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G-Structures, Fluxes and Calibrations in M-Theory

Dario Martelli and James Sparks

Department of Physics
Queen Mary, University of London
Mile End Rd, London E1 4NS, U.K.

d.martelli@qmul.ac.uk , j.sparks@qmul.ac.uk

Abstract

We study the most general supersymmetric warped M-theory backgrounds with non-trivial G-flux of the type $\mathbb{R}^{1,2} \times M_8$ and $\mathrm{AdS}_3 \times M_8$. We give a set of necessary and sufficient conditions for preservation of supersymmetry which are phrased in terms of G-structures and their intrinsic torsion. These equations may be interpreted as calibration conditions for a static "dyonic" M-brane, that is, an M5-brane with self-dual three-form turned on. When the electric flux is turned off we obtain the supersymmetry conditions and non-linear PDEs describing M5-branes wrapped on associative and special Lagrangian three-cycles in manifolds with G_2 and $SU(3)$ structures, respectively. As an illustration of our formalism, we recover the 1/2-BPS dyonic M-brane, and also construct some new examples.

1 Introduction

Recently there has been considerable interest in trying to understand the types of geometries that arise in supersymmetric solutions of supergravity theories. When all fields are turned off, apart from the metric, it has long been known that supersymmetric solutions are described by special holonomy manifolds – for example, Calabi-Yau manifolds or manifolds of G_2 holonomy. However, for many applications one is interested in solutions where the fluxes are turned on. These include important areas of research, such as the AdS/CFT correspondence, or phenomenological models based on string/M-theory compactifications.

Until recently, the study of supersymmetric solutions with non-vanishing fluxes has been based mostly on physically motivated ansatze for the supergravity Killing spinor equations. While this method has led to many interesting results, a more systematic approach is clearly desirable. In [1] it was advocated that the G-structures defined by the Killing spinors provide such a formalism. Subsequent works have used this approach to analyse and classify supersymmetric backgrounds in various supergravity theories [2, 3, 4, 5, 6, 7, 8, 9, 10]. Using the language of G-structures and their "intrinsic torsion" one can rewrite the supersymmetry equations of interest in terms of an equivalent set of first-order equations for a particular set of forms.

Another point, emphasized in [1] (and based on [11]), is the fact that some of the resulting conditions have an interpretation in terms of "generalised calibrations" [12, 13]. This was further elaborated on in [5] and [7]. Generalised calibrations extend to backgrounds with fluxes the original notion of calibrations in special holonomy manifolds [14], and their physical significance is then that supersymmetric probe branes have minimal *energy*. On a more practical level, the formalism based on G-structures can often be very useful for actually finding new solutions in a given supergravity theory. For instance, in [1, 5, 7] new examples were found this way, while in lower dimensions [2, 9, 10] the general form for *all* supersymmetric solutions was given.

In this paper we study M-theory on eight-manifolds – that is, supersymmetric warped M-theory backgrounds of the type $M_3 \times M_8$, with M_3 either Minkowski$_3$ or AdS$_3$ space. Supersymmetric compactifications of M-theory to three dimensions have been considered before in [15, 16, 17, 18, 19, 20]. The types of geometries described in these papers may be thought of as M2-brane solutions where the transverse space is a manifold of special holonomy. Alternatively, one may think of them as compactifications on a special holonomy manifold where one includes some number of space-filling M2-branes in the vacuum.

One of our motivations was to investigate more general types of supersymmetric solutions to M-theory on eight-manifolds. In particular, there should clearly be another way to obtain an $\mathcal{N} = 1$ Minkowski vacuum from M-theory – namely, one may wrap M5-branes over a supersymmetric three-cycle in a G_2-holonomy manifold (times an S^1). After including the backreaction of the M5-brane on the geometry, one no longer expects the eight-manifold to have special holonomy, but rather a more general G_2-structure with intrinsic torsion related to the G-flux. Similarly, M5-branes wrapped on special Lagrangian three-cycles in a Calabi-Yau three-fold yield $\mathcal{N} = 2$ in three dimensions. We will show how these various geometries may be obtained by relaxing the assumptions of [15, 18]; in particular we relax the assumption that the internal spinor is chiral. Furthermore, this generalisation yields supersymmetric AdS_3 compactifications, which were excluded before. The method we use relies on local equations, and thus also covers non-compact geometries; examples of typical interest are solutions describing wrapped branes or brane intersections.

The M-theory five-brane has a self-dual three-form gauge field that propagates on its world-volume. Turning on this field induces an electric coupling to the C-field, and therefore also an M2-brane charge. Thus the backreaction of such a "dyonic" M5-brane should correspond to some more general supersymmetric solution with electric and magnetic G-flux. In fact, we will see how such solutions arise in our formalism. One can argue that the most general supersymmetric solution of the form $M_3 \times M_8$ is of this type, with the M2-brane solutions being a limit in which the M5-brane disappears completely.

The plan of the paper is as follows. In section 2 we give a brief summary of what is known about M-theory on eight-manifolds. This will also allow us to introduce our notations and conventions. We then describe how one extends the analysis to allow for more general supersymmetric solutions with fluxes. The key point is to allow for a generic spinor on the internal space – in particular, we do not impose that it be chiral. Thus, in addition to the M2-brane-type of solutions, one also expects M5-brane-type solutions, including "dyonic" or "interpolating" solutions which have both charges present, and also AdS_3 solutions.

In section 3 we show how the conditions for supersymmetry may be recast into the language of G-structures and intrinsic torsion. In particular, we argue that there is a $G_2 \subset SO(8)$ structure and obtain a simple set of differential conditions on the forms that comprise it. By examining the intrinsic torsion one can show that these conditions are necessary and sufficient for supersymmetry. We also give the Bianchi identity and equations of motion in this formalism and briefly discuss the issue of compact eight-manifolds. When the external manifold is $\mathbb{R}^{1,2}$, a simple inspection of the Einstein equations shows that one cannot have compact manifolds with flux, unless higher order corrections are included.

In section 4 we turn our attention to the physical interpretation of the differential conditions on the G_2-structure. We show how these may be interpreted as generalised calibration conditions for the M5-brane. We argue that the geometries that these equations describe correspond to "dyonic" M5-branes wrapped over associative three-cycles in a G_2-holonomy manifold. Moreover, we show that supersymmetric probe M5-branes saturate a calibration bound on their energy. We find that the M5-brane world-volume theory gives rise not only to an M5-brane type of calibration, but also one gets the M2-brane calibration "for free".

In section 5 we specialise our discussion to the case of "pure" M5-branes (that is, with no electric flux) wrapped on associative and special Lagrangian (SLAG) three-cycles. We recover the results for wrapped NS5-branes in type IIA theory [1] in the special case that the vector constructed as a spinor bilinear is Killing so that one can dimensionally reduce along this direction. We also comment on the relationship of our approach with the work of [21]. In particular, we give the supersymmetry constraints and the non-linear PDEs (following from the Bianchi identity) that one must solve to find solutions describing M5-branes wrapped over associative and SLAG three-cycles. Furthermore, we discuss how our approach may be extended straightforwardly to obtain a similar description of five-branes wrapped on other calibrated cycles.

In section 6 we discuss the case in which the internal (magnetic) G-flux is swithed off. In this case our equations simplify drastically and we are able to give the most general solution. In particular, we show that all AdS_3 solutions may be viewed as AdS_4 solutions, foliated by copies of AdS_3, with a weak G_2-holonomy manifold as internal space. We show how the compactifications of [15, 18, 20] are recovered in a degenerate limit in which the internal spinor becomes chiral and, therefore, the G_2-structure becomes a $Spin(7)$-structure.

As illustration of our formalism in section 7 we give some explicit examples. We easily recover the dyonic M-brane solution of [22]. This solution describes a 1/2-BPS M5/M2 bound state and serves as a simple example of the essential features of our geometries. We discuss also the relevance of our work to the recent "dielectric flow" solutions of [23, 24, 25]. These in fact also lie within our class of geometries. We present a class of singular solutions based on G_2-holonomy manifolds, where the M5-brane is completely smeared over the G_2-manifold.

Appendix A gives a discussion of G_2-structures. Appendix B includes a brief discussion of the Hamiltonian formulation of the M5-brane theory. Appendix C contains some relations useful in the main text.

2 M-theory on eight-manifolds

In this section we begin the analysis of eight-dimensional warped compactifications of M-theory. After summarising the *status quo* regarding the M2-brane-like solutions of [15, 18, 20], we then go on to describe how one extends the analysis to allow for more general supersymmetric solutions with fluxes.

The fields of eleven-dimensional supergravity consist of a metric $\hat{g}_{MN}$, a three-form potential C with field strength $G = \mathrm{d}C$, and a gravitino ψ_M. Supersymmetric backgrounds are those for which the gravitino vanishes and there is at least one solution to the equation

$$\delta\psi_M = \hat{\nabla}_M \eta - \frac{1}{288}\left(G_{NPQR}\hat{\Gamma}^{NPQR}{}_M - 8G_{MNPQ}\hat{\Gamma}^{NPQ}\right)\eta = 0 \ . \tag{2.1}$$

Here η is a spinor of $Spin(1,10)$, and $\hat{\Gamma}_M$ form a representation of the eleven-dimensional Clifford algebra, $\{\hat{\Gamma}_M, \hat{\Gamma}_N\} = 2\hat{g}_{MN}$. We take the spacetime signature to be $(-,+,\ldots,+)$, so that one may take $\hat{\Gamma}_M$ to be hermitian for $M \neq 0$ and anti-hermitian for $M = 0$. Geometrically, (2.1) is a parallel transport equation for a generalised connection, taking values in the full Clifford algebra, whose holonomy lies in $SL(32,\mathbb{R})$ [26]. In our conventions the equations of motion are

$$\begin{aligned} \hat{R}_{MN} - \frac{1}{12}(G_{MPQR}\hat{G}_N{}^{PQR} - \frac{1}{12}\hat{g}_{MN}G_{PQRS}\hat{G}^{PQRS}) &= 0 \qquad (2.2) \\ \mathrm{d}\,\hat{*}\,G + \frac{1}{2}G\wedge G &= 0 \ . \qquad (2.3) \end{aligned}$$

One also has the Bianchi identity $\mathrm{d}G = 0$. Generically the field equations (2.2) and (2.3) receive higher order corrections. In particular, the latter equation has a contribution X_8 on the right hand side, where

$$X_8 \;=\; -\frac{(2\pi)^2}{192}\left(p_1^2 - 4p_2\right) \ . \tag{2.4}$$

Here p_i is the i^{th} Pontryagin form, and we have set the M2-brane tension equal to one.

We will consider supersymmetric geometries with Poincaré or AdS invariance in three external dimensions. Thus a general such ansatz for the metric is of the form

$$d\hat{s}^2_{11} \;=\; \mathrm{e}^{2\Delta}(ds_3^2 + g_{mn}dx^m dx^n) \tag{2.5}$$

and for the G-field we take the maximally symmetric ansatz

$$\begin{aligned} G_{\mu\nu\rho m} &= \epsilon_{\mu\nu\rho}g_m \\ G_{mnpq} & \quad \text{arbitrary} \ , \end{aligned} \tag{2.6}$$

where here, and henceforth, Greek indices run over $0, 1, 2$ and Latin indices run over $3, \ldots 10$ – that is, over the internal manifold. We adopt the standard realisation of the eleven-dimensional Clifford algebra $\mathrm{Cliff}^{\mathrm{even}}(\mathbb{R}^{1,10}) \simeq \mathrm{Mat}(32, \mathbb{R}) \simeq \mathrm{Cliff}\,(\mathbb{R}^{1,2}) \otimes \mathrm{Cliff}\,(\mathbb{R}^{0,8})$, namely

$$\begin{aligned}\hat{\Gamma}_\mu &= \mathrm{e}^{\Delta}(\gamma_\mu \otimes \gamma_9) \\ \hat{\Gamma}_m &= \mathrm{e}^{\Delta}(\mathbb{1} \otimes \gamma_m)\end{aligned} \tag{2.7}$$

A convenient explicit representation of the three-dimensional Clifford algebra is given by $\gamma_0 = i\sigma_1, \gamma_1 = \sigma_2, \gamma_2 = \sigma_3$, where $\{\sigma_k \mid k = 1, 2, 3\}$ are the Pauli matrices. The eight-dimensional gamma-matrices are 16×16 real, symmetric matrices. We have also $\gamma_9^2 = \mathbb{1}$. An eleven-dimensional spinor η is likewise decomposed into three and eight-dimensional spinors as

$$\eta \;=\; \psi \otimes \xi \;. \tag{2.8}$$

The Majorana condition in eleven dimensions then imposes the following reality constraints:

$$\psi^* = \gamma_2 \psi \;, \qquad\qquad \xi^* = \xi \;. \tag{2.9}$$

Thus ψ has two real components, and ξ has *sixteen* real components. The supersymmetry equation of interest (2.1) may now be decomposed into two parts

$$\begin{aligned}\delta\psi_\mu \;&=\; \nabla_\mu \eta + \frac{1}{6}\mathrm{e}^{-3\Delta}(\gamma_\mu \otimes g_m \gamma^m)\eta - \frac{1}{2}(\gamma_\mu \otimes \partial_m \Delta \gamma^m \gamma_9)\eta \\ &\quad - \frac{1}{288}\mathrm{e}^{-3\Delta}(\gamma_\mu \otimes G_{npqr}\gamma^{npqr}\gamma_9)\eta \;=\; 0 \end{aligned} \tag{2.10}$$

$$\begin{aligned}\delta\psi_m \;&=\; \nabla_m \eta + \frac{1}{2}(\mathbb{1} \otimes \gamma_m{}^n \partial_n \Delta)\eta + \frac{1}{12}\mathrm{e}^{-3\Delta}(\mathbb{1} \otimes \gamma_m{}^n g_n \gamma_9)\eta - \frac{1}{6}\mathrm{e}^{-3\Delta}(\mathbb{1} \otimes \gamma_9) g_m \eta \\ &\quad - \frac{1}{288}\mathrm{e}^{-3\Delta}\Big[(\mathbb{1} \otimes G_{pqrs}\gamma^{pqrs}{}_m) - 8(\mathbb{1} \otimes G_{mpqr}\gamma^{pqr})\Big]\eta \;=\; 0 \end{aligned} \tag{2.11}$$

which we refer to as the external and internal equations, respectively.

