Sasaki-Einstein manifolds and volume minimisation

Published version:

DOI:10.1007/s00220-008-0479-4

April 23, 2008

hep-th/yymmnnn
CERN-PH-TH/2006-039
HUTP-06/A0002

Sasaki–Einstein Manifolds and Volume Minimisation

Dario Martelli[1], James Sparks[2,3] and Shing–Tung Yau[2]

1: Department of Physics, CERN Theory Division
1211 Geneva 23, Switzerland

2: Department of Mathematics, Harvard University
One Oxford Street, Cambridge, MA 02138, U.S.A.

3: Jefferson Physical Laboratory, Harvard University
Cambridge, MA 02138, U.S.A.

dario.martelli@cern.ch sparks@math.harvard.edu
yau@math.harvard.edu

Abstract

We study a variational problem whose critical point determines the Reeb vector field for a Sasaki–Einstein manifold. This extends our previous work on Sasakian geometry by lifting the condition that the manifolds are toric. We show that the Einstein–Hilbert action, restricted to a space of Sasakian metrics on a link L in a Calabi–Yau cone X, is the volume functional, which in fact is a function on the space of Reeb vector fields. We relate this function both to the Duistermaat–Heckman formula and also to a limit of a certain equivariant index on X that counts holomorphic functions. Both formulae may be evaluated by localisation. This leads to a general formula for the volume function in terms of topological fixed point data. As a result we prove that the volume of a Sasaki–Einstein manifold, relative to that of the round sphere, is always an algebraic number. In complex dimension $n = 3$ these results provide, via AdS/CFT, the geometric counterpart of a–maximisation in four dimensional superconformal field theories. We also show that our variational problem dynamically sets to zero the Futaki invariant of the transverse space, the latter being an obstruction to the existence of a Kähler–Einstein metric.

Contents

1 Introduction and summary

1.1 Background

The AdS/CFT correspondence [1] is one of the most important advancements in string theory. It provides a detailed correspondence between certain conformal field theories and geometries, and has led to remarkable new results on both sides. A large class of examples consists of type IIB string theory on the background $\mathrm{AdS}_5 \times L$, where L is a Sasaki–Einstein five–manifold and the dual theory is a four–dimensional $\mathcal{N} = 1$ superconformal field theory [2, 3, 4, 5]. This has recently led to considerable interest in Sasaki–Einstein geometry.

Geometry

Recall that a *Sasakian* manifold (L, g_L) is a Riemannian manifold of dimension $(2n-1)$ whose metric cone

$$g_{C(L)} = \mathrm{d}r^2 + r^2 g_L \tag{1.1}$$

is Kähler. (L, g_L) is *Sasaki–Einstein* if the cone (1.1) is also Ricci–flat. It follows that a Sasaki–Einstein manifold is a positively curved Einstein manifold. The canonical example is an odd–dimensional round sphere S^{2n-1}; the metric cone (1.1) is then $\mathbb{C}^n$ with its flat metric.

All Sasakian manifolds have a canonically defined Killing vector field ξ, called the *Reeb* vector field. This vector field will play a central role in this paper. To define ξ note that, since the cone is Kähler, it comes equipped with a covariantly constant complex structure tensor J. The Reeb vector field is then defined to be

$$\xi = J\left(r\frac{\partial}{\partial r}\right) . \tag{1.2}$$

As a vector field on the link[1] $L = \{r = 1\}$ this has norm one, and hence its orbits define a *foliation* of L. Either these orbits all close, or they don't. If they all close, the flow generated by ξ induces a $U(1)$ action on L which, since the vector field is nowhere–vanishing, is locally free. The orbit, or quotient, space is then a Kähler orbifold, which is a manifold if the $U(1)$ action is actually free. These Sasakian metrics are referred to as quasi–regular and regular, respectively. More generally, the generic orbits of ξ might not close. In this case there is no quotient space, and the Kähler structure exists only as a transverse structure. The closure of the orbits of ξ is at least a two–torus, and thus these so–called irregular Sasakian metrics admit at least a two–torus of isometries. This will also be crucial in what follows. Note that Sasakian geometries are then sandwiched between two Kähler geometries: one of complex dimension n on the cone, and one, which is generally only a transverse structure, of dimension $n-1$. For Sasaki–Einstein manifolds, the transverse metric is in fact *Kähler–Einstein*.

Until very recently, the only explicit examples of simply–connected Sasaki–Einstein manifolds in dimension five (equivalently complex dimension $n = 3$) were the round sphere S^5 and the homogeneous metric on $S^2 \times S^3$, known as $T^{1,1}$ in the physics literature. These are both regular Sasakian structures, the orbit spaces being $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ with their Kähler–Einstein metrics. All other Sasaki–Einstein metrics, in dimension five, were known only through existence arguments. The remaining regular metrics are based on circle bundles over del Pezzo surfaces dP_k, $3 \leq k \leq 8$, equipped with their Kähler Einstein metrics – these are known to exist through the work of Tian and Yau [6, 7]. On the other hand, Boyer and Galicki have produced many examples of quasi–regular Sasaki–Einstein metrics using existence results of Kollár

[1] We prefer this choice of terminology to "base of the cone", or "horizon".

and collaborators for Kähler–Einstein metrics on orbifolds. For a review of their work, see [8].

In references [9, 10, 11] infinite families of explicit inhomogeneous Sasaki–Einstein metrics in all dimensions have been constructed. In particular, when $n = 3$ there is a family of cohomogeneity one five–metrics, denoted $Y^{p,q}$ where $q < p$ with p, q positive integers [10]. This has subsequently been generalised to a three–parameter cohomogeneity two family $L^{a,b,c}$ [12, 13, 14]. Provided the integers a, b, c are chosen such that $L^{a,b,c}$ is smooth and simply–connected, these manifolds are all diffeomorphic to $S^2 \times S^3$. Further generalisations in complex dimension $n \geq 4$ have appeared in [15, 16, 17]. The metrics $Y^{p,q}$ are quasi–regular when $4p^2 - 3q^2$ is a square. However, for general $q < p$, they are *irregular*. These were the first examples of irregular Sasaki–Einstein manifolds, which in particular disproved the conjecture of Cheeger and Tian [18] that irregular Sasaki–Einstein manifolds do not exist.

Field theory

In general, the field theory duals of Sasaki–Einstein five–manifolds may be thought of as arising from a stack of D3–branes sitting at the apex $r = 0$ of the Ricci–flat Kähler cone (1.1). Alternatively, the Calabi–Yau geometry may be thought of as arising from the moduli space of the Higgs branch of the gauge theory on the D3–branes. Through simple AdS/CFT arguments, one can show that the symmetry generated by the Reeb vector field and the volume of the Sasaki–Einstein manifold correspond to the R–symmetry and the a central charge of the AdS/CFT dual superconformal field theory, respectively. All $\mathcal{N} = 1$ superconformal field theories in four dimensions possess a global R–symmetry which is part of the superconformal algebra. The a central charge appears as a coefficient in the one–point function of the trace of the energy–momentum tensor on a general background[2], and its value may be computed exactly, once the R–symmetry is correctly identified. A general procedure that determines this symmetry is a–maximisation [19]. One defines a function a_{trial} on an appropriate space of potential (or "trial") R–symmetries. The local maximum of this function determines the R–symmetry of the theory at its superconformal point. Moreover, the critical value of a_{trial} is precisely the central charge a of the superconformal theory. Since a_{trial} is a cubic

[2]The other coefficient is usually called c. However, superconformal field theories with a Sasaki–Einstein dual have $a = c$.

function with rational coefficients, it follows[3] that the R–charges of fields are *algebraic numbers* [19].

The AdS/CFT correspondence relates these R–charges to the volume of the dual Sasaki–Einstein manifold, as well as the volumes of certain supersymmetric three–dimensional submanifolds of L. In particular, we have the relation [20, 21]

$$\frac{a_L}{a_{S^5}} = \frac{\mathrm{vol}[S^5]}{\mathrm{vol}[L]} \ . \tag{1.3}$$

Since the left–hand side is determined by a–maximisation, we thus learn that the volume of a Sasaki–Einstein five–manifold vol$[L]$, relative to that of the round sphere, is an algebraic number. Moreover, in the field theory, this number has been determined by a finite dimensional extremal problem. Our aim in [22], of which this paper is a continuation, was to try to understand, from a purely *geometrical* viewpoint, where these statements are coming from.

Toric geometries and their duals

Given a Sasaki–Einstein manifold, it is in general a difficult problem to determine the dual field theory. However, in the case that the local Calabi–Yau singularity is *toric* there exist techniques that allow one to determine a dual gauge theory, starting from the combinatorial data that defines the toric variety. Using these methods it has been possible to construct gauge theory duals for the infinite family of Sasaki–Einstein manifolds $Y^{p,q}$ [23, 24], thus furnishing a countably infinite set of AdS/CFT duals where both sides of the duality are known explicitly. Indeed, remarkable agreement was found between the geometrical computation in the case of the $Y^{p,q}$ metrics [10, 23] and the a–maximisation calculation [24, 25] for the corresponding quiver gauge theories. Thus the relation (1.3) was confirmed for a non–trivial infinite family of examples. Further developments [26, 27, 28, 29, 30, 31] have resulted in the determination of families of gauge theories that are dual to the wider class of toric Sasaki–Einstein manifolds $L^{a,b,c}$ [28, 32] (see also [33]).

[3]Since we make a similar claim in this paper, we recall here a proof of this fact: suppose we have a vector $v \in \mathbb{R}^s$ which is an isolated zero of a set of polynomials in the components of v with *rational* coefficients. Consider the Galois group Gal$(\mathbb{C}/\mathbb{Q})$. This group fixes the set of polynomials, and thus in particular the Galois orbit of the zeros is finite. An *algebraic number* may be defined as an element of $\mathbb{C}$ with finite Galois orbit, and thus we see that the components of the vector v are algebraic numbers. We thank Dorian Goldfeld for this argument. Recall also that the set of algebraic numbers form a field. Thus, in the present example, the R–charges of fields being algebraic implies that the a–central charge, which is a polynomial function of the R–charges with rational coefficients, is also an algebraic number.

Of course, given this success, it is natural to try to obtain a general understanding of the geometry underlying a–maximisation and the AdS/CFT correspondence. To this end, in [22] we studied a variational problem on a space of toric Sasakian metrics. Let us recall the essential points of [22]. Let (X, ω) be a toric Kähler cone of complex dimension n. This means that X is an affine toric variety, equipped with a conical Kähler metric that is invariant under a holomorphic action of the n–torus $\mathbb{T}^n$. X has an isolated singular point at the tip of the cone, the complement of which is $X_0 = C(L) \cong \mathbb{R}_+ \times L$. A conical metric on X which is Kähler (but in general not Ricci–flat) then gives a Sasakian metric on the link L. The moment map for the torus action exhibits X as a Lagrangian $\mathbb{T}^n$ fibration over a strictly convex rational polyhedral cone[4] $\mathcal{C}^* \subset \mathtt{t}_n^* \cong \mathbb{R}^n$. This is a subset of $\mathbb{R}^n$ of the form

$$\mathcal{C}^* = \{y \in \mathbb{R}^n \mid (y, v_a) \geq 0,\ a = 1, \ldots, D\}\ . \tag{1.4}$$

Thus $\mathcal{C}^*$ is made by intersecting D hyperplanes through the origin in order to make a convex polyhedral cone. Here $y \in \mathbb{R}^n$ are coordinates on $\mathbb{R}^n$ and v_a are the inward pointing normal vectors to the D hyperplanes, or *facets*, that bound the polyhedral cone.

The condition that X is Calabi–Yau implies that the vectors v_a may, by an appropriate $SL(n;\mathbb{Z})$ transformation of the torus, be all written as $v_a = (1, w_a)$. In particular, in complex dimension $n = 3$ we may therefore represent any toric Calabi–Yau cone X by a convex lattice polytope in $\mathbb{Z}^2$, where the vertices are simply the vectors w_a. This is usually called the toric diagram. Note that the cone $\mathcal{C}^*$ may also be defined in terms of its generating edge vectors $\{u_\alpha\}$ giving the directions of the lines going through the origin. When $n = 3$ the projection of these lines onto the plane with normal $(1, 0, 0)$ are the external legs of the so–called pq–web appearing in the physics literature. These are also weight vectors for the torus action and generalise to the non–toric case.

For a toric Kähler cone (X, ω), one can introduce symplectic coordinates (y_i, ϕ_i) where $\phi_i \sim \phi_i + 2\pi$ are angular coordinates along the orbits of the torus action, and y_i are the associated moment map coordinates. These may be considered as coordinates on $\mathtt{t}_n^* \cong \mathbb{R}^n$. The symplectic (Kähler) form is then

$$\omega = \sum_{i=1}^{n} \mathrm{d}y_i \wedge \mathrm{d}\phi_i\ . \tag{1.5}$$

[4]We make a change of notation from our previous paper [22]: specifically, we exchange the roles of cone and dual cone. This is more in line with algebro–geometric terminology, and is more natural in the sense that the moment cone $\mathcal{C}^*$ lives in the dual Lie algebra $\mathtt{t}_n^*$ of the torus.

In this coordinate system, the metric degrees of freedom are therefore entirely encoded in the complex structure tensor J – see [22] for further details.

In [22] we considered the space of all smooth toric Kähler cone metrics on such an affine toric variety X. The space of metrics naturally factors into the space of Reeb vector fields, which live in the interior $\mathcal{C}_0$ of the dual done $\mathcal{C} \subset \mathtt{t}_n \cong \mathbb{R}^n$ to $\mathcal{C}^*$, and then an infinite dimensional space of transverse Kähler metrics. A general Reeb vector field may be written

$$\xi = \sum_{i=1}^{n} b_i \frac{\partial}{\partial \phi_i} \tag{1.6}$$

where $b \in \mathcal{C}_0 \subset \mathbb{R}^n$. In this toric setting, the remaining degrees of freedom in the metric are described by the space of all homogeneous degree 1 functions on $\mathcal{C}^*$ which are smooth up to the boundary (together with a convexity requirement).

The main result of [22] is that the Einstein–Hilbert action on L, restricted to this space of toric Sasakian metrics on L, reduces to the volume function $\mathrm{vol}[L] : \mathcal{C}_0 \to \mathbb{R}$, which depends only on the Reeb vector field in $\mathcal{C}_0$. Moreover, this is essentially just the Euclidean volume of the polytope formed by $\mathcal{C}^*$ and the hyperplane $2(b, y) = 1$. This depends only on b and the toric data $\{v_a\}$. In particular, for $n = 3$, we have the formula

$$V(b) \equiv \frac{\mathrm{vol}[L](b)}{\mathrm{vol}[S^5]} = \frac{1}{b_1} \sum_{a=1}^{D} \frac{(v_{a-1}, v_a, v_{a+1})}{(b, v_{a-1}, v_a)(b, v_a, v_{a+1})} = \frac{1}{b_1} \sum_{a=1}^{D} \frac{(v_{a-1}, v_a, v_{a+1})}{(b, u_a)(b, u_{a+1})} \tag{1.7}$$

for the normalised volume of L. The symbol $(\cdot, \cdot, \cdot)$ denotes a 3×3 determinant, while $(\cdot, \cdot)$ is the usual scalar product on $\mathbb{R}^n$ (or dual pairing between $\mathtt{t}_n$ and $\mathtt{t}_n^*$, whichever the reader prefers). The function $V(b)$ diverges to $+\infty$ at the boundary $\partial\mathcal{C}$ of $\mathcal{C}$ – this is because the Reeb vector field develops a fixed point set in this limit, as will be explained in section 2.

Once the critical Reeb vector field $b = b_*$ is obtained one can compute the volume of the Sasaki–Einstein manifold, as well as the volumes of certain toric submanifolds, without explicit knowledge of the metric[5]. From the explicit form of the Einstein–Hilbert action, it follows that the ratios of these volumes to those of round spheres are in general algebraic numbers. This method of determining the critical Reeb vector field, and the corresponding volume, has been referred to as "Z–minimisation", where Z is just the restriction of the function $V(b)$ to the hyperplane $b_1 = n$. Indeed, the

[5] In [22] the issue of existence of this metric was not addressed. However, the real Monge–Ampère equation derived in [22] has recently been shown to always admit a solution [34], thus solving the existence problem for toric Sasaki–Einstein manifolds.

results of [22], for $n = 3$, were interpreted as the geometric "dual" of a–maximisation for the case that the Sasaki–Einstein manifolds, and hence the superconformal gauge theories, were toric.

An analytic proof of the equivalence of these two optimisation problems was given in the work of [29], modulo certain assumptions on the matter content of the field theory. The key point is that, following on from results in [28], the trial a–function a_{trial} for the gauge theory may be defined in closed form in terms of the toric data, *i.e.* the normal vectors $\{v_a\}$, independently of the precise details of the gauge theory – for example the form of the superpotential. This is *a priori* a function of $D-1$ variables, the trial R–charges, where D is the number of facets of the cone. The global baryonic symmetries are $U(1)_B^{D-3}$ [28]. Once one maximises a_{trial} over this space, one is left with a function of two variables which geometrically are the components of the Reeb vector field. Rather surprisingly, the functions a_{trial} and $1/Z$ are then identically equal[6]. This of course explains why maximising a in the field theory is the same as minimising Z in the geometry. Further work on the relation between a–maximisation and Z–minimisation has appeared in [35, 36, 37].

1.2 Outline

The main result of the present work is to extend to general Sasaki–Einstein manifolds the toric results obtained in [22]. This was initially a technical problem – some of the methods described above simply do not extend when X is not toric. However, in the process of solving this problem, we will also gain further insight into the results of [22].

We begin by fixing a complex manifold X, which is topologically a real cone over a compact manifold L. Thus $X_0 = \mathbb{R}_+ \times L$, where $r > 0$ is a coordinate on $\mathbb{R}_+$, and $r = 0$ is always an isolated singular point of X, unless L is a sphere. For most of the paper it will be irrelevant whether we are referring to X_0 or the singular cone X. This is because we shall mainly be interested in the Sasakian geometry of the link L – the embedding into X is then purely for convenience, since it is generally easier to work with the Kähler geometry of the cone. Since we are interested in Ricci–flat Kähler metrics, we certainly require the canonical bundle of X_0 to be trivial. We implement this by assuming[7] we have a nowhere vanishing holomorphic $(n, 0)$–form Ω on X_0. By

[6]This fact was also observed by two of us (D.M. and J.F.S.) in unpublished work.

[7]Ω so defined is far from unique – one is always free to multiply by a nowhere vanishing holomorphic function. This is an important difference to the case of compact Calabi–Yau manifolds. This degree of freedom can be fixed by imposing a "homogeneous gauge" for Ω, as we discuss later.

definition the singularity X is therefore *Gorenstein*.

We also require there to be a space of Kähler cone metrics on X. The space of orbits of every homothetic vector field $r\partial/\partial r$ in this space is required to be diffeomorphic to L. The closure of the orbits of the corresponding Reeb vector field $\xi = J(r\partial/\partial r)$ defines some torus $\mathbb{T}^m \subset \mathrm{Aut}(X)$ since ξ is holomorphic. Thus, as for the toric geometries above, we fix a (maximal) torus $\mathbb{T}^s \subset \mathrm{Aut}(X)$ and assume that it acts isometrically on our space of Kähler cone metrics on X. The Reeb vector fields in our space of metrics are all required to lie in the Lie algebra of this torus. Note that there is no loss of generality in making these assumptions: the Reeb vector field for a Sasaki–Einstein metric defines some torus that acts isometrically: by going "off–shell" and studying a space of *Sasakian* metrics on which this torus (or a larger torus containing this) also acts isometrically, we shall learn rather a lot about Sasaki–Einstein manifolds, realised as critical points of the Einstein–Hilbert action on this space of metrics.

The first result is that the Einstein–Hilbert action $\mathcal{S}$ on L, restricted to the space of Sasakian metrics, is essentially just the volume functional vol$[L]$ of L. More precisely, we prove that

$$\mathcal{S} = 4(n-1)(1+\gamma-n)\mathrm{vol}[L] \tag{1.8}$$

where one can show that, for any Kähler cone metric, there exists a gauge in which Ω is homogeneous degree γ under $r\partial/\partial r$, where γ is unique. Given any homothetic vector field $r\partial/\partial r$, $cr\partial/\partial r$ is another homothetic vector field for a Kähler cone metric on X, where c is any positive constant[8]. Setting to zero the variation of (1.8) in this direction gives $\gamma = n$, since vol$[L]$ is homogeneous degree $-n$ under this scaling. Thus we may think of $\mathcal{S}$ as the volume functional:

$$\mathcal{S} = 4(n-1)\mathrm{vol}[L] \tag{1.9}$$

provided we consider only metrics for which Ω is homogeneous degree n under $r\partial/\partial r$. This condition is the generalisation of the constraint $b_1 = n$ in the context of toric geometries [22].

The next result is that the volume of the link L depends only on the Reeb vector field ξ, and not on the remaining degrees of freedom in the metric. The first and second

[8]This is a transverse homothety, in the language of Boyer and Galicki [38].

derivatives of this volume function are computed in section 3:

$$\begin{aligned}\mathrm{dvol}[L](Y) &= -n\int_L \eta(Y)\mathrm{d}\mu \\ \mathrm{d}^2\mathrm{vol}[L](Y,Z) &= n(n+1)\int_L \eta(Y)\eta(Z)\mathrm{d}\mu \ .\end{aligned} \tag{1.10}$$

Here Y, Z are holomorphic Killing vector fields in $\mathtt{t}_s$, the Lie algebra of the torus $\mathbb{T}^s$, and $\eta(Y)$ denotes the contraction of Y with the one–form η, the latter being dual to the Reeb vector field. In particular, note that the second equation shows that $\mathrm{vol}[L]$ is strictly convex – one may use this to argue uniqueness of critical points. We shall return to discuss the first equation in detail later. Note that, when the background (L, g_L) is Sasaki–Einstein, the right hand sides of these formulae essentially appeared in [36]. In this context these formulae arose from Kaluza–Klein reduction on $\mathrm{AdS}_5 \times L$. In particular, we see that the first derivative $\mathrm{dvol}[L](Y)$ is proportional to the coefficient τ_{RY} of a two–point function in the CFT, via the AdS/CFT correspondence. This relates the geometric problem considered here to τ–minimisation [39] in the field theory.

Since the torus $\mathbb{T}^s$ acts isometrically on each metric, there is again a moment map and a fixed convex rational polyhedral cone $\mathcal{C}^* \subset \mathtt{t}_s^*$. Any Reeb vector field must then lie in the interior of the dual cone $\mathcal{C}$ to $\mathcal{C}^*$. The space of Reeb vector fields under which Ω has charge n form a convex polytope Σ in $\mathcal{C}_0$ – this is formed by the hyperplane $b_1 = n$ in the toric case [22]. The boundary of $\mathcal{C}$ is a singular limit, since ξ develops a fixed point set there. We again write the Reeb vector field as

$$\xi = \sum_{i=1}^{s} b_i \frac{\partial}{\partial \phi_i} \tag{1.11}$$

where $\partial/\partial\phi_i$ generate the torus action. The volume of the link is then a *function*

$$\mathrm{vol}[L] : \mathcal{C}_0 \rightarrow \mathbb{R} \ . \tag{1.12}$$

At this point, the current set–up is not dissimilar to that in our previous paper [22] – essentially the only difference is that the torus $\mathbb{T}^s$ no longer has maximal possible dimension $s = n$. The crucial point is that, for $s < n$, the volume function (1.12) is no longer given just in terms of the combinatorial data specifying $\mathcal{C}^*$. It should be clear, for example by simply restricting the toric case to a subtorus, that this data is insufficient to determine the volume as a function of b.

The key step to making progress in general is to write the volume functional of L in the form

$$\mathrm{vol}[L] = \frac{1}{2^{n-1}(n-1)!}\int_X e^{-r^2/2}\, e^{\omega} \ . \tag{1.13}$$

The integrand in (1.13) may be interpreted as an equivariantly closed form, since $r^2/2$ is precisely the Hamiltonian function for the Reeb vector field ξ. The right hand side of (1.13) takes the form of the Duistermaat–Heckman formula [40, 41]. This may then be *localised* with respect to the Reeb vector field ξ. Our general formula is:

$$V(b) \equiv \frac{\mathrm{vol}[L](b)}{\mathrm{vol}[S^{2n-1}]} = \sum_{\{F\}} \frac{1}{d_F} \int_F \prod_{\lambda=1}^{R} \frac{1}{(b, u_\lambda)^{n_\lambda}} \left[\sum_{a \geq 0} \frac{c_a(\mathcal{E}_\lambda)}{(b, u_\lambda)^a} \right]^{-1} . \qquad (1.14)$$

Since ξ vanishes only at the tip of the cone $r = 0$, the right hand side of (1.14) requires one to resolve the singular cone X – the left hand side is of course independent of the choice of resolution. This resolution can always be made, and *any* equivariant (orbifold) resolution will suffice[9]. The first sum in (1.14) is over connected components of the fixed point set of the $\mathbb{T}^s$ action on the resolved space W. The u_λ are weights of the $\mathbb{T}^s$ action on the normal bundle to each connected component F of fixed point set. These essentially enter into defining the moment cone $\mathcal{C}^*$. The $c_a(\mathcal{E}_\lambda)$ are Chern classes of the normal bundle to F. The term d_F denotes the order of an orbifold structure group – these terms are all equal to 1 when the resolved space W is completely smooth. Precise definitions will appear later in section 5. We also note that the right hand side of (1.14) is homogeneous degree $-n$ in b, precisely as in the toric case, and is manifestly a rational function of b with rational coefficients, since the weight vectors u_λ and Chern classes $c_a(\mathcal{E}_\lambda)$ (and $1/d_F$) are generally *rational*. From this formula for the volume, which recall is essentially the Einstein–Hilbert action, it follows immediately that the volume of a Sasaki–Einstein manifold, relative to that of the round sphere, is an algebraic number. When the complex dimension $n = 3$, this result is AdS/CFT "dual" to the fact that the central charges of four dimensional superconformal field theories are indeed algebraic numbers.

