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Longevity-linked assets and pre-retirement consumption/portfolio decisions*

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Abstract

We solve the consumption/investment problem of an agent facing a stochastic mortality intensity. The investment set includes a longevity-linked asset, as a derivative on the force of mortality. In a complete and frictionless market, we derive a closed form solution when the agent has Hyperbolic Absolute Risk Aversion preferences and a fixed financial horizon. Our calibrated numerical analysis on US data shows that individuals optimally invest a large fraction of their wealth in longevity-linked assets in the pre-retirement phase, because of their need to hedge against stochastic fluctuations in their remaining life-time at retirement.

Keywords: longevity risk, pre-retirement savings, portfolio choice, HARA preferences.

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1 Introduction

Despite the relevant and increasing hedging need of pension funds and annuity providers, the market for longevity risk, i.e. the risk of unexpected changes in the mortality of a group of individuals, is not sufficiently liquid yet.

Many reasons may have contributed to undermine a rapid development of the market, such as the lack of standardisation, informational asymmetries, and basis risk. Nevertheless, recent developments provide a sound technology for modelling the systematic randomness in mortality (see e.g. Lee and Carter, 1992), for designing and evaluating hedging instruments (Blake et al., 2006 and Denuit et al., 2007) and for managing longevity risk (Barrieu et al., 2012).

Furthermore, the transfer of longevity risk from pension funds to re-insurers has become more and more common, although on an over-the-counter basis. For instance, the volume of outstanding UK longevity swaps has reached 50 billion pounds as of the end of 2014, with a prevalence of very large deals, such as the 16 billion pounds swap between BT Pension Scheme and Prudential and the 12 billion euros Delta Lloyd/RGA Re index-based transaction. Investment banks have been also actively in the transactions. Between 2008 and 2014, alongside reinsurance specialists, JP Morgan, Credit Suisse, Goldman Sachs, Deutsche Bank and Société Générale were involved in longevity deals (Luciano and Regis, 2014).

Longevity-linked products should be of interest to both households and asset managers for at least two reasons: their low correlation to other asset classes (at least in the short run, see Loeys et al., 2007), and their effectiveness in hedging individual investors against the unexpected fluctuations of their subjective discount factors, which take into account lifetime uncertainty (Yaari, 1965, Merton, 1971, Huang et al., 2012). Importantly, from the point of view of individual investors, while traditional insurance products are non-marketable, longevity assets are listed on the market and allow to dynamically hedge against mortality fluctuations.

The aim of this paper is to analyse the optimal consumption and portfolio choices of a consumer/investor subject to longevity risk prior to retirement. The agent can invest in a friction-less, arbitrage free, and complete financial market where both traditional assets (bonds and stocks) and a longevity bond are listed.

An extensive literature has explored consumption and investment decisions when mortality contingent claims are present. In particular, Huang and Milevsky (2008) analyse the decisions of households in the presence of income risk and life insurance. Explicit solutions are also obtained by Pirvu and Zhang (2012) with stochastic asset prices drifts and inflation risk and by Kwak and Lim (2014) with constant relative risk aversion (CRRA) preferences. All these papers consider a deterministic force of mortality, while we model it as a stochastic process. We describe longevity risk by means of a doubly stochastic process whose intensity follows a continuous-time diffusion (as in Dahl, 2004). This process may be correlated with all the other state variables. Because of the stochastic mortality, both individuals and annuity/life insurance sellers are exposed to unexpected changes in the force of mortality, implying under or over reserving. For instance, the optimal investment problem of pension funds in the accumulation phase has been studied extensively, for instance by Battocchio et al. (2007) and Delong et al. (2008).

In this paper, we focus instead on the effects of longevity risk and its hedging on individual's consumption and portfolio decisions. The role of longevity-linked assets in investor's optimal portfolio has been addressed first by Menoncin (2008). In this work, we generalise Menoncin (2008) in many directions: (i) the consumer is endowed with a stochastic labour income, (ii) investor's preferences belong to the Hyperbolic Absolute

Risk Aversion (HARA) family, allowing for a subsistence level of both consumption and final wealth, which significantly affect the inter-temporal behaviour of the optimal asset allocation, (iii) our numerical simulations take into account mean reverting square root processes for both the interest rate and the force of mortality, which allow for a more realistic framework with respect to the simple Ornstein-Uhlenbeck processes, (iv) we focus on pre-retirement decisions (with a finite-horizon), where the final wealth must reach at least a value sufficient for be traded with a life annuity.

We consider a fixed deterministic retirement age (in contrast, for instance, with Farhi and Panageas, 2007 and Dybvig and Liu, 2010 who consider an endogenous retirement choice), which coincides with the time horizon of the problem. Horneff et al. (2010) and Maurer et al. (2013) numerically analyse life-cycle portfolio investment problems with longevity risk and focus on the role of deferred annuities and variable annuities respectively. In the context of a life-cycle model, Cocco and Gomes (2012) analyse the demand for a perfect hedge against shocks in the life expectancy of a CRRA agent. They study the optimal investment in a longevity bond, which is akin to the zero-coupon longevity asset that we use in this work. In their numerical simulations, they find that individuals – at old ages and especially approaching retirement – should invest a relevant fraction of their wealth in the longevity asset. Even if our results align with the findings of this literature, we highlight the importance of individual's systematic longevity risk protection in motivating the holding of longevity-linked securities, rather than consumption smoothing after retirement.

Our main contribution consists in providing a closed form solution to the (finite-horizon) problem of an agent prior to retirement, endowed with a general HARA class of preferences when mortality intensity is stochastic. We also provide a calibrated application, which allows to appreciate the magnitude of the investment in longevity-linked products in the optimal agent's portfolio.

Under reasonable stochastic models and calibration for both mortality intensity and interest rate, we find that individuals should optimally invest a relevant fraction of their wealth in a longevity bond. A 60-year old US male, who wants to retire at 65 optimally invests around 70% of his portfolio in the (zero market price of risk) longevity bond and then progressively decreases this share approaching retirement. Demand of the longevity-linked asset is due to hedging motives, as the individual invests in longevity bonds to protect against the fluctuations in his/her discount factor, which accounts for an uncertain lifetime. Because of the stochastic force of mortality, also the amount of wealth that must be invested, at retirement, to obtain a life annuity, varies over time. We explore the sensitivity of our results to both individual and market characteristics, finding that the optimal demand for longevity bonds: (i) is higher for 60-year old US females than for 60-year old US males; (ii) reduces (but very slightly) when the agent displays a more conservative behaviour (either high risk aversion, or high minimum consumption or high final wealth minimum level); (iii) remains positive over the whole horizon even when longevity risk is not remunerated. These last two results are robust for younger agents.

The outline of the paper is the following. Section 2 presents the model set-up, while Section 3 describes the individual preferences and the maximisation problem. The optimal consumption and portfolio are found in closed form. Section 4 provides a calibrated application based on US data. Finally, Section 5 concludes, and some technical derivations are left to two appendices.

2 The model setup

2.1 State Variables

On a continuously open and friction-less financial market over the time set $[t_0, +\infty[$, the economic framework is described by a set of s state variables $z(t) \in \mathbb{R}^s$ which solve the following (matrix) stochastic differential equation:

$$\underset{s\times 1}{dz(t)} = \underset{s\times 1}{\mu_z(t,z)}dt + \underset{s\times n}{\Omega(t,z)'}\underset{n\times 1}{dW(t)}, \tag{1}$$

where $z(t_0)$ is a deterministic vector that defines the initial state of the system, $W(t)$ is a vector of n independent Wiener processes,[1] and the prime denotes transposition. The usual properties for guaranteeing the existence of a strong solution to (1) are assumed to hold. The vector $z(t)$ can be divided into two components: the financial state variables $z_f(t)$ and the mortality intensity of a group of individuals, which, as customary in the actuarial literature, are assumed to be homogeneous by cohort, i.e. they belong to the same generation:

$$\underbrace{\begin{bmatrix} \underset{(s-1)\times 1}{dz_f(t)} \\ d\lambda(t) \end{bmatrix}}_{dz(t)} = \underbrace{\begin{bmatrix} \underset{(s-1)\times 1}{\mu_{zf}(t,z)} \\ \mu_\lambda(t,z) \end{bmatrix}}_{\mu_z(t,z)} dt + \underbrace{\begin{bmatrix} \underset{(s-1)\times(n-1)}{\Omega_f(t,z)'} & \underset{(s-1)\times 1}{\mathbf{0}} \\ \underset{1\times(n-1)}{\sigma_{f\lambda}(t,z)'} & \sigma_\lambda(t,z) \end{bmatrix}}_{\Omega(t,z)'} \underbrace{\begin{bmatrix} \underset{(n-1)\times 1}{dW_f(t)} \\ dW_\lambda(t) \end{bmatrix}}_{dW(t)}, \tag{2}$$

where $\mathbf{0}$ is a vector of zeros. The diffusion vector $\sigma_{f\lambda}(t,z)$ captures the correlation between the financial state variables $z_f(t)$ and the mortality intensity $\lambda(t)$.

[1] The case with dependent Wiener processes can be easily obtained through the Cholesky's decomposition.

2.2 Financial Market

On the financial market n risky assets are traded. Their prices $S(t) \in \mathbb{R}_+^n$ solve the (matrix) stochastic differential equation

$$\underset{n\times 1}{I_S^{-1}} \underset{n\times 1}{dS(t)} = \underset{n\times 1}{\mu(t,z)}dt + \underset{n\times n}{\Sigma(t,z)'} \underset{n\times 1}{dW(t)}, \tag{3}$$

where I_S is a diagonal matrix containing the elements of vector $S(t)$. The initial asset prices $S(t_0)$ are deterministic. Finally, a risk-less asset exists, whose price $G(t) \in \mathbb{R}_+$ solves the ordinary differential equation

$$G(t)^{-1} dG(t) = r(t,z)\, dt, \tag{4}$$

where $r(t,z) \in \mathbb{R}_+$ is the instantaneously risk-less interest rate. We assume $G(t_0) = 1$, i.e. the risk-less asset is the *numéraire* of the economy. The financial market is assumed to be arbitrage free and complete. In other words, a unique vector of market prices of risk $\xi(t,z) \in \mathbb{R}^n$ exists, such that $\Sigma(t,z)' \xi(t,z) = \mu(t,z) - r(t,z)\mathbf{1}$, where $\mathbf{1}$ is a vector of ones (i.e. $\exists!\ \Sigma(t,z)^{-1}$).

