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A Morse-Smale index theorem for indefinite elliptic systems and bifurcation

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Abstract

The aim of this paper is to publish at least something, which is, however, not that bad...

1 Introduction

The Morse index theorem is a famous result in differential geometry, relating the Morse index of a geodesic on a Riemannian manifold to its number of conjugate points (cf. [**?**]). It was proved by Marston Morse in the forties and since then has been generalized into various directions. After introducing coordinates, the Morse-index theorem turns out to be a result about Dirichlet boundary value problems for strongly elliptic systems of ordinary differential equations of second order. More precisely, the equations are of the form

$$\begin{aligned} -u''(x) + S(x)u(x) = 0, \quad x \in [0,1], \\ u(0) = u(1) = 0 \end{aligned} \tag{1}$$

where $S : [0,1] \to M(n;\mathbb{R})$ is a smooth path of symmetric matrices for some $n \in \mathbb{N}$. If we now set the *Morse index* $\mu_{Morse}(S)$ to be the number of negative eigenvalues of the boundary value problem (1) counted with multiplicities and

$$m(t) = \dim\{u : [0,t] \to \mathbb{R}^n : -u''(x) + S(x)u(x) = 0, u(0) = u(t) = 0\}, \tag{2}$$

then the Morse index theorem states that

$$\mu_{Morse}(S) = \sum_{t\in[0,1]} m(t). \tag{3}$$

Instants $t \in I$ such that $m(t) > 0$ are called *conjugate* and (3) implies in particular that they are finite in number.

. Smale showed in [**?**] that an equality like (3) continues to be true in the more general case of general strongly elliptic second order partial differential equations as follows: let M be a smooth compact manifold with non empty boundary ∂M, E a Riemannian vector bundle over M and $\varphi_t : M \to M$ a continuous curve of smooth embeddings such that $\varphi_0 = id$ and $M_s \subset M_t$ for $s > t$. Let $L : \Gamma_0(E) \to \Gamma(E)$ be a strongly elliptic selfadjoint differential operator of order $2k$, where $\Gamma_0(E)$ denotes the subspace of those elements of $\Gamma(E)$ that vanish on the boundary of M.

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Note that L has a finite Morse index by the strong ellipticity assumption. Moreover, we obtain differential operators $L_t : \Gamma_0(E_{M_t}) \to \Gamma(E_{M_t})$ by restricting L to $E\mid_{M_t}$ and we denote

$$m(t) = \dim\{u \in \Gamma_0(E_t) : L_t u = 0\}.$$

Now Smale's theorem states that under a certain "unique continuation property" of the operators L, the corresponding equality (3) still holds. Later Uhlenbeck [?] and Swanson [?] gave alternative proofs of Smale's result using abstract Hilbert space theory and intersection theory in symplectic Hilbert spaces respectively. However, all the constructions mentiones so far use strongly the assumption of Dirichlet boundary conditions, which permit an extension of functions in the Sobolev spaces $H^k(M_t)$, $t \in (0,1)$ to $H^k(M)$ by 0.
A completely different variation of Morse's classical result is inspired by physical applications and concerns the corresponding statement for geodesics in Lorentzian manifolds, the models of space-time in general relativity theory. Complete results in case of so called light-like and time-like geodesics can be found for example in the book [?]. Fifteen years ago, Helfer studied the same question for the remaining space-like geodesics and, moreover, geodesics in arbitrary semi-Riemannian manifolds [?]. His results yield that it is not even possible to make sense of the values involved in the classical Morse-index theorem (3) in this generality since the ordinary Morse index is infinite and conjugate points may accumulate. Starting with Helfers work, considerable amount of research has been done in order to extend the Morse-index theorem to geodesics in arbitrary semi-Riemannian manifolds (cf. PICCIONE +REFERENCES). A new approach to this problem was proposed by Musso, Pejsachowicz and the first author in [?], where topological tools like spectral flow and the winding number were used in order to give a meaning to the Morse index and the conjugate index in the semi-Riemannian setting. Subsequently, the second author gave an alternative proof of their version of the Morse theorem using the Atiyah-Jänich bundle and K-theoretic methods [?]. This makes the Morse-index theorem reminiscent of the Atiyah-Singer index theorem for selfadjoint elliptic operators.
Recently, Smale's theorem was extended for scalar equations to more general boundary conditions under the additional assumption that the manifold M is a star-shaped domain Ω with respect to the origin in some Euclidean space $\mathbb{R}^N$ and the shrinking φ is the canonical contraction to $0 \in \mathbb{R}^N$. Deng and Jones considered in [?] bounded perturbations of the Laplace equation for boundary value problems which are either similar to the Dirichlet- or the Neumann problem. HOWEVER; THERE IS A GAP! The first author extended their results in collaboration with Dalbono for the classical Dirichlet and Neumann problem to the case of general scalar second order elliptic partial differential equations [DP12]. The novelty in these investigations is that now, except for the case of the classical Dirichlet condition as treated by Smale in [?], conjugate points can accumulate as in the case of semi-Riemannian geodesics and hence the right hand side in [?] does no longer makes sense, while the left hand side is still well defined. Deng and Jones overcome this problem in [?] by using a Maslov index for curves of Lagrangian subspaces in a symplectic Hilbert space consisting of functions on the boundary of Ω.
Note that compared to (1), the equations considered in [?] and [DP12] correspond for Dirichlet boundary conditions to the case of geodesics in one-dimensional manifolds. The aim of this work is to extend their ideas to systems of partial differential equations which are formally of the type (1). More precisely, let $\Omega \subset \mathbb{R}^N$ be a bounded and smooth domain which is star-shaped with respect to the origin $0 \in \mathbb{R}^N$ for some $N \in \mathbb{N}$. We denote henceforth the boundary of Ω by Γ. Let us consider for some $k \in \mathbb{N}$ homogenous systems of second-order differential equations

$$Lu(x) + S(x)u(x) = 0, \tag{4}$$

where $u : \Omega \to \mathbb{R}^k$,

$$L = -J \begin{pmatrix} \Delta & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \Delta \end{pmatrix},$$

J is a diagonal matrix having entries ± 1, and $S : \overline{\Omega} \to \mathcal{S}(k;\mathbb{R})$ is a smooth map of symmetric matrices.
We consider the classical Dirichlet and Neumann problem for the equation (4). Accordingly, we study boundary value problems of the form

$$\begin{aligned} &Lu(x) + S(x)u(x) = 0, \quad x \in \Omega \\ &P^l(u\,|_\Gamma, \partial_n u\,|_\Gamma) = 0, \end{aligned} \tag{5}$$

where

$$P^l u = (u_1, \ldots, u_l, \partial_n u_{l+1}, \ldots, \partial_n u_k)$$

and $\partial_n u_i = \langle n, \nabla u_i \rangle$, where n denotes the outward normal to the boundary $\partial\Omega$.
Note that the boundary value problems (5) are selfadjoint. Our aim is to construct three indices which are obtained by shrinking the domain Ω into a point in the canonical way.
The paper is structured as follows: In the second section we consider the weak formulations of the equations (5) and we introduce the spectral index which is defined by means of the spectral flow. In the third section we define the Maslov index for (5), where we follow the ideas of Deng and Jones from [?]. In the fourth section we state and prove our main theorem on the equality of these indices. In the fifth section we discuss briefly bifurcation for semi-linear systems under shrinking of the star-shaped domain in connection with the non-vanishing of our indices for the linearised equations. Finally, there are two appendices. In the first one we recall some facts about the Fredholm Lagrangian Grassmannian of symplectic Hilbert spaces and the Maslov index. In the second one we recall the definition of the spectral flow for bounded selfadjoint Fredholm operators and a construction of Robbin and Salamon from [RS95] which we strongly use in our proofs.

