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Moving as a circle: Folds and nuances of a mathematical concept

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When we look at a drop falling into a pond, we see many circles originating from the perturbation of the water surface. The circles move as they appear to be expanding radially from the position in which the drop initially hit the surface, in all directions. When a group of children creates a circle, holding their hands and extending their arms so that they enlarge the void space in-between all of them, we observe a movement that brings them far from an imaginary centre.

Both examples bring forth events in which the concept of circle is enmeshed with the perception and experience of movement. But what do we mean when we say 'circle'? Even if we limit ourselves to examples that relate more directly to mathematical definitions than the ones given above, we might think of this concept in different ways. For example, as the locus of points equidistant from a given centre, but also as the material trace of an object that is subjected to two opposing forces. De Freitas and Sinclair (2014) pre- sent this example and suggest that the first vision of the circle realises the possible, while the second actualises the virtual. If thought of in the first way, the circle emerges from the logical constraints given by the proposition (the given rule). The second way sheds light on the generative, mobile activity of forces that might produce the circle. It does not adhere entirely to logical determinations (the idea of a pre-fixed circular shape) and the circle is evoked as a dynamic concept (a point/object moving according to physical forces).

Each vision, in the case described above, brings forth a distinct trait of the circle as a concept; in particular, the second one engenders the circle as a mobile entity, which does not exist until we put something in motion. The philosopher of mathematics Gilles Chatelet (1993) argues that both these visions coexist in mathematics by introducing the idea of physico-mathematical concepts. He considers the physical in the mathematical, rather than seeing the mathematical and the physical as separated. In so doing, Chatelet troubles the ontology of the relationship between mathematics and the physical world, as well as the classical vision of what it means to do mathematics. What is peculiar about this relationship, and sustains the tension between the two visions, is how the concept partakes in the virtual dimension of the material world.

The concept of the virtual does not emphasise the world as we know it but rather its potential to transform itself beyond its actual forms and configurations. Although it is very difficult to define, one way to think of the virtual is to conceive it as the infinite realm that pertains a (mathematical) concept and the forms of engagement with mathematics. It is at play when we reconceive concepts less as static, abstract entities and more in terms of the concepts' power of affecting and being affected, their animating force, their potentiality and mobility, their capacity of giving rise to new configurations. So, for example, the circle can be thought of in terms of the virtual motions that it generates instead of simply being thought of as a static geometrical object. In 'L'enchantement du virtuel', Chatelet (2010) takes the example of the circle when he discusses how points might be considered not as given or simply 'lying' on a plane, but as being possessing the power to algebraically describe sinusoidal functions, when they are dynamically put in motion along a circle. The concept of the virtual is profoundly grounded in an openness to movement rooted in the human body, as described by Chatelet through the interplay between gestures and diagrams. He provides us with the powerful image that the horizon of a diagram is populated by gestures and future alterations that spring from it.

These future alterations are actualised both bodily and in thought experiments. Perception and movement are at play in such processes, and consistently inform the way in which the concept emerge from the mathematical activity. In this article, I pursue a discussion about the concept of circle that is grounded in the virtual mobility of the mathematical concept and inextricably bounded to the bodily movements that actualise it. Starting from a first person experience with a mathematical instrument, which is used to draw a circle, I want to contribute to recent research on the role of the body in mathematical activity, elucidating a path to account for how movement and kinaesthesia play out in the constitutionof mathematical concepts.

Coming full circle

Chorney (2017) discusses the circle proposing that "as withany mathematical object, [a circle] ought not be seen as a reproduction of an ideal form, but rather as a meshwork of materials and forces" (p. 45). He builds upon the tension thatexists in the mathematics classroom between considering the circle either the movement or the trace, or both. Engaging with the concept in a dynamic way entails encounteringthe process of creation of a circle, rather than just focusing on the circle as a (pre-)fixed shape. Mathematics educators have suggested a number of ways in which this could be done in the classroom, with common instruments like penciland string and compass, and less common ones like DGEs and graphing motion devices.