Girsanov's theorem allows us to switch from the historical ($\mathbb{P}$) to the risk-neutral probability $\mathbb{Q}$ by using $dW^{\mathbb{Q}}(t) = \xi(t,z)\, dt + dW(t)$. We recall that a sufficient condition for Girsanov's theorem to hold is the so-called Novikov's condition, that requires that

$$\mathbb{E}_t \left[e^{\int_t^T \xi(u,z)'\xi(u,z)du} \right] < \infty. \tag{5}$$

The value in t_0 of any tradeable cash flow $\Xi(t)$ available at time t can be written as

$$\Xi(t_0) = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\Xi(t)\frac{G(t_0)}{G(t)}\right] = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\Xi(t)\, e^{-\int_{t_0}^{t} r(u,z)du}\right] = \mathbb{E}_{t_0}\left[\Xi(t)\, m(t_0,t)\, e^{-\int_{t_0}^{t} r(u,z)du}\right], \tag{6}$$

where $\mathbb{E}_{t_0}[\bullet]$ and $\mathbb{E}_{t_0}^{\mathbb{Q}}[\bullet]$ are the expected value operators under the historical $(\mathbb{P})$ and the risk neutral $(\mathbb{Q})$ probabilities respectively, conditional on the information set at time t_0, and the martingale $m(t_0,t)$, such that $m(t_0,t_0) = 1$, solves $m(t_0,t)^{-1} dm(t_0,t) = -\xi(t,z)\, dW(t)$.

2.3 Longevity Bonds Market

The mortality intensity (or force of mortality) $\lambda(t,z) \in \mathbb{R}_+$ of a homogeneous group of individuals, which the investor belongs to, is an element of $z(t)$. Following the stochastic mortality approach initiated by Milevsky and Promislow (2001) and Dahl (2004), the death event is modelled as a Poisson process with stochastic intensity. The probability to be alive at time t, given that an agent is alive in t_0, is given by $\mathbb{E}_{t_0}^{\mathbb{P}}\left[e^{-\int_{t_0}^{t}\lambda(u,z)du}\right]$.

The value in t_0 of a financial flow $\Xi(t)$ available in t if an agent is still alive can be written as $\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\Xi(t)\, e^{-\int_{t_0}^{t} r(u,z)+\lambda(u,z)du}\right]$, while the value of the same cash flow available at the death time of an agent is given by $\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\infty}\lambda(s)\,\Xi(s)\, e^{-\int_{t_0}^{s} r(u,z)+\lambda(u,z)du}ds\right]$, where we have assumed that the death time is defined on the interval $[t_0, +\infty[$ (see Lando, 1998).

The financial market described in the previous section is assumed to be complete even with respect to the force of mortality. In other words, we assume that there exists a derivative on $\lambda(t)$, which we will refer to hereafter as the "longevity asset".

Remark 1. Since the market is completed because of the presence of this marketed asset, the exact form we assume for it is immaterial. Indeed, in a complete market any other security whose payoff is linked to mortality intensity can be replicated.

We denote with $\Lambda(t)$ the price of such an asset. The dynamics of asset prices (3) can be rewritten disentangling the (independent) Wiener processes $W_f(t)$ and $W_\lambda(t)$:

$$\underbrace{\begin{bmatrix} \underset{(n-1)\times(n-1)}{I_{S_f}^{-1}} & \underset{(n-1)\times 1}{\mathbf{0}} \\ \underset{1\times(n-1)}{\mathbf{0}'} & \Lambda^{-1} \end{bmatrix}}_{I_S^{-1}} \underbrace{\begin{bmatrix} \underset{(n-1)\times 1}{dS_f(t)} \\ d\Lambda(t) \end{bmatrix}}_{dS(t)} = \underbrace{\begin{bmatrix} \underset{(n-1)\times 1}{\mu_f(t,z)} \\ \mu_\Lambda(t,z) \end{bmatrix}}_{\mu(t,z)} dt + \underbrace{\begin{bmatrix} \underset{(n-1)\times(n-1)}{\Sigma_f(t,z)'} & \underset{(n-1)\times 1}{\mathbf{0}} \\ \underset{1\times(n-1)}{\sigma_{\Lambda f}(t,z)'} & \sigma_{\Lambda\lambda}(t,z) \end{bmatrix}}_{\Sigma(t,z)'} \underbrace{\begin{bmatrix} \underset{(n-1)\times 1}{dW_f(t)} \\ dW_\lambda(t) \end{bmatrix}}_{dW(t)}. \tag{7}$$

3 Investor's maximisation problem

3.1 Investor's Wealth, Consumption and Revenue

The consumer/investor holds $\theta_S(t) \in \mathbb{R}^n$ units of the risky assets and $\theta_G(t) \in \mathbb{R}$ units of the risk-less asset. Thus, at any instant in time, the his/her wealth $R(t)$ is given by the static budget constraint

$$R(t) = \theta_S(t)' S(t) + \theta_G(t) G(t), \tag{8}$$

whose differential is the dynamic budget constraint

$$dR(t) = \theta_S(t)' dS(t) + \theta_G(t) dG(t) + \underbrace{d\theta_S(t)' (S(t) + dS(t)) + d\theta_G(t) G(t)}_{dR_a(t)}. \tag{9}$$

The first two components on the right hand side of (9) account for the changes in prices. The $dR_a(t)$ component, which accounts for the dynamic adjustment of the portfolio allocation: (i) finances the instantaneous consumption $c(t)\,dt$, (ii) discounts the probability of dying between t and $t+dt$, which is measured by $\lambda(t,z)\,dt$, and (iii) is financed by the investor's labour income.

The accumulated labour income from t_0 up to time t is $L(t)$ and solves

$$dL(t) = w(t,z)\,dt + \underbrace{\left[\begin{array}{cc} \underset{1\times(n-1)}{\sigma_{Lf}(t,z)'} & \sigma_{L\lambda}(t,z) \end{array}\right]}_{\sigma_L(t,z)'} \underbrace{\left[\begin{array}{c} \underset{(n-1)\times 1}{dW_f(t)} \\ dW_\lambda(t) \end{array}\right]}_{dW(t)}, \tag{10}$$

where $w(t,z)$ is the (instantaneous) labour income (or wage) of the agent. Thus, the investor's wealth dynamics is

$$dR(t) = \theta_S(t)'\,dS(t) + \theta_G(t)\,dG(t) - c(t)\,dt + dL(t) + \lambda(t,z)\,R(t)\,dt. \tag{11}$$

Once the static budget constraint (8) and the asset differentials (3), (4) and (10) are suitably taken into account, $dR(t)$ becomes

$$\begin{aligned} dR(t) &= \big(R(t)\left(r(t,z) + \lambda(t,z)\right) + \theta_S(t)'\,I_S\left(\mu(t,z) - r(t,z)\,\mathbf{1}\right) + w(t,z) - c(t)\big)\,dt \\ &\quad + \left(\theta_S(t)'\,I_S\Sigma(t,z)' + \sigma_L(t,z)'\right)dW(t)\,. \end{aligned} \tag{12}$$

3.2 Consumer's preferences and objective

The consumer obtains utility from both the inter-temporal consumption, $U_c(c(t)) = (c(t) - c_m)^{1-\delta}/(1-\delta)$, and the wealth at the end of the financial horizon, $U_R(R(T)) = (R(T) - R_m)^{1-\delta}/(1-\delta)$, where $\delta > 0$ and c_m and R_m can be interpreted as the minimum subsistence value of consumption and final wealth, respectively. Both utilities belong to the HARA family. In fact, the Arrow-Pratt absolute risk aversion indexes are $\delta/(c(t) - c_m)$ and $\delta/(R(T) - R_m)$, respectively. Accordingly, the higher c_m (or R_m), the higher the risk aversion: an agent who has to guarantee a higher minimum level of consumption (or final wealth) will choose a safer investment. The case of CRRA preferences is obtained with $c_m = R_m = 0$.

We let R_m coincide with the value of an annuity that is subscribed by the agent at time T for receiving a flow of pensions until the time of death. Since on the financial market there exists a longevity-linked asset,the agent can replicate the annuity through a suitable portfolio.

The value in T of an annuity whose instalment is $p(t)$ is given by

$$R_m(T,z) = \mathbb{E}_T^{\mathbb{Q}}\left[\int_T^{\infty} p(s)\, e^{-\int_T^s r(u)+\lambda(u)du} ds\right]. \tag{13}$$

The value of $R_m(T,z)$ stochastically changes over time because of both the interest rate risk and the longevity risk. Indeed, if the force of mortality at time T is lower than the agent's expectations, then the discounted value of the future pension cash flows is higher.

The consumer/investor chooses the pair $(c(t), \theta_S(t))$ which maximises the inter-temporal utility of his/her wealth and consumption up to time T:

$$\begin{aligned} \max_{\theta_S(t),c(t)} \quad \mathbb{E}_{t_0} & \left[\int_{t_0}^{T} \frac{(c(t)-c_m)^{1-\delta}}{1-\delta} e^{-\int_{t_0}^{t}\rho(u,z)+\lambda(u,z)du} dt \right. \\ & \left. +\pi\, \frac{(R(T)-R_m(T,z))^{1-\delta}}{1-\delta} e^{-\int_{t_0}^{T}\rho(u,z)+\lambda(u,z)du}\right], \end{aligned} \tag{14}$$

where $\rho(t,z)$ is a (possibly stochastic) subjective discount rate, and π is a constant which measures the subjective relevance of the final wealth utility with respect to the intertemporal consumption utility. The higher π the higher the "intensity" associated with the utility obtained from the final wealth.

