An intuitionistic version of Ramsey Theorem and its use in Program Termination

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An intuitionistic version of Ramsey Theorem and its use in Program Termination

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Abstract. Ramsey Theorem for pairs is a fundamental result in combinatorics which cannot be intuitionistically proved. In this paper we present a new form of Ramsey Theorem for pairs we call H-closure Theorem. H-closure is a property of well-founded relations, intuitionistically provable, informative, and simple to use in intuitionistic proofs. Using our intuitionistic version of Ramsey Theorem we intuitionistically prove the Termination Theorem by Poldenski and Rybalchenko. This theorem concerns an algorithm inferring termination for while-programs, and was originally proved from the classical Ramsey Theorem, then intuitionistically, but using an intuitionistic version of Ramsey Theorem different from our one.

Keywords: Intuitionism, Ramsey Theorem, inductive definitions, termination of while-programs.

1 Introduction

In computer science deciding whether a program is terminating on a given input is one of the most studied topics. In general it is a famous undecidable problem, but in some particular case it can be solved. In [1] Podelski and Rybalchenko defined a condition on well-founded relations and they proved that it is equivalent to the termination of transition-based programs. From this result, called Termination Theorem, Cook, Podelski, and Rybalchenko extracted an algorithm taking in input an imperative program made with the instructions **while**, **if** and assignment, and able to decide in some case whether the program is terminating or not, and in some other cases leaving the question open. The authors used in their proof of the Termination Theorem Ramsey Theorem for pairs [2], from now on called just "Ramsey" for short. Ramsey is a classical result that cannot be intuitionistically proved: we refer to [3] for a detailed analysis of the minimal classical principle required to prove Ramsey. According to the Π^0_2-conservativity of Classical Analysis w.r.t. Intuitionistic Analysis [4], the proof of Termination Theorem hides some effective bounds for the while program which the theorem shows to terminate. Our long-term goal is to find them, by first turning the proof of Termination Theorem into an intuitionistic proof. For instance, by using this proof, we can characterize the class of the primitive recursive functions in term of Podelski and Rybalchenko Termination Theorem [5].

Our first step is to formulate a version of Ramsey which has a purely intuitionistic proof, that is, a proof which does not use Excluded Middle, nor Brouwer Thesis. Our version of Ramsey is informative, in the sense that it has no negation, while it has a disjunction. We say that a relation R is H-well-founded if the tree of all R-decreasing transitive sequences is well-founded (w.r.t. the inductive definition of well-foundedness). We express Ramsey as a property of well-founded relations, saying that H-well-founded relations are closed under finite unions. For short we will call this statement the *H-closure Theorem*. Thus, we are able to split the proof of Ramsey into two parts: the intuitionistic proof of the H-closure Theorem, followed by an "easy" (in the sense of the Reverse Mathematics, see [6]) classical proof of the equivalence between Ramsey and the H-closure Theorem.

The result closest to H-closure we could find is by Coquand [7]. Coquand, as Veldman and Bezem did before him [8], considers *almost full relations* (a kind of dual of H-closed relations) and proves that they are closed under finite *intersections*. Veldman and Bezem use Choice Axiom of type 0 (if $\forall x \in \mathbb{N}.\exists y \in \mathbb{N}.C(x, y)$, then $\exists f : \mathbb{N} \to \mathbb{N}.\forall x \in \mathbb{N}.C(x, f(x))$) and Brouwer's thesis. Coquand's proof, instead, is purely intuitionistic, and it may be used to give a purely intuitionistic proof of the Termination Theorem [9]. However, it is not evident what are the effective bounds hidden in Coquand's proof of Termination Theorem. If we compare H-closure with the Almost Full Theorem, in the most recent version by Coquand [7], we find no easy way to intuitionistically deduce one from the other, due to the use of de' Morgan laws to move from the definition of almost full to the definition of H-closure. H-closure is in a sense more similar to the original Ramsey theorem, because it was obtained from it with just one classical step, a contrapositive (see §2), while almost fullness requires one application of de' Morgan Law, followed by a contrapositive. We expect that H-closure, hiding one application less of de' Morgan laws, should be a version of Ramsey simpler to use in intuitionistic proofs and for extracting bounds.

Another motivation for our work is the following. In [10] Lee, Jones and Ber-Amram introduced the notion of size-change termination and they proved the Size-Change Termination Theorem which states that a first order functional program is terminating if and only if it satisfies a property which can be statically verified from the recursive definition of the program. Also in this proof the authors used Ramsey Theorem for pairs. By using Almost Full Theorem in [9] the authors provided an intuitionistic proof of it. In [11] there is a very different proof of it which used H-closure.

Our motivation for producing a new intuitionistic version of Ramsey is to provide a new intuitionistic proof of the termination theorems. We expect that, by analysing these new proofs, we will be able to extract effective bounds from the termination theorems, and possibly, from other concrete applications of Ramsey.

This is the plan of the paper, which is an expanded version of the conference paper [12]. In section 2 we present Ramsey Theorem for pairs and we informally introduce H-closure. In section 3 we formally define inductive well-foundedness and H-well-foundedness, whose main properties are stated in section 4. The goal of section 5 is to present what we call *Intuitionistic Nested Fan Theorem*, which is a part of the proof of the H-closure Theorem, as shown in section 6. In section 7 we intuitionistically prove the Termination Theorem. In section 8 we compare our result with the previous works along the same line and we draw some conclusions. Unless explicitly stated, our proofs use intuitionistic second order arithmetic, without Choice Axiom, Brouwer Thesis, Bar-Induction.

2 From Ramsey Theorem to H-closure

In this section we introduce and discuss a property of well-founded relations we call H-closure. Classically, H-closure is but a variant of Ramsey Theorem for pairs, and therefore it is classically provable. In the rest of the paper we will show that H-closure has an intutionistic proof, and that by using H-closure instead Ramsey Theorem, the Termination Theorem of Podelski and Rybalchenko [1] turns out to be intuitionistic provable.

We first recall the statement of Ramsey Theorem for pairs, just Ramsey for short. Assume given an infinite coloring over the edges of a complete graph with countably many nodes G; i.e. a partition of the edges of G into n-many sets. Then Ramsey says there is an infinite *homogeneous* set; i.e. there exists an infinite subset of the nodes X such that all the edges between any two different $x, y \in X$ have the same color k, for some $k < n$.

Assume $\{x_i : i \in \mathbb{N}\}$ is an injective enumeration of the elements of G. We arbitrarily represent a non-oriented edge between two points x_i, x_j in G with $j < i$ by the pair (i, j). Edges of G are not oriented, therefore the opposite edge from x_j to x_i is the same edge of G, and it is again represented with (i, j). Thus, a partition of edges in n sets $S_0, \ldots, S_{n-1}$ may be represented by a partition of the set $\{(x_i, x_j) : j < i\}$ into n binary relations $S_0, \ldots, S_{n-1}$. Therefore one possible formalization of Ramsey is the following.

Theorem 1 (Ramsey for pairs [2]). *Assume I is a set having some injective enumeration $I = \langle x_i : i \in \mathbb{N}\rangle$. Assume $S_0, \ldots, S_{n-1}$ are binary relations on I which are a partition of $\{(x_i, x_j) \in I \times I : j < i\}$, that is:*

1. *$S_0 \cup \cdots \cup S_{n-1} = \{(x_i, x_j) \in I \times I : j < i\}$*
2. *for all $k < h < n$: $S_k \cap S_h = \emptyset$.*

Then for some $k < n$ there exists a set $Y \subseteq \mathbb{N}$, such that: $\forall i, j \in Y.(j < i \implies x_i S_k x_j)$.

Then the set $X = \{x_i : i \in Y\}$ is the infinite homogeneous set for the graph.

In the statement above three assumptions may be dropped.

1. First of all, we may drop the assumption that $S_0, \ldots, S_{n-1}$ are pairwise disjoint. Suppose we do. Then, if we put $S'_i = S_i \setminus \bigcup\{S'_j : j < i\}$ for any $i < n$ we obtain a partition $S'_0, \ldots, S'_{n-1}$ of $\{(x_i, x_j) : j < i\}$. Therefore by

applying Theorem 1 to the coloring given by the relations S'_i for $i < n$, there exists a $k < n$ and an infinite $Y \subseteq \mathbb{N}$, such that $\forall i, j \in Y.(j < i \implies x_i S'_k x_j)$, and with more reason, $\forall i, j \in Y.(j < i \implies x_i S_k x_j)$.

2. Second, we may drop the assumption "the enumeration is injective" (in this case, $X = \{x_i : i \in Y\}$, may be a finite set). Assume we do. Then, if we set $S'_k = \{(i, j) : x_i S_k x_j\}$ for any $k < n$, we obtain n relations $S'_0, \ldots, S'_{n-1}$ on $\mathbb{N}$, whose union is the set $\{(i, j) \in \mathbb{N} \times \mathbb{N} : j < i\}$. Therefore, thanks to Theorem 1, there exists a $k < n$ and an infinite $Y \subseteq \mathbb{N}$, such that $\forall i, j \in Y.(j < i \implies i S'_k j)$, and with more reason, $\forall i, j \in Y.(j < i \implies x_i S_k x_j)$.
3. Third, we may drop the assumption that $\langle x_i : i \in \mathbb{N}\rangle$ is an enumeration of I. Suppose we do. Then, if we restrict $S_0, \ldots, S_{n-1}$ to $I' = \{x_i : i \in \mathbb{N}\}$, we obtain some binary relations $S'_0, \ldots, S'_{n-1}$ on I' such that $S'_0 \cup \ldots \cup S'_{n-1} = \{(x_i, x_j) \in I' \times I' : j < i\}$. Again, we conclude by Theorem 1 that there exists some $k < n$ and some infinite $Y \subseteq \mathbb{N}$ such that $\forall i, j \in Y.(j < i \implies x_i S'_k x_j)$, and with more reason, $\forall i, j \in Y.(j < i \implies x_i S_k x_j)$.