In the rest of this section we will assume, as in [15, 18], that the internal spinor is chiral. We will briefly review the consequences of this restriction, before lifting it in the rest of the paper. If ξ is chiral, without loss of generality, one may take $\gamma_9 \xi = \xi$. Requiring that $\nabla_\mu \psi = 0$ in (2.11) then implies

$$\left[-\frac{1}{288}\Delta_B^{3/2} G_{mpqr}\gamma^{mpqr} + \frac{1}{6}\Delta_B^{3/2} g_m \gamma^m + \frac{1}{4}\gamma^m \partial_m \log \Delta_B\right]\xi = 0 \tag{2.12}$$

where, for easier comparison with [15, 18], we have defined the warp factor $\Delta = -\frac{1}{2}\log \Delta_B$. Projecting this equation onto its positive and negative chirality parts[1] we obtain

$$g_m = \partial_m \Delta_B^{-3/2} \;, \qquad\qquad G_{mpqr}\gamma^{mpqr}\xi = 0 \;. \tag{2.13}$$

[1] Notice that this projection simplifies somewhat the analysis in the original papers [15, 18].

Upon rescaling the spinor and the internal metric as $\xi = \Delta_B^{-1/4}\tilde{\xi}$ and $g_{mn} = \Delta_B^{3/2}\tilde{g}_{mn}$ respectively, the relations (2.13) allow one to simplify the internal part of the gravitino equation, yielding

$$\tilde{\nabla}_m\tilde{\xi} + \frac{1}{24}\Delta_B^{-3/4}G_{mpqr}\tilde{\gamma}^{pqr}\tilde{\xi} \;=\; 0\;. \tag{2.14}$$

One again notes that the two terms in (2.14) have opposite chirality, and must therefore vanish separately. In particular it follows that the metric $\tilde{g}_{mn}$ has $Spin(7)$ holonomy and the internal flux satisfies

$$G_{mnpq}\gamma^{npq}\xi \;=\; 0\;, \tag{2.15}$$

implying that some, but not all, of the $Spin(7)$ irreducible components of the flux must vanish. Recall that on manifolds with $Spin(7)$ structure four-forms may be decomposed into four irreducible components $\mathbf{70} \to \mathbf{35} + \mathbf{27} + \mathbf{7} + \mathbf{1}$ under $SO(8) \mapsto Spin(7)$ (see e.g. [27]). A convenient way to understand the condition (2.15) is to recast it into a tensorial equation [20]. Multiplying (2.15) on the left with $\xi^{\mathrm{T}}\gamma^r$ one obtains

$$T_{mn} \equiv \frac{1}{3!}G_{mpqr}\Psi^{pqr}{}_n \;=\; 0 \tag{2.16}$$

where Ψ is the Cayley four-form, characterising the $Spin(7)$ structure. A general two-index tensor decomposes into the $SO(8)$ irreducible representations $\mathbf{35} + \mathbf{28} + \mathbf{1}$, which, under $SO(8) \mapsto Spin(7)$, further reduces to $\mathbf{35} + \mathbf{21} + \mathbf{7} + \mathbf{1}$. However, given the representation content of the four-form G, T_{mn} must contain only the irreducible representations $\mathbf{35} + \mathbf{7} + \mathbf{1}$. One therefore concludes that only the $\mathbf{27}$ component of the internal flux is allowed. A characterisation of this representation may also be given as follows

$$G_{\mathbf{27}\,mnpq} \;=\; \frac{3}{2}G_{\mathbf{27}\,rs[mn}\Psi_{pq]}{}^{rs}\;. \tag{2.17}$$

In conclusion, the general solution takes the form

$$\begin{aligned} d\hat{s}^2_{11} &= H^{-2/3}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/3}\tilde{g}_{mn}dx^m dx^n \\ G &= \mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}(H^{-1}) + G_{\mathbf{27}} \end{aligned} \tag{2.18}$$

with the warp factor satisfying the equation

$$\tilde{\Box}H + \frac{1}{2}G_{\mathbf{27}}\wedge G_{\mathbf{27}} \;=\; X_8 \tag{2.19}$$

where $G_{\mathbf{27}}$ is harmonic, and we have not included any explicit space-filling M2-brane sources. Integrating (2.19) over a compact X gives

$$\frac{1}{2}\int_X \frac{G_{\mathbf{27}}}{2\pi}\wedge\frac{G_{\mathbf{27}}}{2\pi} = -\frac{1}{192}\int_X p_1^2 - 4p_4 \;=\; \frac{\chi(X)}{24}\;. \tag{2.20}$$

In general, existence of a nowhere vanishing section of a vector bundle requires that the Euler class of that bundle is zero. Thus existence of a nowhere vanishing positive/negative chirality spinor requires that $\chi(\mathbb{S}_\pm) = 0$, and it is this condition which gives the relation between the topological invariants in the last equality in (2.20) (see, for example, [28]). One then has compact solutions with flux provided the flux is quantised appropriately.

Note that these solutions describe M2-branes where the transverse space is a $Spin(7)$ holonomy manifold. Non-compact examples of such solutions may be found in [29] and [30]. Notice that $G_{\mathbf{27}}$ decouples from the supersymmetry conditions, but it does play a role in the equations of motion, providing the "transgressive" terms [29].

The present analysis is readily extended to cases with more supersymmetry. For example when ξ is a *complex* chiral spinor [15] we have two $Spin(7)$ structures of the same chirality or, equivalently, an $SU(4)$-structure. Repeating the same steps, one shows that the general solution is now of the form (2.18), (2.19) with $\tilde{g}_{mn}$ having $SU(4)$ holonomy. The magnetic flux which drops out of the supersymmetry equations is given by $G_{(2,2)}$ (that is, the four-form has two holomorphic and two anti-holomorphic indices with respect to the corresponding complex structure) where $G_{(2,2)}$ is also primitive, so that taking the wedge-product with the Kähler form gives zero. Again, these solutions are akin to M2-branes transverse to Calabi-Yau four-folds, and the role of the internal flux is to provide an additional source term in the equation for the warp-factor.

Generalisation

As we have summarised, imposing that the internal spinor be chiral leads to M2-brane-type solutions. However, there clearly should be another way to obtain a supersymmetric Minkowski$_3$ vacuum from M-theory: one may wrap space-filling M5-branes over a supersymmetric three-cycle in a special holonomy manifold. Such cycles are calibrated. In particular, one may wrap the M5-branes over an associative three-cycle in a G_2-holonomy manifold (times a circle) to obtain an $\mathcal{N} = 1$ vacuum, or a special Lagrangian cycle in a Calabi-Yau three-fold (times a two-torus) to obtain an $\mathcal{N} = 2$ vacuum. When one includes the back-reaction of the brane on the initial geometry, one no longer has a manifold of special holonomy, but rather some more general geometry with flux. However, the G_2 or $SU(3)$ structures still remain, respectively. Such manifolds admit two (respectively four) invariant Majorana-Weyl spinors, one (respectively two) of *each* chirality. Thus to describe more general supersymmetric solutions with fluxes one has to generalise the form of the internal spinor. We will also find that when one lifts the chirality assumption, one can find supersymmetric AdS_3 solutions, and we will present a simple class of examples in this paper.

From a more mathematical viewpoint, there is no reason to restrict the spinor to be chiral. The M-theory Killing spinor equation is geometrically a parallel transport equation for a supercovariant connection taking values in the Clifford algebra $\mathrm{Cliff}^{\mathrm{even}}(\mathbb{R}^{1,10}) \simeq \mathrm{Mat}(32,\mathbb{R})$. Indeed, in the three/eight split of the eleven-dimensional spinor η, the internal spinor ξ turns out to have 16 real components, *i.e.* it belongs to $Spin(8)_+ \oplus Spin(8)_-$. We are therefore led to consider an internal 16-dimensional spinor of indefinite chirality[2], which in general can be written in the following form

$$\eta \;\; = \;\; \mathrm{e}^{-\frac{\Delta}{2}} \psi \otimes (\xi_+ \oplus \xi_-) \tag{2.21}$$

where $\gamma_9 \xi_\pm = \pm \xi_\pm$ are real chiral spinors in eight dimensions, and ψ is a Majorana spinor in three dimensions. The factor $\mathrm{e}^{-\Delta/2}$ has been inserted for later convenience. For calculational convenience it is useful to introduce the non-chiral 16-dimensional spinors

$$\epsilon^+ \;\; = \;\; \frac{1}{\sqrt{2}}(\xi_+ + \xi_-) \tag{2.22}$$

and $\epsilon^- \equiv \gamma_9 \epsilon^+ = (\xi_+ - \xi_-)/\sqrt{2}$. The advantage of working with $\epsilon^\pm$, as opposed to $\xi_\pm$, is that the former will turn out to have constant norms, which, without loss of generality, we take to be unity, whereas the chiral spinors do not have this desirable property.

Since we wish to allow for AdS_3 compactifications in our analysis, we impose the following condition on the external spinor:

$$\nabla_\mu \psi + m \gamma_\mu \psi \;\; = \;\; 0 \; . \tag{2.23}$$

Writing the G-flux as

$$G = \mathrm{e}^{3\Delta}(F + \mathrm{vol}_3 \wedge f) \tag{2.24}$$

with F and f parameterising the magnetic and electric components, respectively, the supersymmetry conditions may be written in terms of $\epsilon^\pm$ as follows:

$$\nabla_m \epsilon^\pm \pm \frac{1}{24} F_{mpqr} \gamma^{pqr} \epsilon^\pm - \frac{1}{4} f_n \gamma^n{}_m \epsilon^\mp \pm m \, \gamma_m \epsilon^\mp \;\; = \;\; 0 \tag{2.25}$$

$$\frac{1}{2} \partial_m \Delta \gamma^m \epsilon^\pm \mp \frac{1}{288} F_{mpqr} \gamma^{mpqr} \epsilon^\pm - \frac{1}{6} f_m \gamma^m \epsilon^\mp \mp m \, \epsilon^\mp \;\; = \;\; 0 \; . \tag{2.26}$$

These equations are the starting point for our analysis.

[2] For a four-seven decomposition, it was noticed in [31], and more recently also in [32, 33], that in order to have non-trivial G-flux a generic spinor ansatz must be allowed.

3 Supersymmetry and the G_2-structure

In [1] it has been recognised that the notion of G-structures and their intrinsic torsion provides a powerful technique for studying Killing spinor equations in the presence of fluxes. A rigorous account of the mathematics may be found, for example, in [27]. For our purposes, a G-structure in d dimensions is a collection of locally defined G-invariant objects, each in some irreducible representation of the (spin cover of the) tangent space group $Spin(d) \supset G$. Notice that, *a priori*, our equations need only be defined in some open set, which is why we use the term G-structure in this local sense. When the objects in question extend globally over the whole manifold one has a G-structure in the stricter mathematical sense that the principal frame bundle admits a sub-bundle with fibre G. Of course, there may be topological obstructions, and indeed the structure may break down, for example at horizons.

The way that intrinsic torsion enters into the Killing spinor equations is via the fluxes. Exploiting this, one can study a supersymmetric geometry by extracting from the supersymmetry conditions the differential constraints on a set of forms that comprise the structure. These forms may be constructed as spinorial bilinears. The intrinsic torsion is an element of $\Lambda^1 \otimes g^\perp$ (see, for example, [7] or [5] for a brief review), which may be decomposed into irreducible G-modultes, denoted $\mathcal{W}_i$ in this paper. The manifold will have G-holonomy only when all the components vanish.

In the following we apply these methods to the case at hand, showing that one in general has a G_2-structure on the internal eight-manifold. It is also important to establish what other conditions must be imposed on the structure for it to correspond to a solution of the supergravity theory. We address this issue towards the end of the section.

