

AperTO - Archivio Istituzionale Open Access dell'Università di Torino

## Supersymmetric AdS5 solutions of type IIB supergravity

### This is the author's manuscript

*Original Citation:*

*Availability:*

This version is available <http://hdl.handle.net/2318/1739879> since 2020-05-26T13:26:34Z

*Published version:*

DOI:10.1088/0264-9381/23/14/009

*Terms of use:*

Open Access

Anyone can freely access the full text of works made available as "Open Access". Works made available under a Creative Commons license can be used according to the terms and conditions of said license. Use of all other works requires consent of the right holder (author or publisher) if not exempted from copyright protection by the applicable law.

(Article begins on next page)

# Supersymmetric $AdS_5$ Solutions of Type IIB Supergravity

Jerome P. Gauntlett<sup>1</sup>, Dario Martelli<sup>2</sup>, James Sparks<sup>3</sup> and Daniel Waldram<sup>1</sup>

<sup>1</sup>*Blackett Laboratory, Imperial College  
London, SW7 2AZ, U.K.*

<sup>2</sup>*Department of Physics, CERN Theory Division  
1211 Geneva 23, Switzerland*

<sup>3</sup>*Department of Mathematics, Harvard University  
One Oxford Street, Cambridge, MA 02138, U.S.A.  
and  
Jefferson Physical Laboratory, Harvard University  
Cambridge, MA 02138, U.S.A.*

## Abstract

We analyse the most general bosonic supersymmetric solutions of type IIB supergravity whose metrics are warped products of five-dimensional anti-de Sitter space ( $AdS_5$ ) with a five-dimensional Riemannian manifold  $M_5$ . All fluxes are allowed to be non-vanishing consistent with  $SO(4,2)$  symmetry. We show that the necessary and sufficient conditions can be phrased in terms of a local identity structure on  $M_5$ . For a special class, with constant dilaton and vanishing axion, we reduce the problem to solving a second order non-linear ODE. We find an exact solution of the ODE which reproduces a solution first found by Pilch and Warner. A numerical analysis of the ODE reveals an additional class of local solutions.

# 1 Introduction

The AdS/CFT correspondence [1] is one of the most important developments in string theory. It is therefore an important issue to understand the geometric structures underpinning the correspondence. On the one hand such an understanding can lead to new explicit examples where one can make detailed comparisons with the dual field theory and which can also suggest further generalisations. On the other hand, and more generally, a precise statement of the underlying geometry is the foundation for progress without recourse to explicit examples. By analogy, recall that our understanding of Calabi–Yau geometry has been made without a single non-trivial explicit compact Calabi–Yau 3-fold metric having been constructed.

In ref. [2] we analysed the most general kind of solutions of  $D = 11$  supergravity that can be dual to a four-dimensional superconformal field theory. These bosonic supersymmetric solutions have a metric that is a warped product of  $AdS_5$  with a six-dimensional Riemannian manifold  $M_6$ . In order that the  $SO(4, 2)$  isometry group of  $AdS_5$  is a symmetry group of the full solution, the four-form field strength has non-vanishing components only on  $M_6$ . We used the, by now, standard technique of analysing the canonical  $G$ -structure dictated by supersymmetry [3, 4, 5] in order to obtain necessary and sufficient conditions for supersymmetry. We showed that the geometry on  $M_6$  admits a local  $SU(2)$ -structure and that this implies that  $M_6$  is determined, in part, by a one parameter family of Kähler metrics.

We further analysed a special sub-class of solutions by imposing the condition that  $M_6$  is complex and we used the results to construct several new classes of compact examples of  $M_6$  in explicit form. We showed that one sub-class of solutions leads to new type IIA and type IIB solutions with  $AdS_5$  factors, via dimensional reduction and T-duality, respectively. In particular, the type IIB solutions turn out to be direct product backgrounds  $AdS_5 \times X_5$  with  $X_5$  a Sasaki–Einstein manifold and only the self-dual five-form non-vanishing and proportional to the sum of the volume forms on  $AdS_5$  and  $X_5$  – see [6, 7, 8, 9] for a general discussion of such backgrounds. This is an interesting class of solutions since the dual SCFT can be identified as that arising on a stack of D3-branes transverse to the Calabi–Yau three-fold cone based on  $X_5$ . The most well-known examples of five-dimensional Sasaki–Einstein manifolds, and until recently the only explicit examples, are  $S^5$  and  $T^{1,1}$ ; the corresponding IIB solutions are dual to  $N = 4$  super Yang–Mills theory and an  $N = 1$  superconformal field theory discussed in [6, 9], respectively. The solutions found in [2] led to an infinite number of new explicit Sasaki–Einstein metrics on  $S^2 \times S^3$  called  $Y^{p,q}$  [10]. The dual conformal field theories have now been identified [12, 13, 14] and there have been many further checks and developments. The  $Y^{p,q}$  metrics were generalised to all dimensions in [11] and were recently further generalised to the  $L^{a,b,c}$  metrics in [15, 16] (see also [17]).

The analysis of [2] covered  $AdS_5$  geometries in  $D = 11$  supergravity preserving  $N = 1$  supersymmetry. A refinement of this analysis was recently carried out in [18], where the additional conditions imposed by  $N = 2$  supersymmetry were studied.

In this paper we will generalise the M-theory analysis of [2] to type IIB string theory. In particular, we go beyond the Sasaki–Einstein class and analyse the most general bosonic supersymmetric solutions of type IIB supergravity with a metric that

is a warped product of  $AdS_5$  with  $M_5$ . In addition we allow all of the NS-NS and R-R bosonic fields to be non-vanishing consistent with  $SO(4, 2)$  symmetry. Once again, following [3, 4, 5], we analyse the  $G$ -structure defined by the Killing spinors. We find that the most general geometries have a local identity structure, or equivalently a canonically defined frame, and we use this to determine the necessary and sufficient conditions for supersymmetry. The geometries have a canonically defined Killing vector, which corresponds to the  $U(1)$  R-symmetry of the dual SCFT. We also show that for these solutions supersymmetry implies the equations of motion, just as we saw in [2].

To construct explicit solutions we further restrict our considerations to the special case of constant dilaton and vanishing axion with some additional restrictions imposed on the geometry. We can then reduce the entire problem to solving a second order non-linear ODE. We find one solution in closed form, which turns out to be a solution first obtained by Pilch–Warner [19] (constructed by uplifting a solution first found in five-dimensional gauged supergravity [20]). This solution has constant dilaton and vanishing axion, but non-vanishing three-forms and self-dual five-form; it has been identified [21, 22] as being dual to an  $N = 1$  supersymmetric fixed point discovered by Leigh and Strassler [23]. A numerical analysis of our ODE leads to a continuous family of local solutions. We show that they lead to complete metrics on  $S^5$ , but a detailed analysis indicates that neither the three-form fluxes nor the spinors are globally defined. It is not clear to us whether or not these solutions can be given a physical interpretation. It is also possible that other solutions of the ODE lead to interesting solutions, but we leave this for future work.

The plan of the rest of paper is as follows. In section 2 we outline our conventions for type IIB supergravity. Section 3 derives the necessary and sufficient conditions for the most general supersymmetric solutions with  $AdS_5$  factors. For the convenience of the reader, we have summarised the main results, in a somewhat self contained way, in section 3.6. Section 4 continues the analysis by introducing local coordinates. The discussion of the special class of solutions, including vanishing axion and constant dilaton, and the recovery of the Pilch–Warner solution, is presented in section 5. Section 6 briefly concludes. We have relegated some technical material to several appendices.

## 2 Type IIB equations and conventions

We begin by presenting the equations of motion and supersymmetry transformations for bosonic configurations of type IIB supergravity [24, 25] in the conventions given in appendix A. Essentially we are following [24], with some minor changes, including the signature of the metric.

The conditions for a bosonic geometry to preserve some supersymmetry are

$$\begin{aligned} \delta\psi_M &\approx D_M\epsilon - \frac{1}{96} \left( \Gamma_M^{P_1P_2P_3} G_{P_1P_2P_3} - 9\Gamma^{P_1P_2} G_{MP_1P_2} \right) \epsilon^c \\ &\quad + \frac{i}{192} \Gamma^{P_1P_2P_3P_4} F_{MP_1P_2P_3P_4} \epsilon = 0 \quad (2.1) \\ \delta\lambda &\approx i\Gamma^M P_M \epsilon^c + \frac{i}{24} \Gamma^{P_1P_2P_3} G_{P_1P_2P_3} \epsilon = 0 . \end{aligned}$$

We are working in the formalism where  $SU(1,1)$  is realised linearly. In particular there is a local  $U(1)$  invariance and  $Q_M$  acts as the corresponding gauge field. Note that  $Q_M$  is a composite gauge-field with field strength given by

$$dQ = -iP \wedge P^* . \quad (2.2)$$

Also note that  $D$  is the covariant derivative with respect to local Lorentz transformations and local  $U(1)$  transformations. The spinor  $\epsilon$  has  $U(1)$  charge  $1/2$  so that

$$D_M\epsilon = \left( \nabla_M - \frac{i}{2} Q_M \right) \epsilon . \quad (2.3)$$

The field  $P$  has charge 2, while  $G$  has charge 1. We also have the chirality conditions  $\Gamma_{11}\psi = -\psi$ ,  $\Gamma_{11}\lambda = \lambda$  and  $\Gamma_{11}\epsilon = -\epsilon$ .

The equations of motion are<sup>1</sup>

$$\begin{aligned} R_{MN} &= P_M P_N^* + P_N P_M^* + \frac{1}{96} F_{MP_1P_2P_3P_4} F_N^{P_1P_2P_3P_4} \\ &\quad + \frac{1}{8} \left( G_M^{P_1P_2} G_{NP_1P_2}^* + G_N^{P_1P_2} G_{MP_1P_2}^* - \frac{1}{6} g_{MN} G^{P_1P_2P_3} G_{P_1P_2P_3}^* \right) \\ D^P G_{MNP} &= P^P G_{MNP}^* - \frac{i}{6} F_{MNP_1P_2P_3} G^{P_1P_2P_3} \\ D^M P_M &= -\frac{1}{24} G_{P_1P_2P_3} G^{P_1P_2P_3} \\ F &= *_{10}F . \end{aligned} \quad (2.4)$$

We also need to impose the Bianchi identities

$$\begin{aligned} DP &= 0 \\ DG &= -P \wedge G^* \\ dF &= \frac{i}{2} G \wedge G^* . \end{aligned} \quad (2.5)$$

Note that in the usual string theory variables we have, following [26],

$$\begin{aligned} P &= \frac{i}{2} e^\phi dC^{(0)} + \frac{1}{2} d\phi \\ Q &= -\frac{1}{2} e^\phi dC^{(0)} \end{aligned} \quad (2.6)$$

---

<sup>1</sup>The sign in the third equation differs from that of [24]: we fixed it here by studying the integrability conditions for supersymmetry, as discussed in appendix D.

and we observe that the Binachi identity  $DP = 0$  is identically satisfied. In addition

$$G = ie^{\phi/2}(\tau dB - dC^{(2)}) \quad (2.7)$$

(taking into account a sign difference between our  $G$  and that in [26]). In these conventions, according to [26], the  $SL(2, \mathbb{R})$  action is

$$\tau \rightarrow \frac{p\tau + q}{r\tau + s}, \quad \begin{pmatrix} C^{(2)} \\ B \end{pmatrix} \rightarrow \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} C^{(2)} \\ B \end{pmatrix} \quad (2.8)$$

where  $\tau \equiv C^{(0)} + ie^{-\phi}$ , with the Einstein metric and the five-form left unchanged.

### 3 The conditions for supersymmetry in $d = 5$

We consider the most general class of bosonic supersymmetric solutions of type IIB supergravity with  $SO(4, 2)$  symmetry. The  $d = 10$  metric in Einstein frame is taken to be a warped product

$$ds_{10}^2 = e^{2\Delta} [ds^2(AdS_5) + ds_5^2] \quad (3.1)$$

where  $ds^2(AdS_5)$  denotes the metric on  $AdS_5$ , normalised so that its Ricci tensor is  $-4m^2$  times the metric, and  $ds_5^2$  denotes an arbitrary five-dimensional metric on the internal space  $M_5$ .  $\Delta$  is a real function on this space,  $\Delta \in \Omega^0(M_5, \mathbb{R})$ . We also take  $P \in \Omega^1(M_5, \mathbb{C})$ ,  $Q \in \Omega^1(M_5, \mathbb{R})$ ,  $G \in \Omega^3(M_5, \mathbb{C})$  and

$$F = (\text{vol}_{AdS_5} + \text{vol}_5)f \quad (3.2)$$

where  $\text{vol}_5$  denotes the volume form on  $M_5$  and  $f$  is a real constant to ensure that the five-form Bianchi identity (or equation of motion),  $dF = 0$ , is satisfied.