2 Weak formulation and the generalised Morse index

Let us consider a boundary value problem of the form (5). The bilinear form in the weak formulation of (5) is given by

$$B(u,v) = \int_\Omega \sum_{i=1}^N \langle JD_i u(x), D_i u(x)\rangle\,dx + \int_\Omega \langle S(x)u(x), v(x)\rangle\,dx, \quad u,v \in E, \tag{6}$$

where

$$E = H^1_0(\Omega) \oplus \cdots \oplus H^1_0(\Omega) \oplus H^1(\Omega) \oplus H^1(\Omega) \subset H^1(\Omega, \mathbb{R}^k).$$

By assumption the domain Ω is star-shaped with respect to $0 \in \mathbb{R}^N$. In what follows we denote by $\Omega_t := \{tx \in \Omega : x \in \Omega\}$, $t \in (0,1]$, the shrinked domain and we consider the restriction of the boundary value problem (5) to Ω_t, that is,

$$\begin{aligned} &Lu(x) + S(x)u(x) = 0, \quad x \in \Omega_t \\ &P^l(u, \partial_n u)(x) = 0, \quad x \in \partial\Omega_t, \end{aligned} \tag{7}$$

Definition 2.1 *We call $t \in (0,1]$ a conjugate instant for the equation* (5) *if the boundary value problem* (7) *admits a non trivial solution.*

Remark 2.2 *It was already shown by Smale in [Sma65] that conjugate points are isolated for the Dirichlet problem and our theorem 3 below will give a new proof of this fact (cf. also OUR WORK IN PROGRESS). In contrast, it is easily seen that conjugate instants can accumulate for the Neumann problem. Indeed, if there exists $t_0 \in (0,1]$ such that $S\mid_{\Omega_{t_0}} \equiv 0$, then every instant in $(0,t_0]$ is conjugate.*

It is easily seen by rescaling that (7) is equivalent to

$$\begin{aligned} &\frac{1}{t^2} Lu(x) + S(tx)u(x) = 0, \quad x \in \Omega \\ &P(u, \partial_n u(x)) = 0, \quad x \in \partial\Omega \end{aligned} \tag{8}$$

and as in (6) the bilinear forms of the weak formulations are given by

$$B_t(u,v) = \frac{1}{t^2}\int_\Omega \sum_{i=1}^N \langle J D_i u(x), D_i v(x)\rangle\, dx + \int_\Omega \langle S(tx)u(x), v(x)\rangle\, dx, \quad u,v \in E \tag{9}$$

Henceforth we will denote L_t the Riesz representation of B_t, that is, the unique bounded selfadjoint operator L_t on E, such that

$$B_t(u,v) = \langle L_t u, v\rangle_E, \quad u,v \in E.$$

Remark 2.3 *explain why we do not just multiply the equation by t^2.*

Lemma 2.4 *L_t, $t \in (0,1]$, define Fredholm operators on E.*

Proof. Has to be written but is clear....relatively...

Before we can define the spectral index, we need to introduce a technical assumption.

(A) There exist $\nu, t_0 > 0$ such that $\sigma(L_t) \cap (-\nu, 0) = \emptyset$ for all $0 < t < t_0$.

In what follows we denote by C the Riesz representation of the L^2 scalar product on E, i.e.

$$\langle u, v\rangle_{L^2(\Omega,\mathbb{R}^k)} = \langle Cu, v\rangle_E, \quad u,v \in E,$$

and we consider for $\delta \in \mathbb{R}$

$$L_t^\delta(u) = L_t(u) + \delta C u.$$

Note that L_t^δ is a selfadjoint Fredholm operator.

Lemma 2.5 *Let the boundary value problem* (5) *admit only the trivial solution and assume that (A) holds. Then there exists $\delta_0 > 0$ such that $L_1^\delta = L^\delta$ and $L_{t_0}^\delta$ are invertible for all $\delta \in (0,\delta_0)$ and*

$$\operatorname{sf}(L^\delta, [t_0, 1]) = \operatorname{sf}(L^{\tilde\delta}, [t_1, 1])$$

for all $\delta, \tilde\delta \in (0, \delta_0)$ and $0 < t_1 < t_0$.

Proof. Since (5) has no non-trivial solutions, we know that there exists $\delta_0 > 0$ such that L^δ is invertible for all $\delta \in (-\delta_0, \delta_0)$. Moreover, L_{t_1} is non-negative by (A) and hence $L^\delta_{t_1}$ is positive for all $\delta > 0$. This shows the first assertion. The second assertion now follows immediately from the properties of the spectral flow as stated in theorem B.1.

INTRODUCE $\overline{u}$!!!

DEFINITION 2.6 *Assume that the boundary value problem* (5) *admits only the trivial solution and that assumption (A) holds. Then the generalised Morse index of* $\overline{u}$ *is defined by*

$$\mu_{Morse}(\overline{u}) = -\mathrm{sf}\,(h^\delta, [t_0, 1]),$$

where $t_0 \in (0, 1]$ *is chosen as in (A).*

Let us consider the special case that J is the identity; i.e., the differential operator L is positive. Then the Morse index of $L = L_1$ is finite and we obtain the following result. WRITE SOMETHING ON THE COMPONENTS OF THE SPACE OF SELFADJOINT FREDHOLM OPERATORS...

LEMMA 2.7 *If* J *is the identity, assumption (A) holds and* (5) *admits only the trivial solution, then*

$$\mu_{Morse}(\overline{u}) = i_{Morse}(h_1).$$

Proof. .....

3 The spectral index and the Maslov index

The aim of this section is to define the spectral index and the Maslov index for the boundary value problems (5). We introduce in a first section the spectral index which is defined as spectral flow of a path of unbounded selfadjoint Fredholm operators. In a second section we introduce a symplectic Hilbert space β and show that the space of solutions of (4) induces a Lagrangian subspace of β. Finally, we consider (4) on the shrinked domains Ω_t, $t \in (0, 1]$, and obtain a curve of Lagrangian subspaces which we use in order to define the Maslov index of (5).

3.1 The spectral index

We consider the differential equations (7) and define a path of differential operators by

$$\mathcal{A}^\delta_t : D = \{u \in H^2(\Omega, \mathbb{R}^k) : \, P^l(u, \partial_n u) = 0\} \subset L^2(\Omega, \mathbb{R}^k) \to L^2(\Omega, \mathbb{R}^k), \quad t \in (0, 1],$$

defined by

$$(\mathcal{A}^\delta_t u)(x) = \frac{1}{t^2} Lu(x) + S(tx)u(x) + \delta\, u(x).$$

Compact resolvent etc.... We observe at first:

LEMMA 3.1 *If the boundary value problem* (5) *admits only the trivial solution and assumption (A) holds, then there exists* $t_0, \delta_0 > 0$ *such that* $\mathcal{A}_{t_1} + \delta I$ *and* $\mathcal{A}_1 + \delta I$ *are invertible for all* $0 < \delta < \delta_0$, $0 < t_1 << t_0$ *and*

$$\mathrm{sf}\,(\mathcal{A}^\delta, [t_1, 1]) = \mathrm{sf}\,(\mathcal{A}^\delta, [t_0, 1]).$$

Proof.