Summing up, we showed that, classically, we may restate Ramsey Theorem as follows:

For any infinite sequence $\langle x_i : i \in \mathbb{N}\rangle$ of elements of I, if $\forall i, j \in \mathbb{N}.(j < i \implies x_i(S_0 \cup \ldots \cup S_{n-1})x_j)$, then for some $k < n$ there is some infinite $Y \subseteq \mathbb{N}$, such that $\forall i, j \in Y.(j < i \implies x_i S_k x_j)$.

It is likely that even this statement cannot be intuitionistically proved, because Y is akin to an homogeneous set, and there is no effective way to produce homogeneous sets (see for instance [3]). By taking the contrapositive, we obtain the following corollary:

If for all $k < n$, all the sequences $\langle y_i : i \in \mathbb{N}\rangle$ such that $\forall i, j \in \mathbb{N}.(j < i \implies y_i S_k y_j)$ are finite, then all sequences $\langle x_i : i \in \mathbb{N}\rangle$ such that $\forall i, j \in \mathbb{N}.(j < i \implies x_i(S_0 \cup \ldots \cup S_{n-1})x_j)$ are finite.

It is immediate to check that, classically, this is yet another version of Ramsey. We call this property *classical H-closure*.

Given a relation S over I, let $H(S)$ be the set of all finite lists $\langle x_i : i < l\rangle$ of elements of I such that $j < i < l$ implies $x_i S x_j$. Then classical H-closure may be restated as follows: if $S_0, \ldots, S_{n-1}$ are binary relations over some set I, and $H(S_0), \ldots, H(S_{n-1})$ are sets of lists well-founded by one-step extension, then $H(S_0 \cup \ldots \cup S_{n-1})$ is a set of lists well-founded by one-step extension as well. Thus, classical H-closure is a property classically equivalent to Ramsey Theorem, but which deals with well-founded relations. In Proof Theory, there is plenty of examples of classical proofs of well-foundedness which are turned into intuitionistic proofs, and indeed from H-closure we will obtain an intuitionistic version of Ramsey.

There is a last step to be done. We call *intuitionistic H-closure*, or just H-closure for short, the statement obtained by replacing, in classical H-closure, the classical definition of well-foundedness (all decreasing sequences are finite) with the inductive definition of well-foundedness, which is customary in intuitionistic logic. We will recall the inductive definition of well-foundedness in §3.1: thus, for the formal definition of H-well-foundedness we have to wait until §3.3.

For us, the interest of an intutionistic proof of H-closure lies in the fact that it is the combinatorial fragment of Ramsey required in order to intuitionistically prove some results about termination. In the proof of the Termination Theorem by Podelski and Rybalchenko [1], the part of Ramsey which is actually used is H-closure. In §7, by replacing Ramsey Theorem with H-closure, we will obtain an intuitionistic proof of the Termination Theorem. Moreover in [11] by using H-closure we can intuitionistically prove the size-change-termination Theorem (Lee, Jones and Ben-Amram [10]).

3 Well-founded relations

In this section we introduce the main objects we will deal with in this paper: well-founded relations.

We will use $I,\ J,\ \dots$ to denote sets, $R,\ S,\ T,\ U$ will denote binary relations, $X,\ Y,\ Z$ will be subsets, and $x,\ y,\ z,\ t,\ \dots$ elements. We identify the properties $P(\cdot)$ of elements of I with their extensions $X = \{x \in I : P(x)\} \subseteq I$.

Let R be a binary relation on I. Classically $x \in I$ is R-well-founded if there is no infinite decreasing R-chain $\{x_i : i \in \mathbb{N}\}$ from x in I; i.e.

$$\dots x_n R x_{n-1} R \dots x_1 R x_0 = x$$

Classically R is well-founded if and only if every $x \in I$ is R-well-founded. Equivalently we say that R is well-founded if and only if every non-empty subset of I has a minimal element with respect to R.

The inductive definition of well-founded relations is more suitable than the classical one in the intuitionistic proofs. In the first subsection we introduce this definition; in the second one we provide some examples; in the last subsection we present the definition of H-well-foundedness, which is fundamental to state the new intuitionistic form of Ramsey Theorem, as seen in the previous section. For short we write that a relation is "well-founded" to say that it is intuitionistically well-founded.

3.1 Intuitionistic well-founded relations

The intuitionistic definition of well-founded relation is based on the definition of inductive property. Let R be a binary relation on I. A property is R-inductive if whenever it is true for all R-predecessors of a point it is true also for the point. $x \in I$ is R-well-founded if and only if it belongs to every R-inductive property; R is well-founded if every x in I is R-well-founded. Formally:

Definition 1. *Let R be a binary relation on I.*

- *A property $X \subseteq I$ is* R-inductive *if and only if* IND^R_X*; where*

$$\mathrm{IND}^R_X := \forall y. (\forall z. (zRy \implies z \in X) \implies y \in X) .$$

– *An element $x \in I$ is* R-well-founded *if and only if* $\mathrm{WF}^R(x)$*; where*

$$\mathrm{WF}^R(x) := \forall X. \left(\mathrm{IND}_X^R \implies x \in X\right).$$

– R is well-founded *if and only if* $\mathrm{WF}(R)$*; where*

$$\mathrm{WF}(R) := \forall x. \mathrm{WF}^R(x).$$

A binary structure, just a *structure* for short, is a pair (I, R), where R is a binary relation on I. We say that (I, R) is well-founded if R is well-founded.

We need also the notion of co-inductivity. A property X is R-co-inductive in $y \in I$ if it satisfies the inverse property of R-inductive: if the property X holds for a point, then it holds also for all its R-predecessors. Formally:

Definition 2. *Let R be a binary relation on I.*

– *A property X is* R-co-inductive in $y \in I$ *if and only if* $\mathrm{CoIND}_X^R(y)$*; where*

$$\mathrm{CoIND}_X^R(y) := \forall z. (zRy \implies z \in X).$$

– *A property X is* R-co-inductive *if and only if* CoIND_X^R*; where*

$$\mathrm{CoIND}_X^R := \forall y. (y \in X \implies \forall z. (zRy \implies z \in X)).$$

3.2 Some examples of intuitionistic proofs of well-foundedness

The simplest non-trivial example is the intuitionistic proof that $(\mathbb{N}, <)$ is well-founded, where $<$ is the classical order of the natural numbers. Recall that, through this paper, "well-founded" is short for "inductive well-founded".

Example 1. $(\mathbb{N}, <)$ is well-founded.

Proof. In order to prove $\mathrm{WF}(<)$ we need to show that for every $x \in \mathbb{N}$ and for every $<$-inductive property X, $x \in X$. By definition the following holds:

$$\forall y. (\forall z. z < y \implies z \in X) \implies y \in X.$$

So it is sufficient to show that $[0, x] \subseteq X$, for every $x \in \mathbb{N}$. We prove it by Peano induction:

$$(0 \in X \wedge \forall x. [0, x] \subseteq X \implies [0, x+1] \subseteq X) \implies \forall z. [0, z] \subseteq X.$$

We have $0 \in X$, since 0 has no predecessor and X is $<$-inductive. Moreover if $[0, x] \subseteq X$, then for every $y < x + 1$ we have $y \in X$. Since X is $<$-inductive, $x + 1 \in X$. So $[0, x + 1] \subseteq X$.

Now let $<$ be the classical order in $\mathbb{Z}$ and let now consider the following set:

$$\mathbb{Z}^- := \{z \in \mathbb{Z} : z \leq 0\}.$$

Example 2. Every $z \in \mathbb{Z}^-$ is not $<$-well-founded in $\mathbb{Z}^-$.

Proof. Let $X = \emptyset$. Then X is $<$-inductive for every $z \in \mathbb{Z}^-$, since the inductive hypothesis

$$\forall z. (z < x \implies z \in X)$$

is false for $z = x - 1$. So $z \in \mathbb{Z}^-$ is not $<$-well-founded since X is $<$-inductive and $z \notin X$.

In general we will intuitionistically prove that if there exists an infinite decreasing R-chain from x then x is not R-well-founded. Classically, and by using the Axiom of Choice, x is R-well-founded if and only if there are no infinite decreasing R-chains from x, and R is well-founded if and only if there are no infinite decreasing R-chains in I.

3.3 *H*-well-founded relations

In order to define H-well-foundedness we need to introduce some notations. We denote a list on I with $\langle x_0, \ldots, x_{n-1} \rangle$ where $n \in \mathbb{N}$ and $x_i \in I$ for any $i < n$; $\langle \rangle$ is the empty list. We define the operation of concatenation of two lists on I in the natural way as follows:

$$\langle x_0, \ldots, x_{n-1} \rangle * \langle y_0, \ldots, y_{m-1} \rangle = \langle x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1} \rangle.$$

We define the relation of one-step expansion $\succ$ between two lists L, M on the same I, as $L \succ M \iff L = M * \langle y \rangle$, for some y.