For the geometry to preserve supersymmetry it must admit solutions to the Killing spinor equations (2.1). To proceed we construct the most general ansatz for the spinor  $\epsilon$  consistent with minimal supersymmetry in  $AdS_5$ . As explained in detail in appendix A,  $\epsilon$  is constructed from two spinors,  $\xi_i$ , of  $Spin(5)$  combined with a  $Spin(4, 1)$  spinor  $\psi$  satisfying  $\nabla_\mu \psi = \frac{1}{2}m\rho_\mu \psi$  on  $AdS_5$ , where  $\rho_\mu$  generate  $\text{Cliff}(4, 1)$ . After substituting this spinor ansatz into (2.1), one eventually obtains two differential conditions

$$D_m \xi_1 + \frac{i}{4} (e^{-4\Delta} f - 2m) \gamma_m \xi_1 + \frac{1}{8} e^{-2\Delta} G_{mnp} \gamma^{np} \xi_2 = 0 \quad (3.3)$$

$$\bar{D}_m \xi_2 - \frac{i}{4} (e^{-4\Delta} f + 2m) \gamma_m \xi_2 + \frac{1}{8} e^{-2\Delta} G_{mnp}^* \gamma^{np} \xi_1 = 0 \quad (3.4)$$

and four algebraic conditions

$$\gamma^m \partial_m \Delta \xi_1 - \frac{1}{48} e^{-2\Delta} \gamma^{mnp} G_{mnp} \xi_2 - \frac{i}{4} (e^{-4\Delta} f - 4m) \xi_1 = 0 \quad (3.5)$$

$$\gamma^m \partial_m \Delta \xi_2 - \frac{1}{48} e^{-2\Delta} \gamma^{mnp} G_{mnp}^* \xi_1 + \frac{i}{4} (e^{-4\Delta} f + 4m) \xi_2 = 0 \quad (3.6)$$

$$\gamma^m P_m \xi_2 + \frac{1}{24} e^{-2\Delta} \gamma^{mnp} G_{mnp} \xi_1 = 0 \quad (3.7)$$

$$\gamma^m P_m^* \xi_1 + \frac{1}{24} e^{-2\Delta} \gamma^{mnp} G_{mnp}^* \xi_2 = 0 \quad (3.8)$$

where  $\gamma^m$  generate  $\text{Cliff}(5)$  with  $\gamma_{12345} = +1$ .

It is interesting to consider first the special case where one of the two spinors,  $\xi_2$  say, is identically zero. It is then easy to see that the warp factor must be constant and related to  $f$  by  $f = 4me^{4\Delta}$ . Hence the metrics are direct products of  $AdS_5$  with a five-manifold. In addition we deduce that

$$\begin{aligned} G_{mnp}^* \gamma^{np} \xi_1 &= G_{mnp} \gamma^{mnp} \xi_1 = 0 \\ \gamma^m P_m^* \xi_1 &= 0 \\ D_m \xi_1 + i \frac{m}{2} \gamma_m \xi_1 &= 0. \end{aligned} \tag{3.9}$$

The first two conditions imply<sup>2</sup> that  $G = 0$ . Next, the third condition implies  $P^2 = 0$ . Writing this out in terms of the axion and dilaton, using (2.6), the imaginary part says that  $\partial C^{(0)} \cdot \partial \phi = 0$ . The equation of motion for  $C^{(0)}$  then says that it is harmonic. Now on a compact manifold, which is the case of most interest for AdS/CFT applications, we deduce that  $C^0$  is constant. The equation of motion for the dilaton then implies that the dilaton is also constant, for the same reason. The last condition in (3.9) then leads us back to the well known  $AdS_5 \times X_5$  solutions where  $X_5$  is Sasaki–Einstein.

More generally, we can enquire whether it is possible to have solutions preserving supersymmetry with both  $\xi_i$  non-vanishing but linearly dependent. In fact this is not possible as we show in appendix C. Note that this implies that the only solutions with compact  $M_5$  having a local  $SU(2)$  structure (determined by supersymmetry), rather than an identity structure to be considered next, are Sasaki–Einstein.

### 3.1 The identity structure

We now turn to the main focus of the paper: supersymmetric solutions with  $\xi_i$  generically linearly independent. We first note that, in neighbourhoods where  $\xi_i$  are generic, they define, locally, an identity structure<sup>3</sup>, or equivalently, a canonical orthonormal frame  $e^a$ . One way to see this is that the set of spinors  $\{\xi_1, \xi_2, \xi_1^c, \xi_2^c\}$  generically form a complete basis for the spinor representation of  $Spin(5)$ .

Equivalently, this structure can easily be seen by noting that there are six real vectors that can be constructed from two non-vanishing spinors. These can be written as

$$\begin{aligned} K^m &\equiv \bar{\xi}_1^c \gamma^m \xi_2 \\ K_3^m &\equiv \bar{\xi}_2 \gamma^m \xi_1 \\ K_4^m &\equiv \frac{1}{2} (\bar{\xi}_1 \gamma^m \xi_1 - \bar{\xi}_2 \gamma^m \xi_2) \\ K_5^m &\equiv \frac{1}{2} (\bar{\xi}_1 \gamma^m \xi_1 + \bar{\xi}_2 \gamma^m \xi_2) \end{aligned} \tag{3.10}$$

where the first two are complex and the last two are real. Since we are in a five-dimensional space they cannot be linearly independent. Using Fierz identities one

<sup>2</sup>It is the same calculation that is used to derive (3.18) below.

<sup>3</sup>An alternative, but equivalent, point of view is that the spinors  $\xi_1 \otimes \xi_2$  define an  $SU(2) \times SU(2)$  structure on  $TM_5 \oplus T^*M_5$ , in the sense of Hitchin [27] (see also [28, 29]). However, we will not adopt this language in the present paper.

finds that there is a single linear relation

$$\epsilon^{ik}\epsilon^{jl}(\bar{\xi}_i\xi_j)(\bar{\xi}_k\gamma^m\xi_l) - 2\text{Re}(\bar{\xi}_2\xi_1^c)(\bar{\xi}_1^c\gamma^m\xi_2) = 0 \quad (3.11)$$

leaving five independent vectors. From these, given the norms of the vectors, one can build an orthonormal basis  $e^a$  defining the identity structure. The relation between the vectors and a particular useful basis  $e^a$  is given in appendix B. Again by Fierz identities, one can write the norms of the vectors in terms of the six independent scalar bilinears. These can be parameterised as

$$\begin{aligned} A &\equiv \frac{1}{2}(\bar{\xi}_1\xi_1 + \bar{\xi}_2\xi_2) \\ A \sin \zeta &\equiv \frac{1}{2}(\bar{\xi}_1\xi_1 - \bar{\xi}_2\xi_2) \\ S &\equiv \bar{\xi}_2^c\xi_1 \\ Z &\equiv \bar{\xi}_2\xi_1 \end{aligned} \quad (3.12)$$

where the first two are real and the second two are complex. In summary, these vector and scalar bilinears define the identity structure.

We now aim to find the conditions on the identity structure and on the fluxes that are equivalent to supersymmetry. This calculation falls into two parts. First one considers the differential conditions (3.3) and (3.4). This is equivalent to giving the intrinsic torsion, or here since we have an identity structure, the torsion itself, in terms of the flux,  $f$ ,  $m$  and the warp factor  $\Delta$ . The same information is contained in the exterior derivatives of the canonical orthonormal frame  $e^m$ , which is in turn encoded in the exterior derivatives of the vector and the scalar bilinears. The second step is then to find necessary and sufficient constraints on the structure due to the algebraic conditions (3.5)–(3.8).

## 3.2 Torsion conditions

In calculating the torsion conditions it is convenient to work not with the exterior derivatives of a particular orthonormal basis  $e^m$ , but rather the exterior derivatives of the vector and scalar bilinears defined above, which are completely equivalent. The results of appendix B then provide a translation to  $e^m$  if required.

We start by calculating the derivatives of the scalar bilinears. Making use of the algebraic conditions (3.5)–(3.8), one finds first that  $A$  is constant,  $dA = 0$ . Thus we can consistently set

$$A = 1 . \quad (3.13)$$

The remaining scalars then satisfy

$$d(e^{4\Delta} \sin \zeta) = 0 \quad (3.14)$$

$$e^{-4\Delta} d(e^{4\Delta} S) = 3imK \quad (3.15)$$

$$e^{-2\Delta} D(e^{2\Delta} Z) = -PZ^* . \quad (3.16)$$



Next we turn to the vectors. Again with some judicious use of the conditions (3.5)–(3.8), after some work we find that the differential constraints (3.3) and (3.4) imply

$$d(e^{4\Delta}K) = 0 \quad (3.17)$$

$$e^{-6\Delta}D(e^{6\Delta}K_3) = P \wedge K_3^* - 4imW - e^{-2\Delta} * G \quad (3.18)$$

$$e^{-4\Delta}d(e^{4\Delta}K_4) = -2mV \quad (3.19)$$

$$e^{-8\Delta}d(e^{8\Delta}K_5) = e^{-4\Delta}fV - 6mU - \text{Re}(e^{-2\Delta}Z^* * G) \quad (3.20)$$

where we have introduced the two-forms

$$\begin{aligned} iU_{mn} &\equiv \frac{1}{2} (\bar{\xi}_1 \gamma_{mn} \xi_1 + \bar{\xi}_2 \gamma_{mn} \xi_2) \\ iV_{mn} &\equiv \frac{1}{2} (\bar{\xi}_1 \gamma_{mn} \xi_1 - \bar{\xi}_2 \gamma_{mn} \xi_2) \\ W_{mn} &\equiv -\bar{\xi}_2 \gamma_{mn} \xi_1 \end{aligned} \quad (3.21)$$

which can, of course, be rewritten in terms of the basis  $e^m$  (see appendix B) and hence the scalar and vector bilinears. Doing so, or using Fierz identities, and given that  $A = 1$ , one finds the identity

$$\sin \zeta V - U - \frac{1}{2} K^* \wedge K + \text{Re}[iZ^*W] = 0 . \quad (3.22)$$

We note first that the first differential condition (3.17) is in fact implied by the scalar condition (3.15). Next we recall that the six vector bilinears are not independent. The linear relation (3.11) implies that

$$K_5 = \sin \zeta K_4 + \text{Re}[Z^*K_3] - \text{Re}[S^*K] \quad (3.23)$$

where we have again used the fact that  $A = 1$ . Taking the exterior derivative of (3.23) and comparing with  $dK_5$  in (3.20) gives the consistency condition

$$(e^{-4\Delta}f + 2m \sin \zeta)V = 6mU - 4m \text{Re}[iZ^*W] + 3imK^* \wedge K . \quad (3.24)$$

However, the two-forms above are linearly related; in particular they obey the identity (3.22). To be consistent with this identity we require, first, that

$$e^{-4\Delta}f = 4m \sin \zeta \quad (3.25)$$

fixing the integration constant in the differential condition (3.14) (In fact, it is straightforward to show this relation holds, directly from the algebraic constraints (3.5) and (3.6)). Secondly, we also require  $\text{Re}[iZ^*W] = 0$ . Using the explicit expression for  $W$  (see appendix B), it is easy to show that this implies the important condition

$$Z = 0 . \quad (3.26)$$

This condition simplifies considerably the algebraic and differential conditions obeyed by the bilinears.

In summary, the torsion conditions (3.3) and (3.4) are equivalent<sup>4</sup> to

$$e^{-4\Delta}f = 4m \sin \zeta, \quad A = 1, \quad Z = 0 \quad (3.27)$$

together with the differential conditions

$$e^{-4\Delta}d(e^{4\Delta}S) = 3imK \quad (3.28)$$

$$e^{-6\Delta}D(e^{6\Delta}K_3) = P \wedge K_3^* - 4imW - e^{-2\Delta} * G \quad (3.29)$$

$$e^{-8\Delta}d(e^{8\Delta}K_5) = 4m \sin \zeta V - 6mU . \quad (3.30)$$

(We drop the  $dK_4$  condition since the linear dependence means it is implied by the other vector bilinear conditions.) As expected, starting with the work [30] and others following this (in particular, see [5, 31, 32, 33, 34]), we note that these differential conditions are written in a form reminiscent of “generalized calibrations” [35, 36].

### 3.3 Algebraic conditions

Next we turn to the algebraic conditions (3.5)–(3.8). We would like to find the equivalent algebraic conditions relating the identity structure,  $P$ ,  $G$ ,  $f$ ,  $m$ , and  $\Delta$ . The simplest way to do this is to note that, as mentioned above, generically the set  $\eta_\alpha \in \{\xi_1, \xi_2, \xi_1^c, \xi_2^c\}$  form a complete basis in the  $Spin(5)$  spinor representation space. Thus we can construct the identity operator

$$\mathbf{1} = \eta_\alpha (m^{-1})^{\alpha\beta} \bar{\eta}_\beta \quad (3.31)$$

where  $m_{\alpha\beta} = \bar{\eta}_\alpha \eta_\beta$ .