By the previous lemma we obtain that the following definition is possible, because it does not depend on the particular choice of δ and t_0.

DEFINITION 3.2 *Assume that* (5) *has only the trivial solution and (A) holds. Then the spectral index of* $\overline{u}$ *is defined by*

$$\mu_{spec}(\overline{u}) = \mathrm{sf}\,(\mathcal{A}^\delta, [t_0, 1]),$$

where $\delta, t_0 > 0$ *are sufficiently small.*

Let us recall that the Morse index of (5) is the number of negative eigenvalues of this equation counted with multiplicities; i.e., the sum of the dimensions of all subspaces of $L^2(\Omega, \mathbb{R}^k)$ such that

$$Lu(x) + S(x)u(x) = \lambda\, u(x), \quad x \in \Omega$$
$$P^l(u\,|_\Gamma, \partial_n u\,|_\Gamma) = 0$$

for some $\lambda < 0$. If J is the identity then the Morse index is finite and it obviously coincides with the Morse index of the differential operator $\mathcal{A}_1$ as defined in ????APPENDIX.

LEMMA 3.3 *If* J *is the identity, assumption (A) holds and* (5) *admits only the trivial solution, then*

$$\mu_{spec}(\overline{u}) = i_{Morse}(\mathcal{A}_1).$$

3.2 Definition of the Maslov index

Let us consider the differential operator L and recall that L is closed and symmetric on the domain

$$D_{min} = \{u \in H^2(\Omega) : u\,|_\Omega = \partial_n u\,|_\Omega = 0\}.$$

In what follows, we will denote by L_{min} the restriction of L to D_{min} and we denote by $L_{max} := (L_{min})^*$ the adjoint of L_{min}. It is well known (cf. ???) that L_{max} is given by the operator L on the domain

$$D_{max} = \{u \in L^2(\Omega, \mathbb{R}^k) :\ Lu \in L^2(\Omega, \mathbb{R}^k)\}.$$

If we consider the latter space with the graph scalar product

$$\langle u, v\rangle = \langle u, v\rangle_{L^2(\Omega,\mathbb{R}^k)} + \langle L^*u, L^*v\rangle_{L^2(\Omega,\mathbb{R}^k)}, \quad u, v \in D_{max},$$

then this is a Hilbert space and D_{min} is a closed subspace in it. Consequently, the quotient space $\beta = D_{max}/D_{min}$ is a Hilbert space. In what follows, we denote by γ the quotient map from D_{max} to β. We define a bilinear form on β by

$$\omega : \beta \times \beta \to \mathbb{R}, \quad \omega(\gamma(u), \gamma(v)) = \langle L^*u, v\rangle_{L^2(\Omega,\mathbb{R}^k)} - \langle u, L^*v\rangle_{L^2(\Omega,\mathbb{R}^k)}.$$

Note that this is well defined...

LEMMA 3.4 ω *is a symplectic form on* β.

Proof.

Let us now show that our domain space $D \subset H^2(\Omega)$ is a Lagrangian subspace of β......

In what follows, we denote by a slight misuse of notation by S_t the ´bounded operator on $L^2(\Omega, \mathbb{R}^k)$ which is defined by

$$S_t : L^2(\Omega, \mathbb{R}^k) \to L^2(\Omega, \mathbb{R}^k), \quad (S_t u)(x) = t^2\, S(t \cdot x)u(x).$$

We consider $L_t = \gamma(\ker(L^* + S_t)) \subset \beta$, $t \in I$. Note that from the very definition $L_t \cap \gamma(D) \neq \{0\}$ if and only if $\ker \mathcal{A} \not= 0$. The following proposition will give the possibility to define the Maslov index of $\overline{u}$.

PROPOSITION 3.5 L_t *is an element of* $\mathcal{FL}_{\gamma(D)}(\beta)$ *and the path* $L : I \to \mathcal{FL}_{\gamma(D)}(\beta)$ *is smooth.*

Proof. Rather long, but understandable...

4 The main results

After having constructed the indices of the boundary value problems (5), we now can state our main theorem.

THEOREM 1 *Let the equation* (5) *admit only the trivial solution and assume that (A) holds. Then*

$$\mu_{Morse}(\overline{u}) = \mu_{spec}(\overline{u}) = \mu_{Mas}(\overline{u}) \in \mathbb{Z}.$$

We now derive various corollaries of this theorem. Let us at first consider the case that J is the identity matrix. We obtain from Lemma ??? and Lemma ????: HAS TO BE MODIFIED....

THEOREM 2 *Let the equation* (5) *admit only the trivial solution and assume that (A) holds. Then the Morse index of $\overline{u}$ is given by*

$$\mu_{spec}(\overline{u}) = \mu_{Mas}(\overline{u}).$$

In the case that we have in addition Dirichlet boundary conditions, we will prove in addition

THEOREM 3 *Let the equation* (5) *admit only the trivial solution for the Dirichlet problen. Then the Morse index of* (5) *is given by*

$$\mu_{spec}(\overline{u}) = \mu_{Mas}(\overline{u}) = \sum_{t\in I} m(t),$$

where

$$m(t) = \dim\{u \in H_0^1(\Omega_t) :\ u\, solves\, (7)\} = \dim\ker h_t.$$

Note that this in particular implies that there are only finitely many conjugate instants. Moreover, recall that this fact and the equality of the Morse index and the sum in theorem 3 were already obtained by Smale in [Sma65].
GEODESICS IN SEMI RIEMANNIAN AND RIEMANNIAN

5 Proofs

We now prove the assertions in three?????? consecutive steps. Moreover, at the end of the proof we make some comments on possible generalisations of our arguments to broader classes of boundary conditions.

Step 1: Equality of Morse and spectral index

According to the definition we have

$$\mu_{Morse}(\overline{u}) = -\mathrm{sf}\,(h^\delta, [t_0, 1])$$

where

$$h_t^\delta[u] = \int_\Omega \sum_{i=1}^N \langle J D_i u, D_i u\rangle\, d\boldsymbol{x} + \int_\Omega \langle t^2 S(t\boldsymbol{x})\, u, u\rangle\, d\boldsymbol{x} + \delta \|u\|^2_{L^2(\Omega)} \qquad \forall\, t \in [t_0, 1],\ u \in E.$$

The crossing form $\Gamma(h^\delta, t)$ is defined as the restriction of the derivative of h_t^δ with respect to t to the subspace $\ker h_t^\delta$. In particular, for each $u \in \ker h_t^\delta$ it is easy to show that the crossing form is given by

$$\Gamma(\boldsymbol{h}^\delta, t)(z) = -\frac{2}{t^3}\int_\Omega \sum_{i=1}^N \langle JD_i z\,, D_i z\rangle\, d\boldsymbol{x} \;+\; \int_\Omega \langle \dot S(t\boldsymbol{x})z, z\rangle\, d\boldsymbol{x}, \tag{10}$$

Now consider the unbounded selfadjoint operators $\mathcal{A}_t^\delta$ acting on $L^2(\Omega)$ with domains $\mathcal{D}(\mathcal{A}_t^\delta) = D$ that we introduced in ??????. By construction we have

$$h_t^\delta[u] = \langle \mathcal{A}_t^\delta u, u\rangle,\;\; u \in D \tag{11}$$

and hence a function u is a crossing for h_t^δ if and only if it belongs to the kernel of the corresponding unbounded operator $\mathcal{A}_t^\delta$. According to theorem B.3 $\mathcal{A}^\delta$ has only regular crossings for almost all $\delta \in \mathbb{R}$ and it follows from (11) that the crossing forms of $\mathcal{A}^\delta$ coincide with the ones of h^δ. Hence in what follows we can assume without loss of generality that h^δ has only regular crossing points, which accordingly are in particular finite in number. Moreover, we conclude from propositions B.3 and B.2

$$\mu_{spec}(\overline{u}) = -\mathrm{sf}\,(\mathcal{A}^\delta) = \mathrm{sf}\,(h^\delta) = \mu_{Morse}(\overline{u}).$$

CORRECT AND EXTEND THE COMPUTATIONS!!!