When X is toric, that is the torus action is n–dimensional, the formula (1.14) simplifies and reduces to a sum over the fixed points of the torus action over any toric resolution of the Kähler cone:

$$V(b) = \sum_{p_A \in P} \prod_{i=1}^{n} \frac{1}{(b, u_i^A)} . \qquad (1.15)$$

[9] We may in general resolve X, in an equivariant manner, by blowing up the Fano orbifold V associated to any quasi–regular Kähler cone structure on X, as we shall explain later. It is interesting to note that, when constructing the gauge theory that lives on D3–branes probing the conical singularity, one also makes such a resolution. Specifically, an exceptional collection of sheaves on V may then, in principle, be used to derive the gauge theory (see *e.g.* [42, 43]).

Here p_A are the vertices of the polytope P of the resolved toric variety – these are the very same vertices that enter into the topological vertex in topological string theory. The $u_i^A \in \mathbb{Z}^n$, $i = 1, \ldots, n$, are the n primitive edge vectors that describe the $A-$th vertex. The right hand side of formula (1.15) is of course necessarily independent of the choice of resolving polytope P, in order that this formula makes sense. It is a non–trivial fact that (1.15) is equivalent to the previous toric formula for the volume (1.7). For instance, the number of terms in the sum in (1.15) is given by the Euler number of any crepant resolution – that is, the number of gauge groups of the dual gauge theory; while the number of terms in (1.7) is D – that is, the number of facets of the polyhedral cone. However, it can be shown that (1.15) is finite everywhere in the interior $\mathcal{C}_0$ of the polyhedral cone $\mathcal{C}$, and has simple poles at the facets of $\mathcal{C}$, precisely as the expression[10] (1.7).

These formulae show that the volumes of general Sasakian manifolds, as a function of the Reeb vector field, are *topological*. For toric geometries, this topological data is captured by the normal vectors v_a that define $\mathcal{C}^*$. For non–toric geometries, there are additional Chern classes that enter. In fact, we will also show that these formulae may be recovered from a particular limit of an equivariant index on X, which roughly counts holomorphic functions according to their charges under $\mathbb{T}^s$. Specifically, we define

$$C(q, X) = \mathrm{Tr}\{q \mid \mathcal{H}^0(X)\} \ . \tag{1.16}$$

Here $q \in (\mathbb{C}^*)^s$ lives in (a subspace of) the algebraic torus associated to $\mathbb{T}^s$, and the notation denotes a trace of the induced action of this torus on the space of holomorphic functions on X[11]. This equivariant index is clearly a holomorphic invariant, and the volume of the corresponding Sasakian link will turn out to appear as the coefficient of the leading divergent term of this index in a certain expansion:

$$V(b) = \lim_{t \to 0} t^n \ C(q = \exp(-tb), X) \ . \tag{1.17}$$

The character $C(q, X)$ has a pole of order n at $q_i = 1$, $i = 1, \ldots, n$, and this limit picks out the leading behaviour near this pole. For regular Sasaki–Einstein manifolds this relation, with critical $b = b_*$, was noted already in [46] – the essential difference here is that we interpret this relation as a *function* of b, by using the equivariant index rather than just the index. This result is again perhaps most easily described

[10] This follows from the fact that the Duistermaat–Heckman formula reduces to the *characteristic function* [44] of the cone (see also [45]), as we will show later.

[11] We will not worry too much about where this trace converges, as we are mainly interested in its behaviour near a certain pole.

in the toric setting. In this case, the equivariant index counts holomorphic functions on X weighted by their $U(1)^n$ charges. It is known that these are in one–to–one correspondence with integral points inside the polyhedral cone, the $U(1)^n$ charges being precisely the location of the lattice points in $\mathcal{S}_{\mathcal{C}^*} = \mathbb{Z}^n \cap \mathcal{C}^*$. In a limit in which the lattice spacing tends to zero, the distribution of points gives an increasingly better approximation to a volume measure on the cone. The slightly non–obvious point is that this measure in fact reduces to the measure on the Sasakian link L, giving (1.17).

From a physical viewpoint, the equivariant index[12] is counting BPS mesonic operators of the dual gauge theory, weighted by their $U(1)^s$ charges. This is because the set of holomorphic functions on the Calabi–Yau cone correspond to elements of the chiral ring in the dual guage theory. In [47, 48] some indices counting BPS operators of superconformal field theories have been introduced and studied. In contrast to the equivariant index defined here, those indices take into account states with arbitrary spin. On the other hand, the fact that the index considered here is equivariant means that it is twisted with respect to the global flavour symmetries of the gauge theory. Moreover, the equivariant index is a holomorphic invariant, and may be computed without knowledge of the Kaluza–Klein spectrum. Rather interestingly, our results in section 6 may then be interpreted as saying that the trial central charge of the dual gauge theory emerges as an asymptotic coefficient of the generating function of (scalar) BPS operators. It would be very interesting to study in more detail the relation between these results and the work of [47, 48].

Let us now return to the expression for the first derivative of vol$[L]$ in (1.10). This is zero for a Sasaki–Einstein manifold, since Sasaki–Einstein metrics are critical points of the Einstein–Hilbert action. Moreover, for fixed Reeb vector field ξ, this derivative is independent of the metric. Thus, dvol$[L]$ is a linear map on a space of holomorphic vector fields which is also a holomorphic invariant and vanishes identically when ξ is the critical Reeb vector field for a Sasaki–Einstein metric. Those readers that are familiar with Kähler geometry will recognise these as properties of the *Futaki invariant* in Kähler geometry [49]. Indeed, if ξ is quasi–regular, we show that

$$\mathrm{dvol}[L](Y) = -\frac{\ell}{2} \cdot F[J_V(Y_V)] \ . \tag{1.18}$$

Here Y_V is the push down of Y to the Fano orbifold V, J_V is its complex structure tensor, and ℓ is the length of the circle fibre. We will define the Futaki invariant F, and review some of its properties, in section 4. Thus the dynamical problem of finding

[12]We would like to thank S. Benvenuti and A. Hanany for discussions on this.

the critical Reeb vector field can be understood as varying ξ such that the transverse Kähler orbifold V, when ξ is quasi–regular, has zero Futaki invariant. This is a well–known obstruction to the existence of a Kähler–Einstein metric on V [49]. This new interpretation of the Futaki invariant places the problem of finding Kähler–Einstein metrics on Fanos into a more general context. For example, the Futaki invariant of the first del Pezzo surface is well–known to be non–zero. It therefore cannot admit a Kähler–Einstein metric. However, the canonical circle bundle over this del Pezzo surface does admit a Sasaki–Einstein metric [10, 23] – the Kähler–Einstein metric is only a transverse metric[13]. From the point of view of our variational problem, there is nothing mysterious about this: the vector field that rotates the S^1 fibre of the circle bundle is simply not a critical point of the Einstein–Hilbert action.

This also leads to a result concerning the isometry group of Sasaki–Einstein manifolds. In particular, we will argue[14] that the isometry group of a Sasaki–Einstein manifold is a *maximal* compact subgroup $K \supset \mathbb{T}^s$ of the holomorphic automorphism group $\mathrm{Aut}(X)$ of the Kähler cone. The Reeb vector field then lies in the centre of the Lie algebra of K. This gives a rigorous account of the expectation that flavour symmetries of the field theory must be realised as isometries of the dual geometry, and that the R–symmetry does not mix with non–abelian flavour symmetries.

Let us conclude this outline with an observation on the types of algebraic numbers that arise from the volume minimisation problem studied here. We have shown that all Sasaki–Einstein manifolds have a normalised volume, relative to the round sphere, which is an algebraic number, that is the (real) root of a polynomial over the rationals. Let us say that the *degree*, denoted $\deg(L)$, of a Sasaki–Einstein manifold L is the degree of this algebraic number. Thus, for instance, if $\deg(L) = 2$ the normalised volume is quadratic irrational. Recall that the *rank* of a Sasaki–Einstein manifold is the dimension of the closure of the orbits of the Reeb vector field. We write this as $\mathrm{rank}(L)$. We then make the following *conjecture*: for a Sasaki–Einstein manifold L of dimension $2n-1$, the degree and rank are related as follows

$$\deg(L) = (n-1)^{\mathrm{rank}(L)-1} \ . \qquad (1.19)$$

For example, all quasi–regular Sasaki–Einstein manifolds have degree one since the normalised volume is a rational number. By definition they also have rank one. In all the irregular cases that we have examined, which by now include a number of infinite

[13] In general, we may *define* the Futaki invariant for transverse metrics by (1.18).
[14] We will give a rigorous proof only for quasi–regular structures.

families, this relation holds. Although we have obtained explicit expressions for the volume function $V(b)$, it seems to be a non–trivial fact that one obtains algebraic numbers of such low degree obeying (1.19) from extremising this function – *a priori*, the degree would seem to be much larger. It would be interesting to investigate this further, and prove or disprove the conjecture.

2 Sasakian geometry

In this section we present a formulation of Sasakian geometry in terms of the geometry of Kähler cones. This way of formulating Sasakian geometry, although equivalent to the original description in terms of metric contact geometry, turns out to be more natural for describing the problems of interest here. A review of Sasakian geometry, where further details may be found, is contained in [8].

A central fact we use here, that will be useful for later computations, is that the radial coordinate r determines not only the link L in X, but also that r^2 may be interpreted as the Kähler potential for the Kähler cone [22]. A choice of Sasakian metric on L, for fixed complex structure J on X, requires a choice of Reeb vector field $\xi = J(r\partial/\partial r)$ and a choice of *transverse Kähler metric*. We also discuss moment maps for torus actions on the cone. By a result of [50], the image of the moment map is a convex rational polyhedral cone $\mathcal{C}^*$ in the dual Lie algebra $\mathtt{t}_s^*$ of the torus, provided $\xi \in \mathtt{t}_s$. Here we identify elements of the Lie algebra with the corresponding vector fields on X (or L). We show that the space of Reeb vector fields in $\mathtt{t}_s$ lies in the interior $\mathcal{C}_0$ of the dual cone to $\mathcal{C}^*$. Finally, we discuss the existence of certain Killing spinors on Sasakian manifolds and their relation to the Sasakian structure. We will make use of some of these formulae in later sections.

2.1 Kähler cones

A Sasakian manifold is a compact Riemannian manifold (L, g_L) whose metric cone (X, g_X) is Kähler. Specifically,

$$g_X = \mathrm{d}r^2 + r^2 g_L \tag{2.1}$$

where $X_0 = \{r > 0\}$ is diffeomorphic to $\mathbb{R}_+ \times L = C(L)$. We also typically take L to be simply–connected.

The Kähler condition on (X, g_X) means that, by definition, the holonomy group of (X, g_X) reduces to a subgroup of $U(n)$, where $n = \dim_{\mathbb{C}} X$. In particular, this means that there is a parallel complex structure J

$$\nabla^X J = 0 \ , \tag{2.2}$$

where ∇^X is the Levi–Civita connection of (X, g_X). We refer to the vector field $r\partial/\partial r$ as the homothetic vector field. The Reeb vector field is defined[15] to be

$$\xi = J\left(r\frac{\partial}{\partial r}\right) \ . \tag{2.3}$$

A straightforward calculation shows that $r\partial/\partial r$ and ξ are both holomorphic. Moreover, ξ is Killing. A proof of these statements may be found in Appendix A.

Provided L is not locally isometric to the round sphere[16], Killing vector fields on the cone X are in one–to–one correspondence with Killing vector fields on the link L. Since L is compact, the group of isometries of (L, g_L) is a compact Lie group. Since all holomorphic Killing vector fields on (X, g_X) arise from Killing vector fields on the link L, they therefore commute with $r\partial/\partial r$ and thus also commute with $\xi = J(r\partial/\partial r)$. Since ξ is itself Killing, it follows that ξ lies in the centre of the Lie algebra of the isometry group.

We now define the 1–form on X

$$\eta = J\left(\frac{\mathrm{d}r}{r}\right) = \frac{1}{r^2} g_X(\xi, \cdot) \ . \tag{2.4}$$

This is the contact form of the Sasakian structure when pulled back to the link L via the embedding

$$i : L \hookrightarrow X \tag{2.5}$$

that embeds L in X at $r = 1$. Note that η is homogeneous degree zero under $r\partial/\partial r$. In terms of the d^c and ∂ operator on X we have

$$\eta = J\mathrm{d}\log r = \mathrm{d}^c \log r = i(\bar{\partial} - \partial)\log r \tag{2.6}$$

and thus

$$\mathrm{d}\eta = 2i\partial\bar{\partial}\log r \ . \tag{2.7}$$

We may now write the metric g_X on X as

$$g_X = \mathrm{d}r^2 + r^2\left(\eta \otimes \eta + g_T\right) \tag{2.8}$$

[15] An alternative definition using spinors, perhaps more familiar to physicists, will be given later in subsection 2.6.

[16] If L is locally isometric to the round sphere then Killing vector fields on the cone (X, g_X) may be constructed from solutions to Obata's equation [51], which in turn relates to conformal Killing vector fields on the link L.

where one can show that g_T is a Kähler metric on the distribution orthogonal to the span of $r\partial/\partial r$ and ξ. The corresponding transverse Kähler form is easily computed to be

$$\omega_T = \frac{1}{2}\mathrm{d}\eta \ . \tag{2.9}$$

The Kähler form on X is thus

$$\omega = \omega_X = \frac{1}{2}\mathrm{d}(r^2\eta) \ . \tag{2.10}$$

In particular ω is exact due to the homothetic symmetry generated by $r\partial/\partial r$. We may rewrite (2.10) as

$$\omega = \frac{1}{4}\mathrm{d}\mathrm{d}^c r^2 = \frac{1}{2}i\partial\bar{\partial}r^2 \ . \tag{2.11}$$

The function r^2 thus serves a dual purpose: it defines the link $L = X\mid_{r=1}$ and is also the *Kähler potential* that defines the metric.

Note from (2.11) that any holomorphic vector field χ which is tangent to L, $\mathrm{d}r(\chi) = 0$, is automatically Killing. Here we have used the notation $\alpha(\chi)$ for the pairing between a 1–form α and vector field χ. Conversely, recall that all Killing vector fields on (X, g_X) are tangent to L. Thus for any holomorphic Killing vector field Y we have[17]

$$\mathcal{L}_Y\eta = 0 \ . \tag{2.12}$$

A Riemannian manifold (X, g_X) is a cone if and only if the metric takes the form (2.1). We end this subsection by reformulating this in terms of the Kähler form ω on X when (X, J, g_X) is a Kähler cone. Thus, let (X_0, J) be a complex manifold, with a diffeomorphism onto $\mathbb{R}_+ \times L$ with $r > 0$ a coordinate on $\mathbb{R}_+$. We require that $r\partial/\partial r$ be holomorphic with respect to J. We may then simply define ω in terms of r by equation (2.11). One must also ensure that the corresponding metric $g = \omega(\cdot, J\cdot)$ is positive definite – typically we shall use this formulation only for infinitesimal deformations around some fixed background Sasakian metric and thus this will not be an issue. Since $r\partial/\partial r$ is holomorphic, and ω is clearly homogeneous of degree two under $r\partial/\partial r$, the metric g will be also homogeneous degree two under $r\partial/\partial r$. However, this is not sufficient for g to be a cone – one also requires $g(r\partial/\partial r, Y) = 0$ for all vector fields Y tangent to L, $\mathrm{d}r(Y) = 0$. Equivalently, $\omega(\xi, Y) = 0$ where we define $\xi = J(r\partial/\partial r)$. It is simple to check that the necessary condition

$$\mathcal{L}_\xi r = \mathrm{d}r(\xi) = 0 \tag{2.13}$$

[17]In this paper we will use extensively the Lie derivative $\mathcal{L}_Y$ along vector fields Y. It is useful to recall the standard formula for Lie derivatives acting on a differential form α: $\mathcal{L}_Y\alpha = \mathrm{d}(Y \lrcorner \alpha) + Y \lrcorner \mathrm{d}\alpha$. Note in components we have $(Y \lrcorner \alpha)_{\mu_1\ldots\mu_{p-1}} \equiv Y^{\mu_p}\alpha_{\mu_p\mu_1\ldots\mu_{p-1}}$.

that ξ is tangent to L is also sufficient for $g = \omega(\cdot, J\cdot)$ to be a cone. It follows now that $\partial/\partial r$ has unit norm and that the metric g is a Kähler metric which is a cone of the form (2.1).

2.2 The Calabi–Yau condition

So far we have not fixed any Calabi–Yau condition on (X, J). We are interested in finding a Ricci–flat Kähler metric on X, and thus we certainly require $c_1(X_0) = 0$. We may impose this by assuming that there is a nowhere vanishing holomorphic section Ω of $\Lambda^{n,0}X_0$. In particular, Ω is then closed

$$\mathrm{d}\Omega = 0 \ . \tag{2.14}$$

One can regard the $(n, 0)$–form Ω as defining a reduction of the structure group of the tangent bundle of X_0 from $GL(2n;\mathbb{R})$ to $SL(n;\mathbb{C})$. The corresponding almost complex structure is then integrable if (2.14) holds. The conical singularity X, including the isolated singular point $r = 0$, is then by definition a *Gorenstein singularity.*

On a compact manifold, such an Ω is always unique up to a constant multiple. However, when X is non–compact, and in particular a Kähler cone, Ω so defined is certainly not unique – one is free to multiply Ω by any nowhere zero holomorphic function on X, and this will also satisfy (2.14). However, for a *Ricci–flat* Kähler cone, with homothetic vector field $r\partial/\partial r$, we may always choose[18] Ω such that

$$\mathcal{L}_{r\partial/\partial r}\Omega = n\Omega \ . \tag{2.15}$$

In fact, one may construct this Ω as a bilinear in the covariantly constant spinor on X, as we recall in section 2.6. In section 2.7 we show that, for any fixed Kähler cone metric – not necessarily Ricci–flat – with homothetic vector field $r\partial/\partial r$, one can always choose a gauge for Ω in which it is homogeneous degree γ under $r\partial/\partial r$, with γ a unique constant. Then (2.15) will follow from varying the Einstein–Hilbert action on the link L, as we show in section 3.1. However, until section 2.7, we fix (X, Ω) together with a space of Kähler cone metrics on X such that Ω satisfies (2.15) for every metric. This Ω is then unique up to a constant multiple[19]. Given such an Ω, for any Kähler form ω

[18]For the toric geometries studied in [22], this condition is equivalent to $b_1 = n$, as can be seen by writing $r\partial/\partial r = -\sum_{i=1}^n b_i J(\frac{\partial}{\partial \phi_i})$ and using the explicit form of Ω given in [22].

[19]To see this, pick a quasi–regular $r\partial/\partial r$. Any other such holomorphic $(n, 0)$–form is $\alpha\Omega$ where α is a nowhere zero holomorphic function on X. Since α is degree zero under $r\partial/\partial r$, it descends to a holomorphic function on V, where V is the space of orbits of ξ on L. Since V is compact, α is constant.

on X there exists a real function f on X such that

$$\frac{i^n}{2^n}(-1)^{n(n-1)/2}\Omega \wedge \bar{\Omega} = \exp(f)\frac{1}{n!}\omega^n \ . \tag{2.16}$$

A Ricci–flat Kähler metric with Kähler form ω of course has f constant.

2.3 The Reeb foliation

The vector field ξ restricts to a unit norm Killing vector field on L, which by an abuse of notation we also denote by ξ. Since ξ is nowhere–vanishing, its orbits define a foliation of L. There is then a classification of Sasakian structures according to the global properties of this foliation:

- If all the orbits close, ξ generates a circle action on L. If, moreover, the action is free the Sasakian manifold is said to be *regular*. All the orbits have the same length, and L is the total space of a principal circle bundle $\pi : L \to V$ over a Kähler manifold V. This inherits a metric g_V and Kähler form ω_V, where g_V is the push–down to V of the transverse metric g_T.

- More generally, if ξ generates a $U(1)$ action on L, this action will be *locally free*, but not free. The Sasakian manifold is then said to be *quasi–regular*. Suppose that $x \in L$ is a point which has some non–trivial isotropy subgroup $\Gamma_x \subset U(1)$. Thus $\Gamma_x \cong \mathbb{Z}_m$ for some positive integer m. The length of the orbit through x is then $1/m$ times the length of the generic orbit. The orbit space is naturally an orbifold, with L being the total space of an orbifold circle bundle $\pi : L \to V$ over a Kähler orbifold V. Moreover, the point x descends to a singular point of the orbifold with local orbifold structure group $\mathbb{Z}_m$.

- If the generic orbit of ξ does not close, the Sasakian manifold is said to be *irregular*. In this case the generic orbits are diffeomorphic to the real line $\mathbb{R}$. Recall that the isometry group of (L, g_L) is a compact Lie group. The orbits of a Killing vector field define a one–parameter subgroup, the closure of which will always be an abelian subgroup and thus a torus. The dimension of the closure of the generic orbit is called the *rank* of the Sasakian metric, denoted $\mathrm{rank}(L, g_L)$. Thus irregular Sasakian metrics have rank > 1.

A straightforward calculation gives

$$\mathrm{Ric}(g_X) = \mathrm{Ric}(g_L) - (2n-2)g_L = \mathrm{Ric}(g_T) - 2ng_T \tag{2.17}$$

and thus in particular

$$\rho = \rho_T - 2n\omega_T \tag{2.18}$$

where ρ_T denotes the transverse Ricci–form. We also have[20]

$$\rho = i\partial\bar{\partial}\log \|\Omega\|^2_{g_X} \ , \tag{2.19}$$

where we have defined $\|\Omega\|^2_{g_X} = \frac{1}{n!}\Omega\lrcorner\bar{\Omega}$. The Ricci–potential for the Kähler cone is thus $\log \|\Omega\|^2_{g_X}$. Since we assume that Ω is homogeneous degree n under $r\partial/\partial r$, this is homogeneous degree zero *i.e.* it is independent of r, and hence is the pull–back under p^* of a global function on L. Here

$$p : X \to L \tag{2.20}$$

projects points $(r, x) \in \mathbb{R}_+ \times L$ onto $x \in L$. Moreover, since $\mathcal{L}_\xi\Omega = ni\Omega$ and ξ is Killing it follows that the Ricci–potential is basic with respect to the foliation defined by ξ. Recall that a p–form α on L is said to be *basic* with respect to the foliation induced by ξ if and only if

$$\mathcal{L}_\xi\alpha = 0, \qquad\qquad \xi\lrcorner\alpha = 0 \ . \tag{2.21}$$

Thus α has no component along $g(\xi, \cdot)$ and is independent of ξ. It is straightforward to check that the transverse Kähler form ω_T, and its Ricci–form, are also basic.

Suppose now that (L, g_L) is quasi–regular[21]. Thus the space of orbits of ξ is a compact complex orbifold V. The transverse Kähler and Ricci forms push down to ω_V, and ρ_V on V, respectively. Thus (2.18) may be interpreted as an equation on V. The left hand side is $i\partial\bar{\partial}$ exact on the orbifold V, and hence

$$[\rho_V] - 2n[\omega_V] = 0 \in H^2(V; \mathbb{R}) \ . \tag{2.22}$$

In particular this shows that V is Fano, since $c_1(V) = [\rho_V/2\pi]$ is positive.

Note that η satisfies $\mathrm{d}\eta = 2\pi^*(\omega_V)$. The Kähler class of V is then proportional to the first Chern class of the orbifold circle bundle $\pi : L \to V$. By definition, the orbifold is thus *Hodge*. We denote the associated orbifold complex line bundle over V by $\mathcal{L}$. Note from (2.22) that, since $[\omega_V]$ is proportional to the anti–canonical class $c_1(V)$ of V, the orbifold line bundle $\mathcal{L}$ is closely related to the canonical bundle $\mathcal{K} \to V$ over V. To see this more clearly, let $U \subset V$ denote a smooth open subset of V over which $\mathcal{L}$

[20] Note that multiplying Ω by a nowhere zero holomorphic function α on X leaves the right hand side of (2.19) invariant.

[21] We include the regular case when V is a manifold in this terminology.

trivialises. We may then introduce a coordinate ψ on the circle fibre of $\pi : L\mid_U \rightarrow U$ such that on $L\mid_U = \pi^{-1}(U)$ we have

$$\eta = \mathrm{d}\psi + \pi^*(\sigma) \tag{2.23}$$

where σ is a one–form on U with $\mathrm{d}\sigma = 2\omega_V$. Note that, although this cannot be extended to a one–form on all of V, η is globally defined on L – one may cover V by open sets, and the σ and ψ are related on overlaps by opposite gauge transformations. From equation (2.22), we see that if $\psi \sim \psi + 2\pi/n$ then X is the total space of the canonical complex cone over V – that is, the associated line bundle $\mathcal{L}$ to π is $\mathcal{L} = \mathcal{K}$. More generally, we may set

$$\psi \sim \psi + \frac{2\pi\beta}{n}; \qquad -c_1(\mathcal{L}) = \frac{c_1(V)}{\beta} \in H^2_{\mathrm{orb}}(V;\mathbb{Z}) \ . \tag{2.24}$$

Then $\mathcal{L}^\beta \cong \mathcal{K}$. Here we have introduced the integral *orbifold cohomology* $H^*_{\mathrm{orb}}(V;\mathbb{Z})$ of Haefliger, such that orbifold line bundles are classified up to isomorphism by

$$c_1(\mathcal{L}) \in H^2_{\mathrm{orb}}(V;\mathbb{Z}) \ . \tag{2.25}$$

This reduces to the usual integral cohomology when V is a manifold. The maximal integer β in (2.24) is called the *Fano index* of V. If L is simply–connected, then $c_1(\mathcal{L})$ is primitive[22] in $H^2_{\mathrm{orb}}(V;\mathbb{Z})$ and β is then equal to the index of V. For example, if $V = \mathbb{C}P^{n-1}$ then $H^2(V;\mathbb{Z}) \cong \mathbb{Z}$. The line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = -1 \in H^2(V;\mathbb{Z}) \cong \mathbb{Z}$ gives $L = S^{2n-1}$, the complex cone being isomorphic to $\mathcal{L} = \mathcal{O}(-1) \rightarrow \mathbb{C}P^{n-1}$ with the zero section contracted. Note that the total space of $\mathcal{L}$ is the blow–up of $\mathbb{C}^n$ at the origin. On the other hand, the canonical bundle is $\mathcal{K} = \mathcal{O}(-n) \rightarrow \mathbb{C}P^{n-1}$ which gives the link $S^{2n-1}/\mathbb{Z}_n$. Thus the index is equal to n.