Definition 3. *Let R be a binary relation on I.*

– *$H(R)$ is the set of the R-decreasing transitive finite sequences on I:*

$$\langle x_0, \ldots, x_{n-1} \rangle \in H(R) \iff \forall i, j < n. (i < j \implies x_j R x_i)\,.$$

– *R is H-well-founded if $H(R)$ is $\succ$-well-founded.*

Well-founded relations are H-well-founded relations, and the converse holds for transitive relations, as proved by the next result. Later on, we will provide examples of H-well-founded relations which are not well-founded.

Proposition 1. *1. R well-founded implies that R H-well-founded.*
2. R H-well-founded and R transitive imply that R well-founded

Proof. 1. We will prove that

$$X := \{y : \text{ each } L \in H(R) \text{ that ends in } y \text{ is } \succ\text{-well-founded}\}$$

is R-inductive. This guarantees that $\forall x (x \in X)$, that is, all non-empty lists are $\succ$-well-founded. It will follow that the empty list is $\succ$-well founded and that $H(R)$ is $\succ$-well-founded.

Now, assume that
$$\forall z.(zRy \implies z \in X)$$
in order to prove that $\forall L \in H(R)$ such that L ends in y, L is $\succ$-well-founded. By unfolding definition, it means that $\forall L \in H(R)$
$$\forall Y. \left(\mathrm{IND}_Y^{\succ} \implies L \in Y\right).$$
Let $L \in H(R)$ and Y $\succ$-inductive, then we need to prove
$$\forall M.(M \succ L \implies M \in Y).$$
If $M \succ L$ then the last element z of M is such that zRy. This guarantees that M is $\succ$-well-founded, by inductive hypothesis. So $M \in Y$ since M belongs to any inductive property.

2. Assume that $H(R)$ is $\succ$-well-founded and R is transitive. We have to show that
$$\forall x \in I \forall X \subseteq I. \left(\mathrm{IND}_X^R \implies x \in X\right).$$
Assume that $x \in I$ and X is R-inductive, let
$$Y := \{\langle\rangle\} \cup \{L : \text{ the last element } z \text{ of } L \text{ is in } X\}.$$
If we prove that Y is $\succ$-inductive, then $Y = H(R)$ by $H(R)$ $\succ$-well-founded, therefore $\forall x \in I.x \in X$. Hence we will show that Y is $\succ$-inductive.
Assume that $\forall M(M \succ L \implies M \in Y)$ in order to prove $L \in Y$. If $L = \langle\rangle$ then $L \in Y$. Assume $L \neq \langle\rangle$ and let y be the last element of L, so $L \in X$ if and only if $y \in X$. We use the R-inductivity of X:
$$\forall z.(zRy \implies z \in X) \implies y \in X.$$
Let zRy then, by y last element of L and transitivity of R, the list $L{*}\langle Z\rangle$ is R-decreasing and transitive, that is, $L{*}\langle z\rangle \in H(R)$. Then by inductive hypothesis on Y, $L{*}\langle z\rangle \in Y$, this guarantees $z \in X$. So by R-inductivity of X, $y \in X$; hence $L \in Y$.

4 Basic properties of well-founded relations

There are several folk-lore methods to intuitionistically prove that a binary relation R is well-founded by using the well-foundedness of another binary relation S. The most prevalent ones are the followings:

- a subset of a well-founded relation is well-founded;
- if there exists a morphism from a relation R to a relation S and S is well-founded then R is well-founded;
- if there exists a "simulation" relation from R to S, then each point "simulable" in a S- well-founded point is R-well-founded.

The goal of this section is to recall the proofs of these results. In §4.1 we are going to define simulation relations, in §4.2 we introduce some operations which preserve well-foundedness, while in §4.3 we will show the main properties of well-foundedness.

4.1 Simulation relations

A simulation relation is a binary relation which correlates two other binary relations.

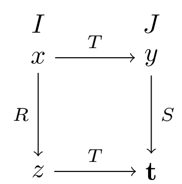
Definition 4. *Let R be a binary relation on I and S be a binary relation on J. Let T be a binary relation on $I \times J$.*

- *Domain of T.* $\mathrm{dom}(T) = \{x \in I : \exists y \in J.xTy\}$.
- *Morphism. $f : (I, R) \to (I, S)$ is a morphism if f is a function such that $\forall x, y \in I.xRy \implies f(x)Sf(y)$.*
- *Simulation. T is a simulation of R in S if and only if it is a relation and*

$$\forall x, z \in I.\forall y \in J.\,((xTy \wedge zRx) \implies \exists t \in J.\,(tSy \wedge zTt))$$

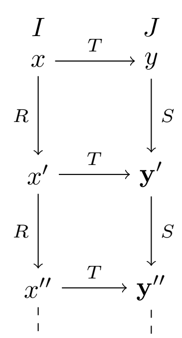
- *Total simulation. A simulation relation T of R in S is total if* $\mathrm{dom}(T) = I$.
- *Simulable. R is simulable in S if there exists a* total *simulation relation T of R in S.*

We may describe the behaviour of a simulation T of R in S by filling the lower right angle of the following diagram.



If we have a simulation T of R in S and xTy holds, we can transform each finite decreasing R-chain in I from x in a finite decreasing S-chain in J from y.

In fact it suffices to complete the lower right angle by following the order $\mathbf{y}'$, $\mathbf{y}''$, ….



By using the Axiom of Choice this result holds also for infinite decreasing R-chains from a point in $\mathrm{dom}(T)$. Then if there are no infinite decreasing S-chains in J there are no infinite decreasing R-chains in $\mathrm{dom}(T)$. If, furthermore, the simulation is total there are no infinite decreasing R-chains in I. By using classical logic and the Axiom of Choice we may conclude that if S is well-founded and T is a total simulation relation of R in S then R is well-founded. In the last subsection of this section we will present an intuitionistic proof of this result which does not use the Axiom of Choice.

We recall some trivial examples of simulation relations.

Example 3. Let R be a binary relation on I, and let S be a binary relation on J. If there exists a morphism $f : (I, R) \to (J, S)$, then

$$T = \{(x, f(x)) : x \in I\}$$

is a total simulation of R in S.

In fact if zRx, then $f(z)Sf(x)$ thanks to the definition of morphism. This guarantees that we may complete the diagram by choosing $t = f(z)$.

Example 4. Let $R,\ S$ be binary relations on I such that $R \subseteq S$. Then

$$T = \{(x, x) : x \in I\}$$

is a total simulation of R in S.

In this case it suffices to complete the diagram by putting

$$t = y = x.$$

We may see binary relations as abstract reduction relations. From now on, by an abstract reduction relation we simply mean a binary relation (for example a rewriting relation). Classically, a reduction relation R is said to be terminating or strongly normalizing if and only if there are no infinite R-chains [13]. Intuitionistically, we require that R is well-founded. Observe that we use simulation to prove well foundedness and this is the same method used for labelled state transition systems [14], except that, for us, the set of labels is always a singleton.

4.2 Some operations on binary structures

In this subsection we introduce some operations mapping binary structures into binary structures. In §4.3 we prove that these operations preserve well-foundedness.

The first operation is the successor operation (adding a top element).

Definition 5. *Let R be a relation on I and let $\top$ be an element not in I. We define the relation $R+1 = R \cup \{(x, \top) : x \in I\}$ on $I+1 = I \cup \{\top\}$. We define the successor structure of (I, R) as $(I, R)+1 = (I+1, R+1)$.*

Another operation on binary structures is the relation defined by components, inspired by the order by components.

Definition 6. *Let R be a binary relation on I, and let S be a binary relation on J. The relation $R \otimes S$ of components R, S is defined as below:*

$$R \otimes S := (R \times \mathrm{Diag}(J)) \cup (\mathrm{Diag}(I) \times S) \cup (R \times S),$$

where $\mathrm{Diag}(X) = \{(x, x) : x \in X\}$.

Equivalently $R \otimes S$ is defined for all $x, x' \in I$ and for all $y, y' \in J$ by:

$$(x, y) R \otimes S (x', y') \iff$$
$$((xRx') \wedge (y = y')) \vee ((x = x') \wedge (ySy')) \vee ((xRx') \wedge (ySy')) .$$

If R, S are orderings then $R \otimes S$ is the componentwise ordering, also called the product ordering. In this case $R \otimes S = R \times S$, while in general $R \otimes S \supseteq R \times S$.

4.3 Properties of well-foundedness

Now we may list the main intuitionistic properties of well-founded relations. Proofs are folk-lore.

Proposition 2. *Let R be a binary relation on I, and let S be a binary relation on J.*

1. *Well-foundedness is both an inductive and a co-inductive property:*

$$x \text{ is } R\text{-well-founded} \iff \forall y.(yRx \implies y \text{ is } R\text{-well-founded}).$$

2. *If R, S are well-founded, then $R \otimes S$ is well-founded.*
3. *If T is a simulation of R in S and if xTy and y is S-well-founded, then x is R-well-founded.*
4. *If T is a simulation of R in S and S is well-founded, then* $\mathrm{dom}(T)$ *is R-well-founded.*
5. *If R is simulable in S and S is well-founded, then R is well-founded.*
6. *Assume that $f : (I, R) \to (J, S)$ is a morphism. If $x \in I$ and $f(x)$ is S-well-founded, then x is R-well-founded. If S is well-founded, then R is well-founded.*
7. *If R is included in S and S is well-founded then R is well-founded.*

Proof. 1. − *Well-foundedness is inductive.* Assume that

$$\forall y.(yRx \implies y \text{ is } R\text{-well-founded}),$$

in order to prove
$$\forall X. \left(\mathrm{IND}^R_X \implies x \in X\right).$$

Let X be such that IND^R_X holds. Then our thesis follows by proving

$$\forall z.(zRx \implies z \in X).$$

In order to prove it, let z be such that zRx, then by hypothesis z is R-well-founded, hence, by unfolding definition, $z \in X$.

