,v) = f_0(v) \qquad (t \geqslant 0, v \in \mathbb{R}^3). \tag{5.2}$$

Moreover, we introduce the following time scaling

$$f_\epsilon(t,v) = h\left(t\epsilon^{-2}, v\right) \qquad \forall t \geqslant 0. \tag{5.3}$$

Using the weak form of the Boltzmann operator provided by (1.15), for a general test function $\varphi(v)$ one gets that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f_\epsilon(t,v)\,\varphi(v)\,\mathrm{d}v = \frac{1}{\epsilon^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\epsilon(t,v)M(v_*)|v-v_*|^\gamma\,\mathrm{d}v\,\mathrm{d}v_*\int_{\mathbb{S}^{d-1}} b_\epsilon(\xi)\big[\varphi(v')-\varphi(v)\big]\,\mathrm{d}n. \tag{5.4}$$

Using a Taylor expansion of φ, one has[2]

$$\varphi(v') = \varphi(v) + \nabla_v\varphi(v)\cdot(v'-v) + \frac{1}{2}\,\mathbb{D}^2\varphi(v)\cdot\big[(v'-v)\otimes(v'-v)\big] + o(|v'-v|^2) \tag{5.5}$$

where $\mathbb{D}^2\varphi$ is the Hessian matrix of φ, with components $\big(\mathbb{D}^2\varphi(v)\big)_{ij} = \dfrac{\partial^2\varphi(v)}{\partial v_i\partial v_j}$ $(i,j=1,2,3)$; hence, taking into account (1.6)

$$\varphi(v') = \varphi(v) - \nabla_v\varphi(v)\cdot(q\cdot n)n + \frac{1}{2}\,\mathbb{D}^2\varphi(v)\cdot\big[|q\cdot n|^2 n\otimes n\big] + o(|v'-v|^2).$$

Let us evaluate integrals over the angular variable n, taking the direction of the relative velocity q as polar axis $(\hat{e}_3)$. It is easy to check that

$$\int_{\mathbb{S}^2} b_\epsilon(\xi)(q\cdot n)n\,\mathrm{d}n = 2q\int_0^\epsilon \xi^3\,\mathrm{d}\xi = \frac{\epsilon^2}{2}\,q\,,$$

(the other components vanishing by parity arguments), while

$$\begin{aligned}\int_{\mathbb{S}^2} b_\epsilon(\xi)|q\cdot n|^2 n\otimes n\,\mathrm{d}n &= |q|^2\int_0^\epsilon \xi^3\Big[(1-\xi^2)(\hat{e}_1\otimes\hat{e}_1+\hat{e}_2\otimes\hat{e}_2) + 2\xi^2\hat{e}_3\otimes\hat{e}_3\Big]\,\mathrm{d}\xi\\ &= |q|^2\left[\left(\frac{\epsilon^2}{4}-\frac{\epsilon^4}{6}\right)(\hat{e}_1\otimes\hat{e}_1+\hat{e}_2\otimes\hat{e}_2) + \frac{\epsilon^4}{6}\hat{e}_3\otimes\hat{e}_3\right].\end{aligned}$$

By inserting all these results into (5.4) we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f_\epsilon(t,v)\,\varphi(v)\,\mathrm{d}v = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\epsilon(t,v)M(v_*)\Big[&-\frac{1}{2}(v-v_*)\cdot\nabla_v\varphi(v)\\ &+\frac{1}{8}\,\mathbb{D}^2\varphi(v)\cdot\Big(|q|^2\mathbf{I}-q\otimes q\Big)\Big]\,\mathrm{d}v\,\mathrm{d}v_* + O(\epsilon^2)\end{aligned} \tag{5.6}$$

where $\mathbf{I}$ is the identity matrix. Set

$$\mathbf{S}(v,v_*) = |v-v_*|^2\mathbf{I} - (v-v_*)\otimes(v-v_*).$$

By considering the last term in (5.6) we see that (here and below we use Einstein's summation convention on repeated indices)

$$\begin{aligned}f_\epsilon(t,v)\mathbb{D}^2\varphi\cdot\Big(|v-v_*|^2\mathbf{I}-(v-v_*)\otimes(v-v_*)\Big) &= f_\epsilon(t,v)\frac{\partial}{\partial v_j}\left(\frac{\partial\varphi}{\partial v_i}\right)\mathbf{S}_{ij}(v,v_*)\\ &= \frac{\partial}{\partial v_j}\left(f_\epsilon(t,v)\mathbf{S}_{ij}(v,v_*)\frac{\partial\varphi(v)}{\partial v_i}\right) - \frac{\partial\varphi(v)}{\partial v_i}\mathbf{S}_{ij}(v,v_*)\frac{\partial f_\epsilon(t,v)}{\partial v_j} - \frac{\partial\varphi(v)}{\partial v_i}\frac{\partial\mathbf{S}_{ij}(v,v_*)}{\partial v_j}f_\epsilon(t,v)\\ &= \nabla_v\cdot\Big(f_\epsilon(t,v)\mathbf{S}(v,v_*)\cdot\nabla_v\varphi(v)\Big) - \nabla_v\varphi(v)\cdot\Big(\mathbf{S}(v,v_*)\nabla_v f_\epsilon(t,v) - 2(v-v_*)f_\epsilon(t,v)\Big)\end{aligned}$$

where we used that $\frac{\partial\mathbf{S}_{ij}(v,v_*)}{\partial v_j} = -(v-v_*)_i\mathbb{1}_{j\neq i}$. Therefore (5.6) may be cast as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f_\epsilon(t,v)\,\varphi(v)\,\mathrm{d}v\\ &\qquad= -\frac{1}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\nabla_v\varphi(v)\cdot\Big[2(v-v_*)f_\epsilon(t,v)+\mathbf{S}(v,v_*)\nabla_v f_\epsilon(t,v)\Big]M(v_*)|v-v_*|^\gamma\,\mathrm{d}v\,\mathrm{d}v_* + O(\epsilon^2).\end{aligned}$$