From the Kodaira–Bailey embedding theorem (V, g_V), with the induced complex structure, is necessarily a *normal projective algebraic variety* [8]. Let $\mathcal{T} \rightarrow V$ be the orbifold holomorphic line bundle over V with first Chern class $c_1(V)/\mathrm{Ind}(V)$, with $\mathrm{Ind}(V)$ being the index of V. In particular, $\mathcal{T}$ is ample and has a primitive first Chern class. By the Kodaira–Bailey embedding theorem, for $k \in \mathbb{N}$ sufficiently large, $\mathcal{T}^k$ defines an embedding of V into $\mathbb{C}P^{N-1}$ via its space of global holomorphic sections. Thus, a basis s_α, $\alpha = 1, \ldots, N$, of $H^0(V;\mathcal{T}^k)$ may be regarded as homogeneous coordinates on $\mathbb{C}P^{N-1}$, with $V \ni p \rightarrow [s_1(p), \ldots, s_N(p)]$ being an embedding. The image is a projective algebraic variety, and thus the zero locus of a set of homogeneous polynomials $\{f_A = 0\}$ in the homogeneous coordinates. If $H = \mathcal{O}(1)$ denotes the hyperplane

[22]That is, there is no $\gamma \in H^2_{\mathrm{orb}}(V;\mathbb{Z})$ and integer $m \in \mathbb{Z}$, $|m| > 1$, such that $m\gamma = c_1(\mathcal{L})$.

bundle on $\mathbb{C}P^{N-1}$ then its pull–back to V is of course isomorphic to $\mathcal{T}^k$. On the other hand, as described above, $\mathcal{L}^* \cong \mathcal{T}^{\mathrm{Ind}(V)/\beta}$ for some positive β, where recall that $\mathcal{T}^{\mathrm{Ind}(V)} \cong \mathcal{K}^{-1}$ is the anti–canonical bundle. By taking the period $\beta = \mathrm{Ind}(V)/k$ in the above it follows that the corresponding cone X is the affine algebraic variety defined by $\{f_A = 0\} \subset \mathbb{C}^N$. The maximal value of β, given by $\beta = \mathrm{Ind}(V)$, is then a k–fold cover of this X. Moreover, by our earlier assumptions, X constitutes an isolated Gorenstein singularity.

We note that there is therefore a natural (orbifold) resolution of any (X, J) equipped with a quasi–regular Kähler cone metric: one simply takes the total space of the orbifold complex line bundle $\mathcal{L} \to V$. The resulting space W has at worst orbifold singularities, and $W \setminus V \cong X_0$ is a biholomorphism. Thus W is birational to X. One might be able to resolve the cone X completely, but the existence of a resolution with at worst orbifold singularities will be sufficient for our needs later.

2.4 Transverse Kähler deformations

In order to specify a Sasakian structure on L, one clearly needs to give the Reeb vector field ξ. By embedding L as a link in a fixed non–compact X, this is equivalent to choosing a homothetic vector field $r\partial/\partial r$ on X. Having determined this vector field, the remaining freedom in the choice of Sasakian metric consists of *transverse Kähler deformations*.

Suppose that we have two Kähler potentials r^2, $\tilde{r}^2$ on X such that their respective homothetic vector fields coincide:

$$r\frac{\partial}{\partial r} = \tilde{r}\frac{\partial}{\partial \tilde{r}} \ . \tag{2.26}$$

This equation may be read as saying that $\tilde{r}$ is a homogeneous degree one function under $r\partial/\partial r$, and thus

$$\tilde{r}^2 = r^2 \exp \phi \tag{2.27}$$

for some homogeneous degree zero function ϕ. Thus ϕ is a pull–back of a function, that we also call ϕ, from the link L under p^*. We compute

$$\tilde{\eta} = \frac{1}{2}\mathrm{d}^c\left(\log r^2 + \phi\right) = \eta + \frac{1}{2}\mathrm{d}^c\phi \ . \tag{2.28}$$

In order that $\tilde{r}$ defines a metric cone, recall from the end of section 2.1 that we require

$$\mathcal{L}_\xi \tilde{r} = 0 \tag{2.29}$$

which implies that ϕ is basic with respect to the foliation induced by ξ, $\mathcal{L}_\xi \phi = 0$. Introducing local transverse CR coordinates, *i.e.* local complex coordinates on the transverse space, $(z_i, \bar{z}_i)$ on L, one thus has $\phi = \phi(z_i, \bar{z}_i)$. Note also that

$$\mathrm{d}\tilde{\eta} = \mathrm{d}\eta + i\partial\bar{\partial}\phi \tag{2.30}$$

so that the transverse Kähler forms $\tilde{\omega_T} = (1/2)\mathrm{d}\tilde{\eta}$, $\omega_T = (1/2)\mathrm{d}\eta$ differ precisely by a transverse Kähler deformation.

When the Sasakian structure is quasi–regular, ϕ pushes down to a global function on the orbifold V, and transformations of the metric of the form (2.30) are precisely those preserving the Kähler class $[\omega_V] \in H^2(V;\mathbb{R})$.

2.5 Moment maps

In this subsection we consider a space of Kähler cone metrics on X such that each metric has isometry group containing a torus $\mathbb{T}^s$. Moreover, the flow of the Reeb vector field is assumed to lie in this torus. For each metric, there is an associated moment map whose image is a convex rational polyhedral cone $\mathcal{C}^* \subset \mathtt{t}_s^* \cong \mathbb{R}^s$. Moreover, these cones are all isomorphic. We show that the space of Reeb vector fields on L is (contained in) the interior $\mathcal{C}_0 \subset \mathtt{t}_s$ of the *dual cone* to $\mathcal{C}^*$.

Suppose then that $\mathbb{T}^s$ acts holomorphically on the cone X, preserving a fixed choice of Kähler form (2.11) on X. Let $\mathtt{t}_s$ denote the Lie algebra of $\mathbb{T}^s$. We suppose that $\xi \in \mathtt{t}_s$ – the torus action is then said to be of *Reeb type* [52]. Let us introduce a basis $\partial/\partial\phi_i$ of vector fields generating the torus action, with $\phi_i \sim \phi_i + 2\pi$. Then we may write

$$\xi = \sum_{i=1}^{s} b_i \frac{\partial}{\partial \phi_i} \ . \tag{2.31}$$

Since

$$\mathcal{L}_{\partial/\partial\phi_i}\omega = 0 \tag{2.32}$$

and X has $b_1(X_0) = 0$, it follows that for each $i = 1, \ldots, s$ there exists a function y_i on X such that

$$\mathrm{d}y_i = -\frac{\partial}{\partial\phi_i} \lrcorner \omega \ . \tag{2.33}$$

In fact, it is simple to verify that

$$y_i = \frac{1}{2} r^2 \eta \left(\frac{\partial}{\partial \phi_i} \right) \tag{2.34}$$

is the homogeneous solution. The functions y_i may be considered as coordinates on the dual Lie algebra $\mathtt{t}_s^*$. This is often referred to as the *moment map*

$$\mu : X \to \mathtt{t}_s^* \tag{2.35}$$

where for $Y \in \mathtt{t}_s$ we have

$$(Y, \mu) = \frac{1}{2} r^2 \eta(Y) \ . \tag{2.36}$$

Under these conditions, [50] proved that the image of X under μ is a *convex rational polyhedral cone* $\mathcal{C}^* \subset \mathtt{t}_s^*$. This is a convex polyhedral cone whose generators are all vectors whose components are rational numbers.

The image of the link $L = X \mid_{r=1}$ is given by

$$2(b, y) = 1 \tag{2.37}$$

as follows by setting $Y = \xi$ in (2.36). The hyperplane (2.37) intersects the cone $\mathcal{C}^*$ to form a compact polytope

$$\Delta(b) = \mathcal{C}^* \cap \{2(b, y) \leq 1\} \tag{2.38}$$

if and only if b lies in the interior of the *dual cone* to $\mathcal{C}^*$, which we denote $\mathcal{C} \subset \mathtt{t}_s$. Note that this analysis is essentially the same as that appearing in the toric context in our previous paper [22]. The only difference is that the torus no longer has maximal possible rank $s = n$, and thus the cone need not be toric. However, the Euclidean volume of $\Delta(b)$ is no longer the volume of the Sasakian metric on L.

The cone $\mathcal{C}$ is a convex rational polyhedral cone by Farkas' Theorem. Geometrically, the limit in which the Reeb vector field ξ approaches the boundary $\partial\mathcal{C}$ of this cone is precisely the limit in which ξ develops a non–trivial fixed point set on X. Recall that ξ has square norm r^2 on X and thus in particular is nowhere vanishing on $X_0 = \{r > 0\}$. Thus the boundary of the cone $\mathcal{C}$ is a singular limit of the space of Sasakian metrics on L. To see this, let $\mathcal{F}_\alpha$ denote the facets of $\mathcal{C} \subset \mathtt{t}_s$, and let the associated primitive inward pointing normal vectors be $u_\alpha \in \mathtt{t}_s^*$. The u_α are precisely the generating rays of the dual cone $\mathcal{C}^*$ to $\mathcal{C}$. Thus we may exhibit $\mathcal{C}^* \subset \mathtt{t}_s^* \cong \mathbb{R}^s$ as

$$\mathcal{C}^* = \left\{ \sum_\alpha t_\alpha u_\alpha \in \mathtt{t}_s^* \mid t_\alpha \geq 0 \right\} \ . \tag{2.39}$$

If $\xi \in \mathcal{F}_\alpha$, for some α, then $(\xi, u_\alpha) = 0$. We may reinterpret this equation in terms of the moment cone $\mathcal{C}^*$. Let $\mathcal{R}_\alpha$ denote the 1–dimensional face, or ray, of $\mathcal{C}^*$ generated by the vector u_α. The inverse image $X_\alpha = \mu^{-1}(\mathcal{R}_\alpha)$ is a $\mathbb{T}^s$–invariant conic symplectic subspace of X [50] and is a vanishing set for the vector field $\xi \in \mathcal{F}_\alpha \subset \mathtt{t}_s$.

It may help to give a simple example. Thus, let $X = \mathbb{C}^n$. Taking the flat metric gives

$$\omega = \sum_{i=1}^{n} \frac{1}{2}\mathrm{d}(\rho_i^2) \wedge \mathrm{d}\phi_i \tag{2.40}$$

where (ρ_i, ϕ_i) are polar coordinates on the ith complex plane of $\mathbb{C}^n$. We have $\mathcal{C}^* = (\mathbb{R}_+)^n$, with coordinates $y_i = \rho_i^2/2 \geq 0$, which happens to be isomorphic to its dual cone, $\mathcal{C} = (\mathbb{R}_+)^n$. This orthant is bounded by n hyperplanes, with primitive inward–pointing normals $u_i = e_i$, where e_i is the ith standard orthonormal basis vector: $e_i^j = \delta_i^j$. The $\{u_i\}$ indeed also generate the moment cone $\mathcal{C}^*$. The inverse image under the moment map of the ray $\mathcal{R}_i$ generated by u_i is the subspace

$$X_i = \{z_j = 0 \mid j \neq i\} \cong \mathbb{C} \subset \mathbb{C}^n \; . \tag{2.41}$$

Any vector field ξ of the form

$$\xi = \sum_{j \neq i} c_j \frac{\partial}{\partial \phi_j} \tag{2.42}$$

clearly vanishes on X_i.

2.6 Killing spinors and the $(n, 0)$–form

We now discuss the existence of certain Killing spinors on Sasakian manifolds, and their relation to the differential forms defining the Sasakian structure which we have already introduced. The use of spinors often provides a quick and elegant method for obtaining various results, as we shall see in section 4. In fact, these methods are perhaps more familiar to physicists.

Recall that all Kähler manifolds admit a gauge covariantly constant spinor. More precisely, the spinor in question is in fact a section of a specific spin^c bundle that is intrinsically defined on any Kähler manifold. We will not give a complete account of this in the following, but simply make a note of the results that we will need in this paper, especially in section 4. For further details, the reader might consult a number of standard references [53, 54]. The spinors on a Sasakian manifold are induced from those on the Kähler cone, again in a rather standard way. This is treated, for Sasaki–Einstein manifolds, in the paper of Bär [55]. The extension to Sasakian manifolds is straightforward.

Let (X, J, g) be a Kähler manifold. The bundle of complex spinors $\mathcal{S}$ does not necessarily exist globally on X, the canonical example being $\mathbb{C}P^2$. However, the spin^c bundle

$$\mathcal{V} = \mathcal{S} \otimes \mathcal{K}_X^{-1/2} \tag{2.43}$$

always exists. Here $\mathcal{K}_X$ denotes the canonical bundle of X, which is the (complex line) bundle $\Lambda^{n,0}(X)$ of forms of Hodge type $(n, 0)$ with respect to J. The idea in (2.43) is that, although neither bundle may exist separately due to $w_2(X) \in H^2(X; \mathbb{Z}_2)$ being non–zero, the obstructing cocycle cancels out in the tensor product and $\mathcal{V}$ exists as a genuine complex vector bundle. The metric on X induces the usual spin connection on $\mathcal{S}$, and the canonical bundle $\mathcal{K}_X$ inherits a connection one–form with curvature $-\rho$, where ρ is the Ricci–form on X. Thus the bundle $\mathcal{V}$ has defined on it a standard connection form.

A key result is that, as a complex vector bundle,

$$\mathcal{V} \cong \Lambda^{0,*}(X) \ . \tag{2.44}$$

In fact, since X is even dimensional, the spin^c bundle $\mathcal{V}$ decomposes into spinors of positive and negative chirality, $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ and

$$\mathcal{V}^+ \cong \Lambda^{0,\text{even}}(X), \qquad\qquad \mathcal{V}^- \cong \Lambda^{0,\text{odd}}(X) \ . \tag{2.45}$$

The connection on $\mathcal{V}$ referred to above is then equal to the standard metric–induced connection on $\Lambda^{0,*}(X)$. In particular, there is always a covariantly constant section of $\Lambda^{0,*}(X)$ – it is just the constant function on X. Via the isomorphism[23] (2.44) this gets interpreted as a gauge covariantly constant spinor Ψ on X, the gauge connection being the standard metric–induced one on $\mathcal{K}_X^{-1/2}$.

We conclude then that there is always a spinor field[24] Ψ on X satisfying (in a local coordinate patch over which $\mathcal{V}$ trivialises)

$$\nabla^X_Y \Psi - \frac{i}{2} A(Y)\Psi = 0 \tag{2.46}$$

where Y is any vector field on X and ∇^X denotes the spin connection on (X, g). The connection one–form $A/2$ on $\mathcal{K}_X^{-1/2}$ satisfies, as mentioned above,

$$\mathrm{d}A = \rho \ . \tag{2.47}$$

[23]This is essentially the same twisting of spinors that occurs on the worldvolumes of D–branes wrapping calibrated submanifolds.

[24]By an abuse of terminology we refer to sections of $\mathcal{V}$ as "spinors".

For our Calabi–Yau cone[25] (X, Ω), $\mathcal{K}_X$ is of course topologically trivial by definition. Thus, topologically, Ψ is a genuine spinor field. However, unless the metric is Ricci–flat, the connection form A will be non–zero. The reduction of Ψ to a spinor on the link $L = X\mid_{r=1}$ is again rather standard [55]. Either spinor bundle $\mathcal{V}^+\mid_{r=1}$ or $\mathcal{V}^-\mid_{r=1}$ is isomorphic to the spinor (or rather spin^c) bundle on L. By writing out the spin connection on the cone in terms of that on the link L, one easily shows that there is a spinor $\theta = \Psi\mid_{r=1}$ on L satisfying

$$\nabla^L_Y\theta - \frac{i}{2}Y\cdot\theta - \frac{i}{2}A(Y)\theta = 0 \ . \tag{2.48}$$

Here $Y\cdot\theta$ denotes Clifford multiplication: $Y\cdot\theta = Y^\mu\gamma_\mu\theta$, and γ_μ generate the Clifford algebra $\mathrm{Cliff}(2n-1,0)$. Thus $\{\gamma_\mu,\gamma_\nu\} = 2g_{L\ \mu\nu}$. It is now simple to check from (2.48) that θ has constant norm, and we normalise it so that $\bar{\theta}\theta = 1$. Then the contact one–form η on L is given by the bilinear

$$\eta = \bar{\theta}\gamma_{(1)}\theta \ . \tag{2.49}$$

Note that one may define the Reeb vector field as the dual of the contact one–form η. Thus, in components, $\xi^\mu = g_L^{\mu\nu}\eta_\nu$. It is then straightforward to check, using (2.48), that $\nabla^L_{(\mu}\eta_{\nu)} = 0$, so that ξ is indeed a Killing vector field. One also easily verifies, again using (2.48), that

$$\mathrm{d}\eta = -2i\bar{\theta}\gamma_{(2)}\theta = 2\omega_T \ . \tag{2.50}$$

We may define an $(n,0)$–form K on X as a bilinear in the spinor Ψ, namely

$$K = \bar{\Psi}^c\gamma_{(n)}\Psi \ . \tag{2.51}$$

It is important to note that this is *different* from the holomorphic $(n,0)$–form Ω on X we introduced earlier. The two are of course necessarily proportional, and in fact are related by

$$\Omega = \exp{(f/2)}K \ . \tag{2.52}$$

Here f is the same function as that appearing in (2.16). Indeed, we can write the Ricci–form on X as $\rho = i\partial\bar{\partial}f$, so that we may take

$$A = \frac{1}{2}\mathrm{d}^c f \ . \tag{2.53}$$

Equivalently, we have

$$f = \log\|\Omega\|^2_{g_X} \ . \tag{2.54}$$

[25]Strictly, one should write X_0 in most of what follows.

From (2.46) we have, as usual on a Kähler manifold,

$$\mathrm{d}K = iA \wedge K = \frac{i}{2}\mathrm{d}^c f \wedge K \ . \tag{2.55}$$

Then

$$\mathrm{d}\Omega = \frac{1}{2}(\mathrm{d}f + i\mathrm{d}^c f) \wedge \Omega = \partial f \wedge \Omega = 0 \ . \tag{2.56}$$

In an orthonormal frame e_μ, $\mu = 1, \ldots, 2n$, for (X, g) note that

$$\begin{aligned} \omega &= e_1 \wedge e_2 + \cdots + e_{2n-1} \wedge e_{2n} \\ K &= (e_1 + ie_2) \wedge \cdots \wedge (e_{2n-1} + ie_{2n}) \end{aligned} \tag{2.57}$$

so that

$$\frac{i^n}{2^n}(-1)^{n(n-1)/2} K \wedge \bar{K} = \frac{1}{n!}\omega^n \ . \tag{2.58}$$

The relation (2.52) is then consistent with the normalisation in (2.16).

Finally, we introduce the space of Reeb vector fields under which Ω has charge n. Thus, we define

$$\Sigma = \{\xi \in \mathcal{C}_0 \subset \mathtt{t}_s \mid \mathcal{L}_\xi \Omega = in\Omega\} \ . \tag{2.59}$$

Clearly, if $\xi' \in \Sigma$ is fixed, any other $\xi \in \Sigma$ is given by $\xi = \xi' + Y$ where Ω is uncharged with respect to Y:

$$\mathcal{L}_Y \Omega = 0 \ . \tag{2.60}$$

The space of all Y satisfying (2.60) forms a vector subspace of $\mathtt{t}_s$. Moreover, this subspace has codimension one, so that the corresponding plane through ξ' forms a finite polytope with $\mathcal{C}$. Thus Σ is an $(r-1)$–dimensional polytope. In the toric language of [22], this is just the intersection of the plane $b_1 = n$ with Reeb polytope $\mathcal{C}$.

2.7 The homogeneous gauge for Ω

Suppose that $X_0 = \mathbb{R}_+ \times L$ is a complex manifold admitting a nowhere vanishing holomorphic section Ω' of $\Lambda^{n,0}(X_0)$. Recall that, in contrast to the case of compact X, Ω' is unique only up to multiplying by a nowhere vanishing holomorphic function. Suppose moreover that g_X is a quasi–regular Kähler cone metric on X, with homothetic vector field $r\partial/\partial r$. The aim in this section is to prove that there always exists a "gauge" in which the holomorphic $(n, 0)$–form is homogeneous of constant degree $\gamma \in \mathbb{R}$ under $r\partial/\partial r$, where γ is unique. We shall use this result in section 3.1 to argue that $\gamma = n$ arises by varying the Einstein–Hilbert action of the link.

Since $r\partial/\partial r$ is a holomorphic vector field

$$\mathcal{L}_{r\partial/\partial r}\Omega' = \kappa\Omega' \tag{2.61}$$

where κ is a holomorphic function. We must then find a nowhere zero holomorphic function α and a constant γ such that

$$\mathcal{L}_{r\partial/\partial r}\log\alpha = \gamma - \kappa \tag{2.62}$$

since then $\Omega \equiv \alpha\Omega'$ is homogeneous degree γ. Let us expand

$$\gamma - \kappa = \sum_{k\geq 0} a_k \tag{2.63}$$

where a_k are holomorphic functions of weight k under the $U(1)$ action generated by $\xi = J(r\partial/\partial r)$. Roughly, we are doing a Taylor expansion in the fibre of $\mathcal{L}^* \to V$, where V is the Kähler orbifold base defined by the quasi–regular Reeb vector field ξ. Here $\mathcal{L}$ is the associated complex line orbifold bundle to $U(1) \hookrightarrow L \to V$. The a_k, in their V dependence, may be considered as sections of $(\mathcal{L}^*)^k \to V$. If $\partial/\partial\nu$, $\nu \sim \nu + 2\pi$, rotates the fibre of L with weight *one*, then we have

$$\frac{\partial}{\partial\nu} = h\xi \tag{2.64}$$

for some positive constant h. Thus

$$\mathcal{L}_\xi a_k = i\frac{k}{h}a_k\ . \tag{2.65}$$

The equation (2.62) is then straightforward to solve:

$$\log\alpha = \sum_{k\geq 1}\frac{h}{k}a_k + \log\delta\ . \tag{2.66}$$

Here we have used the fact that each a_k is holomorphic. Moreover, δ is holomorphic of homogeneous degree a_0 where

$$a_0 = \gamma - \kappa_0 \tag{2.67}$$

and the constant κ_0 is the degree zero part of κ. In order that α be nowhere vanishing, we now require $a_0 = 0$. This is because δ is homogeneous of fixed degree, and thus corresponds to a section of $(\mathcal{L}^*)^m \to V$, where $a_0 = m/h$. However, since $(\mathcal{L}^*)^m$ is a non–trivial bundle for $m \neq 0$, any section must vanish somewhere, unless $m = 0$. Thus $a_0 = 0$, which in fact fixes γ uniquely because of the latter argument. Finally, the resulting expression (2.66) for α is clearly nowhere vanishing, holomorphic, and satisfies (2.62). This completes the proof.

3 The variational problem

In this section we show that the Einstein–Hilbert action on L, restricted to the space of Sasakian metrics on L, is essentially the *volume functional* of L. Moreover, the volume depends only on the choice of Reeb vector field, and not on the remaining degrees of freedom in the metric. We give general formulae for the first and second variations, in particular showing that the volume of L is a strictly convex function. The derivations of these formulae, which are straightforward but rather technical, are relegated to Appendix C. The first variation will be related to the Futaki invariant for quasi–regular Sasakian metrics in section 4. From the second variation formula it follows that there is a unique critical point of the Einstein–Hilbert action in a given Reeb cone $\Sigma \subset \mathcal{C}_0 \subset \mathtt{t}_s$.