– *Well-foundedness is co-inductive.* It suffices to show that the set

$$X := \{y : \forall w \in I.wRy \implies w \text{ is } R\text{-well-founded}\}$$

is R-inductive. In fact if we prove it then the property X will hold for any x R-well-founded.
In order to show that X is R-inductive, assume that

$$\forall z.(zRy \implies z \in X).$$

Then, thanks to the previous point for any z:

$$z \in X \implies z \text{ is } R\text{-well-founded},$$

therefore

$$\forall z.(zRy \implies z \text{ is } R\text{-well-founded}).$$

So $y \in X$ and we are done.

2. In order to prove that $R \otimes S$ is well-founded, it is enough to prove that

$$X := \{x : \forall y \in J.(x, y) \text{ is } R \otimes S\text{-well-founded}\}$$

is R-inductive. Assume that

$$\forall z.(zRx \implies z \in X),$$

we will show that $x \in X$. In order to do it we are going to verify that Y_x is S-inductive; where

$$Y_x := \{y : (x, y) \text{ is } R \otimes S\text{-well-founded}\}.$$

Assume that

$$\forall w.(wSy \implies w \in Y_x),$$

to show $y \in Y_x$, that is, that (x, y) is $R \otimes S$-well-founded. By point 1, this is equivalent to

$$\forall (z, w).((z, w)R \otimes S(x, y) \implies (z, w) \text{ is } R \otimes S\text{-well-founded}).$$

Recall that

$$(z, w)R \otimes S(x, y) \iff (zRx \wedge (w = y \vee wSy)) \vee (z = x \wedge wSy).$$

If zRx then $z \in X$ by inductive hypothesis on X. Then (z, w) is $R \otimes S$-well-founded. If $z = x \wedge wSy$ then $w \in Y_x$, by inductive hypothesis on Y_x. This implies that (z, w) is $R \otimes S$-well-founded and we are done.

3. In order to prove that $\mathrm{dom}(T)$ is R-well-founded, it is enough to prove that

$$Y := \{y \in J : \forall x \in I.xTy \implies x \text{ is } R\text{-well-founded}\}$$

is S-inductive. Assume that

$$\forall w(wSy \implies w \in Y)$$

in order to prove $y \in Y$. Let xTy, we need to show that x is R-well-founded. Thanks to the point 1 above, it suffices to verify that

$$\forall z \in I.zRx \implies z \text{ is } R\text{-well-founded.}$$

If $zRx \wedge xTy$, then by definition of simulation,

$$\exists t \in J.(tSy \wedge zTy).$$

By the inductive hypothesis

$$t \in Y \wedge zTt \implies z \text{ is } R\text{-well-founded.}$$

4. If $x \in \mathrm{dom}(T)$, then there exists $y \in J$ such that xTy. Moreover y has to be S-well-founded, since S is well-founded. By point 3 above x is R-well-founded.
5. By definition of simulable, there exists a simulation T of R in S such that $\mathrm{dom}(T) = I$. Thanks to point 4 above, each $x \in I$ is R-well-founded.
6. It holds thanks points 3 and 5 above, since $T = \{(x, f(x)) : x \in I\}$ is a total simulation of R in S.
7. It holds thanks point 5 above, since $T = \{(x, x) : x \in I\}$ is a total simulation of R in S.

The next remark requires the notion of R-minimal.

Definition 7. *Let R be a binary relation on I. An element $x \in I$ is R-minimal if and only if there are no y such that yRx.*

We may observe that if x is R-minimal then x is R-well-founded by Proposition 2.1: trivially, since it has no R-predecessors.

Example 5. The empty relation V is well-founded, since every element is V-minimal.

Definition 8. *Let R be a binary relation on I, let $x \in I$ and let $n \in \mathbb{N}$. We say that x has R-height n if the longest decreasing R-chain from x has $n + 1$ points.*

Corollary 1. *Let R be a binary relation on I, $n \in \mathbb{N}$, and $x \in I$. If x has R-height n then it is R-well-founded.*

Proof. By induction on n. If $n = 0$ then x is R-minimal, so it is R-well-founded. Assume that it holds for any $m < n$, we will prove it for n. If x has R-height $n + 1$, then every y such that yRx has R-height $\leq n$, so y is R-well-founded by inductive hypothesis. By applying Proposition 2 x is R-well-founded.

Corollary 2. *Let R be a binary relation on I. (I, R) well-founded implies that $(I, R) + 1$ well-founded.*

Proof. $T = \{(x, x) : x \in I\}$ is a total simulation of $(I, R) + 1$ in (I, R). In fact if $x \in I$ and $y(R + 1)x$, then yRx by definition of $R + 1$. So if (I, R) is well-founded, then any $x \in I$ is also $R+1$-well-founded by Proposition 2.4. Moreover $\top$ is well-founded in $(I, R) + 1$ by Proposition 2.1; since $x(R + 1)\top$ implies that $x \in I$, therefore x is well-founded.

Corollary 3. *Let R be a binary relation on I and $x \in I$. If there exists an infinite decreasing R-chain from x, then x is not R-well-founded.*

Proof. Assume that there exists an infinite decreasing R-chain from x:

$$\ldots Rx_2Rx_1Rx_0 = x,$$

then there exists a morphism

$$f : (Z^-, <) \to (I, R)$$
$$-n \mapsto x_n.$$

Suppose by contradiction that x is R-well-founded, then (by Proposition 2.6) 0 should be $<$-well-founded. Contradiction (see Example 2).

So the intuitionistic definition of well-founded intuitionistically implies the classical definition; while the other implication is purely classical.

Now we may see an example of not intuitionistically well-founded set.

Example 6. Each element of $(\mathbb{R}, <)$ is not well-founded, since there exists an infinite decreasing $<$-chain from any real.

Since $(\mathbb{N}, <)$ is well-founded, we may observe that well-foundedness is not preserved by adding elements. Well-foundedness is not preserved also by adding relations over the existing elements. Trivially, $(\mathbb{R}, \emptyset)$, where $\emptyset$ is the empty binary relation, is well-founded, while $(\mathbb{R}, <)$ is not, and $\emptyset \subseteq <$.

When I and R are finite, we may characterize the well-foundedness and the H-well-foundedness in an elementary way.

Definition 9. *Let R be a binary relation on I and $x \in I$. A finite sequence $\langle x_0, \ldots, x_n \rangle$ is an R-cycle from x if $n > 0$ and*

$$x = x_nRx_{n-1}Rx_{n-2}R \ldots Rx_0 = x.$$

If $n = 1$ (that is, if xRx), we call the R-cycle an R-loop.

Proposition 3. *Assume $I = \{x_1, \ldots, x_k\}$ for some $k \in \mathbb{N}$. Let R be any binary relation on I.*

1. *R is well-founded if and only if there are no R-cycles.*
2. *R is H-well-founded if and only if there are no R-loops.*

Proof. 1. Suppose that there are no R-cycles. In no R-chain $\{y_0, \dots, y_{m-1}\}$ we may have $i < j < m$ and $y_i = y_j$ otherwise there exists an R-cycle in I. By the Finite Pigeonhole principle (which is intuitionistically derivable), if it were $m > k$ we would have $i < j < m$ and $y_i = y_j$ contradiction. We deduce that every R-chain has at most k-many points. Then each $x \in I$ has height less than k. Thanks to Corollary 1 R is well-founded.
Now suppose that R is well-founded, in order to prove that there exists no R-cycle from x. If there were an R-cycle from x, there exists an infinite decreasing R-chain from x, hence, by Corollary 3, x is not well-founded. Contradiction.

2. On the one hand, if there exist no R-loops, then there is no decreasing transitive R-chain $\langle x_0, \dots, x_n \rangle$ such that $x_0 = x_n$ and $n > 0$, since this would imply that $x_n R x_0 = x_n$. Then $H(R)$ has no R-cycles, hence the number of the elements in $H(R)$ is at most the number of permutations on I. Then $(H(R), \succ)$ is well-founded since it is a finite structure without $\succ$-cycles.
On the other hand, assume that $H(R)$ is well-founded, in order to prove that there exists no R-loop for x. If there were an R-loop from x, then for every $n \in \mathbb{N}$, the list composed by x repeated n times is transitive and R-decreasing. Hence $H(R)$ is ill founded, contradiction.

Thanks to Proposition 3 we may prove H-closure Theorem if $R_1, \dots, R_n$ are relations over a finite set I. In fact $R = (R_1 \cup R_2 \cup \dots \cup R_n)$ is H-well-founded if and only if there are no R-loops. This is equivalent to: there are no R_i-loops for any $i \in [1, n]$. Hence R is H-well-founded if and only if for each $i \in [1, n]$, R_i is H-well-founded.

Now we want to prove H-closure Theorem for any set I.

5 An intuitionistic version of König

In this section we deal with binary trees. In the first part we introduce binary trees, while in the second part we use binary trees to prove an intuitionistic version of König Lemma for nested binary trees (binary trees whose nodes are themselves binary trees), which we call Nested Fan Theorem. As in the classical case [3], there is a strong link between intuitionistic Ramsey Theorem and Nested Fan Theorem.

5.1 Binary trees

Let R be a binary relation. Then we can define the set of all binary trees where each child node is in relation R with its father node. If R is well-founded, this set will be well-founded with respect to the relation "one-step extension" between trees.

A finite binary tree may be defined in many ways, the most common runs as follows.