Notice that the budget constraint equalises the initial wealth increased by the expected value of all the future revenues to the sum between the final wealth and the whole consumption stream (recall $\mathbb{E}_{t_0}^{\mathbb{Q}}\left[dL\left(t,z\right)\right] = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[w\left(t,z\right) - \sigma_L\left(t,z\right)'\xi\left(t,z\right)\right]dt$)

$$\begin{aligned} R\left(t_0\right) &= \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{T}\left(c\left(s\right) - w\left(s,z\right) + \sigma_L\left(s,z\right)'\xi\left(s,z\right)\right)e^{-\int_{t_0}^{s} r(u,z)+\lambda(u,z)du}ds\right. \\ &\quad \left. + R\left(T\right)e^{-\int_{t_0}^{T} r(u,z)+\lambda(u,z)du}\right], \end{aligned} \tag{15}$$

i.e. the difference between the discounted value of final wealth at T and the initial wealth of the consumer must coincide with the expected value of the discounted flow of risk-adjusted consumption net of labour income.

3.3 The optimal consumption and portfolio

Problem (14) under the constraint (15) can be solved either through dynamic programming (via the so-called Hamilton-Jacobi-Bellman equation) or through the so-called martingale approach. This last method is viable in our framework because of market completeness.

Proposition 1. *The optimal consumption and portfolio solving problem (14) are*

$$c^*\left(t\right) = c_m + \frac{R\left(t\right) - H\left(t,z\right)}{F\left(t,z\right)}, \tag{16}$$

$$\begin{aligned} I_S\theta_S^*\left(t\right) &= -\Sigma\left(t,z\right)^{-1}\sigma_L\left(t,z\right) + \frac{R\left(t\right) - H\left(t,z\right)}{\delta}\Sigma\left(t,z\right)^{-1}\xi\left(t,z\right) \\ &\quad + \frac{R\left(t\right) - H\left(t,z\right)}{F\left(t,z\right)}\Sigma\left(t,z\right)^{-1}\Omega\left(t,z\right)\frac{\partial F\left(t,z\right)}{\partial z} + \Sigma\left(t,z\right)^{-1}\Omega\left(t,z\right)\frac{\partial H\left(t,z\right)}{\partial z}, \end{aligned} \tag{17}$$

where

$$
\begin{aligned}
H(t,z) &= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \left(c_m - w(s,z) + \sigma_L(s,z)'\,\xi(s,z)\right) e^{-\int_t^s r(u,z)+\lambda(u,z)du} ds \right. \\
&\quad \left. + R_m(T,z)\, e^{-\int_t^T r(u,z)+\lambda(u,z)du}\right], \qquad (18)
\end{aligned}
$$

$$
F(t,z) = \mathbb{E}_t^{\mathbb{Q}_\delta}\left[\begin{array}{c} \int_t^T e^{-\int_t^s \left(\frac{\delta-1}{\delta} r(u,z)+\frac{1}{\delta}\rho(u,z)+\lambda(u,z)+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi(u,z)'\xi(u,z)\right)du} ds \\ +\pi^{\frac{1}{\delta}} e^{-\int_t^T \left(\frac{\delta-1}{\delta} r(u,z)+\frac{1}{\delta}\rho(u,z)+\lambda(u,z)+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi(u,z)'\xi(u,z)\right)du} \end{array}\right], \qquad (19)
$$

$$
dW(t)^{\mathbb{Q}_\delta} = \frac{\delta-1}{\delta}\xi(t,z)\,dt + dW(t). \qquad (20)
$$

Proof. See Appendix 5. □

In the solution we used the new probability measure $\mathbb{Q}_\delta$ defined in (20). It has two relevant properties: (i) for a log-utility agent, i.e. $\delta = 1$, the probability $\mathbb{Q}_\delta$ coincides with the historical probability; (ii) when the agent is infinitely risk averse, i.e. $\delta \to +\infty$, the probability $\mathbb{Q}_\delta$ coincides with $\mathbb{Q}$. In fact, we can think of the Wiener processes under $\mathbb{Q}_\delta$ as a linear combination of the Wiener processes under the risk neutral and the historical probabilities:

$$
dW(t)^{\mathbb{Q}_\delta} = \left(1 - \frac{1}{\delta}\right) dW(t)^{\mathbb{Q}} + \frac{1}{\delta} dW(t). \qquad (21)
$$

Such a linear combination is strictly convex if $\delta \geq 1$. In this case, in fact, the weight $\delta^{-1} \in]0,1]$ allows us to interpret $\mathbb{Q}_\delta$ as a weighted average of $\mathbb{Q}$ and $\mathbb{P}$.

The function $H(t,z)$ is the expected value, under $\mathbb{Q}$, of the minimum final wealth $R_m(T,z)$ and of the risk-adjusted minimum consumption level net of wage, appropriately discounted in both actuarial and financial terms. Thus, it represents the net expected balance after financing the minimum consumption and final wealth.

The function $F(t,z)$ is the expected value (under the preference-adjusted measure $\mathbb{Q}_\delta$) of the sum of all the discount factors for both the consumption stream (first term) and the final wealth (second term). We can interpret $F(t,z)$ as a "global" discount factor.

Optimal consumption is equal to the sum of the minimum consumption level c_m and the difference between actual wealth $R(t)$ and the expected value of all the discounted subsistence levels of consumption and terminal wealth $H(t,z)$, divided by the "global" risk and preference-adjusted discount factor $F(t,z)$. We remark that the difference $R(t) - H(t,z)$ is also relevant for computing the optimal portfolio, which depends also on the sensitivities of $H(t,z)$ and $F(t,z)$ with respect to the state variables z.

The role of the longevity asset in (17) can be identified using the decomposition of matrices Ω and Σ as shown in (2) and (7), respectively. The optimal investments in the longevity asset and in the financial assets are, respectively:

$$\Lambda\theta_\Lambda^* = -\frac{\sigma_{L\lambda}}{\sigma_{\Lambda\lambda}} + \frac{R-H}{\delta}\frac{1}{\sigma_{\Lambda\lambda}}\left(\xi_\lambda + \frac{\delta}{F}\sigma'_{f\lambda}\frac{\partial F}{\partial z_f} + \sigma_\lambda\frac{\delta}{F}\frac{\partial F}{\partial\lambda}\right) - \frac{1}{\sigma_{\Lambda\lambda}}\left(\sigma'_{f\lambda}\frac{\partial H}{\partial z_f} + \sigma_\lambda\frac{\partial H}{\partial\lambda}\right), \quad (22)$$

$$I_{S_f}\theta_{S_f}^* = -\Sigma_f^{-1}\sigma_{Lf} + \frac{R-H}{\delta}\Sigma_f^{-1}\left(\xi_f + \frac{\delta}{F}\Omega'_f\frac{\partial F}{\partial z_f}\right) - \Sigma_f^{-1}\Omega'_f\frac{\partial H}{\partial z_f} - \Sigma_f^{-1}\sigma_{\Lambda f}\Lambda\theta_\Lambda^*. \quad (23)$$

We identify four components in the demand for the longevity asset: (i) a speculative component, related to the risk premium ξ_λ, (ii) a hedging component against labour income fluctuations, (iii) a hedging component against the fluctuations of the global discount factor $F(t,z)$, and (iv) a hedging component against the fluctuations of the expected imbalance to finance minimum consumption and wealth $H(t,z)$.

The last two components depend on: (i) the risk aversion of the individual, (ii) the variance-covariance matrix of the state variables, and (iii) the sensitivities of $F(t,z)$ and $H(t,z)$ with respect to changes in the state variables.

The inclusion of the longevity asset in the investment set modifies the individual demand of both the risk-less and the other financial assets. We can interpret the amount of wealth invested in the longevity asset as taken partly from the wealth invested in the risk-less asset and partly from the wealth invested in the financial assets (as in Menoncin, 2008). The proportion taken from the financial assets is given by the ratio between the covariance of the longevity and the financial assets and the variance of the financial assets (in fact, $\left(\Sigma_f'\Sigma_f\right)^{-1}\Sigma_f'\sigma_{\Lambda f} = \Sigma_f^{-1}\sigma_{\Lambda f}$). This means that the higher the (absolute value of the) correlation between a financial asset and the longevity asset, the higher the (absolute value of the) amount of wealth taken from the former to be invested in the latter. If the longevity asset is not correlated to the other financial assets (i.e. $\sigma_{\Lambda f} = 0$), then the amount of money to be invested in it is fully taken from what is invested in the risk-less asset.

4 A numerical application

4.1 State Variables

In order to present a numerical application, a simplified market structure is taken into account. We consider two uncorrelated state variables: the instantaneously risk-less interest rate $r(t)$ and the force of mortality $\lambda(t)$. They solve the following differential equations

$$dr(t) = \alpha_r\left(\beta_r - r(t)\right)dt + \sigma_r\sqrt{r(t)}dW_r(t), \tag{24}$$

$$d\lambda(t) = \alpha_\lambda\left(\underbrace{\frac{1}{\alpha_\lambda}\frac{\partial\gamma(t)}{\partial t} + \gamma(t)}_{\beta_\lambda(t)} - \lambda(t)\right)dt + \sigma_\lambda\sqrt{\lambda(t)}dW_\lambda(t), \tag{25}$$

where $\alpha_r, \alpha_\lambda > 0$ are the strength of the mean reversion effect and $\beta_r, \gamma(t) > 0$ are the long-term means. $\gamma(t)$ is set to be equal to the Gompertz law (with $\lambda(t_0) = \gamma(t_0)$, see Menoncin, 2009). If the Feller conditions $2a_r\beta_r > \sigma_r^2$ and $2\alpha_\lambda\beta_\lambda(t) > \sigma_\lambda^2$ are satisfied for any t, the two processes are always positive. The choice of this mortality model is justified by its good properties: (i) it is a stochastic extension of the Gompertz law, that represents the long-term mean of the process, (ii) it is always non-negative, provided that the Feller condition is satisfied, (iii) it is analytical tractable, and (iv) it fits well the cohort-based mortality patterns, as it will be shown in Section 4.4.