We can construct explicitly a one-form, a three-form, and two four-forms as bilinears in the spinors

$$\begin{aligned}
\bar{K}_m &= \xi_+^{\mathrm{T}}\gamma_m\xi_- \\
\bar{\phi}_{mnp} &= \xi_+^{\mathrm{T}}\gamma_{mnp}\xi_- \\
\bar{\Psi}^\pm_{mnpr} &= \xi_\pm^{\mathrm{T}}\gamma_{mnpr}\xi_\pm \ .
\end{aligned} \tag{3.1}$$

In the calculations it is useful to re-express these in terms of the $\epsilon^\pm$ spinors, and it is also useful to define the following auxiliary bilinear

$$Y_{mnpr} = \epsilon^{\pm\mathrm{T}}\gamma_{mnpr}\epsilon^\pm \ . \tag{3.2}$$

Notice that, for a generic Clifford connection, the corresponding Killing spinors are not in general orthonormal, in contrast to the case of a connection on the $Spin(d)$ bundle [7]. In particular, we have that, using (2.25), $\nabla(\epsilon^{+\mathrm{T}}\epsilon^{+}) = \nabla(\epsilon^{-\mathrm{T}}\epsilon^{-}) = 0$. Thus we can normalise the spinors so as to obey

$$||\epsilon^{+}||^2 = ||\epsilon^{-}||^2 = \frac{1}{2}\left(||\xi_{+}||^2 + ||\xi_{-}||^2\right) = 1 \ . \tag{3.3}$$

On the other hand $\nabla(\epsilon^{+\mathrm{T}}\epsilon^{-}) \neq 0$, and we parameterise this non-trivial function, which takes values in the interval $[-1, 1]$, as

$$\epsilon^{+\mathrm{T}}\epsilon^{-} \ = \ \frac{1}{2}\left(||\xi_{+}||^2 - ||\xi_{-}||^2\right) \equiv \sin\zeta \ . \tag{3.4}$$

It follows that the chiral spinors have norms $||\xi_{\pm}||^2 = 1 \pm \sin\zeta$, and in the limit $\sin\zeta \to \pm 1$ one of the two vanishes.

The stabiliser of each chiral spinor $\xi_{\pm}$ is $Spin(7)_{\pm}$, and their common subgroup is G_2. In order to discuss the supersymmetry conditions in terms of the G-structure it is convenient to introduce rescaled forms, defined as $\phi = (\cos\zeta)^{-1}\bar{\phi}$ and $K = (\cos\zeta)^{-1}\bar{K}$. These are canonically normalised, namely $||K||^2 = 1$, $||\phi||^2 = 7$, and define a $G_2 \subset SO(8)$ structure in eight dimensions. One can give an explicit expression for Y in terms of the other bilinears

$$Y \ = \ -i_K * \phi + \phi \wedge K \sin\zeta \ , \tag{3.5}$$

where here, and henceforth, $*$ denotes the Hodge dual on the internal eight-manifold. The forms are also subject to the constraint

$$i_K\phi \ = \ 0 \ . \tag{3.6}$$

Notice that ϕ defines a unique seven-dimensional metric via the equations

$$\begin{aligned} g^7_{ij} \ &= \ (\det b)^{-\frac{1}{9}}\, b_{ij} \ , \\ b_{ij} \ &= \ -\frac{1}{144}\epsilon^{m_1 \ldots m_7}\phi_{i m_1 m_2}\phi_{j m_3 m_4}\phi_{m_5 m_6 m_7} \end{aligned} \tag{3.7}$$

where $\epsilon^{1234567} = 1$, and we therefore have $g^7_{ij}K^j = 0$. The intrinsic torsion of the structure lives in the space $\Lambda^1 \otimes g_2^{\perp}$ where $g_2 \oplus g_2^{\perp} = so(8)$. The Lie algebra $so(8) \simeq \mathbf{28}$ decomposes as $\mathbf{28} \to 2(\mathbf{7}) + \mathbf{14}$, so the orthogonal complement of the g_2 algebra is given by $g_2^{\perp} = \mathbf{7} + \mathbf{7}$. The intrinsic torsion then decomposes into ten modules

$$\begin{gathered} T \in \Lambda^1 \otimes g_2^{\perp} = \bigoplus_{i=1}^{10} \mathcal{W}_i, \\ (\mathbf{1} + \mathbf{7}) \times (\mathbf{7} + \mathbf{7}) \to 2(\mathbf{1}) + 4(\mathbf{7}) + 2(\mathbf{14}) + 2(\mathbf{27}) \ . \end{gathered} \tag{3.8}$$

It turns out that the ten classes are determined by the exterior derivatives of the forms. These have the following decompositions into irreducible G_2 representations

$$\begin{aligned} \mathrm{d}K &\to \mathbf{7}'' + \mathbf{7}''' + \mathbf{14}' \\ \mathrm{d}\phi &\to \mathbf{1} + \mathbf{1}' + \mathbf{7} + \mathbf{7}' + \mathbf{27} + \mathbf{27}' \\ \mathrm{d} * \phi &\to \mathbf{1}' + \mathbf{7} + \mathbf{7}' + \mathbf{14} + \mathbf{27}' \ . \end{aligned} \tag{3.9}$$

Note that some representations appear more than once, and we have denoted different representations with different numbers of primes. In particular, the representations $\mathbf{1} + \mathbf{7} + \mathbf{14} + \mathbf{27}$ are those relevant to $\mathrm{d}_7\phi$ and $\mathrm{d}_7 *_7 \phi$ discussed in appendix A. Using the identities (A.14) - (A.16) one shows that $\partial_K\phi$ and $\partial_K *_7 \phi$ contain the same representations, denoted with $\mathbf{1}' + \mathbf{7}' + \mathbf{27}'$. Finally, $\mathrm{d}K = \alpha \wedge K + \beta$, with the one-form α corresponding to $\mathbf{7}'''$ and the two-form β to $\mathbf{7}'' + \mathbf{14}'$. Notice that we have an eight-manifold of G_2 holonomy if and only if $\mathrm{d}K = \mathrm{d}\phi = \mathrm{d} * \phi = 0$. Note also that K is Killing if and only if the representations $\mathbf{1}' + \mathbf{7}' + \mathbf{27}'$ vanish. This follows on noticing that the non-trivial components of the Lie derivative $\mathcal{L}_K g$ can be computed from $\mathcal{L}_K\phi = i_K\mathrm{d}\phi$ using equation (3.7).

We can proceed now to analyse the constraints imposed on the structure by the supersymmetry conditions. Rather than presenting all the details of the calculations, we shall instead present a simple illustrative computation. Consider, for instance, $\nabla_r\bar{K}_m$. Using the definition of $\bar{K}$ as a spinor bilinear, together with the Killing spinor equations (2.25), after some straightforward gamma-algebra one calculates

$$\nabla_r\bar{K}_m = \frac{1}{12}F_{rijk}\epsilon^{+\mathrm{T}}\gamma^{ijk}{}_m\epsilon^+ - 2m\sin\zeta\, g_{rm} - \frac{1}{2}f^j\bar{\phi}_{jrm} \ . \tag{3.10}$$

Next, the first identity in appendix C (with the Clifford element $A = \gamma_{rm}$), can be used to compute the antisymmetric part of (3.10), obtaining equation (3.11) below. Similar calculations yield the following constraints on the G_2 structure:

$$\mathrm{d}(\mathrm{e}^{3\Delta}K\cos\zeta) = 0 \tag{3.11}$$

$$K \wedge \mathrm{d}(\mathrm{e}^{6\Delta}i_K * \phi) = 0 \tag{3.12}$$

$$\mathrm{e}^{-12\Delta}\mathrm{d}(\mathrm{e}^{12\Delta}\mathrm{vol}_7\cos\zeta) = -8m\,\mathrm{vol}_7 \wedge K\sin\zeta \tag{3.13}$$

$$\mathrm{d}\phi \wedge \phi\,\cos\zeta = 24m\,\mathrm{vol}_7 - 4 * \mathrm{d}\zeta + 2\cos\zeta\ * f \tag{3.14}$$

where $\mathrm{vol}_7 = \frac{1}{7}\phi \wedge i_K * \phi$. The electric and magnetic components of the flux are then determined as follows

$$\mathrm{e}^{-3\Delta}\mathrm{d}(\mathrm{e}^{3\Delta}\sin\zeta) = f - 4mK\cos\zeta \tag{3.15}$$

$$\mathrm{e}^{-6\Delta}\mathrm{d}(\mathrm{e}^{6\Delta}\phi\cos\zeta) = -\ * F + F\sin\zeta + 4m(i_K * \phi - \phi \wedge K\sin\zeta) \ . \tag{3.16}$$

As we will discuss more extensively in section 4, these equations can be interpreted as generalised calibrations for membranes or fivebranes wrapped on supersymmetric cycles (at least when $m = 0$). An important point to emphasize is that the conditions derived are also sufficient to ensure solutions to the Killing spinor equations. Notice that generically K *is not* a Killing vector. However, we see from (3.11) that it is in fact hypersurface orthogonal or, equivalently, defines an integrable almost product structure [7] which allows us to write the metric in the canonical form

$$d\hat{s}^2_{11} \;=\; \mathrm{e}^{2\Delta(x,y)}(ds^2_3 + g^7_{ij}(x,y)dx^i dx^j) + \frac{1}{\cos^2\zeta(x,y)}\mathrm{e}^{-4\Delta(x,y)}dy^2 \;. \tag{3.17}$$

The remaining conditions may be thought of as putting constraints on the seven-dimensional part of the G_2-structure. Consider, for example, equation (3.12). From this we read off immediately that the $\mathbf{14}$ representation is absent and the $\mathbf{7}$ is given by the Lee-form $W_4 = 18\,\mathrm{d}_7\Delta$. Likewise, equation (3.13) relates $\partial_y \log\sqrt{g^7}$ to $\partial_y\Delta$ and $\partial_y\zeta$, hence fixing the $\mathbf{1}'$ representation. Continuing, the rest of the equations may be used to determine *all* the components of the intrinsic torsion. One can thus construct a connection with non-trivial torsion which preserves the G_2 structure, and in particular preserves two spinors of opposite chirality, corresponding to solutions of the supersymmetry equations. For simplicity we will present some details of the calculation in the case of purely magnetic solutions in section 5.

The four-form flux is completely determined in terms of the structure by (3.15) and (3.16). In fact it is easy to show that there are no components which automatically drop out of the supersymmetry equations (2.25) and (2.26), in contrast to section 2. First let us decompose the four-form flux into $SO(7)$ irreducible representations:

$$F \;=\; F_4 + F_3\wedge K \;. \tag{3.18}$$

We thus want to check if there are G_2 irreducible components whose Clifford action $F_{mnpq}\gamma^{npq}$ annihilates both the spinors $\xi_\pm$, namely $F_{4\,mnpq}\gamma^{npq}\xi_\pm = F_{3\,mnp}\gamma^{np}\xi_\pm = 0$. This would imply that the following tensors vanish

$$\begin{aligned} \frac{1}{2!}F_{3\,mpq}\phi^{pq}{}_n &= 0 \\ \frac{1}{3!}F_{4\,mpqr}(*_7\phi)^{pqr}{}_n &= 0\;. \end{aligned} \tag{3.19}$$

As discussed in appendix A these tensors contain all the components of F_3 and F_4, which should therefore vanish identically. This situation is to be contrasted with the cases where we have spinor(s) of a fixed chirality, as recalled in section 2. Each spinor defines a $Spin(7)$ structure and the $\mathbf{27}$ component of the flux, with respect to that structure, is undetermined by the supersymmetry equations. The existence of two spinors with opposite chirality means that the associated $Spin(7)$ structures have opposite self-duality, and the undetermined flux should therefore simultaneously be in the $\mathbf{27}_+$ and $\mathbf{27}_-$, and hence is trivial.

All the non-zero components of the flux can be extracted from the conditions (3.11) - (3.16). As examples, and for later reference, let us give the expressions for the **1** and **7** components of F_3 (*cf.* appendix A)

$$\begin{aligned}\pi_1(F_3) &= \frac{2}{7}(\partial_K\zeta - 2m)\,\phi \\ \pi_7(F_3) &= -\frac{1}{2}\mathrm{e}^{-3\Delta}\mathrm{d}_7(\mathrm{e}^{3\Delta}\cos\zeta)\lrcorner\, i_K * \phi\end{aligned} \tag{3.20}$$

and of F_4

$$\begin{aligned}\pi_1(F_4) &= \frac{2}{7}\left(4m\sin\zeta - \mathrm{e}^{-3\Delta}\partial_K(\mathrm{e}^{3\Delta}\cos\zeta)\right)\, i_K * \phi \\ \pi_7(F_4) &= \frac{1}{2}\phi\wedge\mathrm{d}_7\zeta\ .\end{aligned} \tag{3.21}$$

A solution will also have to obey the equations of motion and Bianchi identity. Using the above expressions for the fluxes, it is straightforward to show that these reduce to the two equations

$$\mathrm{d}(\mathrm{e}^{3\Delta}F) = 0 \tag{3.22}$$

$$\mathrm{e}^{-6\Delta}\mathrm{d}(\mathrm{e}^{6\Delta} * f) + \frac{1}{2}F\wedge F = 0\ . \tag{3.23}$$

One can now show, using the results of [5], that the Einstein equation is automatically implied as an integrability condition for the supersymmetry conditions, once the G-field equation and Bianchi identity are imposed. It is useful to give explicitly the external part of the Einstein equation:

$$\mathrm{e}^{-9\Delta}\Box_8\mathrm{e}^{9\Delta} - \frac{3}{2}||F||^2 - 3||f||^2 + 72m^2 = 0\ . \tag{3.24}$$

One may use this to prove that, when $m = 0$, there are no compact solutions with electric and/or magnetic flux. Explicitly, one easily integrates (3.24) over the compact manifold X to get

$$\int_X \mathrm{e}^{9\Delta}||F||^2 + 2\int_X \mathrm{e}^{9\Delta}||f||^2 = 0 \tag{3.25}$$

which requires $F = 0$ and $f = 0$. This is a rather general property of supergravity theories [34]. The common lore to evade such "no-go theorems" is to appeal to higher derivative terms, such as the X_8 term mentioned in section 2, although these arguments typically neglect the corresponding terms in the Einstein equations. In our case, a non-zero X_8 seems to allow for the possibility of compact solutions[3]. One must then also satisfy

$$\int_X G_{\text{internal}} \wedge G_{\text{internal}} \quad = \quad 0 \tag{3.26}$$

which is implied by integrating equation (3.23). Here one uses the fact that X_8 integrates to zero. This is so because the existence of two linearly independent spinors of opposite chirality implies that $\chi(\mathbb{S}_\pm) = 0$. Equivalently, the vector K constructed from the spinors is nowhere vanishing, which implies that the Euler number of the eight-manifold is zero.