Definition 10. *A finite binary tree on I is defined inductively as an empty tree, called* Nil*, or a triple composed by one element of I and two trees, called immediate subtrees: so we have* $\mathrm{Tr} = \mathrm{Nil}$ *or* $\mathrm{Tr} = \langle x, \mathrm{Tr}_0, \mathrm{Tr}_1 \rangle$*.*

$$\mathrm{BinTr} = \{\mathrm{Tr} : \mathrm{Tr} \ \textit{is a binary tree}\ \}$$

Let $\mathrm{Tr} = \langle x, \mathrm{Tr}_0, \mathrm{Tr}_1 \rangle$, then we say that

- Tr is a tree with root x;
- if $\mathrm{Tr}_0 = \mathrm{Tr}_1 = \mathrm{Nil}$, Tr is a leaf-tree;
- if $\mathrm{Tr}_0 \neq \mathrm{Nil}$ and $\mathrm{Tr}_1 = \mathrm{Nil}$, Tr has exactly one left child;
- if $\mathrm{Tr}_0 = \mathrm{Nil}$ and $\mathrm{Tr}_1 \neq \mathrm{Nil}$, Tr has exactly one right child;
- if $\mathrm{Tr}_0 \neq \mathrm{Nil}$ and $\mathrm{Tr}_1 \neq \mathrm{Nil}$, Tr has two children: one right child and one left child.

The universe $|\,\mathrm{Tr}\,|$ of a binary tree on I is the set of the elements of I in Tr, formally:

Definition 11. *Let* Tr *be a binary tree. The universe* $|\,\mathrm{Tr}\,|$ *is defined by induction on* BinTr*:*

- $|\,\mathrm{Nil}\,| = \emptyset$*;*
- $|\langle x, \mathrm{Tr}_0, \mathrm{Tr}_1 \rangle| = \{x\} \cup |\,\mathrm{Tr}_0\,| \cup |\,\mathrm{Tr}_1\,|$.

If $L = \langle x_0, \dots, x_{n-1} \rangle$ is a list on I, we define the universe of L as $|L| = \{x_0, \dots, x_{n-1}\}$.

Definition 12. *Let L be a list on I and* Tr *be a binary tree on I. L is covered by* Tr *if and only if* $|L| = |\,\mathrm{Tr}\,|$.

The covering relation will be useful in order to simulate a set of lists in a set of trees. Each list will be associated with a tree with the same universe.

A binary tree may also be define as a labelled oriented graph on I, empty (if $\mathrm{Tr} = \mathrm{Nil}$) or with a special element, called root, which has exactly one path from the root to any node. Each edge is labelled with a color $c \in C = \{0, 1\}$ in such a way that from each node there is at most one edge in each color.

Equivalently we may define firstly colored lists and then the binary trees as sets of some colored lists.

Definition 13. *A colored list (L, f) is a pair, where $L = \langle x_0, \dots, x_{n-1} \rangle$ is a list on I equipped with a list $f = \langle c_0, \dots, c_{n-2} \rangle$ on $C = \{0, 1\}$.* $\mathrm{nil} = (\langle\rangle, \langle\rangle)$ *is the empty colored list and* $\mathrm{ColList}(C)$ *is the set of the colored lists with colors in C.*

We should imagine that the list L is drawn as a sequence of its elements and that for each $i < n-1$ the segment (x_i, x_{i+1}) has color c_i. Observe that if $L = \langle\rangle$ or if $L = \langle x \rangle$, then $f = \langle\rangle$: if there are no edges in L, then there are no colors in (L, f).

We use $\lambda, \mu, \dots$ to denote colored lists in $\mathrm{ColList}(C)$. Let $c \in C$. We define the composition of color c of two colored lists by connecting the last element of the first list (if any) with the first of the second list (if any) with an edge of color c. Formally we set $\mathrm{nil} *_c \lambda = \lambda *_c \mathrm{nil} = \lambda$, and $(L, f) *_c (M, g) = (L * M, f * \langle c \rangle * g)$ whenever $L, M \neq \mathrm{nil}$.

We can define the relation one-step extension on colored lists: $\succ_c$ is the one-step extension of color c and $\succ_{\mathrm{col}}$ is the one-step extension of any color. Assume $C = \{0, 1\}$ and $x \in I$ and $\lambda, \mu \in \mathrm{ColList}(C)$. Then we set:

- $\lambda *_c (\langle x \rangle, \langle\rangle) \succ_c \lambda$.
- $\lambda \succ_{\mathrm{col}} \mu$ if $\lambda \succ_c \mu$ for some $c \in C$.

Now we can equivalently define a binary tree on I as a particular set of some colored lists.

Definition 14. *A binary tree* Tr *is a set of colored lists on I, such that:*

1. nil *is in* Tr*;*
2. *If* $\lambda \in \mathrm{Tr}$ *and* $\lambda \succ_{col} \mu$*, then* $\mu \in \mathrm{Tr}$*;*
3. *Each list in* Tr *has at most one one-step extension for each color* $c \in C$*: if* $\lambda_0, \lambda_1, \lambda \in \mathrm{Tr}$ *and* $\lambda_0, \lambda_1 \succ_c \lambda$*, then* $\lambda_0 = \lambda_1$.

For all sets $\mathcal{L} \subseteq \mathrm{ColList}(C)$ *of colored list,* $\mathrm{BinTr}(\mathcal{L})$ *is the set of binary trees whose branches are all in* $\mathcal{L}$*.*

For instance the empty tree is the set $\mathrm{Nil} = \{\mathrm{nil}\}$. From $(\langle x \rangle, \langle\rangle) \succ_c \mathrm{nil}$ we deduce that there is at most one $(\langle x \rangle, \langle\rangle) \in \mathrm{Tr}$: x is root of Tr. The leaf-tree of root x may be represented as $\{(\langle x \rangle, \langle\rangle), \mathrm{nil}\}$. The tree with only one root x and two children y, z may be represented as

$$\{(\langle x, y \rangle, \langle 0 \rangle), (\langle x, z \rangle, \langle 1 \rangle), (\langle x \rangle, \langle\rangle), \mathrm{nil}\} .$$

The last definition we need is the one-step extension $\succ_T$ between binary tree; $\mathrm{Tr}' \succ_T \mathrm{Tr}$ if Tr' has one leaf more than Tr.

Definition 15 (One-step extension for binary trees). *If* Tr *is a binary tree and* $\lambda \in \mathrm{Tr}$ *and* $\mu \succ_c \lambda$ *and* $\lambda' \succ_c \lambda$ *for no* $\lambda' \in \mathrm{Tr}$*, then*

$$\mathrm{Tr} \cup \{\mu\} \succ_T \mathrm{Tr}$$

5.2 Nested Fan Theorem

König Lemma is a result of classical logic which guarantees that any infinite binary tree has some infinite branch. By taking the contrapositive, this result has the classically equivalent form: if every branch of a binary tree is finite then the tree is finite. This version is called the Fan Theorem.

There exists a corresponding intuitionistic result, intuitionistically weaker than the original one that we may state as follow.

Lemma 1 (Intuitionistic Fan Theorem). *Each well-founded binary tree is finite.*

Here we are interested to an intuitionistic version of Fan Theorem for nested trees (trees whose nodes are trees), that we will call Intuitionistic Nested Fan Theorem, just Nested Fan Theorem for short.

Let consider a tree Tr whose nodes are finite binary trees, and whose father/child relation between nodes is the one-step extension $\succ_T$. Classically we may say: if for each branch of Tr the union of the nodes in this branch is a binary tree with only finite branches, then each branch of Tr is finite: that is, the tree of trees Tr is classically well-founded.

In the intuitionistic proof of the intuitionistic Ramsey Theorem we will use an intuitionistic version of this statement, in which the finitess of the branches is replaced by inductive well-foundedness of branches. Intuitionistic Nested Fan Theorem states that if a set of colored lists $\mathcal{L}$ is well-founded then the set $\mathrm{BinTr}(\mathcal{L})$, of all binary trees whose branches are all in $\mathcal{L}$, is well-founded.

In this subsection $\leq$ will denote the usual relation of prefix between lists on I: $L \leq M \iff \exists N \in \{\text{lists on } I\} . L{*}N = M$.

Lemma 2 (Intuitionistic Nested Fan Theorem). *Let $C = \{0, 1\}$ be a set of colors and let $\mathcal{L} \subseteq \mathrm{ColList}(C)$ be any set of colored lists with all colors in C. Then*

$$(\mathcal{L}, \succ_{col}) \text{ is well-founded} \implies (\mathrm{BinTr}(\mathcal{L}), \succ_T) \text{ is well-founded.}$$

Proof. Assume Tr is any binary tree of root $x \in I$. Let $c \in C$. We define $\pi_c(\mathrm{Tr})$ as the immediate subtree number c of Tr (the unique subtree connected to the root of Tr by an edge of color c). Formally we set $\pi_c(\mathrm{Tr}) = \{\lambda \in \mathrm{ColList}(C) : (\langle x \rangle, \langle\rangle){*_c}\lambda \in \mathrm{Tr}\}$. $\pi_c(\mathrm{Tr})$ is undefined when $\mathrm{Tr} = \mathrm{Nil}$.

If $c \in C$, $\lambda \in \mathrm{ColList}(C)$ and $\mathrm{Tr} \in \mathrm{BinTr}(C)$, we denote with $\lambda {*_c} \mathrm{Tr}$ the set $\{\lambda {*_c} \mu : \mu \in \mathrm{Tr}\}$. For instance $\mathrm{nil} {*_c} \mathrm{Tr} = \mathrm{Tr}$.