3.1 The Einstein–Hilbert action

As is well–known, a metric g_L on L satisfying the Einstein equation

$$\mathrm{Ric}(g_L) = (2n-2)g_L \tag{3.1}$$

is a critical point of the Einstein–Hilbert action

$$\mathcal{S} : \mathrm{Met}(L) \to \mathbb{R} \tag{3.2}$$

given by

$$\mathcal{S}[g_L] = \int_L [s(g_L) + 2(n-1)(3-2n)]\, \mathrm{d}\mu \tag{3.3}$$

where $s(g_L)$ is the scalar curvature of g_L, and $\mathrm{d}\mu$ is the associated Riemannian measure. We would like to restrict $\mathcal{S}$ to a space of *Sasakian* metrics on L. Recall that we require $X_0 = \mathbb{R}_+ \times L$ to be Calabi–Yau, meaning that there is a nowhere vanishing holomorphic $(n,0)$–form Ω. At this stage, we are not assuming that Ω obeys any additional property. The Ricci–form on X is then $\rho = i\partial\bar{\partial} f$ where

$$f = \log \|\Omega\|^2_{g_X} \ . \tag{3.4}$$

Note that of course the Ricci–form is independent of multiplying Ω by any nowhere zero holomorphic function on X. The scalar curvature of (X, g_X) is

$$s(g_X) = \mathrm{Tr}\left(g_X^{-1}\mathrm{Ric}(g_X)\right) = -\Delta_X f \tag{3.5}$$

where Δ_X is the Laplacian on (X, g_X). Using the relation

$$s(g_X) = \frac{1}{r^2}\left[s(g_L) - 2(n-1)(2n-1)\right] \tag{3.6}$$

one easily sees that

$$\mathcal{S}[g_L] = 2(n-1)\left(R + 2\mathrm{vol}[L]\right) \tag{3.7}$$

where R is defined by

$$R = \int_{r\leq 1} s(g_X)\, \frac{\omega^n}{n!} = -\int_{r\leq 1} \Delta_X f\, \frac{\omega^n}{n!}\ . \tag{3.8}$$

Note that this is independent of the gauge choice for Ω, *i.e.* it is independent of the choice of nowhere zero holomorphic multiple. However, in order to relate R to an expression on the link, it is useful to impose a homogeneity property on Ω. This can always be done, as we showed in section 2.7. Strictly speaking, we only proved this for *quasi–regular* Sasakian metrics. However, since the rationals are dense in the reals and $\mathcal{S}$ is continuous, this will in fact be sufficient. Thus, let Ω be homogeneous degree γ and $r\partial/\partial r$ be quasi–regular. It follows that f satisfies

$$\mathcal{L}_{r\partial/\partial r} f = 2(\gamma - n)\ . \tag{3.9}$$

Recall that on a cone

$$\Delta_X = \frac{1}{r^2}\Delta_L - \frac{1}{r^{2n-1}}\frac{\partial}{\partial r}\left(r^{2n-1}\frac{\partial}{\partial r}\right) \tag{3.10}$$

where Δ_L is the Laplacian on (L, g_L). We therefore have

$$R = 2(\gamma - n)\mathrm{vol}[L] - \int_{r\leq 1} \Delta_L f\ r^{2n-3}\mathrm{d}r \wedge \mathrm{d}\mu\ . \tag{3.11}$$

The integrand in the second term is now homogeneous under $r\partial/\partial r$, so we may perform the r integration trivially. Using Stokes' theorem on L we conclude that this term is identically zero. Thus we find that

$$\mathcal{S}[g_L] = 4(n-1)(1+\gamma-n)\mathrm{vol}[L]\ . \tag{3.12}$$

Thus the Einstein–Hilbert action on L is related simply to the volume functional on L. Given any homothetic vector field $r\partial/\partial r$, $cr\partial/\partial r$ is always[26] another homothetic vector field for a Kähler cone metric on X, where c is a positive constant. Since vol$[L]$

[26]That is, the space of homothetic vector fields is itself a cone.

is homogeneous degree $-n$ under this scaling, one may immediately extremise (3.12) in this direction, obtaining

$$\gamma = n \ . \tag{3.13}$$

Note that this is precisely analogous to the argument in our previous paper [22], which sets $b_1 = n$, in the notation there. Thus, provided we restrict to Kähler cone metrics for which $\gamma = n$, *i.e.* there is a nowhere vanishing holomorphic $(n, 0)$–form of homogeneous degree n, the Einstein–Hilbert action is just the volume functional for the link:

$$\mathcal{S}[g_L] = 4(n-1)\mathrm{vol}[L] \ . \tag{3.14}$$

This reduces our problem to studying the volume of the link in the remainder of the paper.

We next show that $\mathcal{S}$ is independent of the choice of transverse Kähler metric. Hence $\mathcal{S}$ is a function on the space of Reeb vector fields, or equivalently, of homothetic vector fields. Thus, consider

$$\mathrm{vol}[L] : \mathrm{Sas}(L) \to \mathbb{R} \ . \tag{3.15}$$

We may write the volume as

$$\mathrm{vol}[L] = \int_L \mathrm{d}\mu = 2n \ \mathrm{vol}[X_1] = 2n \int_{r\leq 1} \frac{\omega^n}{n!} \tag{3.16}$$

where we define $X_1 = X \mid_{r\leq 1}$. Here we have simply written the measure on X in polar coordinates. Note that, since we are regarding r^2 as the Kähler potential, changing the metric also changes the definition of X_1. Let us now fix a background with Kähler potential r^2 and set

$$r^2(t) = r^2 \exp(t\phi) \tag{3.17}$$

where t is a (small) parameter and ϕ is a basic function on $L = X \mid_{r=1}$. Thus

$$\frac{\mathrm{d}\omega}{\mathrm{d}t}(t=0) = \frac{1}{4}\mathrm{dd}^c(r^2\phi) \ . \tag{3.18}$$

To first order in t, the hypersurface $r(t) = 1$ is given by

$$r = 1 - \frac{1}{2}t\phi \ . \tag{3.19}$$

We now write

$$\mathrm{vol}[L] = \int_{r\leq 1} \mathrm{d}(r^{2n}) \wedge \mathrm{d}\mu \ . \tag{3.20}$$

The first order variation in $\mathrm{vol}[L](t)$ from that of $t = 0$ contains a contribution from the domain of integration, as well as from the integrand. The former is slightly more subtle. Consider the expression

$$\int_{r\leq 1-\frac{1}{2}t\phi} \mathrm{d}(r^{2n}) \wedge \mathrm{d}\mu \tag{3.21}$$

which is $\mathrm{vol}[L]$ together with the first order variation due to the change of integration domain. By performing the r integration pointwise over the link L we obtain, to first order in t,

$$\mathrm{vol}[L] - t\ n \int_L \phi \mathrm{d}\mu\ . \tag{3.22}$$

The total derivative of $\mathrm{vol}[L]$ at $t = 0$ is thus

$$\frac{\mathrm{dvol}[L]}{\mathrm{d}t}(t=0) = -n \int_L \phi \mathrm{d}\mu + \frac{n}{2} \int_{r\leq 1} \mathrm{dd}^c(r^2\phi) \wedge \frac{\omega^{n-1}}{(n-1)!} \tag{3.23}$$

where the second term arises by varying the Liouville measure $\omega^n/n!$. We may now apply Stokes' theorem to the second term on the right hand side of (3.23). Using

$$i^*\omega = \omega_T \tag{3.24}$$

together with the equation $\mathrm{d}^c r^2 = 2r^2\eta$ (see eq. (2.6)), we obtain

$$\frac{\mathrm{dvol}[L]}{\mathrm{d}t}(t=0) = -n \int_L \phi \mathrm{d}\mu + n \int_L \phi\eta \wedge \frac{\omega_T^{n-1}}{(n-1)!}\ . \tag{3.25}$$

Notice that the term involving $\mathrm{d}^c\phi$ does not contribute, since ξ contracted into the integrand is identically zero. Indeed, we have

$$\xi \lrcorner \mathrm{d}^c\phi = \mathcal{L}_{r\partial/\partial r}\phi = 0 \tag{3.26}$$

and $\omega_T = (1/2)\mathrm{d}\eta$ is basic. Noting that

$$\mathrm{d}\mu = \eta \wedge \frac{\omega_T^{n-1}}{(n-1)!} \tag{3.27}$$

we have thus shown that

$$\frac{\mathrm{dvol}[L]}{\mathrm{d}t}(t=0) = 0 \tag{3.28}$$

identically for all transverse Kähler deformations[27]. It follows that $\mathrm{vol}[L]$ may be interpreted as a *function*

$$\mathrm{vol}[L] : \mathcal{C}_0 \to \mathbb{R}\ . \tag{3.29}$$

Our task in the remainder of this paper is to understand the properties of this function.

[27]Note that we didn't use the equation $\mathcal{L}_\xi\phi = 0$ anywhere. Any deformation ϕ of the metric not satisfying this equation will preserve the homothetic scaling of the metric on X, but it will no longer be a cone.

3.2 Varying the Reeb vector field

In the previous subsection we saw that vol$[L]$ may be regarded as a function on the space of Reeb vector fields, since the volume is independent of transverse Kähler deformations. We now fix a maximal torus $\mathbb{T}^s \subset \mathrm{Aut}(X)$, acting by isometries on each metric in a space of Sasakian metrics, and consider the properties of the functional vol$[L]$ on this space as we vary the Reeb vector field. In particular, in the remainder of this section we give formulae for the first and second variations.

We would like to differentiate the function vol$[L]$. Thus, we fix an arbitrary background Kähler cone metric, with Kähler potential r^2, and linearise the deformation equations around this. We set

$$\begin{aligned} \xi(t) &= \xi + tY & (3.30)\\ r^2(t) &= r^2(1+t\phi) & (3.31)\end{aligned}$$

where t is a (small) parameter and $Y \in \mathtt{t}_s$ is holomorphic and Killing. *A priori*, ϕ is any function on X. Working to first order in t, the calculation goes much as in the last subsection. The details may be found in Appendix C. We obtain

$$\mathrm{dvol}[L](Y) = -n\int_L \eta(Y)\mathrm{d}\mu \ . \tag{3.32}$$

This is our general result for the first derivative of vol$[L]$. As one can see, it manifestly depends only on the direction Y in which we deform the Reeb vector field, and not on the function ϕ, in accord with the previous section.

Note that the integrand in (3.32) is twice the Hamiltonian function for Y. Indeed, in this case, $\mathrm{d}r(Y) = 0$ and hence $\mathcal{L}_Y\omega = 0$. Since X necessarily has $b_1(X_0) = 0$, there is therefore a function y_Y such that

$$\mathrm{d}y_Y = -\frac{1}{4}Y \lrcorner \mathrm{d}\mathrm{d}^c r^2 \ . \tag{3.33}$$

Using

$$Y \lrcorner \omega = -\frac{1}{2}\left[\eta(Y)\mathrm{d}r^2 - r^2 Y \lrcorner \mathrm{d}\eta\right] \tag{3.34}$$

and (2.12), one finds that the homogeneous solution to this equation is

$$y_Y = \tfrac{1}{2}r^2\eta(Y) \ . \tag{3.35}$$

The Hamiltonian y_Y is then homogeneous degree two under $r\partial/\partial r$. Substituting this into (3.32) we recover the toric formula (3.18) in [22], obtained in a completely different way using convex polytopes.

In order to compute the second variation, we note that, since Y is Killing, it commutes with $r\partial/\partial r$: $[Y, r\partial/\partial r] = 0$. As a result, $\eta(Y)$ is independent of r. Using this property, we may write

$$\mathrm{dvol}[L](Y) = -n(n+1)\int_{r\leq 1}(\mathrm{d}^c r^2)(Y)\frac{\omega^n}{n!} \tag{3.36}$$

where recall that $\mathrm{d}^c r^2 = 2r^2\eta$. We now differentiate again to obtain

$$\mathrm{d}^2\mathrm{vol}[L](Y,Z) = n(n+1)\int_L \eta(Y)\eta(Z)\mathrm{d}\mu\ . \tag{3.37}$$

This is a general form for the second variation of the volume of a Sasakian manifold. The derivation, which is a little lengthy, is contained in Appendix C. Note that (3.37) is manifestly positive definite, and hence the volume is a strictly convex function. Again, for toric geometries, this reduces to a formula in [22].

3.3 Uniqueness of critical points

Using this last result, we may prove uniqueness of a critical point rather simply. We regard the volume as a function

$$\mathrm{vol}[L] : \mathcal{C}_0 \to \mathbb{R}\ . \tag{3.38}$$

The Reeb vector field for a Sasaki–Einstein metric is a critical point of $\mathrm{vol}[L]$, restricted to the subspace Σ for which the holomorphic $(n,0)$–form has charge n. This defines a compact convex polytope $\Sigma \subset \mathtt{t}_s$, and we may hence regard the Einstein–Hilbert action as a function

$$\mathcal{S} : \Sigma \to \mathbb{R}\ . \tag{3.39}$$

Since we have just shown that $\mathcal{S}$ is strictly convex on this space, and Σ is itself convex, standard convexity arguments show that $\mathcal{S}$ has a unique critical point. Thus, assuming a Sasaki–Einstein metric exists in our space of Sasakian metrics on L, its Reeb vector field is unique in $\Sigma \subset \mathtt{t}_s$.

4 The Futaki invariant

In this section, we consider a fixed background Sasakian metric which is *quasi–regular*. We show that the first derivative $\mathrm{dvol}[L]$, as a linear function on the Lie algebra $\mathtt{t}_s$, is closely related to the Futaki invariant of V. This is a well–known [49] obstruction to the existence of a Kähler–Einstein metric on V. Using this relation, together with Matsushima's theorem [56], we show that the group of holomorphic isometries of a quasi–regular Sasaki–Einstein metric on L is a maximal compact subgroup of $\mathrm{Aut}(X)$. We conjecture this to be true also for the more generic irregular case.

4.1 Brief review of the Futaki invariant

Let (V, J_V, g_V) be a Kähler orbifold[28] with Kähler form ω_V and corresponding Ricci–form ρ_V such that

$$[\rho_V] = \lambda[\omega_V] \in H^2(V;\mathbb{R}) \tag{4.1}$$

where λ is a real positive constant. The value of λ is irrelevant since one can always rescale the metric, leaving ρ_V invariant[29]. By the global $i\partial\bar{\partial}$–lemma, there is a globally defined smooth function $f = f_{g_V}$ such that

$$\rho_V - \lambda\omega_V = i\partial\bar{\partial}f \tag{4.2}$$

where, throughout this section, the $\bar{\partial}$ operator is that defined on V. Note that this is the same f as that appearing in section 2. We now define a linear map

$$F : \mathrm{aut}_{\mathbb{R}}(V) \to \mathbb{R} \tag{4.3}$$

from the Lie algebra of real holomorphic vector fields $\mathrm{aut}_{\mathbb{R}}(V)$ on V to the real numbers by assigning to each holomorphic vector field ζ on V the number

$$F[\zeta] = \int_V (\mathcal{L}_\zeta f)\ \frac{\omega_V^{n-1}}{(n-1)!}\ . \tag{4.4}$$

It is more natural, and more standard, to define F on the space of *complex* holomorphic vector fields:

$$F_{\mathbb{C}} : \mathrm{aut}(V) \to \mathbb{C} \tag{4.5}$$

by simply complexifying (4.4).

The functional F, introduced by Futaki in 1983 [49], has the following rather striking properties:

- F is independent of the choice of Kähler metric representing $[\omega_V]$. That is, it is invariant under Kähler deformations. In this sense it is a topological invariant.

[28] One usually works with manifolds. Passing to the larger orbifold category involves no essential differences.

[29] Nevertheless, the value $\lambda = 2n$ is that relevant for Kähler cones in complex dimension n.

- $F_{\mathbb{C}}$ is a Lie algebra homomorphism.
- If V admits a Kähler–Einstein metric, so that for some g_V the function f is constant, the Futaki invariant F clearly vanishes identically.

Because of the second item, Calabi named $F_{\mathbb{C}}$ the *Futaki character* [57] – it is a character because $F_{\mathbb{C}}$ is a homomorphism onto the complex numbers. Let $G = \mathrm{Aut}(V)$ denote the group of holomorphic automorphisms of V, and $\mathtt{g}$ its Lie algebra. Thus $F_{\mathbb{C}} : \mathtt{g} \to \mathbb{C}$. Mabuchi [58] proved that the nilpotent radical of $\mathtt{g}$ lies in $\ker F_{\mathbb{C}}$. Thus if

$$\mathtt{g} = \mathtt{h} \oplus \mathrm{Lie}(\mathrm{Rad}(G)) \tag{4.6}$$

denotes the Levi decomposition of $\mathtt{g}$, it follows that $F_{\mathbb{C}}$ is completely determined by its restriction to the maximal reductive algebra $\mathtt{h}$. Since $F_{\mathbb{C}}$ is a Lie algebra character of $\mathtt{h}$, it vanishes on the derived algebra $[\mathtt{h}, \mathtt{h}]$, and is therefore determined by its restriction to the Lie algebra of the centre of $G = \mathrm{Aut}(V)$. The upshot then is that $F_{\mathbb{C}}$ is non–zero only on the centre of $\mathtt{g}$.

4.2 Relation to the volume

Note that our derivative $\mathrm{dvol}[L] : \mathtt{t}_s \to \mathbb{R}$ is a linear map which is also independent of transverse Kähler deformations. This follows since $\mathrm{vol}[L]$ itself has this property for all Kähler cones. Moreover, if L admits a Sasaki–Einstein metric with Reeb vector field ξ then $\mathrm{dvol}[L](Y) = 0$ for all vector fields $Y \in \mathtt{t}_s$ with $\mathcal{L}_Y \Omega = 0$. This is true simply because Sasaki–Einstein metrics are critical points of the Einstein–Hilbert action, which is equal to $\mathrm{vol}[L]$ on the subspace Σ. Thus, given the exposition in the previous subsection, it is not surprising that $\mathrm{dvol}[L]$ is related to Futaki's invariant. We now investigate this in more detail.

We regard

$$\mathrm{dvol}[L](Y) = -n \int_L \eta(Y) \mathrm{d}\mu \tag{4.7}$$

as a linear map

$$\mathrm{dvol}[L] : \mathtt{t}_s \to \mathbb{R} \tag{4.8}$$

with $Y \in \mathtt{t}_s$. Since all such Y commute with $r\partial/\partial r$ and ξ, Y descends to a holomorphic vector field $Y_V = \pi_* Y$ on V, where recall that $\pi : L \to V$ is the orbifold circle fibration. In fact, when interpreting $\mathrm{vol}[L]$ as the Einstein–Hilbert action, we should consider only those $Y \in \mathtt{t}_s$ such that $\mathcal{L}_Y \Omega = 0$. These form a linear subspace in $\mathtt{t}_s$.

In order to relate $\mathrm{dvol}[L]$ to the Futaki invariant, we shall use the existence of a certain spinor field θ on the Sasakian link L, as discussed in subsection 2.6. We shall also need the Lie derivative, acting[30] on spinor fields, along a Killing vector field[31] Y:

$$\mathcal{L}_Y\theta = \nabla_Y\theta + \frac{1}{4}\mathrm{d}Y^\flat \cdot \theta \ . \tag{4.9}$$

Suppose now that Y is Killing and satisfies

$$\mathcal{L}_Y\theta = i\alpha\theta \ . \tag{4.10}$$

Of course, we are interested in those Y with $\alpha = 0$ since then $\mathcal{L}_Y\Omega = 0$ also, as follows by writing the holomorphic $(n,0)$–form Ω as a bilinear in the spinor field. The equivalence of these two conditions is not quite obvious – a detailed argument proving this is given in Appendix B.

We now compute

$$\begin{aligned}
\frac{1}{2}\int_L \eta(Y)\mathrm{d}\mu &= \int_L \left(-i\bar\theta\nabla_Y\theta - \frac{1}{2}A(Y)\bar\theta\theta\right)\mathrm{d}\mu \\
&= \int_L \bar\theta\left(\alpha\theta + \frac{i}{4}\mathrm{d}Y^\flat\cdot\theta - \frac{1}{2}A(Y)\theta\right)\mathrm{d}\mu \\
&= \alpha\mathrm{vol}[L] - \frac{1}{8}\int_L(\mathrm{d}\eta,\mathrm{d}Y^\flat)\mathrm{d}\mu - \frac{1}{2}\int_L A(Y)\mathrm{d}\mu \ ,
\end{aligned} \tag{4.11}$$

where recall that the spinor is normalised so that $\bar\theta\theta = 1$. Here we have denoted the pointwise inner product between two two–forms A, B as

$$(A,B) \equiv \frac{1}{2}A_{\mu\nu}B^{\mu\nu} \ . \tag{4.12}$$

Next, we obtain

$$\begin{aligned}
-\frac{1}{8}\int_L(\mathrm{d}\eta,\mathrm{d}Y^\flat)\mathrm{d}\mu &= -\frac{1}{8}\int_L(\mathrm{d}^*\mathrm{d}\eta, Y^\flat)\mathrm{d}\mu \\
&= -\frac{1}{8}\int_L(\Delta_L\eta, Y^\flat)\mathrm{d}\mu \\
&= -\frac{n-1}{2}\int_L \eta(Y)\mathrm{d}\mu \ .
\end{aligned} \tag{4.13}$$

Here we have used the fact that $\mathrm{d}^*\eta = 0$ as η is a Killing one–form. Moreover, we have

$$\Delta_L\eta = 2\mathrm{Ric}(g_L)(\eta) = 4(n-1)\eta + 2\mathrm{Ric}(g_X)(\eta) = 4(n-1)\eta \ . \tag{4.14}$$

[30]Recall that the differential forms act on spinors via Clifford multiplication; that is for a p–form A, we have $A\cdot\theta \equiv \frac{1}{p!}A_{\mu_1\ldots\mu_p}\gamma^{\mu_1\ldots\mu_p}\theta$.

[31]We denote the one–form dual to Y by $Y^\flat = g_L(Y,\cdot)$.

The first equality is true for any Killing one–form. In the second we have used equation (2.17) to relate the Ricci curvature of the link L to that of the cone X. Note that from the same equation it also follows that ξ contracted into $\mathrm{Ric}(g_X)$ is zero, which gives the last equality. Finally, substituting (4.13) into (4.11) gives us

$$\mathrm{dvol}[L](Y) = -n \int_L \eta(Y) \mathrm{d}\mu = -2\alpha \mathrm{vol}[L] + \int_L A(Y) \mathrm{d}\mu \ . \tag{4.15}$$

Clearly, to relate to the Futaki invariant (4.4), we must now integrate over the circle fibre of $\pi : L \to V$. Indeed, recall that

$$\rho = \rho_V - 2n\omega_V = \frac{1}{2}\mathrm{dd}^c f \tag{4.16}$$

as an equation on the Fano orbifold V. We thus have

$$A = \frac{1}{2}\pi^*(\mathrm{d}^c f) \ . \tag{4.17}$$

Since f is basic by definition, and Y commutes with ξ, it follows that $A(Y)$ is also a basic function, *i.e.* it is independent of ξ. Thus we may trivially integrate over the circle fibre. Denote the length of this by

$$\ell = \frac{2\pi\beta}{n} \ . \tag{4.18}$$

Then

$$\begin{aligned} \mathrm{dvol}[L](Y) &= -2\alpha \mathrm{vol}[L] + \frac{\ell}{2}\int_V \mathrm{d}^c f(Y_V) \ \frac{\omega_V^{n-1}}{(n-1)!} \\ &= -2\alpha \mathrm{vol}[L] - \frac{\ell}{2}\int_V \left(\mathcal{L}_{J_V(Y_V)} f\right) \ \frac{\omega_V^{n-1}}{(n-1)!} \ . \end{aligned} \tag{4.19}$$

Thus we have shown that

$$\mathrm{dvol}[L](Y) = -2\alpha \mathrm{vol}[L] - \frac{\ell}{2} \cdot F[J_V(Y_V)] \ . \tag{4.20}$$

Of course, for Y preserving Ω we have $\alpha = 0$ and we are done. Note also that, when $Y = \xi$ the spinor has charge $\alpha = n/2$, as follows from a simple calculation. In this case $Y_V = 0$ and (4.20) shows that $\mathrm{vol}[L]$ is homogeneous degree $-n$ under deformations along the Reeb vector field. Later we will see this rather directly from our explicit formula for the volume.

Thus, as expected, the derivative of the volume is directly related to the Futaki invariant of V. The dynamical problem of finding a critical point of the Sasakian volume V may be interpreted as choosing the Reeb vector field in such a way so that the corresponding Kähler orbifold V has zero Futaki invariant. This is certainly a necessary condition for existence of a Kähler–Einstein metric on V, which clearly is also necessary in order for a Sasaki–Einstein metric to exist on L. Of course, this interpretation requires us to stick with quasi–regular stuctures, which is unnatural. Nevertheless, it gives a very interesting new interpretation of the Futaki invariant.

4.3 Isometries of Sasaki–Einstein manifolds

Using the result of the last subsection, together with known properties of the Futaki invariant described above, we may now deduce some additional properties of Sasaki–Einstein manifolds. In particular, we first show that the critical Reeb vector field is in the centre of a maximal compact subgroup $K \subset \mathrm{Aut}(X)$, where $\mathbb{T}^s \subset K$. Then we use this fact to argue that the group of (holomorphic) isometries of a Sasaki–Einstein metric on L is a maximal compact subgroup of $\mathrm{Aut}(X)$.

Let $K \subset \mathrm{Aut}(X)$ be a maximal compact subgroup of the holomorphic automorphism group of X, containing the maximal torus $\mathbb{T}^s \subset K$. Let $\mathtt{z} \subset \mathtt{k}$ denote the centre of $\mathtt{k}$, and write

$$\mathtt{t}_s = \mathtt{z} \oplus \mathtt{t}' \ . \tag{4.21}$$

Pick a basis for $\mathtt{z}$ so that we may identify $\mathtt{z} \cong \mathbb{R}^m$. We may consider the space of Reeb vector fields in this subspace – by the same reasoning as in section 2, these form a cone $\mathcal{C}_0^{(m)}$ where we now keep track of the dimension of the cone. There will be a unique critical point of the Einstein–Hilbert action on this space. Let us suppose that $b_*^{(m)} \in \mathbb{Q}^m$, so that this critical point is quasi–regular. Clearly we have considered only a subspace $\mathtt{z} \subset \mathtt{t}_s$, or $\mathbb{R}^m \subset \mathbb{R}^s$, and the minimum we have found in $\mathbb{R}^m$ might not be a minimum on the larger space. However, using the relation between $\mathrm{dvol}[L]$ and the Futaki invariant we may in fact argue that the critical point $b_*^{(m)} \in \mathbb{R}^m$ is necessarily a critical point of the minimisation problem on $\mathbb{R}^s$.

To see this, note that $\mathtt{t}' \cong \mathbb{R}^{s-m}$ descends to a subalgebra $\mathtt{t}' \subset \mathrm{aut}_{\mathbb{R}}(V)$. Since $b = (b_*^{(m)}, 0) \in \mathbb{R}^m \oplus \mathbb{R}^{s-m}$ is in the centre of $\mathtt{k}$ by construction, in fact the whole of $\mathtt{k}$ descends to a subalgebra of $\mathrm{aut}_{\mathbb{R}}(V)$. Recall now that $F_{\mathbb{C}} : \mathrm{aut}(V) \to \mathbb{C}$ vanishes on the complexification of $\mathtt{t}'$. This is because $F_{\mathbb{C}}$ is non–zero only on the centre of $\mathrm{aut}(V)$. Thus the derivative of $\mathrm{vol}[L]$ in the directions $\mathtt{t}'$ is *automatically zero*. Since the critical point is unique, this proves that $b_* = (b_*^{(m)}, 0) \in \mathbb{R}^m \oplus \mathbb{R}^{s-m}$ is the critical point also for the larger extremal problem on $\mathbb{R}^s$. This argument may of course be made for any $K \supset \mathbb{T}^s$, but in particular applies to a maximal such K. Hence we learn that the critical Reeb vector field, for a Sasaki–Einstein metric, necessarily lies in the centre[32] of the Lie algebra of a maximal compact subgroup of $\mathrm{Aut}(X)$.