In order to preserve the statistical properties of $r(t)$ and $\lambda(t)$ after switching between probabilities, we assume that both the market prices of interest rate risk and mortality risk are proportional to the square root of the respective variable (ϕ_r and ϕ_λ are constant):

$$\xi_r = \phi_r\sqrt{r(t)}, \quad \xi_\lambda = \phi_\lambda\sqrt{\lambda(t)}. \tag{26}$$

The results we are about to present rely on the following proposition.

Proposition 2. *If the variable $x(t)$ solves the stochastic differential equation*

$$dx(t) = \alpha(\beta(t) - x(t))\,dt + \sigma\sqrt{x(t)}dW(t), \tag{27}$$

then, for any constant χ,

$$\mathbb{E}_t\left[e^{-\chi\int_t^T x(u)du}\right] = e^{-\alpha\int_t^T \beta(s)C(s;\chi,\alpha,\sigma,T)ds - C(t;\chi,\alpha,\sigma,T)x(t)}, \tag{28}$$

where

$$C(t;\chi,\alpha,\sigma,T) = 2\chi\frac{1-e^{-\Delta(T-t)}}{\Delta+\alpha+(\Delta-\alpha)e^{-\Delta(T-t)}}, \tag{29}$$

$$\Delta \equiv \sqrt{\alpha^2 + 2\sigma^2\chi}. \tag{30}$$

Proof. See Appendix 5. □

In order to simplify the computations, we assume that the labour income is deterministic (i.e. $\sigma_L = 0$) and the wage w is constant. Of course, our model can accommodate a stochastic labour income, with non-zero correlation with stocks, bonds and/or the longevity-linked security, as presented in Section 2. This would modify the asset allocation strategy, implying additional hedging demand for the correlated assets. [2] Furthermore, the annuity $(p(t))$ that the gent wants to obtain starting form time T is also assumed to be constant (p). Thus, the function $H(t,z)$ can be written as

$$\begin{aligned} H(t,z) &= (c_m - w)\int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r(u,z)du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda(u,z)du}\right]ds \\ &+\mathbb{E}_t^{\mathbb{Q}}\left[R_m(T,z)\,e^{-\int_t^T r(u,z)+\lambda(u,z)du}\right], \end{aligned} \tag{31}$$

and if the value of $R_m(T,z)$ is substituted from (13) and by using the tower property of conditional expected values,

$$\begin{aligned} H(t,z) &= (c_m - w)\int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r(u,z)du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda(u,z)du}\right]ds \\ &+p\int_T^\infty \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r(u)du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda(u)du}\right]ds, \end{aligned} \tag{32}$$

where

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\right] = e^{-\alpha_r^{\mathbb{Q}}\beta_r^{\mathbb{Q}}\int_t^T C\left(s;1,\alpha_r^{\mathbb{Q}},\sigma_r,T\right)ds - C\left(t;1,\alpha_r^{\mathbb{Q}},\sigma_r,T\right)r(t)}, \tag{33}$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \lambda(u)du}\right] = e^{-\alpha_\lambda^{\mathbb{Q}}\int_t^T \beta_\lambda^{\mathbb{Q}}(s)C\left(s;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T\right)ds - C\left(t;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T\right)\lambda(t)}. \tag{34}$$

[2] Importantly, extensive numerical simulation shows that introducing positive correlation with stocks has a negligible impact on the demand for the longevity-linked asset, while tilting the investment choice between stocks and bonds towards the former.

Since the optimal portfolio contains the derivatives of $H(t,z)$ with respect to the state variables (in this framework: $r(t)$ and $\lambda(t)$), we recall that their values are:

$$\begin{aligned}\frac{\partial H(t,z)}{\partial r(t)} = & -(c_m - w)\int_t^T C\left(t;1,\alpha_r^{\mathbb{Q}},\sigma_r,s\right)\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r(u,z)du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda(u,z)du}\right]ds \\ & -p\int_T^{\infty} C\left(t;1,\alpha_r^{\mathbb{Q}},\sigma_r,s\right)\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r(u)du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda(u)du}\right]ds,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial H(t,z)}{\partial \lambda(t)} = & -(c_m - w)\int_t^T C\left(t;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,s\right)\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r(u,z)du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda(u,z)du}\right]ds \\ & -p\int_T^{\infty} C\left(t;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,s\right)\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s r(u)du}\right]\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^s \lambda(u)du}\right]ds,\end{aligned}$$

respectively.

4.2 Traded Assets

Four assets are traded on the financial market:

- the risk-less asset, whose price $G(t)$ evolves as in (4);
- a risky asset (like a stock index) whose price $A(t)$ follows a GBM:

$$A(t)^{-1}dA(t) = \mu dt + \sigma_A dW_A(t) + \sigma_{Ar}dW_r(t), \tag{35}$$

where σ_{Ar} measures the instantaneous covariance between the stock and the risk-less interest rate. We assume ξ_A is constant[3] and thus

$$\begin{aligned} A(t)^{-1} dA(t) &= r(t)\,dt + \sigma_A dW_A^{\mathbb{Q}}(t) + \sigma_{Ar} dW_r^{\mathbb{Q}}(t) \\ &= \left(r(t) + \sigma_A \xi_A + \sigma_{Ar} \phi_r \sqrt{r(t)}\right) dt + \sigma_A dW_A(t) + \sigma_{Ar} dW_r(t); \end{aligned} \tag{36}$$

- a constant time to maturity (T_B) zero-coupon bond whose price is

$$B(t) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+T_B} r(u)du}\right], \tag{37}$$

and whose differential is

$$\begin{aligned} B(t)^{-1} dB(t) &= r(t)\,dt - C\left(0; 1, \alpha_r^{\mathbb{Q}}, \sigma_r, T_B\right) \sigma_r \sqrt{r(t)} dW_r^{\mathbb{Q}}(t) \\ &= r(t)\left(1 - C\left(0; 1, \alpha_r^{\mathbb{Q}}, \sigma_r, T_B\right) \sigma_r \phi_r\right) dt \\ &\quad - C\left(0; 1, \alpha_r^{\mathbb{Q}}, \sigma_r, T_B\right) \sigma_r \sqrt{r(t)} dW_r(t); \end{aligned} \tag{38}$$

- a constant time to maturity (T_Λ) zero-coupon longevity bond whose price is

$$\Lambda(t) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{t+T_\Lambda} r(u)+\lambda(u)du}\right], \tag{39}$$

[3]Because of the form we have chosen for the vector ξ (see also (26)), we can easily check that the Novikov's condition in (5) actually holds.

and whose differential is

$$\begin{aligned}\Lambda(t)^{-1} d\Lambda(t) &= (r(t)+\lambda(t))\,dt - C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_\Lambda\right)\sigma_r\sqrt{r(t)}dW_r^{\mathbb{Q}}(t) \\ &- C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)\sigma_\lambda\sqrt{\lambda(t)}dW_\lambda^{\mathbb{Q}}(t) \\ &= \left(\left(1-C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_\Lambda\right)\sigma_r\phi_r\right)r(t)+\left(1-C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)\sigma_\lambda\phi_\lambda\right)\lambda(t)\right)dt \\ &- C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_\Lambda\right)\sigma_r\sqrt{r(t)}dW_r(t) - C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)\sigma_\lambda\sqrt{\lambda(t)}dW_\lambda(t).\end{aligned} \tag{40}$$

Since the market price of the stock (ξ_A) is constant then the function $F(t)$ can be written as

$$\begin{aligned}F(t) &= \int_t^T e^{-\left(\frac{1}{\delta}\rho+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi_A^2\right)(s-t)}\mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\frac{\delta-1}{\delta}\left(1+\frac{1}{2}\frac{1}{\delta}\phi_r^2\right)\int_t^s r(u)du}\right]\mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\left(1+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\phi_\lambda^2\right)\int_t^s \lambda(u)du}\right]ds \\ &+\pi^{\frac{1}{\delta}}e^{-\left(\frac{1}{\delta}\rho+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi_A^2\right)(T-t)}\mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\frac{\delta-1}{\delta}\left(1+\frac{1}{2}\frac{1}{\delta}\phi_r^2\right)\int_t^T r(u)du}\right]\mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\left(1+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\phi_\lambda^2\right)\int_t^T \lambda(u)du}\right],\end{aligned} \tag{41}$$

where

$$\mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\frac{\delta-1}{\delta}\left(1+\frac{1}{2}\frac{1}{\delta}\phi_r^2\right)\int_t^s r(u)du}\right] = e^{-\alpha_r^{\mathbb{Q}_\delta}\beta_r^{\mathbb{Q}_\delta}\int_t^s C\left(u;\frac{\delta-1}{\delta}\left(1+\frac{1}{2}\frac{1}{\delta}\phi_r^2\right),\alpha_r^{\mathbb{Q}_\delta},\sigma_r,s\right)du}\times e^{-C\left(t;\frac{\delta-1}{\delta}\left(1+\frac{1}{2}\frac{1}{\delta}\phi_r^2\right),\alpha_r^{\mathbb{Q}_\delta},\sigma_r,s\right)r(t)}, \tag{42}$$

$$\mathbb{E}_t^{\mathbb{Q}_\delta}\left[e^{-\left(1+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\phi_\lambda^2\right)\int_t^s \lambda(u)du}\right] = e^{-\alpha_\lambda^{\mathbb{Q}_\delta}\int_t^s \beta_\lambda^{\mathbb{Q}_\delta}(s)C\left(u;\left(1+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\phi_\lambda^2\right),\alpha_\lambda^{\mathbb{Q}_\delta},\sigma_\lambda,s\right)du}\times e^{-C\left(t;\left(1+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\phi_\lambda^2\right),\alpha_\lambda^{\mathbb{Q}_\delta},\sigma_\lambda,s\right)\lambda(t)}. \tag{43}$$