Comparing with the results reviewed in section 2 we see that allowing the internal spinor to be non-chiral has a led to a substantially enlarged number of possible geometries and fluxes. We emphasize the fact that AdS_3 solutions are not ruled out any more, and generically the internal manifold is not conformal to a $Spin(7)$ (or $SU(4)$) holonomy manifold. Note also that the function $\sin\zeta$ plays a role in our equations, and setting it to zero, or constant, rules out many supersymmetric geometries. In particular, from (3.15) it should be clear that $\sin\zeta$ is related to M2-brane charges, as we will see more explicitly in the next section.

4 Generalised calibrations and dyonic M-branes

In this section we show how the supersymmetry constraints on the G-structure are related to a generalised calibration condition for the M5-brane. For simplicity we will restrict our analysis to Minkowksi_3 backgounds, and hence we set $m = 0$ throughout this section. We argue that the supersymmetric geometries we have been describing so far may be thought of as being generated by M5-branes wrapped over an associative three-cycle in a G_2-holonomy manifold. An interesting twist to the story arises from the otherwise mysterious function $\sin\zeta$, introduced in the last section.

Recall that the M-theory fivebrane has a self-dual three-form field strength H propagating on its world-volume, which induces an M2-brane charge on the M5-brane via a Wess-Zumino coupling. The supergravity description of the M5-brane should account for this feature. Thus we expect "dyonic" backgrounds – that is, solutions with non-trivial electric *and* magnetic fluxes. Placing a dyonic M-brane probe in its corresponding background should not then break any further supersymmetry, and in particular a generalised calibration condition for such a probe should exist. We will find that all of the supersymmetry equations (except for

[3]Equation (3.25) receives a correction proportional to $\int_X \mathrm{e}^{3\Delta} \sin\zeta X_8$.

one) may be interpreted as generalised calibration conditions for a probe M5-brane in our background. For example, (3.16) is the generalisation of the associative calibration $\mathrm{d}\phi = 0$ in G_2-holonomy manifolds to dyonic M5-branes in warped backgrounds with flux.

Supersymmetric probes should saturate a generalised calibration bound which minimises their energy. In [35] a calibration bound for the M5-brane was derived. Although some comments were made about general backgrounds the computation there was for a flat space background with zero G-flux. It is easy to extend their analysis to the case of non-zero G-flux, by taking into account the Wess-Zumino terms. In appendix B we use the Hamiltonian formalism of [36] to obtain an expression for the energy of a class of static M5-branes with non-zero background G-flux and world-volume three-form H. This formula may then be used to show that supersymmetric branes are calibrated and saturate a bound on the energy.

The very alert reader may notice an obstacle in carrying out the above program. The calibration bound derived in [35] requires the existence of a time-like Killing vector which in turn one uses to define the energy in a Hamiltonian formulation. Moreover, such a vector should arise as a spinor bilinear. However, the supersymmetric geometries we are considering belong to the "null" class, namely the stabiliser of the spinor η (for any choice of ψ) is $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ and the vector one constructs from it is a null vector [37, 5]. As discussed in [5], in this case the interpretation of the supersymmetry conditions as calibration conditions is less clear. However, by some sleight of hand, we may still use the static formulation of the M5-brane. The key to this is simply that we in fact have *two* linearly independent null spinors, from which we may construct a time-like Killing vector.

As discussed in appendix B, an M5-brane probe will be supersymmetric if, and only if,

$$\mathcal{P}_- \eta \;=\; 0 \tag{4.1}$$

where $\mathcal{P}_-$ is a κ-symmetry projector, and η is the eleven-dimensional supersymmetry parameter. We have two linearly independent null spinors, $\eta_\lambda = \sqrt{2}\mathrm{e}^{-\Delta/2}\psi_\lambda \otimes \epsilon^+$, where ψ_λ, for $\lambda = 1, 2$, are two linearly independent constant spinors on $\mathbb{R}^{1,2}$. With an appropriate choice of ψ_λ, the vectors one constructs from these spinors are $\partial/\partial t \pm \partial/\partial X_1$. Both vectors are null, but their sum $2k = 2\partial/\partial t$ is time-like. Thus we are led to consider the following Bogomol'nyi-type bound:

$$\sum_{\lambda=1,2} ||\mathcal{P}_- \eta_\lambda||^2 \;=\; \sum_{\lambda=1,2} \frac{1}{2}\eta_\lambda^\dagger \mathcal{P}_- \eta_\lambda \;\geq\; 0\;. \tag{4.2}$$

One then rewrites this bound in terms of the energy. From appendix B we have

$$E \;=\; T_{M_5}\left(\mathcal{C}_0 + \mathrm{e}^{\Delta} L_{DBI}\right) \tag{4.3}$$

where T_{M_5} is the M5-brane tension, $\mathcal{C}_0$ is the contribution of a Wess-Zumino-like term to the energy, and L_{DBI} is a Dirac-Born-Infeld action (*cf.* appendix B). The bound may therefore be written

$$\mathrm{e}^{\Delta} L_{DBI}\mathrm{vol}_5 \quad \geq \quad \sum_{\lambda=1,2} j^*\nu_\lambda + j^*\chi_\lambda \wedge H \tag{4.4}$$

where we have defined the space-time forms

$$\begin{aligned} \nu_\lambda &= \frac{1}{(||\eta_1||^2+||\eta_2||^2)}\frac{1}{5!}\eta_\lambda^\dagger\hat{\Gamma}_{0\ M_1\ldots M_5}\eta_\lambda \mathrm{d}X^{M_1}\wedge\ldots\wedge\mathrm{d}X^{M_5} \\ \chi_\lambda &= -\frac{1}{(||\eta_1||^2+||\eta_2||^2)}\frac{1}{2!}\eta_\lambda^\dagger\hat{\Gamma}_{0\ MN}\eta_\lambda \mathrm{d}X^{M}\wedge\mathrm{d}X^{N} \end{aligned} \tag{4.5}$$

and j^* denotes a pull-back to the M5-brane world-volume. Using (4.3) we obtain a bound on the energy density $\mathcal{E} = E\mathrm{vol}_5$:

$$\frac{1}{T_{M_5}}\mathcal{E} \quad \geq \quad \sum_{\lambda=1,2} (j^*\nu_\lambda + j^*\chi_\lambda\wedge H) + \mathcal{C}_0\mathrm{vol}_5 \tag{4.6}$$

where

$$\mathcal{C}_0\mathrm{vol}_5 \quad = \quad i_k C_6 - \frac{1}{2} i_k C\wedge(C-2H) \tag{4.7}$$

and a pull-back is understood on the right-hand side of this equation.

Given a static supersymmetric background, a pair (Σ_5, H), with Σ_5 a 5-cycle and $H = h + j^*C$ a three-form on Σ_5 satisfying $\mathrm{d}H = j^*G$, is said to be *calibrated* if the bound (4.6) is saturated on all tangent planes of Σ_5. As we will show below, such a calibrated M5-brane worldspace then has minimal energy in its equivalence class $[(\Sigma_5, H)]$. Here, a pair (Σ_5', H') is in the same equivalence class as (Σ_5, H) if Σ_5 is homologous to Σ_5' via a six-chain B_6 (that is, $\partial B_6 = \Sigma_5 - \Sigma_5'$) over which H and H' extend to the same three-form, H, satisfying $\mathrm{d}H = j^*G$ on B_6. In fact, since C clearly extends (it is defined over all of space-time), it is enough to extend h over B_6 as a closed form. Now, by Poincaré duality on the M5-brane worldvolume, h defines a two-cycle $\Sigma_2 \subset \Sigma_5$, where $[\Sigma_2]$ is isomorphic to $[h]$ under Poincaré duality. h induces an M2-brane charge via the Wess-Zumino coupling (B.5), and thus Σ_2 may be thought of as the effective M2-brane worldspace, sitting inside the M5-brane.

To prove the calibration bound on the energy the forms χ_λ, ν_λ must obey suitable differential conditions. As we show below, these combine to give the general conditions on the forms defining the $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structures in eleven dimensions [5]. These read[4]

$$\begin{aligned} \mathrm{d}\chi_\lambda &= i_{\omega_\lambda} G \\ \mathrm{d}\nu_\lambda &= i_{\omega_\lambda} \hat{*} G - \chi_\lambda \wedge G \end{aligned} \tag{4.8}$$

where in our case the one-forms

$$\omega_\lambda = \frac{1}{(||\eta_1||^2 + ||\eta_2||^2)} \eta_\lambda^\dagger \hat{\Gamma}_{0\ M} \eta_\lambda \ \mathrm{d}X^M \tag{4.9}$$

are both null. With our choice of ψ_λ, we may take their sum $\omega_1 + \omega_2 = -\mathrm{d}t e^{2\Delta}$. The dual vector is then simply $(\omega_1 + \omega_2)^\# = \partial/\partial t = k$. A calibrated pair (Σ_5, H) therefore obeys

$$\begin{aligned} E(\Sigma_5, H) &= \int_{\Sigma_5} \sum_{\lambda=1,2} (\nu_\lambda + \chi_\lambda \wedge H) + i_k C_6 - \frac{1}{2} i_k C \wedge (C - 2H) \\ &= \int_{B_6} \sum_{\lambda=1,2} (\mathrm{d}\nu_\lambda + \mathrm{d}(\chi_\lambda \wedge H)) + \mathrm{d}(i_k C_6) - \frac{1}{2} \mathrm{d}\left(i_k C \wedge (C - 2H)\right) \\ &\quad + \int_{\Sigma_5'} \sum_{\lambda=1,2} (\nu_\lambda + \chi_\lambda \wedge H') + i_k C_6 - \frac{1}{2} i_k C \wedge (C - 2H') \\ &= 0 + \int_{\Sigma_5'} \sum_{\lambda=1,2} (\nu_\lambda + \chi_\lambda \wedge H') + i_k C_6 - \frac{1}{2} i_k C \wedge (C - 2H') \\ &\leq E(\Sigma_5', H') \end{aligned} \tag{4.10}$$

for any (Σ_5', H') in the same equivalence class as (Σ_5, H). Notice that we have used, for example, $\mathrm{d}(i_k C_6) = -i_k(\mathrm{d}C_6) = -i_k\left(\hat{*}G + \frac{1}{2} C \wedge G\right)$, in order to show that the integral over B_6 vanishes.

Note also that this result holds for all cases where it is possible to construct an appropriate time-like Killing vector from the Killing spinors (not necessarily as a bilinear), and thus it holds in particular for the entire "time-like" class of [5].

It is now a simple matter to relate this to the supersymmetry equations of the last section. Indeed, these are equivalent to (4.8) on rewriting them in terms of the quantities defined in the last section. In particular, we have that

$$\nu_1 + \nu_2 = -\mathrm{vol}_2 \wedge e^{6\Delta} \phi \cos\zeta - \mathrm{d}t \wedge e^{6\Delta} Y \tag{4.11}$$

$$\chi_1 + \chi_2 = +\mathrm{vol}_2 e^{3\Delta} \sin\zeta + \mathrm{d}t \wedge e^{3\Delta} K \cos\zeta \tag{4.12}$$

where $\mathrm{vol}_2 = \mathrm{d}X^1 \wedge \mathrm{d}X^2$ is the spatial two-volume. Thus we have

$$\mathrm{d}(\chi_1 + \chi_2) = \mathrm{vol}_2 \wedge \mathrm{d}\left(e^{3\Delta} \sin\zeta\right) - \mathrm{d}t \wedge \mathrm{d}\left(e^{3\Delta} K \cos\zeta\right) = i_k G = \mathrm{vol}_2 \wedge e^{3\Delta} f \tag{4.13}$$

[4]Our conventions differ from those of [5]. To rectify this, one can simply change the sign of the gamma matrices of [5]. This leads to some extra minus signs when using their results.

which shows the equivalence of (3.11) and (3.15) with the first equation in (4.8), and also

$$\begin{aligned}
\mathrm{d}(\nu_1+\nu_2) &= -\mathrm{vol}_2\wedge \mathrm{d}\left(\mathrm{e}^{6\Delta}\phi\cos\zeta\right)+\mathrm{d}t\wedge\mathrm{d}\left(\mathrm{e}^{6\Delta}Y\right)\\
&= i_k\hat{*}G-(\chi_1+\chi_2)\wedge G\\
&= -\mathrm{vol}_2\wedge\left(-\mathrm{e}^{6\Delta}*F+\mathrm{e}^{6\Delta}\sin\zeta F\right)-\mathrm{d}t\wedge\mathrm{e}^{6\Delta}\cos\zeta F\wedge K\ . \qquad (4.14)
\end{aligned}$$

This equation is clearly equivalent to the condition (3.16) together with

$$\mathrm{e}^{-6\Delta}\mathrm{d}\left(\mathrm{e}^{6\Delta}Y\right) = -F\wedge K\ \cos\zeta\ . \qquad (4.15)$$

On expanding the various terms, this can be shown to be equivalent to (3.12), (3.13), and the contraction of (3.14) with K. The relation (A.17) is useful for establishing this result.