Using this last fact, we may now prove that, for fixed $\mathbb{T}^s \subset K$, the isometry group of a Sasaki–Einstein metric on L with Reeb vector field in $\mathtt{t}_s$ is a maximal compact subgroup of $\mathrm{Aut}(X)$. To see this, note that since the critical Reeb vector field $\xi_* \subset \mathtt{z}$ lies in the centre of $\mathtt{k}$, the whole of K, modulo the $U(1)$ generated by the Reeb vector field, descends to a compact subgroup of the complex automorphisms $G = \mathrm{Aut}(V)$ of the orbifold V. The latter is Kähler–Einstein, and by Matsushima's theorem[33] [56], we learn that K in fact acts *isometrically* on V. Thus K acts isometrically on L, and we are done.

Of course, this is also likely to be true for irregular Sasaki–Einstein metrics. Thus we make a more general conjecture

- The group of holomorphic isometries of a Sasaki–Einstein metric on L is a maximal compact subgroup of $\mathrm{Aut}(X)$.

5 A localisation formula for the volume

In this section we explain that the volume of a Sasakian manifold may be interpreted in terms of the Duistermaat–Heckman formula. The essential point is that the Kähler potential $r^2/2$ may also be interpreted as the Hamiltonian function for the Reeb vector field. By taking any equivariant orbifold resolution of the cone X, we obtain an explicit formula for the volume, as a function of the Reeb vector $\xi \in \mathcal{C}_0$, in terms of topological fixed point data. If we write $\xi \in \mathcal{C}_0 \subset \mathtt{t}_s$ as

$$\xi = \sum_{i=1}^{s} b_i \frac{\partial}{\partial \phi_i} \tag{5.1}$$

then as a result the volume $\mathrm{vol}[L] : \mathcal{C}_0 \to \mathbb{R}$, relative to the volume of the round sphere, is a rational function of $b \in \mathbb{R}^s$ with rational coefficients. The Reeb vector field b_* for a Sasaki–Einstein metric is a critical point of $\mathrm{vol}[L]$ on the convex subspace Σ, formed by

[32]Note this statement is different from the statement that the Reeb vector field for any Sasakian metric lies in the centre of the isometry group: the group of automorphisms of (L, g_L) *i.e.* the isometry group, depends on the choice of Reeb vector field, whereas K depends only on (X, J).

[33]Recall that Matsushima's theorem states that on a Kähler–Einstein manifold V (or, more generally, orbifold) the Lie algebra of holomorphic vector fields $\mathrm{aut}(V)$ is the complexification of the Lie algebra generated by Killing vector fields.

intersecting $\mathcal{C}_0$ with the rational hyperplane of Reeb vectors under which Ω has charge n. It follows that b_* is an algebraic vector – that is, a vector whose components are all algebraic numbers. It thus also follows that the volume $\mathrm{vol}[L](b_*)$ of a Sasaki–Einstein metric, relative to the round sphere, is an algebraic number.

5.1 The volume and the Duistermaat–Heckman formula

There is an alternative way of writing the volume of (L, g_L). In the previous subsections we used the fact that

$$\mathrm{vol}[L] = \int_L \mathrm{d}\mu = 2n \int_{r\leq 1} \frac{\omega^n}{n!} \ . \tag{5.2}$$

This follows simply by writing out the measure on the cone X in polar coordinates and cutting off the r integral at $r = 1$. However, we may also write

$$\mathrm{vol}[L] \quad = \quad \frac{1}{2^{n-1}(n-1)!} \int_X e^{-r^2/2} \, \frac{\omega^n}{n!} \ . \tag{5.3}$$

This is now an integral over the whole cone. Note that the term in the exponent is the Kähler potential, which here acts as a convergence factor.

So far, the function r^2 has played a dual role: it determines the link $L = X\mid_{r=1}$, and is also the Kähler potential. However, the function $r^2/2$ is also the Hamiltonian function for the Reeb vector field. To see this, recall from (3.35) that the Hamiltonian function associated to the vector field Y is $y_Y = \frac{1}{2}r^2\eta(Y)$. Setting $Y = \xi$, we thus see that $r^2/2$ is precisely the Hamiltonian associated to the Reeb vector field. Thus we may suggestively write the volume (5.3) as

$$\mathrm{vol}[L] \quad = \quad \frac{1}{2^{n-1}(n-1)!} \int_X e^{-H} \; e^{\omega} \tag{5.4}$$

where $H = r^2/2$ is the Hamiltonian. The integrand on the right hand side of (5.4) is precisely that appearing in the Duistermaat–Heckman formula [40, 41] for a (non–compact) symplectic manifold X with symplectic form ω. H is the Hamiltonian function for a Hamiltonian vector field. The Duistermaat–Heckman theorem expresses this integral as an integral of local data over the fixed point set of the vector field. Of course, for a Kähler cone, the action generated by the flow of the Reeb vector field is locally free on $r > 0$, since ξ has square norm r^2. The fixed point contribution is therefore, formally, entirely from the isolated singular point $r = 0$. Thus in order to apply the theorem, one must first resolve the singularity. Taking a limit in which the resolved space approaches the cone, we will obtain a well–defined expression for the volume in terms of purely topological fixed point data. The volume is of course independent of the choice of resolution.

5.2 The Duistermaat–Heckman Theorem

In this subsection we give a review of the Duistermaat–Heckman theorem [40, 41] for compact Kähler manifolds. The non–compact case of interest will follow straightforwardly from this, as we shall explain. Since the proof of the Duistermaat–Heckman theorem [41] is entirely differentio–geometric, the result is also valid for orbifolds, with a simple modification that we describe.

Let W be a compact Kähler manifold with Kähler form ω, $\dim_{\mathbb{C}} W = n$, and let $\mathbb{T}^s \subset \mathrm{Aut}(W)$ act on W in a Hamiltonian fashion. Let $\xi \in \mathtt{t}_s$ with Hamiltonian H. Thus

$$\mathrm{d}H = -\xi \lrcorner \omega \ . \tag{5.5}$$

The flow on W generated by ξ will have some fixed point set, which is also the zero set of the vector field ξ. In general, the fixed point set $\{F\}$ will consist of a number of disconnected components F of different dimensions; each component is a Kähler submanifold of W. Let $f : F \hookrightarrow W$ denote the embedding, so that $f^*\omega$ is a Kähler form on F. From now on, we focus on a particular connected component F, of complex codimension k.

The normal bundle $\mathcal{E}$ of F in W is a rank k complex vector bundle over F. The flow generated by ξ induces a linear action on $\mathcal{E}$. Let $u_1, \ldots, u_R \in \mathbb{Z}^s \subset \mathtt{t}_s^*$ be the set of distinct weights of this linear action on $\mathcal{E}$. This splits $\mathcal{E}$ into a direct sum of complex vector bundles

$$\mathcal{E} = \bigoplus_{\lambda=1}^{R} \mathcal{E}_\lambda \ . \tag{5.6}$$

Here $\mathcal{E}_\lambda$ is a complex vector bundle over F of rank n_λ, and hence

$$\sum_{\lambda=1}^{R} n_\lambda = k \ . \tag{5.7}$$

Thus, for example, if $\mathcal{E}$ splits into a sum of complex line bundles then each $n_\lambda = 1$ and $R = k$. We denote the linear action of ξ on $\mathcal{E}$ by $L\xi$. This acts on the λth factor in (5.6) by multiplication by $i(b, u_\lambda)$, where recall that $b \in \mathbb{R}^s$ are the components of the vector field ξ as in (5.1). With respect to the decomposition (5.6), we thus have

$$L\xi = i \ \mathrm{diag}\left(1_{n_1}(b, u_1), 1_{n_2}(b, u_2), \ldots, 1_{n_R}(b, u_R)\right) \tag{5.8}$$

where 1_{n_λ} denotes the unit $n_\lambda \times n_\lambda$ matrix. Hence the determinant of this transformation is

$$\det\left(\frac{L\xi}{i}\right) = \prod_{\lambda=1}^{R} (b, u_\lambda)^{n_\lambda} \ . \tag{5.9}$$

Note this is homogeneous degree k in b.

Finally, choose any $\mathbb{T}^s$–invariant Hermitian connection on $\mathcal{E}$, with curvature $\Omega_{\mathcal{E}}$. The Duistermaat–Heckman theorem then states that

$$\int_W e^{-H}\ \frac{\omega^n}{n!} = \sum_{\{F\}} \int_F \frac{e^{-f^*H}\ e^{f^*\omega}}{\det\left(\frac{L\xi - \Omega_{\mathcal{E}}}{2\pi i}\right)} \ . \tag{5.10}$$

The sum is over each connected component F of the fixed point set. The determinant is a $k \times k$ determinant and should be expanded formally into a differential form of mixed degree. Moreover, the inverse is understood to mean one should expand this formally in a Taylor series. This notation is standard in index theory. We will now examine the right hand side of (5.10) in more detail.

If we let Ω_λ be the curvature of $\mathcal{E}_\lambda$, we may write

$$\det\left(\frac{L\xi - \Omega_{\mathcal{E}}}{2\pi i}\right) = \frac{1}{(2\pi)^k} \det\left(\frac{L\xi}{i}\right) \prod_{\lambda=1}^{R} \det(1 - (L\xi)^{-1}\Omega_\lambda) \ . \tag{5.11}$$

Fix one of the vector bundles $\mathcal{E}_\lambda$. Then

$$\det(1 - (L\xi)^{-1}\Omega_\lambda) = \det\left(1 + w\frac{i\Omega_\lambda}{2\pi}\right) \tag{5.12}$$

where

$$w = \frac{2\pi}{(b, u_\lambda)} \ . \tag{5.13}$$

The right hand side of (5.12) is precisely the *Chern polynomial* of $\mathcal{E}_\lambda$. As a cohomology relation, we have

$$\det\left(1 + w\frac{i\Omega_\lambda}{2\pi}\right) = \sum_{a \geq 0} c_a(\mathcal{E}_\lambda) w^a \in H^*(F; \mathbb{R}) \ . \tag{5.14}$$

Recall that each $\mathcal{E}_\lambda$ has Chern classes

$$c_a(\mathcal{E}_\lambda) \in H^{2a}(F; \mathbb{R}) \tag{5.15}$$

where $0 \leq a \leq n_\lambda$ and $c_0 = 1$. Thus we may write

$$\begin{aligned} \left[\det\left(\frac{L\xi - \Omega_{\mathcal{E}}}{2\pi i}\right)\right]^{-1} &= (2\pi)^k \left[\det\left(\frac{L\xi}{i}\right)\right]^{-1} \sum_{a \geq 0} \beta_a(b) \\ &= \frac{(2\pi)^k}{\prod_{\lambda=1}^{R} (b, u_\lambda)^{n_\lambda}} \sum_{a \geq 0} \beta_a(b) \ . \end{aligned} \tag{5.16}$$

The $\beta_a(b)$ are closed differential forms on F of degree $2a$, with $\beta_0(b) = 1$. The cohomology class of $\beta_a(b)$ in $H^{2a}(F;\mathbb{R})$ is then a polynomial in the Chern classes of the $\mathcal{E}_\lambda$. Moreover, the coefficients are *rational* functions of b, of degree $-a$.

Since f^*H is constant on each connected component of F, it follows that, for each connected component, the integrand on the right hand side of (5.10) is a polynomial in the Chern classes of $\mathcal{E}_\lambda$ and the pulled–back Kähler form $f^*\omega$ on F. The coefficients are memomorphic[34] in b, and analytic provided $(b, u_\lambda) \neq 0$ for all $\lambda = 1, \ldots, R$. Of course, the left hand side of (5.10) is certainly analytic.

As a special case of this result, suppose that F is an isolated fixed point. Thus $k = n$, and $\mathcal{E}$ is a trivial bundle. We may then write the n, *possibly indistinct*, weights as u_λ, $\lambda = 1, \ldots, n$. Neither the Chern classes nor the Liouville measure $\exp(f^*\omega)$ contribute non–trivially, and we are left with

$$(2\pi)^n \; e^{-f^*H} \prod_{\lambda=1}^{n} \frac{1}{(b, u_\lambda)} \; . \tag{5.17}$$

This is the general formula for the contribution of an isolated fixed point to the Duistermaat–Heckman formula (5.10).

The proof of the Duistermaat–Heckman theorem [41] is entirely differentio geometric, and thus the proof also goes through easily for non–compact manifolds, and orbifolds. The proof goes roughly as follows. The integrand on the left hand side of the Duistermaat–Heckman formula (5.10) is exact on W minus the set of fixed points $\{F\}$. One then applies Stokes' theorem to obtain a sum of integrals over boundaries around each connected component F. This boundary is diffeomorphic, via the exponential map, to the total space of the normal sphere bundle to F in W, of radius ϵ. One should eventually take the limit $\epsilon \to 0$. By introducing an Hermitian connection on the normal bundle $\mathcal{E}$ with curvature $\Omega_{\mathcal{E}}$, one can perform the integral over the normal sphere explicitly, resulting in the formula (5.10). Thus we may extend this proof straightforwardly to our non–compact case, and to orbifolds, as follows:

- For non–compact manifolds, provided the fixed point sets are in the interior, and that the measure tends to zero at infinity (so that the boundary at infinity makes no contribution in Stokes' theorem), the formula (5.10) is still valid.

- For orbifolds, we must modify the formula (5.10) slightly. The normal space to a *generic* point in a connected component F is not a sphere, but rather a quotient

[34] We may analytically continue the right hand side to $b \in \mathbb{C}^s$.

S^{2k-1}/Γ, where Γ is a finite group of order d. Thus, when we integrate over this normal space, we pick up a factor $1/d$. For each connected component F we denote this integer, called the order of F, by d_F. The general formula is then almost identical

$$\int_W e^{-H} \frac{\omega^n}{n!} = \sum_{\{F\}} \int_F \frac{1}{d_F} \frac{e^{-f^*H}\ e^{f^*\omega}}{\det\left(\frac{L\xi - \Omega_{\mathcal{E}}}{2\pi i}\right)} \ . \tag{5.18}$$

In the orbifold case, the Chern classes, defined in terms of a curvature form $\Omega_{\mathcal{E}}$ on the vector orbibundle $\mathcal{E}$, are in general *rational*, *i.e.* images under the natural map

$$H^*(F;\mathbb{Q}) \to H^*(F;\mathbb{R}) \ . \tag{5.19}$$

These slight generalisations will be crucial for the application to Sasakian geometry, to which we now turn.

5.3 Application to Sasakian geometry

Let (X,ω) be a Kähler cone, with Kähler potential r^2. The volume of the link $L = X\ |_{r=1}$ may be written as in (5.4). In order to apply the Duistermaat–Heckman theorem, we must first resolve the cone X. In fact in order to compute the volume $\mathrm{vol}[L](b)$ as a function of b, we pick, topologically, a fixed (orbifold) resolution W of X. Thus we have a map

$$\Pi : W \to X \tag{5.20}$$

and an exceptional set E such that $W \setminus E \cong X_0$ is a biholomorphism. Moreover, the map Π should be equivariant with respect to the action of $\mathbb{T}^s$. Thus all the fixed points of $\mathbb{T}^s$ on W necessarily lie on the exceptional set. There is a natural way to do this: we simply choose a (any) quasi–regular Reeb vector field ξ_0 and blow up the Kähler orbifold V_0 to obtain the total space W of the bundle $\mathcal{L} \to V_0$. Note that then $W \setminus V_0 \cong X_0$, and that $\mathbb{T}^s$ acts on W. Thus this resolution is obviously equivariant.

We then assume[35] that, for every Reeb vector field $\xi \in \mathcal{C}_0$, there is a 1–parameter family of $\mathbb{T}^s$–invariant Kähler metrics $g(T)$, $0 < T < \delta$ for some $\delta > 0$, on W such that $g(T)$ smoothly approaches a Kähler cone metric with Reeb vector field ξ as $T \to 0$. We may then apply the Duistermaat–Heckman theorem (5.18) to the Kähler metric $g(T)$ on the orbifold W. In the limit that $T \to 0$, the exceptional set collapses to zero

[35]We assume there is no obstruction to doing this. In any case, we shall also prove the localisation formula (5.25) in the next section using a different relation to an equivariant index on the cone X. This doesn't assume the existence of any metric on W.

volume and we recover the cone X. Since all fixed points of $\mathbb{T}^s$ lie on the exceptional set, the pull–back of the Hamiltonian f^*H tends to zero in this limit, as $H \to r^2/2$ which is zero at $r = 0$. Moreover, the pull–back of the Kähler form $f^*\omega$ is also zero in this limit. Hence the exponential terms on the right–hand side of the Duistermaat–Heckman formula are equal to 1 in the conical limit $T \to 0$. This leaves us with the formula

$$\mathrm{vol}[L](b) = \frac{1}{2^{n-1}(n-1)!}\sum_{\{F\}}\int_F \frac{1}{d_F}\,\frac{(2\pi)^k}{\prod_{\lambda=1}^R (b,u_\lambda)^{n_\lambda}}\sum_{a\geq 0}\beta_a(b)\ . \tag{5.21}$$

This formula is valid for b a generic element of $\mathtt{t}_s$. Then the vanishing set of the Reeb vector field is the fixed point set of $\mathbb{T}^s$. For certain special values of b the set of fixed points changes. For example, when W is obtained by taking a quasi–regular Reeb vector field ξ_0 and blowing up V_0, the fixed point set of ξ_0 is the whole of V_0. Note, however, that $\mathrm{vol}[L](b)$ is still a smooth function of b.

The integral over F picks out the term in the sum of degree $a = (n-k)$. Recall that the ath term is homogeneous in b of degree $-a$. In particular, we may extract the factor $(2\pi)^{n-k}$ and write

$$\mathrm{vol}[L](b) = \frac{2\pi^n}{(n-1)!}\sum_{\{F\}}\beta(b)\prod_{\lambda=1}^R \frac{1}{(b,u_\lambda)^{n_\lambda}}\ . \tag{5.22}$$

Here $\beta(b)$ is a sum of Chern numbers of the normal bundle $\mathcal{E}$ of F in W, with coefficients which are homogeneous degree $-(n-k)$ in b. Specifically,

$$\beta(b) = \int_F \frac{1}{d_F}\prod_{\lambda=1}^R\left[\sum_{a\geq 0}\frac{c_a(\mathcal{E}_\lambda)}{(b,u_\lambda)^a}\right]^{-1}\ . \tag{5.23}$$

The Chern polynomials in (5.23) should be expanded in a Taylor series, and the integral over F picks out the differential form of degree $2(n-k)$. Recall that Chern numbers are defined as integrals over F of wedge products of Chern classes. Thus when W is a manifold the coefficients in $\beta(b)$ are integers. For the more general case of orbifolds, the Chern numbers are rational numbers.

Noting that

$$\mathrm{vol}[S^{2n-1}] = \frac{2\pi^n}{(n-1)!} \tag{5.24}$$

is precisely the volume of the round $(2n-1)$–sphere, we thus have

$$V(b) \equiv \frac{\mathrm{vol}[L](b)}{\mathrm{vol}[S^{2n-1}]} = \sum_{\{F\}}\frac{1}{d_F}\int_F\prod_{\lambda=1}^R\frac{1}{(b,u_\lambda)^{n_\lambda}}\left[\sum_{a\geq 0}\frac{c_a(\mathcal{E}_\lambda)}{(b,u_\lambda)^a}\right]^{-1}\ . \tag{5.25}$$

Here we have defined the normalised volume $V(b)$. The right hand side of (5.25) is homogeneous degree $-n$ in b, and is manifestly a rational function of b with *rational* coefficients, since the weights and Chern numbers are generally rational numbers. This is our general formula for the volume of a Sasakian metric on L with Reeb vector field b. Using this result, we may now prove:

- The volume of a Sasaki–Einstein manifold, relative to that of the round sphere, is an algebraic number.

This follows trivially, since the Reeb vector b_* for a Sasaki–Einstein metric is a critical point of (5.25) on the subspace of vector fields under which the holomorphic $(n,0)$–form Ω has charge n. Thus b_* is an isolated zero of a system of algebraic equations with rational coefficients, and hence is algebraic. The normalised volume $V(b_*)$ is thus also an algebraic number.

We conclude this section by relating the formula (5.25) to the volume of quasi–regular Sasakian metrics. Let ξ be the Reeb vector field for a quasi–regular Sasakian structure, and choose the resolution W above so that V is the Kähler orbifold of the Sasakian structure. ξ generates an action of $U(1)$ on W which is locally free outside the zero section, and fixes the zero section V. Thus in this case $\mathcal{E} = \mathcal{L}$ and the formula (5.25) simplifies considerably. The weight $u = 1$, the codimension $k = 1$, $d = 1$, and hence (5.25) gives

$$\begin{aligned} V(b) &= \frac{1}{b}\int_V \left[1 + b^{-1}c_1(\mathcal{L})\right]^{-1} \\ &= \frac{1}{b^n}\int_V c_1(\mathcal{L}^*)^{n-1} \end{aligned} \tag{5.26}$$

where $c_1(\mathcal{L}) = -c_1(\mathcal{L}^*)$. Recall from section 2 that $\mathcal{L} \to V$ is always given by some root of the canonical bundle over V. The first Chern class of $\mathcal{L}^*$ is then

$$c_1(\mathcal{L}^*) = \frac{c_1(V)}{\beta} \in H^2_{\mathrm{orb}}(V;\mathbb{Z}) \tag{5.27}$$

where $c_1 = c_1(V) \in H^2_{\mathrm{orb}}(V;\mathbb{Z})$ is the first Chern class of the complex orbifold V. The total space L of the associated circle bundle to $\mathcal{L}$ will be simply connected if and only if β is the *maximal* positive integer such that (5.27) is an integer class, which recall is called the index of V. The general volume formula (5.25) gives

$$V(b) = \frac{1}{b^n\beta^{n-1}}\int_V c_1^{n-1}\ . \tag{5.28}$$

We now compute the volume $V(b)$ directly. Recall that, in general, the volume of a *quasi–regular* Sasakian manifold is given by the formula

$$\mathrm{vol}[L] = \frac{2\pi^n\beta}{n^n(n-1)!}\int_V c_1^{n-1} \ . \tag{5.29}$$

To see this, recall from (2.22) that the Kähler class of V is $[\omega_V] = [\rho_V/2n]$, where in the latter equation we have used the fact that the holomorphic $(n,0)$–form Ω is homogeneous degree n under $r\partial/\partial r$. We also have $c_1 = [\rho_V/2\pi]$. Thus the volume of V is given by

$$\mathrm{vol}[V] = \int_V \frac{\omega_V^{n-1}}{(n-1)!} = \frac{\pi^{n-1}}{n^{n-1}(n-1)!}\int_V c_1^{n-1} \ . \tag{5.30}$$

The length of the circle fibre is $2\pi\beta/n$, and hence (5.29) follows. For example, when $V = \mathbb{C}P^{n-1}$ and $\beta = n$ this leads to the formula

$$\mathrm{vol}[S^{2n-1}] = \frac{2\pi^n}{(n-1)!} \tag{5.31}$$

for the volume of the round sphere. Thus from (5.29) we have

$$V(b) = \frac{\beta}{n^n}\int_V c_1^{n-1} \tag{5.32}$$

which is precisely formula (5.28) with

$$b = n/\beta \ . \tag{5.33}$$

This equation for b must be imposed to compare (5.32) to (5.28), since in the former equation we have assumed that Ω has charge n under the Reeb vector field. More precisely, in (5.28) we have written

$$\xi = b\frac{\partial}{\partial\nu} \tag{5.34}$$

where $\nu \sim \nu + 2\pi$, and $\partial/\partial\nu$ rotates the fibre of the line bundle $\mathcal{L}$ with weight *one*. Thus

$$\nu = \frac{n\psi}{\beta} \tag{5.35}$$

where ψ was defined in section 2.3. Recalling that Ω has charge n under $\partial/\partial\psi$, we may thus also impose this on the vector field ξ in (5.34):

$$in\Omega = \mathcal{L}_\xi\Omega = b\mathcal{L}_{\partial/\partial\nu}\Omega = b\frac{\beta}{n}\mathcal{L}_{\partial/\partial\psi}\Omega = ib\beta\Omega \tag{5.36}$$

from which (5.33) follows.

5.4 Sasakian 5–manifolds and an example

The most physically interesting dimension is $n = 3$. Then the AdS/CFT correspondence conjectures that string theory on $AdS_5 \times L$ is dual to an $\mathcal{N} = 1$ superconformal field theory in four dimensions. In this section we therefore specialise the formula (5.25) to complex dimension $n = 3$.

Suppose that $X_0 \cong \mathbb{R}_+ \times L$ is a Calabi–Yau 3–fold, and suppose we have a space of Kähler cone metrics on X which are invariant under a $\mathbb{T}^2$ action. We choose this case since a $\mathbb{T}^3$ action would mean that L were toric, which we treat in section 7.

Resolving the cone X to W, as before, the fixed point sets on V_0 will consist of isolated fixed points $p \in V_0$ and curves $C \subset V_0$. We have already treated the isolated fixed points in (5.17). Let the normal bundle of C in V_0 be $\mathcal{M}$. The total normal bundle of C in W is then $\mathcal{E} = h^*\mathcal{L} \oplus \mathcal{M}$ where $h : C \hookrightarrow V_0$ is the inclusion. Thus the normal bundle $\mathcal{E}$ to C in W splits into a sum of two line bundles. We denote the weights as $u_\lambda \in \mathbb{Z}^2$, $\lambda = 1, 2$ for $\mathcal{L}$ and $\mathcal{M}$, respectively. We then get the following general formula for the volume, in terms of topological fixed point data:

$$V(b) = \sum_{\{p\}} \frac{1}{d_p} \prod_{\mu=1}^{3} \frac{1}{(b, u_\mu)} - \sum_{\{C\}} \frac{1}{d_C} \left[\prod_{\lambda=1}^{2} \frac{1}{(b, u_\lambda)} \right] \int_C \left[\frac{c_1(\mathcal{L})}{(b, u_1)} + \frac{c_1(\mathcal{M})}{(b, u_2)} \right] . \quad (5.37)$$

Here $u_\mu \in \mathbb{Z}^2$, $\mu = 1, 2, 3$ are the weights on the tangent space at p. Note that $V(b)$ is clearly homogeneous degree -3 in b, as it should be.