Chassapis (1998) discusses making a circle with instruments like a circle tracer or a compass, and suggests that both actions and thoughts differ according to the tool used and that each tool leads to a particular reconceptualisation ofthe concept. Beside the users' increasing competence in using a tool and in developing a conception (or sense) for thecircle, the kinaesthetic engagement in the activity also deserves close attention, beyond the procedures they triggerand the meanings they convey. For example, Noble, DiMattia, Nemirovsky and Barros (2006) examine the work of students using a device called a Drawing Machine to createa circle. They account for relationships with the tool that areincorporated bodily and sustained in the experience. While emphasis is often on how the different ways of creating a circle speak directly to certain properties of the concept of circle, *e.g.,* the compass 'embeds' the definition and suggestsdependence of its dimension from its radius (Chassapis, 1998), the micro-perceptual engagement that is at play in drawing or creating a circle is often neglected.

Nevertheless, the actual use of the compass requires attunement to its shape and weight, and this encompasses exploring directions of movement, modulation of speeds, theuse of another hand for controlling the paper, and so forth. Aswe become expert in the use of a tool, these aspects might behidden and a holistic sense permeates the experience, but they still are significant for the creation of the diagram in many ways. Arguing that these dimensions are not accessoryto the creation of meanings for the circle, enmeshed as theyare with the perceptual investments that permeate the coordination between the body and the tool, I propose to study the kinaesthetic and perceptual engagement with a tool as a wayof unravelling some *folds* and *nuances* for the concept of circle. The idea of fold comes from Deleuze's vision of concepts as open-ended and unexhaustive, non-exclusive andunlimited, exterior and infinite (Deleuze & Strauss, 1991). Briefly speaking, for Deleuze, all of the universe is a processof folding and unfolding the outside. Moreover, the fold is not accessible until we unfold it. Then, "unfolding the foldsof a mathematical concept" is the attempt to account for thevirtuality of the mathematical concept. I refer to *folds* for speaking about the emerging properties of concepts that are experienced through movement.

By the term *nuances*, I mean qualities that are analytically disclosed in movement. Speaking of the nuances of a mathematical concept, then, reveals the possibility of investigating its sensuous and intensive dimension, accounting for the qualities that are disclosed in the encounter. Foldsand nuances are a way of describing the texture and dynamics of concepts through their virtuality, in line with de Freitasand Sinclair's (2014) materialist vision of concepts as devices.

Moving as a way of thinking

Among other senses, kinaesthesia is peculiar in that, as a sense, it is distributed across the body and contributes to a recognition of movement as grounding the perception of oneself in the world. Thanks to kinaesthetic engagement we know where our body is, and we feel the direction of our movement, independently from our sight. What is significant about kinaesthesia is not only the identification with a sixth sense, but the fact that, from a physiological perspective, there are several mechanisms that involve receptors, exteroceptors and proprioceptors that belong to this unifyingsense or perception of movement. Sheets-Johnstone (2011)takes on a phenomenological perspective on (human) movement drawing on the concept of kinaesthesia. She argues thatit is not a *positional* sense but a *movement sense*, the experience of which constitutes a specific qualitative dynamic. Byexploring the world in movement, *four primary qualitative structures of movement* are disclosed to the mover: tensional, linear, amplitudinal and projectional qualities. Linear and amplitudinal qualities are related to the spatial aspects of movement, since they capture the direction and extension in space of a movement, whereas tensional and projectional qualities are related to the temporal aspects of movement, since their combination is responsible for the intensive expression of a movement. These qualities are inextricably constitutive of movement. They create a felt qualitative char-acter, which is in fact made up by a constellation of qualitativeaspects and can be only disclosed analytically after the fact (Sheets-Johnstone, 2011). As a consequence, kinaesthetic experiences are not reducible to a mere change in position, instead are a matter of change and variations. To exemplifythis, I propose the reader to engage in an experiment.