4.3 The optimal portfolio

In order to explicitly compute the optimal portfolio from (17), we define the following matrices: $\theta_S(t) = \begin{bmatrix} \theta_A(t) & \theta_B(t) & \theta_\Lambda(t) \end{bmatrix}'$, $dW(t) = \begin{bmatrix} dW_A(t) & dW_r(t) & dW_\lambda(t) \end{bmatrix}'$, $z(t) = \begin{bmatrix} r(t) & \lambda(t) \end{bmatrix}'$, $\xi = \begin{bmatrix} \xi_A & \phi_r\sqrt{r(t)} & \phi_\lambda\sqrt{\lambda(t)} \end{bmatrix}'$,

$$I_S = \begin{bmatrix} A(t) & 0 & 0 \\ 0 & B(t) & 0 \\ 0 & 0 & \Lambda(t) \end{bmatrix}, \Omega' = \begin{bmatrix} 0 & \sigma_r\sqrt{r(t)} & 0 \\ 0 & 0 & \sigma_\lambda\sqrt{\lambda(t)} \end{bmatrix}, \tag{44}$$

$$\Sigma' = \begin{bmatrix} \sigma_A & \sigma_{Ar} & 0 \\ 0 & -C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)\sigma_r\sqrt{r(t)} & 0 \\ 0 & -C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_\Lambda\right)\sigma_r\sqrt{r(t)} & -C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)\sigma_\lambda\sqrt{\lambda(t)} \end{bmatrix}. \tag{45}$$

The optimal portfolio is

$$\begin{aligned} \begin{bmatrix} A(t)\theta_A(t) \\ B(t)\theta_B(t) \\ \Lambda(t)\theta_\Lambda(t) \end{bmatrix} &= \frac{R(t)-H(t)}{\delta}\begin{bmatrix} \frac{\xi_A}{\sigma_A} \\ \frac{1}{C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)}\left(\frac{\sigma_{Ar}\xi_A}{\sigma_A\sigma_r\sqrt{r(t)}} - \frac{\phi_r}{\sigma_r} + \frac{\phi_\lambda C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_\Lambda\right)}{\sigma_\lambda C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)}\right) \\ -\frac{\phi_\lambda}{\sigma_\lambda C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)} \end{bmatrix} \\ &+\frac{R(t)-H(t)}{F(t)}\begin{bmatrix} 0 \\ \frac{1}{C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)}\left(-\frac{\partial F(t)}{\partial r(t)} + \frac{C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_\Lambda\right)}{C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)}\frac{\partial F(t)}{\partial\lambda(t)}\right) \\ -\frac{1}{C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)}\frac{\partial F(t)}{\partial\lambda(t)} \end{bmatrix} \\ &+\begin{bmatrix} 0 \\ \frac{1}{C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)}\left(-\frac{\partial H(t)}{\partial r(t)} + \frac{C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_\Lambda\right)}{C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)}\frac{\partial H(t)}{\partial\lambda(t)}\right) \\ -\frac{1}{C\left(0;1,\alpha_\lambda^{\mathbb{Q}},\sigma_\lambda,T_\Lambda\right)}\frac{\partial H(t)}{\partial\lambda(t)} \end{bmatrix}. \end{aligned} \tag{46}$$

The optimal investment in the risk-less asset is obtained as the difference between the total wealth $R(t)$ and the amount invested in the other assets.

4.4 Stochastic process calibration

All the financial market parameters are estimated from three time series (from January 1st, 1962 to January 1st, 2007, thus taking out the turbulence following the sub-prime crisis): (i) the 3-month US Treasury Bill interest rate (on secondary market) for calibrating $r(t)$, (ii) the 10-year US Bond interest rate (on secondary market) for calibrating $B(t)$ (with $T_B = 10$), and (iii) S&P 500 for calibrating $A(t)$.

The parameters of the risk-free interest rate α_r, β_r and σ_r are gathered in Table 1 (they are calibrated by applying the homoscedastic transformation $2\sqrt{r}(t)$ to (24) and then using ordinary least squares). $r(t_0)$ is set to its long term equilibrium value (i.e. $r_0 = \beta_r$).

The average return on 10-year bonds is about 7.1%, thus

$$\mathbb{E}_t\left[d\ln B(t)\right] = 0.071dt, \tag{47}$$

$$\left(1 - C\left(0; 1, \alpha_r^{\mathbb{Q}}, \sigma_r, T_B\right)\sigma_r\phi_r - \frac{1}{2}C\left(0; 1, \alpha_r^{\mathbb{Q}}, \sigma_r, T_B\right)^2\sigma_r^2\right) r(t) = 0.071, \tag{48}$$

which is solved for $\phi_r = -0.5590635$ (recall that $\alpha_r^{\mathbb{Q}}$ is a function of ϕ_r) where instead of $r(t)$ we use the long term equilibrium value β_r (Table 1).

The variance and the mean of the log-return of S&P 500 are

$$\mathbb{V}\left[d\ln A(t)\right] = \left(\sigma_{Ar}^2 + \sigma_A^2\right)dt = 0.0223dt, \tag{49}$$

$$\mathbb{E}\left[d\ln A(t)\right] = \left(r(t) + \sigma_A\xi_A + \sigma_{Ar}\phi_r\sqrt{r(t)} - \frac{1}{2}\left(\sigma_A^2 + \sigma_{Ar}^2\right)\right)dt = 0.06688dt, \tag{50}$$

where $r(t)$ will be substituted with the long term equilibrium level β_r.

The covariance between the S&P 500 log-return and the return on the 10-year bonds is

$$\mathbb{C}\left[d\ln A(t), d\ln B(t)\right] = -C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)\sigma_{Ar}\sigma_r\sqrt{r(t)}dt = -0.0004552dt, \tag{51}$$

where the interest rate will be again substituted with its long term mean β_r. Thus, we have to solve the following system

$$\begin{cases} \sigma_{Ar}^2 + \sigma_A^2 = 0.0223, \\ \beta_r + \sigma_A\xi_A + \sigma_{Ar}\phi_r\sqrt{\beta_r} - \frac{1}{2}\left(\sigma_A^2 + \sigma_{Ar}^2\right) = 0.06688, \\ -C\left(0;1,\alpha_r^{\mathbb{Q}},\sigma_r,T_B\right)\sigma_{Ar}\sigma_r\sqrt{\beta_r} = -0.0004552, \end{cases} \tag{52}$$

which has a positive and a negative solution for σ_A; we take the positive one as shown in Table 1.

Mortality is estimated by fitting the observed survival probabilities for US males born in 1950, who were aged 60 (t_0) at January 1st, 2010. We fix the observation point to January 1st, 1990 and we obtain the observed survival curve from cohort tables available at the Human Mortality Database using 20 data points. For $t \in \{1, 2, ..., 20\}$ we get the observed survival probability $p(0,t)$. We calibrate the intensity process by minimising the mean squared error between fitted and observed values of the survival probability, imposing at the same time the Feller condition. The initial value of the mortality intensity is

$$\lambda(t_0) = \phi_0 + \frac{1}{b}\left(1 + \frac{1}{\alpha_\lambda}\right)e^{\frac{t_0-m}{b}}. \tag{53}$$

The survival probability between t_0 and T is available in semi-closed form:

$$\hat{p}\left(t_0, T\right) = \mathbb{E}_{t_0}^{\mathbb{P}}\left[e^{-\int_{t_0}^{T}\lambda(u)du}\right] = e^{-\alpha_\lambda \int_{t_0}^{T}\beta_\lambda(s)C(s;1,\alpha_\lambda,\sigma_\lambda,T)ds - C(0;1,\alpha_\lambda,\sigma_\lambda,T)\lambda(t_0)}. \tag{54}$$

We thus determine the parameters of the intensity process (gathered in Table 1), α_λ, σ_λ, ϕ_0, b and m, by minimising the cost function

$$\frac{1}{n}\sqrt{\sum_{t=1}^{n}\left(\hat{p}\left(0,t\right) - p\left(0,t\right)\right)^2}. \tag{55}$$

Figure 1 clearly shows that our stochastic mean reverting process fits accurately the observed survival probabilities. The Rooted Mean Squared Error (RMSE), presented in Table 1, is 1.89×10^{-4}. Furthermore, Figure 2 shows the ratios $\left|\frac{p(0,t)-\hat{p}(0,t)}{p(0,t)}\right|$, from which we see that, for most maturities, the mean reverting process fits better than the standard Gompertz model.[4]

4.5 Base Scenario

We assume the existence of a continuously-rolled over longevity bond with maturity $T_L =$ 10 years whose underlying is the mortality intensity of the cohort of US males born in 1950. Thus, they are individuals aged 60 in 2010. In our base scenario, we set the risk premium for mortality $\phi_\lambda = 0$. The initial wealth of the consumer is set to $R_0 = 100$. Following the estimates in Gourinchas and Parker (2002) and Cocco et al. (2005), that report, for a 60 year old US citizen, an annual wage of about 30000 dollars and a financial wealth of about 200000 dollars, we set the annual (constant) wage to $w = 15$.

[4]Unreported results show that the stochastic mean reverting square root process fits the survival curve better than a non-mean reverting square root process, such as the Feller process in Luciano and Vigna (2008). The better RMSE in our case is due to the presence of three additional parameters.