Interestingly, (3.15) and (3.11) may also be derived from considerations of the M2-brane. In fact [5], the first condition in (4.8) is a generalised calibration condition for the M2-brane world-volume theory. The latter is more straightforward than the M5-brane theory as there is no form-field propagating on the M2-brane. Specifically, there is a simple Nambu-Goto term plus the Wess-Zumino electric coupling to the C-field. In this case, the energy is essentially just the action. Equation (3.15) is then a calibration condition for a space-filling M2-brane, whereas (3.11) is a calibration condition for an M2-brane wrapped over the K-direction. Notice that the remaining component of equation (3.14) did not enter the M5-brane calibration and in fact its eleven dimensional origin is in the equation (2.18) of [5] for the Killing one-form dk. We suspect that this should ultimately be related to a "calibration" for momentum carrying branes, or waves. It would be interesting to understand this point further.

5 M5 branes wrapped on associative and SLAG three-cycles

In this section we specialise our results to the case in which the electric component of the flux f is set to zero as well as the mass m. This situation corresponds to purely magnetic M5-branes wrapping three-cycles inside the transverse eight-manifold, with vanishing world-volume three-form field H. The geometries we consider are then of the form $\mathbb{R}^{1,2}\times M_8$, where M_8 generically admits a G_2 structure corresponding to $\mathcal{N}=1$ in the external Minkowski$_3$ space, or an $SU(3)$ structure corresponding to $\mathcal{N}=2$. We will also briefly discuss how one can easily extend these results to the case of M5-branes wrapping various four-cycles.

Associative calibration and $\mathcal{N}=1$

Specialising the equations of section 3 to the case at hand we get the following set of conditions on the G_2 structure:

$$\begin{aligned} \mathrm{d}(\mathrm{e}^{3\Delta}K) &= 0 && (5.1)\\ K\wedge \mathrm{d}(\mathrm{e}^{6\Delta}i_K * \phi) &= 0 && (5.2)\\ \mathrm{d}(\mathrm{e}^{12\Delta}\mathrm{vol}_7) &= 0 && (5.3)\\ \mathrm{d}\phi\wedge\phi &= 0 && (5.4)\\ \mathrm{e}^{-6\Delta}\mathrm{d}(\mathrm{e}^{6\Delta}\phi) &= - * F\ . && (5.5) \end{aligned}$$

The metric takes the following form

$$d\hat{s}^2_{11} \;=\; \mathrm{e}^{2\Delta}\left(ds^2(\mathbb{R}^{1,2}) + ds^2_7\right) + \mathrm{e}^{-4\Delta}dy^2\ . \qquad (5.6)$$

Notice that equation (5.3) is equivalent to $\partial_y \log\sqrt{g^7} = -12\,\partial_y\Delta$. Thus M5-branes wrapped on associative three-cycles give rise to an almost product structure geometry on the transverse eight-manifold which, at any fixed value of y, admits a G_2 structure of the type $\mathcal{W}_3\oplus\mathcal{W}_4$. Explicit solutions were presented in [38]. The close relation to the results of [1] is of course not accidental. Recall that K is generically not a Killing vector. However, when it is, one can Kaluza-Klein reduce along the y direction (identifying the dilaton as $\Phi=-3\Delta$) to get solutions of the type IIA theory, which describe NS5-branes wrapped on associative three-cycles [1]. Of course, if additional Killing vectors are present in specific solutions one can also reduce along those directions to obtain type II backgrounds which may contain RR fluxes in addition to the NS three-form.

Let us comment here on the relationship of our approach to the work initiated in [21] and expanded upon in a series of papers (see [39] for a review). The strategy in [21] is to write down an appropriate ansatz for the solution and then substitute this into the supersymmetry equations. Eventually one is left with a non-linear PDE for some metric functions which parameterise the ansatz (after imposing the Bianchi identity). It should be clear that using the techniques of G-structures one can easily recover the various constraints obtained using the approach of [21]. As a bonus we have in addition a physical interpretation of the constraints in terms of generalised calibrations[5] and, thanks to the machinery of intrinsic

[5]The relation of the work of [21] to generalised calibrations was noticed in [40, 41, 42, 43] . These papers consider a class of geometries where the internal space is Hermitian. This is related to the fact that these geometries describe M5 or M2 branes wrapped on holomorphic cycles.

torsion, we can apply the technique to more general cases which do not admit complex geometries. The work of [8], using the G-structure approach, recovers the $\mathcal{N}=1$ geometries of [44], corresponding to M5-branes wrapped on Kähler two-cycles in Calabi-Yau three-folds (times S^1), *i.e.* seven-manifolds with $SU(3)$-structure, after including the flux back-reaction. These in turn reduce in type IIA to the complex geometries first described in [45, 46] in the context of Type I/Heterotic, as can easily be checked using the equivalent formulation given in [7]. It is straightforward to see that a similar formulation exists for the $\mathcal{N}=2$ geometry of [21] corresponding to M5-branes wrapped on Kähler two-cycles in seven-manifolds with $SU(2)$-structure. In this case the supersymmetry conditions are exactly those discussed in the type IIA limit in section 6 of [7], with the transverse space $\mathbb{R}^2$ replaced by $\mathbb{R}^3$. Clearly, all the geometries discussed in [7] have a direct counterpart in M-theory as wrapped M5-branes.

Thus, imposing the Bianchi identity on G, we can write down the associative analogue of the non-linear equations of [21], which reads

$$\mathrm{d}_7\left[\mathrm{e}^{-6\Delta} *_7 \mathrm{d}_7(\mathrm{e}^{6\Delta}\phi)\right] + \partial_y^2(\mathrm{e}^{6\Delta} *_7 \phi) \;=\; 0 \tag{5.7}$$

where we have used the following expression for the G field

$$G \;=\; \partial_y(\mathrm{e}^{6\Delta} *_7 \phi) + \mathrm{e}^{-6\Delta} *_7 \mathrm{d}_7(\mathrm{e}^{6\Delta}\phi)\wedge \mathrm{d}y\;. \tag{5.8}$$

This is equivalent to the generalised calibration condition (5.5). Here we do not write down possible source terms. Note that equation (2.3) is automatically satisfied, with $G\wedge G$ and $\mathrm{d}\,\hat{*}\,G$ being separately zero (using (5.4), (5.5), respectively).

Next, as promised in section 3, we address more explicitly the issue of sufficiency of the conditions we have derived. This is ensured by the careful counting of irreducible components of the intrinsic torsion, but it is perhaps instructive to look also at the Killing spinor equations directly. The strategy is essentially to substitute our conditions back into the Killing spinor equations and check that they indeed admit solutions. Substituting the conditions (5.1) - (5.5) into the supersymmetry equations, we find that the external part (2.26) gives

$$-\,3\gamma^i\partial_i\Delta\,\xi_\mp + \frac{1}{12}F_{3\,ijk}\gamma^{ijk}\xi_\mp + \frac{1}{48}F_{4\;ijkl}\gamma^{ijkl}\xi_\pm - 3\mathrm{e}^{3\Delta}\partial_y\Delta\xi_\pm \;=\; 0 \tag{5.9}$$

while the internal part (2.25) gives

$$\nabla^{(7)}_i\xi_\pm + \frac{1}{8}F_{3\,ijk}\gamma^{jk}\xi_\pm + \frac{1}{4}\mathrm{e}^{3\Delta}\partial_y(g^7_{ij})\gamma^j\xi_\mp + \frac{1}{24}F_{4\,ijkl}\gamma^{jkl}\xi_\mp \;=\; 0 \tag{5.10}$$

$$\partial_y\xi_\pm + \frac{1}{4}e^i{}_{[a}\partial_y\, e_{b]i}\gamma^{ab}\xi_\pm \;=\; 0 \tag{5.11}$$

where here the indices run from 1 to 7 and $\nabla^{(7)}$ is the Levi-Civita connection constructed from g^7_{ij}. Next, we can simplify these equations using the fact that

$$\frac{1}{3!}F_{4\,iklm}(*_7\phi)^{klm}{}_j \;=\; -\mathrm{e}^{3\Delta}\partial_y(g^7_{ij}) \tag{5.12}$$

which can be computed from the expression for the flux (5.5) and the conditions (5.3), (5.4). Notice that, as discussed in appendix A, this means that the **7** representation in F_4 vanishes, as is implied by the second equation in (3.21). One can then show that the equations (5.9) and (5.10) reduce respectively to

$$\begin{aligned}\gamma^i\partial_i\Phi\,\xi + \frac{1}{12}F_{3\,ijk}\gamma^{ijk}\xi &= 0\\ \nabla^{(7)}_i\xi + \frac{1}{8}F_{3\,ijk}\gamma^{jk}\xi &= 0\end{aligned} \tag{5.13}$$

where ξ is the unique seven-dimensional spinor corresponding to $\xi_\pm$ in eight dimensions, and we have intentionally used the notation $\Phi = -3\Delta$ to demonstrate that the resulting equations are essentially the dilatino and gravitino equations of type IIA. Thus, by the results of [47, 48, 1], we indeed have a solution. Equation (5.11) is solved by taking the spinor to be y-independent and the ω_{yab} component of the spin-connection to be in the **14** of G_2: this simply corresponds to the standard choice of local frame where ϕ_{abc} has constant coefficients.

SLAG calibration and $\mathcal{N}=2$

Following the same line of reasoning as above, the equations describing M5-branes wrapping SLAG three-cycles in manifolds with an $SU(3)$-structure may almost be extrapolated from those pertaining to NS5-branes wrapping the same cycles obtained in [1]. By repeating the arguments of [1, 7] we have that doubling the amount of supersymmetry yields the presence of two G_2 structures, whose maximal common subgroup gives us an $SU(3)$-structure. One may then carry over the previous analysis by considering a Killing spinor of the type $\psi\otimes(\xi_+\oplus\xi_-)$ where ψ and $\xi_\pm$ are now *complex* spinors. Thus one can also think of $SU(3)$ as arising from two $SU(4)$-structures having opposite chiralities, each defined by a complex Weyl spinor. Notice that this geometry then belongs to both the "null" and "time-like" classes of [5], as $SU(3)$ embeds into $(Spin(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$ as well as into $SU(5)$.

In a real notation, we take our spinors to be

$$\eta^{(a)} \;=\; \mathrm{e}^{-\frac{\Delta}{2}}\psi^{(a)}\otimes(\xi^{(a)}_+\oplus\xi^{(a)}_-) \qquad a=1,2 \tag{5.14}$$

where each of the $\psi^{(a)}$ has two independent real components, thus corresponding to $\mathcal{N}=2$ in three dimensions. To realize the $SU(3)$ structure explicitly one can now construct additional bilinears. We refer to appendix B of [7] for details. Notice that we have two vectors, which in a local frame are given by $K^{(1)}=e^7$, $K^{(2)}=e^8$ and a two-form given by

$$J_{mn} \;=\; \epsilon^{+\mathrm{T}}_{(1)}\gamma_{mn}\epsilon^{+}_{(2)}, \tag{5.15}$$

where, as before, $\epsilon^{+}_{(a)}=(\xi^{(a)}_{+}+\xi^{(a)}_{-})/\sqrt{2}$ and in a local frame we have $J=e^{12}+e^{34}+e^{56}$. There are, of course, other bilinears that one can consider, but this is all we need. In fact, in terms of the associative three-forms, we have

$$\phi^{(a)} \;=\; J\wedge K^{(1)}\pm\mathrm{Im}\,\Omega \tag{5.16}$$

with $\Omega=(e^1+ie^2)\wedge(e^3+ie^4)\wedge(e^5+ie^6)$. The $SU(3)$ structure is given by $K^{(a)}$, J, Ω with the last two defining the structure in its canonical dimension of six, and $i_{K^{(a)}}J=i_{K^{(a)}}\Omega=0$.