Step 2: Equality of spectral and Maslov index

According to our first step of the proof, we can assume without loss of generality that h^δ has only regular crossings. We now fix a regular crossing point $t \in (0,1)$ and our aim is to compute the crossing form (10) more explicitely.
We assume that $z = (z_1, \ldots, z_k) \in \ker h_t^\delta$. In particular, z solves (4), namely

$$-\frac{1}{t^2}\widetilde\Delta z + S(t\boldsymbol{x})z + \delta z = 0 \quad \text{in} \;\; \Omega.$$

For every $s \in (0,1]$ we set

$$z_s^t(\boldsymbol{x}) := z\left(\frac{s}{t}\boldsymbol{x}\right).$$

Clearly $z_t^t = z$ and, for every s, we get

$$-\frac{1}{s^2}(\widetilde\Delta z_s^t)(\boldsymbol{x}) + S(s\,\boldsymbol{x})z_s^t(\boldsymbol{x}) + \delta z_z^t(\boldsymbol{x}) = 0, \qquad \boldsymbol{x}\in\Omega \tag{12}$$

which follows from the fact that $(\frac{1}{s^2}\widetilde\Delta)z_s^t = \frac{1}{t^2}(\widetilde\Delta z_s^t)$ and $S(s\,\boldsymbol{x}) = S(t\,\frac{s\boldsymbol{x}}{t})$.
If we now differentiate the equation (12) with respect to s and evaluate in $s = t$, we obtain

$$\frac{2}{t^3}\widetilde\Delta z(\boldsymbol{x}) \;-\; \frac{1}{t^3}\widetilde\Delta \dot z(\boldsymbol{x}) \;+\; \dot S(t\boldsymbol{x})z(\boldsymbol{x}) \;+\; \frac{1}{t}S(t\,\boldsymbol{x})\,\dot z(\boldsymbol{x}) \;+\; \frac{\delta}{t}\dot z = \,0, \tag{13}$$

where we denote

$$\dot z(\boldsymbol{x}) := \frac{d}{ds}\,|_{s=t}\; z(s\boldsymbol{x}) = (\langle \nabla z_1, x\rangle, \ldots, \langle \nabla z_k, x\rangle). \tag{14}$$

Taking scalar products with z in (13) and integrating over Ω, we infer

$$0 = \frac{2}{t^3}\int_\Omega \langle \widetilde\Delta z, z\rangle\, dx - \frac{1}{t^3}\int_\Omega \langle \widetilde\Delta \dot z, z\rangle\, dx + \int_\Omega \langle \dot S(t\,\boldsymbol{x})z, z\rangle\, dx + \frac{1}{t}\int_\Omega \langle S(tx)\dot z, z\rangle\, dx + \frac{1}{t}\int_\Omega \delta\langle \dot z, z\rangle\, dx.$$

By applying Green's formula three times, we obtain

$$0 = -\frac{2}{t^3}\int_\Omega \sum_{i=1}^N \langle D_i z, D_j z\rangle dx + \int_\Omega \langle \dot S(tx)z, z\rangle\, dx - \frac{1}{t^3}\int_\Omega \langle \widetilde{\Delta} z, \dot z\rangle\, dx + \frac{1}{t}\int_\Omega \langle S(tx)z, \dot z\rangle\, dx$$
$$+ \frac{1}{t}\int_\Omega \delta\langle \dot z, z\rangle\, dx + \frac{1}{t^3}\int_\Gamma \langle \dot z, \partial_n z\rangle\, d\Gamma + \frac{2}{t^3}\int_\Gamma \langle z, \partial_n z\rangle\, d\Gamma - \frac{1}{t^3}\int_\Gamma \langle z, \partial_n \dot z\rangle\, d\Gamma.$$

Since $z \in \ker h_t^\delta$, we conclude

$$\begin{aligned} &-\frac{2}{t^3}\int_\Omega \sum_{i=1}^N \langle D_i z, D_i z\rangle dx + \int_\Omega \langle \dot S(tx)z, z\rangle\, dx \\ &= -\frac{1}{t^3}\int_\Gamma \langle \dot z, \partial_n z\rangle\, d\Gamma - \frac{2}{t^3}\int_\Gamma \langle z, \partial_n z\rangle\, d\Gamma + \frac{1}{t^3}\int_\Gamma \langle z, \partial_n \dot z\rangle\, d\Gamma \end{aligned} \tag{15}$$

and accordingly we obtain

$$\Gamma(h^\delta, t) = -\frac{1}{t^3}\int_\Gamma \langle \dot z, \partial_n z\rangle\, d\Gamma + \frac{1}{t^3}\int_\Gamma \langle z, \partial_n \dot z\rangle\, d\Gamma. \tag{16}$$

Our next aim is to calculate the crossing form $\Gamma(l^\delta, \mu; t)$ of the Maslov index at the regular crossing point t, where μ is the Lagrangian subspace $\gamma(D)$. We will show that the quadratic form $\Gamma(l^\delta, \mu; t)$ coincides up to a positive constant with $\Gamma(\boldsymbol{h}, t)$.
By definition, $l(t) \cap \mu \neq \{0\}$. In order to write the explicit expression of $\Gamma(l, \mu; t)$, we consider a Lagrangian subspace ν which is transversal to $l(t)$. Then there exists a differentiable path of bounded operators $\phi_s : l(t) \to \nu$ so that $l(s) = graph\phi_s$ for every s in a suitable small neighborhood of t. In other words, given $y \in l(t)$, then $\phi_s(y)$ is the unique vector such that

$$\phi_s(y) \in \nu, \qquad y + \phi_s(y) \in l(s).$$

Let us recall that $\Gamma(l, \mu; t)$ is the quadratic form associated with

$$Q(x, y) := \frac{d}{ds}\Big|_{s=t} \omega(x, \phi_s(y)), \qquad \forall x, y \in l(t) \cap \mu.$$

Fix $y \in l(t) \cap \mu$, then $y = \tilde{\mathscr{T}}_t(z) = (z_\Gamma, \partial_n z\, |_\Gamma)$, where z solves the equation (**??**). As before, we can immediately prove that z_s^t solves equation (12). If we define $X(s) := \gamma(z_s^t)$, we note that $X(s) \in l(s)$. Hence, $X(s) = c(s) + \phi_s(c(s))$, with $c(s) \in l(t)$. Observe that $X(t) = y = c(t)$. Taking into account that $\dot c(t) + \phi_t(\dot c(t)) \in l(t)$, we get

$$\omega(X(t), \frac{dX}{ds}(t)) = \omega(y, \dot c(t) + \phi_t(\dot c(t)) + \dot\phi_t(c(t))) = \omega(y, \dot\phi_t(y)).$$

Hence,

$$\begin{aligned} \Gamma(l, \mu; t) &= \frac{d}{ds}\Big|_{s=t} \omega(y, \phi_s(y)) = \omega(X(t), \frac{dX}{ds}(t)) \\ &= \omega\left(\gamma(z), \frac{d}{ds}\Big|_{s=t} \gamma(z_t^s)\right). \end{aligned} \tag{17}$$

From $\frac{d}{ds}\Big|_{s=t} \gamma(z_t^s) = \gamma(\dot z)$ and the fact that $z, \dot z$ are smooth, we conclude

$$\Gamma(l^\delta, \mu; t) = -\frac{1}{t^2}\int_\Gamma \langle \dot z, \partial_n z\rangle + \frac{1}{t^2}\int_\Gamma \langle z, \partial_n \dot z\rangle\, d\Gamma = t\Gamma(h^\delta, t).$$

Accordingly the crossing forms coincide up to the positive factor $\frac{1}{t}$ with $\Gamma(h^\delta, t)$, and by the propositions B.2 and A.13 this shows the equality of the spectral and the Maslov index.