Our representative consumer wants to reach a minimum amount of wealth (R_m) at the retirement age, $T = 65$, which will be annuitised to finance post-retirement consumption. We set the annual instalment of the annuity as a fraction (the replacement rate) of the wage. In our base-case, we assume that the agent wants to reach a total replacement rate of 68% as in Cocco and Gomes (2012). He finances 25% of this pension earning using personal private wealth, consistently with a 51% social security replacement rate. Hence, we set $p = 0.25w\kappa$, where κ is the average total replacement rate at retirement. We set the subsistence consumption in the pre-retirement phase according to the same rule: $c_m = \kappa w$. We set the subjective discount factor $\rho = 0.04$ and the risk aversion parameter $\delta = 2.5$, following the most common choices in the life-cycle literature (Horneff et al. (2015),Gourinchas and Parker (2002)). The parameter π is set to 1 to represent the case of an agent who is interested in both the utility of intertemporal consumption and the utility of final wealth with the same "intensity".

Figure 3 collects 100 paths with optimal strategies computed at monthly intervals. At the beginning of the investment horizon, the individual invests on average: about 28% in the stock, 19% in the bond, and 69% in the longevity-linked asset. He partially funds this investment shorting the risk-free asset, taking a -16% position. While retirement approaches, he/she progressively reduces the investment in all the risky assets, be they equity or bonds, to increase the probability to reach his/her final wealth target, and increases the share invested in the risk-free asset, which reaches around 50% close to the retirement age. For the same reason, he/she reduces consumption, which drops from an initial 30 to an average 24 at T. Even if the magnitude of this effect varies across simulations, such a drop persists and is consistent with the empirical evidence on consumption patterns in the pre-retirement phase.

While retirement age approaches, the discount factors are less and less affected by the changes in the force or mortality, and, accordingly, there is less and less need to hedge against such a risk. This is the intuition behind the fall in the share of the longevity-linked asset in the optimal portfolio. Furthermore, since R_m is affected by the longevity risk too, the share of the longevity-linked asset does not fall below 40% (which is the lowest level reached just at retirement).

The initial allocation in equity may seem smaller than that observed in the classical life-cycle literature. Actually, it is even larger than that obtained in papers which introduce the opportunity to invest in longevity-linked securities (Cocco and Gomes, 2012,Horneff et al., 2015). Furthermore, the small stock share (about 10%) at 65, must be coupled with the more persistent investment in the longevity-linked security, which is indeed another risky asset.

Recall that, in this base scenario, we set $\phi_\lambda = 0$. Such a value for the market price of risk erases the speculative component of the longevity-linked asset demand. Thus, the demand for this asset is entirely due to the hedging motive against the fluctuations of the discount factors.

Figure 3, allows to appreciate also the variability in sample paths. Investment profiles are very much stable across different paths, while the consumption profile is more volatile. This result is consistent with the findings of some works that show how consumption tends to absorb most of the uncertainty in models (Bernasconi et al., 2015, for instance, demonstrate that even uncertainty on fiscal parameters affects consumption but does not alter the decision to evade). Thus, after a financial shock, an agent prefers to vary his/her consumption rather than his/her portfolio allocation.

4.6 Sensitivity Analysis: Longevity Risk Premium

Given the absence of a liquid market for longevity securities at present, estimating a reasonable value for the risk premium is very difficult. The risk premium should be negative, as longevity bonds negatively react to changes in $\lambda(t)$, for which investors would require a compensation (a lower price for the zero-coupon bond).

Obviously, any value for the risk premium different from our baseline value of 0 increases the investment in the longevity asset. Given the small development of the market for longevity-linked securities, it is difficult to find reliable estimates of the longevity risk premium in the literature. Bauer et al. (2010) estimate the Sharpe Ratio, based on a monthly time series of UK pension annuity quotes. To provide an alternative scenario, we set the Sharpe Ratio to the more conservative level estimated by Bauer et al. (2010), i.e.-0.07, which is lower than the one suggested as reasonable by Loeys et al. (2007). The associated $\phi_\lambda = -0.12$ is obtained under the simplifying assumption of a constant force of mortality, equal to the expected average intensity for the individuals of the cohort until age ω. The simulations show a pattern which is similar to the base scenario, but the investment choices are tilted towards the longevity-linked asset, whose "speculative" attractiveness adds to the hedging motive. Initial investment reaches more than 253%, and the additional allocation in the longevity asset is financed by short selling the bond. This result is due to the independence between the mortality intensity and the short rate. While retirement approaches, there is less and less need to hedge against the stochastic change in the interest rate, while the need to hedge against the longevity risk is still relevant. Indeed, the initial wealth invested in the longevity-linked asset and the bond together (88%) is the same that is obtained in the base case.

4.7 Sensitivity Analysis: Individual characteristics

We present here an analysis of the changes in the consumption/portfolio choices due to: (i) the sex of the consumer/investor, (ii) the time horizon (T), (iii) the risk aversion (δ), (iv) the subsistence final wealth (R_m), (v) the subsistence consumption (c_m) and (vi) the replacement rate (κ).

Sex

We calibrate the stochastic mortality model to 60-year old US females, using the same procedure we described in Section 4.4, and we consider the longevity asset to be written on this mortality intensity. Parameters are reported in Table 2. Women should invest a higher fraction of their wealth in the longevity asset (due to their longer lifetime expectancy). The optimal share invested in this asset is initially 90% and drops more slowly than for men. This is due to the higher predictability of mortality rates, because females' mortality intensity shows a higher speed of mean reversion to the Gompertz law. While investment in stocks is unaltered, the share of the longevity-linked asset is higher and the share of the bond is lower than those of the base scenario. This result is due to the higher relative importance of the females' longevity risk with respect to the interest rate risk (which is the same for both males and females).

Time horizon

Analysis of the consumption/investment profile of a younger individual, who is 55 at time t_0 (parameters are reported in Table 2), with same fixed retirement age at 65, reveals that his/her initial demand for the longevity asset is around 80% and it is decreasing over the whole period, from age 55 to 65. Consumption/investment decisions in the second part of the time period (when the agent is aged 60 to 65) are in line with the base scenario. Investment in equity decreases over time, while the share of the risk-free asset increases, from -50% to around 50% at retirement, as in the base scenario. Investments in both bonds and stocks are decreasing, and drop from initial values that are higher than the base case (since the individual is younger at the beginning) to final values that are in line with the base case. In the first 3-year period, instead, consumption stays on average almost constant. Then, it starts decreasing markedly around age 58 and then behaves as in the base case, in line with the hump-shaped profiles highlighted by the life-cycle literature

Risk aversion

A smaller risk aversion ($\delta = 1.5$) obviously increases investment in the stock. The stock share evolves from an initial 48% on average down to 16%. Also, investment in the bond is increased over the whole horizon, with initial share at around 47%. At retirement the bond share is still 25%.

Investment in the longevity bond is in line with the base case and increases only slightly (around 0.5%). Finally, the additional investment in both the stock and the bond is financed by shorting the risk-free asset.

On the contrary, a higher risk aversion ($\delta = 3.5$) decreases investment in the stock (which goes down from 20% at the beginning to about 8% at retirement) but affects very slightly the investment in the longevity-linked asset, which however increases relative to the base case towards the end of the investment period.

Finally, the investment in the bond is initially lower than in the base case (7%), it decreases, but slowly, to reach about 5% close to the age 64, and then, it increases until retirement.

Subsistence final wealth

When investor has no interest in setting a minimum level of final wealth (i.e. $R_m = 0$), the ratio between consumption and wealth progressively increases over time until retirement. Minimum consumption (that we set to $c_m = \kappa w$) is always well beyond actual consumption. The investment profile is riskier, as the individual invests more in the stock (from an initial 36% to a final 30%, with values always above 30%) and progressively increases the bond share, from an initial 20% to 55% at retirement.

With $R_m = 0$, there is no need to hedge against the longevity risk at retirement and, accordingly, the share of the longevity-linked asset is smaller and the optimal consumption is higher than in the base scenario. Indeed, the longevity asset share, that accounts initially for 71% of wealth, decreases exponentially, down to 0% at retirement.

Subsistence consumption

Without any minimum consumption level ($c_m = 0$,) the agents adopts a less conservative behaviour. The initial investment in stock (41%) and bond (39%) are both higher than in the base case, while the share in the risk-less asset (-52%) is more negative, to finance such additional risky investments. Investment in the longevity asset slightly increases at the beginning (72%) and then decreases slightly, but still accounts for 86.5% of the initial wealth. Over time, the optimal portfolio qualitatively behaves as in the base scenario.

Replacement rate

We explore alternative values for the replacement rate as proposed in the literature (Cocco and Gomes, 2012,Campanale et al., 2015). When the replacement rate is higher (80%), the agent's portfolio is more conservative because his/her final minimum wealth (R_m) is higher. Accordingly, he/she invests less in the risky assets: 24% in stocks, 15% in bonds and 68% in the longevity-linked asset. The share of this last asset is the less affected by the change in the replacement rate. Consistently, a lower replacement rate increases the overall portfolio riskiness, raising investments in both the stock (36%) and the bond (28%). Finally, consumption and investment behaviours in time do not vary relative to the base case.

5 Concluding comments

This paper derives the optimal consumption and investment profiles of an individual prior to a fixed retirement age in the presence of longevity risk. The market is complete and a derivative on the mortality intensity of the individual is listed. Investment in such an asset is driven by three motivations: (i) speculation, if the risk premium is attractive, (ii) diversification, if the asset is negatively correlated with other financial assets, and (iii) hedging, as changes in mortality affect the discount factors of the individual. When focusing on this latter factor, we calibrate our model to US real data, and find that the demand for longevity asset might be relevant. The result is robust across different individual types. We explore sensitivity with respect to mortality risk premium, finding that the demand for the longevity asset is positive even when the risk premium is comparable to the one required for stocks. Our results seem to suggest that, alongside reinsurance of longevity risk, which is currently growing in volume and number of transactions, there is potential room for involving individual investors in the longevity market, because of the high optimal demand for longevity-linked securities. We acknowledge that neglecting transaction costs and liquidity risks may substantially affect our analysis.