Let $c \in C$, $\lambda \in \mathrm{ColList}(C)$. We define $\mathrm{BinTr}(\mathcal{L}, \lambda, c)$ as the set of binary trees $\{\mathrm{Tr} \in \mathrm{BinTr}(C) : \lambda {*_c} \mathrm{Tr} \subseteq \mathcal{L}\}$. $\mathrm{BinTr}(\mathcal{L}, \lambda, c)$ is the set of trees occurring in some tree of $\mathrm{BinTr}(\mathcal{L})$, as immediate subtree number c of the last node of the branch λ. For instance, $\mathrm{BinTr}(\mathcal{L}, \mathrm{nil}, c) = \mathrm{BinTr}(\mathcal{L})$.

We will prove that $(\mathrm{BinTr}(\mathcal{L}, \lambda, c), \succ_T)$ is well-founded for all $\lambda \in \mathcal{L}$. The thesis will follow if we set $\lambda = \mathrm{nil}$, $c = 1$ (a dummy value). Since $\mathcal{L}$ is well-founded, we argue by induction over λ.

Let us abbreviate $B = \mathrm{BinTr}(\mathcal{L}, \lambda, c)$. Assume that $\mathrm{Tr} \in B$. We have to prove that Tr is well-founded in $(B, \succ_T)$. We distinguish two cases.

1. Assume Tr has root some $x \in I$. Let us abbreviate $\lambda_x = \lambda {*_c} (\langle x \rangle, \langle\rangle)$ and $B_c = \mathrm{BinTr}(\mathcal{L}, \lambda_x, c)$ for all $c \in C$. We define a simulation S from $(B, \succ_T)$ to $(B_0, \succ_T) \otimes (B_1, \succ_T)$ such that $\mathrm{Tr} \in \mathrm{dom}(S)$. Since by inductive hypothesis on λ_x, both $(B_0, \succ_T)$ and $(B_1, \succ_T)$ are well-founded, the thesis will follow by Proposition 2.2 and Proposition 2.3. The simulation is defined by $\mathrm{Tr}' \, S(\pi_0(\mathrm{Tr}'), \pi_1(\mathrm{Tr}'))$ whenever Tr' has root x. S is well-defined because $\mathrm{Tr}' \neq \mathrm{Nil}$, hence $\pi_0(\mathrm{Tr}'), \pi_1(\mathrm{Tr}')$ are well-defined. We have $\mathrm{Tr} \in \mathrm{dom}(S)$ because Tr has root x. We have $\pi_c(\mathrm{Tr}') \in B_c$ by definition. Whenever $\mathrm{Tr}' \succ_T \mathrm{Tr}''$, then Tr'' is obtained from Tr' adding one node either

in the first or in the second immediate subtree of Tr'. In the first case we have $\pi_0(\mathrm{Tr}') \succ_T \pi_0(\mathrm{Tr}'')$ and $\pi_1(\mathrm{Tr}') = \pi_1(\mathrm{Tr}'')$, in the second case we have $\pi_0(\mathrm{Tr}') = \pi_0(\mathrm{Tr}'')$ and $\pi_1(\mathrm{Tr}') \succ_T \pi_1(\mathrm{Tr}'')$. In both cases, the pair $(\pi_0(\mathrm{Tr}'), \pi_1(\mathrm{Tr}'))$ is related to $(\pi_0(\mathrm{Tr}''), \pi_1(\mathrm{Tr}''))$ by $\succ_T \otimes \succ_T$. Thus, S is a simulation such that $\mathrm{Tr} \in \mathrm{dom}(S)$, as we wished to show.
2. Assume $\mathrm{Tr} = \mathrm{Nil}$. Then all one-step extensions of Tr in B are not empty, therefore they are well-founded by point 1 above. Thus, Tr is well-founded by Proposition 2.1.

6 An intuitionistic form of Ramsey Theorem

In this section we present a new intuitionistic version of Ramsey Theorem, the H-closure Theorem. In the first part of the section we state it and we prove the easy classical equivalence between it and Ramsey Theorem, in the second part we prove the H-closure Theorem.

6.1 Stating an intuitionistic form of Ramsey Theorem

In [3] we proved that the first order fragment of Ramsey Theorem is equivalent to the purely classical principle Σ^0_3- LLPO [15], so it is not an intuitionistic result. The H-closure Theorem is a version of Ramsey Theorem intuitionistically valid.

Theorem 2. *[H-closure Theorem] The H-well-founded relations are closed under finite unions:*

$$(R_1, \ldots, R_n \text{ } H\text{-well-founded}) \implies ((R_1 \cup \cdots \cup R_n) \text{ } H\text{-well-founded}).$$

H-closure Theorem is classically true, because there exists a simple classical proof of the equivalence between Ramsey Theorem and H-closure Theorem. This is one reason for finding an intuitionistic proof of H-closure Theorem: it splits the proof of Ramsey Theorem into two parts, one intuitionistic and the other classical but simple (where simple means it could be provable in RCA_0 [6], and it could be proved using the sub-classical principle LLPO-3 [3]). Let us see a short classical proof of this equivalence.

Proposition 4. *Classically:*

$$\textit{Ramsey Theorem} \iff H\textit{-closure Theorem.}$$

Proof. $\Rightarrow$ Let $S_0, \ldots, S_{n-1}$ be symmetric relations on $\mathbb{N}$ such that

$$S_0 \cup \cdots \cup S_{n-1} = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x \neq y\}.$$

We need to prove that there exists an infinite homogeneous set $X \subseteq \mathbb{N}$. For any $k < n$, put

$$R_k := \{(x, y) : xS_ky \wedge x > y\}$$

Then

$$R_0 \cup \cdots \cup R_{n-1} = \{(x, y) : x > y\},$$

and $\{n : n \in \mathbb{N}\}$ is an infinite transitive decreasing $(R_0 \cup \cdots \cup R_{n-1})$-chain. By applying H-closure Theorem we obtain there exists an infinite transitive chain $X = \{x_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$ for $H(R_k)$ for some $k < n$. Hence for any $i < j$ in $\mathbb{N}$ $x_j R_k x_i$ and by definition this implies that $x_j S_k x_i$ and $x_i S_k x_j$ by S symmetric. X is an infinite homogeneous subset of $\mathbb{N}$.

$\Leftarrow$ Suppose that there exists an infinite transitive decreasing $(R_0 \cup \cdots \cup R_{n-1})$-chains:

$$C := \{x_n : n \in \mathbb{N}\}.$$

For any $k < n$, put

$$S_k := \{(i, j) : (i < j \wedge x_j R_k x_i) \vee (j < i \wedge x_i R_k x_j)\},$$

Since C is transitive for any $i,\ j \in \mathbb{N}$ we have

$$i < j \implies x_j (R_0 \cup \cdots \cup R_{n-1}) x_i,$$

then $(S_0 \cup \cdots \cup S_{n-1}) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i \neq j\}$. Thanks to Ramsey Theorem we have there exists an infinite homogeneous set $X \subseteq \mathbb{N}$ for some S_k, for some $k < n$. Then

$$\forall i, j \in X.(i < j \implies x_j S_k x_i),$$

so, by definition of S_k, X is an infinite transitive decreasing R_k-chain. Hence $H(R_k)$ is ill founded.

6.2 Proving the Intuitionistic form Ramsey Theorem

We introduce a particular set of colored lists: the (R_0, R_1)-colored lists. This set will be well-founded if R_0, R_1 are H-well-founded. Let (L, f) be a colored list. We say that (L, f) is a (R_0, R_1)-colored list if for every segment (x_i, x_{i+1}) of (L, f), if it has color $k \in \{0, 1\}$ then x_i is R_k-greater than all the elements of L that follows it. Informally, a sequence is a (R_0, R_1)-colored list if whenever the sequence decreases w.r.t. R_i, then it remains smaller w.r.t. to R_i. Formally:

Definition 16. $(L, f) \in \mathrm{ColList}(C)$ *is a* R_0, R_1*-colored list if either* $L = \langle\rangle$ *and* $f = \langle\rangle$ *or* $L = \langle x_0, \ldots, x_{n-1}\rangle$, $f = \langle c_0, \ldots, c_{n-2}\rangle$, *and*

$$\forall i < n - 1.(c_i = k \implies (\forall j < n. i < j \implies (x_j R_k x_i))).$$

$\mathrm{ColList}(R_0, R_1) \subseteq \mathrm{ColList}(C)$ *is the set of* (R_0, R_1)*-colored lists.*

We may think of a (R_0, R_1)-colored list as a simultaneous construction of one R_0-decreasing transitive list and one R_1-decreasing transitive list. We call an *Erdős-tree over* R_0, R_1, a (R_0, R_1)-tree for short, any binary tree whose branches are all in $\mathrm{ColList}(R_0, R_1)$. Erdős-trees are inspired by the trees used first by Erdős then by Jockusch in their proofs of Ramsey [16], hence the name. We may think of a (R_0, R_1)-tree as a simultaneous construction of many R_0-decreasing transitive lists and many R_1-decreasing transitive lists.

$\mathrm{BinTr}(\mathrm{ColList}(R_0, R_1))$ is the set of all (R_0, R_1)-trees. We will considering the one-step extension $\succ_{\mathrm{col}}$ on colored lists in $\mathrm{ColList}(R_0, R_1)$, and the one-step extension $\succ_T$ on binary trees in $\mathrm{BinTr}(\mathrm{ColList}(R_0, R_1))$.