Noticing that $\nabla_{v_*}\cdot\mathbf{S}(v,v_*) = 2(v-v_*)$ while $\mathbf{S}(v,v_*)(v-v_*)=0$, we check that, for both $\gamma=0,1$, it holds

$$\nabla_{v_*}\cdot\left(|v-v_*|^\gamma\,\mathbf{S}(v,v_*)\right) = 2\,|v-v_*|^\gamma\,(v-v_*).$$

[2] Given two vectors $w,v\in\mathbb{R}^3$, we use in the sequel the tensor notation $w\otimes v$ to denote the matrix with entries w_iv_j $i,j=1,2,3$. In particular, the matrix product $\mathbb{D}^2\varphi(v)\cdot\big[(v'-v)\otimes(v'-v)\big]$ simply denotes $(v'-v)(\mathbb{D}^2\varphi)(v'-v)^\top$.

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f_\epsilon(t,v)\,\varphi(v)\,\mathrm{d}v = -\frac{1}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \nabla_v\varphi(v)\cdot\Big[f_\epsilon(t,v)\nabla_{v_*}\cdot\left(|v-v_*|^\gamma\,\mathbf{S}(v,v_*)\right) \\ + |v-v_*|^\gamma\mathbf{S}(v,v_*)\nabla_v f_\epsilon(t,v)\Big]M(v_*)\,\mathrm{d}v\,\mathrm{d}v_* + O(\epsilon^2),$$

which, performing the integration with respect to v_* and setting

$$\mathbf{D}_\gamma(v) = \frac{1}{8}\int_{\mathbb{R}^3} |v-v_*|^\gamma\mathbf{S}(v,v_*)\,M(v_*)\,\mathrm{d}v_*\,, \tag{5.7}$$

yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f_\epsilon(t,v)\,\varphi(v)\,\mathrm{d}v = &-\int_{\mathbb{R}^3}\nabla_v\varphi(v)\cdot\mathbf{D}_\gamma(v)\nabla_v f_\epsilon(t,v)\,\mathrm{d}v \\ &-\frac{1}{8}\int_{\mathbb{R}^3} f_\epsilon(t,v)\nabla_v\varphi(v)\cdot\left(\int_{\mathbb{R}^3}\nabla_{v_*}\cdot\left(|v-v_*|^\gamma\,\mathbf{S}(v,v_*)\right)M(v_*)\,\mathrm{d}v_*\right) \\ = &-\frac{1}{8}\int_{\mathbb{R}^3}\nabla_v\varphi(v)\cdot\mathbf{D}_\gamma(v)\nabla_v f_\epsilon(t,v)\,\mathrm{d}v \\ &+\frac{1}{8}\int_{\mathbb{R}^3} f_\epsilon(t,v)\nabla_v\varphi(v)\cdot\int_{\mathbb{R}^3}|v-v_*|^\gamma\,\mathbf{S}(v,v_*)\nabla_{v_*}M(v_*)\,\mathrm{d}v_* + O(\epsilon^2).\end{aligned}$$

Since $\nabla_{v_*}M(v_*) = -\,\frac{v_*-u_0}{\theta}\,\mathcal{M}(v_*)$ and $\mathbf{S}(v,v_*)\cdot(v_*-v)=0$, we recognize that

$$\frac{1}{8}\int_{\mathbb{R}^3}|v-v_*|^\gamma\,\mathbf{S}(v,v_*)\nabla_{v_*}M(v_*)\,\mathrm{d}v_* = \frac{1}{\theta}\mathbf{D}_\gamma(v)(v-u_0).$$

Finally, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f_\epsilon(t,v)\,\varphi(v)\,\mathrm{d}v = \int_{\mathbb{R}^3}\varphi(v)\nabla_v\cdot\left\{\mathbf{D}_\gamma(v)\left[\nabla_v f_\epsilon(t,v) + \frac{v-u_0}{\theta}f_\epsilon(t,v)\right]\right\}\,\mathrm{d}v\, + O(\epsilon^2). \tag{5.8}$$

In particular, one expects the limit $f(t,v) = \lim_{\epsilon\to0} f_\epsilon(t,v)$ to satisfy the Fokker–Planck equation

$$\partial_t f(v) = \nabla_v\cdot\left\{\mathbf{D}_\gamma(v)\cdot\left[\nabla_v f(v) + \frac{v-u_0}{\theta}f(v)\right]\right\}. \tag{5.9}$$

where the diffusion coefficient $\mathbf{D}_\gamma(v)$ is defined in (5.7).

The above computations are clearly formal. Nevertheless, they can be made rigorous following the lines of Lods and Toscani (2004) (see also Goudon (1997); Desvillettes (1992) for similar considerations for the nonlinear Boltzmann equation) to get the following

Proposition 5.1. *Let $f_0 \in L^1(\mathbb{R}^3,(1+|v|^2)\,\mathrm{d}v)$ be a nonnegative probability distribution. For $\gamma\in\{0,1\}$ and $\epsilon\in(0,1)$, let $\mathcal{L}_\epsilon$ denote the linear Boltzmann operator with collision kernel given by (5.1) and let $h_\epsilon(t,\cdot)$ be the unique solution to (5.2). Set $f_\epsilon(t,v) = h(t\epsilon^{-2},v)$ for any $t\geqslant 0$, $v\in\mathbb{R}^3$. Then, there exists a subsequence, still denoted $(f_\epsilon(t))_\epsilon$ such that*

$$f_\epsilon \underset{\epsilon\to0}{\rightharpoonup} f \qquad \text{weakly in } L^1_{\mathrm{loc}}([0,\infty),L^1(\mathbb{R}^3))$$

where $f=f(t,v)$ is the unique solution to the Fokker-Planck equation (5.9) with initial datum $f(0)=f_0$.