Here, we have numerically computed the relevance of a zero-coupon longevity bond, but the optimal design of longevity securities needs to be investigated more deeply (as suggested by Cocco and Gomes (2012)). Further research is needed as well in order to explore the impact of basis risk, i.e. the possibility that the longevity asset is imperfectly correlated with the mortality intensity of the individual.

Appendix

Proof of Proposition 1

We solve problem (14) following the martingale approach. Its Lagrangian function under constraint (15) is:

$$\begin{aligned}\mathcal{L} &= \mathbb{E}_{t_0}\left[\int_{t_0}^{T}\frac{(c(s)-c_m)^{1-\delta}}{1-\delta}e^{-\int_{t_0}^{s}\rho(u)+\lambda(u)du}ds+\pi\frac{(R(T)-R_m(T,z))^{1-\delta}}{1-\delta}e^{-\int_{t_0}^{T}\rho(u)+\lambda(u)du}\right]\\ &\quad+\kappa\left(R(t_0)-\mathbb{E}_{t_0}\left[\int_{t_0}^{T}\left(c(s)-w(s)+\sigma_L(s)'\xi(s)\right)m(t_0,s)\,e^{-\int_{t_0}^{s}r(u)+\lambda(u)du}ds\right.\right.\\ &\quad\left.\left.+\;R(T)\,m(t_0,T)\,e^{-\int_{t_0}^{T}r(u)+\lambda(u)du}\right]\right), \qquad (56)\end{aligned}$$

where the functional dependencies on z have been omitted for the sake of simplicity, κ is the (constant) Lagrangian multiplier, and all the expected values have been written under the historical probability. The first order condition on consumption

$$\frac{\partial\mathcal{L}}{\partial c(s)}=\mathbb{E}_{t_0}\left[\int_{t_0}^{T}\left((c(s)-c_m)^{-\delta}e^{-\int_{t_0}^{s}\rho(u)+\lambda(u)du}-\kappa m(t_0,s)\,e^{-\int_{t_0}^{s}r(u)+\lambda(u)du}\right)ds\right]=0, \qquad (57)$$

must hold for any state of the world and, accordingly, the optimal consumption at any time s is

$$c^*(s)=c_m+\left(\kappa m(t_0,s)\,e^{-\int_{t_0}^{s}r(u)du}e^{\int_{t_0}^{s}\rho(u)du}\right)^{-\frac{1}{\delta}}. \qquad (58)$$

The same approach on final wealth gives the first order condition

$$\frac{\partial \mathcal{L}}{\partial R(T)} = \mathbb{E}_{t_0}\left[\pi \left(R(T) - R_m(T,z)\right)^{-\delta} e^{-\int_{t_0}^{T}\rho(u)+\lambda(u)du} - \kappa m(t_0,T)\, e^{-\int_{t_0}^{T} r(u)+\lambda(u)du}\right] = 0, \tag{59}$$

and the optimal final wealth

$$R^*(T) = R_m(T,z) + \left(\frac{\kappa}{\pi} m(t_0,T)\, e^{-\int_{t_0}^{T} r(u)du} e^{\int_{t_0}^{T}\rho(u)du}\right)^{-\frac{1}{\delta}}. \tag{60}$$

When the constraint is rewritten at time t (instead of t_0) as follows

$$\begin{aligned} R(t) &= \mathbb{E}_t\left[\int_t^T \left(c(s) - w(s) + \sigma_L(s)'\xi(s)\right) m(t,s)\, e^{-\int_t^s r(u)+\lambda(u)du} ds \right. \\ &+ \left. R(T)\, m(t,T)\, e^{-\int_t^T r(u)+\lambda(u)du}\right], \end{aligned} \tag{61}$$

and the optimal consumption and final wealth are both substituted in it, we obtain the following expression:

$$R(t) = \left(\kappa m(t_0,t)\, \frac{e^{-\int_{t_0}^{t} r(u)du}}{e^{-\int_{t_0}^{t}\rho(u)du}}\right)^{-\frac{1}{\delta}} F(t,z) + H(t,z) \tag{62}$$

where

$$H(t,z) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \left(c_m - w(s) + \sigma_L(s)'\xi(s)\right) e^{-\int_t^s r(u)+\lambda(u)du} ds + R_m(T,z)\, e^{-\int_t^T r(u)+\lambda(u)du}\right], \tag{63}$$

$$\begin{aligned} F(t,z) &= \mathbb{E}_t\left[\int_t^T m(t,s)^{1-\frac{1}{\delta}}\, e^{-\int_t^s \left(\frac{\delta-1}{\delta} r(u) + \frac{1}{\delta}\rho(u) + \lambda(u)\right)du} ds \right. \\ &\quad \left. + \pi^{\frac{1}{\delta}} m(t,T)^{1-\frac{1}{\delta}}\, e^{-\int_t^T \left(\frac{\delta-1}{\delta} r(u) + \frac{1}{\delta}\rho(u) + \lambda(u)\right)du}\right]. \end{aligned} \tag{64}$$

While $m(t,s)^{1-\frac{1}{\delta}}$ is not a martingale, $m(t,s)^{1-\frac{1}{\delta}} e^{\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\int_{t_0}^{t}\xi'\xi ds}$ is:

$$\left(m(t,s)^{1-\frac{1}{\delta}} e^{\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\int_{t_0}^{t}\xi'\xi ds}\right)^{-1} d\left(m(t,s)^{1-\frac{1}{\delta}} e^{\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\int_{t_0}^{t}\xi'\xi ds}\right) = -\frac{\delta-1}{\delta}\xi(s)\,dW(s). \quad (65)$$

Accordingly, we define the new probability

$$dW(t)^{\mathbb{Q}_\delta} = \frac{\delta-1}{\delta}\xi(t)\,dt + dW(t), \quad (66)$$

and write

$$F(t,z) = \mathbb{E}_t^{\mathbb{Q}_\delta}\left[\begin{array}{l} \int_t^T e^{-\int_t^s\left(\frac{\delta-1}{\delta}r(u,z)+\frac{1}{\delta}\rho(u,z)+\lambda(u,z)+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi(u,z)'\xi(u,z)\right)du}ds \\ +\pi^{\frac{1}{\delta}}e^{-\int_t^T\left(\frac{\delta-1}{\delta}r(u,z)+\frac{1}{\delta}\rho(u,z)+\lambda(u,z)+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi(u,z)'\xi(u,z)\right)du} \end{array}\right]. \quad (67)$$

The differential of (62), through Ito's lemma, is (the drift term is neglected since it is immaterial to replication):

$$\begin{aligned} dR(t) &= (...)\,dt + \frac{1}{\delta}\left(\kappa m(t_0,t)\,e^{-\int_{t_0}^t r(u)du}e^{\int_{t_0}^t \rho(u)du}\right)^{-\frac{1}{\delta}} F(t,z)\,\xi(t,z)'\,dW(t) \\ &\quad + \left(\kappa m(t_0,t)\,e^{-\int_{t_0}^t r(u)du}e^{\int_{t_0}^t \rho(u)du}\right)^{-\frac{1}{\delta}} F_z(t,z)'\,\Omega(t,z)'\,dW(t) \\ &\quad + H_z(t,z)'\,\Omega(t,z)'\,dW(t), \end{aligned} \quad (68)$$

where the subscripts on $F(t,z)$ and $H(t,z)$ indicate partial derivatives. Once the following relationship

$$\frac{R(t)-H(t,z)}{F(t,z)} = \left(\kappa m(t_0,t)\,e^{-\int_{t_0}^t r(u,z)du}e^{\int_{t_0}^t \rho(u,z)du}\right)^{-\frac{1}{\delta}}, \quad (69)$$

is suitably taken into account, the differential equation becomes

$$dR(t) = (...)\,dt + \tag{70}$$
$$+\left(\frac{R(t)-H(t,z)}{\delta}\xi(t,z)' + \frac{R(t)-H(t,z)}{F(t,z)}F_z(t,z)'\,\Omega(t,z)' + H_z(t,z)'\,\Omega(t,z)'\right)dW(t).$$

When $\Sigma(t,z)\,I_S\theta_S(t) + \sigma_L(t,z)$ is set equal to the diffusion term of (70), the optimal portfolio in Proposition 1 is found.

Proof of Proposition 2

By using Itô's lemma, we know that the function

$$Y(t,x(t)) = \mathbb{E}_t\left[e^{-\chi\int_t^T x(s)ds}\right],$$

must satisfy

$$\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial x}\alpha(\beta(t)-x(t)) + \frac{1}{2}\frac{\partial^2 Y}{\partial x^2}\sigma^2 y = \chi x(t)\,Y,$$

with the final condition

$$Y(T,x(T)) = 1,$$

Now, we use the guess function

$$Y(t,x) = e^{-A(t)-C(t)x},$$

where the function A and C must be computed in order to solve the previous differential equation. The boundary condition translates into the following conditions:

$$A(T) = 0,$$
$$C(T) = 0.$$

Once the partial derivatives of Y are substituted into the differential equation we obtain[5]

$$0 = \frac{1}{2}C^2\sigma^2 x - \chi x - \frac{\partial A}{\partial t} - \frac{\partial C}{\partial t}x - C\alpha\left(\beta(t) - x\right),$$

which is an ordinary differential equation in A and C. Since this equation must hold for any value of x then we can split it into two ordinary differential equations as follows

$$\begin{cases} 0 = \frac{\partial A}{\partial t} + C\alpha\beta(t), \\ 0 = \frac{\partial C}{\partial t} + \chi - \alpha C - \frac{1}{2}C^2\sigma^2. \end{cases} \tag{71}$$

We immediately see that the value of function $C(t)$ can be computed from the second equation. With the suitable boundary condition the only solution of the differential equation for $C(t)$ is given by

$$C(t) = 2\chi\frac{1 - e^{-(T-t)\sqrt{\alpha^2+2\sigma^2\chi}}}{\sqrt{\alpha^2 + 2\sigma^2\chi} + \alpha + \left(\sqrt{\alpha^2 + 2\sigma^2\chi} - \alpha\right)e^{-(T-t)\sqrt{\alpha^2+2\sigma^2\chi}}}.$$

The values of all the other function can be written in terms of $C(t)$:

$$\frac{\partial A(t)}{\partial t} = -C(t)\,\alpha\beta(t),$$

[5]For the sake of simplicity, we have omitted the functional dependencies (except for the function $\beta(t)$).

with the boundary condition $A(T) = 0$. The only solution of this equation is

$$A(t) = \int_t^T C(s)\,\alpha\beta(s)\,ds.$$

Finally, the result of the proposition follows.