Using the Killing spinor equations, after some calculations one arrives at the following set of conditions:

$$\begin{aligned}
\mathrm{d}(\mathrm{e}^{3\Delta}K^{(a)}) &= 0 && (5.17)\\
\mathrm{d}(\mathrm{e}^{3\Delta}J) &= 0 && (5.18)\\
K^{(1)}\wedge K^{(2)}\wedge\mathrm{d}(\mathrm{e}^{3\Delta}\mathrm{Re}\,\Omega) &= 0 && (5.19)\\
\mathrm{d}(\mathrm{Im}\,\Omega)\wedge\mathrm{Im}\,\Omega &= 0 && (5.20)\\
\mathrm{e}^{-6\Delta}\mathrm{d}(\mathrm{e}^{6\Delta}\mathrm{Im}\,\Omega) &= -\,*F\,. && (5.21)
\end{aligned}$$

The two vectors give rise to an almost product metric structure of the form

$$d\hat{s}^2_{11} \;=\; \mathrm{e}^{2\Delta}\left(ds^2(\mathbb{R}^{1,2})+ds^2_6\right)+\mathrm{e}^{-4\Delta}(dy^2+dz^2)\,. \tag{5.22}$$

As discussed in [7] the six-dimensional slices at fixed y and z have an $SU(3)$ structure with intrinsic torsion lying in the class $\mathcal{W}_2\oplus\mathcal{W}_4\oplus\mathcal{W}_5$ with warp-factor $6\,\mathrm{d}_6\Delta=-W_4=W_5$ (see [49, 7] for details about the intrinsic torsion of $SU(3)$ structures). Notice that these geometries are *not* Hermitian, which mirrors the fact that the M5-branes wrap SLAG three-cyles: equation (5.21) is the corresponding generalised calibration condition. Explicit solutions of this type were presented in [50]. The proof that the above equations are also sufficient to ensure the existence of four solutions to the Killing spinor equations amounts to the observation that with these one can construct two G_2 structures, as in the previous subsection, each of which corresponds to two Killing spinors with opposite chiralities.

As in the previous case, let us write down the equation implied by the Bianchi identity $\mathrm{d}G = 0$. This is the SLAG-3 analogue of the equations of [21] and reads

$$\mathrm{d}_6 \left[\mathrm{e}^{-9\Delta} *_6 \mathrm{d}_6 (\mathrm{e}^{6\Delta} \mathrm{Im}\, \Omega) \right] + \triangle_{yz} (\mathrm{e}^{3\Delta} \mathrm{Re}\, \Omega) \;\; = \;\; 0 \; . \tag{5.23}$$

Where $\triangle_{yz} = \partial_y^2 + \partial_z^2$ is the flat Laplacian in the transverse directions. To derive this equation we have made use of the conditions above to rewrite the flux in the following form

$$G = -\mathrm{e}^{-9\Delta} *_6 \mathrm{d}_6 (\mathrm{e}^{6\Delta} \mathrm{Im}\, \Omega) \wedge \mathrm{d}y \wedge \mathrm{d}z + \partial_z (\mathrm{e}^{3\Delta} \mathrm{Re}\, \Omega) \wedge \mathrm{d}y - \partial_y (\mathrm{e}^{3\Delta} \mathrm{Re}\, \Omega) \wedge \mathrm{d}z \; . \tag{5.24}$$

The G equation of motion (2.3) is again automatically satisfied.

More wrapped M5-branes

We have presented the general conditions on the geometry of M5-branes wrapped on associative and SLAG three-cycles, giving explicitly the non-linear PDE which results from imposing the Bianchi identity. M5-branes wrapped on Kähler two-cycles in Calabi-Yau two-folds and three-folds were described in [21, 44], and in [8] from the point of view of G-structures. Consulting the tables in [7] one realises that to complete the analysis of wrapped M5-branes one needs to consider four-cycles, yielding geometries of the type $\mathbb{R}^{1,1} \times M_9$. Clearly, it is straightforward to extend our analysis to cover all the remaining cases of M5-brane configurations wrapping supersymmetric cycles. These will essentially be the M-theory lifts of the conditions derived in [7] for all possible wrapped NS5-branes in the type IIA theory. For instance, we anticipate that, for static purely magnetic M5-branes, the flux is given by the generalised calibration condition

$$*_9 F \;\; = \;\; \mathrm{e}^{-6\Delta} \mathrm{d} (\mathrm{e}^{6\Delta} \Xi) \tag{5.25}$$

where Ξ is the relevant calibrating form. Thus when fivebranes wrap coassociative four-cycles in G_2-manifolds (times $\mathbb{T}^2$) we have $\Xi = *_7 \phi$; for Kähler four-cycles $\Xi = \frac{1}{2} J \wedge J$, and so on. Imposing the Bianchi identity gives the corresponding non-linear PDE. Notice that the "time-like" case in [5] covers the case of M5-branes wrapped on SLAG five-cycles in Calabi–Yau five-folds, and the resulting $SU(5)$ structure is described there in detail.

6 All purely electric solutions

In this section we discuss supersymmetric solutions with no internal components of the flux; namely, we set $F = 0$. Suppose first that $m \neq 0$. In this case, setting to zero the **1** and **7** components of the flux in (3.20) and (3.21) one can solve for K, f and Δ in terms of the function ζ, which one may take as a coordinate on the internal space, thus obtaining

$$\begin{aligned} K &= \frac{1}{2m}\mathrm{d}\zeta \\ f &= 3\sec\zeta\,\mathrm{d}\zeta \\ \mathrm{e}^{-\Delta} &= \cos\zeta\ . \end{aligned} \tag{6.1}$$

Using these, one finds that the supersymmetry conditions (3.11) - (3.16) reduce to the single equation

$$\mathrm{d}(\mathrm{e}^{3\Delta}\phi) \;=\; 4m\,\mathrm{e}^{4\Delta}i_K * \phi\ . \tag{6.2}$$

We can now define a conformally rescaled three-form $\tilde{\phi} = \mathrm{e}^{-3\Delta}\phi$, and the corresponding four-form and metric $\tilde{*}_7\tilde{\phi} = \mathrm{e}^{-4\Delta} *_7 \phi$ and $\tilde{g}_{mn} = \mathrm{e}^{-2\Delta}g_{mn}$, in terms of which equation (6.2) becomes

$$\mathrm{d}\tilde{\phi} \;=\; 4m\,\tilde{*}_7\tilde{\phi}\ . \tag{6.3}$$

The genereral solution is therefore given by

$$\begin{aligned} d\hat{s}^2_{11} &= \sec^2\zeta\left(ds^2_3(\mathrm{AdS}_3) + \frac{1}{4m^2}d\zeta^2\right) + d\tilde{s}^2_7 \\ G &= 3\mathrm{vol}_3 \wedge \sec^4\zeta\,\mathrm{d}\zeta \end{aligned} \tag{6.4}$$

where the seven-dimensional metric has weak G_2 holonomy, as dictated by (6.3). Notice that the G equation of motion (3.23) is automatically satisfied since $\mathrm{e}^{6\Delta} * f = 6m\,\tilde{\mathrm{vol}}_7$.

Compactifications of M-theory on weak G_2 manifolds were studied extensively in the 1980's (see, for example, [51]). The simplest example is the well-known $\mathrm{AdS}_4 \times S^7$ compactification, which is in fact maximally supersymmetric. Indeed, by a suitable change of coordinates, one can check that the solution (6.4) is of the form $\mathrm{AdS}_4 \times M_7$, where M_7 has weak G_2 holonomy. Setting $\sec\zeta = \cosh(2mr)$, the eleven-dimensional metric becomes

$$d\hat{s}^2_{11} \;=\; \cosh^2(2mr)\,ds^2(\mathrm{AdS}_3) + dr^2 + d\tilde{s}^2_7\ . \tag{6.5}$$

The four dimensional piece is the metric on AdS_4 with radius $l = 1/2m$, foliated with copies of AdS_3. The seven-metric $d\tilde{s}^2_7$ is a weak G_2 manifold, with metric normalised such that the Ricci tensor satisifies $\mathrm{Ric} = 6m^2\tilde{g}$.

Let us consider briefly the case when $m = 0$, so that the three-dimensional external space is flat $\mathbb{R}^{1,2}$. In this case, setting to zero the components of the internal flux (3.20) and

(3.21) implies that $\sin\zeta = \pm 1$. This is the limit in which one of the chiral spinors vanishes, leaving only the spinor of opposite chirality. The one-form K and the three-form ϕ are then identically zero, while there is only one independent four-form, Ψ^+ or Ψ^-. This defines a $Spin(7)$-structure in the usual way.

Although this case has been reviewed already in section 2 let us check that one correctly recovers it from our equations. In taking the limit one needs to be careful and consider only those equations obtained from spinor bilinears with four gamma matrices as these are the only equations which are non-trivial. In fact, as written, the conditions on the G_2 structure in section 3 are, naively, all trivial in the limit $\sin\zeta \to \pm 1$. This is just because they are written in G_2-invariant form, whereas in this limit there is no G_2 structure at all. An appropriate combination to consider is in fact equation (4.15) which we encountered in section 4. This reduces to the condition $\mathrm{d}(\mathrm{e}^{6\Delta}\Psi^\pm) = 0$ when $\sin\zeta \to \pm 1$, and determines the internal space to be conformal to a $Spin(7)$ manifold, as in section 2. The electric flux reduces accordingly to

$$G_{\text{electric}} \quad = \quad \pm \mathrm{vol}_3 \wedge \mathrm{d}(\mathrm{e}^{3\Delta}) \ . \tag{6.6}$$

Notice that in fact we have set to zero only the irreducible G_2 components **1** and **7** of the magnetic flux, and in principle some components are still allowed. Indeed, we recover the constraint on the magnetic flux from equation (3.10) which reduces to (2.16), requiring the flux to be in the $\mathbf{27}_+$ or $\mathbf{27}_-$ of $Spin(7)_\pm$, respectively.

Note that taking the $Spin(7)$ manifold to be a cone over a weak G_2 manifold and choosing the harmonic function $\mathrm{e}^{-3\Delta} = 1/(mr)^6$ one again obtains $\mathrm{AdS}_4 \times M_7$ solutions, although now AdS_4 is foliated by $\mathbb{R}^{1,2}$ horospheres, with metric

$$d\hat{s}^2_{11} \quad = \quad \mathrm{e}^{-4ym} ds^2(\mathbb{R}^{1,2}) + dy^2 + d\tilde{s}^2_7 \ . \tag{6.7}$$

To summarise, we have shown that warped supersymmetric solutions with purely electric flux are of only two types: the AdS_3 compactifications are in fact more naturally written as AdS_4 compactifications, foliated by copies of AdS_3, with the transverse space being weak G_2 holonomy. On the other hand in $\mathrm{Minkowski}_3$ compactifications the internal manifold must be conformal to a $Spin(7)$-holonomy manifold, as discussed in [18], with a single chiral spinor. Note that in the AdS_3 slicing case, the internal manifold M_8 provides a simple realisation of a space whose spinor "interpolates" between two spinors of opposite chirality.

7 Examples

In this section we demonstrate that the formalism we have developed may be useful for finding supersymmetric solutions. In particular, we easily recover the dyonic M-brane solution of [22]. This describes a 1/2-BPS M5/M2 bound state. We also argue that the recently discovered dyonic solutions of [24, 25] lie within this class, although we will not attempt to rederive these solutions here. Indeed, all of these solutions involve M5-branes with an M2-brane sitting inside. Finally, we present some simple solutions to the equations of section 5.

The dyonic M-brane

As explained in section 4, equation (3.16) is a generalised calibration condition for an M5-brane wrapping an associative three-cycle in a G_2 manifold. Presently we shall regard $\mathbb{T}^3 \oplus \mathbb{R}^4$ as a G_2 holonomy space[6] in which M5-branes wrap the three-torus $\mathbb{T}^3$. The remaining three unwrapped world-volume directions span a $\mathbb{R}^{1,2}$ Minkowski space, and we accordingly set $m = 0$. Thus, it is natural to write down the following simple metric ansatz describing such a wrapped brane:

$$d\hat{s}^2_{11} = \mathrm{e}^{2\Delta}\left(ds^2(\mathbb{R}^{1,2}) + A\ d\mathbf{u}.d\mathbf{u} + H\ d\mathbf{x}.d\mathbf{x}\right) . \tag{7.1}$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ are coordinates on the three-torus and $\mathbf{x} = (x_1, \ldots, x_5)$ are coordinates on the Euclidean five-space transverse to the M5-brane. At this point Δ, A and H are arbitrary functions on the internal eight-manifold. It is convenient to choose the following orthonormal frame for the latter

$$\begin{aligned} e^{2+i} &= A^{1/2}\mathrm{d}u_i \\ e^{5+\bar{a}} &= H^{1/2}\mathrm{d}x_{\bar{a}} \end{aligned} \tag{7.2}$$

where $i = 1, 2, 3$ and $\bar{a} = 1, \ldots, 5$. We then take the following G_2 structure on this eight-manifold

$$\begin{aligned} \phi &= -e^{345} - e^3 \wedge (e^{67} - e^{89}) - e^4 \wedge (e^{68} + e^{79}) - e^5 \wedge (e^{69} - e^{78}) \\ K &= e^{10} . \end{aligned} \tag{7.3}$$

Thus we have written $\mathbb{R}^8 = \mathrm{Im}\mathbb{H} \oplus \mathbb{H} \oplus \mathbb{R}$, where $\mathrm{Im}\mathbb{H} \oplus \mathbb{H}$ denotes the G_2-structure in its canonical dimension of seven, and $\mathbb{R}$ is the K-direction. This appears to break the invariance

[6] One may also consider the universal covering space $\mathbb{R}^7$, and wrap the brane over $\mathbb{R}^3$.

of the space transverse to the fivebrane under the five-dimensional Euclidean group, but in fact the solution we shall obtain respects this invariance – it is simply not manifest in the above notation.