Step 3: Conjugate instants of the Dirichlet problem

For the Dirichlet problem we obtain from (16)

Check and explain!

$$\Gamma(h_D^\delta, t)[z] = -\frac{1}{t^3}\int_\Gamma \langle \dot z, \partial_n z\rangle \, d\Gamma = -\frac{1}{t^3}\int_\Gamma \langle\langle x, \nabla z\rangle, \partial_n z\rangle \, d\Gamma$$
$$= -\frac{1}{t^3}\int_\Gamma \|x\|\|\partial_n z\|^2 \, d\Gamma < 0$$

Accordingly, the crossing form $\Gamma(h_D^\delta, t)$ is negative-definite for any δ and so in particular for $\delta = 0$. Hence any crossing t of h_D is regular, the crossings are isolated and the contribution to the spectral index at each crossing is precisely the dimension of the kernel, i.e. the multiplicity $m(t)$ of the conjugate instant.

A remark on the proof

It is clear that the results (15) and (17) can also be used to show the equality of the crossing forms for the Maslov index and the spectral index for more general boundary conditions. However, in this case the unbounded operators in the first step of our proof have varying domains and the perturbation result B.3 is no longer valid. Hence one can only show the equality of corresponding spectral and Maslov indices under the additional assumption that all crossings are regular.

Geodesics

6 Bifurcation

Let $f : \Omega \times \mathbb{R}^k \to \mathbb{R}^k$ be a $\mathscr{C}^\infty$ function. We assume henceforth that there are constants a, b and r, with $r > 1$, such that

$$|f(\boldsymbol{x}, u)| \le a + b|u|^r \qquad \text{and} \qquad |\partial_u f(\boldsymbol{x}, u)| \le a + b|u|^r, \qquad \forall\, (\boldsymbol{x}, u) \in \Omega \times \mathbb{R}^k.$$

Consider the functional $\mathscr{E} : H_0^1(\Omega) \to \mathbb{R}$ given by

$$\mathscr{E}(u) := \frac{1}{2} h_D(u) - \int_\Omega F(\boldsymbol{x}, u)\, dx$$

where CHECK THIS CAREFULLY/ IT DOES NOT MAKE SENSE HERE $F(x, s) = \int_0^s \langle f(\boldsymbol{x}, \sigma)\sigma, \sigma\rangle \, d\sigma$.

Lemma 6.1 *The functional $\mathscr{E}$ is $\mathscr{C}^2$ on the Hilbert space $H_0^1(\Omega)$.*

Proof. The proof is standard and follows by the fact that the embedding $H_0^1(\Omega) \to L^q(R)$ is continuous for $q \ge 1$. (It is enough to apply [AP95, Theorem 2.6] to f and $\partial_u f$).

□

We now consider the Dirichlet problem for the semi-linear equation

$$\begin{aligned} -\widetilde{\Delta} u(x) + S(x)u(x) + f(x, u(x)) &= 0, \quad x \in \Omega \\ u(x) &= 0, \quad x \in \Gamma \end{aligned} \tag{18}$$

and its linearisation at $0 \in H_0^1(\Omega)$

$$\begin{aligned} -\widetilde{\Delta} u(x) + S(x)u(x) + G(x)u(x) &= 0, \quad x \in \Omega \\ u(x) &= 0, \quad x \in \Gamma, \end{aligned} \tag{19}$$

where $G(x) =$.

LEMMA 6.2 *The critical points of $\mathscr{E}$ are the weak solutions $u \in H_0^1(\Omega)$ of the problem* (18).

Proof. In fact by an elementary computation, we infer

$$\langle \nabla \mathscr{E}(u), v \rangle = h_D(u,v) - \int_\Omega \langle f(x,u), v \rangle dx, \qquad \forall\, u, v \in H_0^1(\Omega).$$

□

We now consider the family of domains Ω_t, $t \in (0,1]$ induced by shrinking the star-shaped domain Ω and the induced family of boundary value problems

$$\begin{aligned} -\widetilde{\Delta} u(x) + S(x)u(x) + f(x,u(x)) = 0, &\quad x \in \Omega_t \\ u(x) = 0, &\quad x \in \Gamma_t \end{aligned} \tag{20}$$

and we assume henceforth that $f(x,0) = 0$, $x \in \Omega$.

DEFINITION 6.3 *We call $t^* \in (0,1]$ a bifurcation point if there exists a sequence $\{(t_n,u_n)\}_{n\in\mathbb{N}} \in (0,1] \times H_0^1(\Omega_{t_n})$ such that u_n solves* (20) *for $t = t_n$, $u_n \neq 0 \in H_0^1(\Omega_{t_n})$, $n \in \mathbb{N}$, and $\|u_n\|_{H_0^1(\Omega_{t_n})} \to 0$, $n \to \infty$.*

The main result of this section reads as follows.

THEOREM 6.4 *Assume that the linearised boundary value problem* (19) *admits only the trivial solution. If one of the indices in theorem 3 does not vanish, then there exists a bifurcation point $t^* \in (0,1]$ for the equation* (18). *Moreover, the bifurcation points are precisely the conjugate points.*

After rescaling, the boundary value problem (20) can be written as

$$\begin{aligned} -\widetilde{\Delta} u(x) + t^2 S(t\,x)u(x) + t^2 f(t\,x,u(x)) = 0, &\quad x \in \Omega \\ u(x) = 0, &\quad x \in \Gamma \end{aligned} \tag{21}$$

and we note that t^* is a bifurcation point for (20) if and only if there exists a sequence $\{(t_n,u_n)\}_{n\in\mathbb{N}} \in (0,1] \times H_0^1(\Omega)$ such that $u_n \neq 0$ solves (21) and $u_n \to 0$, $n \to \infty$.
As above one can check that solutions of (21) are precisely the critical points of the functional

$$\mathscr{E}_t(u) = \frac{1}{2} h_{D,t}(u) + F(t\boldsymbol{x},u), \quad t \in (0,1]$$

and that the Hessians at the critical point $0 \in H_0^1(\Omega)$ are given by

$$hess(0)_t[\xi,\eta] = h_{D,t}(\xi,\eta) - \int_\Omega \langle g(t\boldsymbol{x},0)\xi,\eta \rangle\, dx.$$

We now infer from theorem 3 that $h_{D_{t^*}}$ is non-invertible if and only if t^* is a conjugate point and, since we are considering Dirichlet boundary conditions, that $\mu_{Morse}(hess(0)_{t^*+\varepsilon}) \neq \mu_{Morse}(hess(0)_{t^*-\varepsilon})$ for any sufficiently small $\varepsilon > 0$. According to [FPR99, Corollary 2], this implies that t^* is a bifurcation point of the equation (21) and hence for (20) as well.