Next, using the fact that  $\gamma_{mnp}G^{mnp} = -3 * G_{mn}\gamma^{mn}$  one rewrites the algebraic conditions in the form

$$e^{-2\Delta} * G_{mn}\gamma^{mn}\eta_\alpha = \Sigma_\alpha{}^\beta \eta_\beta . \quad (3.32)$$

Using the completeness relation (3.31), we see that the algebraic conditions are equivalent to an operator equation

$$e^{-2\Delta} * G_{mn}\gamma^{mn} = \Sigma_\alpha{}^\gamma (m^{-1})^{\alpha\beta} \eta_\gamma \bar{\eta}_\beta . \quad (3.33)$$

Performing Fierz identities on  $\eta_\gamma \bar{\eta}_\beta$  one gets three types of relations. From the  $\mathbf{1}$  coefficient one finds

$$i_{K_3^*}P = 2i_{K_3}d\Delta . \quad (3.34)$$

The  $\gamma^m$  coefficient gives three additional conditions

$$i_{K_5}d\Delta = 0 \quad (3.35)$$

$$i_{K_5}P = 0 \quad (3.36)$$

$$e^{-4\Delta}f = 4m \sin \zeta \quad (3.37)$$

---

<sup>4</sup>Note we have also used some of the algebraic conditions (3.5)–(3.8).

where the final expression has already appeared as a consistency condition (3.25). The  $\gamma^{mn}$  coefficient meanwhile gives an expression for the flux  $*G$

$$\begin{aligned} & (\cos^2 \zeta - |S|^2) e^{-2\Delta} *G \\ &= 2P \wedge K_3^* - (4d\Delta + 4imK_4 - 4im \sin \zeta K_5) \wedge K_3 \\ &+ 2 * (P \wedge K_3^* \wedge K_5 - 2d\Delta \wedge K_3 \wedge K_5) . \end{aligned} \quad (3.38)$$

In deriving this last expression one uses the identities

$$\begin{aligned} S^* \xi_1^c \gamma_{(2)} \xi_1 &= (1 + \sin \zeta) W - (K_4 + K_5) \wedge K_3 \\ S^* \xi_2^c \gamma_{(2)} \xi_2 &= (1 - \sin \zeta) W^* - (K_4 - K_5) \wedge K_3^* . \end{aligned} \quad (3.39)$$

For completeness, we also note that, with  $Z = 0$ , the two-forms  $U$ ,  $V$  and  $W$  are given by

$$\begin{aligned} U &= \frac{1}{2(\cos^2 \zeta - |S|^2)} (i \sin \zeta K_3 \wedge K_3^* + iK \wedge K^* - 2 \operatorname{Im} S^* K \wedge K_5), \\ V &= \frac{1}{2 \sin \zeta (\cos^2 \zeta - |S|^2)} (i \sin \zeta K_3 \wedge K_3^* \\ &\quad + i[\sin^2 \zeta + |S|^2] K \wedge K^* - 2 \operatorname{Im} S^* K \wedge K_5), \\ W &= \frac{1}{\sin \zeta (\cos^2 \zeta - |S|^2)} (\cos^2 \zeta K_5 + \operatorname{Re} S^* K + i \sin \zeta \operatorname{Im} S^* K) \wedge K_3. \end{aligned} \quad (3.40)$$

In summary, the algebraic conditions (3.34)–(3.38), together with (3.27) and (3.28)–(3.30), are equivalent to the Killing spinor equations (3.3)–(3.8).

### 3.4 The Killing vector $K_5$

We now show that  $K_5$  is a Killing vector and, moreover, it generates a symmetry of the full solution. This corresponds to the fact that the dual  $D = 4$  superconformal field theories have a global  $U(1)_R$  symmetry. While the Killing condition is implied by the necessary and sufficient conditions (3.28)–(3.30) and (3.34)–(3.38), the simplest derivation is directly from the spinor conditions (3.3)–(3.8). Calculating  $\nabla K_5$ , one can easily show that the symmetric part vanishes and hence  $K_5$  is Killing.

We next compute its action on the remaining bosonic fields. From (3.35) we immediately see that

$$\mathcal{L}_{K_5} \Delta = 0 \quad (3.41)$$

and hence, by (3.37),  $\mathcal{L}_{K_5} \zeta = 0$ . From (3.36), given the expression (2.6) for  $P$ , one also immediately has

$$\mathcal{L}_{K_5} \phi = \mathcal{L}_{K_5} C^{(0)} = 0 \Leftrightarrow \mathcal{L}_{K_5} P = 0 . \quad (3.42)$$

Finally, we need to consider  $\mathcal{L}_{K_5} G$ . This can be calculated directly from the expression (3.38) for  $*G$ . To do so we need to know the action of the Lie derivative  $\mathcal{L}_{K_5}$  on

the scalar and vector bilinears. One finds that all the bilinears are invariant except for

$$\mathcal{L}_{K_5} S = -3imS , \quad (3.43)$$

(and hence also  $\mathcal{L}_{K_5} K = -3imK$ ). This implies  $\mathcal{L}_{K_5}(SS^*) = 0$  and thus, from (3.38),

$$\mathcal{L}_{K_5} G = 0 . \quad (3.44)$$

We thus see that the Killing vector  $K_5$  does indeed generate a symmetry of the full solution.

### 3.5 Equations of motion

We now show that the conditions we have derived for supersymmetry automatically imply the equations of motion and the Bianchi identities.

We first recall that  $DP = 0$  follows automatically from the expression for  $P$  in terms of the variables (2.6). Also, our ansatz has  $dF = 0$  by construction. Next, from (3.29) (and using (3.37)) we find

$$D(e^{4\Delta} * G) = e^{4\Delta} P \wedge *G^* - ifG \quad (3.45)$$

which is just the  $G$  equation of motion. The easiest way to show that the  $G$  Bianchi identity is also satisfied is to derive a differential condition for  $W$  directly from spinor conditions (3.3)–(3.8). One finds

$$D(e^{6\Delta} W) = -e^{6\Delta} P \wedge W^* + (f/4m)G . \quad (3.46)$$

Taking a derivative with  $D$  then reproduces the Bianchi identity for  $G$ .

In appendix D we consider the integrability conditions for the Killing spinor equations. Assuming that the  $P$ ,  $G$  and  $F$  Bianchi identities, together with the  $G$  equation of motion are satisfied, one finds that a supersymmetric background necessarily satisfies the  $P$  equations of motion. Moreover, all but one component of Einstein's equations is automatically satisfied. In appendix D we also show that this component is satisfied in the present case, so we can conclude that:

*For the class of solutions with metric of the form (3.1) and fluxes respecting  $SO(4, 2)$  symmetry, all the equations of motion and Bianchi identities are implied by supersymmetry.*

A similar situation was found to hold for the supersymmetric  $AdS_5$  M-theory solutions of [2]. Clearly, this is very useful for constructing solutions. In fact, it is often the Bianchi identities that are the difficult equations to satisfy.

### 3.6 Summary

Let us end by summarising the necessary and sufficient conditions for the generic supersymmetric solution with metric of the form (3.1) and fluxes respecting  $SO(4, 2)$

symmetry.  $M_5$  must admit an identity structure defined by two spinors  $\xi_1, \xi_2$  which determine the preserved supersymmetry. The scalars  $A$  and  $Z$  defined in (3.12) are given by  $A = 1$  and  $Z = 0$ . The identity structure can then be specified by a real vector  $K_5$ , and two complex vectors  $K, K_3$  defined in (3.10), along with a real scalar  $\zeta$  and a complex scalar  $S$  defined in (3.12). These satisfy the following conditions

$$e^{-4\Delta} d(e^{4\Delta} S) = 3imK \quad (3.47)$$

$$e^{-6\Delta} D(e^{6\Delta} K_3) = P \wedge K_3^* - 4imW - e^{-2\Delta} * G \quad (3.48)$$

$$e^{-8\Delta} d(e^{8\Delta} K_5) = 4m \sin \zeta V - 6mU \quad (3.49)$$

and the additional algebraic constraint

$$i_{K_3^*} P = 2 i_{K_3} d\Delta . \quad (3.50)$$

The five-form flux is given by (3.2) with

$$f = 4me^{4\Delta} \sin \zeta \quad (3.51)$$

while the three-form flux is given by

$$\begin{aligned} & (\cos^2 \zeta - |S|^2) e^{-2\Delta} * G \\ &= 2P \wedge K_3^* - (4d\Delta + 4imK_4 - 4im \sin \zeta K_5) \wedge K_3 \\ &+ 2 * (P \wedge K_3^* \wedge K_5 - 2d\Delta \wedge K_3 \wedge K_5) . \end{aligned} \quad (3.52)$$

The metric can be written (using results in appendix B)

$$\begin{aligned} ds^2 = & \frac{(K_5)^2}{\sin^2 \zeta + |S|^2} + \frac{K_3 \otimes K_3^*}{\cos^2 \zeta - |S|^2} + \frac{|S|^2}{\cos^2 \zeta - |S|^2} (\text{Im } S^{-1} K)^2 \\ & + \frac{|S|^2}{\sin^2 \zeta} \frac{\sin^2 \zeta + |S|^2}{\cos^2 \zeta - |S|^2} \left( \text{Re } S^{-1} K + \frac{1}{\sin^2 \zeta + |S|^2} K_5 \right)^2 . \end{aligned} \quad (3.53)$$

The conditions imply that  $K_5$  is a Killing vector field that generates a symmetry of the full solution:  $\mathcal{L}_{K_5} \Delta = i_{K_5} P = \mathcal{L}_{K_5} G = 0$ . Furthermore, all equations of motion and the Bianchi identities are satisfied.

## 4 Reducing the conditions

It is now useful to introduce some convenient local coordinates and hence reduce the conditions to a simpler set. We will first reduce on the Killing direction  $K_5$  and then use the condition (3.47) to write the resulting four-dimensional metric as a product of a one-dimensional metric and a three-dimensional metric  $\tilde{g}$ . The problem then reduces to a set of conditions on the local identity structure on  $\tilde{g}$ .

We begin by choosing a coordinate  $\psi$  that is adapted to the Killing direction  $K_5$ . As a vector, we write

$$K_5^\# = 3m \frac{\partial}{\partial \psi} \quad (4.1)$$

and therefore as a one-form

$$K_5 = \frac{1}{3m} \cos^2 \eta (d\psi + \rho) , \quad (4.2)$$

where  $\cos \eta$  is the norm of  $K_5$ , given by  $\cos^2 \eta = \sin^2 \zeta + |S|^2$ . (Note that in the conventions of appendix B  $\eta = 2\phi$ .)

Let us now turn to (3.47). The dependence of  $S$  on  $\psi$  is given by the Lie derivative (3.43) which is solved by  $S = e^{-i\psi} \hat{S}$ , where the complex scalar  $\hat{S}$  is independent of  $\psi$ . Noting that there is a gauge freedom in shifting the coordinate  $\psi$  by a function of the remaining coordinates, we take  $\hat{S}$  to be real, without loss of generality. It is then natural to introduce a coordinate  $\lambda$  such that

$$S = \sin \zeta \lambda e^{-i\psi} , \quad (4.3)$$

where the factor of  $\sin \zeta$  is added for convenience so that (3.47) now reads, given (3.51),

$$S^{-1}K = -\frac{1}{3m} (d\psi + i d \ln \lambda) . \quad (4.4)$$

Note that in these coordinates we have

$$\sin \zeta = \frac{\cos \eta}{(1 + \lambda^2)^{1/2}} , \quad (4.5)$$

and it is convenient to switch to  $\eta$ ,  $\psi$  and  $\lambda$  instead of the scalars  $\zeta$  and  $S$ .

For convenience let us also define a new complex one-form  $\sigma$  by

$$K_3 = \sigma / 3m . \quad (4.6)$$

Using the results contained in appendix B, one can write the underlying orthonormal frame as

$$\begin{aligned} 3m e^1 &= \cos \eta (d\psi + \rho) , \\ 3m e^2 &= \lambda \cot \eta \rho , \\ 3m(-ie^3 + e^4) &= \frac{1}{\sin \eta} \sigma , \\ 3m e^5 &= \frac{\cot \eta}{(1 + \lambda^2)^{1/2}} d\lambda . \end{aligned} \quad (4.7)$$

Now since  $\mathcal{L}_{K_5} \rho = \mathcal{L}_{K_5} \sigma$  we can always choose coordinates  $(\psi, \lambda, x^i)$  such that  $\rho$  and  $\sigma$  are independent of  $d\lambda$ . (One first reduces on the Killing direction to a four-dimensional metric, independent of  $\psi$ , spanned by  $e^2, e^3, e^4$  and  $e^5$ . Then given  $e^5 \sim d\lambda$  one can always make a four-dimensional coordinate transformation such that there are no cross-terms  $d\lambda dx^i$  in the metric.) Thus the five-dimensional metric has the form

$$9m^2 ds_5^2 = \cos^2 \eta (d\psi + \rho)^2 + \frac{\cot^2 \eta}{1 + \lambda^2} d\lambda^2 + \tilde{g}_{ij}(\lambda, x^i) dx^i dx^j , \quad (4.8)$$

where the three-dimensional metric  $\tilde{g}$  is given in terms of  $\sigma$  and  $\rho$

$$\tilde{g}_{ij}(\lambda, x^i) dx^i dx^j = \lambda^2 \cot^2 \eta \rho^2 + \frac{\sigma \otimes \sigma^*}{\sin^2 \eta} . \quad (4.9)$$

In summary, we have reduced the problem to a three-dimensional metric  $\tilde{g}$  with a local identity structure given by  $(\rho, \sigma, \sigma^*)$  which also depends on the coordinate  $\lambda$ . In addition, there is one remaining scalar  $\eta$ . (Note that  $\Delta$  is given in terms of  $\eta$  and  $\lambda$  using (3.51) and (4.5).). In making this reduction we have used (3.47) and the fact that  $K_5$  is Killing. It remains to translate the remaining conditions (3.48) and (3.49) into conditions on  $\rho$  and  $\sigma$ .