Example: As an example of formula (5.37), let us calculate the volume of Sasakian metrics on the complex cone over the first del Pezzo surface. Of course, this is toric, so one can use the toric methods developed in [22]. However, the point here is that we will rederive the result using *non–toric* methods. Specifically, we'll use the formula (5.37).

We think of the del Pezzo as the first Hirzebruch surface, $\mathbb{F}_1$. This is a $\mathbb{C}P^1$ bundle over $\mathbb{C}P^1$, which may be realised as the projectivisation

$$\mathbb{F}_1 = \mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(-1)) \to \mathbb{C}P^1 . \quad (5.38)$$

We take W to be the total space of the canonical bundle $\mathcal{K} \to \mathbb{F}_1$, and the $\mathbb{T}^2$ to act by rotating the fibre of $\mathcal{K}$ and the fibre $\mathbb{C}P^1$ of $\mathbb{F}_1$. The fixed point set of this $\mathbb{T}^2$ action consists of two curves on $V_0 = \mathbb{F}_1$, which are the north and south pole sections of $\mathbb{F}_1$. We denote these by H and E, which are two copies of $\mathbb{C}P^1$. In fact, $[H]$ is the hyperplane class on dP_1, and $[E]$ is the exceptional divisor. The normal bundles over

H and E are

$$\begin{aligned} H &: \mathcal{E} = \mathcal{O}(-3) \oplus \mathcal{O}(1) \\ E &: \mathcal{E} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \ . \end{aligned} \tag{5.39}$$

Note that the Chern numbers sum to -2 in each case, as is necessary to cancel the Chern number $+2$ of $\mathbb{C}P^1$.

Write $\mathbb{T}^2 = U(1)_1 \times U(1)_2$. Let $U(1)_1$ rotate the fibre of $\mathcal{L} = \mathcal{K}$ with weight one, and let $U(1)_2$ rotate the fibre $\mathbb{C}P^1$ of $\mathbb{F}_1$ with weight one, where we take the canonical lifting of this action to the canonical bundle $\mathcal{K}$. We write

$$\xi = \sum_{i=1}^{2} b_i \frac{\partial}{\partial \phi_i} \tag{5.40}$$

We need to compute the weights $u_\lambda \in \mathbb{Z}^2 \subset \mathtt{t}_2^*$ of the $\mathbb{T}^2$ action on the normal bundles (5.39). Here $\lambda = 1, 2$ denote the two line bundles in the splitting (5.39). We compute the weight of each $U(1) \subset \mathbb{T}^2$ in turn. $U(1)_1$ has weights $[1, 0]$ with respect to the splitting (5.39) for both H and E since this simply rotates the fibre of $\mathcal{K}$ with weight one. As for $U(1)_2$, note that $\mathcal{K}$ restricted to any point x on the base $\mathbb{C}P^1$ of $\mathbb{F}_1$ is a copy of $T^*\mathbb{C}P^1 = \mathcal{O}(-2) \to \mathbb{C}P^1$. By definition, $U(1)_2$ rotates this fibre $\mathbb{C}P^1$ with weight one, and thus fixes its north and south poles. Thus as we vary x on the base $\mathbb{C}P^1$, we sweep out H and E, respectively. The weights of $U(1)_2$ on the tangent space to x in $T^*\mathbb{C}P^1$ are thus $[-1, 1]$ and $[1, -1]$, respectively – the opposite signs appear because we have the cotangent bundle, rather than the tangent bundle. These also give the weights for $U(1)_2$ acting on H and E, with respect to the decomposition (5.39), respectively.

Thus, to summarise, the weights $u_\lambda \in \mathbb{Z}^2 \subset \mathtt{t}_2^*$ are

$$\begin{aligned} H &: u_1 = (1, -1), \qquad u_2 = (0, 1) \\ E &: u_1 = (1, 1), \qquad u_2 = (0, -1) \ . \end{aligned} \tag{5.41}$$

The formula (5.37) thus gives

$$\begin{aligned} V(b) &= \frac{1}{(b_1 - b_2) b_2} \left(\frac{3}{b_1 - b_2} + \frac{-1}{b_2} \right) + \frac{1}{(b_1 + b_2)(-b_2)} \left(\frac{1}{b_1 + b_2} + \frac{1}{-b_2} \right) \\ &= \frac{8 b_1 + 4 b_2}{(b_1^2 - b_2^2)^2} \ . \end{aligned} \tag{5.42}$$

One can verify that, on setting $b_1 = 3$, corresponding to the holomorphic $(3,0)$–form Ω having charge 3, the remaining function of b_2 has a critical point, inside the Reeb cone, at $b_2 = -4 + \sqrt{13}$. The volume at the critical point is then

$$V_* = \frac{43 + 13\sqrt{13}}{324} \tag{5.43}$$

which is indeed the correct result [10, 23].

The reason that we get the correct result here is that any circle that rotates the *base* $\mathbb{C}P^1$ of $\mathbb{F}_1$ is not in the centre of the Lie algebra of the compact group $K = U(1)^2 \times SU(2)$ acting on the cone X. From the results on the Futaki invariant in section 4, it follows that $b_3 = 0$ necessarily at any critical point. Indeed, the Lie algebra $\mathtt{k} = \mathtt{t}_2 \oplus \mathtt{su}(2)$. The reason that we do not need to extremise over $\mathtt{t}_3$ is that $\mathtt{k} = \mathtt{z} \oplus \mathtt{t}'$ where $\mathtt{t}' \subset \mathtt{su}(2)$. Thus the derivative of $\mathrm{vol}[L]$ is automatically zero in the direction $\mathtt{t}'$, provided $\xi \in \mathtt{z} = \mathtt{t}_2$. The isometry algebra of the metric is then, according to our conjecture, the maximal $\mathtt{k} = \mathtt{t}_2 \oplus \mathtt{su}(2)$, which indeed it is [10, 23]. The formula (5.42) may indeed be recovered from the toric result on setting $b_3 = 0$, as we show in section 7.

6 The index–character

In this section we show that the volume of a Sasakian manifold, as a function of b, is also related to a limit of the equivariant index of the Cauchy–Riemann operator $\bar{\partial}$ on the cone X. This equivariant index essentially counts holomorphic functions on X according to their charges under the $\mathbb{T}^s$ action. The key to this relation is the Lefschetz fixed point theorem for the $\bar{\partial}$ operator. In fact, by taking a limit of this formula, we will recover our general formula (5.25) for the volume in terms of fixed point data.

6.1 The character

Recall that $\bar{\partial}$ is the Cauchy–Riemann operator on X. We may consider the elliptic complex

$$0 \longrightarrow \Omega^{0,0}(X) \xrightarrow{\bar{\partial}} \Omega^{0,1}(X) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{0,n}(X) \longrightarrow 0 \tag{6.1}$$

on X. Here $\Omega^{0,p}(X)$ denotes the differential forms of Hodge type $(0,p)$ with respect to the complex structure of X. We denote the cohomology groups of this sequence as $\mathcal{H}^p(X) \cong H^{0,p}(X;\mathbb{C})$. In fact, the groups $\mathcal{H}^p(X)$, for $p > 0$, are all zero. This follows since X may be realised as the total space of a negative complex line bundle $\mathcal{L}$ over a compact Fano orbifold V. The property is then inherited from the Fano V. On the other hand, $\mathcal{H}^0(X)$ is clearly infinite dimensional, in contrast to the compact case.

The action of $\mathbb{T}^s$ on X, since it is holomorphic, commutes with $\bar{\partial}$. Hence there is an induced action of $\mathbb{T}^s$ on the cohomology groups of $\bar{\partial}$. The equivariant index, or holomorphic Lefschetz number, for an element $q \in \mathbb{T}^s$, is defined to be

$$L(q, \bar{\partial}, X) = \sum_{p=0}^{n} (-1)^p \ \mathrm{Tr}\{q \mid \mathcal{H}^p(X)\} \ . \tag{6.2}$$

Here the notation $\mathrm{Tr}\{q \mid \mathcal{H}^p(X)\}$ means one should take the trace of the induced action of q on $\mathcal{H}^p(X)$. The index itself, given by setting $q = 1$, is clearly infinite: the action of q is trivial and the trace simply counts holomorphic functions on X. However, the equivariant index is well–defined, provided the trace converges. In fact, we may analytically continue (6.2) to $q \in \mathbb{T}^s_{\mathbb{C}}$. The singular behaviour at $q = 1$ will then show up as a pole. Note we have not imposed any type of boundary conditions on $\bar{\partial}$. We shall henceforth write the equivariant index as

$$C(q, X) = L(q, \bar{\partial}, X) = \mathrm{Tr}\{q \mid \mathcal{H}^0(X)\} \tag{6.3}$$

and refer to it simply as the character.

6.2 Relation to the ordinary index

Suppose that we have a *regular* Sasakian structure on L, and consider the corresponding circle action on X. Holomorphic functions on the cone X of charge k under this circle action may be identified with holomorphic sections of the bundle

$$(\mathcal{L}^*)^k \to V \tag{6.4}$$

where recall that $\mathcal{L}^*$ is an ample line bundle over V whose dual is the associated complex line bundle to the projection $\pi : L \to V$. Canonically, we may take $\mathcal{L} = \mathcal{K}$, the canonical bundle over V.

The trace of $q \in \mathbb{C}^*$ on the space of holomorphic functions of charge k is given by

$$\mathrm{Tr}\{q \mid \mathcal{H}^0(X)_k\} = q^k \dim H^0\left(V; (\mathcal{L}^*)^k\right) \ . \tag{6.5}$$

The right hand side is given by the Riemann–Roch theorem. Indeed, we have

$$\dim H^0\left(V; (\mathcal{L}^*)^k\right) = \chi\left(V, (\mathcal{L}^*)^k\right) = \int_V e^{-kc_1(\mathcal{L})} \cdot \mathrm{Todd}(V) \ . \tag{6.6}$$

In the first equality we have used the fact that $\dim H^i\left(V; (\mathcal{L}^*)^k\right) = 0$ for $i > 0$ since V is Fano and hence $\mathcal{L}^*$ is ample. The second equality is the Riemann–Roch theorem[36].

[36] This does *not* generalise as straightforwardly to orbifolds as one might have hoped. We shall make some comments on the (equivariant) Riemann–Roch theorem for orbifolds at the end of this section.

The Todd class is a certain polynomial in the Chern classes of V, whose precise form we won't need.

It follows then that the character, for a regular $U(1)$ action, is simply given by

$$C(q, X) = \sum_{k\geq 0} q^k \int_V e^{-kc_1(\mathcal{L})} \cdot \mathrm{Todd}(V) \ . \tag{6.7}$$

More generally, we can interpret the character $C(q, X)$ in terms of the equivariant index theorem for $\mathbb{T}^{s-1}$ on V. However, it is easier to keep things defined on the cone.

The relation between $C(q, X)$ defined in (6.7) and the volume of a regular Sasakian (–Einstein) manifold has been noted before in [46]. The key in this section is to extend this to the equivariant case. Then the relation of the volume to the equivariant index becomes a function of the Reeb vector field.

6.3 Localisation and relation to the volume

In the above discussion we have been slightly cavalier in defining $C(q, X)$ as a trace over holomorphic functions on X, since X is *singular*. Recall from section 2 that X is an affine algebraic variety with an isolated Gorenstein singularity at $r = 0$, that is defined by polynomial equations $\{f_1 = 0, \ldots, f_S = 0\} \subset \mathbb{C}^N$. The space of holomorphic functions on X that we want is then given by elements of the coordinate ring of X,

$$\mathbb{C}[X] = \mathbb{C}[z_1, \ldots, z_N]/\langle f_1, \ldots, f_S\rangle \ , \tag{6.8}$$

where $\mathbb{C}[z_1, \ldots, z_N]$ is simply the polynomial ring on $\mathbb{C}^N$ and $\langle f_1, \ldots, f_S\rangle$ is the ideal generated by the functions $\{f_A, A = 1, \ldots, S\}$. Also as discussed in section 2, we may always resolve X to a space W with at worst orbifold singularities: for example, one can take any quasi–regular Sasakian structure and take W to be the total space of $\mathcal{L} \to V_0$. It is important that the resolution is equivariant with respect to the torus action. Thus, in general, we have a $\mathbb{T}^s$–equivariant birational map

$$f : W \to X \tag{6.9}$$

which maps some exceptional set $E \subset W$ to the singular point $p = \{r = 0\} \in X$. In particular, $f : W \setminus E \to X_0 = X \setminus \{p\}$ is a biholomorphism. The fixed point set of $\mathbb{T}^s$ on W is then necessarily supported on E. The character is conveniently computed on W, as in the previous section, and is independent of the choice of resolution. In fact this set–up is identical to that in section 5 – the resulting formula is rather similar to (5.10), although for orbifolds the equivariant index theorem is rather more involved than for manifolds[37].

We claim that the volume $V(b)$ is given in terms of the character $C(q,X)$ by the simple formula

$$V(b) = \lim_{t\to 0}\; t^n\; C(\exp(-tb), X)\ . \tag{6.10}$$

Recall that $C(q,X)$ is singular at $q=1$. By defining $q_i = \exp(-tb_i)$, and sending $t\to 0$, we are essentially picking out the leading singular behaviour. As we shall explain, the leading term in t is always a pole of order n.

Let W be a completely smooth resolution of the cone X. For example, if X admits any *regular* Sasakian structure, as in the previous section we make take W to be the total space of $\mathcal{L}\to V_0$ with V_0 a *manifold*. With definitions as in the last section, the equivariant index theorem in [59] gives

$$C(q,W) = \sum_{\{F\}}\int_F \frac{\mathrm{Todd}(F)}{\prod_{\lambda=1}^{R}\prod_j\left(1-q^{u_\lambda}e^{-x_j}\right)}\ . \tag{6.11}$$

Here the x_j are the basic characters[38] for the bundle $\mathcal{E}_\lambda\to F$. These are defined via the splitting principle. This says that, for practical calculations, we may assume that $\mathcal{E}_\lambda$ splits as a direct sum of complex line bundles

$$\mathcal{E}_\lambda = \bigoplus_{j=1}^{n_\lambda}\mathcal{L}_j\ . \tag{6.12}$$

Then

$$x_j = c_1(\mathcal{L}_j)\in H^2(F;\mathbb{Z})\ . \tag{6.13}$$

For a justification of this, the reader might consult reference [60]. The Chern classes of $\mathcal{E}_\lambda$ may then be written straightforwardly in terms of the basic characters. For example,

$$c(\mathcal{E}_\lambda) = \prod_j(1+x_j) \tag{6.14}$$

so that $c_1 = \sum_j x_j$. The Chern character is given by $\mathrm{ch}(\mathcal{E}_\lambda) = \sum_j \exp(x_j)$.

To illustrate this, we note that one defines the Todd class as

$$\mathrm{Todd} = \prod_a \frac{x_a}{1-\exp(-x_a)} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2+c_2)+\dots\ . \tag{6.15}$$

[37] The additional technicalities for orbifolds drop out on taking the limit to obtain the volume.

[38] We suppress the λ–dependence for simplicity of notation.

Here x_a are the basic characters for the complex tangent bundle of F. We have also expanded the Todd class in terms of Chern classes of F. It will turn out that the Todd class does not contribute to the volume formula in the limit (6.10). The denominator in (6.11), and (6.15), are again understood to be expanded in formal a Taylor expansion.

Before proceeding to the limit (6.10), we note that one can recover (6.7) from (6.11) rather straighforwardly. The fixed point set of the free $U(1)$ action is the zero section, or exceptional divisor, V. The normal bundle is then $\mathcal{L}$, and the weight $u = 1$. Hence (6.11) gives

$$\begin{aligned} C(q, X) = C(q, W) &= \int_V \frac{\mathrm{Todd}(V)}{1 - q e^{-c_1(\mathcal{L})}} \\ &= \int_V \sum_{k \geq 0} q^k e^{-k c_1(\mathcal{L})} \cdot \mathrm{Todd}(V) \ . \end{aligned} \tag{6.16}$$

We now turn to proving (6.10). Set $q = \exp(-tb)$ with t a (small) real positive number. The denominator in (6.11) is then, to leading order in t, given by

$$\prod_{\lambda=1}^{R} \prod_{j=1}^{n_\lambda} [t(b, u_\lambda) + x_j] = t^k \left[\prod_{\lambda=1}^{R} (b, u_\lambda)^{n_\lambda} \right] \prod_{\lambda=1}^{R} \prod_{j=1}^{n_\lambda} [1 + z x_j] \tag{6.17}$$

where

$$z = \frac{1}{t(b, u_\lambda)} \ . \tag{6.18}$$

The higher order terms in x_j will not contribute at leading order in t, once one integrates over the fixed point set, which is why they do not appear in (6.17). Recalling the definition of the Chern polynomial, we thus see that, to leading order in t, (6.11) is given by

$$\begin{aligned} C(e^{-tb}, W) &\sim \sum_{\{F\}} t^{-k} \prod_{\lambda=1}^{R} \frac{1}{(b, u_\lambda)^{n_\lambda}} \int_F \frac{\mathrm{Todd}(F)}{\prod_{\lambda=1}^{R} \det\left(1 + z i \frac{\Omega_\lambda}{2\pi}\right)} \\ &= \sum_{\{F\}} t^{-k} \prod_{\lambda=1}^{R} \frac{1}{(b, u_\lambda)^{n_\lambda}} \int_F \frac{\mathrm{Todd}(F)}{\prod_{\lambda=1}^{R} \sum_{a \geq 0} c_a(\mathcal{E}_\lambda) z^a} \ . \end{aligned} \tag{6.19}$$

Again, it is simple to verify that the Todd class (6.15) contributes only the constant term 1 to leading order in t. Moreover, the integral over F picks out the differential form of degree $2(n-k)$, the coefficient of which is homogeneous degree $-(n-k)$ in t. Thus we have

$$C(e^{-tb}, W) \sim t^{-n} \sum_{\{F\}} \prod_{\lambda=1}^{R} \frac{1}{(b, u_\lambda)^{n_\lambda}} \beta(b) \tag{6.20}$$

where

$$\beta(b) = \int_F \prod_{\lambda=1}^{R} \left[\sum_{a\geq 0} \frac{c_a(\mathcal{E}_\lambda)}{(b, u_\lambda)^a} \right]^{-1} . \tag{6.21}$$

Thus we have shown that the expression

$$\lim_{t\to 0} \; t^n \; C(e^{-tb}, W) \tag{6.22}$$

gives precisely the earlier volume formula (5.25). We have recovered it here by taking a limit of the equivariant index theorem for the $\bar{\partial}$ operator. One can argue that, for a *fixed quasi–regular* Sasakian metric, this limit of the index gives the volume – such an argument was given in [46] and is similar to that at the end of section 5.3. Since the rationals are dense in the reals, this proves (6.10) in general, as a *function* of the Reeb vector field.

We finish this subsection with some comments on the extension of this result to orbifold resolutions of X – that is, resolutions with at worst orbifold singularities. Unfortunately, the Lefschetz formula (6.11) is *not* true for orbifolds. Recall that, for the Duistermaat–Heckman theorem, the only essential difference was the order d of the fixed point set which enters into the formula. For the character $C(q, W)$ the difference is more substantial. The order again appears, but the integral over a connected component of the fixed point set F is replaced by an integral over the associated orbifold $\hat{F}$ to F. Moreover, there are additional terms in the integrand. For a complete account, we refer the reader to the (fairly recent) original paper [61]. We shall not enter into the details of the general equivariant index theorem for orbifolds, since we do not need it. Instead, we simply note that the orbifold version of Duistermaat–Heckman may be recovered from an expression for $C(q, W)$ using the general results of [61] in much the same way as the smooth manifold case treated here.

7 Toric Sasakian manifolds

In this section we turn our attention to toric Sasakian manifolds. In this case the Kähler cone X is an affine toric variety. The equivariant index, which is a character on the space of holomorphic functions, may be computed as a sum over integral points inside the polyhedral cone $\mathcal{C}^*$. The toric setting also allows us to obtain a "hands–on" derivation of the volume function from the index–character. For the purpose of being self–contained, we begin by recalling the well–known correspondence between the combinatorial data of an affine toric variety and the set of holomorphic functions defined on it.

7.1 Affine toric varieties

When X is toric, that is the torus has maximal possible rank $s = n$, it is specified by a convex rational polyhedral cone $\mathcal{C}^* \subset \mathbb{R}^n$. Let $\mathcal{S}_{\mathcal{C}^*} = \mathcal{C}^* \cap \mathbb{Z}^n$. As is well–known, $\mathcal{S}_{\mathcal{C}^*} \subset \mathbb{Z}^n$ is an abelian semi–group, which by Gordan's lemma is finitely generated. This means that there are a finite number of generators $m_1, \ldots, m_N \in \mathcal{S}_{\mathcal{C}^*}$, such that every element of $\mathcal{S}_{\mathcal{C}^*}$ is of the form

$$a_1 m_1 + \cdots + a_N m_N \ , \qquad a_A \in \mathbb{N} \ . \tag{7.1}$$

To $\mathcal{S}_{\mathcal{C}^*}$ there is an associated semi–group algebra, denoted $\mathbb{C}[\mathcal{S}_{\mathcal{C}^*}]$, given by the characters $w^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined as

$$w^m = \prod_{i=1}^{n} w_i^{m^i} \tag{7.2}$$

with multiplication rule

$$w^m \cdot w^{m'} = w^{m+m'} \ . \tag{7.3}$$

Notice that $\mathbb{C}[\mathcal{S}_{\mathcal{C}^*}]$ is generated by the elements $\{w^{m_A} | m_A \text{ generate } \mathcal{S}_{\mathcal{C}^*}\}$. In algebraic geometry, the toric variety $X_{\mathcal{C}^*}$ associated to a strictly convex rational polyhedral cone $\mathcal{C}^*$ is defined as the maximal spectrum[39]

$$X_{\mathcal{C}^*} = \mathrm{Spec}_{\mathrm{m}}\ \mathbb{C}[\mathcal{S}_{\mathcal{C}^*}] \tag{7.4}$$

of the semi–group algebra $\mathbb{C}[\mathcal{C}^* \cap \mathbb{Z}^n]$. In general, one can show that there exist suitable binomial[40] functions $f_A \subset \mathbb{C}^N$, where N is the number of generators of $\mathcal{S}_{\mathcal{C}^*}$, such that

$$\mathbb{C}[\mathcal{S}_{\mathcal{C}^*}] = \mathbb{C}[z_1, \ldots, z_N] / \langle f_1, \ldots, f_S \rangle \ . \tag{7.5}$$

Then, more concretely,

$$X_{\mathcal{C}^*} = \{f_1 = 0, \ldots, f_S = 0\} \subset \mathbb{C}^N \tag{7.6}$$

[39] The maximal spectrum of an algebra A is defined to be the set $\mathrm{Spec}_{\mathrm{m}}\ A = \{\text{maximal ideals in } A\}$ equipped with the Zariski topology. An ideal I in A is said to be maximal if $I \neq A$ and the only proper ideal in A containing I is I itself.

[40] A binomial is a difference of two monomials. Then the affine toric variety is defined by equations of the type 'monomial equals monomial'.

presents $X_{\mathcal{C}^*}$ as an affine variety, with ring of holomorphic functions given precisely by (7.5).

To exemplify this, let us describe briefly the conifold singularity. A set of generators of $\mathcal{S}_{\mathcal{C}^*}$ is given, in this case, by the four outward primitive edge vectors that generate the polyhedral cone $\mathcal{C}^*$, which we will present shortly, see (7.27). Denoting $w = (x, y, z)$, the corresponding generators of $\mathbb{C}[\mathcal{S}_{\mathcal{C}^*}]$ are given by

$$Y = y \ , \quad W = xz^{-1} \ , \quad X = xy^{-1} \ , \quad Z = z \tag{7.7}$$

respectively. It then follows, as is well–known, that the conifold singularity can be represented as the single equation

$$f = XY - ZW = 0 \subset \mathbb{C}^4 \tag{7.8}$$

and the coordinate ring is simply $\mathbb{C}[X, Y, Z, W]/\langle XY - ZW\rangle$. The vanishing of $XY - ZW$ follows from the relation $m_1 + m_3 = m_2 + m_4$ between the generators of $\mathcal{C}^* \cap \mathbb{Z}^3$, and the ideal $\langle XY - ZW\rangle$ is determined by the integer linear relation among these generators. While this is a rather trivial example, it is important to note that in general to construct the monomial ideal one has to include all the generators of $\mathcal{S}_{\mathcal{C}^*}$ (otherwise the resulting variety is not normal) and these are generally many more than the generating edge vectors of $\mathcal{C}^*$. For instance, for the complex cone over the first del Pezzo surface, whose link is the Sasaki–Einstein manifold $Y^{2,1}$, there are 9 generators of $\mathcal{S}_{\mathcal{C}^*}$, so that $N = 9$, while there are 20 relations among them, so that $S = 20$. As a result this is not a complete intersection. Some discussion illustrating these points in the physics literature can be found in [63, 64].

7.2 Relation of the character to the volume

As we have explained, when X is toric, by construction a basis of holomorphic functions on X is given by the w^m above. Thus counting holomorphic functions on X according to their charges under $\mathbb{T}^n$ is equivalent to counting the elements of the semi–group $\mathcal{S}_{\mathcal{C}^*}$.

The character[41] $C(q, X)$ is thus given by

$$C(q, X) = \sum_{m \in \mathcal{S}_{\mathcal{C}^*}} q^m \ . \tag{7.9}$$

We are again tacitly assuming here that the series defining $C(q, X)$ converges.

As we proved in the last section, in general the normalised volume $V(b)$ is related to the character $C(q, X)$ by the simple formula

$$V(b) = \lim_{t \to 0} \ t^n \ C(e^{-tb}, X) \ . \tag{7.10}$$

We again remind the reader that the notation $q = e^{-tb}$ is shorthand for defining the components

$$q_i = e^{-tb_i} \ . \tag{7.11}$$

In this section we shall prove this relation more directly, using the formula (7.9). The limit $t \to 0$ may be understood as a Riemann integral, with the limit giving the volume formula (5.3).