Imagine yourself in a corridor of your house. You are standing on your feet; your hands and arms are relaxed on the sides of your body. You start walking. As your right legmoves forward, the left follows and your arms dangle slightly, for a few steps. We now perform *free variations* onthis walking movement. For example, you can walk quickly,or change your speed as you go forward, gradually accelerating or brutally changing rhythm, and all these aspects are instances of the manifold possibilities for the temporality ofthis (and any) movement. There are also manifold possibilitiesregarding the tensional aspects. You can move powerfully, with great tension in your steps; you can clump down the corridor; you can play around with the intensities of your movement, alternating or modulating them as you go for- ward. You can change the ways in which to project force. You can lift your right leg with initial great force, and leaveyour foot touching the ground without control; or you can perform the sequence of steps in a sustained but constant manner. You can also initiate your movement by projectingyour head forward, while the rest of your body moves after (as if you were losing balance) or you can shift forward as ifyour pelvis was initiating and guiding the step, while your torso is dragged along by it. You can similarly vary the movement spatially, in both a linear and amplitudinal sense.You can emphasise the rotatory movement of your legs or zigzag in the corridor; you can make big or little steps and emphasise or restrict your arms' swing.

The above experiment illustrates some dynamic variations that can characterise the movement of a person walking down a corridor and shows how we can turn attention to them through the four qualitative structures we have under- scored. Sheets-Johnstone (2011) proposes that the global kinetic qualitative nature that is experienced in movement constitutes the process of 'thinking in movement'. In this process it is not that thoughts about movements and movements are overlapping, or the former precede the latter. Rather, they compose and co-constitute each other in the process of moving oneself.

To think is first of all to be caught up in a dynamic flow; thinking is itself, by its very nature, kinetic. It moves forward, backward, digressively, quickly, slowly, narrowly, suddenly, hesitantly, blindly, confusedly, penetratingly. What is distinctive about thinking in movement is not that the flow of thought is kinetic, but that the thought itself is. It is motional through and through; at once spatial, temporal, dynamic. (p. 421)

Movement and kinaesthesia are gates for a deep understanding of our being in the world. They expand on a spectrum of variations that are proprioceptively experienced, inform and constitute the texture of meaning-making. Following this line of thought, I propose that investigating the ways in which concepts are generated through movement can give insights on the ways in which we think (of) them. This approach profoundly resonates with the process of perceptuomotor integration that Nemirovsky and colleagues (2013) discuss in terms of gaining fluency with a tool but I want to expand the discourse towards describing how the concept is at play in multimodal engagement.

To study the process of unfolding of the mathematical concept through movement, I try to account for the lived-through dynamic realities of the experience (Sheets-Johnstone, 2011), highlighting nuances that are disclosed in the process, and for the remixing of matter and meaning in mathematical activity (de Freitas & Sinclair, 2014), through the idea of folds. The specific activity involves the drawing of a circle using a graphing motion software named WiiGraph.

WiiGraph and the circle

WiiGraph was developed by a team led by Ricardo Nemirovsky. It leverages two remote controllers, each of which can be held by a person (or both of them can be heldby one person only), to create graphical representations based on the controllers' distance from a LED bar. When theremotes are moved in an interaction space in front of the LED bar, the software creates (in real time) a time versus distance graph for each of them (*Line* modality). *Versus* modality plots an ordered pair of the positions of the two controllers over time, leaving time implicit. *Versus* producesthen a single graph but requires the presence of both remotes. The resulting graph depends on both their movements (vertical displacement in the graph corresponds to oneremote's movement, horizontal displacement to the other remote's movement). A session in *Versus* has no time limitbut can be restarted or toggled to freeze processing. The most interesting challenges with *Versus* graphs demand theproduction of plane figures, like rectangles, rhombuses andcircles that are to be composed by the remotes' movements(Figure 1). These tasks might have a collaborative nature, since, when two users hold the controllers, the task of creating a specific figure implies that they have to coordinate their movements over time in a joint effort (de Freitas, Fer-rara & Ferrari, 2019). Here I focus on a first-person experience with the tool, so that the controllers are both heldby one person (the author) and coordination must be established between hands. Nevertheless, bodily movements underpin drawing experiences with WiiGraph in a fundamental way, since the whole body is involved in the creationof mathematical representations, which are given by real time outputs based on the movements of the remotes.