Let us first split

$$d = \tilde{d} + d\lambda \frac{\partial}{\partial \lambda} + d\psi \frac{\partial}{\partial \psi} \quad (4.10)$$

where  $\tilde{d} = dx^i \partial / \partial x^i$ . Similarly we write

$$P = \tilde{P} + P_\lambda d\lambda, \quad (4.11)$$

recalling that  $i_{K_5} P = 0$ . Writing  $\partial_\lambda = \partial / \partial \lambda$ , the condition (3.49) is equivalent to

$$\begin{aligned} \partial_\lambda \rho &= -\frac{2(1 + 2 \sin^2 \eta) \lambda}{3 \sin^2 \eta (1 + \lambda^2)} \rho, \\ \tilde{d}\rho &= -\frac{i}{3 \sin^2 \eta \cos \eta (1 + \lambda^2)^{1/2}} \sigma \wedge \sigma^*. \end{aligned} \quad (4.12)$$

Similarly the condition (3.48) reduces to

$$\begin{aligned} \sin^2 \eta e^{-6\Delta} D_\lambda (e^{6\Delta} \sigma) &= \left( 4\partial_\lambda \Delta - \frac{4\lambda \cos^2 \eta}{3(1 + \lambda^2)} \right) \sigma - (1 + \cos^2 \eta) P_\lambda \sigma^* \\ &\quad - \frac{2 \cos^2 \eta}{\sin \eta (1 + \lambda^2)^{1/2}} \tilde{*} \left( 2\tilde{d}\Delta \wedge \sigma - \tilde{P} \wedge \sigma^* \right), \\ \sin^2 \eta e^{-6\Delta} \tilde{D} (e^{6\Delta} \sigma) &= 4\tilde{d}\Delta \wedge \sigma - (1 + \cos^2 \eta) \tilde{P} \wedge \sigma^* \\ &\quad - 2i\lambda (1 + \lambda^2)^{1/2} \cos \eta \rho \wedge (2\partial_\lambda \Delta \sigma + P_\lambda \sigma^*). \end{aligned} \quad (4.13)$$

The only remaining condition is the algebraic relation (3.50) which reads

$$i_{\sigma^*} P = 2i_\sigma d\Delta. \quad (4.14)$$

In summary, one needs to solve (4.12) and (4.13) subject to (4.14). This concludes our analysis of the most general  $AdS_5$  geometries arising in type IIB supergravity.

## 5 A simplifying ansatz

In order to find explicit solutions to these equations we will now make a particular, very natural, ansatz. First we assume that the dilaton is constant and the axion zero,  $P = 0$ . Then we assume that the one-forms  $(\rho, \sigma)$  are (locally) proportional to the left-invariant one-forms on  $S^3$ , that is

$$\begin{aligned} \rho &= A\sigma_3, \\ \sigma &= B(\sigma_2 - i\sigma_1). \end{aligned} \quad (5.1)$$

(Note that with this choice  $(\sigma_1, \sigma_2, \sigma_3)$  define the same orientation as  $(e^2, e^3, e^4)$ .) Explicitly we can introduce coordinates  $\sigma_3 = dy - \cos \alpha d\beta$  and  $\sigma_1 = -\sin y d\alpha - \cos y \sin \alpha d\beta$ ,  $\sigma_2 = \cos y d\alpha - \sin y \sin \alpha d\beta$ . In addition we assume that the functions  $A$ ,  $B$  and  $\eta$  all depend only on  $\lambda$ . As we will see this ansatz means the metric has a local  $SU(2) \times U(1) \times U(1)$  isometry group.

We find that the entire analysis then boils down to solving a second-order non-linear ordinary differential equation. Furthermore, we find one exact solution to this ODE which after a change of coordinates turns out to be precisely a solution first found by Pilch and Warner [19]. Our numerical investigations of the ODE lead to a one parameter family of local solutions, which do not extend to globally defined solutions, as we will discuss.

We start by introducing two functions

$$\begin{aligned} h &= -A(1 + \lambda^2) , \\ g &= \frac{1}{\sin \zeta} = \frac{(1 + \lambda^2)^{1/2}}{\cos \eta} . \end{aligned} \quad (5.2)$$

To satisfy the  $\tilde{d}$ -equation in (4.12) one requires

$$B = \left[ \frac{3h(g^2 - 1 - \lambda^2)}{2g^3} \right]^{1/2} . \quad (5.3)$$

This implies that the metric takes the form

$$\begin{aligned} 9m^2 ds_5^2 &= \frac{1 + \lambda^2}{g^2} \left( d\psi - \frac{h}{1 + \lambda^2} \sigma_3 \right)^2 \\ &+ \frac{1}{g^2 - 1 - \lambda^2} \left( d\lambda^2 + \frac{\lambda^2}{1 + \lambda^2} h^2 \sigma_3^2 \right) + \frac{3h}{2g} (\sigma_1^2 + \sigma_2^2) , \end{aligned} \quad (5.4)$$

and it is clear that the metric has a local  $SU(2) \times U(1) \times U(1)$  isometry group. (Note that  $\sigma_1^2 + \sigma_2^2$  is just the round metric on  $S^2$ .) The  $\partial_\lambda$ -equations in (4.12) and the conditions (4.13) all reduce to a pair of coupled first-order differential equations for  $g$  and  $h$ , namely

$$\begin{aligned} \dot{h} &= -\frac{2\lambda h}{3} \frac{1}{g^2 - 1 - \lambda^2} , \\ \dot{g} &= \frac{1}{\lambda h} (g^2 - 1 - \lambda^2) . \end{aligned} \quad (5.5)$$

where the dot denotes  $\partial_\lambda$ . These are equivalent to a second order ODE for  $g$ , which reads

$$\ddot{g} \lambda (g^2 - 1 - \lambda^2) + \dot{g} (g^2 - 1 + \frac{1}{3} \lambda^2 - 2\lambda g \dot{g}) = 0 . \quad (5.6)$$

Any solution to these equations gives rise to a (local) supersymmetric solution with an  $AdS_5$  factor and non-trivial 3-form flux.



For completeness we note that the flux is given by

$$G = \left(\frac{f}{4m}\right)^{1/2} mg^{1/2} \left[ \frac{4\lambda^2 gh - 3(1 + \lambda^2)(g^2 - 1 - \lambda^2)}{\lambda(1 + \lambda^2)^{1/2} gh} e^{15} - 4 \frac{(g^2 - 1 - \lambda^2)^{1/2}}{g(1 + \lambda^2)^{1/2}} e^{25} + 3i \frac{g^2 - 1 - \lambda^2}{\lambda h} e^{12} \right] \wedge (e^4 - ie^3). \quad (5.7)$$

One can also integrate this expression to give the complex potential  $A$  in a relatively simple form (E.7). Note also that the equations (5.5) are symmetric under  $\lambda \rightarrow -\lambda$ ,  $g \rightarrow -g$  and  $h \rightarrow -h$ .

## 5.1 Pilch–Warner solution

We managed to find a single analytic solution to (5.5) given by

$$\begin{aligned} g &= 1 + \frac{1}{\sqrt{3}}\lambda \\ h &= 2 \left(1 - \frac{1}{\sqrt{3}}\lambda\right). \end{aligned} \quad (5.8)$$

We now show that this is locally the same solution first found by Pilch and Warner [19]. To see this we take the range of  $\lambda$  to be  $0 \leq \lambda \leq \sqrt{3}$  and change coordinates via

$$\begin{aligned} \lambda &= \sqrt{3} \sin^2 \theta \\ \psi &= 2\phi \\ y &= \gamma + 2\phi. \end{aligned} \quad (5.9)$$

Also define corresponding set of left-invariant forms  $\hat{\sigma}_3 = d\gamma - \cos \alpha d\beta$  and  $\hat{\sigma}_1 = -\sin \gamma d\alpha - \cos \gamma \sin \alpha d\beta$ ,  $\hat{\sigma}_2 = \cos \gamma d\alpha - \sin \gamma \sin \alpha d\beta$ .

Then the metric can be written as

$$\begin{aligned} (9m^2)ds_5^2 &= 6d\theta^2 + \frac{6 \sin^2(2\theta)}{(3 - \cos(2\theta))^2} \hat{\sigma}_3^2 + \frac{6 \cos^2 \theta}{3 - \cos(2\theta)} (\hat{\sigma}_1^2 + \hat{\sigma}_2^2) \\ &+ 4 \left[ d\phi + \frac{2 \cos^2 \theta}{3 - \cos(2\theta)} \hat{\sigma}_3 \right]^2 \end{aligned} \quad (5.10)$$

which is the form of the metric as written by Pilch and Warner. Note that in the new coordinates the canonical Killing vector takes the form  $\partial_\psi = (1/2)\partial_\phi - \partial_\gamma$ . The warp factor is given by

$$e^{2\Delta} = \left(\frac{f}{4m}\right)^{1/2} \frac{(3 - \cos 2\theta)^{1/2}}{\sqrt{2}}. \quad (5.11)$$

In the new orthonormal frame with  $\hat{e}^1 \propto [d\phi + \dots]$ ,  $\hat{e}^2 \propto \hat{\sigma}_3$ ,  $\hat{e}^3 \propto \hat{\sigma}_1$ ,  $\hat{e}^4 \propto \hat{\sigma}_2$ , the flux is given by

$$G = m\sqrt{3}e^{2\Delta}e^{2i\phi} \left( \hat{e}^1 + i \frac{\sqrt{2} \sin 2\theta}{\sqrt{3}(3 - \cos 2\theta)} \hat{e}^5 \right) (\hat{e}^5 - i\hat{e}^2)(\hat{e}^4 - i\hat{e}^3) \quad (5.12)$$

which coincides with [19] (up to possible factors).

The fibration defined by  $\partial_\phi$  defines, locally, a four-dimensional base space. However, it is easy to see that there is no choice of the range of coordinates  $\phi$  and  $\hat{\psi}$  that makes it a regular four-dimensional manifold. The same is true of the base space defined by the foliation using  $\partial_y$ . We therefore introduce a new set of coordinates defined by

$$\begin{aligned}\phi &= \delta \\ \gamma &= \gamma' + \delta.\end{aligned}\tag{5.13}$$

Then the metric takes the form

$$\begin{aligned}9m^2 ds_5^2 &= 6d\theta^2 + \frac{12 \sin^2 2\theta}{35 - 3 \cos^2 2\theta} (\sigma'_3)^2 + \frac{3(1 + \cos 2\theta)}{3 - \cos 2\theta} [(\sigma'_1)^2 + (\sigma'_2)^2] \\ &\quad + \frac{2(35 - 3 \cos^2 2\theta)}{(3 - \cos 2\theta)^2} (d\delta + A)^2,\end{aligned}\tag{5.14}$$

where  $\sigma'_i$  are the left invariant one-forms  $\hat{\sigma}_i$  above with  $\gamma$  replaced with  $\gamma'$  and the one-form  $A$  is given by

$$A = \frac{(1 + \cos 2\theta)(11 - 3 \cos(2\theta))}{(35 - 3 \cos^2(2\theta))} \sigma'_3.\tag{5.15}$$

If we choose the period of  $\gamma'$  to be  $4\pi$  then it is not difficult to see that the four-dimensional base orthogonal to  $\partial_\delta$  is diffeomorphic to  $\mathbb{C}P^2$ . In particular at  $\theta = 0$  the metric has a two-sphere bolt, with normal neighbourhood being that of the chiral spin bundle of  $S^2$ , while at  $\theta = \pi/2$  the metric has a NUT, *i.e.* it smoothly approaches  $\mathbb{R}^4$ . The full space is obtained by gluing these together which gives  $\mathbb{C}P^2$ . Furthermore, we note that the single non-trivial two-cycle of the base space is represented by the two-sphere bolt at  $\theta = 0$ . We next analyse the fibre direction  $\partial_\delta$ . First note that the norm of this Killing vector field is nowhere vanishing. The one-form  $A$  is a bona-fide connection one-form; its first Chern class, defined by the integral of  $dA/(2\pi)$  over the two-sphere bolt, is one. After recalling the Hopf fibration of  $S^5$  over  $\mathbb{C}P^2$ , we conclude that if we choose the period<sup>5</sup> of  $\delta$  to be  $2\pi$  the topology of the five-dimensional space is in fact  $S^5$ .

## 5.2 Numerical analysis

A numerical investigation of the ODE seems to reveal a continuous family of solutions containing the PW solution and all with topology  $S^5$ . We summarise the main points first and then discuss how the three-form flux and the spinors are not globally defined.

Following on from our discussion of the PW solution, we first consider the general coordinate transformation

$$\begin{aligned}\psi &= 2\delta \\ y &= \gamma' + c\delta.\end{aligned}\tag{5.16}$$

---

<sup>5</sup>For completeness we note that the periodicities of  $\delta$  and  $\gamma'$  imply that  $y$  is a periodic coordinate with period  $4\pi$  while the range of  $\psi$  is  $4\pi$ .