A Fredholm Lagrangian Grassmannian and Maslov index

In this section we recall some facts about the *Fredholm Lagrangian Grassmannian* and the construction of the Maslov index in the infinite dimensional setting. Our basic references are [Fur04, DN06, DN08, DJ10].

Let H be a real separable Hilbert space of infinite dimension equipped with a (strong) symplectic form, i.e. a skew-symmetric, bounded bilinear form ω which is non-degenerate in the sense that it induces an isomorphism between H and its dual space H^*.

Definition A.1 *A subspace L of the symplectic space (H,ω) is called* isotropic *if $\omega|_L \equiv 0$, i.e. $\omega(p,q)=0$ for all $p,q \in L$. A* Lagrangian subspace *is a maximal closed isotropic subspace of H.*

Let $\Lambda(H)$ denote the set of all *Lagrangian subspaces* of H which is called the *Lagrangian Grassmannian*. $\Lambda(H)$ is an infinite dimensional Banach manifold modeled on the Banach space of all bounded selfadjoint operators (cf. [Fur04]).

Definition A.2 *Given two closed subspaces μ,η of H, the pair (μ,η) is called a* Fredholm pair *if*

$$\dim(\mu \cap \eta) < +\infty \quad \textit{and} \quad \operatorname{codim}(\mu+\eta) < +\infty. \tag{22}$$

Note that many authors require in the definition of a Fredholm pair also the sum $\mu+\eta \subset H$ to be closed. However, it is not hard to show that this property follows from (22) (cf. KATO).

Definition A.3 *The* Fredholm Lagrangian Grassmannian *with respect to the Lagrangian subspace $\mu \in \Lambda(H)$ is defined as*

$$\mathscr{F}\mathscr{L}_\mu(H) := \{\eta \in \Lambda(H) : (\mu,\eta) \textit{ is a Fredholm pair}\},$$

and the subset

$$\mathscr{M}_\mu(H) := \{\eta \in \mathscr{F}\mathscr{L}_\mu(H) : \eta \cap \mu \neq \{0\}\},$$

is called the Maslov cycle *with respect to μ.*

We denote by $Gl_c(H)$ the *Fredholm group* of H consisting of all compact perturbations of the identity which are invertible. Moreover, we define the *Fredholm symplectic group $Sp_c(H)$* as the group of all $\Psi \in Gl_c(H)$ such that $\Psi^*\omega = \omega$ (cf. [Swa78a, Swa78b] for further details).

Definition A.4 *Let $L \subset H$ be a closed subspace. We define the* reduced Fredholm Grassmannian $\mathrm{Fred}_{res}(L)$ *as the orbit of L under the action of the group $Gl_c(H)$. Moreover, if $L \in \Lambda(H)$ is a Lagrangian subspace, then the* reduced Fredholm Lagrangian Grassmannian $\mathscr{F}\mathscr{L}_{res}(L)$ *is the orbit of L under the group $Sp_c(H)$.*

It is easily seen from the definition that $\mathrm{Fred}_{res}(L) \subset \mathrm{Fred}_L(H)$ for any closed subspace $L \subset H$ and, accordingly, $\mathscr{F}\mathscr{L}_{res}(L) \subset \mathscr{F}\mathscr{L}_L(H)$ if $L \in \Lambda(H)$.

A.1 Some useful criteria

The aim of this section is to collect some useful facts about the (reduced) Lagrangian Grassmannian which we frequently use in our arguments.
We begin by recalling some elementary facts about Lagrangian subspaces which can be found in [DJ10, Section 3].

Lemma A.5 *Let (H,ω) be a symplectic Hilbert space.*

i) Let $\xi \in \Lambda(H)$ be a Lagrangian subspace of H. If $\mu \in \mathrm{Fred}_{res}(\xi)$ and $\nu \in \mathrm{Fred}_{res}(\mu)$, then $\nu \in \mathrm{Fred}_{res}(\xi)$.

ii) For $\mu,\eta \in \Lambda(H)$, if $\dim(\mu\cap\eta)$ is finite dimensional and $\mu+\eta$ is closed, then (μ,η) is a Fredholm pair.

iii) Let $\xi \in \Lambda(H)$ and $\eta \in \mathrm{Fred}_{res}(\xi)$. If ω vanishes on η, then $\eta \in \Lambda(H)$.

For the following result we also refer to [DJ10, Section 3].

Lemma A.6 *Let μ,η be two closed subspaces of H. If (η,μ) is a Fredholm pair and $\xi \in \mathrm{Fred}_{res}(\eta)$, then (ξ,μ) is a Fredholm pair.*

The following well known lemma is often helpful for checking that an isotropic subspace is actually Lagrangian (cf. BOOSS).

LEMMA A.7 *Let $L_0, L_1 \subset H$ be isotropic subspaces of the symplectic Hilbert space H such that $H = L_0 + L_1$. Then $L_0, L_1 \in \Lambda(H)$.*

We fix a direct sum decomposition $H = H_+ \oplus H_-$, where H_+, H_- are infinite-dimensional orthogonal closed subspaces. The following result can be found in [DN06, Lemma6].

LEMMA A.8 *A closed subspace $H_1 \subset H$ belongs to $\mathrm{Fred}_{res}(H_-)$ if and only if there exists a (not necessarily invertible) compact perturbation of the identity $A = Id + K$ and $H_2 \in \mathrm{Fred}_{res}(H_-)$ such that:*

- $\mathrm{im}\,(A) + H_2 = H$,
- $A^{-1}(H_2) = H_1$.

Finally, for the differentiability of curves in $\mathrm{Fred}_{res}(H_-)$ we need the following lemma. For its proof we refer to [DN06, Lemma 9].

LEMMA A.9 *Let $I \ni \lambda \mapsto S(\lambda) := I + K(\lambda) \in \mathscr{L}(H)$ be of class $\mathscr{C}^k$ for some $k \in \mathbb{N}$, where $K(\lambda)$ is a compact operator for each λ and assume that*

$$\mathrm{im}\,(S(\lambda)) + H_- = H, \qquad \forall\,\lambda \in I.$$

Then we have

1. *$S(\lambda)^{-1}(H_-) \in \mathrm{Fred}_{res}(H_-)$, for each $\lambda \in I$;*
2. *The map $\Psi : I \ni \lambda \mapsto \Psi(\lambda) := S(\lambda)^{-1}(H_-) \in \mathrm{Fred}_{res}(H_-)$ is of class $\mathscr{C}^k$.*

A.2 The infinite dimensional version of the Maslov index

The Maslov index was introduced in ????? for finite dimensional symplectic vector spaces (V, ω) and it provides an explicite isomorphism between the fundamental group $\pi_1(\Lambda(V))$ and the integers. Heuristically, the Maslov index COUNTS SOMETHING AND SO IS A QUITE POOR GUY! In contrast, it can be shown from Kuiper's theorem [Kui65] that $\Lambda(H)$ is a contractible space if H is an infinite dimensional symplectic Hilbert space (cf. e.g. [Nic95]) and hence no non-trivial homotopy invariants can arise. However, it can be shown that $\pi_1(\mathscr{F}\mathscr{L}_\mu(H)) \cong \mathbb{Z}$ for any $\mu \in \Lambda(H)$ and an explicite isomorphism is provided by the *Maslov index*, which we now want to introduce briefly. Our reference in this section is [Fur04].
Let (H, ω) be a symplectic Hilbert space and let $\mu \in \Lambda(H)$ be a fixed Lagrangian subspace. We cannot give a detailed account on the construction of the Maslov index, which is quite elaborate, but instead we introduce it axiomatically. To this aim we fix a generator $\tilde{l} \in \pi_1(\mathscr{F}\mathscr{L}_\mu(H))$.