5.2. **Logarithmic Sobolev inequality.** We recall here some well-known features about the long-time behavior of the solution $f(t,v)$ to the Fokker-Planck equation (5.9) with initial datum $f(0,v)=f_0$, $f_0\in L^1(\mathbb{R}^3,(1+|v|^2)\,\mathrm{d}v)$ being a nonnegative probability distribution. It is very well known that the Maxwellian M is the unique steady state with unit mass to the Fokker-Planck equation and the convergence of $f(t,v)$ towards M (as $t\to\infty$) can be made explicit by the use of entropy methods (see e.g. Arnold et al. (2008, 2001); Calogero (2012)). Let us explain more in detail the general strategy (we follow here the introduction of Calogero (2012)). Introduce the change of unknown

$$g(t,v) = \frac{f(t,v)}{M(v)}, \qquad t\geqslant 0, v\in\mathbb{R}^3$$

Then the relative entropy $\mathcal{H}(f(t)|M)$ can be rewritten as

$$\mathcal{H}(f(t)|M) = \int_{\mathbb{R}^3} f(t,v)\log\left(\frac{f(t,v)}{M(v)}\right)\,\mathrm{d}v = \int_{\mathbb{R}^3} g(t,v)\log g(t,v)M(v)\,\mathrm{d}v$$

where $\mathrm{d}\mu(v) = M(v)\,\mathrm{d}v$ is the invariant measure associated to the Fokker-Planck operator. It is straightforward to check that $g(t,v)$ satisfies now the drift-diffusion equation:

$$\partial_t g(t,v) = \nabla \cdot (\mathbf{D}_\gamma(v)\nabla g(t,v)) - \frac{v-u_0}{\theta} \cdot (\mathbf{D}_\gamma(v)\nabla g(t,v))$$

where $\nabla = \nabla_v$. One can compute the time derivative of $\mathcal{H}(f|M) = H(Mg|M)$ by using this to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f(t)|M) = -\int_{\mathbb{R}^3} \frac{(\mathbf{D}_\gamma(v)\nabla g(t,v)) \cdot \nabla g(t,v)}{g(t,v)}\,\mathrm{d}\mu(v) =: -\mathcal{J}_\gamma(f(t)|M) \qquad \forall t \geqslant 0. \tag{5.10}$$

In particular, $\mathbf{D}_\gamma$ being positive definite, one sees that $\mathcal{J}_\gamma(f|M) \geqslant 0$. If we find $\lambda > 0$ such that the logarithmic Sobolev inequality

$$\lambda\mathcal{H}(f|M) \leqslant \mathcal{J}_\gamma(f|M) \tag{5.11}$$

holds for all probability densities f, then this can be immediately used in (5.10) to deduce exponential convergence to equilibrium in the entropy sense for solutions to (5.9). If the diffusion matrix $\mathbf{D}_\gamma$ is the identity then this is the Gaussian logarithmic Sobolev inequality (1.25), which holds for $\lambda = 2/\theta$. However, the matrix $\mathbf{D}_\gamma$ we obtained through the limiting procedure in Section 5.1 is not the identity; we now consider what can be said regarding the inequality (5.11) in the cases $\gamma = 0$ (Maxwell molecules) and $\gamma = 1$ (hard spheres).

The case $\gamma = 0$. For $\gamma = 0$ the matrix $\mathbf{D}_0(v)$ can be explicitly computed, giving

$$\mathbf{D}_0(v) = \frac{1}{8}\left(\mathbf{S}(v,u_0) + 2\theta\,\mathbf{I}\right). \tag{5.12}$$

In particular, since the matrix $\mathbf{S}(v,u_0)$ is nonnegative, one sees from the definition (5.10) that

$$\mathcal{J}_0(f|M) \geqslant \frac{\theta}{4}\int_{\mathbb{R}^3} \frac{|\nabla g(v)|^2}{g(v)} M(v)\,\mathrm{d}v = \frac{\theta}{4} I(f|M) \geqslant \frac{1}{2}\mathcal{H}(f|M),$$

for any probability density $f \in L^1(\mathbb{R}^3)$, where we have used the Stam-Gross inequality (1.25). That is,

$$\mathcal{J}_0(f|M) \geqslant \frac{1}{2}\mathcal{H}(f|M), \tag{5.13}$$

which is inequality (5.11) for $\gamma = 0$ and $\lambda = 1/2$. From this one deduces that, if $f(t)$ denotes the unique solution to (5.9) then

$$\mathcal{H}(f(t)|M) \leqslant \exp\left(-\frac{1}{2}t\right)\mathcal{H}(f_0|M) \quad \forall t \geqslant 0. \tag{5.14}$$

We do not know whether $1/2$ is the optimal constant here, since we have disregarded one of the terms in (5.12).

The case $\gamma = 1$. For $\gamma = 1$, the matrix $\mathbf{D}_1(v)$ is given by

$$\mathbf{D}_1(v) = \frac{1}{8}\int_{\mathbb{R}^3} \Big[|v-v_*|^3\,\mathbf{I} - |v-v_*|(v-v_*)\otimes(v-v_*)\Big] M(v_*)\,\mathrm{d}v_*$$

With this matrix it is not obvious whether the logarithmic Sobolev inequality (5.11) holds for some $\lambda > 0$.