The metric then takes the form

$$(9m^2)ds^2 = A[d\delta + D\sigma'_3]^2 + \frac{d\lambda^2}{g^2 - 1 - \lambda^2} + Q(\sigma'_3)^2 + \frac{3h}{2g}((\sigma'_1)^2 + (\sigma'_2)^2) \quad (5.17)$$

where

$$\begin{aligned} A &= \frac{-4\lambda^4 - 8\lambda^2 + 4\lambda^2g^2 + 4\lambda^2hc + 4g^2 - 4 + 4hc + h^2c^2g^2 - 4hcg^2 - h^2c^2}{(g^2 - 1 - \lambda^2)g^2} \\ Q &= \frac{4\lambda^2h^2}{-4\lambda^4 - 8\lambda^2 + 4\lambda^2g^2 + 4\lambda^2hc + 4g^2 - 4 + 4hc + h^2c^2g^2 - 4hcg^2 - h^2c^2} \\ D &= \frac{(2\lambda^2 - 2g^2 + 2 - hc + hcg^2)h}{-4\lambda^4 - 8\lambda^2 + 4\lambda^2g^2 + 4\lambda^2hc + 4g^2 - 4 + 4hc + h^2c^2g^2 - 4hcg^2 - h^2c^2} . \end{aligned} \quad (5.18)$$

where  $c$  is an arbitrary constant.

Now at  $\lambda = 0$  we have the two parameter family of approximate solutions

$$\begin{aligned} g &= 1 + \beta l^p + \dots \\ h &= \frac{2}{p} - \frac{2}{3p(2-p)\beta} l^{2-p} + \dots \end{aligned} \quad (5.19)$$

for  $0 < p < 2$ . This includes the exact solution when  $p = 1$  and  $\beta = 1/\sqrt{3}$ .

For these solutions, near  $\lambda = 0$  we get

$$9m^2ds^2 \approx 4\frac{(p-c)^2}{p^2}[d\delta + \frac{1}{c-p}\sigma'_3]^2 + \frac{1}{2\beta\lambda^p}d\lambda^2 + \frac{3}{p}[(\sigma'_1)^2 + (\sigma'_2)^2] + \frac{2\lambda^{2-p}}{(p-c)^2\beta}(\sigma'_3)^2 . \quad (5.20)$$

We see that, for a given solution specified by  $p, \beta$ , this is regular, with a two-sphere bolt, provided that the period of  $\gamma'$  is correlated with  $c$ . For example, it will be useful to observe shortly that the period of  $\gamma'$  can be taken to be  $4\pi$  provided that  $c = 3p - 4$  or  $4 - p$ .

In order to mimic the PW solution, we would like to match these solutions onto solutions with  $h(l_c) = 0$  for some  $l_c$ . Consider then the one parameter family of solutions<sup>6</sup>

$$\begin{aligned} g &= (1 + l_c^2)^{1/2} - \frac{2l_c}{3(1 + l_c^2)^{1/2}}\epsilon + \frac{3 - l_c^2}{27(1 + l_c^2)^{3/2}}\epsilon^2 + \dots \\ h &= \frac{(1 + l_c^2)^{1/2}}{l_c}\epsilon + \frac{3 - l_c^2}{18l_c^2(1 + l_c^2)^{1/2}}\epsilon^2 + \dots \end{aligned} \quad (5.22)$$

---

<sup>6</sup>Note that there is also a two parameter family:

$$\begin{aligned} g &= (1 + l_c^2)^{1/2} - \frac{6}{5A}\epsilon^{5/3} + \dots \\ h &= A\epsilon^{1/3} + \dots, \end{aligned} \quad (5.21)$$

but  $h/g$ , and hence the size of the two-sphere, diverges at  $\epsilon = 0$ .

with  $\epsilon = l_c - l$ , which also includes the exact solution when  $l_c = \sqrt{3}$ .

Now consider the behaviour of the metric for these solutions (5.22) near  $\lambda = \lambda_c$ . We get, for all  $\alpha$ ,

$$(9m^2)ds^2 = 4[d\delta + D\sigma'_3]^2 + \frac{3}{2\lambda_c\epsilon}d\epsilon^2 + \frac{3\epsilon}{2\lambda_c}[(\sigma'_1)^2 + (\sigma'_2)^2 + (\sigma'_3)^2] \quad (5.23)$$

with  $D \propto \epsilon$ . This is regular provided that we take the period of  $\gamma'$  to be  $4\pi$ . We can now numerically integrate these back to  $\lambda = 0$ . The numerical analysis indicates that they map onto a one parameter subset of the solutions (5.19), with  $3/2 < h(0) \leq 2$  *i.e.*  $4/3 > p \geq 1$ . Thus we see that if we choose  $c = 3p - 4$  or  $4 - p$ , then the base of the fibration defined by  $\partial_\delta$  is regular and has the same topology as  $\mathbb{C}P^2$  (see the discussion above for the Pilch–Warner solution). Furthermore, the numerical analysis reveals that  $A$  never vanishes and that  $D$  only vanishes at  $\lambda_c$  in the way described above. We next note that integrating  $(1/2\pi)d(D\sigma'_3)$  over the two-sphere bolt at  $\lambda = 0$  gives  $2D(0) = \pm 1/(p - 2)$ . Thus by choosing the period of  $\delta$  to be  $2\pi/(2 - p)$  we deduce that the topologies of this one-parameter family of solutions, generalising the Pilch–Warner solution, are all  $S^5$ .

There are some problems with this family of solutions, however. Firstly, from (4.3), the spinor bilinear  $S$  satisfies  $S = (\lambda/g)e^{-2i\delta}$ . For this to be well defined we need the period of  $\delta$  to be an integer times  $\pi$ . However, for  $4/3 > p \geq 1$  this is only possible for the PW solution with  $p = 1$ . The second problem concerns the expression for the flux (5.7). For this to be globally well defined we should be able to write it in terms of globally defined one-forms  $d\delta$  and  $\sigma'_i$ . Note, however,  $e^4 - ie^3 \sim \sigma_2 - i\sigma_1 = e^{ic\delta}(\sigma'_2 - i\sigma'_1)$ , which requires  $c$  to be an integer to be globally defined, and for  $4/3 > p \geq 1$  this is again only possible for the PW solution which has  $c = -1$  or  $3$ . Note that this phase in the expression for the complex three-form flux cancels out in the energy momentum tensor and this is consistent with the fact that the metric is globally defined on  $S^5$ . We note that in appendix F, blindly ignoring these problems, we have calculated the central charge of the putative dual conformal field theories by determining the effective five-dimensional Newton's constant.

Thus, to summarise, we conclude that these numerical solutions for the ODE give rise to a regular metric on  $S^5$  but they do not give rise to a globally defined solution since the three-form flux is not globally defined. Moreover, the Killing spinors are also not globally defined. As discussed in appendix F, the five-dimensional Newton's constant is, remarkably, analytic for this family. In particular, the Newton's constant is a monotonic decreasing function of  $p$  for  $1 \leq p < 4/3$ . However, the results of  $\alpha$ -maximisation [43] in four-dimensional superconformal field theories imply that the central charges are always algebraic numbers. Indeed, in the current setting it is natural to expect a quantisation condition on  $p$  to come from imposing well-definedness of the spinors and flux. As we have shown, there are in fact no solutions to these conditions. It seems possible that, nevertheless, there is some physical interpretation of these solutions. Alternatively, we hope that by slightly relaxing our assumptions new globally defined solutions can be found.

## 6 Conclusions

The main result of this paper is a determination of the necessary and sufficient conditions on supersymmetric solutions of type IIB supergravity that can be dual to four-dimensional superconformal field theories. The ten-dimensional metric is taken to be a warped product of  $AdS_5$  with a five-dimensional Riemannian metric and we allowed for the most general fluxes consistent with  $SO(4, 2)$  symmetry. Excluding the well known  $AdS_5 \times X_5$  solutions where  $X_5$  is Sasaki–Einstein and only the self-dual five-form is non-vanishing, we showed that the generic compact  $M_5$  admits a canonical local identity structure. We showed how supersymmetry restricts the torsion of this structure and how it determines the fluxes.

By imposing some additional restrictions, including that the dilaton is constant and the axion vanishes, we reduced the conditions to solving a second order non-linear ODE. We managed to find an analytic solution of this ODE and showed that it reproduces a solution found previously in [19]. It would be nice to find the general solution of this ODE but our numerical analysis is not encouraging that there are further globally defined solutions in this class. It would be interesting to know if the local solutions that we found have a physical interpretation. More generally, it may well be possible to find new exact solutions by slightly relaxing some of our assumptions.

While this work was being completed, two papers appeared where new classes of  $AdS_5$  solutions of type IIB were discovered. In fact the construction of these solutions was one of the original motivations of this work. In [37] numerical evidence for a family of solutions interpolating between the PW solution and the  $AdS_5 \times T^{1,1}$  solution were found. In [38] a powerful technique to generate new  $AdS_5$  solutions from old ones, which describe the so-called  $\beta$ -deformations of the original conformal field theory, was presented. It would be interesting to see how these solutions fit into the formalism presented here. It would be particularly interesting if the results of this paper could be used to find  $AdS_5$  solutions corresponding to exactly marginal deformations more general than the  $\beta$ -deformations.

We only considered solutions preserving minimal  $N = 1$  supersymmetry. This includes geometries preserving  $N = 2$  supersymmetry as a special case, but it would be interesting to determine the additional restrictions on the identity structure that are imposed by  $N = 2$  supersymmetry. Hopefully, these will be strong enough that further exact solutions can be found. Recall that in the context of D=11 supergravity, the analysis of [2] covered  $AdS_5$  geometries preserving  $N = 1$  supersymmetry. A refinement of this analysis was carried out in [18], where the additional conditions imposed by  $N = 2$  supersymmetry were studied. It is interesting to note that a double wick rotation of these geometries in [18] were shown to be related to quite different physical phenomena, and this may also be the case for the analogous type IIB supergravity solutions.

Our analysis has focused on the local identity  $G$ -structure on  $M_5$ , as this is most useful for obtaining explicit solutions. Of course the category of families of solutions that can ultimately be found in explicit form is presumably quite small. We also view our work as providing the foundation for studying more general aspects of conformal

field theories with type IIB duals. For example, it would be interesting to know what topological restrictions supersymmetry imposes on  $M_5$ . To tackle this, one could try to determine the global  $G$ -structure that  $M_5$  admits. A converse result of the form that  $M_5$  satisfying certain topological restrictions always admits a solution would be most desirable. It would be also very interesting to see if there is a generalisation of  $Z$ -minimisation [39] (see also [40, 41, 42]), a geometrical version of  $a$ -maximisation [43] in the toric Sasaki–Einstein setting, to the more general class of geometries analysed here.

## Acknowledgments

Most of this work was carried out whilst DM and JFS were postdoctoral fellows at Imperial College, London. In particular DM was funded by a Marie Curie Individual Fellowship under contract number HPMF-CT-2002-01539, while JFS was supported by an EPSRC mathematics fellowship. At present JFS is supported by NSF grants DMS-0244464, DMS-0074329 and DMS-9803347. DW is supported by a Royal Society University Research Fellowship and thanks the Aspen Center for Physics for hospitality while this work was being completed. JPG acknowledges partial support by the National Science Foundation under Grant No. PHY99-0794 while visiting the KITP, Santa Barbara.

## A Conventions and useful formulae for reduction

The ten-dimensional metric has signature  $(-, +, \dots, +)$ . The ten-dimensional gamma matrices  $\Gamma^A$  satisfy

$$[\Gamma^A, \Gamma^B]_+ = 2\eta^{AB} \quad (\text{A.1})$$

and generate the  $D = 10$  Clifford algebra  $\text{Cliff}(9, 1)$ , where  $A, B = 0, 1, \dots, 10$  are frame indices. We define  $\Gamma_{11} \equiv \Gamma_0 \Gamma_1 \dots \Gamma_9$ .