I draw on my own experience of planning how to make acircle and my subsequent attempts to create it using Wii- Graph (*Versus*). I present an introspective analysis of my personal experience, describing first the planning of movements and then the actual creation of a circle with WiiGraph.The two phases might not be so distinct in the usual way ofhandling the software, however, I separate them here for thesake of clarity. I have no video-recording of the actual experience, so the episode relies on personal memory and re-elaborations made afterwards.

Planning the movements

I hold the two remotes and I stand in front of the sensor bar,facing the computer screen where the software is open, andready to use *Versus*. I decide to try making a circle and I startplanning what I will do. What follows is more or less a paththrough my own thinking of a plan. I know that the circle would emerge from the composition of two parametrical functions that depend on my movements (Figure 2). For example, I would have to move the remotes as if I had to create, at the same time, two sinusoidal curves on a position × time Cartesian plane. Before I start moving, I decide to takeas reference for my movement three different positions in space. That is, I focus on a line that I imagine to be projectedfrom the LED bar to my body and I approximately fix threedifferent but ordinated positions on that line, one close to thesensor, the second far from it and the third halfway. The twoextreme positions would establish and constitute the limits for the remotes' movement. In my planning of action, this division of space would allow me to adjust the relative positions of my hands during my movements. Moreover, it willhelp me modulate speed. For each hand, speed will be at itsmaximum as I get closer to the central position, while it willbe at its minimum as I get closer to one extreme position. Therefore, starting from the extreme position that is close tothe sensor and going far from the sensor, I will be accelerating towards the central position and, once past it, I will begindecelerating towards the farthest extreme position. Both theremotes should follow the same sequence of movements, but

the coordination required would not imply that the remotes should move in an identical manner (i.e., always be in the same position at the same time, or moving at the same speed). In fact, I should take into account that the movements (as the sinusoidal curves) need to be shifted with respect to each other, and that means that (1) I should start with the remotes in different positions and (2) one remote must always be 'chasing' the other, passing through the same positions with some delay. Looking closely at two sinusoidal functions that I have sketched on a sheet of paper,and focusing on the intersections between the two curves, Ithen realise that there will actually exist some instants at which the remotes will be in the same position at the same time. In the meanwhile, the two remotes will meet while myarms are leading their movements towards different directions, that is, swapping their positions.

Creating a circle

At this point I am quite satisfied with the plan I want to implement in my experiment. I stand right in front of the LED bar and the screen, and I place my left hand close to thebar, while my right hand is in the central position (Figure 3).I hold one remote in each hand. In the following, I will describe the actual movement of making a circle with Wii- Graph as I experienced it, given that a graph is being produced in real time by the software while I move. Idescribe the experience paying specific attention to the qualitative dimension of my own movement.

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Since I have to move my arms in different directions, mywhole initial planning presupposed that I would be standingstill in the same position, while my arms are moving on theside of my body, which faces the LED bar. Walking back and forth while preserving the reference positions would be use- less for my purpose and unnecessarily complex. As I start the session, I move my hands with impetus, left chasing right. I move the remotes back and forth two or three times,trying to supervise my arms' movement by looking at the remotes. I am not quite satisfied with the result (a messy shape on the screen) as I soon lose coordination of my hands. Starting a new session, I realise that the coordinationbetween my arms is achieved more easily as I go faster and I avoid looking at the remotes, but I simply focus my gaze on the screen and on the feeling driven by the rotational movement in which I feel caught up. It is as if constraining the remotes' movements along a straight line was limiting my freedom to modulate speed in the course of the remotes' motion. Still, I struggle with keeping the remotes pointed tothe LED bar, which requires a major tension to be sustainedby my arms. I feel immersed in a kind of circular motion, which I perceive as odd, probably since it seems to be morerotational than I had planned, but I enjoy that in its repetition, as I partially achieve the intended complex rhythm. I struggle with maintaining that rhythm, though, since my arms fall into a recursive pattern of alternation after a few oscillations that creates a non-circular shape on the screen. Each time, I do not stop after creating one almost circular trace, but I go on moving, working hard to reach and maintain coordination of my hands. After few trials, the line continues to wrap the initial circle, creating a thicker and jagged line, but more or less circular. My entire body is involved in my arms' movement, as my head and torso also oscillates back and forth. My head follows the leading hand/remote, my torso follows my head smoothly and rhythmically. My legs, which I did not care about in my planning,are at hip distance, with knees slightly bent. Without planningit, I realise that my left leg is positioned slightly in front of theright one, probably to facilitate balance in the asymmetric movement of the upper part of my body. The entire movement alternates smooth phases to abrupt transitions, localized in the extreme positions established in my initial plan, that is, where the leading hand has to change direction and my bodyis split up from a concordant to an odd movement. This element brings forth the asymmetry of my movement, which issustained by the slightly rotational trajectory of my hands, sothat when one arm moves towards the sensor it is also slightlyhigher than when it moves backwards.