We now solve the equations of section 3. Let us start with equation (3.11) for K which is solved by taking

$$\mathrm{e}^{3\Delta} H^{1/2} \cos\zeta \;=\; c_1 \tag{7.4}$$

where c_1 is a constant. Equation (3.12) gives the conditions

$$\begin{aligned} \mathrm{e}^{6\Delta} A H &= c_2^2 &(7.5)\\ \mathrm{d}\left(\mathrm{e}^{6\Delta} H^2\right) \wedge \mathrm{d}x_{12345} &= 0\ . &(7.6)\end{aligned}$$

One may solve the latter by taking $H = H(\mathbf{x})$, $\Delta = \Delta(\mathbf{x})$, which is natural as the solution should depend only on the coordinates transverse to the brane. Using these relations one computes

$$A \;=\; \left(\frac{c_2}{c_1}\right)^2 \cos^2\zeta\ . \tag{7.7}$$

Equation (3.13) is now automatically satisfied. One also computes

$$\mathrm{d}\left(\mathrm{e}^{6\Delta}\phi\cos\zeta\right) \;=\; \frac{c_2^3}{c_1}\mathrm{d}u_{123} \wedge \mathrm{d}\left(H^{-1}\cos^2\zeta\right) \tag{7.8}$$

which implies that $\mathrm{d}\phi \wedge \phi = 0$. Thus (3.14) gives

$$f \;=\; 2\sec\zeta\mathrm{d}\zeta \tag{7.9}$$

and inserting this into the definition of f (3.15) yields the following relation

$$H^{1/2}\tan\zeta \;=\; c_4\ . \tag{7.10}$$

We now set $c_2 = 1$ without loss of generality (by rescaling the coordinates u_i). The magnetic flux is obtained from (3.16) and reads

$$\mathrm{e}^{3\Delta} F \;=\; -c_4\,\mathrm{d}u_{123} \wedge \mathrm{d}(AH^{-1}) + c_1\,\tilde{*}_5 \mathrm{d}H \tag{7.11}$$

where $\tilde{*}_5$ denotes the Hodge dual with respect to the metric $d\mathbf{x}.d\mathbf{x}$. Thus the Bianchi identity (3.22) imposes

$$\tilde{\Box} H \;=\; 0\ . \tag{7.12}$$

That is, H is an harmonic function on the five flat transverse directions. One may easily check that the equation of motion (3.23) is identically satisfied. It appears that we now have a solution with two free parameters, but this is not so: one can remove c_1 by rescaling the coordinates $x_{\bar{a}}$. However, to recover[7] the solution of [22] we in fact need to set

$$c_4 = -\tan\xi \qquad\qquad c_1 = \cos\xi \ . \tag{7.13}$$

We can choose $c_4 = -\tan\xi$ for some angle ξ without loss of generality, and then setting $c_1 = \cos\xi$ corresponds to a specific choice of normalisation for the harmonic function. In conclusion, the metric takes the following form [22]

$$d\hat{s}^2_{11} = H^{-\frac{2}{3}}\left(\sin^2\xi + H\cos^2\xi\right)^{\frac{1}{3}}\left[ds^2(\mathbb{R}^{1,2}) + \frac{H}{\sin^2\xi + H\cos^2\xi}d\mathbf{u}.d\mathbf{u} + Hd\mathbf{x}.d\mathbf{x}\right] \ . \tag{7.14}$$

Notice that the function ζ is given by

$$\tan^2\zeta \;\; = \;\; \frac{1}{H}\tan^2\xi \tag{7.15}$$

and that the M2-brane and M5-brane are recovered in the limits $\xi \to \pi/2$ and $\xi \to 0$, respectively.

Note that the solution actually preserves 16 Killing spinors [22], as for the ordinary flat M5 brane. However, we have shown that the existence of a G_2 structure of the type we have been discussing is enough information to derive the full solution straightforwardly[8].

"Dielectric flow" solutions

The solutions recently constructed in [23, 24, 25] fall in our general class of "dyonic" solutions. Indeed they have a warped Minkowski$_3$ factor times an internal eight-manifold, and most importantly have non-trivial electric and magnetic fluxes turned on. Thus they may be thought of as some M5-brane distribution with induced space-filling M2-branes. Note that the solution of [24], in particular, admits sixteen supersymmetries – as many as the dyonic M-brane of [22]. In principle one should be able to recover these solutions in much the same way as we did for the standard dyonic M-brane solution above. All one has to do is to provide an ansatz for the three-form ϕ, or equivalently for the metric. Thus as shown in section 3 the fluxes are determined by the supersymmetry constraints, and one is left finally

[7] We disagree by factor of 6 with their expression for the flux. However, this appears to be a simple typographical error in taking the M-theory lift.

[8] By a circle reduction to type IIA, followed by T-duality, one obtains D-brane bound states in type IIB. The supersymmetry of the D5/D3 bound state [52] is discussed in [53].

with a non-linear PDE to be solved. Indeed, we have turned the problem into "*algebraic*" equations for the fluxes. While the solutions of [22, 24] preseve sixteen supercharges, and that of [25] eight, our equations describe the most general "dyonic" solution, which admits at least two Killing spinors with opposite chiralities. Thus these might be used to look for more general examples.

Smeared solutions

Here we show that one may derive a simple class of solutions to the equations of section 5. One can think of these as describing M5-branes wrapped on an associative three-cycle and completely smeared over a G_2-manifold. Unfortunately, these solutions are singular. Of course, many of the singularities of supergravity solutions are "resolved" in M-theory. It would be interesting to know if this were the case here.

One makes the ansatz

$$\phi \;=\; \mathrm{e}^{-3A(y)}\phi_0 \tag{7.16}$$

where ϕ_0 is the associative three-form for a G_2-holonomy manifold, and assume in addition $\Delta = \Delta(y)$. Thus, geometrically, we have a family of G_2-holonomy manifolds fibred over the y-direction. One finds that all of the differential equations for the structure are satisfied automatically, apart from one, which imposes

$$\mathrm{d}\left(\mathrm{e}^{12\Delta}\mathrm{vol}_7\right) = 0 \quad \Leftrightarrow \quad 12\Delta(y) = 7A(y) + c\ . \tag{7.17}$$

Notice that one may set $c = 0$ by redefining ϕ_0. Thus it remains to satisfy the Bianchi identity (5.7). This imposes

$$\mathrm{e}^{-6\Delta/7} \;=\; a + by \tag{7.18}$$

where a and b are constants. Thus the solution is

$$d\hat{s}^2_{11} \;=\; (a+by)^{-7/3}ds^2\left(\mathbb{R}^{1,2}\right) + (a+by)^{14/3}dy^2 + (a+by)^{5/3}ds^2(G_2) \tag{7.19}$$

where $ds^2(G_2)$ is any G_2-holonomy metric, and the G-flux is given by

$$G \;=\; b(*_7\phi)_0 \tag{7.20}$$

where $(*_7\phi)_0$ is the coassociative four-form on the G_2-manifold. Setting $b = 0$ gives $\mathbb{R}^{1,3}$ times a G_2-manifold. For $b \neq 0$ one may make a change of variables to write the metric as

$$d\hat{s}^2_{11} \;=\; dr^2 + r^{1/2}ds^2(G_2) + r^{-7/10}ds^2(\mathbb{R}^{1,2})\ . \tag{7.21}$$

Clearly this is singular at $r = 0$, although it is a perfectly regular supersymmetric solution everywhere else.

8 Outlook

In this paper we have studied the most general warped supersymmetric M-theory geometry of the type $M_3 \times M_8$, with the external space M_3 being either Minkowski$_3$ or AdS$_3$. The key ingredient which allowed us to extend the analysis of [15, 18, 20] was to allow for an internal Killing spinor of indefinite chirality. This is in fact the most general form compatible with the three-eight decomposition and the Majorana condition in eleven dimensions. The geometries were shown to admit a particular G_2-structure. This is a special case of the most general eleven-dimensional geometry of the "null" type, for which the corresponding structure is $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ [37, 5].

One of our motivations was to extend the analysis of [15, 18] to more general supersymmetric geometries. However, it is a rather general result that, in the case of Minkowski$_3$ vacua, ignoring higher order corrections or singularities rules out compact solutions. We have noticed that such corrections allow, in principle, compact geometries. It would be interesting to see if compact examples can be constructed.

We have found that the supersymmetry constraints also have a physical interpretation in terms of generalised calibrations [11, 5, 7]. In particular, we have shown most of the conditions arise as generalised calibrations for "dyonic" M5-branes, namely M5-branes with M2-brane charge induced on the world-volume by the three-form. We have shown that when there is a suitable time-like Killing vector, one can construct a Bogomol'nyi bound in the presence of background G-flux. This applies for the entire class of geometries considered here, and also to the "time-like" class of [5]. It would be interesting to understand more precisely the relation of generalised calibrations to the supersymmetry conditions in the general case of a $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, when the Killing vector is null.

The generality of our method implies that the conditions we have derived apply to a variety of situations. Thus, apart from "compactifcations", one can use the same results to describe non-compact geometries of physical interest. Typical examples are wrapped branes or intersecting branes. In these, as in all other cases, the supersymmetry constraints are relatively easy to implement, while ensuring that the Bianchi identity is satisfied is often a challenging task. One generically obtains non-linear PDEs whose explicit solutions are typically beyond reach. In any case, as illustrated in section 5, it should be clear that our approach is suitable for generalising the work of [21]. In particular, we have given the conditions and PDEs describing M5-branes wrapped on associative and SLAG three-cycles. In the last case one can show that the Calabi-Yau three-fold becomes a non-Hermitian manifold after allowing for the backreaction. This is to be contrasted with the case where M5-

branes (or NS5-branes in type II) wrap holomorphic cycles. Here the holomorphic structure of the manifold is preserved [45, 46, 21, 44, 7, 8].

Rewriting the Killing spinor equations in terms of the underlying G-structure provides an elegant organisational principle, and sheds light on the geometry of supersymmetric solutions. Namely, it turns out that the geometrical interpretation of the fluxes is given by the intrinsic torsion. Much physical insight comes from the interpretation of these in terms of branes and calibrations. On the other hand, the complication that arises from solving the equations implied by the Bianchi identitity seems to be a limitation on the method for finding new solutions. It is conceivable that using the geometrical and physical insights of our approach in combination with other techniques, such as those related to gauged supergravities, will improve the situation. Some ideas in this direction have already appeared (see, *e.g.*, [25]) and it would be interesting to elaborate on them further.

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A G_2-structures

A G_2-structure on a seven-dimensional manifold is specified by an associative three-form ϕ, which in a local frame may be written

$$\phi = e^{246} - e^{235} - e^{145} - e^{136} + e^{127} + e^{347} + e^{567} \ . \tag{A.1}$$

This defines uniquely a metric $g_7 = (e^1)^2 + \cdots + (e^7)^2$ and an orientation $\mathrm{vol}_7 = e^1 \wedge \cdots \wedge e^7$. We then have

$$*\phi = e^{1234} + e^{1256} + e^{3456} + e^{1357} - e^{1467} - e^{2367} - e^{2457} \ . \tag{A.2}$$

The adjoint representation of $SO(7)$ decomposes as $\mathbf{21}\to\mathbf{7}+\mathbf{14}$ where $\mathbf{14}$ is the adjoint representation of G_2. We therefore have $g_2^\perp \simeq \mathbf{7}$. The intrinsic torsion then decomposes into four modules [54]:

$$\begin{aligned} T \in \Lambda^1 \otimes g_2^\perp &= \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4, \\ \mathbf{7}\times\mathbf{7} &\to \mathbf{1}+\mathbf{14}+\mathbf{27}+\mathbf{7} \ . \end{aligned} \tag{A.3}$$

The components of T in each module $\mathcal{W}_i$ are encoded in terms of $\mathrm{d}\phi$ and $\mathrm{d} * \phi$ which decompose as

$$\begin{aligned} \mathrm{d}\phi \in \Lambda^4 &\cong \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \\ \mathbf{35} &\to \mathbf{1} + \mathbf{27} + \mathbf{7} \\ \mathrm{d} * \phi \in \Lambda^5 &\cong \mathcal{W}_2 \oplus \mathcal{W}_4 \\ \mathbf{21} &\to \mathbf{14} + \mathbf{7}\ . \end{aligned} \tag{A.4}$$

Note that the W_4 component in the $\mathbf{7}$ representation appears in both $\mathrm{d}\phi$ and $\mathrm{d} * \phi$. It is the Lee form, given by

$$W_4 \equiv \phi \lrcorner \, \mathrm{d}\phi = - * \phi \lrcorner \, \mathrm{d} * \phi. \tag{A.5}$$

The $\mathcal{W}_1$ component in the singlet representation can be written as

$$W_1 \equiv *(\phi \wedge \mathrm{d}\phi). \tag{A.6}$$

The remaining components of $\mathrm{d}\phi$ and $\mathrm{d}*\phi$ encode W_3 and W_2, respectively. The G_2 manifold has G_2 holonomy if and only if the intrinsic torsion vanishes, which is equivalent to $\mathrm{d}\phi = \mathrm{d} * \phi = 0$. Note that G_2-structures of the type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ are called integrable as one can introduce a G_2 Dolbeault cohomology [55].