For the configurations that are a warped product of  $AdS_5$  with  $M_5$ , it is useful to decompose  $\text{Cliff}(9, 1)$  by writing

$$\begin{aligned} \Gamma^a &= \rho^a \otimes 1 \otimes \sigma^3 \\ \Gamma^i &= 1 \otimes \gamma^i \otimes \sigma^1 \end{aligned} \quad (\text{A.2})$$

where  $a, b = 0, 1, \dots, 4$  and  $i, j = 1, 2, \dots, 5$  are frame indices on  $AdS_5$  and  $M_5$  respectively, and we have

$$[\rho^a, \rho^b]_+ = 2\eta^{ab}, \quad [\gamma^i, \gamma^j]_+ = 2\delta^{ij} \quad (\text{A.3})$$

with  $\eta^{ab} = \text{diag}(-1, 1, 1, 1, 1)$ . The  $\rho^a$  satisfy  $\rho_{01234} = i$  and generate  $\text{Cliff}(4, 1)$ , while the  $\gamma^m$  satisfy  $\gamma_{12345} = 1$  and generate  $\text{Cliff}(5)$ . In addition,  $\sigma^i$ ,  $i = 1, 2, 3$ , are the Pauli matrices. We then have

$$\Gamma_{11} = 1 \otimes 1 \otimes \sigma^2. \quad (\text{A.4})$$

Let us work out a consistent set of conventions for the various intertwiner operators in the relevant dimensions (see *e.g.* [44]). The  $A$ -intertwiners operate as follows:

$$\begin{aligned} A_{10}\Gamma^M A_{10}^{-1} &= \Gamma^{M\dagger} \\ A_{1,4}\rho^\mu A_{1,4}^{-1} &= -\rho^{\mu\dagger} \\ A_5\gamma^i A_5^{-1} &= \gamma^{i\dagger} \end{aligned} \quad (\text{A.5})$$

and can be chosen to be Hermitian:

$$A_\bullet^\dagger = A_\bullet . \quad (\text{A.6})$$

The charge conjugation matrices, or  $C$ -intertwiners, operate as follows:

$$\begin{aligned} C_{10}^{-1}\Gamma^M C_{10} &= -\Gamma^{MT} \\ C_{1,4}^{-1}\rho^\mu C_{1,4} &= \rho^{\mu T} \\ C_5^{-1}\gamma^i C_5 &= \gamma^{iT} \end{aligned} \quad (\text{A.7})$$

and in the given dimensions are all antisymmetric:

$$C_\bullet = -C_\bullet^T . \quad (\text{A.8})$$

Finally, we have the following  $D$ -intertwiners

$$\begin{aligned} \Gamma^{*A} &= \tilde{D}_{10}^{-1}\Gamma^A\tilde{D}_{10} & \tilde{D}_{10}\tilde{D}_{10}^* &= 1 \\ \gamma^{*m} &= \tilde{D}_5^{-1}\gamma^m\tilde{D}_5 & \tilde{D}_5\tilde{D}_5^* &= -1 \\ \rho^{*a} &= -D_{1,4}^{-1}\rho^a D_{1,4} & D_{1,4}D_{1,4}^* &= -1 . \end{aligned} \quad (\text{A.9})$$

Also recall that, by definition,  $D_{10} = C_{10}A_{10}^T$  and that  $\tilde{D}_{10} = \Gamma_{11}D_{10}$ . It turns out that one can take the following decompositions:

$$\begin{aligned} A_{10} &= 1 \otimes 1 \otimes \sigma^1 \\ C_{10} &= C_{1,4} \otimes C_5 \otimes \sigma^2 \\ \tilde{D}_{10} &= D_{1,4} \otimes \tilde{D}_5 \otimes \sigma^1 \end{aligned} \quad (\text{A.10})$$

with

$$\begin{aligned} A_{1,4} &= 1, & C_{1,4} &= D_{1,4} \\ A_5 &= 1, & C_5 &= \tilde{D}_5 . \end{aligned} \quad (\text{A.11})$$

We now consider decomposing a  $D = 10$  Majorana–Weyl spinor  $\epsilon'$  as  $\epsilon' = \psi \otimes \chi \otimes \theta$ . The chirality condition in  $D = 10$  is

$$\Gamma_{11}\epsilon' = -\epsilon' \quad (\text{A.12})$$

which implies

$$\sigma^2\theta = -\theta . \quad (\text{A.13})$$

Moreover,  $\epsilon'^c = \tilde{D}_{10}\epsilon'^*$ , which now reads

$$\epsilon'^c = \psi^c \otimes \chi^c \otimes \sigma^1 \theta^* \quad (\text{A.14})$$

where

$$\begin{aligned} \psi^c &= C_{1,4}\psi^* \\ \chi^c &= C_5\chi^* \end{aligned} \quad (\text{A.15})$$

and we note that  $\psi^{cc} = -\psi$  and  $\chi^{cc} = -\chi$ . To impose the Majorana condition in  $D = 10$ ,  $\epsilon'^c = \epsilon'$ , we take

$$\theta = \sigma^1 \theta^* \quad (\text{A.16})$$

which we note is consistent with the chirality condition already imposed on  $\theta$ .

We now want to construct the most general spinor ansatz that is consistent with minimal supersymmetry in  $AdS_5$ . Since type IIB supersymmetry is parametrised by two  $D = 10$  Majorana–Weyl spinors,  $\epsilon_i$ , we take

$$\epsilon_i = \psi \otimes \chi_i \otimes \theta + \psi^c \otimes \chi_i^c \otimes \theta \quad (\text{A.17})$$

where we assume that the spinor  $\psi$  satisfies

$$\nabla_\mu \psi = \frac{1}{2} m \rho_\mu \psi \quad (\text{A.18})$$

to ensure that supersymmetry is preserved on  $AdS_5$ . Notice that  $\psi^c$  then satisfies this equation with  $m \mapsto -m$ . Notice also that each spinor has 16 real components, realised as the real part of 4 complex times 4 complex components. We may then complexify

$$\epsilon \equiv \epsilon_1 + i\epsilon_2 \equiv \psi \otimes \xi_1 \otimes \theta + \psi^c \otimes \xi_2^c \otimes \theta \quad (\text{A.19})$$

where  $\xi_1 = \chi_1 + i\chi_2$ ,  $\xi_2^c = \chi_1^c + i\chi_2^c$ . Then

$$\epsilon^c = \psi \otimes \xi_2 \otimes \theta + \psi^c \otimes \xi_1^c \otimes \theta . \quad (\text{A.20})$$

In fact, to derive (3.3)–(3.8), we rescaled by a convenient power of the warp factor. Indeed the ansatz we used is:

$$\epsilon \equiv \psi \otimes e^{\Delta/2} \xi_1 \otimes \theta + \psi^c \otimes e^{\Delta/2} \xi_2^c \otimes \theta . \quad (\text{A.21})$$

After substituting this into the Killing spinor equations (2.1), one finds a number of equations.

In order to analyse the Killing spinor equations, we shall also need the following result. Suppose we have two complex vector spaces  $V$  and  $W$ , such that  $V$  comes equipped with an anti-unitary operation  $c$ , mapping  $v \in V$  to  $v^c \in V$ , with  $(av)^c = a^*v^c \forall a \in \mathbb{C}$ , which also squares to  $-1$ :  $v^{cc} = -v$ . Then for non-zero  $v \in V$ , we have  $v \otimes w_1 + v^c \otimes w_2 = 0$  implies that  $w_1 = w_2 = 0$ . To see this, first note that



$v \otimes w_1 + v^c \otimes w_2 = 0$  implies that either  $w_1 = w_2 = 0$ , or else  $v = av^c$ ,  $w_1 = w_2/a$  for some  $a \in \mathbb{C}^*$ . Suppose the latter case holds. Taking the conjugate of  $v - av^c = 0$  gives  $v^c + a^*v = 0$ . These two equations imply  $1 + |a|^2 = 0$ , which is impossible. Hence the result. Using this algebraic lemma one can show that the equations are then a sum of two terms, each of which is separately zero.

Much of our analysis of the supersymmetry conditions (3.3)–(3.8) come from analysing bilinears that can be constructed from  $\xi_i$ . Note that there are two kinds of bilinears that can be constructed

$$\begin{aligned}\bar{\chi}\gamma_{(n)}\xi &= \chi^\dagger\gamma_{(n)}\xi \\ \bar{\chi}^c\gamma_{(n)}\xi &= \chi^T C^{-1}\gamma_{(n)}\xi\end{aligned}\tag{A.22}$$

where we have used  $A_5 = 1$ , defined  $C \equiv C_5 = \tilde{D}_5$  and  $\gamma_{(n)}$  is the antisymmetrised product of  $n$  gamma matrices. For convenience we record once again, for reference:

$$\begin{aligned}C^* &= -C^{-1} \\ C^T &= -C\end{aligned}\tag{A.23}$$

and

$$\begin{aligned}\gamma^i &= \gamma^{i\dagger} \\ C^{-1}\gamma^i C &= \gamma^{iT}.\end{aligned}\tag{A.24}$$

Finally we note that the Fierz identity for Cliff(5) reads:

$$\bar{\xi}_1\xi_2\bar{\xi}_3\xi_4 = \frac{1}{4}\bar{\xi}_1\xi_4\bar{\xi}_3\xi_2 + \frac{1}{4}\bar{\xi}_1\gamma_m\xi_4\bar{\xi}_3\gamma^m\xi_2 - \frac{1}{8}\bar{\xi}_1\gamma_{mn}\xi_4\bar{\xi}_3\gamma^{mn}\xi_2.\tag{A.25}$$

## B Algebraic conditions satisfied by the bilinears

There are a number of algebraic conditions satisfied by the various bilinears that we use in the main text that can be derived using Fierz identities. However, we find it most useful to construct them using a convenient basis of  $\gamma$ -matrices of Cliff(5). Specifically, we start by taking

$$\begin{aligned}\gamma^1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathbf{1} \\ \gamma^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1} \\ \gamma^a &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \tau^a\end{aligned}\tag{B.1}$$

where  $\tau^a = -i\sigma^a$  and  $\sigma^a$  are the Pauli matrices. The intertwiner  $\gamma^{*m} = \tilde{D}^{-1}\gamma^m\tilde{D}$ , used in the definition  $\xi^c = \tilde{D}\xi^*$ , is given by  $D = \mathbf{1} \otimes -i\sigma^2$ . The corresponding basis of one-forms are labelled  $e^i$ .

We write the two spinors as  $\xi_i = s_i \otimes \theta_i$ . We demand that the two vectors,  $K_4$  and  $K_5$  defined in (3.10) are chosen to lie in the  $(e^1 - e^2)$ -plane. This requires  $\bar{\xi}_i \gamma^a \xi_i = 0$  and constrains the  $s_i$ . If in addition we require  $K_5$  to be parallel to  $e^1$  we find

$$s_1 = \sqrt{2} \begin{pmatrix} \cos \theta \cos \phi \\ -\sin \theta \sin \phi \end{pmatrix} \quad s_2 = \sqrt{2} \begin{pmatrix} \sin \theta \cos \phi \\ \cos \theta \sin \phi \end{pmatrix} \quad (\text{B.2})$$

where, without loss of generality, we have  $\bar{\theta}_i \theta_i = 1$ . Note that we have imposed (3.13):

$$\bar{\xi}_1 \xi_1 + \bar{\xi}_2 \xi_2 = 2. \quad (\text{B.3})$$

One then finds that the  $\theta$  and  $\phi$  functions are related to the scalar bilinears  $\sin \zeta$ ,  $Z$  and  $S$  defined in (3.10), by

$$\begin{aligned} \sin \zeta &= \cos 2\theta \cos 2\phi \\ Z &= \sin 2\theta \cos 2\phi \bar{\theta}_2 \theta_1 \\ S &= \sin 2\theta \cos 2\phi \bar{\theta}_2^c \theta_1. \end{aligned} \quad (\text{B.4})$$

The vectors defined in (3.10) are given by

$$\begin{aligned} K_5 &= (\cos 2\phi) e^1 \\ K_4 &= (\cos 2\theta) e^1 - (\sin 2\theta \sin 2\phi) e^2 \\ K_3 &= (\sin 2\theta \bar{\theta}_2 \theta_1) e^1 + (\cos 2\theta \sin 2\phi \bar{\theta}_2 \theta_1) e^2 + (\sin 2\phi) \bar{\theta}_2 \tau_a \theta_1 e^a \\ K &= (\sin 2\theta \bar{\theta}_1^c \theta_2) e^1 + (\cos 2\theta \sin 2\phi \bar{\theta}_1^c \theta_2) e^2 - (\sin 2\phi) \bar{\theta}_1^c \tau_a \theta_2 e^a. \end{aligned} \quad (\text{B.5})$$

It is similarly straightforward to write out the two-forms. In particular, we find

$$\begin{aligned} W &= Z(\csc 2\theta \tan 2\phi) e^{12} + (\cos 2\theta \sin 2\phi e^1 - \sin 2\theta e^2) \wedge \bar{\theta}_2 \tau_a \theta_1 e^a \\ &\quad + \frac{1}{2} (\sin 2\theta \cos 2\phi) \epsilon_{abc} \bar{\theta}_2 \tau^c \theta_1 e^{ab}. \end{aligned} \quad (\text{B.6})$$

We then find that  $\text{Re}[iZ^*W] = 0$  implies  $Z = 0$  as claimed in the text.

We now put  $Z = 0$  by setting  $\bar{\theta}_1 \theta_2 = 0$ . Choosing  $K_3$  to suitably lie just within the  $(e^3 - e^4)$ -plane, we can choose

$$\theta_1 = \begin{pmatrix} e^{i\alpha} \\ 0 \end{pmatrix} \quad \theta_2 = \begin{pmatrix} 0 \\ e^{i\alpha} \end{pmatrix} \quad (\text{B.7})$$

and hence

$$\begin{aligned} K_5 &= (\cos 2\phi) e^1 \\ K_4 &= (\cos 2\theta) e^1 - (\sin 2\theta \sin 2\phi) e^2 \\ K_3 &= (\sin 2\phi) (e^4 - i e^3) \\ e^{-2i\alpha} K &= (\sin 2\theta) e^1 + (\cos 2\theta \sin 2\phi) e^2 - i(\sin 2\phi) e^5 \end{aligned} \quad (\text{B.8})$$

where  $\bar{\theta}_1^c \theta_2 = e^{2i\alpha}$ , which is the phase of  $S$ , so that the scalars are now given by

$$\begin{aligned} \sin \zeta &= \cos 2\theta \cos 2\phi \\ S &= -\sin 2\theta \cos 2\phi e^{2i\alpha}. \end{aligned} \quad (\text{B.9})$$

Similarly, the two-forms are

$$\begin{aligned}
U &= -\sin 2\theta \sin 2\phi e^{15} - \cos 2\theta e^{25} + \cos 2\theta \cos 2\phi e^{34} \\
V &= -\cos 2\phi e^{25} + e^{34} \\
W &= (\cos 2\theta \sin 2\phi e^1 - \sin 2\theta e^2 + i \sin 2\theta \cos 2\phi e^5) \wedge (e^4 - ie^3)
\end{aligned} \tag{B.10}$$

and this leads to a quick derivation of (3.40).