Let us briefly review the Bakry-C mery criterion for studying this kind of inequality. We introduce the vector field

$$\mathbf{X}g(v) := \mathbf{D}_1(v)\nabla\kappa(v)\nabla g(v), \qquad \text{where} \quad \kappa(v) = \sqrt{\det(\mathbf{D}_1(v))}\exp\left(-\frac{|v-u_0|^2}{2\theta}\right)$$

and the Riemannian manifold $\Sigma := (\mathbb{R}^3, \mathbf{D}_1(v)^{-1})$ with $\mathbf{D}_\gamma(v)^{-1}$ as covariant metric tensor. It is known (Bakry and Émery, 1985; Bakry, 1994; Arnold et al., 2001) that, if there exists some $\alpha > 0$ such that

$$\mathbf{Ric}^\Sigma - \nabla^\Sigma\mathbf{X} \geqslant \alpha\mathbf{D}_1^{-1} \tag{5.15}$$

(where $\mathbf{Ric}^\Sigma$ and $\nabla^\Sigma\mathbf{X}$ denote respectively the Ricci curvature and the Levi-Civita connection of Σ) then it holds:

$$2\alpha\mathcal{H}(f|M) \leqslant \mathcal{J}_1(f|M) \tag{5.16}$$

for any nonnegative $f \in L^1(\mathbb{R}^3)$ with unit mass.

In the $\gamma = 1$ case the application of the Bakry–Émery criterion is delicate but one can still apply (5.15) to deduce that for $\alpha = \frac{7}{24}\sqrt{\frac{2\theta}{\pi}}$ we have

$$\mathcal{J}_1(f|M) \geqslant 2\alpha\mathcal{H}(f|M) \tag{5.17}$$

for any probability density $f \in L^1(\mathbb{R}^3)$. Details are reported in Appendix B.

5.3. **Entropy dissipation for grazing collisions kernels.** We consider here the entropy dissipation functionals associated to grazing collision kernels as introduced in Section 5.1. For simplicity we discuss only the case of dimension $d = 3$, though our analysis can be extended (with more cumbersome calculations) to $d \geqslant 2$. We consider here *grazing hard-spheres collision kernels* of the form $B_\epsilon(|q|, \xi) = \beta(|q|)b_\epsilon(\xi)$ with

$$\beta(|q|) = |q|, \tag{5.18}$$

$$b_\epsilon(\xi) = \frac{\xi}{\|b_\epsilon\|}\mathbb{1}_{[0,\epsilon]}(\xi), \qquad \|b_\epsilon\| := |\mathbb{S}^{d-1}| \int_0^\epsilon \xi(1-\xi^2)^{\frac{d-3}{2}} \,\mathrm{d}\xi, \tag{5.19}$$

for some $\epsilon \in (0, 1]$. Notice that the case $\epsilon = 1$ corresponds to hard-spheres interactions for which $B(|q|, \xi) = \|b_1\|^{-1}|q \cdot n|$. We show now that for such collision kernels Assumption 3.6 is met, thus extending Proposition 3.5 to include them:

Proposition 5.2 (Comparison of dissipations for grazing hard-spheres)**.** *Take $\epsilon > 0$ and let $B_\epsilon(|q|, \xi) = \beta(|q|)b_\epsilon(\xi)$ be the grazing hard-spheres kernel given by (5.18)–(5.19), in dimension $d = 3$. We denote by $\tilde{B}_\epsilon(\xi) = b_\epsilon(\xi)$ the associated normalized Maxwellian collision kernel. There exists some number $C > 0$ (*independent *of ϵ) such that*

$$D_\Phi^{B_\epsilon}(f) \geqslant C D_\Phi^{\tilde{B}_\epsilon}(f) \tag{5.20}$$

for any convex function $\Phi : [0, \infty) \to [0, \infty)$ (where D_Φ is defined in Section 3).

Proof. As for Proposition 3.5, the proof consists in checking that Assumption 3.6 is met by the kernels $\beta(|q|)$ and $b_\epsilon(\xi)$. First, for a given $\epsilon > 0$, one sets

$$\delta_\epsilon := \frac{\sqrt{1-\epsilon^2}}{\epsilon}.$$

For a given $s > 0$ and a given $\bar{v} \in \mathbb{R}^2$, the numerator of (3.6) reads

$$\int_{\mathbb{R}^2} \beta\left(\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2}\right) b_\epsilon\left(\frac{s}{\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2}}\right) M(\bar{v}_*) \,\mathrm{d}\bar{v}_* = \frac{s}{\|b_\epsilon\|} \int_{\mathbb{R}^2} \mathbb{1}_{\{|\bar{v}-\bar{v}_*| \geqslant s\delta_\epsilon\}} M(\bar{v}_*) \,\mathrm{d}\bar{v}_*$$

while the denominator is simply

$$\frac{s}{\|b_\epsilon\|} \int_{\mathbb{R}^2} \frac{M(\bar{v}_*)}{\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2}} \mathbb{1}_{\{|\bar{v}-\bar{v}_*| \geqslant s\delta_\epsilon\}} \,\mathrm{d}\bar{v}_*.$$

Hence to prove (3.6), it suffices clearly to show that there exists $C > 0$ such that

$$\int_{\{|\bar{v}-\bar{v}_*| \geqslant s\delta_\epsilon\}} M(\bar{v}_*) \,\mathrm{d}\bar{v}_* \geqslant C \int_{\{|\bar{v}-\bar{v}_*| \geqslant s\delta_\epsilon\}} \frac{M(\bar{v}_*)}{\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2}} \,\mathrm{d}\bar{v}_*$$

for any $\bar{v} \in \mathbb{R}^2$ and any $s > 0$. If $s\delta_\epsilon \geqslant 1$ we can directly bound $|\bar{v} - \bar{v}_*| \geqslant 1$ and the inequality is obviously true with $C = 1$. If $s\delta_\epsilon < 1$ we have

$$\int_{\{|\bar{v}-\bar{v}_*| \geqslant s\delta_\epsilon\}} M(\bar{v}_*) \,\mathrm{d}\bar{v}_* \geqslant \int_{|\bar{v}_* - \bar{u}_0| \geqslant 1} M(\bar{v}_*) \,\mathrm{d}\bar{v}_* =: C_1 > 0$$

while

$$\int_{\{|\bar{v}-\bar{v}_*| \geqslant s\delta_\epsilon\}} \frac{M(\bar{v}_*)}{\sqrt{|\bar{v} - \bar{v}_*|^2 + s^2}} \,\mathrm{d}\bar{v}_* \leqslant \int_{\mathbb{R}^2} \frac{M(\bar{v}_*)}{|\bar{v} - \bar{v}_*|} \,\mathrm{d}\bar{v}_* \leqslant C_2 < \infty.$$