Now the crucial remark is that each one-step step extension in a $R_0 \cup R_1$-decreasing transitive list may be simulated as a one-step step extension of some Erdős-tree on (R_0, R_1), that is, as a one-step extension either of one R_0-decreasing transitive list or of one R_1-decreasing transitive list, among those associated to the branches of the (R_0, R_1)-tree. From the well-foundedness of R_0 and R_1 we prove the well-foundedness of $(H(R_0) \times H(R_1), \succ \otimes \succ)$, then of the set of (R_0, R_1)-colored lists (as shown in the first part of next lemma), and so of the set of (R_0, R_1)-Erdős-trees as well. Hence, by using the second part of the next lemma, we will also derive the H-well-foundedness of $R_0 \cup R_1$, getting our intuitionistic version of Ramsey Theorem.

Lemma 3. *(Simulation) Let R_0, R_1 be binary relations on a set I.*

1. $(\mathrm{ColList}(R_0, R_1), \succ_{col})$ *is simulable in* $(H(R_0) \times H(R_1), \succ \otimes \succ) + 1$.
2. $H(R_0 \cup R_1, \succ)$ *is simulable in* $(\mathrm{BinTr}(\mathrm{ColList}(R_0, R_1)), \succ_T)$.

Proof. 1. Define the relation S on $\mathrm{ColList}(R_0, R_1) \times (H(R_0) \times H(R_1)) + 1$ as follows:

- $(\langle\rangle, \langle\rangle) S \top$;
- If $L = \langle x_0, \ldots, x_n \rangle$ and $f = \langle c_0, \ldots, c_{n-1} \rangle$, then $(L, f) S (L_0, L_1)$ if for any $i < 2$ L_i is the sublist of L composed by all the $x_j \in L$ such that $c_j = i$.

We need to prove that S is a total simulation.

- S is a relation on $\mathrm{ColList}(R_0, R_1) \times (H(R_0) \times H(R_1)) + 1$. In fact, since (L, f) is a (R_0, R_1)-colored list, if $j' < j$ are indexes of L_i, then $c_j = i$, and then $x_{j'} R_i x_j$ by definition of (R_0, R_1)-colored list. Thus, L_i is an R_i-decreasing transitive list.
- S is a total relation by definition.
- S is a simulation. Let $\xi \in (H(R_1) \times H(R_2)) + 1$.
  - Assume that $(L', f') \succ_{\mathrm{col}} (\langle\rangle, \langle\rangle)$ and $(\langle\rangle, \langle\rangle) S \theta$: then $\theta = \top$. Since $L' \succ \langle\rangle$, then L' has an element. Hence there exist L_0, L_1, such that $(L', f') S (L'_0, L'_1)$. Trivially, by definitions of $\top$ and of $((\succ \otimes \succ) + 1)$, $(L'_0, L'_1)((\succ \otimes \succ) + 1)\top$.
  - Suppose $(L', f') \succ_{\mathrm{col}} (L, f)$ and $(L, f) S \theta$ for some $L \neq \langle\rangle$. By definition of $\succ_{\mathrm{col}}$ and by $L \neq \langle\rangle$ we get $(L', f') = (L * \langle y \rangle, f * \langle c \rangle)$ for some $y \in I$ and some $c \in C = \{1, 2\}$. Moreover $\theta = (L_0, L_1)$, with L_0, L_1 defined as above. Thanks to the definition of S there exists L'_0 and L'_1 such that $(L * \langle y \rangle, f * \langle c \rangle) S (L'_0, L'_1)$. By definition of L'_0 and L'_1, if $c = 0$ then $L'_0 = L_0 * \langle y \rangle$ and $L'_1 = L_1$, otherwise (if $c = 1$) $L'_0 = L_0$ and $L'_1 = L_1 * \langle y \rangle$. In both the cases $(L'_0, L'_1)(\succ \otimes \succ)(L_0, L_1)$, then it holds also $(L'_0, L'_1)(\succ \otimes \succ) + 1(L_0, L_1)$.

2. Define S on $H(R_0 \cup R_1) \times \mathrm{BinTr}(\mathrm{ColList}(R_0, R_1))$ as $LS\,\mathrm{Tr}$ if and only if $|L| = |\,\mathrm{Tr}\,|$. We need to prove that S is a total simulation. We will show it by induction on the length of the list. Firstly observe that $\langle\rangle S\,\mathrm{nil}$, since $|\langle\rangle| = \emptyset = |\,\mathrm{nil}\,|$. Now assume that $L = \langle x_0, \ldots, x_{n-1}\rangle$, $L * \langle y\rangle \in H(R_0 \cup R_1)$ and that there exists $\mathrm{Tr} \in \mathrm{BinTr}(\mathrm{ColList}(R_0, R_1))$ such that $|L| = |\,\mathrm{Tr}\,|$ (induction hypothesis). We want to prove that there exists $\mathrm{Tr}' \succ_T \mathrm{Tr}$ such that

   $$|L * \langle y\rangle| = |\,\mathrm{Tr}'\,| = |\,\mathrm{Tr} \cup \{y\}|.$$

   If $\mathrm{Tr} = \mathrm{nil}$, then Tr' will be the leaf-tree with root y. Otherwise, if $\mathrm{Tr} \neq \mathrm{nil}$, then $n \geq 1$. Let x_0 be the root of Tr. Observe that, since $L \in H(R_0 \cup R_1)$,

   $$\forall j < n \exists i < 2. y R_i x_j.$$

   Hence there exists $h : n \to 2$ such that

   $$\forall j < n. y R_{h(j)} x_j.$$

   Now we define a set $J \subseteq \mathrm{ColList}(R_0, R_1)$ which contains many lists with last element y. We will prove that there is some list in J which we may add to Tr in order to obtain some $\mathrm{Tr}' \succ_T \mathrm{Tr}$. J contains colored lists of the form

   $$(M * \langle y\rangle, g * \langle h(j_{m-1})\rangle) = (\langle x_{j_0}, \ldots, x_{j_m-1}, y\rangle, \langle h(j_0), \ldots, h(j_{m-1})\rangle),$$

   for some $j_0, \ldots, j_{m-1} < n$ (even with repetitions) and such that

   $$(M, g) = (\langle x_{j_0}, \ldots, x_{j_{m-1}}\rangle, \langle h(j_1), \ldots, h(j_{m-2})\rangle) \in \mathrm{Tr}\,.$$

   Observe that
   - J is not empty since $(\langle x_0, y\rangle, \langle h(x_0)\rangle \in J$.
   - $J \subseteq \mathrm{ColList}(R_0, R_1)$. Let $(M * \langle y\rangle, g * \langle h(j_{m-1})\rangle) \in J$, then $(M, g) \in \mathrm{Tr}$ and so $(M, g) \in \mathrm{ColList}(R_0, R_1)$. Moreover $\forall k < m. y R_{h(x_{j_k})} x_{j_k}$, by the choice of h.
   - Given any $(M^*\langle y\rangle, g^*\langle h(j_{m-1})\rangle) \in J$, then we may find some

     $$(M' * \langle y\rangle, g' * \langle h(j_l)\rangle) \in J$$

     such that

     $$\mathrm{Tr}' = \mathrm{Tr} \cup \{(M' * \langle y\rangle, g' * \langle h(j_l)\rangle)\} \succ_T \mathrm{Tr}$$

     and $\mathrm{Tr}' \in \mathrm{BinTr}(\mathrm{ColList}(R_0, R_1))$. We prove it by inverse induction on the length of $(M, g) \in \mathrm{Tr}$. In fact Tr is finite then there is a finite maximal length.

     Let $(M * \langle y\rangle, g * \langle h(j_{m-1})\rangle) \in J$. If (M, g) has no extension in color $h(j_{m-1})$ in Tr, then we may extend Tr with $(M * \langle y\rangle, g * \langle h(j_{m-1})\rangle)$. Otherwise, (M, g) has some extension $(M * \langle x_{j_m}\rangle, g * \langle h(j_{m-1})\rangle)$ in Tr. In this case J contains

     $$(M' * \langle y\rangle, g' * \langle h(j_m)\rangle) = (M * \langle x_{j_m}, y\rangle, f * \langle h(j_{m-1}), h(j_m)\rangle).$$

Since $(M' * \langle y \rangle, g' * \langle h(j_m) \rangle)$ is associated to the list

$$(M * \langle x_{j_m} \rangle, g * \langle h(j_{m-1}) \rangle) \in \mathrm{Tr}$$

that is longer than (M, g), we may apply the inductive hypothesis and we are done.

So we may consider an element in J, which exists by the first observation; then we find, thanks to the last observation, a $\mathrm{Tr}' \in \mathrm{BinTr}(\mathrm{ColList}(R_0, R_1))$ such that $\mathrm{Tr}' \succ_T \mathrm{Tr}$ and $|Tr'| = |Tr| \cup \{y\}$.

Corollary 4. *Let R_0, R_1 be binary relations H-well-founded on a set I.*

1. *The set $(\mathrm{ColList}(R_0, R_1), \succ_{col})$ of R_0, R_1-colored lists is well-founded.*
2. *The set $(\mathrm{BinTr}(\mathrm{ColList}(R_0, R_1)), \succ_T)$ is well-founded.*

Proof. 1. $(H(R_0) \times H(R_1), \succ \otimes \succ)$ is well-founded by Proposition 2.2, since its components are. By Corollary 2, $(H(R_0) \times H(R_1), \succ \otimes \succ) + 1$ is well-founded. Since $(\mathrm{ColList}(R_0, R_1), \succ_{\mathrm{col}})$ is simulable in $(H(R_0) \times H(R_1), \succ \otimes \succ) + 1$ by Lemma 3, then it is well-founded by Proposition 2.5.