## C Absence of solutions with $\xi_i$ linearly dependent and $\xi_i \neq 0$

Let us consider the possibility

$$\xi_2 = u\xi_1 + v\xi_1^c \tag{C.1}$$

for some functions  $u, v$ , which defines a local  $SU(2)$  structure in five dimensions. We will use the conditions (3.22) and (3.23) that can be derived directly from Fierz relations and in particular do not rely on any aspects of the identity structure that we considered in the text. We also use the differential conditions (3.17)–(3.20). Recall from section 3.2 that we can then deduce (3.25) and  $\text{Re}[iZ^*W] = 0$ .

A calculation shows that  $\text{Re}[iZ^*W] = 0$  implies that

$$2|u|^2 \bar{\xi}_1 \gamma_{(2)} \xi_1 + uv^* \bar{\xi}_1^c \gamma_{(2)} \xi_1 + u^* v \bar{\xi}_1 \gamma_{(2)} \xi_1^c = 0. \tag{C.2}$$

In the special case that  $v = 0$  we deduce, for non-trivial  $\xi_1$ , that  $u = 0$  and we return to the Sasaki–Einstein case analysed just before the start of section 3.1. We continue, therefore, with  $v \neq 0$ . To proceed we derive the following expression for  $V$ :

$$V = -\frac{1 + |u|^2 + |v|^2}{2} i \bar{\xi}_1 \gamma_{(2)} \xi_1. \tag{C.3}$$

Next observe that  $K = -SK_5$  and using (3.15) we get

$$-3imK_5 = d[\ln(e^{4\Delta} S)] \tag{C.4}$$

and hence  $dK_5 = 0$ . We also have  $K_4 = \sin \zeta K_5$  and then using (3.37) we get

$$e^{4\Delta} K_4 = \frac{f}{4m} K_5. \tag{C.5}$$

From (3.19) we then conclude that  $V = 0$ . However, from (C.3) we see that this is not possible unless  $\xi_1 = 0$ .

## D Integrability of IIB supersymmetry conditions

Writing the variation of the gravitino appearing in (2.1) as  $\delta\psi_M = \mathcal{D}_M\epsilon$ , we calculate that

$$\mathcal{D}_{[M}\mathcal{D}_{N]}\epsilon = I_{MN}^{(1)}\epsilon + I_{MN}^{(2)}\epsilon^c \quad (\text{D.1})$$

where

$$\begin{aligned} I_{MN}^{(1)} = & \frac{1}{8}R_{MNS_1S_2}\Gamma^{S_1S_2} - \frac{1}{2}P_{[M}P_{N]}^* + \frac{i}{192}D_{[M}F_{N]S_1S_2S_3S_4}\Gamma^{S_1S_2S_3S_4} \\ & + \frac{1}{768}(F_{MNS_1}{}^{R_1R_2}F_{S_2S_3S_4R_1R_2}\Gamma^{S_1S_2S_3S_4} - 2F_{[M|S_1|}{}^{R_1R_2R_3}F_{N]S_2R_1R_2R_3}\Gamma^{S_1S_2}) \\ & + \frac{1}{9216}(-\Gamma_{MN}{}^{S_1S_2S_3S_4S_5S_6}G_{S_1S_2S_3}G_{S_4S_5S_6}^* - 9\Gamma_{MN}{}^{S_1S_2S_3S_4}G_{S_1S_2}{}^R G_{S_3S_4R}^* \\ & + 12\Gamma_{[M}{}^{S_1S_2S_3S_4S_5}G_{N]S_1S_2}G_{S_3S_4S_5}^* - 6\Gamma_{[M}{}^{S_1S_2S_3S_4S_5}G_{|S_1S_2S_3|}G_{N]S_4S_5}^* \\ & + 18\Gamma_{MN}{}^{S_1S_2}G_{S_1}{}^{R_1R_2}G_{S_2R_1R_2}^* + 36\Gamma_{[M}{}^{S_1S_2S_3}G_{|S_1S_2|}{}^R G_{N]S_3R}^* \\ & + 72\Gamma_{[M}{}^{S_1S_2S_3}G_{N]S_1}{}^R G_{S_2S_3R}^* + 72\Gamma^{S_1S_2S_3S_4}G_{[M|S_1S_2|}G_{N]S_3S_4}^* \\ & - 18\Gamma^{S_1S_2S_3S_4}G_{S_1S_2S_3}G_{MNS_4}^* + 18\Gamma^{S_1S_2S_3S_4}G_{MNS_1}G_{S_2S_3S_4}^* \\ & + 6\Gamma_{MN}G^{R_1R_2R_3}G_{R_1R_2R_3}^* - 72\Gamma_{[M}{}^S G_{N]}{}^{R_1R_2}G_{SR_1R_2}^* \\ & + 36\Gamma_{[M}{}^S G_{|S|}{}^{R_1R_2}G_{N]R_1R_2}^* - 288\Gamma^{S_1S_2}G_{[M|S_1|}{}^R G_{N]S_2R}^* \\ & - 54\Gamma^{S_1S_2}G_{S_1S_2}{}^R G_{MNR}^* + 54\Gamma^{S_1S_2}G_{MN}{}^R G_{S_1S_2R}^* \\ & - 144G_{[M}{}^{R_1R_2}G_{N]R_1R_2}^*) \end{aligned} \quad (\text{D.2})$$

and

$$\begin{aligned} I_{MN}^{(2)} = & \frac{1}{96}(\Gamma_{[M}{}^{S_1S_2S_3}D_{N]}G_{S_1S_2S_3} + 9\Gamma^{S_1S_2}D_{[M}G_{N]S_1S_2}) \\ & - \frac{i}{1536}(3G_{MN}{}^R F_{RS_1S_2S_3S_4}\Gamma^{S_1S_2S_3S_4} + 6G_{[M}{}^{R_1R_2}F_{R_1R_2S_1S_2S_3}\Gamma_{N]}{}^{S_1S_2S_3} \\ & - 6G_{S_1}{}^{R_1R_2}F_{[M|R_1R_2S_2S_3|}\Gamma_{N]}{}^{S_1S_2S_3} - 6G_{S_1S_2}{}^R F_{MNR S_3S_4}\Gamma^{S_1S_2S_3S_4} \\ & - 12G_{[M}{}^{R_1R_2}F_{N]R_1R_2S_1S_2}\Gamma^{S_1S_2} + 4G^{R_1R_2R_3}F_{[M|R_1R_2R_3|}\Gamma_{N]}{}^S \\ & + 4G^{R_1R_2R_3}F_{MNR_1R_2R_3}) \end{aligned} \quad (\text{D.3})$$

and we have used the self-duality of  $F$ . Note that this result is consistent with that in [45].

Setting this to zero, contracting with  $8\Gamma^N$  we deduce that

$$\begin{aligned}
& 2\left[-R_{MS} + P_M P_S^* + P_S P_M^* + \frac{1}{96}F_M^{R_1 R_2 R_3 R_4} F_{S R_1 R_2 R_3 R_4} \right. \\
& \quad \left. + \frac{1}{8}(G_M^{R_1 R_2} G_{S R_1 R_2}^* + G_S^{R_1 R_2} G_{M R_1 R_2}^* - \frac{1}{6}g_{MS} G^{R_1 R_2 R_3} G_{R_1 R_2 R_3}^*)\right] \Gamma^S \epsilon \\
& - \frac{i}{48} *_{10} [dF - \frac{i}{2}G \wedge G^*]_{S_1 S_2 S_3 S_4} \Gamma_M^{S_1 S_2 S_3 S_4} \epsilon + \frac{i}{12} *_{10} [dF - \frac{i}{2}G \wedge G^*]_{M S_1 S_2 S_3} \Gamma^{S_1 S_2 S_3} \epsilon \\
& - \frac{1}{96}(DG + P \wedge G^*)_{S_1 S_2 S_3 S_4} \Gamma_M^{S_1 S_2 S_3 S_4} \epsilon^c + \frac{1}{8}(DG + P \wedge G^*)_{M S_1 S_2 S_3} \Gamma^{S_1 S_2 S_3} \epsilon^c \\
& - \frac{1}{8}(D_R G^R_{S_1 S_2} + \frac{i}{6}G_{R_1 R_2 R_3} F^{R_1 R_2 R_3}_{S_1 S_2} - P^R G^*_{S_1 S_2 R}) \Gamma_M^{S_1 S_2} \epsilon^c \\
& + \frac{3}{4}(D_R G^R_{MS} + \frac{i}{6}G_{R_1 R_2 R_3} F^{R_1 R_2 R_3}_{MS} - P^R G^*_{MSR}) \Gamma^S \epsilon^c \\
& = -\frac{i}{24} G^*_{S_1 S_2 S_3} \Gamma_M^{S_1 S_2 S_3} \delta\lambda + \frac{3i}{8} G^*_{M S_1 S_2} \Gamma^{S_1 S_2} \delta\lambda + 4i P_M \delta\lambda^* . \tag{D.4}
\end{aligned}$$

Similarly, again using the self-duality of  $F$ , a calculation reveals that

$$\begin{aligned}
& i(D_M P^M + \frac{1}{24}G^{M_1 M_2 M_3} G_{M_1 M_2 M_3}) \epsilon^c + iD_{[S_1} P_{S_2]} \Gamma^{S_1 S_2} \epsilon^c \\
& + \frac{i}{24}(D_{[S_1} G_{S_2 S_3 S_4]} + P_{[S_1} G^*_{S_2 S_3 S_4]}) \Gamma^{S_1 S_2 S_3 S_4} \epsilon \\
& + \frac{i}{8}(D_M G^M_{S_1 S_2} - P_M G^*_{S_1 S_2} + \frac{i}{6}F_{S_1 S_2 M_1 M_2 M_3} G^{M_1 M_2 M_3}) \Gamma^{S_1 S_2} \epsilon \\
& = \Gamma^S D_S \delta\lambda - \frac{i}{960} \Gamma^{S_1 S_2 S_3 S_4 S_5} F_{S_1 S_2 S_3 S_4 S_5} \delta\lambda - i\Gamma^M \Gamma^S P_S \delta\psi_M^* \\
& - \frac{i}{24} \Gamma^M \Gamma^{S_1 S_2 S_3} G_{S_1 S_2 S_3} \delta\psi_M . \tag{D.5}
\end{aligned}$$

Suppose we have a supersymmetric configuration satisfying  $\delta\psi_M = \delta\lambda = 0$ . If we demand that it also satisfies the equation of motion and the Bianchi identity for  $G$ , the Bianchi identity for  $P$  and the Bianchi identity for the self-dual five-form  $F$ , then we conclude from (D.5) that the equation of motion for  $P$  is automatically satisfied and from (D.4) that

$$E_{MS} \Gamma^S \epsilon = 0 \tag{D.6}$$

where  $E_{MS} = 0$  is equivalent to Einstein's equations. Now the vector bilinear,  $K^M \equiv \bar{\epsilon} \Gamma^M \epsilon$  that can be constructed from a spinor of  $Spin(9, 1)$  is null. We can use it to set up a frame

$$ds^2 = 2e^+ e^- + e^a e^a \tag{D.7}$$

for  $a = 1, \dots, 9$  with  $K$ , as a one-form, equal to  $e^+$ . Following the argument of section 2.3 of [5], we conclude that (D.6) implies that the only component of  $E_{MS}$  that is not automatically zero is  $E_{++}$ , which thus is the only extra condition that needs to be imposed in order to get a full supersymmetric solution to the equations of motion.

For the class of solutions considered in the text, we have  $E_{++} = 0$ . To see this, we first note that the spinor ansatz (A.21) implies that the vector  $K^M$  only has non-vanishing components in the  $AdS_5$  directions. Next we observe that the Ricci tensor of the ten-dimensional metric has components in the  $AdS_5$  directions given by

$$R_{\mu\nu} = -\bar{g}_{\mu\nu}(4m^2 + 8(\bar{\nabla}\Delta)^2 + \bar{\nabla}^2\Delta) \quad (\text{D.8})$$

where  $\bar{g}$  is the metric on  $AdS_5$ . In addition the right hand side of the Einstein equation in (2.1) is also proportional to  $\bar{g}$ . Since  $\bar{g}_{++} = 0$ , in the frame (D.7), we conclude that  $E_{++} = 0$  is trivially satisfied.