THEOREM A.10 *There exists precisely one integer valued map i_{Mas} on the set of all paths $\gamma : [a, b] \to \mathscr{F}\mathscr{L}_\mu(H)$ having ends outside $\mathscr{M}_\mu$ which satisfies the following properties:*

1. *$i_{\mathrm{Mas}}(\tilde{l}, \mu, [a, b]) = 1$.*
2. *if $\gamma(t) \notin \mathscr{M}_\mu(H)$ for each $t \in [a, b]$, then $i_{\mathrm{Mas}}(\gamma, \mu, [a, b]) = 0$;*
3. *if $\gamma_1 : [a, c] \to \mathscr{F}\mathscr{L}_\mu(H)$ and $\gamma_2 : [c, b] \to \mathscr{F}\mathscr{L}_m u(H)$ are two continuous curves such that the concatenation $\gamma_1 * \gamma_2$ exists, then*

$$i_{\mathrm{Mas}}(\gamma_1 * \gamma_2, \mu, [a, b]) = i_{\mathrm{Mas}}(\gamma_1, \mu, [a, b]) + i_{\mathrm{Mas}}(\gamma_2, \mu, [a, b])$$

   where $*$ denotes the concatenation.

4. *If* $H : [0,1] \times [a,b] \to \mathscr{FL}_\mu(H)$ *is continuous and* $H(\lambda, a), H(\lambda, b) \notin \mathscr{M}_\mu$ *for all* $\lambda \in [0,1]$, *then* $i_{\mathrm{Mas}}(H(0,\cdot), \mu, [a,b]) = i_{\mathrm{Mas}}(H(1,\cdot), \mu, [a,b])$.

5. *Let* $\xi \in \mathscr{FL}_{res}(\mu)$. *Then* $i_{\mathrm{Mas}}(\gamma, \mu, [a,b]) - i_{\mathrm{Mas}}(\gamma, \xi, [a,b])$ *depends only on the endpoints.*

The interpretation of $i_{\mathrm{Mas}}(l, \mu, [a,b])$ is as in the finite case the number of intersections of l with $\mathscr{M}_\mu$.

Definition A.11 *Let* $\gamma : [a,b] \to \mathscr{FL}_\mu(H)$ *be a* $\mathscr{C}^1$ *path. We say that* $t^* \in [a,b]$ *is a* crossing instant *for the curve* γ, *if* $\gamma(t^*) \in \mathscr{M}_\mu(H)$.

Let t^* ba a crosiing for the path $\gamma : [a,b] \to \mathscr{FL}_\mu(H)$ and let ν be a Lagrangian subspace which is transversal to $\gamma(t^*)$. Since transversality is an open condition, there exists $\varepsilon > 0$ such that the Lagrangian subspace $\gamma(t)$ is transversal to ν for each $|t - t^*| < \varepsilon$. Therefore, we can find a $\mathscr{C}^1$-family of bounded operators $\phi_t : \gamma(t^*) \to \nu$ so that

$$\gamma(t) = \mathrm{Graph}(\phi_t), \quad t \in (t^* - \varepsilon, t^* + \varepsilon).$$

The *crossing form* $Q(t^*)$ at the instant $t = t^*$ is the bilinear form on $\gamma(t^*) \cap \mu$, defined by

$$Q(t^*)(x,y) := \frac{d}{dt}\Big|_{t=t^*} \omega(x, \phi_t(y)), \qquad x, y \in \gamma(t^*) \cap \mu.$$

It can be shown that $Q(t^*)$ does not depend on the choice of ν. Crossing forms are fundamental for us, since they give a way to compute the local contribution to the Maslov index as we will introduce now.

Definition A.12 *The crossing* $t^* \in (a,b)$ *will be called* regular *if* $Q(t^*)|_{\gamma(t^*)\cap\mu}$ *is non-degenerate. A curve* $\gamma : [a,b] \to \mathscr{FL}_\mu(H)$ *is called* regular *if each crossing instant is regular.*

It is easy to see that regular crossings are isolated and hence on a compact interval are finite in number.

Proposition A.13 (Localization property) *Let* $\gamma : [a,b] \to \mathscr{FL}_\mu(H)$ *be a* $\mathscr{C}^1$ *path. If* $t^* \in (a,b)$ *is a regular crossing instant of* γ, *then there exists* $\delta > 0$ *such that*

$$i_{\mathrm{Mas}}(\gamma, \mu, [t^* - \delta, t^* + \delta]) = \mathrm{sign}\, Q(t^*),$$

where sign *denotes the signature.*

Note that accordingly the Maslov index of a regular $\mathscr{C}^1$ curve $\gamma : [a,b] \to \mathscr{FL}_\mu(H)$ having ends outside $\mathscr{M}_\mu$ is given by

$$i_{\mathrm{Mas}}(\gamma, \mu, [a,b]) = \sum_{t^* \in \gamma^{-1}(\mathscr{M}_\mu)} \mathrm{sign}\, Q(t^*). \tag{23}$$

B The spectral flow

In this section we introduce the concept of spectral flow of paths of Fredholm quadratic forms acting on a Hilbert space as in [MPP05, FPR99] and for a certain class of paths of unbounded selfadjoint Fredholm operators as in [RS95]. We will give axiomatic definitions as for the Maslov index in the previous section.
We consider a bounded quadratic form $q \colon H \to \mathbb{R}$ and we let $b = b_q \colon H \times H \to \mathbb{R}$ be the bounded symmetric bilinear form such that $q(u) = b(u,u)$, $u \in H$. By the Riesz representation theorem there exists a bounded selfadjoint operator $A_q : H \to H$ such that $b_q(u,v) = \langle A_q u, v\rangle$, $u, v \in H$. We call $q \colon H \to \mathbb{R}$ a *Fredholm quadratic form* if A_q is Fredholm; i.e., $\ker A_q$ is of

finite dimension and $\mathrm{Ran} A_q$ is closed. Recall that the space $Q(H)$ of bounded quadratic forms is a Banach space with respect to the norm $\|q\| = \sup\{|q(u)| : \|u\| = 1\}$. The subset $Q_F(H)$ of all Fredholm quadratic forms is an open subset of $Q(H)$ which is stable under perturbations by weakly continuous quadratic forms. A quadratic form $q \in Q_F(H)$ is called *non-degenerate* if the corresponding Riesz representation A_q is invertible. The following characterisation of the spectral flow can be found in ????? (cf. also [**?**]).