2. Since $(\mathrm{ColList}(R_0, R_1), \succ_{\mathrm{col}})$ is well-founded thanks to the previous point, $(\mathrm{BinTr}(\mathrm{ColList}(R_0, R_1)), \succ_T)$ is well-founded by Lemma 2.

Let $\emptyset$ be the empty binary relation on I. Then $H(\emptyset)$ does not contain lists of length greater or equal than 2. Hence $H(\emptyset) = \{\langle x \rangle : x \in I\} \cup \{\langle\rangle\}$. $H(V)$ is $\succ$-well-founded since each $\langle x \rangle$ is $\succ$-minimal, and $\langle\rangle$ has height less or equal than 1. Thus, the empty relation is H-well-founded.

Theorem 3. *Let $n \in \mathbb{N}$. If $R_0, \ldots, R_{n-1}$ H-well-founded then $(R_0 \cup \cdots \cup R_{n-1})$ is H-well-founded.*

Proof. We may prove it by induction on $n \in \omega$. If $n = 0$ we need to prove the empty relation is H-well-founded: we already considered this case. Assume that $n > 0$, and that the thesis holds for any $m < n$. Then $R_0 \cup \cdots \cup R_{n-2}$ is H-well-founded. Thus, in order to prove that $(R_0 \cup \cdots \cup R_{n-1})$ is H-well-founded, it is enough to consider the case $n = 2$.

By Corollary 4.2, $(\mathrm{BinTr}(\mathrm{ColList}(R_0, R_1)), \succ_T)$ is well-founded. Then, by Lemma 3, $(H(R_0 \cup R_1), \succ)$ is simulable in $(\mathrm{BinTr}(\mathrm{ColList}(R_0, R_1)), \succ_T)$, then well-founded by Proposition 2.5.

Corollary 5. *Let $n \in \mathbb{N}$. $R_0, \ldots, R_{n-1}$ are H-well-founded if and only if $(R_0 \cup \cdots \cup R_{n-1})$ is H-well-founded.*

Proof. $\Rightarrow$ Theorem 3.

$\Leftarrow$ If R and S are binary relations such that $R \subseteq S$, then S is H-well-founded implies that R is H-well-founded. In fact we have $H(R) \subseteq H(S)$; so by Proposition 2.7, if $(H(S), \succ)$ is well-founded then $(H(R), \succ)$ is well-founded. Since $\forall i < n. R_i \subseteq R_0 \cup \cdots \cup R_{n-1}$, then R_i is H-well-founded.

7 Podelski and Rybalchenko's Termination Theorem

H-closure Theorem is useful in order to intuitionistically prove some results about termination, since it contains the combinatorial fragment of Ramsey Theorem required to prove them. In this last section we prove that the Termination Theorem [1, Theorem 1] is intuitionistically valid. For all details we refer to this paper: here we only include the definitions of program, computation, transition invariant and disjunctively well-founded relations that Podelski and Rybalchenko used.

Definition 17 (Transition Invariants). *As in [1]:*

- *A program $P = (W, I, R)$ consists of:*
  - *W: a set of states,*
  - *I: a set of starting states, such that $I \subseteq W$,*
  - *R: a transition relation, such that $R \subseteq W \times W$.*
- *A computation is a maximal sequence of states $s_0, s_1, \dots$ such that*
  - *$s_0 \in I$,*
  - *$(s_i, s_{i+1}) \in R$ for all $i \geq 0$.*
- *The set* Acc *of accessible states consists of all states that appear in some computation.*
- *A transition invariant T is a superset of the transitive closure of the transition relation R restricted to the accessible states* Acc*. Formally,*

$$R^+ \cap (\mathrm{Acc} \times \mathrm{Acc}) \subseteq T.$$

- *The program P is terminating if and only if $R \cap (Acc \times Acc)$ is well-founded.*
- *A relation T is disjunctively well-founded if it is a finite union $T = T_0 \cup \dots \cup T_{n-1}$ of well-founded relations.*

Lemma 4. *If $T = R \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded then $U = R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded.*

Proof. Assume that T is well-founded, we will prove that

$$(y \text{ is } T\text{-well-founded}) \implies (y \text{ is } U\text{-well-founded})$$

by induction on y, T.

Recall that (y is U-well-founded) $\iff \forall z(zSy \implies z$ is U-well-founded).

Assume that zUy, then we have two possibilities:

- zTy, then z is T-well-founded, hence by inductive hypothesis z is U-well-founded.
- $zUx \wedge xTy$ for some $x \in$ Acc. In fact x has to be in Acc since it can be reached after some R-steps from $z \in$ Acc. So x is T-well-founded and, by inductive hypothesis, it is U-well-founded. This implies that (since zUx) also z is U-well-founded.

So for each zUy , z is U-well-founded. This implies that y is U-well-founded.

Theorem 4 (Termination). *The program P is terminating if and only if there exists a disjunctively well-founded transition invariant for P.*

Proof. $\Leftarrow$ Let $T = T_0 \cup \cdots \cup T_{n-1} \supseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ with $T_0, \ldots, T_{n-1}$ well-founded. Then T_i is H-well-founded by Proposition 1 and by H-closure Theorem 3 also T is H-well-founded. Therefore, since H-well-foundedness is preserved between subsets, $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is H-well-founded. Moreover it is transitive then, thanks to the Proposition 1, we obtain $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded, so P is terminating.

$\Rightarrow$ Let P be terminating then $R \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded. By Lemma 4 $R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$ is well-founded. Then we are done.

Another Termination Theorem which turns out to be provable by using H-closure [11] is the one by Lee, Jones and Ben-Amram [10].

8 Related works and conclusions

In [3] we studied how much Excluded Middle is needed to prove Ramsey Theorem. The answer was that the first order fragment of Ramsey Theorem is equivalent in HA to Σ^0_3-LLPO, a classical principle strictly between Excluded Middle for 3-quantifiers arithmetical formulas and Excluded Middle for 2-quantifiers arithmetical formulas [15]. Σ^0_3-LLPO may be interpreted as König's Lemma restricted to trees definable by some Δ^0_3-predicate (see again [15]).

However, Ramsey Theorem in the proof of the Termination Theorem [1] may be replaced by H-closure, obtaining a fully intuitionistic proof. It is worth noticing that we obtained the result of H-closure by analyzing the proof of Termination Theorem, not by building over any existing intuitionistic interpretation.

We could not find any evident connection with the intuitionistic interpretations by Bellin, Oliva and Powell. Bellin [17] applied the no-counterexample interpretation to Ramsey theorem, while Oliva and Powell [18] used the dialectica interpretation. They approximated the homogeneous set by a set which may stand any test for some initial segment (a segment dependent by the try itself). Instead we proved a well-foundedness result.

Instead, we found interesting connections with the intuitionistic interpretations expressing Ramsey Theorem as a property of well-founded relations. This research line started in 1974: the very first intuitionistic proof used Bar Induction. We refer to §10 of [8] for an account of this earlier stage of the research. Until 1990, all intuitionistic versions of Ramsey were negated formulas, hence non-informative. In 1990 [8] Veldman and Bezem proved, using Choice Axiom and Brouwer thesis, the first intuitionistic negation-free version of Ramsey: *almost full relations are closed under finite intersections*, from now on the *Almost-Full Theorem*.

We explain the Almost-full theorem. *Brouwer thesis* says: a relation R is inductively well-founded if and only if all R-decreasing sequences are finite. Brouwer thesis is classically true, yet it is not provable using the rules of intuitionistic natural deduction. In [7] (first published in 1994, updated in 2011)

Coquand showed that we may bypass the need of Choice Axiom and Brouwer thesis in the Almost Full Theorem, provided we take as definition of well-founded directly the inductive definition of well-founded (as we do in this paper).

In [8], a binary relation R over a set is *almost full* if for all infinite sequences $x_0, x_1, x_2, \ldots, x_n, \ldots$ on I there are *some* $i < j$ such that $x_i R x_j$. We claim that, classically, the set of almost full relations R is the set of relations such that *the complement of the inverse of* R is H-well-founded. Indeed, let $\neg R^{-1}$ be the complement of the inverse of R: then, classically, $\neg R^{-1}$ almost full means that in all infinite sequences we have $x_i \neg R^{-1} x_j$ for some $i < j$, that is, $x_j \neg R x_i$ for some $i < j$, that is, all sequences such that $x_j R x_i$ for all $i < j$ are finite. Classically, this is equivalent to H-well-foundedness of R. The fact that the relationship between H-well-founded and almost full requires a complement explains why we prove closure under finite unions, while Veldman, Bezem and Coquand proved the closure under finite *intersections*.

For the future, we plan to use our proof to extract some effective bounds for the Termination Theorem. For example, in [19] the proof of the Termination Theorem presented paper is used in order to prove that a program has a transition invariant of height whose relations are primitive recursive and have height ω then it computes a primitive recursive function. Moreover, as already proved in [20] by using a miniaturization of the Dickson Lemma, by using an argument based on H-closure Theorem we have that if a program has a transition invariant composed of k-many relations of height ω, then the transition relations is in $\mathrm{Ack}(k + 1)$ [6]. Where $\mathrm{Ack}(k)$ represents the level k of the primitive recursive hierarchy.

Another possible challenge is to extract the bounds implicit in the intuitionistic proof [9], which, as we said, uses Ramsey Theorem in the form: “almost full relations are closed under intersection”, and to compare the two bounds.

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