## E Central charges

Recall that the central charge of the conformal field theory dual is determined by the five-dimensional Newton's constant [46]. The type IIB action takes the form

$$S = \frac{1}{2\kappa_{10}^2} \int_M \sqrt{G} [R(G) + \dots] , \quad (\text{E.1})$$

where  $G$  is the ten-dimensional metric. Assuming  $M = M_{4,1} \times M_5$  and using our warped-product ansatz (3.1), we get

$$S = \frac{1}{2\kappa_{10}^2} \int_{M_5} \sqrt{g_5} e^{8\Delta} \int_{M_{4,1}} \sqrt{g_E} [R(g_E) + 12m^2 \dots] , \quad (\text{E.2})$$

where  $g_E$  and  $g_5$  are the metrics on  $M_{4,1}$  and  $M_5$  respectively. The  $m^2$  term in the integrand appears since our  $AdS_5$  metric on  $M_{4,1}$  is normalised so that  $\text{Ric}(g_E) = -4m^2 g_E$ . We will also need the quantisation condition on the five-form which reads

$$\int_{M_5} F - \frac{i}{2} A \wedge G^* = c_0 N_{D3} \quad (\text{E.3})$$

where  $N_{D3}$  is the number of  $D3$ -branes and  $c_0$  is a constant the precise form of which we will not need.

Let us first consider the simplest case of where  $M_5$  is Sasaki–Einstein. We then have  $e^{8\Delta} = (f/4m)^2$ , and the quantisation condition gives  $f = c_0 N_{D3} m^5 / \text{vol}'(M_5)$  where  $\text{vol}'(M_5)$  is the volume of the Sasaki–Einstein metric normalised so that  $\text{Ric}(g_5) = 4g_5$ . In particular  $\text{vol}'(S^5) = \pi^3$ . Thus the type IIB dimensionally reduced action reads

$$S = \frac{1}{16\pi G_5^{\text{SE}}} \int_{M_{4,1}} \sqrt{g_E} [R(g_E) + 12m^2 \dots] , \quad (\text{E.4})$$

where

$$G_5^{\text{SE}} = \frac{2\kappa_{10}^2 \text{vol}'(M_5)}{\pi m^3 c_0^2 N_{D3}^2} . \quad (\text{E.5})$$

Next we consider the special class of solutions that we discussed in section 5. We first observe

$$\int_{M_5} \sqrt{g_5} e^{8\Delta} = -\frac{2f^2}{16^2 \times 27m^7} \int_{M_5} \partial_\lambda(h^2) d\lambda d\psi dy \sigma_1 \wedge \sigma_2 \quad (\text{E.6})$$

noting the remarkable fact that the integrand can be trivially integrated. Equally remarkable is the fact that the quantisation condition on the flux also takes a simple form. To carry out the integral, we first observe that a two-form potential for the three-form flux  $G$  is given by

$$A = \frac{1}{3m^2} \left( \frac{f}{4m} \right)^{1/2} \frac{h^{1/2}(g^2 - 1 - \lambda^2)^{1/2}}{\sqrt{6}\lambda g^2} (-\text{id}\lambda + g\lambda d\psi) \wedge (\sigma_2 - \text{i}\sigma_1) . \quad (\text{E.7})$$

It is then straightforward to deduce that

$$f^{-1} = \frac{1}{27 \times 8m^5 c_0 N_{D3}} \int_{M_5} \partial_\lambda(h^2/g^2) d\lambda d\psi dy \sigma_1 \wedge \sigma_2 . \quad (\text{E.8})$$

Substituting this into (E.6) gives an analytic expression for the Newton's constant for this class of solutions.

We finally focus on the local solutions that we found numerically that include the PW solution as a special case. In particular  $0 \leq \lambda \leq \lambda_c$  with  $h(\lambda_c) = 0$  and  $g(0) = 1$ . In this case we get

$$G_5^{(\text{sec5})} = -\frac{\kappa_{10}^2 h^2(0)}{4 \times 27\pi m^3 c_0^2 N_{D3}^2} \int d\psi dy \sigma_1 \wedge \sigma_2 . \quad (\text{E.9})$$

For the numerical solutions,  $\sigma_1^2 + \sigma_2^2$  gives the round metric on  $S^2$  so  $\int \sigma_1 \wedge \sigma_2 = 4\pi$ , while  $d\psi \wedge dy = 2d\delta \wedge d\gamma'$  and so when integrated contributes a factor of  $16\pi^2/(2-p)$ . Recall that  $h(0) = 2/p$  and that the value of  $p$  is determined by  $\lambda_c$  which specifies the numerical solution and that for the PW solution we have  $p = 1$ .

As a check on these formulae we can calculate the ratio of the central charges of the theories dual to  $AdS_5 \times S^5$  and the PW solution. We get

$$\frac{G_5^{\text{PW}}}{G_5^{S^5}} = \frac{32}{27} \quad (\text{E.10})$$

in agreement with [20].

More generally, the expression (E.9) shows that the ratio of central charges of two solutions in our new family of local solutions depends on the ratio of their values of  $1/p^2(2-p)$ . As this is not constant it indicates that the local solutions could not possibly represent exactly marginal deformations of the PW solution. Furthermore, if the local solutions were to somehow make physical sense, some restrictions on  $p$  would have to be imposed to ensure an algebraic central charge as implied by the general results on  $a$ -maximisation [43].

## References

- [1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [arXiv:hep-th/9711200].
- [2] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, “Supersymmetric AdS(5) solutions of M-theory,” *Class. Quant. Grav.* **21** (2004) 4335 [arXiv:hep-th/0402153].
- [3] J. P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, “G-structures and wrapped NS5-branes,” *Commun. Math. Phys.* **247** (2004) 421 [arXiv:hep-th/0205050].
- [4] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, “All supersymmetric solutions of minimal supergravity in five dimensions,” *Class. Quant. Grav.* **20** (2003) 4587 [arXiv:hep-th/0209114].
- [5] J. P. Gauntlett and S. Pakis, “The geometry of D = 11 Killing spinors”, *JHEP* **0304** (2003) 039 [arXiv:hep-th/0212008].
- [6] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” *Nucl. Phys. B* **536** (1998) 199 [arXiv:hep-th/9807080].
- [7] J. M. Figueroa-O’Farrill, “Near-horizon geometries of supersymmetric branes”, [arXiv:hep-th/9807149].
- [8] B. S. Acharya, J. M. Figueroa-O’Farrill, C. M. Hull and B. Spence, “Branes at conical singularities and holography,” *Adv. Theor. Math. Phys.* **2** (1999) 1249 [arXiv:hep-th/9808014].
- [9] D. R. Morrison and M. R. Plesser, “Non-spherical horizons. I,” *Adv. Theor. Math. Phys.* **3** (1999) 1 [arXiv:hep-th/9810201].
- [10] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, “Sasaki–Einstein metrics on  $S^2 \times S^3$ ,” *Adv. Theor. Math. Phys.* **8** (2004) 711 [arXiv:hep-th/0403002].
- [11] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, “A new infinite class of Sasaki–Einstein manifolds,” to appear in *Adv. Theor. Math. Phys.* [arXiv:hep-th/0403038].
- [12] D. Martelli and J. Sparks, “Toric Geometry, Sasaki–Einstein Manifolds and a New Infinite Class of AdS/CFT Duals,” to appear in *Commun. Math. Phys.* [arXiv:hep-th/0411238].
- [13] M. Bertolini, F. Bigazzi, and A. Cotrone, “New checks and subtleties for AdS/CFT and a-maximization”, *JHEP* **0412**, 024 (2004) [arXiv:hep-th/0411249].



- [14] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, “An Infinite Family of Superconformal Quiver Gauge Theories with Sasaki–Einstein Duals,” *JHEP* **0506**, 064 (2005) arXiv:hep-th/0411264.
- [15] M. Cvetič, H. Lu, D. N. Page and C. N. Pope, “New Einstein-Sasaki spaces in five and higher dimensions,” arXiv:hep-th/0504225.
- [16] M. Cvetič, H. Lu, D. N. Page and C. N. Pope, “New Einstein-Sasaki and Einstein spaces from Kerr-de Sitter,” arXiv:hep-th/0505223.
- [17] D. Martelli and J. Sparks, “Toric Sasaki-Einstein metrics on  $S^2 \times S^3$ ,” *Phys. Lett. B* **621**, 208 (2005) [arXiv:hep-th/0505027].
- [18] H. Lin, O. Lunin and J. Maldacena, “Bubbling AdS space and 1/2 BPS geometries,” *JHEP* **0410**, 025 (2004) arXiv:hep-th/0409174.
- [19] K. Pilch and N. P. Warner, “A new supersymmetric compactification of chiral IIB supergravity,” *Phys. Lett. B* **487** (2000) 22 [arXiv:hep-th/0002192].
- [20] A. Khavaev, K. Pilch and N. P. Warner, “New vacua of gauged  $N = 8$  supergravity in five dimensions,” *Phys. Lett. B* **487** (2000) 14 [arXiv:hep-th/9812035].
- [21] A. Karch, D. Lust and A. Miemiec, “New  $N = 1$  superconformal field theories and their supergravity description,” *Phys. Lett. B* **454** (1999) 265 [arXiv:hep-th/9901041].
- [22] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, “Renormalization group flows from holography supersymmetry and a c-theorem,” *Adv. Theor. Math. Phys.* **3** (1999) 363 [arXiv:hep-th/9904017].
- [23] R. G. Leigh and M. J. Strassler, “Exactly marginal operators and duality in four-dimensional  $N=1$  supersymmetric gauge theory,” *Nucl. Phys. B* **447** (1995) 95 [arXiv:hep-th/9503121].
- [24] J. H. Schwarz, “Covariant Field Equations Of Chiral  $N=2$   $D = 10$  Supergravity,” *Nucl. Phys. B* **226** (1983) 269.
- [25] P. S. Howe and P. C. West, “The Complete  $N=2$ ,  $D = 10$  Supergravity,” *Nucl. Phys. B* **238** (1984) 181.
- [26] S. F. Hassan, “T-duality, space-time spinors and R-R fields in curved backgrounds,” *Nucl. Phys. B* **568** (2000) 145 [arXiv:hep-th/9907152].
- [27] N. Hitchin, “Generalized Calabi-Yau manifolds,” *Quart. J. Math. Oxford Ser.* **54**, 281 (2003) [arXiv:math.dg/0209099].
- [28] C. Jeschek and F. Witt, “Generalised  $G_2$ -structures and type IIB superstrings,” *JHEP* **0503**, 053 (2005) [arXiv:hep-th/0412280].

- [29] M. Grana, J. Louis and D. Waldram, “Hitchin functionals in  $N = 2$  supergravity,” arXiv:hep-th/0505264.
- [30] J. P. Gauntlett, N. Kim, D. Martelli and D. Waldram, “Fivebranes wrapped on SLAG three-cycles and related geometry,” JHEP **0111** (2001) 018 [arXiv:hep-th/0110034].
- [31] J. P. Gauntlett, D. Martelli and D. Waldram, “Superstrings with intrinsic torsion,” Phys. Rev. D **69**, 086002 (2004) [arXiv:hep-th/0302158].
- [32] D. Martelli and J. Sparks, “G-structures, fluxes and calibrations in M-theory,” Phys. Rev. D **68**, 085014 (2003) [arXiv:hep-th/0306225].
- [33] E. J. Hackett-Jones and D. J. Smith, “Type IIB Killing spinors and calibrations,” JHEP **0411**, 029 (2004) [arXiv:hep-th/0405098].
- [34] J. F. G. Cascales and A. M. Uranga, “Branes on generalized calibrated submanifolds,” JHEP **0411**, 083 (2004) [arXiv:hep-th/0407132].
- [35] J. Gutowski and G. Papadopoulos, “AdS calibrations,” Phys. Lett. B **462** (1999) 81 [arXiv:hep-th/9902034].
- [36] J. Gutowski, G. Papadopoulos and P. K. Townsend, “Supersymmetry and generalized calibrations,” Phys. Rev. D **60** (1999) 106006 [arXiv:hep-th/9905156].
- [37] N. Halmagyi, K. Pilch, C. Romelsberger and N. P. Warner, “Holographic duals of a family of  $N = 1$  fixed points,” arXiv:hep-th/0506206.
- [38] O. Lunin and J. Maldacena, “Deforming field theories with  $U(1) \times U(1)$  global symmetry and their gravity duals,” JHEP **0505** (2005) 033 [arXiv:hep-th/0502086].
- [39] D. Martelli, J. Sparks and S. T. Yau, “The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds,” arXiv:hep-th/0503183.
- [40] A. Butti and A. Zaffaroni, “R-charges from toric diagrams and the equivalence of a-maximization and Z-minimization,” arXiv:hep-th/0506232.
- [41] Y. Tachikawa, “Five-dimensional supergravity dual of a-maximization,” arXiv:hep-th/0507057.
- [42] E. Barnes, E. Gorbatov, K. Intriligator and J. Wright, “Current correlators and AdS/CFT geometry,” arXiv:hep-th/0507146.
- [43] K. Intriligator and B. Wecht, “The exact superconformal R-symmetry maximizes a,” Nucl. Phys. B **667** (2003) 183 [arXiv:hep-th/0304128].
- [44] M. F. Sohnius, “Introducing Supersymmetry,” Phys. Rept. **128** (1985) 39.

- [45] G. Papadopoulos and D. Tsimpis, “The holonomy of IIB supercovariant connection,” *Class. Quant. Grav.* **20** (2003) L253 [arXiv:hep-th/0307127].
- [46] M. Henningson and K. Skenderis, “The holographic Weyl anomaly,” *JHEP* **9807** (1998) 023 [arXiv:hep-th/9806087].