Therefore, when $s\delta_\epsilon < 1$, the result holds with $C = C_1/C_2$. This shows that (3.6) holds true with $\tilde{C}_\theta = \max\{1, \frac{C_1}{C_2}\}$ independent of ϵ (and also of ϱ_0 in this case). □

Let us explain now, in a rather informal way, how the above result together with Theorem 2.1 allows us to use (1.21) to give a proof of the logarithmic Sobolev inequality (5.11) for $\gamma = 0$ or $\gamma = 1$.

The case $\gamma = 0$. Since $\tilde{B}_\epsilon$ is a normalized Maxwellian collision kernel, for the special choice of the convex function $\Phi(x) = x\log x - x + 1$, if we denote simply by $\mathcal{D}_{\max,\epsilon}$ the associated entropy dissipation functional, Theorem 2.1 asserts that

$$\mathcal{D}_{\max,\epsilon}(f) \geqslant \gamma_\epsilon \mathcal{H}(f|M)$$

for any probability distribution $f \in L^1(\mathbb{R}^d)$ with $\gamma_\epsilon := \gamma_{b_\epsilon} = \dfrac{\int_0^\epsilon \xi^3 \left(1-\xi^2\right)^{\frac{d-3}{2}} \mathrm{d}\xi}{\int_0^\epsilon \xi \left(1-\xi^2\right)^{\frac{d-3}{2}} \mathrm{d}\xi}$. In particular, in dimension $d = 3$, one has

$$\gamma_\epsilon = \frac{\epsilon^2}{2}.$$

Therefore, in dimension $d = 3$, the solution $h_\epsilon(t,\cdot)$ to (5.2) with $\gamma = 0$ satisfies

$$\mathcal{H}(h_\epsilon(t)|M) \leqslant \exp\left(-\frac{\epsilon^2}{2}t\right)\mathcal{H}(f_0|M) \qquad \forall t \geqslant 0$$

which, in terms of the rescaled function $f_\epsilon(t,\cdot)$ defined in (5.3) reads

$$\mathcal{H}(f_\epsilon(t)|M) \leqslant \exp\left(-\frac{t}{2}\right)\mathcal{H}(f_0|M) \qquad \forall t \geqslant 0.$$

In particular, from Proposition 5.1, taking the limit as $\epsilon \to 0$, we get that

$$\mathcal{H}(f(t)|M) \leqslant \exp\left(-\frac{t}{2}\right)\mathcal{H}(f_0|M) \qquad \forall t \geqslant 0$$

for any solution $f(t)$ to the Fokker-Planck equation (5.9) (associated to the diffusion matrix $\mathbf{D}_0(\cdot)$). We recover in this way (5.14) which, as well-known (Arnold et al., 2001), is equivalent to (5.13), i.e.,

$$\mathcal{J}_0(f|M) \geqslant \frac{1}{2}\mathcal{H}(f|M).$$

This shows that the log-Sobolev inequality (5.13) can be recovered from (1.21) in the limit of grazing collisions and, as explained in the introduction, this can be seen as providing a microscopic ground for these particular log-Sobolev inequalities.

The case $\gamma = 1$. In the same way, if one considers now grazing collisions for hard-spheres interactions $B_\epsilon(|q|,\xi)$, the same reasoning can be applied and using Proposition 5.2 we see that any solution $h_\epsilon(t,\cdot)$ to (5.2) satisfies

$$\mathcal{H}(h_\epsilon(t)|M) \leqslant \exp\left(-C\frac{\epsilon^2}{2}t\right)\mathcal{H}(f_0|M) \qquad \forall t \geqslant 0$$

for some explicitly computable constant $C > 0$. In terms of the rescaled function f_ϵ we get now

$$\mathcal{H}(f_\epsilon(t)|M) \leqslant \exp\left(-\frac{C}{2}t\right)\mathcal{H}(f_0|M) \qquad \forall t \geqslant 0$$

which, at the limit $\epsilon \to 0$, yields now

$$\mathcal{H}(f(t)|M) \leqslant \exp\left(-\frac{C}{2}t\right)\mathcal{H}(f_0|M) \qquad \forall t \geqslant 0$$

for any solution $f(t)$ to the Fokker-Planck equation (5.9) (associated to the diffusion matrix $\mathbf{D}_1(\cdot)$). In particular, one deduces from this inequality that the functional inequality

$$\mathcal{J}_1(f|M) \geqslant \frac{C}{2}H(f|M)$$

holds true for any probability density $f \in L^1(\mathbb{R}^3)$ with finite entropy. Again, in the grazing collisions limit, Theorem 1.1 allows us to prove the nontrivial log-Sobolev inequality (5.11) for $\gamma = 1$ — with a not necessarily optimal constant.