On a manifold with a G_2-structure forms decompose into irreducible G_2 represenations. In particular, we have the following decompositions of the spaces of two-forms and three-forms:

$$\begin{aligned} \Lambda^2 &= \Lambda^2_7 \oplus \Lambda^2_{14} \\ \Lambda^3 &= \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^2_{27}\ . \end{aligned} \tag{A.7}$$

The Hodge dual spaces Λ^5 and Λ^4 decompose accordingly. For applications in the main part of the paper, it is useful to write down explicitly the decompositions of the three-forms and four-forms. A three-form $\Omega \in \Lambda^3$ is decomposed into G_2 irreducible representations as

$$\Omega \;=\; \pi_1(\Omega) + \pi_7(\Omega) + \pi_{27}(\Omega) \tag{A.8}$$

where the projections are given by

$$\begin{aligned} \pi_1(\Omega) &= \frac{1}{7}(\Omega \lrcorner \, \phi)\, \phi \\ \pi_7(\Omega) &= -\frac{1}{4}(\Omega \lrcorner * \phi) \lrcorner * \phi \\ \pi_{27}(\Omega)_{ijk} &= \frac{3}{2}\hat{Q}_{r[i}\phi^r{}_{jk]} \end{aligned} \tag{A.9}$$

and $\hat{Q}_{ij}$ is the traceless symmetric part of the tensor $Q_{ij} = \frac{1}{2!}\Omega_{ikr}\phi^{kr}{}_j$, namely

$$Q_{ij} \;=\; \frac{3}{7}(\Omega \lrcorner \, \phi) g_{ij} - \frac{1}{2}\phi_{ij}{}^k(\Omega \lrcorner * \phi)_k + \hat{Q}_{ij}\ . \tag{A.10}$$

Similarly, a four-form $\Xi \in \Lambda^4$ decomposes into G_2 irreducible represenations as

$$\Xi = \pi_1(\Xi) + \pi_7(\Xi) + \pi_{27}(\Xi) \tag{A.11}$$

where the preojections are given by

$$\begin{aligned}
\pi_1(\Xi) &= \frac{1}{7}(\Xi \lrcorner * \phi) * \phi \\
\pi_7(\Xi) &= -\frac{1}{4}(\phi \lrcorner \Xi) \wedge \phi \\
\pi_{27}(\Xi)_{ijkm} &= -2\hat{U}_{r[i} * \phi^r{}_{jkm]}
\end{aligned} \tag{A.12}$$

and $\hat{U}_{ij}$ is the traceless symmetric part of the tensor $U_{ij} = \frac{1}{3!}\Xi_{ikrm} * \phi^{krm}{}_j$, namely

$$U_{ij} = -\frac{4}{7}(\Xi \lrcorner * \phi) g_{ij} - \frac{1}{2}\phi_{ij}{}^k (\phi \lrcorner \Xi)_k + \hat{U}_{ij} \ . \tag{A.13}$$

Consider an infinitesimal variation of the associative three-form $\delta\phi$ and the induced variations of the metric δg_{ij}, and coassociative four-form $\delta * \phi$. Using the various identities obeyed by the G_2 structure, we obtain an explicit decomposition of $\delta\phi$, namely

$$\begin{aligned}
\pi_1(\delta\phi) &= \frac{3}{7}\delta \log\sqrt{g}\,\phi \\
\pi_7(\delta\phi) &= -\frac{1}{4}(\delta\phi \lrcorner * \phi) \lrcorner * \phi \\
\pi_{27}(\delta\phi)_{ijk} &= \frac{3}{2}\delta g_{r[i}\phi^r{}_{jk]} - \frac{3}{7}\delta \log\sqrt{g}\,\phi_{ijk} \ .
\end{aligned} \tag{A.14}$$

The irreducible components of $\delta * \phi$ are similarly given by

$$\begin{aligned}
\pi_1(\delta * \phi) &= \frac{4}{7}\delta \log\sqrt{g} * \phi \\
\pi_7(\delta * \phi) &= -\frac{1}{4}(\phi \lrcorner \delta * \phi) \wedge \phi \\
\pi_{27}(\delta * \phi)_{ijkm} &= 2\delta g_{r[i} * \phi^r{}_{jkm]} - \frac{4}{7}\delta \log\sqrt{g}\ * \phi_{ijkm} \ .
\end{aligned} \tag{A.15}$$

The following relations also hold

$$\begin{aligned}
\frac{1}{2!}\delta\phi_{(i|kr}\phi^{kr}{}_{j)} &= \delta g_{ij} + g_{ij}\delta \log\sqrt{g} \\
\frac{1}{3!}\delta * \phi_{(i|krm} * \phi^{krm}{}_{j)} &= -\delta g_{ij} - 2g_{ij}\delta \log\sqrt{g} \\
\phi \lrcorner \delta * \phi &= -\delta\phi \lrcorner * \phi \\
\pi_{27}(\delta * \phi) &= - * \pi_{27}(\delta\phi) \ .
\end{aligned} \tag{A.16}$$

Using these expressions one can derive the following useful equation

$$\delta * \phi = - * \delta\phi + \delta \log\sqrt{g} * \phi + \frac{1}{2}(\delta\phi \lrcorner * \phi) \wedge \phi \ . \tag{A.17}$$

B The M5-brane Hamiltonian

In this appendix we present a brief discussion of the Hamiltonian formulation of the M5-brane world-volume theory [36]. We use this to obtain an expression for the energy of a class of static M5-branes, which, in the main text, is shown to satisfy a Bogomol'nyi-type inequality. We also recall some details of the M5-brane κ-symmetry.

The action of the M5-brane is complicated by the presence of a self-dual three-form H which propagates on the world-volume. This requires one to introduce an auxilliary scalar field a (see [56] for a review), with a normalised "field strength" $v_i = \partial_i a/\sqrt{-(\partial a)^2}$. One then has an additional gauge invariance that one may use to gauge-fix a, at the expense of losing manifest spacetime covariance. However, the Hamiltonian treatment requires one to make a choice of time coordinate. Using the symmetries of the M5-brane action, one may then choose the "temporal gauge" $a = \sigma^0 = t$, where $\sigma^i = (\sigma^0, \sigma^a)$ are world-volume coordinates ($a = 1, \ldots, 5$), and the background spacetime is assumed to take the static form $d\hat{s}^2_{11} = -\mathrm{e}^{2\Delta} dt^2 + ds^2_{10}$. One then proceeds with the Hamiltonian approach [36], which yields the constraints

$$\begin{aligned} \tilde{P}^2 + T^2_{M_5} L^2_{DBI} &= 0 \\ \partial_a X^M \tilde{P}_M &= 0 \; . \end{aligned} \tag{B.1}$$

Here $X^M = (t, X^I)$ are the embedding coordinates, T_{M_5} is the M5-brane tension, $L_{DBI} = \sqrt{\det\left(\delta_a{}^b + H^{*b}_a\right)}$ is a Born-Infeld-like term, and

$$\tilde{P}^M = P^M + T_{M_5}\left(V^a \partial_a X^M - \mathcal{C}^M\right) \; . \tag{B.2}$$

We have that

$$V_c = \frac{1}{4} H^{*ab} H_{abc} \tag{B.3}$$

where the two-form $H^* = *_5 H$ is the *world-space* dual of H (the H_{0ab} components of H will not contribute to the energy) and the term $\mathcal{C}_M$ is a contribution from the Wess-Zumino couplings of the M5-brane, namely

$$\mathcal{C}_M = *_5 \left[i_M C_6 - \frac{1}{2} i_M C \wedge (C - 2H) \right] \tag{B.4}$$

where i_M denotes interior contraction with the vector field $\partial/\partial X^M$. Recall that the Wess-Zumino coupling of the M5-brane is given by

$$I_{WZ} = \int_W C_6 + \frac{1}{2} C \wedge H \tag{B.5}$$

where H is the three-form field strength on the five-brane, coupled to the background C-field

$$H = h + j^* C \ . \tag{B.6}$$

Here h is closed, and locally of the form $h = \mathrm{d}\, b$ for some two-form potential b. Notice that $\mathrm{d}H = j^* G$, where j is the M5-brane embedding map.

We may now use the Hamiltonian and momentum constraints (B.1) to obtain an expression for the energy density. We consider static configurations with $\tilde{P}^I = 0$. This is sufficient to satisfy the momentum constraint, but not in general necessary. One could extend our analysis to the general case (with more effort), but we will not do this here – the class of static configurations we consider will be general enough for our purposes. One defines the energy in the usual way

$$E = -P^M k^N \hat{g}_{MN} = -P_0 = \mathrm{e}^{2\Delta} P^0 \tag{B.7}$$

where k is the time-like Killing vector field $\partial/\partial t$. The Hamiltonian constraint now allows one to solve for the energy

$$E \ = \ T_{M_5} \left(\mathcal{C}_0 + \mathrm{e}^{\Delta} L_{DBI} \right) \ . \tag{B.8}$$

In addition to the energy, the other ingredient we use in the main text is the κ-symmetry and supersymmetry transformations of the fermions. These combine to give

$$\delta\theta \ = \ \mathcal{P}_+ \kappa + \eta \tag{B.9}$$

where $\mathcal{P}_\pm = \frac{1}{2}(1 \pm \tilde{\Gamma})$ are projector operators. η is the background supersymmetry $Spin(1,10)$ spinor, and $\tilde{\Gamma}$ is a traceless Hermitian product structure, that is, $\mathrm{tr}\ \tilde{\Gamma} = 0$, $\tilde{\Gamma}^2 = 1$, $\tilde{\Gamma}^\dagger = \tilde{\Gamma}$. Explicitly, we have

$$\tilde{\Gamma} \ = \ \frac{1}{L_{DBI}} \mathrm{e}^{-\Delta} \hat{\Gamma}_0 \left[V \cdot \tilde{\gamma} + \frac{1}{2} \tilde{\gamma}^{ab} H^*_{ab} + \frac{1}{5!} \tilde{\gamma}_{a_1 \ldots a_5} \epsilon^{a_1 \ldots a_5} \right] \tag{B.10}$$

where $\tilde{\gamma}^a$ are the pull-backs of the eleven-dimensional Clifford matrices to the M5-brane world-space. If we consider static configurations with a rest frame that has zero momentum, then $V_a = 0$. This is the form of the projector used in the main text. One can show [57] that the variation (B.9) vanishes if, and only if,

$$\mathcal{P}_- \eta \ = \ 0 \tag{B.11}$$

which therefore characterises bosonic supersymmetric configurations.

C Useful relations

Given the supersymmetry equations (2.26), and using the symmetry properties of the gamma matrices, one can derive some useful identities which we have used extensively in deriving our results. For the reader's convenience we list them here:

$$\frac{1}{288}F_{pqrs}\epsilon^{\pm\mathrm{T}}[\gamma^{pqrs},A]_-\epsilon^{\pm} \mp \frac{1}{2}\partial_m\Delta\epsilon^{\pm\mathrm{T}}[\gamma^m,A]_-\epsilon^{\pm} + m(\epsilon^{\mp\mathrm{T}}A\epsilon^{\pm} - \epsilon^{\pm\mathrm{T}}A\epsilon^{\mp}) \mp\frac{1}{6}f_m\epsilon^{\pm\mathrm{T}}A\gamma^m\epsilon^{\mp} \pm \frac{1}{6}f_m\epsilon^{\mp\mathrm{T}}\gamma^m A\epsilon^{\pm} = 0 \tag{C.1}$$

$$\frac{1}{288}F_{pqrs}\epsilon^{\pm\mathrm{T}}[\gamma^{pqrs},A]_+\epsilon^{\pm} \mp \frac{1}{2}\partial_m\Delta\epsilon^{\pm\mathrm{T}}[\gamma^m,A]_+\epsilon^{\pm} + m(\epsilon^{\mp\mathrm{T}}A\epsilon^{\pm} + \epsilon^{\pm\mathrm{T}}A\epsilon^{\mp}) \pm\frac{1}{6}f_m\epsilon^{\pm\mathrm{T}}A\gamma^m\epsilon^{\mp} \pm \frac{1}{6}f_m\epsilon^{\mp\mathrm{T}}\gamma^m A\epsilon^{\pm} = 0 \tag{C.2}$$

$$\frac{1}{288}F_{pqrs}\epsilon^{+\mathrm{T}}[\gamma^{pqrs},A]_\pm\epsilon^{-} - \frac{1}{2}\partial_m\Delta\epsilon^{+\mathrm{T}}[\gamma^m,A]_\mp\epsilon^{-} + m(\epsilon^{-\mathrm{T}}A\epsilon^{-} \pm \epsilon^{+\mathrm{T}}A\epsilon^{+}) \mp\frac{1}{6}f_m\epsilon^{+\mathrm{T}}A\gamma^m\epsilon^{+} + \frac{1}{6}f_m\epsilon^{-\mathrm{T}}\gamma^m A\epsilon^{-} = 0 \tag{C.3}$$

where $[\cdot,\cdot]_\pm$ refers to an anticommutator or commutator, and A is a general Clifford matrix.

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