We first discuss a toy example – the generalisation will be straightforward. Consider the following limit

$$\lim_{t \to 0} \frac{t}{1 - e^{-tb}} = \frac{1}{b} \ . \tag{7.12}$$

Now, let us expand the fraction in a Taylor series. The radius of convergence of this series is precisely 1, so that for $b > 0$, we require that t also be positive. We will be particularly interested in isolating the singular behaviour as $t \to 0$. We claim that one can deduce the above limit via

$$\lim_{t \to 0} \sum_{m=0}^{\infty} t e^{-tmb} = \int_0^{\infty} e^{-yb} \ \mathrm{d}y = \frac{1}{b} \ . \tag{7.13}$$

The integral arises simply from the definition of the Riemann integral. In particular, we subdivide the interval $[0, \infty]$ into intervals of length t, and sum the contributions of the function e^{-yb} evaluated at the end–points of each interval $y_m = mt$. The limit $t \to 0$ is then precisely a definition of the Riemann integral.

[41] In fact, the character is also very closely related to the *Ehrhart polynomial* when L admits a regular Sasakian structure, with Fano V. In this case V is also toric and is thus associated to a convex lattice polytope Δ in $\mathbb{R}^{n-1}$. The Ehrhart polynomial $E(\Delta, k)$ is defined to be the number of lattice points inside the dilated polytope $k\Delta$. One can then show that $E(\Delta, k)$ is a polynomial in k of degree $n - 1$. The coefficient of the leading term is precisely the volume of the polytope Δ. This is analogous to the relation we discussed in section 6. For a nice account of this, and the relation to toric geometry, see David Cox's notes [62].

This easily generalises, and we obtain

$$\lim_{t\to 0} t^n\ C(e^{-tb}, X) = \lim_{t\to 0} \sum_{m\in\mathcal{S}_{\mathcal{C}^*}} t^n\ e^{-t(b,m)} = \int_{\mathcal{C}^*} e^{-(b,y)}\ \mathrm{d}y_1 \ldots \mathrm{d}y_n\ . \tag{7.14}$$

The term appearing on the right hand side is called the *characteristic function* of the cone, and was introduced in [44]. Of course, the interpretation of this function as the volume of a toric Sasakian manifold is new. Specialising (5.3) to the toric case, we can relate this to the volume of the Reeb polytope. Thus, from (5.3), we have

$$\int_{\mathcal{C}^*} e^{-(b,y)}\ \mathrm{d}y_1 \ldots \mathrm{d}y_n = n! 2^n \operatorname{vol}(\Delta(b)) = \frac{(n-1)!}{2\pi^n} \operatorname{vol}[L](b)\ . \tag{7.15}$$

Putting (7.15) together with (7.14), we have thus shown that the volume of L follows from a simple limit of the index–character

$$\lim_{t\to 0} t^n\ C(e^{-tb}, X) = \frac{(n-1)!}{2\pi^n} \operatorname{vol}[L](b) \tag{7.16}$$

in a very direct manner. Note that here we did not use any fixed point theorems. One can verify (7.16) directly in some simple cases. Consider, for instance, $X = \mathbb{C}^3$, with the canonical basis $v_i = e_i$ for the toric data. We then have

$$\begin{aligned} \lim_{t\to 0} \frac{t^3}{(1-e^{-tb_1})(1-e^{-tb_2})(1-e^{-tb_3})} &= \int_{\mathcal{C}^*} e^{-(b,y)}\ \mathrm{d}y_1 \ldots \mathrm{d}y_n \\ &= \frac{1}{b_1 b_2 b_3} = 48 \operatorname{vol}(\Delta(b)) \end{aligned} \tag{7.17}$$

where the last equality is computed using the formulae in [22].

Finally, we can also give an independent proof of (7.15) by induction, which uses the particular structure of polyhedral cones. First, we note that (7.15) can be proved by direct calculation for $n = 2$: without loss of generality, we can take the primitive normals to the cone to be (v_1, v_2) and $(0, -1)$ respectively. Then the evaluation of the integral yields

$$\begin{aligned} \int_{\mathcal{C}^*} e^{-(b,y)}\ \mathrm{d}y_1 \mathrm{d}y_2 &= \int_0^\infty \mathrm{d}y_1 \int_0^{-\frac{v_1}{v_2}y_1} \mathrm{d}y_2\, e^{-(b_1 y_1 + b_2 y_2)} \\ &= \frac{v_1}{b_1(v_1 b_2 - v_2 b_1)} = 8 \operatorname{vol}(\Delta(b)) \end{aligned} \tag{7.18}$$

where the last equality follows by calculating the area of the triangular region $\Delta(b)$.

A result in [65] shows that the integral of an exponential of a linear function on a polyhedral cone (more generally on a polytope) can be reduced to integrals over its facets $\mathcal{C}_a^*$, namely

$$b\int_{\mathcal{C}^*} e^{-(b,y)}\ \mathrm{d}y_1\dots\mathrm{d}y_n = \sum_{a=1}^{d}\frac{v_a}{|v_a|}\int_{\mathcal{C}_a^*} e^{-(b,y)}\ \mathrm{d}\sigma \tag{7.19}$$

for any $b\in\mathbb{R}^n$. Now we proceed by induction, where the hypothesis is that (7.15) holds at the $(n-1)$–th step. Using the first component of (7.19), one obtains

$$\int_{\mathcal{C}^*} e^{-(b,y)}\ \mathrm{d}y_1\dots\mathrm{d}y_n = \frac{2^{n-1}(n-1)!}{b_1}\sum_{a=1}^{d}\frac{1}{|v_a|}\mathrm{vol}(\mathcal{F}_a)\ , \tag{7.20}$$

which upon using the relation (2.91) of [22] gives

$$2^n n!\mathrm{vol}(\Delta(b)) \tag{7.21}$$

concluding the proof.

Notice that this expression for the volume allows one to compute derivatives in b straightforwardly:

$$\frac{\partial^k}{\partial b_{i_1}\cdots\partial b_{i_k}}V(b) = (-1)^k\int_{\mathcal{C}^*} y_{i_1}\dots y_{i_k}e^{-(b,y)}\ \mathrm{d}y_1\dots\mathrm{d}y_n \tag{7.22}$$

thus generalising in a natural way the formulae in [22]. In particular, convexity of $V(b)$ is now immediate from this form of the volume.

7.3 Localisation formula

In the case of toric cones X, the general fixed point formula (6.11) for the character has a very simple presentation. Recall that every toric X may be completely resolved by intersecting the cone $\mathcal{C}^*$ with enough hyperplanes in generic position. Specifically, the primitive normal vectors $v_a\in\mathbb{Z}^n$ that define the cone $\mathcal{C}^*$ may, by an $SL(n;\mathbb{Z})$ transformtion, be put in the form $v_a=(1,w_a)$ where each $w_a\in\mathbb{Z}^{n-1}$. The convex hull of $\{w_a\}$ in $\mathbb{R}^{n-1}$ defines a convex lattice polytope. Each interior point in this polytope defines a normal vector to a hyperplane in $\mathbb{R}^n$. If all such hyperplanes are included, in generic position[42], it is well–known that the corresponding toric manifold is in fact completely smooth.

Let $W=X_P$ be the resolved toric Calabi–Yau manifold[43] corresponding to the resulting non–compact polytope P. Thus $P\subset\mathbb{R}^n$ is the image of X_P under the

[42]The positions are the Fayet–Iliopoulos parameters, in the language of gauged linear sigma models.

[43] The construction described above ensures that the resolution is Calabi–Yau. However, more generally there is no need to impose this condition in order to compute the character. For example, one can compute the character for the canonical action of $\mathbb{T}^2$ on $\mathbb{C}^2$ by blowing up origin of the latter to give $\mathcal{O}(-1)\to\mathbb{C}P^1$. This is not a Calabi–Yau manifold. Similarly, the character for the conifold may be computed by resolving to $\mathcal{O}(-1,-1)\to\mathbb{C}P^1\times\mathbb{C}P^1$.

moment map for the $\mathbb{T}^n$ action. The vertices of P are precisely the images of the fixed points under the $\mathbb{T}^n$ action on X_P. Denote these as p_A. Since each p_A corresponds to a smooth point, it follows that there are n primitive edge vectors $u_i^A \in \mathbb{Z}^n \subset \mathtt{t}_n^*$, $i = 1, \ldots, n$, meeting at p_A, which moreover span $\mathbb{Z}^n$ over $\mathbb{Z}$. In particular, this ensures that a small neighbourhood of p_A is equivariantly biholomorphic to $\mathbb{C}^n$. The action of $q \in (\mathbb{C}^*)^n$ on complex coordinates $(z_1, \ldots, z_n)$ in this neighbourhood is given by

$$q : (z_1, \ldots, z_n) \to (q^{u_1^A} z_1, \ldots, q^{u_n^A} z_n) \ . \tag{7.23}$$

We may then define the character:

$$C(q, \mathbb{C}^n; \{u_i^A\}) = \prod_{i=1}^{n} \frac{1}{(1 - q^{u_i^A})} \ . \tag{7.24}$$

The fixed point theorem for the equivariant index of $\bar{\partial}$ on X_P is now very simple to state. It says that

$$C(q, X_P) = \sum_{p_A \in P} C(q, \mathbb{C}^n; \{u_i^A\}) \tag{7.25}$$

where the $\{u_i^A\}$ are the outward–pointing primitive edge vectors at each vertex $p_A \in P$.

As explained earlier, $C(q, X) = C(q, X_P)$. Thus we may in fact choose *any* toric resolution X_P. One can prove invariance directly in dimension $n = 3$ as follows. For dimension $n = 3$, any toric resolution of the cone X may be reached from any other by a sequence of local toric flop transitions. Since each flop is a local modification of the formula (7.25), one needs to only focus on the relevant vertices p near the flop at each step. One can show that the formula for the conifold is itself invariant under the flop transition, as we discuss in the next subsection. This proves rather directly that the fixed point formula is invariant under toric flops, in complex dimension $n = 3$.

It is now simple to take the limit (7.10), giving

$$V(b) = \sum_{p_A \in P} \prod_{i=1}^{n} \frac{1}{(b, u_i^A)} \tag{7.26}$$

where again the u_i^A are the outward–pointing primitive normals at the vertex p_A. Clearly, this is a special case of our general result (5.25). The number of fixed points in the sum (7.26) is given by the Euler number $\chi(X_P)$ of the resolution X_P. One can deduce this simply from the Lefschetz fixed point formula: one applies the equivariant index theorem to the de Rham complex. This expresses the Euler number of the resolved space X_P as a sum of the Euler numbers of the fixed point sets. The Euler number of each fixed point contributes 1 to the total Euler number, and the result follows.

7.4 Examples

In this subsection, we compute explicitly the character $C(q, X)$ in a number of examples, and verify that we correctly reproduce the volume $V(b)$ of the Sasakian metric as a limit. In particular, we consider three smooth resolutions of simple toric Gorenstein singularities. In order to demonstrate that in general we only need to consider orbifold resolutions, we recover the Sasakian volume $V(b)$ for the $Y^{p,q}$ singularities by applying our more general orbifold localisation formula to a partial resolution of the singularity obtained by blowing up a Fano.

7.4.1 The conifold

We take the toric data $w_1 = (0,0)$, $w_2 = (0,1)$, $w_3 = (1,1)$, $w_4 = (1,0)$. The outward

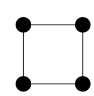

Figure 1: Toric diagram for the conifold.

primitive edge vectors for the polyhedral cone are then easily determined to be

$$(0,1,0), \quad (1,0,-1), \quad (1,-1,0), \quad (0,0,1)\ . \tag{7.27}$$

Each vector has zero dot products with precisely two of the $v_a = (1, w_a)$, and positive dot products with the remaining two. We must now choose a resolution of the cone. There are two choices, related by the flop transition. There are two vertices in each case.

First resolution: We choose the following resolution:

$$\begin{aligned} p_1 &: \ u_1^{(1)} = (0,1,0),\ u_2^{(1)} = (0,0,1),\ u_3^{(1)} = (1,-1,-1) \\ p_2 &: \ u_1^{(2)} = (1,-1,0),\ u_2^{(2)} = (1,0,-1),\ u_3^{(2)} = (-1,1,1)\ . \end{aligned} \tag{7.28}$$

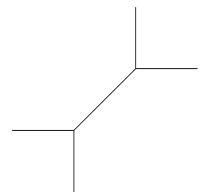

Figure 2: A small resolution of the conifold.

The fixed point formula thus gives

$$
\begin{aligned}
C(q,X) &= \sum_{p_\alpha}\prod_{i=1}^{3}\frac{1}{1-q^{u_i^{(\alpha)}}} \\
&= \frac{1}{(1-q_2)(1-q_3)(1-q_1q_2^{-1}q_3^{-1})}+\frac{1}{(1-q_1q_2^{-1})(1-q_1q_3^{-1})(1-q_1^{-1}q_2q_3)} \\
&= \frac{1-q_1}{(1-q_2)(1-q_3)(1-q_1q_2^{-1})(1-q_1q_3^{-1})}\ . \qquad (7.29)
\end{aligned}
$$

This is the general result for the character[44]. We may now set $q=\exp(-tb)$ and take the limit

$$V(b)=\lim_{t\to 0}\ t^3\ C(e^{-tb},X)=\frac{b_1}{b_2b_3(b_1-b_2)(b_1-b_3)}\ . \qquad (7.30)$$

This indeed correctly reproduces the result of [22] for the volume.

Second resolution: The other small resolution of the conifold has fixed points

$$
\begin{aligned}
p_1 &:\ u_1^{(1)}=(1,-1,0),\ u_2^{(1)}=(0,0,1),\ u_3^{(1)}=(0,1,-1) \\
p_2 &:\ u_1^{(2)}=(0,1,0),\ u_2^{(2)}=(1,0,-1),\ u_3^{(2)}=(0,-1,1)\ . \qquad (7.31)
\end{aligned}
$$

The fixed point formula thus gives

$$
\begin{aligned}
C(q,X) &= \sum_{p_\alpha}\prod_{i=1}^{3}\frac{1}{1-q^{u_i^{(\alpha)}}} \\
&= \frac{1}{(1-q_1q_2^{-1})(1-q_3)(1-q_2q_3^{-1})}+\frac{1}{(1-q_2)(1-q_1q_3^{-1})(1-q_2^{-1}q_3)} \\
&= \frac{1-q_1}{(1-q_2)(1-q_3)(1-q_1q_2^{-1})(1-q_1q_3^{-1})}\ . \qquad (7.32)
\end{aligned}
$$

Of course, as expected, this is the same as (7.29).

[44]Similar computations in related contexts have appeared before in [66, 67, 68].

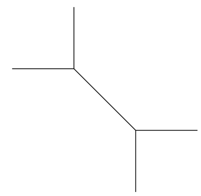

Figure 3: The other small resolution of the conifold.

7.4.2 The first del Pezzo surface

Recall that this singularity is the lowest member of the $Y^{p,q}$ family of toric singularities [23]. Here we take the toric data $w_1 = (-1,-1)$, $w_2 = (-1,0)$, $w_3 = (0,1)$, $w_4 = (1,0)$. The outward primitive edge vectors for the polyhedral cone are then easily determined

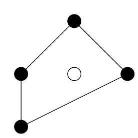

Figure 4: Toric diagram for the complex cone over dP_1.

to be

$$(1,1,0), \quad (1,1,-1), \quad (1,-1,-1), \quad (1,-1,2)\ . \tag{7.33}$$

We resolve the cone by simply blowing up the del Pezzo surface, corresponding to including the interior point $w = (0,0)$. This leads to four vertices, with edges:

$$\begin{aligned} p_1 \ &: \ u_1^{(1)} = (1,1,0),\ u_2^{(1)} = (0,0,-1),\ u_3^{(1)} = (0,-1,1) \\ p_2 \ &: \ u_1^{(2)} = (1,1,-1),\ u_2^{(2)} = (0,0,1),\ u_3^{(2)} = (0,-1,0) \\ p_3 \ &: \ u_1^{(3)} = (1,-1,-1),\ u_2^{(3)} = (0,0,1),\ u_3^{(3)} = (0,1,0) \\ p_4 \ &: \ u_1^{(4)} = (1,-1,2),\ u_2^{(4)} = (0,0,-1),\ u_3^{(4)} = (0,1,-1)\ . \end{aligned} \tag{7.34}$$

The fixed point formula gives, after some algebra:

$$C(q,X) = \frac{N(q)}{(1-q_1q_2)(1-q_1q_2q_3^{-1})(1-q_1q_2^{-1}q_3^{-1})(1-q_1q_2^{-1}q_3^{2})} \tag{7.35}$$

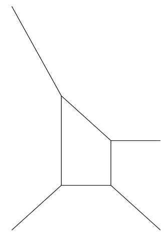

Figure 5: Canonical bundle over dP_1.

where the numerator is given by

$$\begin{aligned} N(q) &= 1 + q_1 + q_1 q_3 + q_1 q_3^{-1} + q_1 q_2^{-1} + q_1 q_2^{-1} q_3 \\ &\quad - q_1^2 (1 + q_1 + q_2 + q_3 + q_3^{-1} + q_2 q_3^{-1}) \ . \end{aligned} \tag{7.36}$$

Either by taking a limit of this expression, or else using (7.26) directly, one obtains

$$V(b) = \lim_{t \to 0} t^3 \ C(e^{-tb}, X) = \frac{2(4b_1 + 2b_2 - b_3)}{(b_1 + b_2)(b_1 - b_2 + 2b_3)(b_1 - b_2 - b_3)(b_1 + b_2 - b_3)} \ . \tag{7.37}$$

This is indeed correct, although we have chosen a different basis from that of [22]. Note also that, setting $b_3 = 0$, we recover the formula (5.42) derived earlier without using toric geometry.

7.4.3 The second del Pezzo surface

We take the toric data $w_1 = (-1, -1)$, $w_2 = (-1, 0)$, $w_3 = (0, 1)$, $w_4 = (1, 0)$, $w_5 = (0, -1)$.

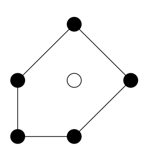

Figure 6: Toric diagram for the complex cone over dP_2.

This is the blow–up of the first del Pezzo surface, introducing an exceptional divisor corresponding to w_5. The outward primitive edge vectors for the polyhedral cone are then easily determined to be

$$(1,1,0), \quad (1,1,-1), \quad (1,-1,-1), \quad (1,-1,1), \quad (1,0,1) \ . \tag{7.38}$$

We resolve the cone by simply blowing up the del Pezzo surface, corresponding to including the interior point $w_5 = (0,0)$. This leads to five vertices, with edges:

$$\begin{aligned}
p_1 &: \ u_1^{(1)} = (1,1,0), \ u_2^{(1)} = (0,0,-1), \ u_3^{(1)} = (0,-1,1) \\
p_2 &: \ u_1^{(2)} = (1,1,-1), \ u_2^{(2)} = (0,0,1), \ u_3^{(2)} = (0,-1,0) \\
p_3 &: \ u_1^{(3)} = (1,-1,-1), \ u_2^{(3)} = (0,0,1), \ u_3^{(3)} = (0,1,0) \\
p_4 &: \ u_1^{(4)} = (1,-1,1), \ u_2^{(4)} = (0,0,-1), \ u_3^{(4)} = (0,1,0) \\
p_5 &: \ u_1^{(5)} = (1,0,1), \ u_2^{(5)} = (0,-1,0), \ u_3^{(5)} = (0,1,-1) \ .
\end{aligned} \tag{7.39}$$

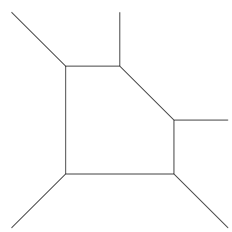

Figure 7: Canonical bundle over dP_2.

Rather than give the full character, we simply state the result for the volume:

$$V(b) = \frac{7b_1^2 + 2b_1b_2 + 2b_1b_3 - b_2^2 - b_3^2 + 2b_2b_3}{(b_1+b_2)(b_1+b_2-b_3)(b_1-b_2-b_3)(b_1-b_2+b_3)(b_1+b_3)} \ . \tag{7.40}$$

Setting $b_1 = 3$, it is straightforward to determine that the critical point, inside the Reeb cone, lies at

$$b_{*2} = b_{*3} = \frac{-57 + 9\sqrt{33}}{16} \tag{7.41}$$

and that the volume at the critical point is

$$V(b_*) = \frac{59 + 11\sqrt{33}}{486} \ . \tag{7.42}$$

7.4.4 An orbifold resolution: $Y^{p,q}$ singularities

Recall that the $Y^{p,q}$ singularities are affine toric Gorenstein singularities generated by four rays, with toric data $w_1 = (0, 0)$, $w_2 = (p - q - 1, p - q)$, $w_3 = (p, p)$, $w_4 = (1, 0)$ [23]. This includes our earlier example of the complex cone over the first del Pezzo surface, although here we use a different basis for convenience. The outward edge

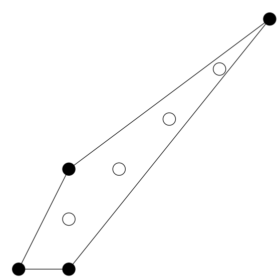

Figure 8: Toric diagram for $Y^{5,3}$.

vectors for the polyhedral cone are then easily determined to be

$$(0, p - q, -p + q + 1), \quad (p, q, -1 - q), \quad (p, -p, p - 1), \quad (0, 0, 1) \ . \tag{7.43}$$

We now *partially resolve* the cone by blowing up the Fano corresponding to the interior point[45] $w = (1, 1)$. This leads to a non–compact polytope $P \subset \mathbb{R}^3$ with four vertices, with outward–pointing edge vectors:

$$\begin{aligned}
p_1 \ &: \ u_1^{(1)} = (0, p - q, -p + q + 1), \ u_2^{(1)} = (0, -1, 1), \ u_3^{(1)} = (1, -p + q + 1, p - q - 2) \\
p_2 \ &: \ (p - 1)u_1^{(2)} = (p, q, -1 - q), \ (p - 1)u_2^{(2)} = (-1, p - q - 1, -p + q + 2), \\
&\quad\ (p - 1)u_3^{(2)} = (0, -p + 1, p - 1) \\
p_3 \ &: \ (p - 1)u_1^{(3)} = (p, -p, p - 1), \ (p - 1)u_2^{(3)} = (0, p - 1, -p + 1), \\
&\quad\ (p - 1)u_3^{(3)} = (-1, 1, 0) \\
p_4 \ &: \ u_1^{(4)} = (0, 0, 1), \ u_2^{(4)} = (1, -1, 0), \ u_3^{(4)} = (0, 1, -1) \ .
\end{aligned} \tag{7.44}$$

The normalisations here ensure that we correctly get the corresponding weights for the torus action that enter the orbifold localisation formula. Note that they are generally *rational* vectors, for vertices p_2 and p_3. Indeed, it is straightforward to show that the

[45] Note that *any* interior point would do.

three primitive outward–pointing edge vectors at these vertices span $\mathbb{Z}^3$ over $\mathbb{Q}$, but not over $\mathbb{Z}$. In both cases, $\mathbb{Z}^3$ modulo this span is isomorphic to $\mathbb{Z}_{p-1}$. This immediately implies that these vertices are $\mathbb{Z}_{p-1}$ orbifold singularities, and thus the orders of these fixed points are $d_{p_2} = d_{p_3} = p-1$. On the other hand, the vertices p_1 and p_4 are smooth, and thus $d_{p_1} = d_{p_4} = 1$. We must now use the localisation formula for orbifolds, which

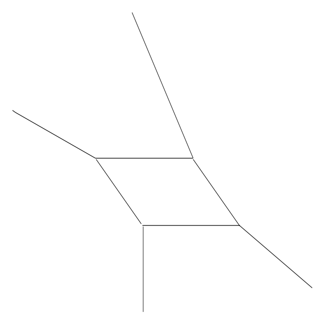

Figure 9: Partially resolved polytope P for $Y^{5,3}$.

includes the inverse orders $1/d_p$ at each vertex p as a multiplicative factor. This is straightforward to compute:

$$\begin{aligned} V(b) &= \sum_{p_A, A=1,\ldots,4} \frac{1}{d_{p_A}} \prod_{i=1}^{3} \frac{1}{(b, u_i^{(A)})} \\ &= \frac{p\left[p(p-q)b_1 + q(p-q)b_2 + q(2-p+q)b_3\right]}{b_3(pb_1 - pb_2 + (p-1)b_3)((p-q)b_2 + (1-p+q)b_3)(pb_1 + qb_2 - (1+q)b_3)} \end{aligned}$$

which is indeed the correct expression [22].

Acknowledgments

We would like to thank S. Benvenuti, D. Goldfeld, P. A. Grassi, A. Hanany, J. F. Morales and R. Thomas for interesting discussions. D. M. and J. F. S would also like to thank the Institute for Mathematical Sciences, Imperial College London, for hospitality during the final stages of this work. J. F. S. is supported by NSF grants DMS–0244464, DMS–0074329 and DMS–9803347. S.–T. Y. is supported in part by NSF grants DMS–0306600 and DMS–0074329.

A The Reeb vector field is holomorphic and Killing

In this appendix we give a proof that $\xi = J(r\partial/\partial r)$ is both Killing and holomorphic. This fact is well–known in the literature, although it seems the derivation is not (however, see [69]). Thus, for completeness, we give one here.