Appendix A. Proof of Theorem 1.3

We give in this Appendix a quick proof of Theorem 1.3, which states that the linear Boltzmann operator $\mathcal{L}$ does not satisfy the log-Sobolev inequality (1.24). That is, if we define the quadratic form associated to $\mathcal{L}$ as

$$\mathscr{E}(g) := -\int_{\mathbb{R}^d} \frac{1}{M(v)}\, g(v)\mathcal{L}g(v)\, dv,$$

we show it is not possible to find a positive $\lambda_0 > 0$ such that

$$\mathscr{E}\left(\sqrt{M}\sqrt{f}\right) \geqslant \lambda_0 \mathcal{H}(f|M) \tag{A.1}$$

for any probability density $f \in L^1(\mathbb{R}^d, \mathrm{d}v)$. The proof is based on the well-known fact that (A.1) is equivalent to Nelson's hypercontractivity. In order to apply directly Nelson's hypercontractivity, one shall reformulate the problem in some equivalent way to define the Markov semigroup associated to $\mathcal{L}$. Namely, we introduce the probability measure

$$\mathrm{d}\mu(v) = M(v)\,\mathrm{d}v$$

and the Markov operator

$$\mathbf{L}(h) = M^{-1}\mathcal{L}(h\,M), \qquad \forall h \in \mathscr{D}(\mathbf{L})$$

where $\mathscr{D}(\mathbf{L})$ denotes the domain of $\mathbf{L}$ in the space $L^2(\mathbb{R}^d, \mathrm{d}\mu)$. Notice that $\mathbf{L}$ is a Markov operator in the sense of Bakry et al. (2014) since

$$\int_{\mathbb{R}^d} \mathbf{L}h(v)\,\mathrm{d}\mu(v) = 0 \qquad \text{and} \qquad \mathbf{L}(1) = 0.$$

Notice that, with such notations and using the terminology of Ané et al. (2000); Bakry et al. (2014), it holds

$$\mathcal{H}(f|M) = \mathbf{Ent}_\mu\left(fM^{-1}\right), \qquad \mathscr{E}(g) = -\int_{\mathbb{R}^d} \frac{g}{M}\mathbf{L}\left(\frac{g}{M}\right)\mathrm{d}\mu =: \mathcal{E}_\mu(gM^{-1})$$

so that (A.1) reads equivalently

$$\mathcal{E}_\mu(\sqrt{h}) \geqslant \lambda_0 \mathbf{Ent}_\mu(h), \qquad h = fM^{-1}$$

which is the classical Log-Sobolev inequality for the measure μ and the associated Dirichlet form $\mathcal{E}_\mu$ (see Ané et al. (2000); Bakry et al. (2014) for details). Considering then the Markov semigroup $(\mathcal{S}_t)_t$ generated by $\mathbf{L}$, we recall the following result (see Ané et al. (2000, Theorem 2.8.2)):

Lemma A.1 (Gross (1975))**.** *If* (A.1) *holds true with* $\lambda_0 > 0$ *then*

$$\|\mathcal{S}_t h\|_{L^{q(t)}(\mathbb{R}^d,\,\mathrm{d}\mu)} \leqslant \|h\|_{L^2(\mathbb{R}^d,\,\mathrm{d}\mu)} \qquad \forall t > 0$$

where $q(t) = 1 + \exp(4t/\lambda_0)$ *for any* $t \geqslant 0$ *and* $\mathrm{d}\mu(v) = M(v)\,\mathrm{d}v$ *is the invariant measure associated to* $\mathcal{L}$.

Remark A.2. *Notice that the above Lemma is valid for any Markov semigroup whose invariant measure is reversible. This is the case for the semigroup* $(\mathcal{S}_t)_t$ *associated to* $\mathbf{L}$ *by virtue of the detailed balance principle* (1.17).

In particular, if (A.1) holds true, then, for h_0 in $L^2(\mathbb{R}^d, \mathrm{d}v)$ and for some $t > 0$, there exists $p > 2$ such that

$$\|h(t)\|_{L^p(\mathbb{R}^d,\,\mathrm{d}\mu)} \leqslant \|h_0\|_{L^2(\mathbb{R}^d,\,\mathrm{d}\mu)} \tag{A.2}$$

where $h(t) = \mathcal{S}_t h_0$ is the unique solution to $\partial_t h(t) = \mathbf{L}(h)$ with initial condition h_0 (in other words, $f(t) = Mh(t)$ is the unique solution to eq. (1.4) with initial data Mh_0).

We show that such a $L^2 - L^p$ regularizing property of $(\mathcal{S}_t)_t$ cannot hold. Using the representation (1.16), one sees that

$$\mathcal{L}f(v) \geqslant -\sigma(v)f(v) \qquad \forall f \geqslant 0$$

where $\sigma(v)$ is the collision frequency (depending on the collision kernel $B(|q|, \xi)$) [3]. This translates obviously into

$$\mathbf{L}h(v) \geqslant -\sigma(v)h(v) \qquad \forall h \geqslant 0. \tag{A.3}$$

[3] We recall that if $B(|q|, \xi) = |q \cdot n|$ (corresponding to hard-spheres interactions) then $\sigma(v) \geqslant c(1 + |v|)$ for some $c > 0$; while if $B(|q|, \xi) = c_d\xi$ (corresponding to a normalized Maxwellian collision kernel) then $\sigma(v) = 1$ for any $v \in \mathbb{R}^d$.

Let us now consider $h_0 \in L^2(\mathbb{R}^d, \mathrm{d}\mu)$ nonnegative and let $h(t) = \mathcal{S}_t h_0$ while $g(t,v)$ denotes the unique solution to

$$\partial_t g(t,v) = -\sigma(v) g(t,v) \qquad g(t=0,v) = h_0(v).$$

Clearly

$$g(t,v) = \exp(-\sigma(v)t) h_0(v) \tag{A.4}$$

and (A.3) implies that

$$h(t,v) \geqslant g(t,v) \qquad \forall t \geqslant 0.$$

Now, it is clear from (A.4) that the above equation for $g(t,v)$ has no regularizing effect; i.e., if $h_0 \notin L^p(\mathbb{R}^d, \mathrm{d}\mu)$ then $g(t,v) \notin L^p(\mathbb{R}^d, \mathrm{d}\mu)$ for any $t \geqslant 0$. One deduces from this that, if $h_0 \notin L^p(\mathbb{R}^d, \mathrm{d}\mu)$ then $h(t,v) \notin L^p(\mathbb{R}^d, \mathrm{d}\mu)$. Therefore, inequality (A.2) cannot hold true for any $h_0 \in L^2(\mathbb{R}^d, \mathrm{d}\mu)$ and Gross' Theorem shows that (A.1) cannot hold true.