The movement, which at the beginning I perceived as (and I planned to be) smooth and happening on a line, has overall different qualities, which I would characterise with unevenness and circularity. Anyway, while moving, I slowlythink of my arms' movements as a unified one. It somehowstands on its own and is distinct from the two individual movements. I feel like this movement is unfolding in harmony with the originating circle on the screen. Looking at the screen and seeing the circle originating is accompanied both with relief and satisfaction and gives me the confidenceto go on following the same rhythm.

As I move, it becomes apparent that some qualitative dynamics of my movement now hide others, or better, that they emerge and are predominant in the entire experience. For example, the chasing of arms in the movement is replaced by the uneven oscillation. The marked positions are not as relevant anymore as they were in the initial planning,as the change in direction is marked by the rhythmic oscillation more than by the actual existence of reference points. Italso appears to me that instead of holding the remotes, I amgrabbing on to them, as if part of my coordination depends on being moving something other than my body, that is nowextending my own body.

Discussion

This experiment is a highly situated experience of a relatively expert person who uses WiiGraph to create a circle.Therefore, I am not arguing that my own experience is trans-ferable to other people, nor that it is the only and unique wayof experiencing the creation of a circular shape with Wii-Graph. Instead, I propose that elucidating details of the kinaesthetic engagement might allow us to study tool use interms of the properties of mathematical concepts as theyemerge out of these distinct kinds of activities. When weturn to movement, we are forced to question what is significant, that is, considering movement *enlarges the field* ofsignificance—a trembling hand, a sudden step might be relevant if we are going down a staircase, holding a cup of coffee.

In my example, I fleshed out my own whole-body effort during the experiment, which was not restricted to the limbs' coordination but rather it was distributed through kinaesthesia among different body parts and senses, like theexperienced variation of tension in the arms, the gaze that focuses more to the screen rather than the arms, the felt circularity of the arms' to-and-fro movement, the trunk's smooth oscillation, and so forth (see Figure 3). I purposely focused on the qualitative emerging structure of my own movement, as a way of enriching the potential meanings forthe circle that emerge from the specific practice of tool use.In particular the *folds*, or the emerging properties of the circle that I experienced through movement, speaks directly toearly expectations from the initial planning and recollections from the past experiences. Starting from knowing howthe sinusoidal functions behave, then choosing the three reference points and thinking and moving as if one hand were 'chasing' the other, constitutes one possible fold, since it relates the circle to the sinusoidal functions via a relation- ship between them (rather than with the circle in the first place). Moreover, it is a huge challenge to achieve a bodilypattern to draw a circle. This might be due to the fact that the required coordination escapes recursive and 'stable' pat-tern of alternation, as it is in the case of the square or rectangle (for which it is possible to just move one remote at a time, at any speed). If we think of a habitual pattern of movement, like walking, we may note that it entails a rather stable alternation and exchange, for each step, betweenwhich leg is carrying the weight, which one is in the front position, and so forth. Instead, the kind of movement that allows for the creation of a circle requires different speeds for each remote (hand) to be combined with varying intensity. The arms are not simply exchanging their positions, but shifting, chasing each other, modifying slowly and ceaselessly the effort and rhythm. This, in turn, speaksdirectly to another fold: the circle as a complex coordination of gradients of speeds, which are responsible for the 'bending' of the line into a curvilinear shape.