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Table 1: Parameters for the base scenario, calibrated on the S&P 500, 3-month Treasury Bills, and 10-year Bonds time series (between January 1st 1962 and January 1st 2007).

Interest rate/Bond	Stock	Wealth/Preferences	Mortality/Longevity
$\alpha_r = 0.0904668$	$\sigma_A = 0.14926$	$R_0 = 100$	$\alpha_\lambda = 0.561$
$\beta_r = 0.0621328 = r_0$	$\sigma_{Ar} = 0.0046306$	$w = 10$	$\sigma_\lambda = 0.0352$
$\sigma_r = 0.0543625$	$\xi_A = 0.1108301$	$T = 65$	$\phi_0 = 0.0009944$
$\phi_r = -0.5590635$		$\rho = 0.01$	$b = 12.9374$
$T_B = 10$		$R_m = 100$	$m = 86.4515$
		$c_m = 0$	$t_0 = 60$
		$\delta = 2.5$	$T_L = 10$
		$\pi = 1$	RMSE=0.000189

Table 2: Calibrated parameters of the mortality processes of individuals outside the base case scenario.

Parameter	60-yr old females	55-year old males
α_λ	5.6863	0.5659
σ_λ	0.0277	0.0243
ϕ_0	$6.12 \cdot 10^{-17}$	0.0002
b	13.0579	17.0836
m	91.9838	90.9065

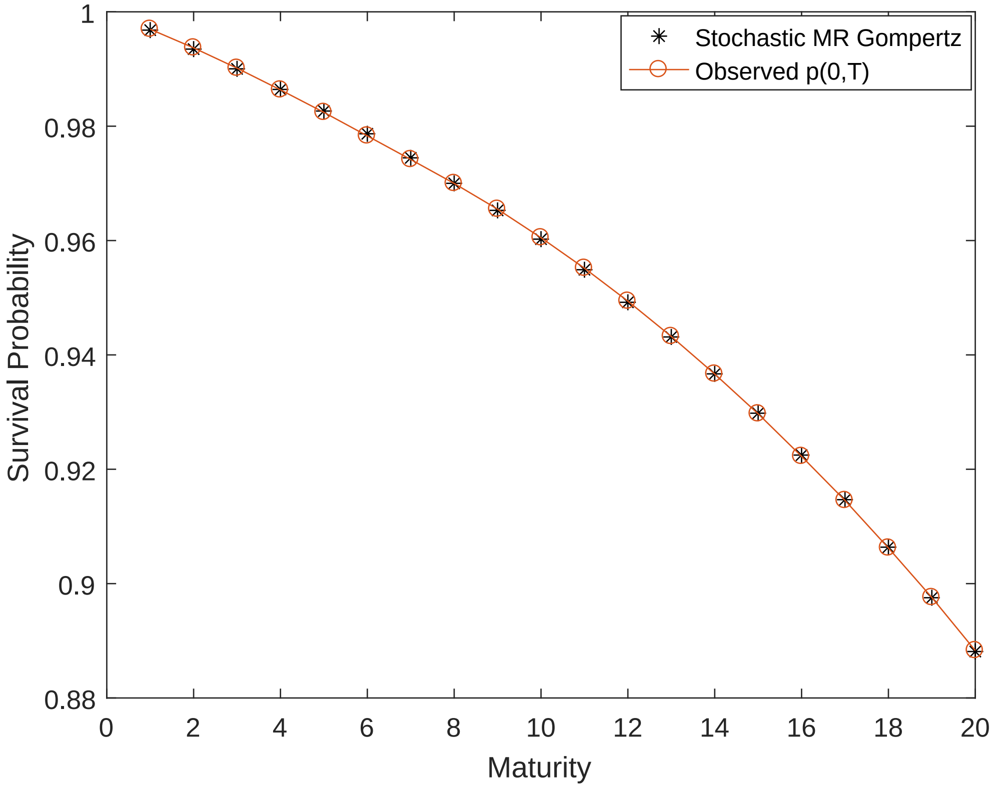

Figure 1: Observed survival probabilities (circles), $p(0,T)$, used to calibrate our stochastic mortality model to 60-year old US individuals, versus fitted $\hat{p}(0,T)$ (asterisks), using our stochastic MR Gompertz model. Observed $p(0,T)$ are the observed survival probabilities for the individuals belonging to the cohort of people aged 60 at t_0 = January 1st, 2010, conditional on survivorship at age 40. The x-axis reports the maturities of the survival probabilities, in years.

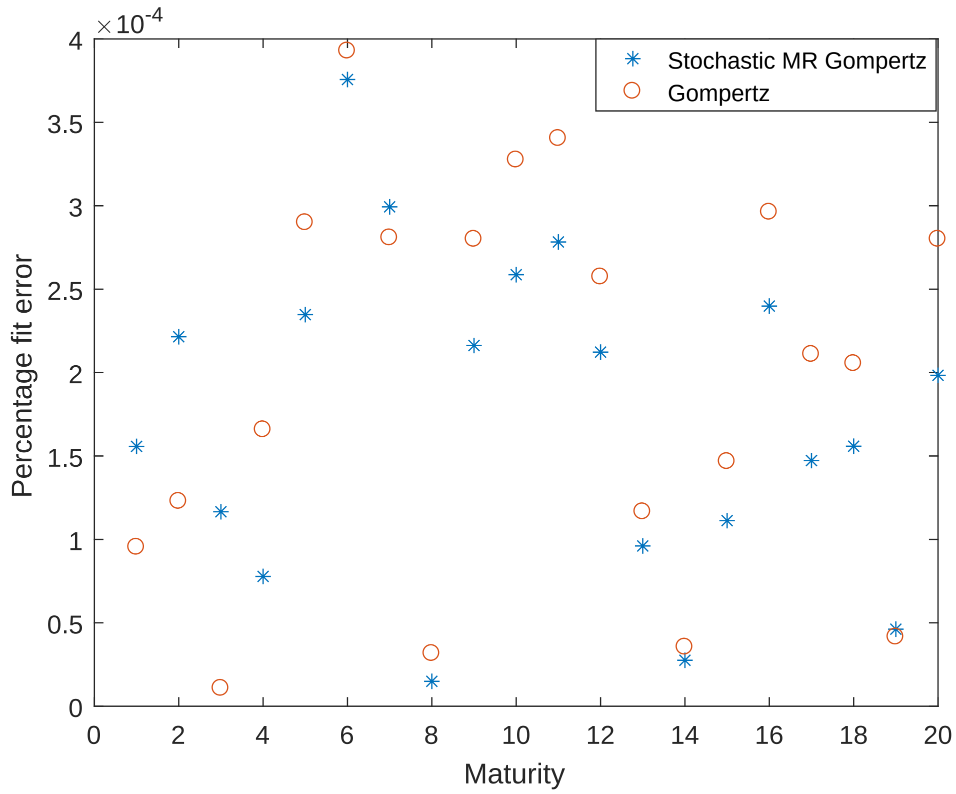

Figure 2: Relative fit error of the Stochastic MR Gompertz model vs. the deterministic Gompertz law by maturity. For each survival probability used in the calibration, the figure reports the percentage (absolute) error of the deterministic Gompertz vs. the Stochastic MR Gompertz model.

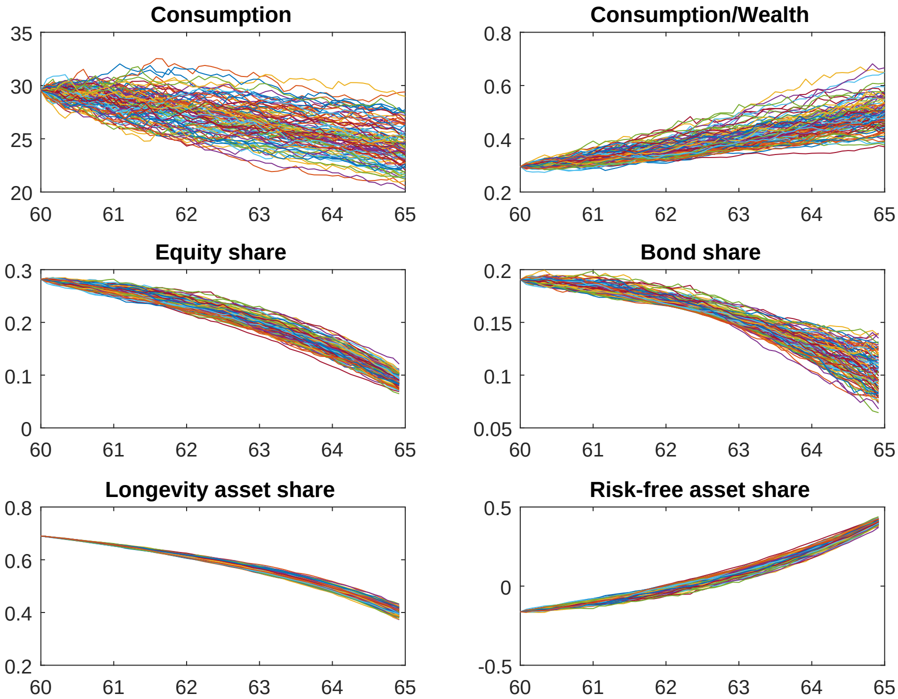

Figure 3: Sample path of optimal portfolio with the base-case values gathered in Table 1.