Appendix B. Bakry-Émery criterion for hard-spheres interactions

We give here a direct proof of the logarithmic Sobolev inequality (5.11) in the case $\gamma = 1$ using the Bakry-C mery criterion (see (5.15)). We use the notation of Section 5.2. In the hard-spheres case ($\gamma = 1$) the diffusion matrix of the associated Fokker–Planck equation reads as

$$\mathbf{D}_1(v) = \frac{1}{8} \int_{\mathbb{R}^3} \Big[|v - v_*|^3 \, \mathbf{I} - |v - v_*| (v - v_*) \otimes (v - v_*) \Big] M(v_*) \, \mathrm{d}v_* \tag{B.1}$$

Since $M(v_*) = \left(\frac{1}{2\pi\theta}\right)^{3/2} \exp\left(-\frac{|v_* - u_0|^2}{2\theta}\right)$, the diffusion matrix may be cast as

$$\mathbf{D}_1(v) = \frac{\gamma^{3/2}}{8\,\pi^{3/2}} \int_{\mathbb{R}^3} \Big[|w|^3 \, \mathbf{I} - |w| w \otimes w \Big] \exp\left(-\gamma |w + a|^2\right) \, \mathrm{d}w$$

where

$$\gamma = \frac{1}{2\theta}\,, \qquad\qquad a = u_0 - v\,.$$

It can be directly checked (see Bisi and Spiga (2011)) that

$$\int_{\mathbb{R}^3} |w|^3 \, \mathrm{e}^{-\gamma|w+a|^2} \, \mathrm{d}w = \frac{\pi}{\gamma^3} \left\{ \mathrm{e}^{-\gamma|a|^2} \left(\gamma |a|^2 + \frac{5}{2} \right) + \frac{\sqrt{\pi}\,\mathrm{erf}\left(\gamma^{1/2}|a|\right)}{\gamma^{1/2}|a|} \left(\gamma^2 |a|^4 + 3\gamma |a|^2 + \frac{3}{4} \right) \right\} \tag{B.2}$$

and, analogously,

$$\begin{aligned}
\int_{\mathbb{R}^3} & |w| (w \otimes w) \mathrm{e}^{-\gamma|w+a|^2} \, \mathrm{d}w \\
&= \frac{\pi}{\gamma^3} \mathbf{I} \left\{ \mathrm{e}^{-\gamma|a|^2} \left(\frac{1}{2} + \frac{1}{4\gamma |a|^2} \right) + \frac{\sqrt{\pi}\,\mathrm{erf}\left(\gamma^{1/2}|a|\right)}{\gamma^{1/2}|a|} \left(\frac{1}{2}\gamma |a|^2 + \frac{1}{2} - \frac{1}{8\,\gamma |a|^2} \right) \right\} \\
&+ \frac{\pi}{\gamma^3} \frac{a \otimes a}{|a|^2} \left\{ \mathrm{e}^{-\gamma|a|^2} \left(\gamma |a|^2 + 1 - \frac{3}{4\gamma |a|^2} \right) + \frac{\sqrt{\pi}\,\mathrm{erf}\left(\gamma^{1/2}|a|\right)}{\gamma^{1/2}|a|} \left(\gamma^2 |a|^4 + \frac{3}{4}\gamma |a|^2 - \frac{3}{4} + \frac{3}{8\,\gamma |a|^2} \right) \right\}
\end{aligned}$$

where erf denotes the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2} \, \mathrm{d}t$, $x \geqslant 0$. Consequently,

$$\mathbf{D}_1(v) = \frac{1}{8\sqrt{\pi}\,\gamma^{3/2}} \left\{ \mathrm{C}\left(\gamma^{1/2}|a|\right) \mathbf{I} + \mathrm{T}\left(\gamma^{1/2}|a|\right) \frac{|a|^2 \mathbf{I} - a \otimes a}{|a|^2} \right\} \tag{B.3}$$

where

$$\mathrm{C}(|x|) = \mathrm{e}^{-|x|^2} \left(1 + \frac{1}{2|x|^2} \right) + \frac{\sqrt{\pi}\,\mathrm{erf}(|x|)}{|x|} \left(\frac{7}{4} |x|^2 + 1 + \frac{1}{4|x|^2} \right)$$

and

$$\mathrm{T}(|x|) = \mathrm{e}^{-|x|^2} \left(|x|^2 + 1 - \frac{3}{4|x|^2} \right) + \frac{\sqrt{\pi}\,\mathrm{erf}(|x|)}{|x|} \left(|x|^4 + \frac{3}{4} |x|^2 - \frac{3}{4} + \frac{3}{8|x|^2} \right).$$

By resorting also to Taylor expansions

$$\mathrm{e}^{-|x|^2} = 1 - |x|^2 + \frac{\xi^4}{2}\,, \qquad \mathrm{erf}(|x|) = \frac{2}{\sqrt{\pi}} \left(|x| - \frac{|x|^3}{3} + \frac{\eta^5}{10} \right)$$

(for suitable $\xi, \eta \in [0, |x|]$), it can be checked that $\mathrm{C}(|x|) > 0$ and $\mathrm{T}(|x|) \geqslant 0$ for any x. Consequently, the matrix $T\left(\gamma^{1/2}|a|\right) \frac{|a|^2\mathbf{I} - a\otimes a}{|a|^2}$ appearing in (B.3) is positive definite and to derive an estimate like (5.16) one can neglect its contribution and consider only the diffusion matrix

$$\frac{1}{8\sqrt{\pi}\,\gamma^{3/2}}\, d(v)\mathbf{I} := \frac{1}{8\sqrt{\pi}\,\gamma^{3/2}}\, \mathrm{C}\left(\gamma^{1/2}|a|\right)\mathbf{I}.$$