I also tried to highlight the *nuances* or qualities that wereexperienced in movement. These are recovered only after the experience and are based on analytical descriptions of pervasive feelings during the graphing session. For example,I evidenced the unevenness and circularity of movement which was striving against the smoothness and linearity in the initial planning, and tensions towards (or falling into) habitual patterns of alternation. These nuances are populatedby affective tones (enjoyment, harmony, relief but also oddity and dissatisfaction) that permeates the experience and sustain the mathematical activity.

I argue that both *folds* and *nuances* are constitutive of theconcept of circle during the activity. They are not already'inscripted' in the concept but instead emerge in/out ofmovement and unfolds meanings for it. As we becomeexpert practitioners of any activity, a sense of unity growsand the movement itself ceases to be inherent the specific practice but stands on its own. After we learn how to walk,the complex bodily coordination that is required to make asimple step is independent from mechanism of explicitthought but adapts itself to the environment. Nevertheless, focussing on the qualities of a movement brings forth peculiarfeelings or perceptions for the mathematical concept of circle. The process of thinking in movement is suffused with,and sustained by, affective bonds, perceptions, surprises andnew discoveries, and every experience creates new mean-ings and possibilities through movement. Concepts are notreality-detached entities but are implicated in the moving hands and the speaking mouth and traversed by streams ofaffect as they flow within and throughout assemblages ofhuman and non-human bodies. More investigations can bedone along this line of thought, widening the discussion about *how* mathematical concepts *matter*.

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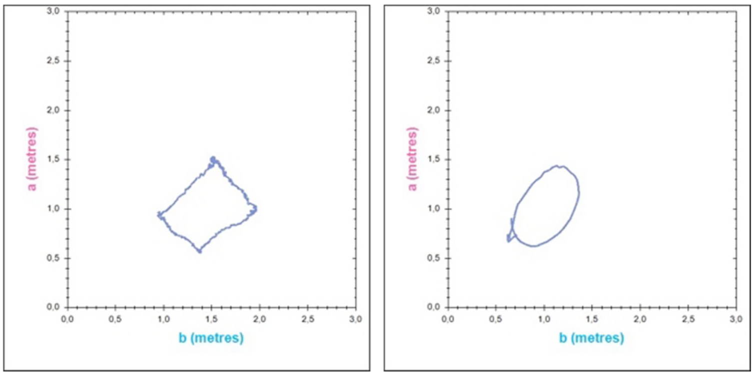

Figure 1. *A rhombus and a circle created in Versus.*

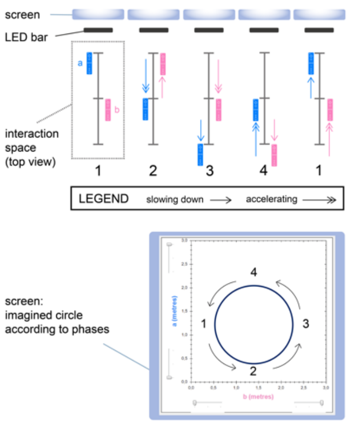

Figure 2. *Drawing a circle in WiiGraph (planning).*

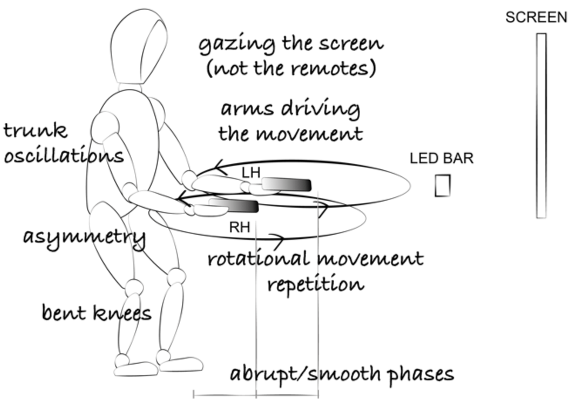

Figure 3. *Creating a circle in WiiGraph, constellation ofqualitative kinaesthetic dynamics*