THEOREM B.1 *There exists prcisely one integer values map* sf *, which is defined for any path* $q : [a, b] \to Q(H)$ *having non-degenerate ends, such that*

i) If $h : [0, 1] \times [a, b] \to Q_F(H)$ *is such that* $h(\lambda, 0)$ *and* $h(\lambda, 1)$ *are non-degenerate for all* $\lambda \in [0, 1]$*, then*

$$\mathrm{sf}\,(h(0, \cdot), [a, b]) = \mathrm{sf}\,(h(1, \cdot), [a, b])$$

.

ii) If q_t *is non-degenerate for all* $t \in [a, b]$*, then* $\mathrm{sf}\,(q, [a, b]) = 0$*.*

iii) Let $q^1, q^2 : [a, b] \to Q_F(H)$ *be two paths having invertible ends such that* $q^1_b = q^2_a$*. Then*

$$\mathrm{sf}\,(q^1 * q^2, [a, b]) = \mathrm{sf}\,(q^1, [a, b]) + \mathrm{sf}\,(q^2, [a, b]).$$

iv) If $\mu_{Morse}(q_t) < \infty$ *for all* $t \in [a, b]$*, then*

$$\mathrm{sf}\,(q, [a, b]) = \mu_{Morse}(q_a) - \mu_{Morse}(q_b).$$

As for the Maslov index, the spectral flow can be calculated explicitely for paths which are sufficiently regular. If a path $q\colon [a, b] \to Q_F(H)$ is differentiable at t, then the derivative $\dot q(t)$ with respect to t is again a quadratic form. We call t a crossing instant if $q(t)$ is degenerate and we say that the crossing instant t is regular if the *crossing form* $\Gamma(q, t)$, defined by

$$\Gamma(q, t) := \dot q(t)|_{\ker q(t)},$$

is non-degenerate.

PROPOSITION B.2 *If all crossing instants* t_i *of the path* q *are regular, then they are finite in number and*

$$\mathrm{sf}\,(q, [a, b]) = \sum_i \mathrm{sign}\; \Gamma(q, t_i). \tag{24}$$

Finally, we want to introduce briefly the construction of the spectral flow of Robbin and Salamon in [RS95], which is in particular important for us due to a perturbation result that we will state below. We assume as before that H is a searable Hilbert space. Let $W \subset H$ be a Hilbert space in its own right with a compact and dense injection $W \hookrightarrow H$. Let $\mathcal{FS}(W, H) \subset \mathcal{L}(W, H)$ denote the space of all bounded operators which are selfadjoint when regarded as unbounded operators on H with dense domain W. Note that each operator in $\mathcal{FS}(W, H)$ has a compact resolvent and hence is in particular Fredholm. In [RS95] the spectral flow $\mathrm{sf}\,(\mathcal{A}, [a, b])$ for paths $\mathcal{A} : [a, b] \to \mathcal{FS}(W, H)$ having invertible ends is constructed by intersection theory and it is characterised axiomatically as in theorem B.1. For details we refer to [RS95].

Now let $\mathcal{A} : [a, b] \to \mathcal{FS}(W, H)$ be a $\mathscr{C}^1$ path having invertible ends. As before, we call $t^* \in [a, b]$ a *crossing instant* of $\mathcal{A}$ if $\ker \mathcal{A}_{t^*} \neq \{0\}$. Moreover, t^* is called a *regular crossing* if the crossing form

$$\Gamma(\mathcal{A}, t^*)u := \langle \dot{\mathcal{A}}_{t^*} u, u\rangle_H, \quad u \in \ker \mathcal{A}_{t^*}$$

is non-degenerate on $\ker \mathcal{A}$.

The following result can be found in [RS95, Theorem 4.2, Lemma 4.7].

THEOREM B.3 *Let $\mathcal{A} : [a, b] \to \mathcal{FS}(W, H)$ be a $\mathscr{C}^1$ path having invertible ends.*

i) For almost all $\delta \in \mathbb{R}$ the path $\mathcal{A} + \delta Id$ has only regular crossings.

ii) If $\mathcal{A}$ has only regular crossings, then they are finite in number and the spectral flow is given by

$$\mathrm{sf}\,(\mathcal{A}, [a, b]) = \sum_{t \in I} \mathrm{sign}\,\Gamma(\mathcal{A}, t).$$

References

[AP95] AMBROSETTI, A.; PRODI, G., A primer of nonlinear analysis. Corrected reprint of the 1993 original. Cambridge Studies in Advanced Mathematics, **34**. Cambridge University Press, Cambridge, 1995.

[Arn67] Arnol'd, V.I. On a characteristic class entering into conditions of quantization. *Funkcional. Anal. i Priložen.* **1** (1967), 1–14.

[CDP10] Capietto, A.; Dalbono, F.; Portaluri, A. A multiplicity result for a class of strongly indefinite asymptotically linear second order systems. *Nonlinear Anal.* **72** (2010), no. 6, 2874–2890.

[DP12] Dalbono, F.; Portaluri, A. Morse-Smale index theorems for elliptic boundary deformation problems To appear in *J. Differential Equations*.

[DN06] Deng, J.; Nii, S. Infinite-dimensional Evans function theory for elliptic eigenvalue problems in a channel. *J. Differential Equations* **225** (2006), no. 1, 57–89.

[DN08] Deng, J.; Nii, S. An infinite-dimensional Evans function theory for elliptic boundary value problems. *J. Differential Equations* **244** (2008), no. 4, 753–765.

[DJ10] Deng, J.; Jones, C.K.R.T. Multi-dimensional Morse index theorems and a symplectic view of elliptic boundary value problems. *Trans. Amer. Math. Soc.* **363** (2011), no. 3, 1487–1508.

[Dui76] Duistermaat, J.J. On the Morse index in variational calculus. *Advances in Math.* **21** (1976) no. 2, 173–195.

[FPR99] Fitzpatrick, P. M.; Pejsachowicz, J.; Recht, L. Spectral flow and bifurcation of critical points of strongly-indefinite functionals. I. General theory. *J. Funct. Anal.* **162** (1999), no. 1, 52–95.

[Fur04] Furutani, K. Fredholm Lagrangian Grassmannian and the Maslov index. *J. Geom. Phys.* **51**, (2004), no. 3, 269–331.

[Hel94] Helfer, Adam D. Conjugate points on space like geodesics or pseudo-self-adjoint Morse-Sturm-Liouville systems. *Pacific J. Math.*, **164**, 321-340, 1994.

[Kui65] KUIPER, N. H., The homotopy type of the unitary group of Hilbert space. Topology **3** 1965 19–30.

[MPP05] Musso, M.; Pejsachowicz, J.; Portaluri, A. A Morse index theorem for perturbed geodesics on semi-Riemannian manifolds. *Topol. Methods Nonlinear Anal.* **25** (2005), no. 1, 69–99.

[Nic95] NICOLAESCU, L. I. The Maslov index, the spectral flow, and decompositions of manifolds. Duke Math. J. **80** (1995), no. 2, 485–533.

[RS95] Robbin, J.; Salamon, D. The spectral flow and the Maslov index. *Bull. London Math. Soc.* **27** (1995), no.1, 1–33.

[Sim68] Simons, J. Minimal varieties in riemannian manifolds. *Ann. of Math.* (2) **88** (1968), 62–105.

[Sma65] Smale, S. On the Morse index theorem. *J. Math. Mech.* **14** (1965), 1049–1055.

[Swa78a] Swanson, R. C. Fredholm intersection theory and elliptic boundary deformation problems. I. *J. Differential Equations* **28** (1978), no. 2, 189–201.

[Swa78b] Swanson, R. C. Fredholm intersection theory and elliptic boundary deformation problems. II. *J. Differential Equations* **28** (1978), no. 2, 202–219.

[Uhl73] Uhlenbeck, K., The Morse index theorem in Hilbert space. *J. Differential Geometry* **8** (1973), 555–564.

[Wat12] Waterstraat, N. , A K-theoretic proof of the Morse index theorem in semi-Riemannian geometry. *Proc. Amer. Math. Soc.* **140** (2012), no. 1, 337–349.

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