In this case, the Bakry-Émery curvature condition (5.15) simply reads, writing $E(v) := \frac{|v-u_0|^2}{2\theta}$, (see Bakry (1994); Arnold et al. (2001)):

$$-\frac{1}{4}\frac{\nabla d(v)\otimes\nabla d(v)}{d(v)} + \frac{1}{2}\left(\Delta d(v) - \nabla d(v)\cdot\nabla E(v)\right)\mathbf{I} \\ + d(v)\mathbb{D}^2 E(v) + \frac{\nabla d(v)\otimes\nabla E(v) + \nabla E(v)\otimes\nabla d(v)}{2} - \mathbb{D}^2 d(v) \geqslant \alpha\, 8\sqrt{\pi}\,\gamma^{3/2}\,\mathbf{I} \quad \text{(B.4)}$$

(in the sense of positive matrices) for any $v \in \mathbb{R}^3$, where we recall that $\mathbb{D}^2$ denotes the Hessian matrix while $E(v) = \frac{|v-u_0|^2}{2\theta}$. Since

$$\nabla d(v) = \mathrm{C}'\left(\gamma^{1/2}|a|\right)\gamma^{1/2}\frac{v-u_0}{|v-u_0|}\,, \qquad \Delta d(v) = \mathrm{C}''\left(\gamma^{1/2}|a|\right)\gamma\,,$$

$$\mathbb{D}^2 d(v) = \mathrm{C}''\left(\gamma^{1/2}|a|\right)\gamma\,\frac{(v-u_0)\otimes(v-u_0)}{|v-u_0|^2} + \mathrm{C}'\left(\gamma^{1/2}|a|\right)\gamma^{1/2}\left[\frac{1}{|v-u_0|}\mathbf{I} - \frac{(v-u_0)\otimes(v-u_0)}{|v-u_0|^3}\right],$$

formula (B.4) becomes

$$A\left(\gamma^{1/2}|a|\right)\mathbf{I} + B\left(\gamma^{1/2}|a|\right)\sum_{i,j=1}^{3}\frac{y_i y_j}{|y|^2}\,\frac{(v_i-u_{0i})(v_j-u_{0j})}{|v-u_0|^2} \geqslant \alpha\, 8\sqrt{\pi}\,\gamma \quad \text{(B.5)}$$

for $v \in \mathbb{R}^3$, $y \in \mathbb{R}^3\setminus\{0\}$, where

$$A(|x|) = \frac{1}{2}\mathrm{C}''(|x|) - \left(|x| + \frac{1}{|x|}\right)\mathrm{C}'(|x|) + 2\,\mathrm{C}(|x|)\,,$$

$$B(|x|) = \mathrm{C}''(|x|) - \left(2\,|x| + \frac{1}{|x|}\right)\mathrm{C}'(|x|) + \frac{1}{4}\,\frac{\left[\mathrm{C}'(|x|)\right]^2}{\mathrm{C}(|x|)}.$$

Since

$$0 \leqslant \sum_{i,j=1}^{3}\frac{y_i y_j}{|y|^2}\,\frac{(v_i-u_{0i})(v_j-u_{0j})}{|v-u_0|^2} \leqslant 1\,,$$

a (non–optimal) estimate for α is provided by $\alpha = \frac{1}{8\sqrt{\pi\,\gamma}}\min\{\alpha_1, \alpha_2\}$ with α_1, α_2 such that

$$A(|x|) \geqslant \alpha_1\,, \qquad A(|x|) - B(|x|) \geqslant \alpha_2\,, \qquad \forall x \in \mathbb{R}^3. \quad \text{(B.6)}$$

Now we have

$$A(|x|) = \mathrm{e}^{-|x|^2}\left(-3\,|x|^2 + 1 + \frac{3}{2|x|^2} + \frac{9}{2|x|^4}\right) + \frac{\sqrt{\pi}\,\mathrm{erf}(|x|)}{|x|}\left(\frac{7}{4}\,|x|^2 + \frac{5}{4} + \frac{3}{4|x|^2} - \frac{9}{4|x|^4}\right);$$

by using Taylor expansion (for $|x| \leqslant 1$) and by studying the derivative $A'(|x|)$ (for higher $|x|$) we get that $A(|x|) \geqslant \alpha_1 = \frac{143}{60} \simeq 2.38$. On the other hand, for $A(|x|) - B(|x|)$ we use the estimate

$$\left|\frac{\mathrm{C}'(|x|)}{\mathrm{C}(|x|)}\right| \leqslant \min\left\{1, \frac{1}{|x|}\right\}\,,$$

and we check that for $|x| > 1$ the quantity $A(|x|) - B(|x|)$ turns out be bounded from below by a constant greater than α_1 (so we skip details here) while, for $|x| \leqslant 1$,

$$A(|x|) - B(|x|) \geqslant \mathrm{e}^{-|x|^2}\left(3\,|x|^2 + \frac{5}{4} - \frac{5}{8|x|^2} - \frac{3}{|x|^4}\right) + \frac{\sqrt{\pi}\,\mathrm{erf}(|x|)}{|x|}\left(\frac{41}{16}\,|x|^2 + \frac{3}{4} + \frac{11}{16|x|^2} + \frac{3}{2|x|^4}\right)$$

that by Taylor expansion turns out to be greater than $\alpha_2 = \frac{7}{3}$. In conclusion, $\alpha = \frac{7}{24}\sqrt{\frac{2\theta}{\pi}}$.

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