We begin with the following simple formulae for covariant derivatives on X:

$$\nabla_{r\partial/\partial r}\left(r\frac{\partial}{\partial r}\right) = r\frac{\partial}{\partial r}, \quad \nabla_{r\partial/\partial r}Y = \nabla_Y\left(r\frac{\partial}{\partial r}\right) = Y$$
$$\nabla_Y Z = \nabla^L_Y Z - g_X(Y,Z)r\frac{\partial}{\partial r}\ , \tag{A.1}$$

which may easily be checked by computing the Christoffel symbols of the metric $g_X = \mathrm{d}r^2 + r^2 g_L$. Here ∇ denotes the Levi–Civita connection on (X, g_X), ∇^L is that on (L, g_L), and Y, Z are vector fields on L, viewed as vector fields on $X_0 = \mathbb{R}_+ \times L$. A straightforward calculation, using $\nabla J = 0$, then shows that ξ is in fact a Killing vector field on X:

$$g_X(\nabla_Y \xi, Z) = g_X(\nabla_Y \left[J(r\tfrac{\partial}{\partial r})\right], Z) = g_X(J(\nabla_Y r\tfrac{\partial}{\partial r}), Z) = g_X(JY, Z) \tag{A.2}$$

where Z and Y are any two vector fields on L. The last term is skew in Z and Y. Similarly,

$$g_X(\nabla_{r\partial/\partial r}\xi, Y) = g_X(\xi, Y)\ , \tag{A.3}$$

and

$$g_X(\nabla_Y \xi, r\partial/\partial r) = -g_X(JY, r\partial/\partial r) = -g_X(Y, \xi)\ , \tag{A.4}$$

so that (A.3) is also skew; while the diagonal element

$$g_X(\nabla_{r\partial/\partial r}\xi, r\partial/\partial r) = 0 \tag{A.5}$$

clearly vanishes. Thus we conclude that $g_X(\nabla_U \xi, V)$ is skew in U, V for any vector fields U and V on X, and hence ξ is Killing. One can similarly check that ξ pushes forward to a unit Killing vector on L, where we identify $L = X\mid_{r=1}$.

In fact $r\partial/\partial r$ and ξ are both holomorphic vector fields. Indeed, for any vector fields U and V, we have the general formula

$$(\mathcal{L}_U J)V = (\nabla_U J)V + J\nabla_V U - \nabla_{JV} U \tag{A.6}$$

relating the Lie derivative to the covariant derivative. Using this and the fact that J is covariantly constant, $\nabla_U J = 0$, one now easily sees that

$$\mathcal{L}_{r\partial/\partial r} J = 0\ , \qquad \mathcal{L}_\xi J = 0\ . \tag{A.7}$$

B More on the holomorphic $(n, 0)$–form

In the main text we used the fact that $\mathcal{L}_Y\Omega = 0$ is equivalent to $\mathcal{L}_Y\Psi = 0$, where Y is holomorphic, Killing, and commutes with ξ. Although intuitively clear, one has to do a little work to prove this. We include the details here for completeness.

Suppose that $\mathcal{L}_Y\Psi = 0$, where Ψ is the canonically defined spinor on the Kähler cone X and Y is a holomorphic Killing vector field that commutes with ξ. This is true of all $Y \in \mathtt{t}_s$ in the main text. The restriction of Ψ to L is the spinor θ. Writing $\Omega = \exp(f/2)K$, we of course have $\mathcal{L}_Y K = 0$. Thus we must prove that $\mathcal{L}_Y f = 0$. Since $i\partial\bar{\partial}f = \rho = J\,\mathrm{Ric}(g_X)$, and Y is both holomorphic and Killing, we immediately have

$$i\partial\bar{\partial}\mathcal{L}_Y f = 0\ . \tag{B.1}$$

Recall that $f = \log\|\Omega\|^2_g$ is degree zero under $r\partial/\partial r$ and basic with respect to ξ. Thus the same is true of $\mathcal{L}_Y f$. We may then interpret (B.1) as a transverse equation, or, when L is quasi–regular, as an equation on the Fano V. In the latter case it follows immediately that $\mathcal{L}_Y f = c$, a constant. In the irregular case, the paper [70] also claims that a transverse $i\partial\bar{\partial}$ lemma holds in general.

We now compute

$$\begin{aligned}\frac{i^n}{2^n}(-1)^{n(n-1)/2}c\Omega\wedge\bar{\Omega} &= \frac{i^n}{2^n}(-1)^{n(n-1)/2}\mathcal{L}_Y(\Omega\wedge\bar{\Omega}) = \mathrm{d}\left[e^f Y\lrcorner\frac{\omega^n}{n!}\right]\\ &= -\mathrm{d}\left(e^f\right)\wedge\mathrm{d}y_Y\wedge\frac{\omega^{n-1}}{(n-1)!}\ .\end{aligned} \tag{B.2}$$

Here we have used the fact that Ω is closed, and recall that $\mathrm{d}y_Y = -Y\lrcorner\omega$ where y_Y is the Hamiltonian function for Y. The right hand side of (B.2) is clearly exact. Hence we may integrate this equation over $r \leq 1$ and use Stokes' Theorem to deduce that

$$\frac{i^n}{2^n}(-1)^{n(n-1)/2}c\int_{r\leq 1}\Omega\wedge\bar{\Omega} = -\int_L e^f\mathrm{d}y_Y\wedge\frac{\omega_T^{n-1}}{(n-1)!}\ . \tag{B.3}$$

Now, every term in the integrand on the right hand side of this equation is basic with respect to ξ. In particular, ξ contracted into the integrand is zero[46]. However, this means that the integral is itself zero. Since the integral of $\Omega\wedge\bar{\Omega}$ is certainly non–zero, we conclude that $c = 0$ and hence that $\mathcal{L}_Y f = 0$, as desired.

[46] To see that $\mathrm{d}y_Y$ is basic, simply notice that $\mathrm{d}y_Y(\xi) = \mathcal{L}_\xi y_Y = \frac{1}{2}\mathcal{L}_\xi(r^2\eta(Y)) = 0$, the last equality following from the fact that ξ preseves η, Y and r, as discussed at various points in the text.

Conversely, suppose that $\mathcal{L}_Y\Omega = 0$. From equation (2.16) we immediately deduce now that $\mathcal{L}_Y f = 0$ since Y is holomorphic and Killing by assumption. Thus $\mathcal{L}_Y K = 0$. Hence

$$0 = \mathcal{L}_Y K = (\mathcal{L}_Y \bar{\Psi}^c)\gamma_{(n)}\Psi + \bar{\Psi}^c\gamma_{(n)}\mathcal{L}_Y\Psi = 2\bar{\Psi}^c\gamma_{(n)}\mathcal{L}_Y\Psi \ . \tag{B.4}$$

Consider now $\mathcal{L}_Y\Psi$. In fact this must be proportional to Ψ. An easy way to see this is to go back to the isomorphism (2.44). The splitting of $\Lambda^{0,*}(X)$ into differential forms of different degrees is realised on the space of spinors $\mathcal{V}$ via the Clifford action of the Kähler form ω. The latter splits the bundle $\mathcal{V}$ into eigenspaces

$$\mathcal{V} = \bigoplus_{a=0}^{n} \mathcal{V}_a \tag{B.5}$$

where $\mathcal{V}_a$ is an eigenspace of $\omega\cdot$ with eigenvalue $i(n-2a)$. Moreover, $\dim \mathcal{V}_a = \binom{n}{a}$ and

$$\mathcal{V}_a \cong \Lambda^{0,a}(X) \ . \tag{B.6}$$

Recall now that Ψ corresponds to a section of $\Lambda^{0,0}(X)$ under the isomorphism (2.44); hence Ψ has eigenvalue in under the Clifford action of ω. Indeed, one can check this eigenvalue rather straighforwardly, without appealing to the isomorphism (2.44).

We may now consider $\mathcal{L}_Y\Psi$. Since Y is holomorphic and Killing, it preserves ω. Thus the Clifford action commutes past the Lie derivative, and we learn that $\mathcal{L}_Y\Psi$ has the same eigenvalue as Ψ. But since this eigenbundle is one–dimensional, they must in fact be proportional: $\mathcal{L}_Y\Psi = F\Psi$ for some function F. Thus (B.4) says that

$$0 = 2F\bar{\Psi}^c\gamma_{(n)}\Psi = 2FK \ . \tag{B.7}$$

Since K is certainly non–zero, we conclude that $F = 0$, and we are done.

C Variation formulae

In this appendix we derive the first and second variation formulae (3.32), (3.37).

C.1 First variation

Recall that we linearise the equations for deforming the Reeb vector field around a given background Kähler cone with Kähler potential r^2. We set

$$\begin{aligned} \xi(t) &= \xi + tY && \text{(C.1)} \\ r^2(t) &= r^2(1+t\phi) \ . && \text{(C.2)} \end{aligned}$$

We work to first order in t. Note that contracting (C.1) with J we have

$$r(t)\frac{\partial}{\partial r(t)} = r\frac{\partial}{\partial r} - tJ(Y) \ . \tag{C.3}$$

Expanding

$$\mathcal{L}_{r(t)\partial/\partial r(t)} r^2(t) = 2r^2(t) \tag{C.4}$$

to first order in t gives

$$\mathcal{L}_{r\partial/\partial r}\phi = 2\mathcal{L}_{J(Y)} \log r = -2\eta(Y) \ . \tag{C.5}$$

Recall we also require

$$\mathcal{L}_{\xi(t)} r^2(t) = 0 \tag{C.6}$$

which gives, again to first order,

$$\mathcal{L}_{\xi}\phi = -2\mathrm{d}\log r(Y) = 0 \ . \tag{C.7}$$

In particular, note that when $Y = 0$ we recover that ϕ should be homogeneous degree zero and basic, and thus gives a transverse Kähler deformation. Note also that the right hand side of (C.7) is zero if and only if Y is a holomorphic Killing vector field of the background.

We may use these equations to compute the derivative of the volume vol$[L]$, which we think of as vol$[\xi]$, in the direction Y. We write this as dvol$[L](Y)$. Arguing much as in section 3.1, we obtain

$$\mathrm{dvol}[L](Y) = -n\int_L \phi \mathrm{d}\mu + \frac{n}{2}\int_{r\leq 1} \mathrm{d}\mathrm{d}^c(r^2\phi) \wedge \frac{\omega^{n-1}}{(n-1)!} \ . \tag{C.8}$$

The first term arises from the variation of the domain, as in equations (3.21), (3.22). However, one must be careful to note that here $\phi = \phi(r, x)$ is a function of both r and the point $x \in L$, where L is the unperturbed link $L = X \mid_{r=1}$. We must integrate up to the hypersurface $r(t) = 1$, which one can check is, to first order in t, given by $r = 1 - (1/2)t\phi(r = 1, x)$. Thus one should replace ϕ by $\phi(r = 1)$ in (3.21).

Using Stokes' theorem on the second term on the right hand side of (C.8), the first term is cancelled, as before, leaving

$$\mathrm{dvol}[L](Y) = \frac{n}{2}\int_L \mathrm{d}^c\phi \wedge \frac{\omega_T^{n-1}}{(n-1)!} \ . \tag{C.9}$$

Now

$$\xi \lrcorner \mathrm{d}^c\phi = \mathcal{L}_{r\partial/\partial r}\phi = -2\eta(Y) \tag{C.10}$$

where in the last equality we have used the linearised equation. We thus have

$$\mathrm{dvol}[L](Y) = -n \int_L \eta(Y) \mathrm{d}\mu \tag{C.11}$$

which is formula (3.32) in the main text.

C.2 Second variation

We now take the second variation of the volume. We write

$$\mathrm{dvol}[L](Y) = -n(n+1) \int_{r\leq 1} (\mathrm{d}^c r^2)(Y) \frac{\omega^n}{n!} \tag{C.12}$$

as described in the main text. We now again deform

$$\begin{aligned} r^2(t) &= r^2(1+t\psi) \\ r(t)\frac{\partial}{\partial r(t)} &= r\frac{\partial}{\partial r} - tJ(Z) \end{aligned} \tag{C.13}$$

giving linearised equations

$$\begin{aligned} \mathcal{L}_{r\partial/\partial r}\psi &= -2\eta(Z) \\ \mathcal{L}_{\xi}\psi &= -2\mathrm{d}\log r(Z) = 0 \ . \end{aligned} \tag{C.14}$$

In fact the second equation again will not be used. The derivative of (C.12) gives

$$\begin{aligned} \frac{1}{n(n+1)}\mathrm{d}^2\mathrm{vol}[L](Y,Z) &= \int_L \psi\eta(Y)\mathrm{d}\mu - \int_{r\leq 1} (\mathrm{d}^c(r^2\psi))(Y)\frac{\omega^n}{n!} \\ &- \int_{r\leq 1} 2r^2\eta(Y)\frac{1}{4}\mathrm{d}\mathrm{d}^c(r^2\psi)\wedge\frac{\omega^{n-1}}{(n-1)!} \ . \end{aligned} \tag{C.15}$$

The three terms occur from the variation in domain, the variation of $\mathrm{d}^c(r^2)$, and the variation of the measure, respectively. The last term in (C.15) may be integrated by parts, with respect to d, giving the two terms

$$-\frac{1}{2}\int_L \eta(Y)\mathrm{d}^c(r^2\psi)\wedge\frac{\omega_T^{n-1}}{(n-1)!} - \int_{r\leq 1}(Y \lrcorner \omega)\wedge \mathrm{d}^c(r^2\psi)\wedge\frac{\omega^{n-1}}{(n-1)!} \ , \tag{C.16}$$

where we have used $2Y \lrcorner \omega = -\mathrm{d}(r^2\eta(Y))$. Expanding the first term in (C.16), with respect to d^c, gives

$$-\int_L \psi\eta(Y)\mathrm{d}\mu + \int_L \eta(Y)\eta(Z)\mathrm{d}\mu \ . \tag{C.17}$$

To produce the form of the second term in (C.17), we have used the trick (C.10) of writing the linearised equation (C.14) as $(\mathrm{d}^c\psi)(\xi) = -2\eta(Z)$. Now, the first term in (C.17) precisely cancels the first term in (C.15). Hence we are left with

$$\frac{1}{n(n+1)}\mathrm{d}^2\mathrm{vol}[L](Y,Z) = \int_L \eta(Y)\eta(Z)\mathrm{d}\mu - \int_{r\leq 1} (\mathrm{d}^c(r^2\psi))(Y)\frac{\omega^n}{n!} + Y\lrcorner\omega \wedge \mathrm{d}^c(r^2\psi) \wedge \frac{\omega^{n-1}}{(n-1)!} \ . \qquad (C.18)$$

Finally, we note the identity

$$\begin{aligned} 0 &= Y\lrcorner\left[\mathrm{d}^c(r^2\psi)\wedge\frac{\omega^n}{n!}\right] \\ &= (\mathrm{d}^c(r^2\psi))(Y)\frac{\omega^n}{n!} - \mathrm{d}^c(r^2\psi)\wedge Y\lrcorner\omega\wedge\frac{\omega^{n-1}}{(n-1)!} \ . \end{aligned} \qquad (C.19)$$

Thus we have shown that

$$\mathrm{d}^2\mathrm{vol}[L](Y,Z) = n(n+1)\int_L \eta(Y)\eta(Z)\mathrm{d}\mu \qquad (C.20)$$

which is equation (3.37) in the main text.

References

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)], [arXiv:hep-th/9711200].

[2] A. Kehagias, "New type IIB vacua and their F-theory interpretation," Phys. Lett. B **435**, 337 (1998), [arXiv:hep-th/9805131].

[3] I. R. Klebanov and E. Witten, "Superconformal field theory on threebranes at a Calabi-Yau singularity," Nucl. Phys. B **536**, 199 (1998), [arXiv:hep-th/9807080].

[4] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull and B. Spence, "Branes at conical singularities and holography," Adv. Theor. Math. Phys. **2** (1999) 1249, [arXiv:hep-th/9808014].

[5] D. R. Morrison and M. R. Plesser, "Non-spherical horizons. I," Adv. Theor. Math. Phys. **3**, 1 (1999), [arXiv:hep-th/9810201].

[6] G. Tian, "On Kähler–Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$," Invent. Math. **89** (1987) 225-246.

[7] G. Tian and S.T. Yau, "On Kähler–Einstein metrics on complex surfaces with $C_1 > 0$," Commun. Math. Phys. **112** (1987) 175-203.

[8] C.P. Boyer and K. Galicki, "Sasakian Geometry, Hypersurface Singularities, and Einstein Metrics," Supplemento ai Rendiconti del Circolo Matematico di Palermo Serie II. Suppl 75 (2005), 57-87, [arXiv: math.DG/0405256].

[9] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Supersymmetric AdS(5) solutions of M-theory," Class. Quant. Grav. **21**, 4335 (2004) [arXiv:hep-th/0402153].

[10] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Sasaki-Einstein metrics on $S^2 \times S^3$," Adv. Theor. Math. Phys. **8**, 711 (2004) [arXiv:hep-th/0403002].

[11] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, "A new infinite class of Sasaki-Einstein manifolds," Adv. Theor. Math. Phys. **8**, 987 (2006) [arXiv:hep-th/0403038].

[12] M. Cvetic, H. Lu, D. N. Page and C. N. Pope, "New Einstein-Sasaki spaces in five and higher dimensions," Phys. Rev. Lett. **95**, 071101 (2005) [arXiv:hep-th/0504225].

[13] D. Martelli and J. Sparks, "Toric Sasaki-Einstein metrics on $S^2 \times S^3$," Phys. Lett. B **621**, 208 (2005) [arXiv:hep-th/0505027].

[14] M. Cvetic, H. Lu, D. N. Page and C. N. Pope, "New Einstein-Sasaki and Einstein spaces from Kerr-de Sitter," [arXiv:hep-th/0505223].

[15] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Supersymmetric AdS Backgrounds in String and M-theory," Proceedings of the 73rd Meeting between Physicists and Mathematicians "(A)dS/CFT correspondence", Strasbourg, September 11-13, 2003, [arXiv:hep-th/0411194].

[16] W. Chen, H. Lu, C. N. Pope and J. F. Vazquez-Poritz, "A note on Einstein-Sasaki metrics in $D \geq 7$," Class. Quant. Grav. **22**, 3421 (2005) [arXiv:hep-th/0411218].

[17] H. Lu, C. N. Pope and J. F. Vazquez-Poritz, "A new construction of Einstein-Sasaki metrics in $D \geq 7$," [arXiv:hep-th/0512306].

[18] J. Cheeger, G. Tian, "On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay," Invent. Math. 118 (1994), no. 3, 493-571.

[19] K. Intriligator and B. Wecht, "The exact superconformal R-symmetry maximizes a," Nucl. Phys. B **667**, 183 (2003), [arXiv:hep-th/0304128].

[20] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," JHEP **9807**, 023 (1998) [arXiv:hep-th/9806087].

[21] S. S. Gubser, "Einstein manifolds and conformal field theories," Phys. Rev. D **59**, 025006 (1999) [arXiv:hep-th/9807164].

[22] D. Martelli, J. Sparks, S.–T. Yau, "The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds," to appear in Commum. Math. Phys. [arXiv:hep-th/0503183].

[23] D. Martelli and J. Sparks, "Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals," Commun. Math. Phys. **262**, 51 (2006) [arXiv:hep-th/0411238].

[24] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, "An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals," JHEP **0506**, 064 (2005) [arXiv:hep-th/0411264].

[25] M. Bertolini, F. Bigazzi and A. L. Cotrone, "New checks and subtleties for AdS/CFT and a-maximization," JHEP **0412**, 024 (2004), [arXiv:hep-th/0411249].

[26] B. Feng, A. Hanany and Y. H. He, "D-brane gauge theories from toric singularities and toric duality," Nucl. Phys. B **595**, 165 (2001), [arXiv:hep-th/0003085].

[27] S. Franco, A. Hanany, K. D. Kennaway, D. Vegh and B. Wecht, "Brane dimers and quiver gauge theories," JHEP **0601**, 096 (2006) [arXiv:hep-th/0504110].

[28] S. Franco, A. Hanany, D. Martelli, J. Sparks, D. Vegh and B. Wecht, "Gauge theories from toric geometry and brane tilings," JHEP **0601**, 128 (2006) [arXiv:hep-th/0505211].

[29] A. Butti and A. Zaffaroni, "R-charges from toric diagrams and the equivalence of a-maximization and Z-minimization," JHEP **0511** (2005) 019 [arXiv:hep-th/0506232].

[30] A. Hanany and D. Vegh, "Quivers, tilings, branes and rhombi," [arXiv:hep-th/0511063].

[31] S. Franco and D. Vegh, "Moduli spaces of gauge theories from dimer models: Proof of the correspondence," [arXiv:hep-th/0601063].

[32] A. Butti, D. Forcella and A. Zaffaroni, "The dual superconformal theory for $L^{p,q,r}$ manifolds," JHEP **0509**, 018 (2005) [arXiv:hep-th/0505220].

[33] S. Benvenuti and M. Kruczenski, "From Sasaki-Einstein spaces to quivers via BPS geodesics: $L(p,q|r)$," [arXiv:hep-th/0505206].

[34] A. Futaki, H. Ono, G. Wang, "Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds," [arXiv:math.DG/0607586].

[35] Y. Tachikawa, "Five-dimensional supergravity dual of a-maximization," Nucl. Phys. B **733**, 188 (2006) [arXiv:hep-th/0507057].

[36] E. Barnes, E. Gorbatov, K. Intriligator and J. Wright, "Current correlators and AdS/CFT geometry," Nucl. Phys. B **732**, 89 (2006) [arXiv:hep-th/0507146].

[37] S. Lee and S. J. Rey, "Comments on anomalies and charges of toric-quiver duals," JHEP **0603**, 068 (2006) [arXiv:hep-th/0601223].

[38] C. P. Boyer, K. Galicki, P. Matzeu, "On Eta-Einstein Sasakian Geometry," [arXiv:math.DG/0406627].

[39] E. Barnes, E. Gorbatov, K. Intriligator, M. Sudano and J. Wright, "The exact superconformal R-symmetry minimizes τ_{RR}," Nucl. Phys. B **730**, 210 (2005) [arXiv:hep-th/0507137].

[40] J. J. Duistermaat, G. Heckman, "On the variation in the cohomology of the symplectic form of the reduced space," Inv. Math. **69**, 259-268 (1982).

[41] J. J. Duistermaat, G. Heckman, Addendum, Inv. Math. **72**, 153-158 (1983).

[42] C. P. Herzog and R. L. Karp, "Exceptional collections and D-branes probing toric singularities," JHEP **0602**, 061 (2006) [arXiv:hep-th/0507175].

[43] A. Hanany, C. P. Herzog and D. Vegh, "Brane tilings and exceptional collections," JHEP **0607**, 001 (2006) [arXiv:hep-th/0602041].

[44] E. B. Vinberg, "The theory of convex homogeneous cones," Trans. Moscow Math. Soc. **12** (1963), 303-358.

[45] T. Oda, "Convex bodies and algebraic geometry," Springer–Verlag, 1988.

[46] A. Bergman, C. P. Herzog, " The Volume of some Non-spherical Horizons and the AdS/CFT Correspondence," JHEP 0201 (2002) 030, [arXiv:hep-th/0108020].

[47] C. Romelsberger, "An index to count chiral primaries in $N = 1$ $d = 4$ superconformal field theories," [arXiv:hep-th/0510060].

[48] J. Kinney, J. Maldacena, S. Minwalla and S. Raju, "An index for 4 dimensional super conformal theories," [arXiv:hep-th/0510251].

[49] A. Futaki, "An obstruction to the existence of Einstein Kähler metrics," Invent. Math., **73** (1983), 437-443.

[50] S. Falcão B. de Moraes, C. Tomei, "Moment maps on symplectic cones," Pacific J. Math. **181** (2) (1997), 357-375.

[51] M. Obata, "Certain conditions for a Riemannian manifold to be isometric with a sphere," J. Math. Soc. Japan **14** (1962), 333-340.

[52] C. P. Boyer and K. Galicki, "A Note on Toric Contact Geometry," J. Geom. Phys. **35** No. 4 (2000) 288-298, [arXiv:math.DG/9907043].

[53] A. L. Besse, "Einstein Manifolds," Spinger–Verlag, 1987.

[54] H. B. Lawson, M.–L. Michelsohn, "Spin Geometry," Princeton University Press, 1989.

[55] C. Bär, "Real Killing spinors and Holomony," Commun. Math. Phys. **154**, 509–521 (1993).

[56] Y. Matsushima, "Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kaehlérienne," Nagoya Math. J. **11** (1957), 145-150.

[57] E. Calabi, "Extremal Kähler Metrics II," in Differential Geometry and Complex Analysis (ed. I. Chavel and H. M. Farkas), Springer–Verlag, 1985.

[58] T. Mabuchi, “An Algebraic Character associated with the Poisson Brackets,” Advanced Studies in Pure Mathematics 18-I, 1990, Recent Topics in Differential and Analytic Geometry, 339-358.

[59] M. F. Atiyah, I. M. Singer, “The index of elliptic operators III,” Ann. Math. **87**, 546-604 (1968).

[60] R. Bott, L. Tu, “Differential forms in algebraic topology,” Springer–Verlag, 1982.

[61] M. Vergne, “The equivariant index formula on orbifolds,” Duke Math. J. **82** (1996) 637-652.

[62] D. Cox, “Minicourse on Toric Varieties,”

obtainable from `http://www.amherst.edu/~dacox/`

[63] D. Berenstein, C. P. Herzog, P. Ouyang and S. Pinansky, “Supersymmetry breaking from a Calabi-Yau singularity,” JHEP **0509**, 084 (2005) [arXiv:hep-th/0505029].

[64] S. Pinansky, “Quantum deformations from toric geometry,” JHEP **0603**, 055 (2006) [arXiv:hep-th/0511027].

[65] J. B. Lasserre, “Integration and homogeneous functions,” Proceedings of the American Mathematical Society **127**, 813 (1999).

[66] N. Nekrasov and S. Shadchin, “ABCD of instantons,” Commun. Math. Phys. **252**, 359 (2004) [arXiv:hep-th/0404225].

[67] P. A. Grassi and J. F. Morales Morera, “Partition functions of pure spinors,” Nucl. Phys. B **751**, 53 (2006) [arXiv:hep-th/0510215].

[68] P. A. Grassi and G. Policastro, “Curved beta-gamma systems and quantum Koszul resolution,” [arXiv:hep-th/0602153].

[69] C. P. Boyer and K. Galicki, “3-Sasakian Manifolds,” Surveys Diff. Geom. **7**, 123 (1999) [arXiv:hep-th/9810250].

[70] A. El Kacimi–Alaoui, “Opérateurs transversalement elliptiques sur un feuilletage Riemannien et applications,” Compositio Mathematica **79** (1990) 57–106.